REPRESENTATION OF RATIONAL NUMBERS THROUGH A LISSAJOUS'S GEOMETRIC FIGURE

by

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In this study we will analyze the properties of a geometric form of rational numbers representation.

We will see that there is a bi- univocal correspondence between each rational and its representative figure.

We will divide the set of rational numbers in four subsets one of them is reducible to some of the remaining subsets, which are different each other by a fundamental property of the shape of the terminal point.

We will also study other characteristics of this representation geometric system which we would be enhanced when they are necessary.

Given an orthogonal Cartesian axis system, and a non null arbitrary segment S_u, we define the rational number $\frac{n_i}{d_i}$ in the following way.

Take n_i in the ordinate and in the abscissa d_i consecutive times of S_u, name the extremes with B and A, and with $y_1, y_2, \dots, y_i$ and $x_1, x_2, \dots, x_i$, the determined points over each axis respectively. We will refer to the last ones as cuts and we will symbolize them as C_r.

From A, B and C_r, we will trace the normals which will form a net. That net will be composed by $m_i \, x \, d_i$ elemental squares. The normals in B and A have the intersection in the point C.

We will name E(corners) to the points A, B, C and C_r(crosses) to the intersections of the remaining normals.

The square diagonal divides the angles, in which it has their extremes, into equal parts that have a measure of $\frac{\pi}{4}$ radians each.

If we based us on this property and trace a line with an angle of $\frac{\pi}{4}$ radians from the origin0, this will pass for the elemental squares which belong to the quadrant bisectrix, and it will effect $\overline{BC}, \overline{AC}$ or C, according to $n_i < d_i$, $n_i > d_i$, $n_i = d_i$ respectively.

In the last case, the line will finish in C and will be the geometric representation of $\frac{n_i}{d_i} = l$.

In the remaining cases, every time the line find a C we will apply the reflection or mirror law, the times that were necessary to reach consequently an E. Then it will be formed a figure that will be defined as a geometric representation of this rational number.

One particularity of the shape of this figure is that it has been formed as follows:

Theorem 1: Every $\frac{n_i}{d_i}$ rational number which has been traced from 0, finished in one E.

Proof: Since the proof is clean by considering geometric arguments we will not give the analytical one.

The following consequence is derived from the previous theorem:

The line cannot pass through a crossing with the same slope twice.

A question appears now: Is any number able to have more than one representation?

We find the answer as follows:

Theorem 2: Every rational number has only one figure.

Proof: Given that the first diagonal is unique, we obtain only one direction afterC_1 by the mirror law. If we repeat this procedure many times as it is necessary in a net that is unique by construction, we will obtain only one shape because all of their consecutive elements are unique.

Consider two rational numbers at random that are not equal between them, so they have different nets and then their representative shapes will be different too. In this way, it is proved the following result:

Theorem 3: Two unequal rational numbers have different shapes.

Caution: All of non equal equivalent rational numbers have a shape with the same form but with different dimensions.

We are ready to state the:

Theorem 4: One figure can be the geometric representation of only one couple of values $n_i$and d_i.

Proof: Suppose that a shape represent two different rational numbers. Different traces correspond to that numbers, as we proved in the previous theorem. We are in front of an absurd which proceed from a false assumption. □

Because of the analysis we have made, we can say that there is bi- univocal correspondence between the rational numbers and their representation.

If we number the consecutive S_u over each axis, we obtain the $S_{xi}$and S_{yi} that will give the position for all the C_c.

Then it will result the types $S_p$and S_l, according to the sums of the values $S_{xi} + S_{yi}$of each C_e, even or odd respectively.

It is obvious that if two C_c have a common side, they belong to different types, and if they have only one vertex in common so they belong to the same type.

We will see a useful property of the exposed types in the:

Theorem 5: In the C_e the type S_p, the diagonal slope which passes through them is positive, and in the S_l is negative.

Proof: Take a net as in the shape, the representation of any rational, by definition, starts in 0 and the first trace will be the quadantbisectrix and will belong to the type S_p, and we already know that the slope of the line is positive. When we reach one side of the net, we can see that the line changes from the S_p to the S_l because it passes through two C_e which has one side in common, in that point the line slope also changes by the mirror law.

We have seen the change of the slope is accomplished by the type change of the C_e. If we apply two consecutive times, the change we will come back to the slope and type of previous C_e.

As in the S_p the slope is positive, because comes always from the first trace, in the S_l the slope will be lower than zero.

Then the theorem is demonstrated.

We called nodes and designate with N to the 0 set, the C_i, the C_r and the E. Subdivide the last one in N_p and N_l types, according to the sum of the coordinates values, for each one, result even or odd.

We will define as vertices the following points: of origin, of change of slope and the line terminal corner.

We will see that property have the V which is exposed in the:

Theorem 6: The V of any shape belong to $N_p$type.

Proof: It is evident that following a normal to any of the axis, we will find N_p and $N_l$alternatively.

If, on the other hand, we will follow a diagonal and apply in each C_l the mirror law, we will find only $N_p$and N_l.

As the starts in 0, that is a N_p, then all of V will result N_p.

A consequence of the theorem is that the shape does not pass for any N_l.

We will divide the rational numbers set in to three subsets, according to the values of$\frac{n_i}{d_i}$ are for:

$$\alpha\text{: } odd/even$$

$$\beta\text{: } even/odd$$

$$\gamma\text{: } odd/even$$

Eliminate the fourth even/ even set because this is always reducible to some of the previous ones.

It is easy to see the only E which belongs to the N_p in $\alpha$is A; in β is B; and in γ is C. This is, as we said, due to the type of each N is given by the even or odd quality of their co-ordinates sum.

Theorem 7: The final point of the rational number geometric representation which belong to α subset A.

Proof: We have just seen that all the N which belong to a shape are N_p, and as the only E of this type which is in this subset is the A, the line have to conclude necessarily at this point.

In the same way, the following theorems are proved.

Theorem 8: The determination of the shape to one βsubset rational number is B.

Theorem 9: The terminal corner of every γsubset representation is C.

The three previous theorems have implicit the symmetry form which differentiate them.

αsubset has, as we demonstrated, its final point in the A corner and we can anticipate without a strictnees demonstration, that these shapes will have their symmetry axis in the middle of the $\overline{OA}$ segment and it will be parallel to the ordinate.

In the subconjunctβ, which has ending in B, the symmetry axis is parallel to the abscissa and cuts to $\overline{OB}$ by the middle.

In the remaining γ, as 0 and C are symmetry respect to the net geometric centre, the subset shapes will not have an axis but a symmetry centre, which will coincide with the net centre.

Now we will see the relationship of some net and shape components.

Call V_x to the vertices of the sides $\overline{OA}$ and $\overline{BC}$, and 0 to the origin point; and V_y to the conjunct of the sides $\overline{OB}$and $\overline{AC}$, and to the shape terminal point, we will have that:

$$N - C_r = 2(n_i + d_i) = 2n_i + 2d_i$$

From that, it results:

$$V_x = d_i$$

$$V_y = n_i$$

only if we consider from the two remaining E that one belongs to n_i and the other to d_i.

Besides, as $V = V_x + V_y$result $V = n_i + d_i$, which does not need demonstration because it is evident.

From the previous one, we obtain:

$$C = N - 2v$$

and as $N = (n_i + 1):(d_i + 1)$we will have

$$C = (n_i + 1):(d_i + 1) - 2(n_i + d_i)$$

$$C = (n_i + 1):(d_i - 1)$$

And if we name $X = (n_i - 1), Y = (d_i - 1)$we have

$$C = X{:}Y$$

That is to say, the cross is equal to the numerator minus one, which is multiplied by the denominator, minus the unit.

If we define Zas the shape's external right angles (all of the scheme), we can divide them into the Z_y with the opposite side parallel to the ordinate, the $Z_x$which are opposite to that axis and its parallel.

We see that $Z_y = n - 1$, and $Z_x = d - 1$; then

$$Z = n_i + d_i - 2$$

and being $V = n_i + d_i$ it results

$$Z = V - 2$$

Designate C' to the points where the line is cut itself, and A' to the square that form the same shape inside them and will observe that:

$$A' = C'$$

Because of they are mutually corresponded, although what was said does not implicate a demonstration.

As C' corresponds to the C even class, it will maintain the relation that was stated before.

$$C' = \frac{C_r}{2} = A'$$

We could go away in the relations of the entities that we worked with, but we will not do it because they are arithmetic relations which are out of the work object.

All that has been exposed previously is acceptable to the negative rational numbers if it is worked in the second and fourth quadrant.

Also this study is realizable for a different angle of $\frac{\lambda}{4}$ radians, f the unitary segments which are located in the two axis are different; but considering the angle tangent as the relation between the different unitary segments, the whole theory will continue valid. It is possible to generalize what was transcribed. Instead of to use the reflection law, it could be employed another law where the incidence angle and the reflection angle are leagued by any function.

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