CHARACTERIZATIONS OF MORSE QUASI-GEODESICS VIA SUPERLINEAR DIVERGENCE AND SUBLINEAR CONTRACTION

GOULNARA N. ARZHANTSEVA, CHRISTOPHER H. CASHEN, DOMINIK GRUBER, AND DAVID HUME

Abstract. We introduce and begin a systematic study of sublinearly contracting projections.

We give two characterizations of Morse quasi-geodesics in an arbitrary geodesic metric space. One is that they are sublinearly contracting; the other is that they have completely superlinear divergence.

We give a further characterization of sublinearly contracting projections in terms of projections of geodesic segments.

1. Introduction

This paper initiates a systematic study of *contracting projections*. The aim is to clarify and quantify ways in which a subspace of a geodesic metric space can 'behave like' a convex subspace of a hyperbolic space.

The definition of hyperbolicity captures the notion that a space is uniformly negatively curved on all sufficiently large scales. Following Gromov's seminal paper [21], hyperbolic groups and spaces have been intensively studied and many generalizations of this notion have been considered.

One particular collection of ideas focus on finding 'hyperbolic directions', geodesics that have some of the features exhibited by geodesics in hyperbolic spaces, for instance, those that satisfy the *Morse lemma*, have *superlinear divergence* or satisfy some *contraction* hypothesis. These ideas find application to Mostow rigidity in rank 1 [29], the Rank Rigidity Conjecture for CAT(0) spaces [4, 8, 11], and hyperbolicity of the curve complex of a hyperbolic surface [24, 22]. Recently, the concept of *strongly contracting projection* has been a topic of intense interest in relation to mapping class groups and outer automorphisms of free groups [1, 7], acylindrically hyperbolic groups [16, 28], and contracting/Morse boundaries [30, 31, 13, 14, 25].

We introduce a more general version of contracting projection than has been previously studied. Our main result is that this new version of contraction is equivalent to the Morse property and to a certain superlinear divergence property. We give quantitative links between these various geometric properties. We also generalize several fundamental theorems about stronger versions of contraction to our new, more general, context.

In this paper we establish fundamental results in a very general setting, so that they will be broadly applicable. Indeed, the novel version of contracting projections we introduce here is essential in a subsequent paper [3], in which we explore the geometry of finitely generated graphical small cancellation groups, a class that includes the Gromov monster groups as notorious examples. In that paper we engineer finitely generated groups with Cayley graphs that mimic the surprising geometry of our examples from Section 3. In particular, the new spectrum of contracting behaviors in geodesic metric spaces that we discover here does appear in the setting of Cayley graphs of finitely generated groups. We also, in [3], use the equivalence between sublinear contraction and the Morse property established here in Theorem 1.4 to characterize Morse geodesics in certain families of graphical small cancellation groups.

Since the preprint version of this article appeared there have already been other applications of our results, including work of Cordes and Hume [15] and Cashen and Mackay [12] on Morse boundaries of

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finitely generated groups and work of Aougab, Durham, and Taylor [2] on cocompact subgroups of mapping class groups and $\mathrm{Out}(F_n)$.

We give detailed introductions to the three main geometric properties in Sections 1.1, 1.2, and 1.3 and make precise statements of our results in Sections 1.4 and 1.5.

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1.1. **Contracting projections.** Let Y be a subspace of a geodesic metric space X, and let $\epsilon \geqslant 0$. The *ϵ–closest point projection of X to Y* is the map $\pi_Y^\epsilon \colon X \to 2^Y$ sending a point $x \in X$ to the set:

$$\pi_Y^\epsilon(x) := \{y \in Y \mid d(x,y) \leqslant d(x,Y) + \epsilon\} \subset Y$$

We do not assume the sets $\pi_Y^\epsilon(x)$ have uniformly bounded diameter. Note that given any $x \in X$, $\emptyset \neq Y \subset X$, and $\epsilon > 0$, the set $\pi_Y^\epsilon(x)$ is non-empty.

Definition 1.1. The ϵ–closest point projection $\pi_Y^\epsilon \colon X \to 2^Y$ is (ρ_1, ρ_2)*–contracting* if the following conditions are satisfied.

- The empty set is not in the image of π_Y^ϵ.
- The functions[1] ρ_1 and ρ_2 are non-decreasing and eventually non-negative.
- The function ρ_1 is unbounded and $\rho_1(r) \leqslant r$.
- For all x, $x' \in X$, if $d(x,x') \leqslant \rho_1(d(x,Y))$ then:
$$\operatorname{diam} \pi_Y^\epsilon(x) \cup \pi_Y^\epsilon(x') \leqslant \rho_2(d(x,Y))$$
- $\lim_{r\to\infty} \frac{\rho_2(r)}{\rho_1(r)} = 0$.

We say that Y *is* (ρ_1,ρ_2)*–contracting* if there exists $\epsilon \geqslant 0$ such that π_Y^ϵ is (ρ_1,ρ_2)–contracting, see Definition 6.4. We say a collection of subspaces $\{Y_i\}_{i\in\mathcal{I}}$ is *uniformly contracting* if there exist ρ_1 and ρ_2 such that for all $i \in \mathcal{I}$, the subspace Y_i is (ρ_1,ρ_2)–contracting.

The rough idea is that, asymptotically as x gets far from Y, if B is a ball centered at x and disjoint from Y then the diameter of its projection is negligible compared to the diameter of B. More accurately, this is true at a specific scale — when the radius of B is $\rho_1(d(x,Y))$. We claim no finer control of the projection diameter when B has smaller radius.

For a simple, but conceptually useful, example, consider a circle X and an arc $Y \subset X$. Take $\rho_1(r) := r$, and let ρ_2 be the constant function whose value is the distance between the endpoints of Y. Then π_Y^0 is (ρ_1,ρ_2)–contracting. There is a unique point $x \in X$ farthest from Y. The ball B of radius $\rho_1(d(x,Y))$ about x is all of $X \setminus Y$, and $\pi_Y^0(B) = \pi_Y^0(x)$ has diameter $\rho_2(d(x,Y))$.

The simplest example that is not (ρ_1,ρ_2)–contracting for any choice of ρ_1 and ρ_2 is to take X to be the Euclidean plane and take Y to be a geodesic. Then the diameter of π_Y^0 of any ball is equal to the diameter of the ball, so we cannot satisfy $\lim_{r\to\infty} \frac{\rho_2(r)}{\rho_1(r)} = 0$.

The simplest contracting example with Y unbounded is to take X to be a tree and Y to be an unbounded convex subspace. Then $\operatorname{diam} \pi_Y^0(B_{d(x,Y)}(x)) = 0$ for every x, so π_Y^0 is $(r,0)$–contracting. In more general δ–hyperbolic spaces, ϵ–closest point projection to a geodesic is (r,D)–contracting for some D depending only on δ and ϵ. Such a case, when $\rho_1(r) := r$ and ρ_2 is bounded, is called *strongly contracting.*

[1] The term 'function' always refers to a real valued function whose domain, unless otherwise noted, is the non-negative real numbers.

Pseudo-Anosov axes in Teichmüller space are strongly contracting [24], as are iwip axes in the Outer Space of the outer automorphism group of a free group [1] and axes of rank 1 isometries of CAT(0) spaces [4, 8].

We say that π_Y^ϵ is *semi-strongly contracting* if it is (ρ_1, ρ_2)–contracting for $\rho_1(r) := r/2$ and ρ_2 bounded. Related notions have been considered in the context of the mapping class group of a hyperbolic surface [22, 5, 19].

We say that π_Y^ϵ is *sublinearly contracting* if it is (ρ_1, ρ_2)–contracting for $\rho_1(r) := r$. In this case the definition implies ρ_2 is a sublinear function, see Definition 2.1. Similarly, π_Y^ϵ is *logarithmically contracting* if it is (ρ_1, ρ_2)–contracting for $\rho_1(r) := r$ and ρ_2 logarithmic.

A schematic diagram of different contracting behaviors is given in Figure 1. A wide range of examples are presented in Section 3.

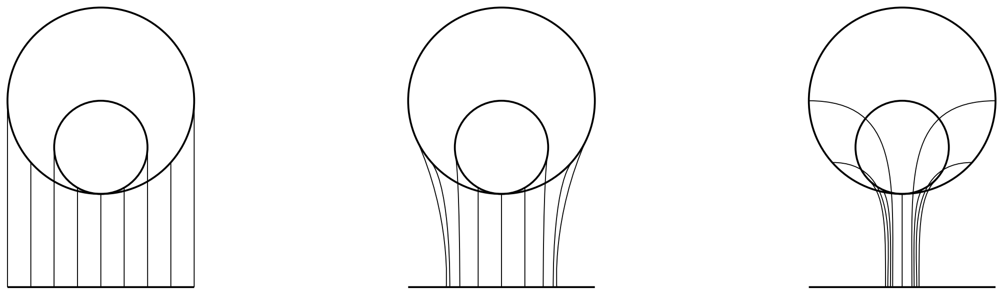

FIGURE 1. Types of contraction

1.2. **The Morse property.**

Definition 1.2. A subspace Y of a geodesic metric space X is μ*–Morse* for a function $\mu\colon [1,\infty) \times [0,\infty) \to [0,\infty)$ if for every $L \geqslant 1$ and $A \geqslant 0$, every (L, A)–quasi-geodesic γ with endpoints on Y remains within distance $\mu(L, A)$ of Y.

The subspace Y is called *Morse*, or is said to *have the Morse property*, if it μ–Morse for some function μ. A collection of subspaces $\{Y_i\}_{i\in\mathcal{I}}$ is said to be *uniformly Morse* if there exists a function μ such that for all $i \in \mathcal{I}$ the subspace Y_i is μ–Morse.

Morse quasi-geodesics have been intensively studied: they play a key role in boundary theory for hyperbolic and relatively hyperbolic groups. Recently, the Charney school [30, 31, 13, 14, 25] has been generalizing such boundary theories to arbitrary proper geodesic metric spaces using the so called 'Morse boundary' consisting of asymptotic equivalence classes of Morse rays.

Morse quasi-geodesics have been characterized[2] in terms of cut-points in asymptotic cones [17]: a quasi-geodesic q in X is Morse if and only if every point x in the limit $\mathfrak{q}$ of q in any asymptotic cone $\mathcal{C}$ of X is a cut-point separating ends of $\mathfrak{q}$; that is, $\mathcal{C} \setminus \{x\}$ has at least two connected components containing points of $\mathfrak{q}$. Cut-points in asymptotic cones are a key element of the proof of the quasi-isometry invariance of relatively hyperbolic (asymptotically tree-graded) spaces [18]. It remains a very important open question to determine whether a space in which every asymptotic cone admits a cut-point necessarily admits a Morse quasi-geodesic.

As a result, it is of great interest to find and classify Morse quasi-geodesics. If a solvable group admits a Morse quasi-geodesic then it is virtually cyclic, and the same holds for any other group satisfying a non-trivial law, for instance, a torsion group with bounded exponent [18]. At the other extreme, every quasi-geodesic in a hyperbolic space is Morse. There are non-trivial classifications of Morse quasi-geodesics for relatively hyperbolic groups [27] and CAT(0) spaces [4, 8, 31]. We use the tools of this paper to perform such a classification for graphical small cancellation groups in [3].

[2]See also the related "middle recurrence" characterization of the Morse property in [17].

1.3. **Divergence.** Closely related to the study of Morse quasi-geodesics is the notion of *divergence*. The definition is technical, so we postpone it until Definition 5.1. The idea is that the divergence of a quasi-geodesic γ in a space X is a function whose value at r is the minimal length of a path in X circumventing a ball of radius r centered on γ. In our version of divergence we allow the forbidden ball to be centered at different points of γ for different values of r. Some authors require the balls to have fixed center at $\gamma(0)$.

Morse geodesics were used to produce cut points in asymptotic cones. Divergence can be used to rule them out [17]: if G is a finitely generated group then no asymptotic cone of G admits a cut point if and only if there exists a constant K such that for any finite geodesic $[a, b]$ with midpoint c, there is a path from a to b avoiding the ball centered at c with radius $d(a, b)/4 - 2$ of length at most $Kd(a, b) + K$. The interplay between divergence and Morse quasi-geodesics is explored in [17] and [6].

Morally, for a quasi-geodesic γ the Morse property and linear divergence are opposites. The Morse property says good (quasi-geodesic) paths between points of γ stay close to γ, and linear divergence says it is easy for a path between points of γ to stray far from γ. However, there are some subtleties. There are groups that admit quasi-geodesics with superlinear divergence, yet have an asymptotic cone with no cut point, and therefore no Morse quasi-geodesics [26]. By construction, for each of these groups there is an unbounded sequence (r_n) such that the divergence is linear (it satisfies the above conditions for a fixed K) whenever $d(a, b) = r_n$ for some n. We say a geodesic metric space has *completely superlinear divergence* if no such unbounded sequence exists. We show in Theorem 1.5 that this is the precise divergence property that characterizes Morse quasi-geodesics.

1.4. **Main theorems.** Restricted to quasi-geodesics, our main results say:

Theorem 1.3. *Let X be a geodesic metric space. Let γ be a quasi-geodesic in X. The following are equivalent:*

(1) γ is sublinearly contracting.
(2) γ is Morse.
(3) γ has completely superlinear divergence.

Special cases of this theorem have appeared before. If X is hyperbolic then these conditions are well-known properties of arbitrary quasi-geodesics, and conditions (1) and (3) can be strengthened to 'strongly contracting' and 'at least exponential divergence', respectively. If X is CAT(0) and γ is a geodesic then this is a recent theorem of Charney and Sultan [13]. In that case, conditions (1) and (3) can be strengthened to 'strongly contracting' and 'at least quadratic divergence', respectively. Our theorem establishes these equivalences in full generality.

The Morse and contraction properties make sense for *subspaces* of X, not just quasi-geodesics. Our main theorem is:

Theorem 1.4. *Let Y be a subspace of a geodesic metric space X. Let $\epsilon \geqslant 0$ be such that π_Y^ϵ does not contain the empty set in its image. The following are equivalent:*

(1) There exists $\mu\colon [1, \infty) \times [0, \infty) \to [0, \infty)$ such that Y is μ–Morse.
(2) There exists $\mu'\colon [1, \infty) \to [0, \infty)$ such that every continuous $(L, 0)$–quasi-geodesic with endpoints on Y remains in the $\mu'(L)$-neighborhood of Y.
(3) There exists ρ such that π_Y^ϵ is (r, ρ)–contracting.
(4) There exist ρ_1 and ρ_2 such that π_Y^ϵ is (ρ_1, ρ_2)–contracting.

Moreover, in each implication we bound the parameters of the conclusion in terms of the parameters of the hypothesis, independent of Y.

Divergence, on the other hand, is specialized to quasi-geodesics.

Theorem 1.5. *Let γ be a quasi-geodesic in a geodesic metric space X. The following are equivalent:*

(1) γ is Morse.
(2) γ has completely superlinear divergence.

Moreover, the Morse function can be bounded in terms of the divergence function, independent of γ.

We mention a further characterization of Morse quasi-geodesics: It can be shown fairly easily that a quasi-geodesic $\gamma\colon I \to X$ is Morse if and only if the collection of its subsegments $\{\gamma_J \mid J \text{ is a subinterval of } I\}$ is uniformly Morse. Moreover, the Morse functions for γ and for the subsegments can be bounded in terms of one another and the quasi-geodesic constants of γ. The quantitative nature of the equivalences in Theorem 1.4 then implies that γ is Morse if and only if the collection of its subsegments is uniformly contracting.

1.5. **Further applications.** We consider several important theorems about strongly contracting projections that have appeared in the literature, and generalize them by proving sublinear analogues.

The first of these results is the 'Bounded Geodesic Image Property', cf [23, 8]. This says that if π_Y^ϵ is strongly contracting then there exist constants A and B such that if γ is a geodesic segment with $d(\gamma, Y) \geqslant A$, then $\operatorname{diam} \pi_Y^\epsilon(\gamma) \leqslant B$. In fact, this property is equivalent to strong contraction. We prove, in Theorem 7.1, that π_Y^ϵ is (r, ρ)–contracting if and only if there exist a constant A and a function $\rho' \asymp \rho$ such that if γ is a geodesic segment with $d(\gamma, Y) \geqslant A$ then

$$\operatorname{diam} \pi_Y^\epsilon(\gamma) \leqslant \rho'(\max\{d(x, Y), d(x', Y)\}),$$

where x and x' are the endpoints of γ.

The second strong contraction result is one of the 'Projection Axioms' of Bestvina, Bromberg, and Fujiwara [7]. It says that if π_Y^ϵ and $\pi_{Y'}^{\epsilon'}$ are both strongly contracting, and if Y and Y' are sufficiently far apart, then $\operatorname{diam} \pi_Y^\epsilon(Y')$ and $\operatorname{diam} \pi_{Y'}^{\epsilon'}(Y)$ are bounded in terms of the contraction constants. In Proposition 8.2 we prove that if 'strongly contracting' is weakened to '(r, ρ)–contracting' then $\operatorname{diam} \pi_Y^\epsilon(Y')$ and $\operatorname{diam} \pi_{Y'}^{\epsilon'}(Y)$ are bounded by an affine function of $\rho(d(Y, Y'))$. This is the best that can be expected, since even for a single point x we can only conclude $\operatorname{diam} \pi_Y^\epsilon(x) \leqslant \rho(d(x, Y))$.

Finally, a theorem of Masur and Minsky [22] says, approximately and in our language, that if for every pair of points in a geodesic metric space X there exists a path between them such that these paths all admit semi-strongly contracting projections, with contraction constants uniform over the family of paths, then the space X is hyperbolic. Our Corollary 8.4 says the conclusion still holds if 'semi-strongly contracting' is weakened to 'sublinearly contracting'.

1.6. **Robustness.** In Section 6 we investigate the following question: Let π_Y^ϵ be (ρ_1, ρ_2)–contracting. What effect does changing ρ_1, ϵ, or Y have on this property, in terms of ρ_2?

We obtain optimal answers when $\rho_1(r) = r$, see Lemma 6.2 and Lemma 6.3. It would be interesting to have good quantitative results in more general cases.

The Morse property is invariant under quasi-isometry, so, by Theorem 1.5, the property of being sublinearly contracting is also a quasi-isometry invariant. Very little is known, however, about how the contraction parameters vary under quasi-isometry. In a subsequent paper [3] we demonstrate that strong contraction is not preserved by quasi-isometries.

2. Preliminaries

Let $N_r(y) := \{x \in X \mid d(x, y) < r\}$ and $\overline{N}_r(y) := \{x \in X \mid d(x, y) \leqslant r\}$. If Y is a subspace of X, let $N_r(Y) := \cup_{y \in Y} N_r(y)$, and $\overline{N}_r(Y) := \cup_{y \in Y} \overline{N}_r(y)$.

Let $\operatorname{diam} Y := \sup\{d(y, y') \mid y,\, y' \in Y\}$.

A *geodesic* is an isometric embedding of an interval. A metric space X is *geodesic* if for every pair of points $x, x' \in X$ there exists a geodesic connecting them.

The *Hausdorff distance* between non-empty subspaces Y and Z of X is the infimal C such that $Y \subset \overline{N}_C(Z)$ and $Z \subset \overline{N}_C(Y)$. Two subspaces are *$C$–Hausdorff equivalent* if the Hausdorff distance between them is at most C.

Given $L \geqslant 1$ and $A \geqslant 0$, a map $\phi\colon X \to Y$ between metric spaces is an *(L, A)–quasi-isometric embedding* if $\frac{1}{L} d(x, x') - A \leqslant d(\phi(x), \phi(x')) \leqslant L d(x, x') + A$ for every $x,\, x' \in X$. It is an *(L, A)–quasi-isometry* if, in addition, $Y = \overline{N}_A(\phi(X))$.

An *(L, A)–quasi-geodesic* is an (L, A)–quasi-isometric embedding of an interval.

Definition 2.1. A function f is *sublinear* if it is non-decreasing, eventually non-negative, and $\lim_{r\to\infty}\frac{f(r)}{r}=0$.

We write $f \preceq g$ if there exist constants $C_1>0$, $C_2>0$, $C_3\geqslant 0$, and $C_4\geqslant 0$ such that $f(r)\leqslant C_1 g(C_2 r+C_3)+C_4$ for all r. This partial order gives an equivalence relation $f\asymp g$ if $f\preceq g$ and $f\succeq g$. If $f\asymp g$ we say f and g are *asymptotic*.

3. Examples of contraction

We begin with a classical example.

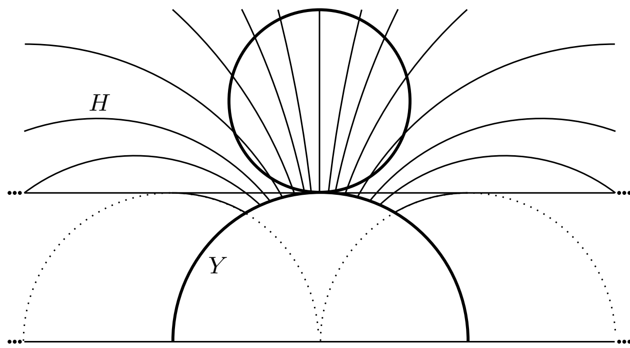

Figure 2. Contraction in $\mathbb{H}^2$.

Example 3.1. Let X be the hyperbolic plane, with the upper half-space model, and let Y be the geodesic that is the upper half of the unit circle, see Figure 2. Pick any point $x\notin Y$. Up to isometry, we may assume x sits on the y–axis above Y. The ball of radius $d(x,Y)$ about x is contained in the horoball $H:=\{(a,b)\in\mathbb{R}^2\mid b\geqslant 1\}$. The closest point projection of H to Y has diameter $\ln(3+2\sqrt{2})$. Thus, π_Y^0 is $(r,\ln(3+2\sqrt{2}))$–contracting.

We now construct examples exhibiting a wider range of contracting behaviors than have appeared previously in the literature.

Example 3.2. Let $\rho=\rho_1\colon [0,\infty)\to[0,\infty)$ be an unbounded function such that $\rho(r)\leqslant r$, $\mathrm{Id}-\rho$ is unbounded, and there exists an $A\geqslant 0$ with $\rho(A)>0$ such that $0\leqslant\rho(a+b)-\rho(a)<b$ for all $a\geqslant A$ and $b>0$. We construct a space X and $Y\subset X$ such that π_Y^0 is $(\rho,2)$–contracting but not strongly contracting.

The map $\phi\colon [A,\infty)\to[A-\rho(A),\infty) : x\mapsto x-\rho(x)$ is a bijection by assumption. We set $\sigma(0):=\phi(A)$ and, for $i\in\mathbb{N}$, recursively define[3] $\sigma(i+1):=\phi^{-1}(\sigma(i))$. This is well-defined since $[A,\infty)\subset[A-\rho(A),\infty)$. Rearranging this expression yields $\rho(\sigma(i+1))=\sigma(i+1)-\sigma(i)$. Note that $\sigma(i+1)-\sigma(i)\geqslant\rho(A)>0$ for every $i\in\mathbb{N}\cup\{0\}$, whence, in particular, $\sigma(i)\to\infty$ as $i\to\infty$.

Let $Y:=[0,\infty)$ be a ray. For $i\in\mathbb{N}\cup\{0\}$, let Z_i be a segment of length $\sigma(i)$ with endpoints labelled y_i and z_i. Identify y_i with the point i in Y. Let W_i be a segment of length $\sigma(i+1)-\sigma(i)+1$ connecting z_i to z_{i+1}. Let X be the resulting geodesic metric space. See Figure 3.

Let x_i be the point of W_i at distance $1/2$ from z_{i+1}. Clearly $\operatorname{diam}\pi_Y^0(x_i)=1$. It is easy to see that each complementary component of $X\setminus(Y\cup\{x_i\}_{i\in\mathbb{N}\cup\{0\}})$ projects to a single point of Y. Now consider the ball of radius $\rho(d(x,Y))$ about some x. First assume $x\in W_i$ for some i. Our assumptions on ρ yield:

$$\overline{N}_{\rho(d(x,Y))}(x)\subseteq W_i\cup\overline{N}_{\rho(\sigma(i))+1/2}(z_i)\cup N_{\rho(\sigma(i+1))+1/2}(z_{i+1})$$

[3] An *Abel function* for f is a function α such that $\alpha(f(x))=\alpha(x)+1$. The function σ is the inverse of an Abel function for ϕ^{-1}.

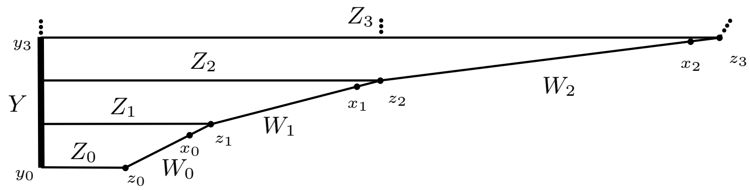

FIGURE 3. $(\rho_1, 2)$–contraction

The latter may contain x_i and x_{i+1} but no other x_j. If, on the other hand, x is in some Z_i, then $\overline{N}_{\rho(d(x,Y))}(x)$ is contained in $Z_i \cup \overline{N}_{\rho(d(z_i,Y))}(z_i)$. Therefore, for any $x \in X$, we have that $\pi^0_Y(\overline{N}_{\rho(d(x,Y))}(x))$ has diameter at most 2.

Observe that $\overline{N}_{d(z_i,Y)}(z_i)$ contains $\{z_j, z_{j+1}, \dots z_i\}$ for $0 \leqslant i - j \leqslant \sigma(j)$. Since $\sigma(i) \to \infty$ as $i \to \infty$, this implies that Y is not strongly contracting.

Concrete examples include:

- $\rho(r) := 2\sqrt{r} - 1$ and $A = 1$ and $\sigma(r) := r^2$.
- $\rho(r) := r/2$ and $A = 2$ and $\sigma(r) := 2^r$. This is an example of semi-strong contraction.
- $\rho(r) := \min\{r, r - \log_2 r\}$ and $A = 2$ and $\sigma(r) := 2 \uparrow\uparrow r$.

In Knuth's 'up-arrow notation' $2 \uparrow\uparrow r$ denotes tetration, so that $2 \uparrow\uparrow r = \underbrace{2^{\cdot^{\cdot^{2}}}}_{r \text{ times}}$ when $r \in \mathbb{N} \cup \{0\}$.

The following proposition shows that it is sometimes possible to 'trade' between the input and output contraction functions, so we can use Example 3.2 to demonstrate further examples of (ρ_1, ρ_2)–contraction conditions.

Proposition 3.3. *Suppose that π^ϵ_Y is (ρ_1, B)–contracting, where B is a constant and $\rho = \rho_1$ is a non-decreasing, non-negative, unbounded function such that* $\mathrm{Id} - \rho$ *is unbounded and such that there exists a constant A such that $\rho(A) > 0$ and $0 \leqslant \rho(a+b) - \rho(a) < b$ for all $a \geqslant A$ and $b > 0$. Define $A' := A - \rho(A)$. For $x \in [A', \infty)$ define*[4] *$\alpha(x)$ to be the minimal non-negative integer such that* $(\mathrm{Id} - \rho)^{\alpha(x)}(x) \in [A', A)$. *Then π^ϵ_Y is $(r - A, \rho_2)$–contracting for some $\rho_2 \asymp \alpha$.*

Proof. Observe as in Example 3.2 that the map $\phi\colon x \mapsto x - \rho(x)$ is a bijection $[A, \infty) \to [A', \infty)$ and that, since ϕ is strictly increasing for $x \geqslant A$, the collection $\{[\phi^k(A'), \phi^{k-1}(A')) \mid k \leqslant 0\}$ is a partition of $[A', \infty)$.

We show that $\rho_2(r) := B\alpha(r)$ will suffice. It follows from unboundedness of ρ that ρ_2 is sublinear: we have $\rho_2 \asymp \alpha$. The map α is a step function with steps of height 1, so it is sufficient to show that the lengths of the steps go to infinity, ie $\phi^{-n-1}(A') - \phi^{-n}(A') \to \infty$ as $n \to \infty$. As computed in Example 3.2, we have $\rho(\phi^{-n-1}(A')) = \phi^{-n-1}(A') - \phi^{-n}(A')$. Since $\phi^{-n-1}(A') \to \infty$ as $n \to \infty$ as argued in Example 3.2 and since ρ goes to infinity, sublinearity follows.

Let x and y be points of X such that $d(x,y) \leqslant d(x,Y) - A$. Define $r_0 := d(x,Y)$ and while $r_0 - r_i \leqslant d(x,y)$, define $r_{i+1} := \phi(r_i)$. Note that this is well-defined, ie $r_i \geqslant A$, since $r_0 - r_i \leqslant d(x,y) \leqslant r_0 - A$. Let k be the largest index such that $r_0 - r_k \leqslant d(x,y)$. Then the fact that $\phi^{\alpha(r_0)}(r_0) < A$ and the observation we just made shows $k < \alpha(r_0)$.

Fix a geodesic from x to y and for $0 \leqslant i \leqslant k$ define x_i to be the point at distance $r_0 - r_i$ from x along this geodesic. Define $x_{k+1} := y$. For $0 \leqslant i \leqslant k$ we have $d(x_{i+1}, x_i) \leqslant \rho(d(x_i, Y))$ by construction, whence:

$$\operatorname{diam} \pi^\epsilon_Y(x) \cup \pi^\epsilon_Y(y) \leqslant \sum_{i=0}^{k} \operatorname{diam} \pi^\epsilon_Y(x_i) \cup \pi^\epsilon_Y(x_{i+1}) \leqslant B\alpha(r_0)$$

Thus, π^ϵ_Y is $(r - A, \rho_2)$–contracting. $\square$

Applying Proposition 3.3 to the concrete examples in Example 3.2 we see:

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[4]The function $\alpha\colon [A', \infty) \to \mathbb{N} \cup \{0\}$ is an Abel function for $(\mathrm{Id} - \rho)^{-1}$. For instance, take α to be the inverse of $\sigma\colon \mathbb{N} \cup \{0\} \to \sigma(\mathbb{N} \cup \{0\})$ from Example 3.2 extended to all of $[A', \infty)$ by a rounding-off function.

- $(2\sqrt{r}-1,2)$–contracting implies $(r-1,\rho_2)$–contracting for $\rho_2 \asymp \sqrt{\cdot}$.
- $(r/2,2)$–contracting implies $(r-2,\rho_2)$–contracting for $\rho_2 \asymp \log_2$.
- Finally, $(r-\log_2 r,2)$–contracting implies $(r-2,\rho_2)$–contracting for $\rho_2 \asymp \mathrm{superlog}_2$.

That the converse to Proposition 3.3 can fail follows from the next example.

Example 3.4. Let ρ_2 be a sublinear function such that $0<\rho_2(r)<r$. Let Y be a line. Choose a collection of disjoint intervals $\{I_i\}_{i\in\mathbb{N}}$ of Y such that $|I_i|=\rho_2(i)$ and let y_i be the center of I_i. Connect the endpoints of I_i by attaching a segment J_i of length $4i$, and let x_i be the center of this segment. Let X be the resulting geodesic space, see Figure 4. We claim π^0_Y is (r,ρ_2)–contracting.

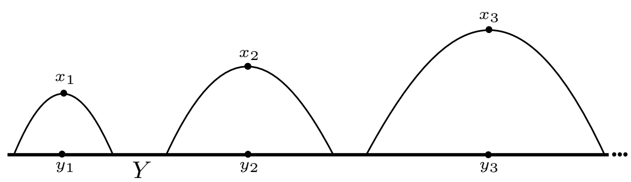

FIGURE 4. (r,ρ_2)–contracting

Suppose that $x\in J_i\subset X$ and $d(x,Y)<i$. Then $d(x,x_i)>d(x,Y)$, and $\operatorname{diam}\pi^0_Y(\overline{N}_{d(x,Y)}(x))=0$. For $x\in J_i\subset X$ with $d(x,Y)\geqslant i$ we have $d(x,x_i)\leqslant d(x,Y)$ and:

$$\operatorname{diam}\pi^0_Y(\overline{N}_{d(x,Y)}(x))=\operatorname{diam}\pi^0_Y(x_i)=\rho_2(i)\leqslant\rho_2(d(x,Y))$$

This proves the claim. Furthermore, ρ_2 is optimal, in the following sense: Since $\operatorname{diam}\pi^0_Y(x_i)=\rho_2(d(x_i,Y)/2)=\rho_2(i)$, if ρ'_1 and ρ'_2 are some other functions such that π^0_Y is (ρ'_1,ρ'_2)–contracting then $\rho_2(i)\leqslant\rho'_2(2i)$ for $i\in\mathbb{N}$.

4. The Morse property

The following two propositions establish our main result, Theorem 1.4.

Proposition 4.1. *Let Y be a subspace of a geodesic metric space X. Suppose π^ϵ_Y is (ρ_1,ρ_2)–contracting. There exists a function μ, depending only on ϵ, ρ_1, and ρ_2, such that Y is μ–Morse.*

Proof. Given L' and A' there exist L, A, and C such that every (L',A')–quasi-geodesic is C–Hausdorff equivalent to a continuous (L,A)–quasi-geodesic with the same endpoints [10, Lemma III.H.1.11]. Thus, it suffices to show there exists a bound B, depending only on ϵ, ρ_1 and ρ_2, such that every continuous (L,A)–quasi-geodesic connecting points on Y is contained in $N_B(Y)$. Then we set $\mu(L',A'):=B+C$.

Let γ be a continuous (L,A)–quasi-geodesic with endpoints on Y. Take E to be sufficiently large so that $\rho_1(E)>3A$ and for all $r\geqslant E$ we have $\frac{\rho_2(r)}{\rho_1(r)}<\frac{1}{3L^2}$.

Suppose $\gamma\not\subseteq N_E(Y)$, and let $[a,b]$ be a maximal subinterval of the domain of γ such that $\gamma|_{[a,b]}\subset X\setminus N_E(Y)$. We show there exists a T independent of γ and Y such that $b-a\leqslant T$. We conclude by setting $B:=E+L\cdot\frac{T}{2}+A$.

Let $t_0:=a$. Supposing we have defined $t_0,\dots,t_i$, if $d(\gamma(t_i),\gamma(b))>\rho_1(d(\gamma(t_i),Y))$ define t_{i+1} to be the first time that $d(\gamma(t_i),\gamma(t_{i+1}))=\rho_1(d(\gamma(t_i),Y))$. Such a t_{i+1} exists because γ is continuous. Since $d(\gamma,Y)\geqslant E$ we have $d(\gamma(t_i),\gamma(t_{i+1}))=\rho_1(d(\gamma(t_i),Y))\geqslant\rho_1(E)>0$, so after finitely many steps we reach an index k such that $d(\gamma(t_k),\gamma(b))\leqslant\rho_1(d(\gamma(t_k),Y))$. Applying the contraction condition to the points $\gamma(t_i)$, we see:

$$\operatorname{diam}\pi^\epsilon_Y(\gamma(a))\cup\pi^\epsilon_Y(\gamma(b))\leqslant\sum_{i=0}^{k}\rho_2(d(\gamma(t_i),Y))$$

This allows us to estimate:

$$\begin{aligned}
d(\gamma(a),\gamma(b)) &\leqslant d(\gamma(a),\pi_Y^\epsilon(\gamma(a))) \\
&\qquad + \operatorname{diam}\pi_Y^\epsilon(\gamma(a))\cup\pi_Y^\epsilon(\gamma(b)) + d(\gamma(b),\pi_Y^\epsilon(\gamma(b))) \\
&\leqslant 2(E+\epsilon)+\sum_{i=0}^{k}\rho_2(d(\gamma(t_i),Y)) \qquad (1)
\end{aligned}$$

On the other hand, since γ is a (L,A)–quasi-geodesic, we have:

$$\begin{aligned}
Ld(\gamma(a),\gamma(b))+LA \geqslant b-a &= b-t_k+\sum_{i=0}^{k-1}(t_{i+1}-t_i) \\
&\geqslant \frac{1}{L}(d(\gamma(b),\gamma(t_k))-A)+\sum_{i=0}^{k-1}\frac{1}{L}(d(\gamma(t_{i+1}),\gamma(t_i))-A) \\
&= \frac{1}{L}(d(\gamma(b),\gamma(t_k))-\rho_1(d(\gamma(t_k),Y))) \\
&\qquad +\sum_{i=0}^{k}\frac{1}{L}(\rho_1(d(\gamma(t_i),Y))-A) \\
&\geqslant \frac{-d(\gamma(b),Y)}{L}+\sum_{i=0}^{k}\frac{1}{L}(\rho_1(d(\gamma(t_i),Y))-A) \\
&= -\frac{E}{L}+\sum_{i=0}^{k}\frac{1}{L}(\rho_1(d(\gamma(t_i),Y))-A)
\end{aligned}$$

Combining this with the previous inequality and rearranging terms, we have:

$$\sum_{i=0}^{k}\Big(\rho_1(d(\gamma(t_i),Y))-L^2\rho_2(d(\gamma(t_i),Y))-A\Big)\leqslant E+L^2A+2L^2(E+\epsilon)$$

Now, left hand side is at least $L^2\sum_{i=0}^{k}\rho_2(d(\gamma(t_i),Y))$, by our choice of E; combined with (1), this gives us:

$$d(\gamma(a),\gamma(b))\leqslant\frac{E}{L^2}+A+4(E+\epsilon)$$

This estimate and the fact that γ is a quasi-geodesic give us a bound for $b-a$. □

Proposition 4.2. *Let Y be a subspace of a geodesic metric space X. Suppose there is a non-decreasing function μ such that every continuous $(L,0)$-quasi-geodesic with endpoints on Y is contained in the closed $\mu(L)$–neighborhood of Y. Suppose the empty set is not in the image of π_Y^ϵ. Then there is a function ρ', depending only on μ and ϵ, such that π_Y^ϵ is (r,ρ')–contracting.*

We remark that since an $(L,0)$–quasi-geodesic is also an $(L',0)$–quasi-geodesic for any $L'>L$, there is no loss in requiring the Morse function to be non-decreasing.

Proof. Consider the optimal contraction function:

$$\rho(r):=\sup_{d(x,y)\leqslant d(x,Y)\leqslant r}\operatorname{diam}\pi_Y^\epsilon(x)\cup\pi_Y^\epsilon(y)\leqslant 4r+2\epsilon$$

Our goal is define a function ρ' depending on μ and ϵ that is non-negative, non-decreasing, and sublinear and such that ρ' is an upper bound for ρ.

Define $\rho'(r):=0$ if $\epsilon=0$ and $\mu\equiv 0$. In this case ρ' clearly has the first three properties. Otherwise, we first replace μ by $s\mapsto\inf_{t>s}\mu(s)$. The new μ still satisfies the hypotheses of the proposition and has that additional property that it is right continuous: $\lim_{t\to s^+}\mu(t)=\mu(s)$ for all $s\geqslant 1$. Define $\rho'(0):=2\epsilon$ and for $r>0$ define:

$$\rho'(r):=\sup\left\{s\leqslant 4r+2\epsilon \mid s\leqslant 18\mu\left(\frac{3(4r+2\epsilon)}{s}\right)+12\epsilon\right\}$$

If $\mu\equiv 0$ then ρ' increases linearly from 2ϵ to 12ϵ and then remains constant, so it is non-negative, non-decreasing, and sublinear.

If $\mu\not\equiv 0$ then $\rho'(r)>0$ when $r>0$, and the conditions on μ ensure ρ' is actually a maximum. The fact that it is non-decreasing then follows by observing that $\rho'(r)$ participates in the supremum defining $\rho'(r')$ when $0\leqslant r<r'$. To see ρ' is sublinear, we suppose that $\limsup_{r\to\infty}\rho'(r)/r>0$ and derive a contradiction. Suppose that there exists some $\delta>0$ and a sequence (r_i) of positive numbers increasing without bound such that $\rho'(r_i)>\delta r_i$ for all i. By definition of ρ', for each i there exists $\delta r_i<s_i\leqslant 4r_i+2\epsilon$ such that:

$$s_i\leqslant 18\mu\left(\frac{3(4r_i+2\epsilon)}{s_i}\right)+12\epsilon\leqslant 18\mu\left(\frac{3(4r_i+2\epsilon)}{\delta r_i}\right)+12\epsilon$$

This is a contradiction, since the left-hand side grows without bound while the right-hand side is bounded above by $18\mu(\frac{12}{\delta}+1)+12\epsilon$ once i is sufficiently large.

Now we must show $\rho(r)\leqslant\rho'(r)$. It suffices to check this for those r such that $\rho(r)>0$. The idea of the proof is to choose, for each such r, points x and y such that $d(x,y)\leqslant d(x,Y)\leqslant r$ whose projection diameters nearly realize $\rho(r)$. Take a path γ that is a concatenation of geodesics from a projection point of x to x, then from x to y, then from y to a projection point of y. For $L:=\frac{3(4r+2\epsilon)}{\rho(r)}\geqslant 3$ we show that we can make γ into an $(L,0)$–quasi-geodesic γ' by introducing at most two shortcuts in a particular way. The Morse hypothesis implies that γ' is contained in the $\mu(L)$–neighborhood of Y. We then argue that the condition $d(x,y)\leqslant d(x,Y)$ implies:

$$\rho(r)<18\mu(L)+12\epsilon \tag{2}$$

In the case that $\epsilon=0$ and $\mu\equiv 0$, this gives a contradiction, which means that there is no r for which ρ takes a positive value, and we have $\rho(r)=\rho'(r)=0$ for all r. Otherwise, plugging the value of L into (2), we conclude that $\rho(r)$ participates in the supremum defining $\rho'(r)$, whence $\rho(r)\leqslant\rho'(r)$.

First we show how to produce quasi-geodesics. Consider points x, y, $p_x\in\pi_Y^\epsilon(x)$, and $p_y\in\pi_Y^\epsilon(y)$. Let $\gamma:=[p_x,x][x,y][y,p_y]$ be a concatenation of three geodesics. Let $[p,q]_\gamma$ denote the subsegment of γ from p to q, and let $|[p,q]_\gamma|$ denote its length. For this part of the argument we may use any $L\geqslant\frac{|\gamma|}{d(p_x,p_y)}\geqslant 1$. Consider the continuous function $D(p,q):=Ld(p,q)-|[p,q]_\gamma|$ defined on points $(p,q)\in\gamma\times\gamma$ such that p precedes q on γ. The restriction on L implies that $D(p_x,p_y)\geqslant 0$. We conclude that if $[p,q]_\gamma$ is a subsegment of γ that is maximal with respect inclusion among subsegments for which D takes non-positive values on the endpoints, then $Ld(p,q)=|[p,q]_\gamma|$. We consider several cases. Each carries the additional assumption that we are not in one of the previous cases.

Case 0: D is non-negative. Set $\gamma':=\gamma$, which is an $(L,0)$–quasi-geodesics by definition of D.

Case 1: D takes a non-positive value on $[p_x,x]_\gamma\times[y,p_y]_\gamma$. In this case there exist points $x'\in[p_x,x]$ and $y'\in[p_y,y]$ such that the segment $[x',y']_\gamma$ is maximal with respect to inclusion among subsegments of γ with the property that D takes non-positive values on endpoints. Define γ' by replacing $[x',y']_\gamma$ by some geodesic segment with the same endpoints; $\gamma':=[p_x,x']_\gamma[x',y'][y',p_y]_\gamma$. We claim that γ' is an $(L,0)$–quasi-geodesic. Since γ' is a concatenation of geodesic segments, it suffices to check that points on distinct segments are sufficiently far apart. We check distances between arbitrary points $x''\in[p_x,x']_{\gamma'}$, $z\in[x',y']_{\gamma'}$, and $y''\in[y',p_y]_{\gamma'}$.

Suppose, for contradiction, that $Ld(x'',y'')<|[x'',y'']_{\gamma'}|$. Since $[x',y']_\gamma$ has been replaced by a geodesic segment, $Ld(x'',y'')<|[x'',y'']_{\gamma'}|\leqslant|[x'',y'']_\gamma|$, so $D(x'',y'')<0$. Since $D(x',y')=0$ we have $x''\in[p_x,x')_\gamma$ or $y''\in(y',p_y]_\gamma$, but then $[x'',y'']_\gamma$ is a subsegment of γ strictly containing $[x',y']_\gamma$ such that D takes a non-positive value on its endpoints. This contradicts maximality of $[x',y']_\gamma$ among such subsegments, so $d(x'',y'') \geq \frac{|[x'',y'']_{\gamma'}|}{L}$.

Suppose, for contradiction, that $Ld(x'',z) < |[x'',z]_{\gamma'}|$. This implies $x'' \neq x'$, because x' and z lie on a geodesic subsegment of γ'. We estimate:

$$\begin{aligned} d(x'',y') &\leqslant d(x'',z) + d(z,y') \\ &< \frac{|[x'',z]_{\gamma'}|}{L} + d(z,y') \\ &= \frac{d(x'',x') + d(x',z)}{L} + d(x',y') - d(x',z) \\ &= \frac{|[x'',x']_\gamma|}{L} + \frac{|[x',y']_\gamma|}{L} - \left(\frac{L-1}{L}\right) d(x',z) \\ &\leqslant \frac{|[x'',y']_\gamma|}{L} \end{aligned}$$

Since $x'' \in [p_x, x')_\gamma$, we have exhibited a subsegment $[x'',y']_\gamma$ strictly containing $[x',y']_\gamma$ such that D takes a non-positive value on its endpoints. This contradicts maximality of $[x',y']_\gamma$ among such subsegments, so $d(x'',z) \geqslant \frac{|[x'',z]_{\gamma'}|}{L}$.

A symmetric argument shows $d(y'',z) \geqslant \frac{|[y'',z]_{\gamma'}|}{L}$, so γ' is an $(L,0)$–quasi-geodesic.

Case 2: D takes a non-positive value on an element of $[p_x,x]_\gamma \times (x,y]_\gamma$. Let $[x',q_x]_\gamma$ be a subsegment of γ maximal with respect to inclusion among subsegments for which D takes non-positive values on endpoints, with $x' \in [p_x,x]_\gamma$. Since we are not in Case 1, $q_x \in (x,y)_\gamma$. Now consider whether or not $[q_x,p_y]_\gamma$ is an $(L,0)$–quasi-geodesic. If so, define $\gamma' := [p_x,x']_\gamma[x',q_x][q_x,p_y]_\gamma$. Otherwise, D takes a negative value on an element of $[q_x,y)_\gamma \times (y,p_y]_\gamma$. Let $[q_y,y']_\gamma$ be a maximal subsegment of $[q_x,p_y]_\gamma$, with $q_y \in [q_x,y)_\gamma$ and $y' \in (y,p_y]$ on which D takes non-positive values on endpoints. We claim that $q_y \in (q_x,y)_\gamma$ and $D(q_y,y') = 0$, because if $D(q_y,y) < 0$ and $q_y \neq q_x$ then we can enlarge the subsegment, contradicting maximality, while if $q_y = q_x$ then $D(x',y') \leqslant 0$, contradicting the assumption that we are not in Case 1.

In either of these cases, we claim γ' is an $(L,0)$–quasi-geodesic. This follows by verifying that the distance between points in distinct geodesic components of γ' have distance at least equal to the length of the subsegment of γ' they bound divided by L. The strategy is to suppose D attains a strictly negative value and then either derive a contradiction to maximality of $[x',q_x]_\gamma$ or $[q_y,y']_\gamma$ or to the assumption that we are not Case 1. The arguments are substantially similar to the computations in Case 1 and are left to the reader.

Case 3: D takes a non-positive value on an element of $[x,y)_\gamma \times [y,p_y]_\gamma$. The argument here is symmetric to the subcase of Case 2 in which only a corner at x is cut short.

We have shown how to produce an $(L,0)$–quasi-geodesic γ' from γ. We now proceed to show $\rho(r) \leqslant \rho'(r)$ for any r such that $\rho(r) > 0$. Since $\rho(r) > 0$ there exist x and y such that $d(x,y) \leqslant d(x,Y) \leqslant r$ and $\operatorname{diam} \pi_Y^\epsilon(x) \cup \pi_Y^\epsilon(y) > \frac{2}{3}\rho(r)$. Choose $p_x \in \pi_Y^\epsilon(x)$, $p_y \in \pi_Y^\epsilon(y)$ such that $d(p_x,p_y) > \frac{2}{3}\rho(r)$.

Let $\gamma := [p_x,x][x,y][y,p_y]$. Let $L := \frac{12r+6\epsilon}{\rho(r)} \geqslant 2\frac{|\gamma|}{d(p_x,p_y)}$, and let γ' be the $(L,0)$–quasi-geodesic produced from γ as above. By the Morse hypothesis, γ' is contained in the $\mu(L)$–neighborhood of Y.

Case a: γ' comes from Case 0 or Case 3. In this case $x \in \gamma'$, so $d(x,Y) \leqslant \mu(L)$, so $\rho(r) < \frac{3}{2}d(p_x,p_y) \leqslant \frac{3}{2}(4\mu(L)+2\epsilon)$.

Case b: γ' comes from Case 1. In this case $p_x \in \pi_Y^\epsilon(x')$ and $p_y \in \pi_Y^\epsilon(y')$, so $d(x',p_x) \leqslant \mu(L)+\epsilon$ and $d(y',p_y) \leqslant \mu(L)+\epsilon$. Also, by definition of L we have:

$$d(x',y') = \frac{|[x',y']_\gamma|}{L} \leqslant \frac{|\gamma|}{L} \leqslant \frac{4r+2\epsilon}{\frac{3(4r+2\epsilon)}{\rho(r)}} = \frac{\rho(r)}{3}$$

Since $d(p_x,p_y) > \frac{2}{3}\rho(r)$, we conclude $d(x',p_x) + d(y',p_y) > \frac{\rho(r)}{3}$, so that $\rho(r) < 6\mu(L) + 6\epsilon$.

Case c: γ' comes from Case 2. In this case γ' contains a geodesic segment from a point $x' \in [p_x, x]_\gamma$ to a point $q_x \in [x,y]_\gamma$. As in the previous case, $d(x', q_x) = \frac{|[x',q_x]_\gamma}{L} \leqslant \frac{|\gamma|}{L} \leqslant \frac{\rho(r)}{3}$. Consider a point $w \in \pi_Y^\epsilon(q_x)$. Since $d(x,y) \leqslant d(x,Y)$, we have $d(q_x, y) \leqslant d(q_x, Y) \leqslant \mu(L)$, which implies $d(w, p_y) \leqslant 4\mu(L) + 2\epsilon$. Thus $d(p_x, w) > \frac{2}{3}\rho(r) - (4\mu(L) + 2\epsilon)$. We also have $d(x', Y) \leqslant \mu(L)$ and $d(q_x, Y) \leqslant \mu(L)$, since both these points belong to γ', so:

$$\frac{\rho(r)}{3} \geqslant d(x', q_x) \geqslant d(p_x, w) - d(x', p_x) - d(q_x, w) > \frac{2}{3}\rho(r) - (6\mu(L) + 4\epsilon)$$

The resulting bound on $\rho(r)$ is the largest of the three cases, and establishes the bound of (2), completing the proof. □

5. Divergence

In this section we relate divergence to contraction and the Morse property, thereby proving Theorem 1.5.

There is a link between the Morse property and superlinear divergence via asymptotic cones [17]. Although this principle is well-known, there are competing definitions of 'superlinear' and 'divergence', so we give a detailed proof of Theorem 1.5 in terms of our definitions. Our analysis actually yields more. In the introduction we claimed that for a quasi-geodesic the Morse property, hence, sublinear contraction, is morally the opposite of high divergence. We prove a precise technical formulation of this claim in Proposition 5.5. Roughly speaking, the result we obtain is that if divergence of a quasi-geodesic γ is greater than a function f then almost closest point projection to γ is (r, f^{-1})–contracting.

Definition 5.1. Let X be a geodesic metric space and let $\gamma\colon \mathbb{R} \to X$ be an (L, A)–quasi-geodesic. Let $\lambda \in (0,1]$, and let $\kappa \geqslant L + A$. Let $\Lambda_\gamma(r, s; L, A, \lambda, \kappa)$ be the infimal length of a path from $\gamma(s-r)$ to $\gamma(s+r)$ that is disjoint from the ball of radius $\lambda(L^{-1}r - A) - \kappa$ centered at $\gamma(s)$, or ∞ if no such path exists. The (L, A, λ, κ)–*divergence* of γ evaluated at r is $\Delta_\gamma(r; L, A, \lambda, \kappa) := \inf_s \Lambda_\gamma(r, s; L, A, \lambda, \kappa)$.

Notice that if γ is a geodesic, $\lambda := 1/2$, and $\kappa := 2$ we recover the definition of divergence we gave in the introduction.

We make the convention that $\infty \leqslant \infty$.

In light of the following lemma, γ has a well defined divergence, up to equivalence of functions, and we use $\Delta_\gamma(r)$ to denote the equivalence class of $\Delta_\gamma(r; L, A, \lambda, \kappa)$.

Lemma 5.2. *Let γ be an (L, A)–quasi-geodesic. Suppose γ is also an (L', A')–quasi-geodesic. Let $\lambda, \lambda' \in (0,1]$, $\kappa \geqslant L + A$, and $\kappa' \geqslant L' + A'$. Then $\Delta_\gamma(r; L, A, \lambda, \kappa) \asymp \Delta_\gamma(r; L', A', \lambda', \kappa')$.*

Proof. Take $0 < M < 1$ small enough that $\frac{\lambda}{L} - \frac{\lambda'}{L'}M > 0$. Then for any sufficiently large $C \geqslant 0$ the affine function $\theta\colon r \mapsto Mr - C$ satisfies:

$$\lambda'((L')^{-1}\theta(r) - A') - 2\kappa' \leqslant \lambda(L^{-1}r - A) - \kappa$$

Fix $s \in \mathbb{R}$ and let P be any path from $\gamma(s-r)$ to $\gamma(s+r)$ that is disjoint from the ball of radius $\lambda(L^{-1}r - A) - \kappa$ centered at $\gamma(s)$. By the above inequality it is also disjoint from the ball of radius $\lambda'((L')^{-1}\theta(r) - A') - 2\kappa'$ about $\gamma(s)$.

Let $\{x_0, x_1, \ldots, x_l\}$ be the set $[s-r, s-\theta(r)] \cap (\mathbb{Z} \cup \{s-r, s-\theta(r)\})$ in descending order and let P_- be the path from $\gamma(s-\theta(r))$ to $\gamma(s-r)$ obtained by concatenating geodesics $[\gamma(x_i), \gamma(x_{i+1})]$. Define a path P_+ from $\gamma(s+r)$ to $\gamma(s+\theta(r))$ similarly. Since $\kappa' \geqslant L' + A'$, the paths P_- and P_+ are disjoint from the ball of radius $\lambda'((L')^{-1}\theta(r) - A') - \kappa'$ centered at $\gamma(s)$.

Define P' to be the path from $\gamma(s-\theta(r))$ to $\gamma(s+\theta(r))$ obtained by concatenating P_-, P, and P_+.

Now, for each r choose s and P so that $|P| \leqslant 1 + \Delta_\gamma(r; L, A, \lambda, \kappa)$. Then $\Delta_\gamma(\theta(r); L', A', \lambda', \kappa') \leqslant |P| + 2(L(r - \theta(r)) + A)$. Since γ is quasi-geodesic, $r \leqslant L|P| + LA$, so the right-hand side can be bounded by an affine function of $\Delta_\gamma(r; L, A, \lambda, \kappa)$. This proves one direction of the equivalence. The other follows immediately by reversing the roles in the above argument. □

We first give an example of the relationship between divergence and contraction.

Example 5.3. Let $f(r) \geqslant r$ be an increasing, invertible function. Consider the space X constructed in Example 3.4, but this time take $|I_i| := 2i$ and $|J_i| := f(i)$ for $i \in \mathbb{N}$. Let γ be a geodesic whose image is Y. Then $\Lambda_\gamma(i, \gamma^{-1}(y_i); 1, 0, 1, 1) = f(i)$, and this is optimal for radius i, so $\Delta_\gamma \asymp f$. On the other hand, the computation of Example 3.4 shows that $\operatorname{diam} \pi^0_Y(x_i) = 2f^{-1}(4r)$. Thus, π^0_Y is sublinearly contracting if and only if f^{-1} is sublinear, and, in this case, it is (r, ρ)–contracting for $\rho \asymp f^{-1}$.

Our next proposition proves the implication (2) $\implies$ (1) of Theorem 1.5. It also gives a quantitave link between high divergence and contraction.

Definition 5.4. We say a function g is *completely super–f* if for every choice of $C_1 > 0$, $C_2 > 0$, $C_3 \geqslant 0$, and $C_4 \geqslant 0$ the collection of $r \in [0, \infty)$ such that $g(r) \leqslant C_1 f(C_2 r + C_3) + C_4$ is bounded.

Proposition 5.5. *Let γ be a quasi-geodesic in a geodesic metric space X. Suppose the empty set is not in the image of π^ϵ_γ. Let $f(r) \geqslant r$ be an increasing, invertible function. If γ has completely super–f divergence, then there exists a function ρ such that π^ϵ_γ is (r, ρ)–contracting and $\lim_{r\to\infty} \frac{\rho(r)}{f^{-1}(r)} = 0$.*

In particular, if γ has completely superlinear divergence then there exists a sublinear function ρ such that π^ϵ_γ is (r, ρ)–contracting.

Proof. Let γ be an (L, A)-quasi-geodesic. Define:

$$\rho(r) := \sup_{d(x,y) \leqslant d(x,\gamma) \leqslant r} \operatorname{diam} \pi^\epsilon_\gamma(x) \cup \pi^\epsilon_\gamma(y)$$

To see that π^ϵ_γ is (r, ρ)–contracting we must show that ρ is sublinear. Since $f(r) \geqslant r$, it suffices to prove the second claim:

$$\lim_{r\to\infty} \frac{\rho(r)}{f^{-1}(r)} = 0$$

Suppose for a contradiction that $\limsup_{r\to\infty} \frac{\rho(r)}{f^{-1}(r)} > 0$. Then there exist $c > 0$; sequences (x_n) and (y_n) with x_n, $y_n \in X$, $d(x_n, \gamma) \geqslant n$, and $d(x_n, y_n) \leqslant d(x_n, \gamma)$; and $x'_n \in \pi^\epsilon_\gamma(x_n)$ and $y'_n \in \pi^\epsilon_\gamma(y_n)$ such that:

$$cf^{-1}(d(x_n, \gamma)) \leqslant d(x'_n, y'_n) \tag{3}$$

Let a_n and b_n be such that $\gamma(a_n - b_n) = x'_n$ and $\gamma(a_n + b_n) = y'_n$. Define $m_n := \gamma(a_n)$ and $R_n := \frac{b_n}{L} - A$. Since γ is an (L, A)–quasi-geodesic, $d(m_n, \{x'_n, y'_n\}) \geqslant R_n$ and $b_n \geqslant \frac{d(x'_n, y'_n) - A}{2L}$. By (3) and the facts that f^{-1} is unbounded and increasing, $\lim_{n\to\infty} R_n = \infty$.

Choose $0 < \lambda < \frac{1}{4}$ and $\kappa := L + A$.

If there is a geodesic from x_n to y_n containing a point z such that $d(z, m_n) \leqslant \lambda R_n$, then:

$$\begin{aligned}
R_n &\leqslant d(y'_n, m_n) \\
&\leqslant d(y'_n, y_n) + d(y_n, z) + d(z, m_n) \\
&\leqslant d(y_n, \gamma) + \epsilon + d(y_n, z) + d(z, m_n) \\
&\leqslant \epsilon + 2(d(y_n, z) + d(z, m_n)) \\
&\leqslant \epsilon + 2\lambda R_n + 2d(y_n, z) \\
&= \epsilon + 2\lambda R_n + 2(d(x_n, y_n) - d(z, x_n)) \\
&\leqslant \epsilon + 2\lambda R_n + 2(d(x_n, \gamma) - (d(x_n, \gamma) - \lambda R_n)) \\
&= \epsilon + 4\lambda R_n
\end{aligned}$$

Thus, $R_n \leqslant \frac{\epsilon}{1-4\lambda}$.

If there is a geodesic from x_n to x'_n or from y_n to y'_n containing a point z such that $d(z, m_n) \leqslant \lambda R_n$, then a similar argument shows $R_n \leqslant \frac{\epsilon}{1-2\lambda}$.

Since $R_n \to \infty$, for all sufficiently large n and any choice of path p_n that is a concatenation of geodesics $[x'_n, x_n]$, $[x_n, y_n]$, $[y_n, y'_n]$, the path p_n remains outside the ball of radius λR_n about m_n. This gives us a path of length at most $4d(x_n, \gamma) + 2\epsilon$ from $\gamma(a_n - b_n)$ to $\gamma(a_n + b_n)$ that remains outside the ball of radius $\lambda \left(\frac{b_n}{L} - A\right)$ about $\gamma(a_n)$.

On the other hand, (3) implies:

$$d(x_n,\gamma) \leqslant f\left(\frac{1}{c}d(x'_n,y'_n)\right) \leqslant f\left(\frac{2b_nL+A}{c}\right)$$

We conclude that for all sufficiently large n the (L,A,λ,κ)–divergence of γ evaluated at b_n is at most $2\epsilon+4f\left(\frac{2b_nL+A}{c}\right)$, which contradicts the hypothesis that the divergence is completely super–f. □

The previous result can be strengthened to the statement:

Proposition 5.6. *Let f be an increasing, invertible, completely superlinear function satisfying the following additional condition:*

($*$) *For every C there exists some D such that for all $r>1$ and $k>D$ we have $f(kr) > Cf(Cr+C)+C$.*

If the divergence of γ is at least f then γ is (r,ρ)–contracting for some function $\rho \preceq f^{-1}$.

Proof. For a contradiction we suppose that $\rho \not\preceq f^{-1}$ and replace (3) with $d(x'_n,y'_n) \geqslant nf^{-1}(d(x_n,\gamma))$. Using the same method as in the proof of Proposition 5.5, we deduce that for all sufficiently large n the (L,A,λ,κ)–divergence of γ evaluated at b_n is at most $2\epsilon+4f\left(\frac{2b_nL+A}{n}\right)$. Thus, $f(b_n) \leqslant 2\epsilon+4f\left(\frac{2b_nL+A}{n}\right)$. Let $c_n := b_n/n$ and $M := \max\{2\epsilon, 4, 2L, A\}$. Then, since f is increasing:

$$f(nc_n) \leqslant Mf(Mc_n+M)+M \tag{4}$$

The left-hand side is unbounded as n grows, so we immediately obtain a contradiction if the sequence $(c_n)_{n\in\mathbb{N}}$ is bounded. If the sequence is unbounded then, by passing to a subsequence, we may assume $c_n>1$ for all n. In this case the inequality (4) holds for all n, which contradicts condition ($*$). □

Suitable functions f for Proposition 5.6 include $f(r) := r^d$, $r^d/\log(r)$, $r\log(r)$ and d^r for any $d>1$. The function $f(r) := 2^{2^{2^{1+\lfloor \log_2\log_2 r\rfloor}}}$ is completely superlinear, but does not satisfy ($*$), since $f(n2^{2^{n-1}}) = f(2^{2^{n-1}})$ for all $n\in\mathbb{N}$.

Corollary 5.7. *If a quasi-geodesic γ has divergence at least r^k then γ is $(r,r^{1/k})$–contracting. If it has exponential divergence, then γ is logarithmically contracting. Finally, if it has infinite divergence, then it is strongly contracting.*

Here infinite divergence means $\Delta_\gamma(r)=\infty$ for all r large enough. Example 5.3 shows these conclusions are optimal.

We now address the implication (1) $\implies$ (2) of Theorem 1.5. In this direction we can show that the Morse property implies completely superlinear divergence, but we do not get explicit control of the divergence function in terms of the Morse function, see Proposition 5.10.

There is one special case in which we can say more. Charney and Sultan [13] recently gave a proof[5] that if α is a Morse geodesic in a CAT(0) space then α has at least quadratic divergence. Essentially the same argument gives a general result:

Proposition 5.8. *Let α be a geodesic in a geodesic metric space X. If α is (ρ_1,ρ_2)–contracting with ρ_2 bounded, then $\Delta_\alpha(r) \succeq r\rho_1(r)$.*

Lemma 5.9. *Let X be a geodesic metric space. Let $a,b,c,d\in X$ and $r>0$ satisfy the following conditions:*

(1) $d(a,d)\geqslant r$
(2) There exists a path γ from a to d passing through b and c such that the length of γ is at most Cr and such that $[a,b]_\gamma$, $[b,c]_\gamma$, and $[c,d]_\gamma$ are continuous $(L,0)$–quasi-geodesics.
(3) The path γ does not contain a point within distance λr of e, where e is the midpoint of a geodesic from a to d.

[5] The original proof of this fact is due to Behrstock and Druţu [6], by different methods.

For any $L' > \max\{L, C, C/\lambda\} \geqslant 1$ there exists a continuous $(L', 0)$–quasi-geodesic γ' from a to d of length at most $|\gamma|$ such that γ' does not contain a point within distance $\lambda r/2$ of e.

Proof. The construction of γ' is exactly as in Proposition 4.2 with L replaced by L'. This involves finding points p and q on γ such that $L'd(p, q) = |[p, q]_\gamma|$ and replacing $[p, q]_\gamma$ by a geodesic with the same endpoints. Now, $d(p, q) \leqslant |\gamma|/L' < \lambda r$, so for any point z on a newly introduced geodesic segment we have $d(z, e) \geqslant d(\gamma, e) - d(p, q)/2 > \lambda r/2$. □

Proposition 5.10. *Let γ be a Morse quasi-geodesic in a geodesic metric space X. Then the divergence of γ is completely superlinear.*

Proof. We prove the contrapositive. Let γ be an (L, A)–quasi-geodesic and suppose its divergence is not completely superlinear. Then there exists $C > 0$ for which there exists an unbounded sequence of numbers $r_n \geqslant 1$ and paths p_n such that:

(1) There exists a sequence of real numbers s_n such that the endpoints of p_n are $x_n = \gamma(s_n - r_n)$ and $y_n = \gamma(s_n + r_n)$.
(2) $|p_n| \leqslant Cr_n$.
(3) p_n does not intersect the $(\frac{r_n}{2L} - A)$–neighborhood of $\gamma(s_n)$.

We may assume all $r_n \geqslant 4AL$ so point (3) can be replaced by:

3′. p_n does not intersect the $(\frac{r_n}{4L})$–neighborhood of $m_n := \gamma(s_n)$.

Our goal is to construct uniform quasi-geodesics γ_n from x_n to y_n that avoid increasingly large balls around m_n.

Set $x_{n,0} := x_n$ and define $x_{n,1}$ to be the last point on p_n for which we have $d(x_{n,0}, x_{n,1}) = r_n/8L$.

Similarly define $x_{n,i}$ to be y_n if $d(x_{n,i-1}, y_n) < r_n/4L$ or to be the last point on p_n satisfying $d(x_{n,i-1}, x_{n,i}) = r_n/8L$ otherwise.

Note that $y_n = x_{n,k_n}$ for some $k_n \leqslant 8CL$. By construction, if $i \neq j$ then $d(x_{n,i}, x_{n,j}) \geqslant r_n/8L$.

Let γ_n^1 be a concatenation of geodesics $[x_{n,0}, x_{n,1}] \ldots [x_{n,k_n-1}, y_n]$. We have that $|\gamma_n^1| \leqslant Cr_n$ and $d(\gamma_n^1, m_n) > r_n/8L$.

Applying Lemma 5.9 for each $1 \leqslant i \leqslant \lfloor k_n/3 \rfloor$ there are $(L_2, 0)$–quasi-geodesics (where L_2 does not depend on n) from $x_{n,3(i-1)}$ to $x_{n,3i}$ such that the concatenation γ_n^2 of these with $[x_{n,3\lfloor k_n/3 \rfloor}, y_n]_{\gamma_n^1}$ satisfies $d(\gamma_n^2, m_n) > r_n/16L$.

Repeating this procedure at most $d = \lceil \log_3 8CL \rceil$ times we obtain an $(L_d, 0)$ quasi-geodesic γ_n^d from x_n to y_n satisfying $d(\gamma_n^d, m_n) > r_n/(2^{d+2}L)$. Again, L_d does not depend on n.

If γ is μ–Morse, then the γ_n^d are μ'–Morse for some μ' that does not depend on n. Then $d(\gamma_n^d, m_n) \leqslant \mu'(K, C)$, which is bounded, contradicting the lower bound above. □

A finitely generated group is called *constricted* if all of its asymptotic cones have cut points [18].

Corollary 5.11. *Suppose there exists a quasi-geodesic γ with completely superlinear divergence in a geodesic metric space X. In every asymptotic cone of X every point of the ultralimit of γ is a cut point.*

In particular, a finitely generated group is constricted if one of its Cayley graphs contains a quasi-geodesic with completely superlinear divergence.

Olshanskii, Osin, and Sapir [26, Corollary 6.4] build a group that has an asymptotic cone with no cut point such that the group has a Cayley graph with geodesics of superlinear divergence. These geodesics are therefore not Morse. They explicitly state that their construction yields geodesics that are not completely superlinear. Corollary 5.11 shows that this will be the case in any such construction.

6. Robustness

Suppose that π_Y^ϵ is (ρ_1, ρ_2)–contracting. In this section we investigate the extent to which ρ_2 is affected by changes to ρ_1, ϵ, or Y.

Clearly π_Y^ϵ is (ρ_1', ρ_2)–contracting for $\rho_1' \leqslant \rho_1$. From Theorem 1.4 we know that π_Y^ϵ is (r, ρ_2')–contracting for some ρ_2' depending on ρ_1 and ρ_2. For this ρ_2', it follows that π_Y^ϵ is (ρ_1', ρ_2')–contracting for every $\rho_1 \leqslant \rho_1' \leqslant r$.

In general ρ_2 and ρ_2' are not asymptotic. For example, if π_Y^ϵ is $(r/2, B_1)$–contracting it is (r, ρ_2)–contracting for $\rho_2 \asymp \log_2$, as in Proposition 3.3, but not necessarily (r, B_2)–contracting for some constant B_2, by Example 3.2. One well-known special case is that $(r/M, B_1)$–contracting for $M > 1$ and B_1 bounded implies $(r/2, B_2)$–contracting for some bounded B_2, see, eg, [30].

The output contraction functions are asymptotic when the input function is changed by an additive constant:

Lemma 6.1. *If π_Y^ϵ is (ρ_1, ρ_2)–contracting for $\rho_1(r) = \rho_1'(r) - C$, with $\rho_1'(r) \leqslant r$ and $C \geqslant 0$, then π_Y^ϵ is (ρ_1', ρ_2')–contracting for some $\rho_2' \asymp \rho_2$.*

Proof. Let $C' := \sup\{r \mid \rho_1(r) \leqslant C\}$. Suppose that x and y are points with $d(x, y) \leqslant \rho_1'(d(x, Y))$. If $d(x, y) \leqslant \rho_1(d(x, Y)) = \rho_1'(d(x, Y)) - C$ then we have $\operatorname{diam} \pi_Y^\epsilon(x) \cup \pi_Y^\epsilon(y) \leqslant \rho_2(d(x, Y))$. Otherwise, let z be a point on a geodesic from x to y such that $d(x, z) = \rho_1(d(x, Y))$. This implies $d(y, z) \leqslant C$. Now:

$$\begin{aligned}\operatorname{diam} \pi_Y^\epsilon(x) \cup \pi_Y^\epsilon(y) &\leqslant \operatorname{diam} \pi_Y^\epsilon(x) \cup \pi_Y^\epsilon(z) + \operatorname{diam} \pi_Y^\epsilon(z) \cup \pi_Y^\epsilon(y)\\ &\leqslant \rho_2(d(x, Y)) + \operatorname{diam} \pi_Y^\epsilon(z) \cup \pi_Y^\epsilon(y)\end{aligned}$$

If $d(z, y) > \rho_1(d(z, Y))$ then $d(z, Y) \leqslant C'$, so $\operatorname{diam} \pi_Y^\epsilon(z) \cup \pi_Y^\epsilon(y) \leqslant 2(C + C' + \epsilon)$. If $d(z, y) \leqslant \rho_1(d(z, Y))$ then $\operatorname{diam} \pi_Y^\epsilon(z) \cup \pi_Y^\epsilon(y) \leqslant \rho_2(d(z, Y)) \leqslant \rho_2(2d(x, Y))$. Combining these cases, we see that $d(x, y) \leqslant \rho_1(d(x, Y))$ implies:

$$\operatorname{diam} \pi_Y^\epsilon(x) \cup \pi_Y^\epsilon(y) \leqslant \rho_2(d(x, Y)) + \rho_2(2d(x, Y)) + 2(C + C' + \epsilon)$$

Thus, it suffices to take $\rho_2'(r) := 2\rho_2(2r) + 2(C + C' + \epsilon)$. □

Next, consider changes to the projection parameter.

Lemma 6.2. *Suppose ϵ_0 and ϵ_1 are constants such that the empty set is neither in the image of $\pi_Y^{\epsilon_0} \colon X \to 2^Y$ nor in the image of $\pi_Y^{\epsilon_1} \colon X \to 2^Y$. If $\pi_Y^{\epsilon_0}$ is (ρ_1, ρ_2)–contracting then there exist ρ_1' and ρ_2' such that $\pi_Y^{\epsilon_1}$ is (ρ_1', ρ_2')–contracting. If $\epsilon_1 \leqslant \epsilon_0$ or if $\rho_1(r) := r$ then we can take $\rho_1' = \rho_1$ and $\rho_2' \asymp \rho_2$.*

Proof. When $\epsilon_1 \leqslant \epsilon_0$ we have $\pi_Y^{\epsilon_1}(x) \subset \pi_Y^{\epsilon_0}(x)$, so the result is clear. In this case $\rho_1' = \rho_1$ and $\rho_2' = \rho_2$ will suffice.

The fact that $\pi_Y^{\epsilon_1}$ is sublinearly contracting follows from Theorem 1.4, since Y is Morse. It remains only to prove the asymptotic statement in the case that $\rho_1(r) := r$, so suppose $\pi_Y^{\epsilon_0}$ is (r, ρ_2)–contracting.

For any $x \in X \setminus Y$ and each $i \in \{0, 1\}$, consider a point $x_i \in \pi_Y^{\epsilon_i}(x)$ and a point z_i on a geodesic from x to x_i with $d(x, z_i) = d(x, Y)$. Then:

$$\begin{aligned}d(x_0, x_1) &\leqslant d(x_0, z_0) + d(z_0, \pi_Y^{\epsilon_0}(z_0)) + \operatorname{diam} \pi_Y^{\epsilon_0}(z_0) \cup \pi_Y^{\epsilon_0}(x)\\ &\qquad + \operatorname{diam} \pi_Y^{\epsilon_0}(x) \cup \pi_Y^{\epsilon_0}(z_1) + d(\pi_Y^{\epsilon_0}(z_1), z_1) + d(z_1, x_1)\\ &\leqslant \epsilon_0 + 2\epsilon_0 + \rho_2(d(x, Y)) + \rho_2(d(x, Y)) + \epsilon_0 + \epsilon_1 + \epsilon_1\\ &= 4\epsilon_0 + 2\epsilon_1 + 2\rho_2(d(x, Y))\end{aligned}$$

If $d(x, y) \leqslant d(x, Y)$ then:

$$\begin{aligned}\operatorname{diam} \pi_Y^{\epsilon_1}(x) \cup \pi_Y^{\epsilon_1}(y) &\leqslant \operatorname{diam} \pi_Y^{\epsilon_1}(x) \cup \pi_Y^{\epsilon_0}(x) + \operatorname{diam} \pi_Y^{\epsilon_0}(x) \cup \pi_Y^{\epsilon_0}(y)\\ &\qquad + \operatorname{diam} \pi_Y^{\epsilon_0}(y) \cup \pi_Y^{\epsilon_1}(y)\\ &\leqslant 4\epsilon_0 + 2\epsilon_1 + 2\rho_2(d(x, Y)) + \rho_2(d(x, Y))\\ &\qquad + 4\epsilon_0 + 2\epsilon_1 + 2\rho_2(d(y, Y))\end{aligned}$$

Since $d(y, Y) \leqslant 2d(x, Y)$, this means that $\pi_Y^{\epsilon_1}$ is (r, ρ_2')–contracting for:

$$\rho_2'(r) := 8\epsilon_0 + 4\epsilon_1 + 3\rho_2(r) + 2\rho_2(2r) \asymp \rho_2(r)$$

□

Finally, consider changes to the target of the projection map.

Lemma 6.3. *Let Y and Y' be subspaces of a geodesic metric space X at bounded Hausdorff distance from one another. Suppose that π^ϵ_Y is (ρ_1,ρ_2)–contracting. Then $\pi^\epsilon_{Y'}$ is (r,ρ'_2)–contracting for some ρ'_2. If $\rho_1(r)=r$ then we can take $\rho'_2\asymp\rho_2$.*

Proof. Let C be the Hausdorff distance between Y and Y'.

For every $x\in X$ we have $\pi^\epsilon_{Y'}(x)\subset\overline{N}_C(\pi^{\epsilon+2C}_Y(x))$. The result now follows easily from Lemma 6.2. □

In light of Lemma 6.2, we can speak of the set Y being a contracting set if some ϵ–closest point projection to Y is contracting.

Definition 6.4. We say Y *is (ρ_1,ρ_2)–contracting* if there exists an $\epsilon\geqslant 0$ such that the ϵ–closest point projection $\pi^\epsilon_Y\colon X\to 2^Y$ is (ρ_1,ρ_2)–contracting.

Equivalently, Y *is (ρ_1,ρ_2)–contracting* if for all sufficiently small $\epsilon\geqslant 0$, if π^ϵ_Y does not have the empty set in its image, then π^ϵ_Y is (ρ_1,ρ_2)–contracting.

7. Geodesic image theorem

In this section we give an additional characterization of sublinear contraction in terms of projections of geodesic segments.

Theorem 7.1. *Let Y be a subspace of a geodesic metric space X. Suppose the empty set is not in the image of π^ϵ_Y. The following are equivalent:*

(1) There exist a sublinear function ρ and a constant $C\geqslant 0$ such that for every geodesic segment $\gamma\subset X$, with endpoints denoted x and y, if $d(\gamma,Y)\geqslant C$ then $\operatorname{diam}\pi^\epsilon_Y(\gamma)\leqslant\rho(\max\{d(x,Y),d(y,Y)\})$.
(2) There exist a sublinear function ρ' and a constant $C'\geqslant 0$ such that for every geodesic segment $\gamma\subset X$, if $d(\gamma,Y)\geqslant C'$ then $\operatorname{diam}\pi^\epsilon_Y(\gamma)\leqslant\rho'(\max_{z\in\gamma}d(z,Y))$.
(3) There exists a sublinear function ρ'' such that π^ϵ_Y is (r,ρ'')–contracting.

Moreover, $\rho\asymp\rho'\asymp\rho''$.

See Figure 3, letting γ be a subsegment of $\cup_i W_i$.

The case that Y is strongly contracting, that is, ρ'' is bounded, recovers the well-known 'Bounded Geodesic Image Property', cf [23, 8].

Corollary 7.2. *If Y is strongly contracting, $R_2\geqslant 1$ is a constant greater than twice the bound on the contraction function for Y, and γ is a geodesic segment that does not enter the R_2–neighborhood of Y then $\operatorname{diam}\pi^\epsilon_Y(\gamma)$ is bounded, with bound depending only on ϵ and ρ''.*

Alternatively, one could read Theorem 7.1 as saying that if π^ϵ_Y is sublinearly contracting and γ is a geodesic ray that is far from Y, but such that $\pi^\epsilon_Y(\gamma)$ is large, then $d(\gamma(t),Y)$ grows superlinearly with respect to $\operatorname{diam}\pi^\epsilon_Y(\gamma([0,t]))$.

Proof of Theorem 7.1.
(1) $\implies$ (3): Define $\rho_1(r):=r-C$ and $\rho_2(r)=\rho(2r-C)$. By Lemma 6.1, it suffices to show that π^ϵ_Y is (ρ_1,ρ_2)–contracting.

Suppose x and y are points of X with $d(x,y)\leqslant\rho_1(d(x,Y))$, and let γ be a geodesic from x to y. Then γ remains outside the C–neighborhood of Y, by the definition of ρ_1, so:

$$\begin{aligned}\operatorname{diam}\pi^\epsilon_Y(x)\cup\pi^\epsilon_Y(y)&\leqslant\operatorname{diam}\pi^\epsilon_Y(\gamma)\\&\leqslant\rho(\max\{d(x,Y),d(y,Y)\})\\&\leqslant\rho(2d(x,Y)-C)=\rho_2(d(x,Y))\end{aligned}$$

This proves (1) $\implies$ (3), and a similar argument proves (2) $\implies$ (3).

Now assume (3). If $d(x,y)\leqslant d(x,Y)+d(y,Y)$ then both (1) and (2) follow easily, so assume not. Let z_0 be the point of γ at distance $d(x,Y)$ from x. Our assumption says $d(z_0,y)>d(y,Y)$. Define points

z_{i+1} inductively as follows: if $d(z_i, y) > d(y, Y) + d(z_i, Y)$ define z_{i+1} to be the point of γ between z_i and y at distance $d(z_i, Y)$ from z_i. Let k be the last index so defined. From these choices we estimate:

$$\begin{aligned} \operatorname{diam} \pi_Y^\epsilon(\gamma) &\leqslant \operatorname{diam} \pi_Y^\epsilon(\overline{N}_{d(x,Y)}(x)) + \sum_{i=0}^{k} \operatorname{diam} \pi_Y^\epsilon(\overline{N}_{d(z_i,Y)}(z_i)) \\ &\quad + \operatorname{diam} \pi_Y^\epsilon(\overline{N}_{d(y,Y)}(y)) \\ &\leqslant 2\left(\rho''(d(x,Y)) + \sum_{i=0}^{k} \rho''(d(z_i,Y)) + \rho''(d(y,Y))\right) \end{aligned} \tag{5}$$

Since γ is a geodesic:

$$\begin{aligned} d(x,y) &= d(x, z_0) + \sum_{i=0}^{k-1} d(z_i, z_{i+1}) + d(z_k, y) \\ &= d(x,Y) + \sum_{i=0}^{k-1} d(z_i, Y) + d(z_k, y) \end{aligned} \tag{6}$$

We can also bound $d(x, y)$ in terms of the projections to Y:

$$\begin{aligned} d(x,y) &\leqslant d(x, \pi_Y^\epsilon(x)) + \operatorname{diam} \pi_Y^\epsilon(x) \cup \pi_Y^\epsilon(y) + d(\pi_Y^\epsilon(y), y) \\ &\leqslant d(x, \pi_Y^\epsilon(x)) + \operatorname{diam} \pi_Y^\epsilon(x) \cup \pi_Y^\epsilon(z_0) + \sum_{i=0}^{k-1} \operatorname{diam} \pi_Y^\epsilon(z_i) \cup \pi_Y^\epsilon(z_{i+1}) \\ &\quad + \operatorname{diam} \pi_Y^\epsilon(z_k) \cup \pi_Y^\epsilon(y) + d(\pi_Y^\epsilon(y), y) \\ &\leqslant d(x,Y) + \epsilon + \rho''(d(x,Y)) + \sum_{i=0}^{k-1} \rho''(d(z_i,Y)) \\ &\quad + \rho''(d(z_k,Y)) + \rho''(d(y,Y)) + d(y,Y) + \epsilon \end{aligned} \tag{7}$$

Combining (6) and (7) gives us the estimate:

$$\begin{aligned} \sum_{i=0}^{k-1} d(z_i, Y) - \rho''(d(z_i,Y)) &\leqslant \\ 2\epsilon + \rho''(d(x,Y)) &+ \rho''(d(z_k,Y)) + \rho''(d(y,Y)) + d(y,Y) - d(z_k, y) \end{aligned} \tag{8}$$

Define $R_n \geqslant 0$ such that for all $r \geqslant R_n$ we have $0 \leqslant \rho''(r) \leqslant r/n$. Suppose that $d(\gamma, Y) \geqslant R_2$ so that $d(z_i, Y) - \rho''(d(z_i, Y)) \geqslant \rho''(d(z_i, Y))$ for all i. These bounds, along with (8), (5), and $E := d(z_k, y) - d(y, Y)$ give:

$$\operatorname{diam} \pi_Y^\epsilon(\gamma) \leqslant 2\left(2\big(\epsilon + \rho''(d(x,Y)) + \rho''(d(z_k,Y)) + \rho''(d(y,Y))\big) - E\right)$$

By construction, $E > 0$, so to prove (2) it suffices to take $C' := R_2$ and $\rho(r) := 4\epsilon + 12\rho''(r)$.

To prove (1) we suppose $d(\gamma, Y) \geqslant C := R_4 \geqslant R_2$ and bound $2\rho''(d(z_k, Y)) - E$ in terms of $\rho''(d(y, Y))$. There are two cases to consider. If $d(z_k, Y) \leqslant 4d(y, Y)$ then $2\rho''(d(z_k, Y)) - E \leqslant 2\rho''(4d(y, Y))$. Otherwise, $d(z_k, Y) > 4d(y, Y)$ implies $E > d(z_k, Y)/2$, so:

$$2\rho''(d(z_k,Y)) - E < 2\frac{d(z_k,Y)}{4} - \frac{d(z_k,Y)}{2} = 0$$

Thus, it suffices to take $\rho'(r) := 4\epsilon + 12\rho''(4r)$. □

8. Further applications

First, we prove a general result.

Proposition 8.1. *Let X be a geodesic metric space. Suppose subspaces Y and Y' of X are μ–Morse. Let $\epsilon \geqslant 0$ be a constant such that there exist points $p \in Y$ and $p' \in Y'$ such that $d(p, p') \leqslant d(Y, Y') + \epsilon$. Then there exist a constant B and a sublinear function ρ, each depending only on μ and ϵ, satisfying the following conditions:*

- *If $d(Y, Y') \leqslant 2\mu(4, 0)$ then $Y \cup Y'$ is B–quasi-convex.*
- *If $d(Y, Y') > 2\mu(4, 0)$ then for every geodesic α from Y to Y' with $|\alpha| \leqslant d(Y, Y') + \epsilon$ and every geodesic γ from Y to Y' we have $d(\alpha, \gamma) < \rho(d(Y, Y'))$.*

Proof. Take geodesics α and γ as hypothesized. Let β be a geodesic from α to γ with $|\beta| = d(\alpha, \gamma)$. See Figure 5. Let $\delta := [p, x]_\alpha \beta [y, q]_\gamma$ and $\delta' := [p', x]_\alpha \beta [y, q']_\gamma$. (Recall that $[p, x]_\alpha$ denotes the subsegment

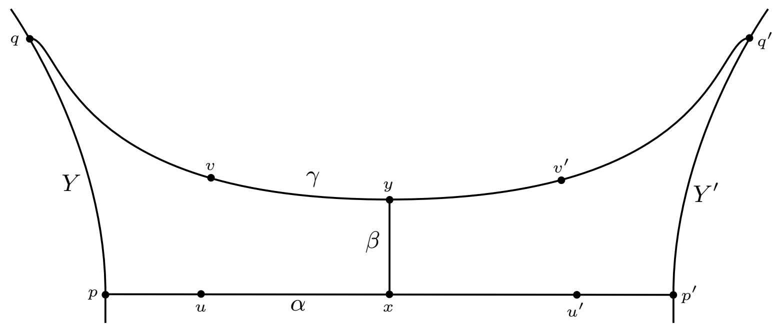

Figure 5. Setup for Proposition 8.1

of α from p to x.) Suppose that δ fails to be a $(k, 0)$–quasi-geodesic for some $k > 3$. Both $[p, x]_\alpha \beta$ and $\beta[y, q]_\gamma$ are $(3, 0)$–quasi-geodesics, by minimality of $d(x, y)$, so there exist points $u \in [p, x]_\alpha$ and $v \in [y, q]_\gamma$ such that $kd(u, v) < d(u, x) + d(x, y) + d(y, v)$. Now, $d(v, y) \leqslant d(v, u) + d(u, x) + d(x, y)$, so:

$$(k-1)d(x, y) \leqslant (k-1)d(u, v) < 2(d(u, x) + d(x, y))$$

Whence:

$$d(\alpha, \gamma) = d(x, y) < \frac{2d(u, x)}{k-3} \leqslant \frac{2|\alpha|}{k-3} \leqslant \frac{2(d(Y, Y') + \epsilon)}{k-3} \tag{9}$$

If $d(Y, Y') \leqslant 2\mu(4, 0)$ and δ is not a $(4, 0)$–quasi-geodesic then $d(\alpha, \gamma) < 4\mu(4, 0) + 2\epsilon$, by (9). This means $[y, q]_\gamma$ is a geodesic with one endpoint on Y and one within distance $6\mu(4, 0) + 2\epsilon$ of Y. Since Y is μ–Morse there is a B_0 depending on μ such that such a geodesic segment is contained in the B_0–neighborhood of Y.

If δ is a $(4, 0)$–quasi-geodesic it is contained in the $\mu(4, 0)$–neighborhood of Y.

The same arguments apply for δ', and $\gamma \subset \delta \cup \delta'$, so if $d(Y, Y') \leqslant 2\mu(4, 0)$ then $Y \cup Y'$ is B–quasi-convex for $B := \max\{B_0, \mu(4, 0)\}$.

Now suppose $d(Y, Y') > 2\mu(4, 0)$. Then δ and δ' cannot both be $(4, 0)$–quasi-geodesics. By (9):

$$\begin{aligned} d(\alpha, \gamma) &< \frac{2(d(Y, Y') + \epsilon)}{\sup\{k \in \mathbb{R} \mid \delta \text{ or } \delta' \text{ is not a } (k, 0)\text{–quasi-geodesic}\} - 3} \\ &\leqslant \frac{2(d(Y, Y') + \epsilon)}{\sup\{k \in \mathbb{R} \mid d(Y, Y') > 2\mu(k, 0)\} - 3} \end{aligned}$$

Define:

$$\rho(r) := \frac{2(r + \epsilon)}{\sup\{k \in \mathbb{R} \mid r > 2\mu(k, 0)\} - 3}$$

We interpret $\rho(r)$ to be 0 if $\{2\mu(k, 0)\}_{k \in \mathbb{R}}$ is bounded above by r. For $r \geqslant \epsilon$ we have:

$$\frac{\rho(r)}{r} \leqslant \frac{4}{\sup\{k \in \mathbb{R} \mid r > 2\mu(k, 0)\} - 3}$$

The denominator is unbounded and non-decreasing as a function of r, so we have $\lim_{r \to \infty} \frac{\rho(r)}{r} = 0$. $\square$

We first give an application of the second part of Proposition 8.1.

Proposition 8.2. *Let X be a geodesic metric space and let Y and Y' be μ–Morse subspaces of X. Let $\epsilon \geqslant 0$ be a constant such that the image of π_Y^ϵ does not contain the empty set, and such that there exist points $p \in Y$ and $p' \in Y'$ such that $d(p, p') \leqslant d(Y, Y') + \epsilon$.*

Suppose $d(Y, Y') > 2\mu(6, 0)$. Then there is a sublinear function ρ depending only on μ such that $\operatorname{diam} \pi_Y^\epsilon(Y') \leqslant \rho(d(Y, Y'))$.

Proof. Since Y is μ–Morse, there is a sublinear function ρ' depending only on μ such that Y is (r, ρ')–contracting, by Proposition 4.2.

Note that $p \in \pi_Y^\epsilon(p')$. Choose $q' \in Y'$ and $q \in \pi_Y^\epsilon(q')$. Let γ be a geodesic from q to q', let α be a geodesic from p to p', and let $x \in \alpha$ and $y \in \gamma$ be points such that $d(x, y) = d(\alpha, \gamma)$. The setup is the same as in Proposition 8.1, and we make the corresponding definitions of δ, δ', etc.

Suppose δ' is not a $(5, 0)$–quasi-geodesic. Define u' and v' as in Proposition 8.1, so that $d(u', x) + d(x, y) + d(y, v') > 5d(u', v')$. We have $p \in \pi_Y^\epsilon(u')$ and $q \in \pi_Y^\epsilon(v')$. By definition of x and y, we know $d(x, y) \leqslant d(u', v')$, so $d(u', x) + d(y, v') > 4d(u', v')$. In particular, we have $2d(u', v') < d(u', x)$ or $2d(u', v') < d(v', y)$. We suppose the former, the other case being similar.

First, suppose that $d(u', Y) < \epsilon$. Then:

$$\begin{aligned} d(p, q) &\leqslant d(p, v') + d(v', q) \\ &\leqslant 2d(p, v') + \epsilon \\ &\leqslant 2(d(p, u') + d(u', v')) + \epsilon \\ &\leqslant 2d(p, u') + d(u', x) + \epsilon \\ &\leqslant 3(d(u', Y) + \epsilon) + \epsilon < 7\epsilon \end{aligned}$$

Otherwise, if $d(u', Y) \geqslant \epsilon$, then we have:

$$d(u', v') < \frac{1}{2} d(u', x) \leqslant \frac{1}{2}(d(u', Y) + \epsilon) \leqslant d(u', Y)$$

By the contraction property:

$$d(p, q) \leqslant \operatorname{diam} \pi_Y^\epsilon(u') \cup \pi_Y^\epsilon(v') \leqslant \rho'(d(u', Y)) \leqslant \rho'(d(Y, Y') + \epsilon)$$

Suppose instead that δ' is a $(5, 0)$–quasi-geodesic. Then δ is not a $(6, 0)$–quasi-geodesic, since $d(Y, Y') > 2\mu(6, 0)$. By (9) we have:

$$d(x, y) < \frac{2}{3}(d(x, u)) \leqslant \frac{2}{3}(d(x, Y) + \epsilon)$$

If $d(x, Y) \leqslant 2\epsilon$ it follows that $d(x, y) \leqslant 2\epsilon$. Thus $d(y, Y) \leqslant d(y, x) + d(x, Y) \leqslant 4\epsilon$, and:

$$d(p, q) \leqslant d(q, y) + d(y, x) + d(x, p) \leqslant d(y, Y) + \epsilon + 2\epsilon + d(x, Y) + \epsilon \leqslant 10\epsilon$$

Otherwise $d(x, Y) > 2\epsilon$ and it follows that $d(x, y) \leqslant d(x, Y)$. We then use the contraction property to see:

$$d(p, q) \leqslant \operatorname{diam} \pi_Y^\epsilon(x) \cup \pi_Y^\epsilon(y) \leqslant \rho'(d(x, Y)) \leqslant \rho'(d(Y, Y') + \epsilon)$$

Since q' was an arbitrary point in Y' and q was an arbitrary point of $\pi_Y^\epsilon(q')$, we conclude $\operatorname{diam} \pi_Y^\epsilon(Y') \leqslant 2(\rho'(d(Y, Y') + \epsilon) + 10\epsilon)$. □

We also have the following applications of the first part of Proposition 8.1:

Corollary 8.3. *A geodesic triangle in which two of the sides are μ–Morse is δ–thin, with δ depending only on μ.*

Corollary 8.4. *Suppose X is a geodesic metric space and $\mathcal{P}$ is a collection of (ρ_1, ρ_2)–contracting paths such that for every pair of points x, $y \in X$ there exists a $\gamma \in \mathcal{P}$ with endpoints x and y. Then X is δ–hyperbolic, with δ depending only on ρ_1 and ρ_2.*

Corollary 8.4 is an analogue of [22, Theorem 2.3], which is roughly the same statement when the paths in $\mathcal{P}$ are all semi-strongly contracting with uniform contraction parameters.

Corollary 8.5. *Let G be a group generated by a finite set $\mathcal{S}$. Suppose there exist functions ρ_1 and ρ_2 and, for each $g \in G$, a path α_g from 1 to g in $\mathrm{Cay}(G, \mathcal{S})$ that is (ρ_1, ρ_2)–contracting. Then G is hyperbolic.*

We must assume uniform contraction in Corollary 8.5, even for finitely presented groups. Druţu, Mozes, and Sapir [17] show that if H is a finitely generated subgroup of a finitely generated group G and $h \in H$ is a Morse element in G, that is, $\langle h \rangle$ is Morse in some, hence, every, Cayley graph of G, then h is a Morse element in H. Thus, if H is a finitely generated subgroup of a torsion-free hyperbolic group then every element of H is Morse. However, Brady [9] constructed an example of a finitely presented subgroup H of a torsion-free hyperbolic group G such that H is not hyperbolic.

Fink [20] claims that if all geodesics in a homogeneous proper geodesic metric space are Morse, then the space is hyperbolic. First is an assertion, [20, Proposition 3.2], that if every geodesic is Morse then the collection of geodesics is uniformly Morse, ie, there exists a μ such that every geodesic is μ–Morse. Then an asymptotic cone argument is used to conclude the space is hyperbolic. This second step can now be accomplished via our Corollary 8.4 without resort to the asymptotic cone machinery.

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Faculty of Mathematics, University of Vienna, Oskar-Morgenstern-Platz 1, 1090 Vienna, Austria
E-mail address: goulnara.arzhantseva@univie.ac.at

Faculty of Mathematics, University of Vienna, Oskar-Morgenstern-Platz 1, 1090 Vienna, Austria
E-mail address: christopher.cashen@univie.ac.at

Institut de Mathématiques, Université de Neuchâtel, Rue Emile-Argand 11, 2000 Neuchâtel, Switzerland
Current address: Department of Mathematics, ETH Zürich, Rämistrasse 101, 8092 Zürich, Switzerland
E-mail address: dominik.gruber@math.ethz.ch

Mathematical Institute, University of Oxford, Woodstock Road, Oxford Ox2 6GG, United Kingdom
E-mail address: david.hume@maths.ox.ac.uk