Matricial Potentiation

By

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Abstract

In this short note we introduce the potentiation of matrices of the same size. We study some simple properties and some example.

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Introduction

We introduce the matrix potentiation. The problem was pose byGuametti.

Consider two matrices$A$, of size$mxn$and B, mxn. We wish to define the potentiation

$$C = A^B$$

For this purpose. We take the logarithm

$$ln \quad C = B \quad ln \quad A,$$

This is valid if $C$and$A$ are not singular. From now on we assome that when we take aln the argument is not singular. The $ln \quad C$ is well defined for the matrices $A$and B real or complex. In the second case we have that it is a multivalued funchon. This assuming that he $ln \quad A$ is a converging sequence. Consider the matrix $D = A - I$ where I stands for the identity matrix. Then

$$ln \quad A = ln \quad (D + I) = D - \frac{D^2}{2} + \frac{D^3}{3} - \frac{D^4}{4} + \frac{D^5}{5} + (-)\frac{{}^{i+1}D^i}{i}$$

From here, it is inmediated that if $m = n$

$$C = e^{B \quad ln \quad A}$$

can be well defined. The only condition that is necessary the convergence of $ln \quad A$, or

$$ln \quad C = B \quad ln \quad A = B \quad [(A - I) - \frac{(A - I)^2}{2} + \frac{(A - I)^3}{3} \dots \dots \dots \dots$$
$$= B \quad \left[\sum_{i=1}^{\infty} (-)^{i+1} \quad \left(\frac{A - I}{i} \right)^i \right] = B \quad \left[\sum_{i=1}^{\infty} \frac{(-)^{i+1}}{i} \quad \sum_{j=0}^{i} \binom{i}{j} \quad A^i (-I)^{i-j} \right]$$

If $A - I$is diagonalizable then

$$A - I = Q^{+1} \quad \Lambda \quad Q^{-1}$$

where the matrix Λ is diagonal with all the eigenvalues in the main diagonal, and Q is formed by the ligenvectors.

Therefore

$$ln \quad C = B \quad ln \quad A = B \quad Q \quad \left[\sum_{i=1}^{\infty} \frac{(-)^{i+1}}{i} \quad \Lambda^i \right] Q^{-1}$$

or

$$C = e^{B \quad ln \quad A} = e^{B \quad Q \left[\sum_{i=1}^{\infty} \frac{(-)^{i+1}}{i} \quad \Lambda^i\right] Q^{-1}} = \sum_{t=0}^{\infty} \frac{1}{t!} [B \quad Q \quad K \quad (A) \quad Q^{-1}]^t$$

where

$$K \quad (A) = \sum_{i=1}^{\infty} \frac{(-)^{i+1}}{i} \Lambda^i$$

We have

$$ln \quad A = \sum_{o=1}^{\infty} \frac{(-)^{i+1}}{i} \sum_{j=0}^{i} \binom{i}{j} \quad (-)^{i-j} \quad A^j$$

and it easy to see that A is diagonalizable in the following way

$$A = Q \quad (\Lambda + I) \quad Q^{-1}$$

Therefore

$$\begin{aligned} ln \quad A &= \sum_{i=1}^{\infty} \frac{(-)^{i+1}}{i} \sum_{j=0}^{i} (-)^{i-j} \quad \binom{i}{j} \quad Q \quad (\Lambda + I)^j \quad Q^{-1} \\ &= \sum_{i=1}^{\infty} \frac{(-)^{i+1}}{i} \quad Q \quad \left\{ \sum_{j=0}^{i} (-)^{i-j} \quad \binom{i}{j} \quad (\Lambda + I)^j \right\} \quad Q^{-1} \end{aligned}$$

As an example if A is diagonal: $A = diag(x_1, \quad x_2, \quad \cdots \cdots, \quad x_m)$ them $A^j = diag(x_1, \quad x_2, \quad \cdots \cdots, \quad x_m)$. replacing we see that $ln \quad A$ is also diagonal

$$\begin{aligned} ln \quad A \quad (r,r) &= \sum_{i=1}^{\infty} \frac{(-)^{i+1}}{i} \sum_{j=0}^{i} (-)^i \quad \binom{i}{j} \quad (\lambda_r + 1)^j = \sum_{i=1}^{\infty} (\lambda_r)^i \quad \frac{(-)^{i=1}}{i} \\ &= \sum_{i=1}^{\infty} (\lambda_r + \downarrow - 1)^i \quad \frac{(-)^i}{i} = lm \quad \lambda_s \end{aligned}$$

$$ln \quad A \quad (r,s) = r \neq s$$

them$ln \quad A$ is diagonal and it is converging

$$ln \quad \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \lambda_m \end{pmatrix} = \begin{pmatrix} ln \quad \lambda_1 & & 0 \\ & ln \quad \lambda_2 & \\ 0 & & ln \quad \lambda_m \end{pmatrix}$$

Next case, we have that A is diagonalizable

$$A = P \quad \Lambda \quad P^{-1}$$

Them the

$$ln \quad A=\sum_{i=1}^{\infty}\frac{(-)^{i+1}}{i}p^{+1}\Lambda^{i}p^{-1}=p^{+1}\sum_{i=1}^{\infty}\frac{(-)^{i=1}}{i}(\Lambda)^{i}p^{-1}=p^{-1}\left[\sum_{i=1}^{\infty}\frac{(-)^{i+1}}{i}\quad(\Lambda)^{i}\right]p^{-1}$$

But$Q=p^{1}$

$$Q\Lambda^{i}=\lambda_1^{i}\begin{bmatrix}q_{11}\\q_{21}\\q_{m1}\end{bmatrix}+\lambda_2^{i}\begin{bmatrix}q_{12}\\q_{22}\\q_{m2}\end{bmatrix}+\cdots=\sum_{r=1}^{n}\lambda_r^{i}\begin{bmatrix}q_{1r}\\q_{2r}\\q_{nr}\end{bmatrix}$$

therefore

$$Q\Lambda \quad P \quad (r,s)=\sum_{k=1}^{n}\lambda_k\, q_{rk}p_{ks}$$

and replacing into equation (1),it twins out that

$$\begin{aligned}(ln \quad A)\quad(r,s)&=\sum_{i=1}^{\infty}\frac{(-)^{i+1}}{i}\quad\left(\sum_{k=1}^{n}(\lambda_k-1)\right)^{i}\quad q_{rk}\quad p_{ks})\\&=\sum_{i=1}^{\infty}\frac{(-)^{i+1}}{i}\quad\left(\sum_{k=1}^{n}(\lambda_k-1)^{i}\right)\quad q_{rk}\quad p_{ks}\\&=\sum_{k=1}^{n}q_{rk}\quad\left[\sum_{i=1}^{\infty}\frac{(-)^{i+1}}{i}\quad(\lambda_k-1)^{i}\right]\quad p_{ks}=\sum_{k=1}^{n}q_{rk}\quad(ln \quad \lambda_k)\quad p_{ks}\\&=[p^{+1}\quad(ln \quad \Lambda)\quad p^{-1}]\quad(r,s)\end{aligned}$$

As a conclusion we have for $A=p^{+1}\quad\Lambda\quad p^{-1}$ diagonalizable with the eigenvalues for Fubini$0<\lambda_r<2$ cocient

$$ln \quad A=p^{+1}\quad(ln \quad \Lambda)\quad p^{-1}$$

and

$$ln \quad \Lambda=\left(\begin{pmatrix}ln \quad \lambda_1 & & c\\ & \ddots & \\ 0 & & ln \quad \lambda_n\end{pmatrix}\right)$$

Then we have proved the following result.

<u>Theorem</u>: Given two diagonalizables matrices $mxn: A$ and B , where $A-I$ has eigenvalues$x_r: 0<x_r$, then the matrix

$$C=A^{B}$$

is well defines and it has the form

$$C = \sum_{t=0}^{\infty} \frac{(B\bar{A})^t}{t!}$$

where

$$\bar{A} = ln \quad A$$

This we have been successful in the definition of matrix potentiation.

<u>Properties</u>

In this section we are going to for diagonalizable matrices some the first one, already proved is

$$A^B \leftrightarrow exp(B \quad ln \quad A)$$

Now we study

$$(A^B)^C$$

We have

$$(A^B)^C = D^C = exp(C \quad ln \quad D) = exp(C \quad ln \quad A^B) = exp(C \quad B \quad ln \quad A) = A^{CB}$$

wher$D = A^B$.

On the other hand

$$A^B A^C = exp(B \quad ln \quad A)exp(C \quad ln \quad A) = exp(B \quad ln \quad A + C \quad ln \quad A)$$
$$= exp((B + C) \quad ln \quad A) = A^{B+C}$$

We follow with

$$exp(ln \quad A) = A$$

We know

$$e^x = \sum_k \frac{x^k}{k^1}$$

an if A_h^k diagonalizable we have

$$ln \quad A = P \quad ln \quad \Lambda \quad p^{-1}$$

where the diagonal matrix Λ is formed by the eigenvalues by the eigevector as columns. Then

$$ln \quad (\exp A) = ln \quad \left(\sum_{k=0}^{\infty} \frac{P \quad \Lambda^k \quad p^{-1}}{k!}\right) = ln \quad \left(P \quad \sum_{k=0}^{\infty} \frac{\Lambda^k}{k!} \quad P^{-1}\right)$$
$$= \sum_{k=1}^{\infty} \frac{(-)^{k-1}}{k} (P^{-1} \quad exp(\Lambda) \quad P)^k = \sum_{k=1}^{\infty} \frac{(-)^{k-1}}{k} \quad P^{-1} \quad (exp\Lambda)^k \quad P$$
$$= P^{-1} \left(\sum_{k=1}^{\infty} \frac{(-)^{k-1}}{k} \quad (exp\Lambda)^k\right) P = P^{-1} \quad \Lambda \quad P = A$$

and in this way we have proved the property.

On the other hand, we have, another basic property, for diagonalizable matrices namely.

Others properties are

$$exp(A+B) = \exp A \exp B$$

Consider

$$e^{(A+B)} = \sum_{k=0}^{\infty} \frac{(A+B)^k}{k!} = \sum_{k=0}^{\infty} \sum_{t=0}^{k} \frac{\binom{k}{t} \quad A^t \quad B^{k-t}}{k!} = \sum_{k=0}^{\infty} \sum_{t=0}^{k} \frac{A^t \quad B^{k-t}}{t! \quad (k-t)!}$$

And now by Fubini property and the standardamegament

$$= \sum_{i=0}^{\infty} \sum_{k=t}^{\infty} \frac{A^t \quad B^{k-t}}{t! \quad (k-t)!}$$

and by a variable change $k - t = j$ and $t = i$ the

$$= \sum_{i=0}^{\infty} \sum_{k=t}^{\infty} \frac{A^i \quad B^j}{i! \quad j!} = \left(\sum_{i=0}^{\infty} \frac{A^i}{i}\right) \quad \left(\sum_{j=0}^{\infty} \frac{B^j}{j!}\right) = \begin{matrix} \exp A \\ \exp B \end{matrix}$$

and is this way the property is proved.

On the other hand, we now prove

$$ln \quad (AB) = ln \quad A + ln \quad B$$

always in the case that $AB = BA$, or they comments let

$$ln \quad (AB) = C$$

$$\exp C = \exp(ln \quad AB) = AB$$

On the other hand if we call

$$D = ln \quad A + ln \quad B$$

then

$$\exp D = \exp(ln \quad A + ln \quad B) = \exp(ln \quad A) . \exp(ln \quad B) = AB$$

then $C = D$.

Now we present

$$A^B A^C = A^{B+C}$$

Let

$$A^B A^C = \exp(B \quad ln \quad A) . \exp(C \quad ln \quad A) = \exp(B \quad ln \quad A + C \quad ln \quad A)$$
$$= \exp((B + C) \quad ln \quad A)) = \exp ln \quad (A)^{B+C} = A^{B+C}$$

Next

$$(A^B)^C = A^{B+C}$$

consider the first term. Calling $D = A^B$ then

$$(A^B)^C = D^C = \exp(C \quad ln \quad D) = \exp(C \quad ln \quad A^B) = \exp(C \quad B \quad ln \quad A) = A^{CB}$$

Next we consider a property about the determining namely

$$\det(ln \quad A) = \det(ln \quad \Lambda)$$

When $A = P^{-1} \quad \Lambda \quad P$ which is inmediate by the decomposition.

Consider the transperise of $A : A^t$, then

$$\exp(A^t) = (\exp A)^t$$

Let $A = P^{-1} \quad \Lambda \quad P$ then

$$A^t = (P^{-1} \quad \Lambda \quad P)^t = P^t \quad \Lambda \quad (P^{-1})^t = P^t \quad \Lambda \quad (P^t)^{-1}$$

On the other hand

$$\exp(A^t) = \sum_{i=0}^{\infty} \frac{(A^t)^{-1}}{i!} = P^t \quad \left(\sum_{i=0}^{\infty} \frac{\Lambda^i}{i!}\right) \quad (p^t)^{-1} = \left(P^{-1} \quad \sum_{i=0}^{\infty} \frac{\Lambda^i}{i!} \quad P\right)^t = (e^A)^t$$

Now we consider another property, namely

$$\det \exp A = \exp \operatorname{tra} \Lambda$$

Let

$$e^{A}=P^{-1}\quad\left(\sum_{k=0}^{\infty}\frac{\Lambda^{i}}{i!}\right)\quad P$$

$$\det(e^{A})=\det\left(\sum_{i=0}^{\infty}\frac{\Lambda^{i}}{i!}\right)=\det(e^{\Lambda})=\prod_{j=1}^{n}e^{\lambda i}=e^{\lambda_1}e^{\lambda_2}\ldots\ldots\ldots e^{\lambda_n}=e^{\lambda_1+\lambda_2\ldots\ldots+\lambda_n}=e^{\operatorname{tra}\Lambda}$$
$$=e^{\operatorname{tra}A}$$

where tra, indicate the trace of the matriz.

Now we consider an example.

$$\begin{pmatrix}\displaystyle\sum_{i=1}^{\infty}\frac{(-)^{i-1}}{i} & \displaystyle\sum_{i=1}^{\infty}\frac{(-)^{i-1}}{i}\quad\left(\frac{1}{2}\right)^{i}\\ \displaystyle 2\sum_{i=1}^{\infty}\frac{(-)^{i-1}}{i} & \displaystyle\sum_{i=1}^{\infty}\frac{(-)^{i-1}}{i}\quad\left(\frac{1}{2}\right)^{i}\end{pmatrix}=ln\quad C=\begin{pmatrix}0 & ln\quad {}^{1}/_{2}\\ 0 & ln\quad {}^{1}/_{2}\end{pmatrix}$$

Ej. 2

$$I+\frac{ln\quad C}{1!}+\frac{(ln\quad C)^{2}}{2!}+\frac{(ln\quad C)^{3}}{3!}$$

$$\begin{pmatrix}0 & ln\quad {}^{1}/_{2}\\ 0 & ln\quad {}^{1}/_{2}\end{pmatrix}\begin{pmatrix}0 & ln\quad {}^{1}/_{2}\\ 0 & ln\quad {}^{1}/_{2}\end{pmatrix}=\begin{pmatrix}0 & (ln\quad {}^{1}/_{2})^{2}\\ 0 & (ln\quad {}^{1}/_{2})^{2}\end{pmatrix}$$

$$(ln\quad C)^{n}=\begin{pmatrix}0 & (ln\quad {}^{1}/_{2})^{n}\\ 0 & (ln\quad {}^{1}/_{2})^{n}\end{pmatrix}\begin{pmatrix}0 & (ln\quad {}^{1}/_{2})^{n}\\ 0 & (ln\quad {}^{1}/_{2})^{n}\end{pmatrix}\begin{pmatrix}0 & ln\quad {}^{1}/_{2}\\ 0 & ln\quad {}^{1}/_{2}\end{pmatrix}$$
$$=\begin{pmatrix}0 & (ln\quad {}^{1}/_{2})^{nH}\\ 0 & (ln\quad {}^{1}/_{2})^{nH}\end{pmatrix}$$

$$C=e^{ln\quad C}=I+\begin{pmatrix}0 & ln\quad {}^{1}/_{2}\\ 0 & ln\quad {}^{1}/_{2}\end{pmatrix}+\begin{pmatrix}0 & (ln\quad {}^{1}/_{2})^{2}\\ 0 & (ln\quad {}^{1}/_{2})^{2}\end{pmatrix}$$

$$\begin{pmatrix} 1 & \sum_{n=1}^{\infty} \frac{\left(ln \quad {}^{1}\!/_{2}\right)^{n}}{n^{1}} \\ 0 & \sum_{n=1}^{\infty} \frac{\left(ln \quad {}^{1}\!/_{2}\right)^{n}}{n^{1}} \end{pmatrix} = \begin{pmatrix} 1 & e^{ln \quad {}^{1}\!/_{2}} \overline{-1} \\ 0 & {}^{1}\!/_{2} \end{pmatrix}$$

$$1 + \sum = e^{ln \quad {}^{1}\!/_{2}} \qquad = \begin{pmatrix} 1 & -{}^{1}\!/_{2} \\ 0 & {}^{1}\!/_{2} \end{pmatrix}$$

<u>Example</u>

We wish to comput

$$C = A^{B}$$

Consider as example

$$A = \begin{pmatrix} 1 & 2 \\ 0 & {}^{1}\!/_{2} \end{pmatrix} \text{and} B = \begin{pmatrix} 1 & 0 \\ 2 & 2 \end{pmatrix}$$

knouing

$$ln \quad C = B \quad ln \quad A = B \quad ln \quad \mathrm{P} \quad \Lambda \quad P^{-1} = B \quad P \quad ln \quad \Lambda \quad P^{-1}$$

them

$$ln \quad \Lambda = \begin{pmatrix} ln \quad 1 & 0 \\ 0 & ln \quad {}^{1}\!/_{2} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & -ln \quad 2 \end{pmatrix}$$

and where

$$P = \begin{pmatrix} 1 & -4 \\ 0 & 1 \end{pmatrix}$$

Therefore

$$ln \quad A = P \quad ln \quad \Lambda \quad P^{-1} = \begin{pmatrix} 0 & ln \quad 16 \\ 0 & -ln \quad 2 \end{pmatrix}$$

hence

$$ln \quad C = B \quad ln \quad A = \begin{pmatrix} 1 & 0 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} 0 & ln \quad 16 \\ 0 & -ln \quad 2 \end{pmatrix} = \begin{pmatrix} 0 & ln \quad 16 \\ 0 & ln \quad 64 \end{pmatrix}$$

from here

$$C = e^{B \quad ln \quad A} = \sum_{I=0}^{\infty} \frac{(B \quad ln \quad A)^i}{i!} = \sum_{i}^{\infty} \frac{1}{i!} \begin{pmatrix} 0 & 2^{i+i} & 3^{i-1} & ln & 2 \\ 0 & & 3^i & ln & 2 \end{pmatrix}$$

$$= ln \quad 2 \begin{pmatrix} 0 & \sum_{i}^{\infty} \frac{2 \quad 2^i \quad 3^i \quad 3^{-1}}{i1} \\ 0 & \sum_{i}^{\infty} \frac{3^i}{i1} \end{pmatrix}$$

$$C = \begin{pmatrix} 0 & {}^{2}/_{3} & e^6 ln & 2 \\ 0 & e^3 & ln & 2 \end{pmatrix}$$

Bibliografía

Burden R.L. and J. DougasFaires: AnálisisNumérico. Congage 2001.

Poole D: Algebra lineal. Thompson 2004.

Wayne L. Winson: Investigación de Operaciones 4ta Ed. Thompson 2004.

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