RIPS CONSTRUCTION WITHOUT UNIQUE PRODUCT

GOULNARA ARZHANTSEVA AND MARKUS STEENBOCK

ABSTRACT. Given a finitely presented group Q, we produce a short exact sequence $1 \to N \hookrightarrow G \twoheadrightarrow Q \to 1$ such that G is a torsion-free Gromov hyperbolic group without the unique product property and N is without the unique product property and has Kazhdan's Property (T). Varying Q, we show a wide diversity of concrete examples of Gromov hyperbolic groups without the unique product property. As an immediate application, we obtain Tarski monster groups without the unique product property.

1. INTRODUCTION

A group G has the *unique product property* (or said to be a *unique product group*) whenever for all pairs of non-empty finite subsets A and B of G the set of products AB has an element $g \in G$ with a unique representation of the form $g = ab$ with $a \in A$ and $b \in B$. Unique product groups are torsion-free. They satisfy the outstanding Kaplansky zero-divisor conjecture [Kap57, Kap70], which states that the group ring of a torsion-free group over an integral domain has no zero-divisors. Rips and Segev [RS87] gave the first examples of torsion-free groups without the unique product property. In [Ste15], the second author has generalized their examples, proved that the (generalized) Rips-Segev groups are Gromov hyperbolic, and provided an uncountable family of non unique product groups. Other examples of torsion-free groups without the unique product property can be found in [Pro88, Car14].

Our goal is to construct new concrete examples of non unique product groups with diverse algebraic and geometric properties. We realize this by extending further our construction of generalized Rips-Segev groups and by showing that every finitely presented group is a non-trivial quotient of a torsion-free Gromov hyperbolic non unique product group.

Theorem 1.1. *Let Q be a finitely presented group. Then there exists a short exact sequence*

$$1 \to N \hookrightarrow G \twoheadrightarrow Q \to 1$$

such that

- *G is a torsion-free Gromov hyperbolic group without the unique product property,*
- *N is a 2-generated subgroup of G.*

2010 *Mathematics Subject Classification.* 20F06, 20F67.
Key words and phrases. Graphical small cancellation theory, unique product property, Kaplansky's zero-divisor conjecture, free products of groups, Cayley graphs with prescribed subgraphs.

The assumption on finite presentation of Q can be relaxed and our method still provides a non unique product group G. In such a general setting, G is not any more Gromov hyperbolic, although it is a direct limit of those (in fact, of graphical small cancellation groups, see more details in Section 3).

Theorem 1.2. *Let Q be a finitely generated group. Then there exists a short exact sequence*

$$1 \to N \hookrightarrow G \twoheadrightarrow Q \to 1$$

such that

- *G is a torsion-free group without the unique product property which is a direct limit of Gromov hyperbolic groups,*
- *N is a 2-generated subgroup of G.*

Varying Q in these theorems, we obtain many new groups without the unique product property that have various algebraic and algorithmic properties, see Section 3.2.

We extend our construction further and produce strongly non-amenable examples.

Theorem 1.3. *Let Q be a finitely generated group. Then there exists a short exact sequence*

$$1 \to N \hookrightarrow G \twoheadrightarrow Q \to 1$$

such that

- *G is a torsion-free group without the unique product property which is a direct limit of Gromov hyperbolic groups,*
- *N is a subgroup of G with Kazhdan's Property (T) and without the unique product property.*

We provide, in particular, first examples of Property (T) groups without the unique product property.

Corollary 1.4. *There are torsion-free Gromov hyperbolic groups with Kazhdan's Property (T) and without the unique product property.*

Theorem 1.3 generalises the result on Rips short exact sequence with Kazhdan's Property (T) kernel from [OW07]. An alternative (T)-Rips construction is in [BO08].

Our approach combines three constructions: the famous Rips construction [Rip82], the construction by Rips-Segev of torsion-free groups without the unique product property [RS87], and Gromov's groundbreaking construction of graphical small cancellation groups with Property (T), cf. [Gro03, 1.2.A, 4.8.(3)], based on his spectral characterization of this property [Sil03, OW07].

An essential technical point in our proofs is that we explain all three constructions using the graphical small cancellation theory over the free product of groups. This viewpoint is novel and of independent interest as it provides a new route for building group presentations with Cayley graphs containing (subdivisions of) prescribed subgraphs. We show, in particular, that Gromov's probabilistic construction of graphs defining groups with Property (T) is flexible under taking edge subdivisions.

Theorem 1.5. *For all $m > 64$, there exists a finite connected graph $\mathcal{T}$ labeled by $\{a_1, \dots, a_m\}$ such that the labeling satisfies the $Gr'_*(1/6)$–small cancellation condition over the free product $\langle a_1 \rangle * \dots * \langle a_m \rangle$, the labeling satisfies the $Gr'(1/6)$–small cancellation condition with respect to the word length metric, and the group with $a_1, \dots, a_m$ as generators and the labels of the cycles of $\mathcal{T}$ as relators has Property (T).*

The graph $\mathcal{T}$ is produced by assigning to every edge of an expander graph a letter and an orientation independently uniformly at random. It is an interesting technical outcome that the small cancellation conditions over the free group and over the free product can be combined in such graphs, see Section A. This flexibility in the small cancellation condition is useful for constructing new groups with exotic properties.

Our examples are in a strong contrast with the previously known constructions of torsion-free groups without the unique product property, alternative to the Rips-Segev groups [Pro88, Car14]. Indeed, all those constructions yield infinite groups with the Haagerup property[1] (= a-T-menable groups, in the terminology of Gromov, see [CCJ$^+$01]), and, hence, groups which do not have Property (T).

Another way to get more non-unique product groups is to use free products of our Gromov hyperbolic non-unique product groups with suitable groups. We can then obtain required quotients of such free products. For instance, an alternative proof of our Corollary 1.4, although without probabilistic and, hence, genericity aspects underlying Theorem 1.5, can be using the small cancellation theory over hyperbolic groups.

Theorem 1.6 (Ol'shanskii, cf. [Ol′93, Th. 2])**.** *Let $G = H_1 * H_2$ be the free product of two non-elementary torsion-free Gromov hyperbolic groups and $M \subseteq H_1$ be a finite subset. Then G has a non-elementary torsion-free Gromov hyperbolic quotient $\overline{G}$ such that the canonical projection $G \twoheadrightarrow \overline{G}$ is surjective on H_2 and injective on M.*

This result, together with our main Theorem 1.1, indeed yields Corollary 1.4. Take for H_1 our torsion-free Gromov hyperbolic group without the unique product property for the sets A and B produced by Theorem 1.1. Take for H_2 a Gromov hyperbolic group with Property (T) (e.g. a discrete subgroup of finite covolume in $Sp(n, 1)$) and for M a finite subset of H_1 containing A, B, and AB. By Theorem 1.6, we get a torsion-free Gromov hyperbolic group $\overline{G}$ with Property (T) and without the unique product property[2].

Our two ways to construct groups in Corollary 1.4 have two distinct outcomes. The first approach, using Theorem 1.3, yields the existence of graphical small cancellation presentations of such groups. Moreover, it gives their genericity as well as a recursive procedure to build such a generic graphical small cancellation presentation. In contrast, the second approach, based on Theorem 1.6, provides first explicit examples. However, such examples are not graphical small cancellation presentations and to make them

[1] Groups in [Pro88] are solvable, hence, a-T-menable; groups in [Car14] are a-T-menable as they have $\mathbb{Z}^k \times \mathbb{F}_m$ as a finite index subgroup.

[2] Similarly, take $G_{\mathcal{RS}} * \mathbb{F}_m$, with $G_{\mathcal{RS}}$ a Rips-Segev group from [Ste15], and use the small cancellation theory over hyperbolic groups to get a group $\widetilde{H}_1$ with properties as in Theorem 1.1, where N is a quotient of $G_{\mathcal{RS}} * \mathbb{F}_m$ [BO08]; due to a referee, this is in contrast to our graphical over the free product approach.

explicit one needs to use a rather involved general method of graded van Kampen diagrams over hyperbolic groups, developed in [Ol′93].

A further strong consequence of our results is the existence of Tarski monster groups without the unique product property.

Corollary 1.7. *There are torsion-free Property (T) groups G without the unique product property such that all proper subgroups of G are cyclic.*

Indeed, it follows from [Ol′93, Th. 2] that every non-cyclic torsion-free Gromov hyperbolic group G has a non-abelian torsion-free quotient $\widetilde{G}$ such that all proper subgroups of $\widetilde{G}$ are cyclic, and that $G \twoheadrightarrow \widetilde{G}$ is injective on any given finite subset of G [Ol′93, Cor. 1]. Applied to a finite subset containing A, B, and AB in a group G given by Theorem 1.1, this immediately yields Tarski monster groups without the unique product property.

In particular, we obtain the first examples of groups without the unique product property all of whose proper subgroups are unique product groups. Again, explicit recursive presentations are available for such new monster groups.

Our constructions are of particular interest also in the context of the following two important open problems.

Open problem 1.8. *Do the Rips-Segev groups without the unique product property satisfy the Kaplansky zero-divisor conjecture?*

Combining recent deep results [Sch14, LOS12, Ago13], we observe that the Kaplansky zero-divisor conjecture holds for all torsion-free CAT(0)-cubical[3] Gromov hyperbolic groups over the field of complex numbers. Our groups from Corollary 1.4 are not CAT(0)-cubical as they are infinite Property (T) groups. Thus, it follows from our results that the CAT(0)-cubulation cannot solve the Kaplansky zero-divisor conjecture for all Gromov hyperbolic groups without the unique product property.

Open problem 1.9. *Is every Gromov hyperbolic group residually finite?*

If Q is finite then N in our construction is normal of finite index and without the unique product property. Every residually finite Gromov hyperbolic group has a finite index subgroup with the unique product property by a result of Delzant [Del97]. Then the following questions arise naturally.

- Does there exist a Gromov hyperbolic group all of whose normal finite index subgroups are without the unique product property?
- Does there exist a Gromov hyperbolic group all of whose subgroups of index at most k, for a given $k \geqslant 2$, are without the unique product property?

The last question has recently been answered in affirmative [GMS15], via a further application of the generalized Rips-Segev graphs.

Acknowledgments. Both authors are partially supported by the ERC grant ANALYTIC no. 259527 of G. Arzhantseva and by the Austria-Romania research cooperation grant GALS-2014-15. The first author was partially supported by the Simons Foundation and by EPSRC grant no EP/K032208/1, while visiting the Isaac Newton

[3]A group is *CAT(0)-cubical* if it admits a proper cocompact action on a CAT(0)-cubical complex.

Institute (Cambridge). The second author is a recipient of the DOC fellowship of the Austrian Academy of Sciences and was partially supported by the University of Vienna research grant 2013.

We thank the Erwin Schrödinger International Institute for Mathematical Physics in Vienna for supporting the workshop "Geometry and Computation in Groups" and participants A. Minasyan and F. Dahmani, who, after they learned from our results, pointed out that Ol'shanskii's small cancellation theory helps to produce more Property (T) groups without the unique product property.

2. RIPS CONSTRUCTION VIA THE FREE PRODUCT OF GROUPS

In this section, we review the original Rips construction [Rip82] but regard it in the context of small cancellation theory over the free product of groups. This allows us to explicit the choice of group relators in an easier way and hence, to provide concrete group presentations of the middle group, both in the original and in our new short exact sequences of groups, see Theorems 1.1 and 1.3.

Let $Q = \langle x_1, \dots, x_m \mid r_1, \dots, r_n \rangle$ be a finitely presented group.

Let $a, b \notin \{x_1^{\pm 1}, \dots, x_m^{\pm 1}\}$. We consider the free product $\langle x_1, \dots, x_m, a\rangle * \langle b \rangle$, endowed with the *free product length* $|\cdot|_*$, also known as the *syllable-length* [LS77].

Let H be a group defined by a presentation $\langle x_1, x_2, \dots, x_m, a, b \mid R\rangle$, where the set R of relators consists of the following $n + 4m$ words:

(1)
$r_i a^{10i-9} b a^{10i-8} b a^{10i-7} b \cdots b a^{10i} b$, $1 \leqslant i \leqslant n$;

(2)
$x_j^{-1} a x_j a^{10(n+j)-9} b a^{10(n+j)-8} b a^{10(n+j)-7} b \cdots b a^{10(n+j)} b$, $1 \leqslant j \leqslant m$,
$x_j a x_j^{-1} a^{10(m+n+j)-9} b a^{10(m+n+j)-8} b a^{10(m+n+j)-7} b \cdots b a^{10(m+n+j)} b$, $1 \leqslant j \leqslant m$;

(3)
$x_j^{-1} b x_j a^{10(2m+n+j)-9} b a^{10(2m+n+j)-8} b a^{10(2m+n+j)-7} b \cdots b a^{10(2m+n+j)} b$, $1 \leqslant j \leqslant m$,
$x_j b x_j^{-1} a^{10(3m+n+j)-9} b a^{10(3m+n+j)-8} b a^{10(3m+n+j)-7} b \cdots b a^{10(3m+n+j)} b$, $1 \leqslant j \leqslant m$,

and the length on H is the free product length induced from $\langle x_1, \dots, x_m, a\rangle * \langle b\rangle$.

In the terminology of the *small cancellation theory over the free product* [LS77, Ch. V. 9], the *pieces* in these relators have length (= the free product length) at most 3 and the relators have length 19. Hence, this presentation of H satisfies the classical free product $C'(1/6)$–small cancellation condition over $\langle x_1, \dots, x_m, a\rangle * \langle b\rangle$.

It follows that H is torsion-free [LS77, Th. 10.1, Ch. V] and Gromov hyperbolic [Pan99]. Let N be the subgroup generated by a and b. The relators (2) and (3) guarantee that N is normal in H. Thus, N coincides with the kernel of the epimorphism $H \twoheadrightarrow Q$ which maps $a \mapsto 1$, $b \mapsto 1$ and $x_i \mapsto x_i$ for all i. We conclude the following significant result of Rips.

Proposition 2.1. [Rip82] *Let Q be a finitely presented group. Then there exists a short exact sequence*

$$1 \to N \hookrightarrow H \twoheadrightarrow Q \to 1$$

such that

- *H is a torsion-free Gromov hyperbolic group,*
- *N is a non-trivial normal 2-generator subgroup of H.*

3. RIPS CONSTRUCTION WITHOUT UNIQUE PRODUCT

3.1. **Combining Rips construction and Rips-Segev construction.** We define here our group G required by Theorem 1.1. We begin with definitions.

Let $\mathcal{G}$ be a (finite or infinite) graph with directed edges. A labeling ℓ of $\mathcal{G}$ by $\{x_1, \ldots, x_n, a, b\}$ assigns to every edge e a letter y equal to x_i, a or b, so that going along e in positive direction we read y and going along e in negative direction we read y^{-1}.

A path in $\mathcal{G}$ is *reduced* if it has no backtracking. Every reduced labeled path in $\mathcal{G}$ bears as label a word in letters from $\{x_1, \ldots, x_n, a, b\}$. On the other hand, for every such a word w there is a reduced path labeled by w.

Let e_1 and e_2 be edges in $\mathcal{G}$ with a common vertex v which are either both directed towards v or both directed away from v, and such that $\ell(e_1) = \ell(e_2)$. A *Stallings folding* (briefly, a folding) of $\mathcal{G}$ is the identification of two such edges. A labeling of $\mathcal{G}$ is *reduced* if it does not admit any foldings.

A *(graphical) piece* in a reduced labeled graph $\mathcal{G}$ is a reduced labeled path which has at least two distinct immersions into $\mathcal{G}$.

The labeling satisfies the $Gr'(1/6)$*–graphical small cancellation condition* whenever for all pieces p in $\mathcal{G}$ we have

$$|\ell(p)| < \frac{1}{6}\min\{|\ell(c)| \mid c \text{ is a non-trivial cycle in } \mathcal{G}\},$$

where $|\cdot|$ denotes the usual word length metric on the free group on the free generating set $\{x_1, \ldots, x_n, a, b\}$.

The labeling satisfies the $Gr'_*(1/6)$*–graphical small cancellation condition* whenever for all pieces p in $\mathcal{G}$ we have

$$|\ell(p)|_* < \frac{1}{6}\min\{|\ell(c)|_* \mid c \text{ is a non-trivial cycle in } \mathcal{G}\}.$$

A *generalized Rips-Segev graph $\mathcal{RS}$*, associated to given non-empty finite subsets A and B of elements in $\langle a\rangle * \langle b\rangle$, is a connected finite reduced graph labeled by $\{a, b\}$, whose labeling satisfies the $Gr_*(1/6)$–graphical small cancellation condition, and whose cycles are labeled by words expressing the non unique product property of AB in a group generated by a, b subject to these relator words [Ste15]. We explain the construction of such graphs on a concrete example, which we use then to prove our theorems.

Let $c_i := b^i a^{10^{J+10i-9}} b \cdots b a^{10^{J+10i-5}}$ for $i \in \{1, \ldots, K\}$ and some integers $K, J \geqslant 1$ to be specified below. Let

$$A := \bigsqcup_{i=1}^{K} \left\{c_i, c_i a, c_i a^2, \ldots, c_i a^{10^{J+10i-1}}\right\} \text{ and } B := \{1, a, b, ab\}.$$

We now produce a graph encoding the non unique product property of AB.

We first choose a finite connected regular covering graph Θ of the (oriented) bouquet of 4 cycles labeled by the letters y_1, y_2, y_3, y_4 such that the length of non-trivial cycles in

Θ is at least 19. Such a covering graph does exist as the free group (= the fundamental group of the bouquet) is residually finite. Indeed, for every ball $B(r)$ of radius r in the Cayley graph of the free group there exists a finite index normal subgroup N such that $B(r) \cap N$ is the identity. The covering graph corresponding to N is a finite connected graph with vertex degree 8 and girth at least $2r$. Taking $r \geqslant 10$ yields the required covering graph Θ.

Let us enumerate the vertices of Θ by $1, \dots, i, \dots K$. As it is the covering graph of the bouquet labeled by y_1, y_2, y_3, y_4, for all $1 \leqslant j \leqslant 4$ each vertex i has an edge $y_{ij} := (l_{ij}, i)$ and an edge $z_{ij} := (i, k_{ij})$ such that $z_{ij} = y_{k_{ij}j}$. These are the "in" and "out" edges at vertex i labeled by y_j if we consider the labeling of Θ induced from the bouquet by the covering map.

We define a new labeling L of Θ as follows. For each i, we set

$$\begin{aligned} L(y_{i1}) :=& ba^{-10^{J+10i-4}} \\ L(y_{i2}) :=& a^{10^{J+10l_{i2}}} ba^{-10^{J+10i-3}} \\ L(z_{i3}) :=& a^{10^{J+10i-2}} ba^{-10^{J+10k_{i3}}} \\ L(z_{i4}) :=& a^{10^{J+10i-1}} b. \end{aligned}$$

The graph Θ coincides with the graph used by Rips-Segev in their original construction [RS87].

Now we subdivide and label edges of Θ according to L, so that an edge bears a letter from $\{a, b\}$ and has the respective orientation. For instance, the edge y_{i1} of Θ becomes a path of length $10^{J+10i-4} + 1$ all of whose vertices but the endpoints are of degree 2 and whose label is $ba^{-10^{J+10i-4}}$. The resulting labeled graph, denoted by Θ', is not reduced. For our purposes, we further reduce the graph (that is, make all possible foldings). Let us denote such a reduced graph by Θ''. Note that certain vertices of Θ'' are identified with vertices $1, \dots, i, \dots, K$ of Θ: the vertices of Θ we started with have not been identified while producing Θ''. Note that the free product length of non-trivial cycles in Θ'' is at least 19.

Observe that a non-zero exponent P_i in words

$$a^Q b^{\varepsilon_1} a^{P_i} b^{\varepsilon_2} a^{Q'} \text{ with } \varepsilon_t = \pm 1,$$

read on reduced paths in Θ'' (hence the reduction of the label on a path $(l_{ij}, i), (i, k_{is})$ in Θ') is unique among the numbers

$$10^{J+10i-4}9, \dots, 10^{J+10i-4}9999, 10^{J+10i-3}9, \dots, 10^{J+10i-1}9, 10^{J+10i-4}, \dots, 10^{J+10i},$$

that is, all P_i's are pairwise distinct.

We now use Θ'' to produce a new graph. Namely, for each i we glue a path γ_i labeled by c_i to Θ'' so that vertex i of Θ'' is identified with the endpoint of γ_i. The starting points of all γ_i's are all identified with a new vertex, denoted by 0, and γ_i has no vertices other than its endpoint in common with Θ''. Let us fold this new graph and denote the result by $\mathcal{RS}$. We see, as above, that the exponents P of a in paths labeled by $b^{\varepsilon_1} a^P b^{\varepsilon_2}$ are pairwise distinct.

The reduced graph $\mathcal{RS}$ is our generalized Rips-Segev graph. In particular, the graphical pieces in $\mathcal{RS}$ consist of paths a^P, $a^P b^{\varepsilon} a^{P'}$, and $a^P b^{2\varepsilon} a^{P'}$, where P and P' are possibly zero. Therefore, the free product length of the pieces in $\mathcal{RS}$ is at most 3. By construction, the free product length of all "new" cycles in $\mathcal{RS}$ is at least 19. Thus, the free product length of all non-trivial cycles in $\mathcal{RS}$ is at least 19. It follows that the labeling of $\mathcal{RS}$ satisfies the $Gr'_*(1/6)$–graphical small cancellation condition over the free product $\langle x_1, \ldots, x_m, a\rangle * \langle b\rangle$.

If we choose J such that $9(n+4m) < 10^J$, then P_i is not among the numbers $1, \ldots, 9(n+4m)$ for all i.

Let $\mathcal{G}$ be the disjoint union of $\mathcal{RS}$ and of the $(n+4m)$ cycles labeled by words (1), (2), and (3) defined above. The graphical pieces in $\mathcal{G}$ are of length at most 3 and non-trivial cycles are labeled by words whose free product length is at least 19 in $\langle x_1, \ldots, x_m, a\rangle * \langle b\rangle$. Therefore, the labeling of $\mathcal{G}$ satisfies the $Gr'_*(1/6)$–graphical small cancellation condition over the free product $\langle x_1, \ldots, x_m, a\rangle * \langle b\rangle$.

Let G be a group generated by $x_1, \ldots, x_m, a, b$, subject to relators defined by $\mathcal{G}$,

$$G := \langle x_1, \ldots, x_m, a, b \mid \text{labels on reduced cycles of } \mathcal{G}\rangle,$$

also denoted briefly $G = \langle x_1, \ldots, x_m, a, b \mid \mathcal{G}\rangle$. The following general results ensure that G is torsion-free and Gromov hyperbolic.

Theorem 3.1. [Ste15, Th. 1] *Let $G_1, \ldots, G_n$ be finitely generated groups. Let $\mathcal{G}$ be a family of finite connected graphs edge-labeled by $G_1 \cup \ldots \cup G_n$ so that the $Gr'_*(1/6)$–graphical small cancellation condition with respect to the free product length on the free product $G_1 * \cdots * G_n$ is satisfied. Let G be the group given by the corresponding graphical presentation, that is, the quotient of $G_1 * \cdots * G_n$ subject to the relators being the words read on the cycles of $\mathcal{G}$.*

Then G satisfies a linear isoperimetric inequality with respect to the free product length. Moreover, G is Gromov hyperbolic whenever $G_1, \ldots, G_n$ are Gromov hyperbolic and $\mathcal{G}$ is finite.

The graph $\mathcal{G}$ injects into the Cayley graph of G with respect to $G_1 \cup \ldots \cup G_n$.

Furthermore, G is torsion-free whenever $G_1, \ldots, G_n$ are torsion-free and $\mathcal{G}$ satisfies the $Gr'_*(1/8)$–graphical small cancellation condition[4].

Applying this theorem to $G = \langle x_1, \ldots, x_m, a, b \mid \mathcal{G}\rangle$ we obtain that G is Gromov hyperbolic. Torsion-freeness of our G, and an alternative proof for the graph injection, is obtained in the more general framework of the $C(6)$–graphical small cancellation theory over the free group [Gru15, Cor. 2.19, Lem. 4.1]: in our case $G_i \cong \mathbb{Z}$ and so the $Gr'_*(1/6)$-condition implies the $C(6)$-condition.

Thus, it suffices to check that G does not have the unique product property. This, together with Proposition 2.1, will then imply our main result, Theorem 1.1.

Theorem 3.2. *The group G does not satisfy the unique product property. Namely, the sets A and B embed into G and do not have the unique product property in G.*

Proof. The set R_G of relators of G is the set of all words read on the reduced cycles of $\mathcal{G}$. By definition, R_G is the disjoint union of the above defined set R and the words

[4]We believe that the torsion-freeness holds under the $Gr'_*(1/6)$–condition as well, though we do not require it here despite of our choice of $\mathcal{G}$ with the $Gr'_*(1/6)$–condition. Also, we can produce a graph $\mathcal{G}$ with the $Gr'_*(1/8)$-condition, and then have both hyperbolicity and torsion-freeness by Theorem 3.1.

read on the cycles of $\mathcal{RS}$. The latter words encode the non unique product property of finite sets A and B of cyclically reduced words in the letters $a^{\pm 1}$ and $b^{\pm 1}$ (in the group $\langle a, b \mid \mathcal{RS} \rangle$ defined by $\mathcal{RS}$). Denote by A' and B' the image of A and B in G. Let us show that the maps $\iota_A \colon A \to A'$ and $\iota_B \colon B \to B'$ are injective. By construction of $\mathcal{RS}$, it will follow that A and B do not have the unique product property in G.

For each $\overline{a} \in A$ the generalized Rips-Segev graph $\mathcal{RS}$ contains a reduced path labeled by the word $\overline{a}$, whose starting vertex is 0 and whose terminal vertex is uniquely determined by the label $\overline{a}$ as $\mathcal{RS}$ is reduced. Suppose that $a_1 \neq a_2 \in A$ and $a_1 =_G a_2$ in G. Let p be a reduced path in $\mathcal{RS}$ connecting the vertices a_1 and a_2 (that is, connecting the endpoints of two paths starting at vertex 0 and labeled by a_1 and a_2, respectively). Let x be the label of p. Then $x =_G 1$. If $a_1 = c_i a^k$, $a_2 = c_j a^l$, then $x = c_i a^{k-l} c_j^{-1}$ is one such label. But $x =_G 1$, then also $b^{-j} c_i a^{k-l} c_j^{-1} b^j =_G 1$. Now $|b^{-j} c_i a^{k-l} c_j^{-1} b^j|_* \leqslant 18$. That is, there is a relation in G of free product length at most 18.

This is a contradiction as the $Gr'_*(1/6)$–graphical small cancellation condition over the free product $\langle x_1, \ldots, x_m, a \rangle * \langle b \rangle$ implies that the free product length of non-trivial relations in G has to be at least 19 (=the girth of the defining graph $\mathcal{G}$).

This follows from the analysis of van Kampen diagrams over free products, see [Ste15, Lemma 1.3], and the observation that the label of each face in a van Kampen diagram equals to the label of a non-trivial reduced cycle in the graph. Therefore, $\iota_A \colon A \to A'$ is injective.

Now for $B = \{1, a, b, ab\}$. If $b_1 \neq b_2 \in B$ and $b_1 =_G b_2$, then $|b_1 b_2^{-1}|_* < 3$. As before we conclude that $\iota_B \colon B \to B'$ is injective as well.

Thus, G does not satisfy the unique product property. □

Remark 3.3. Alternatively, the injectivity of the above maps $\iota_A \colon A \to A'$ and $\iota_B \colon B \to B'$ is immediate from Theorem 3.1. Indeed, for A, by construction the terminal vertices of the two reduced paths labeled by a_1 and a_2 are distinct in $\mathcal{RS}$, the graph $\mathcal{RS}$ injects into $\mathcal{G}$, and our labeling of $\mathcal{G}$ satisfies the $Gr'_*(1/6)$–condition. Theorem 3.1 ensures that $\mathcal{G}$ injects into the Cayley graph of G and, hence, A injects into G. The argument for B is analogous.

Corollary 3.4. *Each connected component of the graph $\mathcal{G}$ injects into the Cayley graph of the group G it defines.*

The groups G we have just constructed are the only known examples of torsion-free Gromov hyperbolic groups without the unique product property. It remains unknown whether these torsion-free groups without the unique product property do satisfy the Kaplansky zero-divisor conjecture.

The proof of Theorem 1.2 is straightforward by the arguments above applied to an infinite generalized Rips-Segev graph, that is, to an infinite disjoint union of finite Rips-Segev graphs. Several explicit constructions of such infinite families of finite graphs can be found in [Ste15]. A finite subunion of such a family yields a torsion-free Gromov hyperbolic group without the unique product property as above, whence the direct limit of such groups for the resulting group G in Theorem 1.2.

3.2. **More examples of torsion-free groups without unique product.** We now vary the quotient group Q in our new short exact sequence above, Theorem 1.1. This provides a wide diversity of new Gromov hyperbolic torsion-free groups without the unique product property which satisfy many unusual algebraic, geometric, and algorithmic properties. In particular, all our examples below are non-trivial in the sense that they are not isomorphic to a free product. The following results are immediate generalizations of [Rip82].

Proposition 3.5. *For each of the following, there exists a torsion-free Gromov hyperbolic group G without the unique product property, which is not isomorphic to a free product, and such that:*

- *G has unsolvable generalized word problem;*
- *there are finitely generated subgroups P_1, P_2 of G such that $P_1 \cap P_2$ is not finitely generated;*
- *there is a finitely generated but not finitely presented subgroup of G;*
- *for any $r \geqslant 3$ there is an infinite strictly increasing sequence of r–generated subgroups of G.*

More algorithmic properties in the context of Rips construction are investigated in [BMS94]. Applied to our situation they yield the following.

Proposition 3.6. *For each of the following, there exists a torsion-free Gromov hyperbolic group G without the unique product property, which is not isomorphic to a free product, and such that there is no algorithm to determine*

- *the rank of G;*
- *whether an arbitrary finitely generated subgroup of G has finite index;*
- *whether an arbitrary finitely generated subgroup of G is normal;*
- *whether an arbitrary finitely generated subgroup of G is finitely presented;*
- *whether an arbitrary finitely generated subgroup S of G has a finitely generated second integral homology group $H_2(S, \mathbb{Z})$.*

The proofs are by choosing a group Q with the required property, which then allows to pullback the property to the group H and then (immediately, for the above algorithmic properties) to G. The presentations of Q and, hence, of H and of G can be given explicitly as in the previous section.

3.3. **(T)-Rips-construction without unique product.** Our viewpoint on Rips construction via the free product of groups allows to combine the original, now classical, arguments with further geometric properties by adding suitable new relators to the presentation of H. In our Theorem 1.1, these new relators encode the non unique product property and are given by the generalized Rips-Segev graph. Another famous property that can be encoded by graphs is Kazhdan's Property (T). For instance, Gromov's spectral characterization of Property (T) can be used to get finitely presented groups with Kazhdan's Property (T) given by the graphical $Gr'(1/6)$–small cancellation presentations [Gro03, Sil03]. Mixing the original Rips construction and this Gromov's result, Ollivier and Wise [OW07] obtain a short exact sequence $1 \to N \hookrightarrow G \twoheadrightarrow Q \to 1$, where G is a torsion-free group defined by a finite graphical $Gr'(1/6)$–small cancellation presentation and N has Kazhdan's Property (T). We extend their result as follows.

Theorem 3.7. *Let Q be a finitely presented group. Then there exists a short exact sequence*

$$1 \to N \hookrightarrow G \twoheadrightarrow Q \to 1$$

such that

- *G is a torsion-free non-elementary Gromov hyperbolic group,*
- *N has Kazhdan's Property (T) and does not satisfy the unique product property.*

Proof. Let $\langle x_1, \ldots, x_m \mid r_1, \ldots, r_n \rangle$ be a presentation of Q. Let $a_1, \ldots, a_{35}, b$ be distinct and different from each of $x_1, \ldots, x_m$. Let $\mathcal{T}$ be the finite graph provided by our Theorem A.1 below, with a labeling by $\{a_1, \ldots, a_{35}, b\}$ such that the group $\langle a_1, \ldots, a_{35}, b \mid \mathcal{T} \rangle$ defined by this graph satisfies Property (T) and the labeling of $\mathcal{T}$ satisfies the $Gr'_*(1/6)$–graphical small cancellation condition. Let M exceed the largest exponent of a_1 in the labeling of $\mathcal{T}$. We take the following new explicit Rips relators, see our version of the Rips construction in Section 2.

$$r_i a_1^{10i-9+M} b a_1^{10i-8+M} b \cdots b a_1^{10i+M} b \text{ for all } 1 \leqslant i \leqslant n, \tag{4}$$

$$\begin{aligned} &x_j^{-1} a_k x_j a_1^{10(n+(k-1)m+j)-9+M} b a_1^{10(n+(k-1)m+j)-8+M} b \cdots \\ &\qquad \cdots b a_1^{10(n+(k-1)m+j)+M} b, \text{ for all } 1 \leqslant j \leqslant m, 1 \leqslant k \leqslant 35, \\ &x_j a_k x_j^{-1} a_1^{10(n+(k+34)m+j)-9+M} b a_1^{10(n+(k+34)m+j)-8+M} b \cdots \\ &\qquad \cdots b a_1^{10(n+(k+34)m+j)+M} b, \text{ for all } 1 \leqslant j \leqslant m, 1 \leqslant k \leqslant 35, \end{aligned} \tag{5}$$

$$\begin{aligned} &x_j^{-1} b x_j a_1^{10(71m+n+j)-9+M} b a_1^{10(71m+n+j)-8+M} b \cdots b a_1^{10(71m+n+j)+M} b \\ &\qquad \text{for all } 1 \leqslant j \leqslant m, \\ &x_j b x_j^{-1} a_1^{10(72m+n+j)-9+M} b a_1^{10(72m+n+j)-8+M} b \cdots b a_1^{10(72m+n+j)+M} b \\ &\qquad \text{for all } 1 \leqslant j \leqslant m. \end{aligned} \tag{6}$$

Let $\mathcal{R}$ be the disjoint union of $72m + n$ cycles, each labeled by one of these Rips relators. Let $M' := 10(72m + n) + M$. Take a generalized Rips-Segev graph $\mathcal{RS}$ for $a := a_1$ and b, where J is chosen such that $M' < 10^J$.

Our new group G is defined by the following graphical presentation,

$$G := \langle x_1, \ldots, x_m, a_1, \ldots, a_{35}, b \mid \mathcal{T} \sqcup \mathcal{R} \sqcup \mathcal{RS} \rangle.$$

The relators of G are the labels of the reduced cycles of $\mathcal{T} \sqcup \mathcal{R} \sqcup \mathcal{RS}$. Let N be the subgroup of G generated by $a_1, \ldots, a_{35}, b$. This subgroup N is normal in G by our Rips relators (5) and (6) read on $\mathcal{R}$. The map $G \to Q$, defined by $x_i \mapsto x_i$ and $a_i, b \mapsto 1$, is an epimorphism. The kernel of this map is generated by $a_1, \ldots, a_{35}, b$ and therefore coincides with N.

The labeling of $\mathcal{T} \sqcup \mathcal{R} \sqcup \mathcal{RS}$ satisfies the $Gr'_*(1/6)$–graphical small cancellation condition over $\langle x_1, \ldots, x_m, a_1 \rangle * \langle a_2 \rangle * \cdots * \langle a_{35} \rangle * \langle b \rangle$, that is, with respect to the free product length $|\cdot|_*$ in $\langle x_1, \ldots, x_m, a_1 \rangle * \langle a_2 \rangle * \cdots * \langle a_{35} \rangle * \langle b \rangle$. Indeed, the reduced non-trivial cycles in $\mathcal{T}$, $\mathcal{R}$, and $\mathcal{RS}$ have the free product length at least 20. The immersed subpaths common in $\mathcal{T}$, $\mathcal{R}$ and $\mathcal{RS}$ are of free product length at most 3, by our choice of the a_1-exponents.

Theorem 3.1 implies that G is torsion-free and Gromov hyperbolic. The proof of Theorem 3.2, applied to the graph $\mathcal{T} \sqcup \mathcal{R} \sqcup \mathcal{RS}$, shows that G is without the unique product property.

As a subgroup, the group N injects into G. Therefore, given two words w_1 and w_2 in $a_1, \ldots, a_{35}, b$ with $w_1 \neq_N w_2$ in N, we have that $w_1 \neq_G w_2$ in G. This yields

$$N = \langle a_1, \ldots, a_{35}, b \mid \mathcal{T} \sqcup \mathcal{RS} \sqcup \text{ relations in } G \text{ in letters } a_1^{\pm 1}, \ldots, a_{35}^{\pm 1}, b^{\pm 1} \rangle.$$

Thus, N is without the unique product property. Indeed, the sets A and B defining $\mathcal{RS}$ are contained in N and the relations read on $\mathcal{RS}$ imply that N does not have the unique product property for A and B. The group N is a quotient of the group $\langle a_1, \ldots, a_{35}, b \mid \mathcal{T} \rangle$. This group has Property (T) by Theorem A.1. Thus, N has Kazhdan's Property (T) as well. □

Observe that our use of the free product language above simplifies the arguments of [OW07, Prop. 2.2] and the proof of [OW07, Th. 1.1] which justify that relators in the Rips construction can be added to the relators defined by the graph $\mathcal{T}$.

Appendix A. Small cancellation labellings and Property (T)

Given a graph $\mathcal{G}$ labeled by $\{a_1, \ldots, a_m\}$, we denote by $G(\mathcal{G})$ the group defined by $\langle a_1, \ldots, a_m \mid \mathcal{G} \rangle$. Our aim is to prove the following result.

Theorem A.1. *For all $m > 35$, there exists a finite connected graph $\mathcal{T}$ labeled by $\{a_1, \ldots, a_m\}$ such that the labeling satisfies the $Gr'_*(1/6)$–graphical small cancellation condition over the free product $\langle a_1 \rangle * \ldots * \langle a_m \rangle$ and such that $G(\mathcal{T})$ has Property (T).*

This generalizes the following result of Gromov [Gro03, 1.2.A, 4.8.(3)], see also [Sil03] and [OW07].

Theorem A.2 ([OW07, Prop. 7.1])**.** *If $m \geqslant 2$, there exists a finite connected graph $\mathcal{T}$ labeled by $\{a_1, \ldots, a_m\}$ such that the labeling satisfies the $Gr'(1/6)$–graphical small cancellation condition with respect to the word length on the free group on $a_1, \ldots, a_m$ and the group $G(\mathcal{T})$ has Property (T).*

Our proof of Theorem A.1 proceeds as the proof of Theorem A.2 of [OW07, Sec. 7] but with appropriate technical and quantitative adjustments required by the free product setting. Moreover, we show that the graph $\mathcal{T}$ satisfies the conclusions of both Theorem A.1 and Theorem A.2:

Theorem A.3. *For all $m > 64$, there exists a finite connected graph $\mathcal{T}$ labeled by $\{a_1, \ldots, a_m\}$ such that the labeling satisfies the $Gr'_*(1/6)$–graphical small cancellation condition over the free product $\langle a_1 \rangle * \ldots * \langle a_m \rangle$, the labeling satisfies the $Gr'(1/6)$–graphical small cancellation condition with respect to the word length metric, and the group $G(\mathcal{T})$ has Property (T).*

It is not surprising that $\mathcal{T}$ satisfies the conclusions of both Theorem A.1 and Theorem A.2, for a large enough m. The intuition is that the free product length in $\langle a_1 \rangle * \ldots * \langle a_m \rangle$ approximates the word length on the free group on $a_1, \ldots, a_m$ as $m \to \infty$. Indeed, the minimal cycle length in the free product length bounds the length of the minimal cycles in the word length from below. Pieces are words of finite length chosen uniformly at random. Let us evaluate the probability that the word length and the free product length of such a random word in letters $a_1^{\pm 1}, \ldots, a_m^{\pm 1}$ coincide. Such a word is of word length equal to n if it is $a_{i_1}^{P_1} a_{i_2}^{P_2} \ldots a_{i_j}^{P_j}$ with all coefficients $P_i \neq 0$, $a_{i_j} \neq a_{i_{j+1}}$, and $\sum_{i=1}^{j} P_i = n$. Its free product length equals to n if, in addition, all exponents $P_i = \pm 1$. The probability that all $P_i = \pm 1$ in such a word is given by $\left(\frac{2m-2}{2m}\right)^{n-1}$, which tends to 1 as $m \to \infty$.

We provide an explicit value $m = 64$, for which the approximation of the word length by the free product length is sufficient to conclude Theorem A.3.

A.1. **Ollivier-Wise's proof of Theorem A.2.** First, we give a quantitative explanation of Ollivier-Wise's proof of Theorem A.2. Then we extend this proof to our general free product setting.

Given an expander graph, we endow it with a labeling chosen uniformly at random and extract from [Sil03] and [OW07] explicit bounds on the probability that the group defined by such a labeled graph has Property (T) and the corresponding presentation satisfies the graphical small cancellation condition. We put an emphasis on the combination of the estimates on the occurring probabilities.

Let $\mathcal{G}$ be a finite connected graph with a vertex set $V(\mathcal{G})$ and a set of undirected edges $E(\mathcal{G})$. We denote by $\lambda(\mathcal{G})$ the spectral gap of $\mathcal{G}$. The girth, denoted by $\operatorname{girth}(\mathcal{G})$, is the minimal number of edges in a shortest non-trivial cycle of $\mathcal{G}$.

A labeling ℓ of $\mathcal{G}$ by $\{x_1, \ldots, x_n, a, b\} \times \{\pm 1\}$ assigns to each edge a letter x_i, a, or b, and an orientation. We keep the notation $\mathcal{G}$ for the resulting directed graph labeled by $\{x_1, \ldots, x_n, a, b\}$. We say $\mathcal{G}$ is *reduced*, whenever $\mathcal{G}$ and its folding coincide.

Given $m > 1$, we denote by $\widetilde{\mathcal{G}}$ the graph $\mathcal{G}$ labeled by $\{a_1, \ldots, a_m\} \times \{\pm 1\}$ uniformly at random. We denote the corresponding folded labeled directed graph by $\operatorname{Fold}(\widetilde{\mathcal{G}})$.

The *j-subdivision* $\mathcal{G}^j$ of $\mathcal{G}$ is the graph $\mathcal{G}$ with every edge replaced by j edges. Consequently, $G\left(\widetilde{\mathcal{G}^j}\right)$ denotes the group defined by the j-subdivision of $\mathcal{G}$ labeled uniformly at random.

The probability that $G\left(\widetilde{\mathcal{G}^j}\right)$ has Property (T) is denoted by P_T. The probability that the map *folding*: $\widetilde{\mathcal{G}^j} \to \operatorname{Fold}\left(\widetilde{\mathcal{G}^j}\right)$ is a local quasi-isometric embedding is denoted by P_{qi}, and the *conditional* probability that the labeling of $\operatorname{Fold}\left(\widetilde{\mathcal{G}^j}\right)$ satisfies the $Gr'(\alpha)$–small cancellation condition with respect to the word length metric, under the condition that the folding is a local quasi-isometric embedding, is denoted by P_{sc}. Thus, the probability that the labeling of $\operatorname{Fold}\left(\widetilde{\mathcal{G}^j}\right)$ satisfies the $Gr'(\alpha)$–small cancellation condition is at least $P_{qi}P_{sc}$.

We extract explicit lower bounds for P_T, P_{qi} and P_{sc} from [Sil03, OW07]. This allows to estimate the probability that the labeling of $\operatorname{Fold}\left(\widetilde{\mathcal{G}^j}\right)$ satisfies the $Gr'(\alpha)$–small cancellation condition and $G\left(\widetilde{\mathcal{G}^j}\right)$ has Property (T). This probability is at least $P_T + P_{qi}P_{sc} - 1$. For certain infinite families of graphs $(\mathcal{S}_i)_i$, we then show that the probability that $(\mathcal{S}_i^j)_i$ satisfies these properties converges to 1 as $i \to \infty$. This provides the existence of graphs that define groups with Property (T) and whose labeling satisfies the graphical small cancellation condition.

Let S_l be the number of words of length l in letters $a_1^{\pm 1}, \ldots, a_m^{\pm 1}$ that reduce to the identity in the free group on free generators $a_1, \ldots, a_m$. The *gross cogrowth* of the free group is defined by

$$\eta(m) := \lim_{l \to \infty} \frac{\log_{2m}(S_{2l})}{2l}.$$

The limit exists as $S_{l+l'} \geqslant S_l S_{l'}$ and, hence, $\log_{2m}(S_{2l})$ is superadditive.

The spectral radius of the simple random walk on the free group of m generators equals to $(2m)^{\eta(m)-1}$. By a result of Kesten [Kes59, Th. 3],

$$(2m)^{\eta(m)} = 2\sqrt{2m-1}.$$

The gross cogrowth satisfies $1/2 < \eta(m) < 1$ and $\eta(m) \to 1/2$ as $m \to \infty$. See e.g. [Oll04, Sec. 1.2] for basic properties of the gross cogrowth.

We extract from [Sil03, Cor. 2.19 p. 164] the following estimate on P_T. We denote by $\lambda(G)$ the smallest non-zero eigenvalue of the graph laplacian Δ, while in [Sil03] $\lambda(G)$ denotes the maximal eigenvalue of $1-\Delta$, cf. [OW07, comment to Prop. 7.3] and [Sil03, Def. in Lem.2.11 p. 154 & p.151]. (We denote the number of generators by m instead of k used by [Sil03].)

Proposition A.4. *For all $m \geqslant 2$, $d \geqslant 3$, $\lambda_0 > 0$, and $j \in \mathbb{N}$, there exists a number*

$$g_0 = g_0(m, \lambda_0, j)$$

such that if the graph $\mathcal{G}$ satisfies

(1) $\operatorname{girth}(\mathcal{G}) \geqslant g_0$,
(2) $\lambda(G) \geqslant \lambda_0$ *for all* i,
(3) $3 \leqslant \deg(v) \leqslant d$ *for all* $v \in V(\mathcal{G})$,

then $G(\widetilde{\mathcal{G}^j})$ has Property (T) with probability

$$P_T \geqslant 1 - a(m, d, \lambda_0, j) e^{-b(m,d,\lambda_0,j)|V(\mathcal{G})|},$$

where a and b are positive numbers which do not depend on $\operatorname{girth}(\mathcal{G})$ *or* $|V(\mathcal{G})|$.

We express the probability P_T in terms of $\operatorname{girth}(\mathcal{G}) \leqslant |V(\mathcal{G})|$:

$$P_T \geqslant 1 - a(m, d, \lambda_0, j) e^{-b(m,d,\lambda_0,j)\operatorname{girth}(\mathcal{G})}.$$

The following proposition allows to compare the edge length of an immersed path p in $\widetilde{\mathcal{G}^j}$ with the word length of the labeling of p.

A (c_1, c_2, c_3)*–local quasi-isometric embedding* between metric spaces (X, d_X) and (Y, d_Y) is a map $f : X \to Y$ such that, whenever $d_X(a_1, x_2) \leqslant c_3$, we have

$$c_1^{-1} d_X(a_1, x_2) - c_2 \leqslant d_Y(f(a_1), f(x_2)) \leqslant c_1 d_X(a_1, x_2) + c_2.$$

We use the proof of [OW07, Prop. 7.8] to obtain the following.

Lemma A.5. *For all $m \geqslant 2$, $\beta > 0$, $d \in \mathbb{N}$, $j \geqslant 1$, if $\deg(v) \leqslant d$ for all $v \in V(\mathcal{G})$, then the folding $\widetilde{\mathcal{G}^j} \to \operatorname{Fold}\left(\widetilde{\mathcal{G}^j}\right)$ is a $\left(\frac{\eta(m)}{1-\eta(m)}, \beta j \operatorname{girth}(\mathcal{G}), \operatorname{girth}(\mathcal{G}) j\right)$–local quasi-isometric embedding with probability*

$$P_{qi} \geqslant 1 - j^2 d^{\operatorname{diam}(\mathcal{G}) + \operatorname{girth}(\mathcal{G})} (2m)^{-(1-\eta(m))\beta \operatorname{girth}(\mathcal{G}) j}.$$

In particular, if the folding is a local quasi-isometric embedding, then it maps non-trivial cycles to non-trivial cycles.

We extract the following estimate from [OW07, Proof of Prop. 7.4, the small cancellation part].

Lemma A.6. *For all* $m \geqslant 2$, $\beta > 0$ *such that* $\frac{1-\eta(m)}{\eta(m)} - \beta > 0$, $d \in \mathbb{N}$, $\alpha > 0$ *such that* $\frac{1-\eta(m)}{2\eta(m)} - \beta > \alpha$, $j \geqslant 1$ *if*

(1) $\deg(v) \leqslant d$ *for all* $v \in V(\mathcal{G})$,

(2) *folding* : $\widetilde{\mathcal{G}^j} \to \operatorname{Fold}\left(\widetilde{\mathcal{G}^j}\right)$ *is a* $\left(\frac{\eta(m)}{1-\eta(m)}, \beta \operatorname{girth}(\mathcal{G})j, \operatorname{girth}(\mathcal{G})j\right)$*–local quasi-isometric embedding,*

then the labeling of $\operatorname{Fold}\left(\widetilde{\mathcal{G}^j}\right)$ *satisfies the* $Gr'(\alpha)$*–small cancellation condition with respect to the word length metric on the free group with probability*

$$P_{sc} \geqslant 1 - j^4 d^{2\operatorname{diam}(\mathcal{G})+2\operatorname{girth}(\mathcal{G})}(2m)^{-(1-\eta(m))2\operatorname{girth}(\mathcal{G})j\alpha\left(\frac{1-\eta(m)}{\eta(m)}-\beta\right)}.$$

As P_{sc} is a conditional probability, where the condition is that the folding is a local quasi-isometric embedding, we conclude:

Proposition A.7. *For all* $m \geqslant 2$, $\beta > 0$ *such that* $\frac{1-\eta(m)}{2\eta(m)} - \beta > 0$, $d \in \mathbb{N}$, $\alpha > 0$, *such that* $\frac{1-\eta(m)}{2\eta(m)} - \beta > \alpha$, $j \geqslant 1$ *if* $\deg(v) \leqslant d$ *for all* $v \in V(\mathcal{G})$*, then the labeling of* $\operatorname{Fold}\left(\widetilde{\mathcal{G}^j}\right)$ *satisfies the* $Gr'(\alpha)$*–small cancellation condition with respect to the word length metric on the free group with probability at least*

$$P_{sc}(m, \beta, d, \alpha, j) P_{qi}(m, \beta, d, j).$$

Let us now consider the Selberg family of graphs $\mathcal{S} := (\mathcal{S}_i)_i$ [Lub94]:

(1) for all vertices v in $\mathcal{S}$, $3 \leqslant \deg(v) \leqslant d$ for some fixed $d \in \mathbb{N}$,
(2) $\lambda(\mathcal{S}_i) \geqslant \lambda_0 > 0$ uniformly over all i for some constant λ_0,
(3) $\operatorname{girth}(\mathcal{S}_i)_i \to \infty$ as $i \to \infty$,
(4) there is $C > 1$ such that $\operatorname{diam}(\mathcal{S}_i) \leqslant C \operatorname{girth}(\mathcal{S}_i)$ for all i.

Choose $\beta > 0$ such that $\frac{1-\eta(m)}{2\eta(m)} - \beta > 0$. For all $\alpha > 0$ such that $\frac{1-\eta(m)}{2\eta(m)} - \beta > \alpha$, the probability that the labeling of $\widetilde{\mathcal{S}_i^j}$ satisfies the $Gr'(\alpha)$–small cancellation condition and $G\left(\widetilde{\mathcal{S}_i^j}\right)$ has Property (T) is at least

$$P_T(m, d, \lambda_0, j)(i) + P_{sc}(m, d, \alpha, \beta, j)(i) P_{qi}(m, d, \alpha, \beta, j)(i) - 1.$$

There exists j_0 so that for all $j \geqslant j_0$ we have that

$$d^{2(C+1)}(2m)^{-(1-\eta(m))\beta j} < 1 \text{ and } d^{2(C+1)}(2m)^{-(1-\eta(m))2j\alpha\left(\frac{1-\eta(m)}{\eta(m)}-\beta\right)} < 1.$$

Then, $P_{sc}P_{qi}$ converges to 1 exponentially as $i \to \infty$. Simultaneously, the probability P_T converges to 1 exponentially as $i \to \infty$.

A labelling satisfying the $Gr'(\alpha)$–small cancellation condition clearly satisfies the $Gr'(\alpha')$–small cancellation condition for all $\alpha' \geqslant \alpha$. Theorem A.2 follows.

A.2. **Proof of Theorem A.1 and Theorem A.3.** We extend the proof from [OW07], in particular, Lemmas A.5 and A.6 to the free product setting. We view $\widetilde{\mathcal{G}^j}$ with the edge length and $\mathrm{Fold}(\widetilde{\mathcal{G}^j})$ with the free product length over $\langle a_1\rangle * \ldots * \langle a_m\rangle$. The probability that the map *folding* : $\widetilde{\mathcal{G}^j} \to \mathrm{Fold}\left(\widetilde{\mathcal{G}^j}\right)$ is a local quasi-isometric embedding is denoted by P^*_{qi}, and the *conditional* probability that the labeling of $\mathrm{Fold}\left(\widetilde{\mathcal{G}^j}\right)$ satisfies the $Gr'_*(\alpha)$–small cancellation condition over $\langle a_1\rangle * \ldots * \langle a_m\rangle$, under the condition that the folding is a local quasi-isometric embedding, is denoted by P^*_{sc}. That is, the probability that the labeling of $\mathrm{Fold}\left(\widetilde{\mathcal{G}^j}\right)$ satisfies the $Gr'_*(\alpha)$–small cancellation condition is at least $P^*_{qi}P^*_{sc}$.

We derive lower bounds for this probabilities. Our results then require a careful analysis of the obtained estimates.

Lemma A.8. *Let W_l be a word of length l in $2m$ letters chosen uniformly at random. Then*

$$P(|W_l|_* \leqslant L) \leqslant (2m-1)^{\frac{L}{2}} \left(\frac{lm}{2m-1}\right)^{\frac{1}{2}} \left(\frac{\sqrt{2}}{(2m)^{1-\eta(m)}}\right)^{l}.$$

Proof. Let B_l be the ball of radius l with respect to the word length metric in the free group on m generators. Let p^l_x denote the probability that $W_l = x$ where the equality is in the free group.

The number of elements x in B_l such that $|x|_* = k$ is at most

$$\sum_{k\leqslant l'\leqslant l} \binom{l'-1}{k-1} 2m(2m-2)^{k-1} \leqslant l\binom{l-1}{k-1} 2m(2m-1)^{k-1}.$$

Hence

$$\sum_{x\in B_l} (2m-1)^{-|x|_*} \leqslant \sum_{1\leqslant k\leqslant l} l\binom{l-1}{k-1} 2m(2m-2)^{k-1}(2m-1)^{-k} \leqslant l\frac{2m}{2m-1}2^{l-1}. \tag{7}$$

We compute the expected value,

$$\mathbb{E}\left((2m-1)^{-\frac{1}{2}|W_l|_*}\right) = \sum_{x\in B_l} (2m-1)^{-\frac{1}{2}|x|_*} p^l_x.$$

By the Cauchy-Schwartz inequality, this is bounded by

$$\leqslant \left(\sum_{x\in B_l} (2m-1)^{|x|_*}\right)^{\frac{1}{2}} \left(\sum_{x\in B_l} (p^l_x)^2\right)^{\frac{1}{2}}.$$

The right term $\sum_{x\in B_l}(p^l_x)^2$ is the return probability of the simple random walk on the free group of rank m at time $2l$. This probability is at most $(2m)^{-(1-\eta(m))2l}$. Applying inequality (7), we have that

$$\mathbb{E}\left((2m-1)^{-\frac{1}{2}|W_l|_*}\right) \leqslant \left(l\frac{2m}{2m-1}2^{l-1}\right)^{\frac{1}{2}}(2m)^{-(1-\eta(m))l}$$
$$= \left(\frac{lm}{2m-1}\right)^{\frac{1}{2}}\left(\frac{\sqrt{2}}{(2m)^{1-\eta(m)}}\right)^{l}.$$

The result now follows using Markov's inequality,

$$P(|W_l|_* \leqslant L) = P\left((2m-1)^{-L/2} \leqslant (2m-1)^{-\frac{1}{2}|W_l|_*}\right)$$
$$\leqslant (2m-1)^{-L/2}\mathbb{E}\left((2m-1)^{-\frac{1}{2}|W_l|_*}\right).$$

□

Lemma A.9. *For all* $m \geqslant 2$*,* $d \in \mathbb{N}$*,* $\beta > 0$*,* $j \geqslant 1$*, if* $\deg(v) \leqslant d$ *for all* $v \in V(\mathcal{G})$*, then the folding map from* $\widetilde{\mathcal{G}^j}$*, equipped with the edge length, to* $\operatorname{Fold}\left(\widetilde{\mathcal{G}^j}\right)$*, equipped with the free product length in* $\langle a_1\rangle * \ldots * \langle a_m\rangle$*, is a* $\left(\frac{1}{2(1-\eta(m))}, \beta \operatorname{girth}(\mathcal{G})j, \operatorname{girth}(\mathcal{G})j\right)$*–local quasi-isometric embedding with probability* P^*_{qi}*, which is*

$$\geqslant 1-(jd)^2 d^{\operatorname{diam}(\mathcal{G})+\operatorname{girth}(\mathcal{G})}\left(\operatorname{girth}(\mathcal{G})j\frac{m}{2m-1}\right)^{\frac{1}{2}} 2^{\frac{\operatorname{girth}(\mathcal{G})j}{2}}\left(\frac{\sqrt{2}(2m)^{\eta(m)}}{2m}\right)^{\beta \operatorname{girth}(\mathcal{G})j}.$$

Proof. Let $g := \operatorname{girth}(\mathcal{G})$. Choose a path p of edge length $\beta gj + \ell \leqslant gj$ in $\widetilde{\mathcal{G}^j}$. It suffices to show that $|\ell(p)|_* > 2(1-\eta(m))\ell$, where $|\cdot|_*$ denotes the free product length on the folded graph $\operatorname{Fold}\left(\widetilde{\mathcal{G}^j}\right)$.

The probability that a random labeling of p, that is, a word in $\beta gj + \ell$ letters chosen uniformly at random, has the free product length at most $2(1-\eta(m))\ell$ has been estimated in Lemma A.8. It is at most

$$(2m-1)^{(1-\eta(m))\ell}\left(\frac{(\beta gj+\ell)m}{2m-1}\right)^{\frac{1}{2}}\left(\frac{\sqrt{2}}{(2m)^{1-\eta(m)}}\right)^{\beta gj+\ell}.$$

There are at most $(jd)^2 d^{\operatorname{diam}(\mathcal{G})+\operatorname{girth}(\mathcal{G})}$ paths of length $\leqslant gj$ in $\mathcal{G}^j$. Indeed, there are at most $d^{\operatorname{diam}(\mathcal{G})}$ starting vertices for a simple path in $\mathcal{G}$. There are at most $d^{\operatorname{diam}(\mathcal{G})+l}$ possibilities of paths of length $\leqslant l$ in $\mathcal{G}$. A path in $\mathcal{G}^j$ of edge length l' is traveling along l'/j vertices in $\mathcal{G}$ with at most jd possibilities to choose the starting/terminal vertex. Therefore, there are at most $(jd)^2 d^{\operatorname{diam}(\mathcal{G})+l'/j}$ possibilities for paths of length l' in $\mathcal{G}^j$.

We combine both estimates to complete the proof. □

Compared to the estimate of P_{qi} in Lemma A.5, we have a new subexponential term and a new exponential term $2^{\operatorname{girth}(\mathcal{G})j/2}2^{\beta \operatorname{girth}(\mathcal{G})j/2}$ in our estimate of P^*_{qi}. To obtain the required results we therefore need a more careful analysis than above.

Proposition A.10. *For all* $m > 35$*,* $d \in \mathbb{N}$*, there is* $j_0 > 0$ *such that for all* $j > j_0$ *the folding from* $\widetilde{\mathcal{S}_i^j}$*, equipped with the word length metric, to* $\operatorname{Fold}\left(\widetilde{\mathcal{S}_i^j}\right)$*, equipped with the* *free product length in* $\langle a_1\rangle * \ldots * \langle a_m\rangle$*, is a* $\left(\frac{1}{2(1-\eta(m))}, 1/3\,\mathrm{girth}(\mathcal{G})j, \mathrm{girth}(\mathcal{G})j\right)$*–local quasi-isometric embedding with probability tending to* 1 *exponentially as* $i \to \infty$*.*

Compared to Lemma A.9, we have specified $\beta = 1/3$.

Proof. The claim follows when $2^{\frac{1}{2}}\left(\frac{\sqrt{2}(2m)^{\eta(m)}}{2m}\right)^{1/3} < 1$. Then there is j_0 such that for all $j > j_0$ we have that

$$d^{(C+1)}\left(2^{\frac{1}{2}}\left(\frac{\sqrt{2}(2m)^{\eta(m)}}{2m}\right)^{1/3}\right)^{j} < 1.$$

By Lemma A.9, $P^*_{qi} \to 1$ exponentially as $i \to \infty$.

By a result of Kesten [Kes59, Th. 3], $(2m)^{\eta(m)} = 2\sqrt{2m-1}$, so we need that

$$\left(\frac{\sqrt{2}\sqrt{2m-1}}{m}\right)^{1/3} < \frac{1}{\sqrt{2}}.$$

That is, $0 < m^2 - 32m + 16$. This holds for all $m > 35$. Note that $\eta(m) < \frac{2}{3}$ if $m > 35$. We therefore have

$$1 - \eta(m) > \frac{1}{3}.$$

We conclude as before. □

Lemma A.11. *For all* $m \geqslant 2$*,* $d \in \mathbb{N}$*,* β *such that* $1 - \eta(m) - \beta > 0$*,* $\alpha > 0$ *such that* $\alpha < (1-\eta(m)) - \beta$*,* $j \geqslant 1$*, if*

1. *the vertices of the graph* $\mathcal{G}$ *have degree at most* d*,*
2. *folding* : $\widetilde{\mathcal{G}^j} \to \mathrm{Fold}\left(\widetilde{\mathcal{G}^j}\right)$ *is a* $\left(\frac{1}{2(1-\eta(m))}, \beta\,\mathrm{girth}(\mathcal{G})j, \mathrm{girth}(\mathcal{G})j\right)$*–local quasi-isometric embedding, where* $\mathrm{Fold}\left(\widetilde{\mathcal{G}^j}\right)$ *is with the free product length in* $\langle a_1\rangle * \ldots * \langle a_m\rangle$*,*

then the labeling of $\mathrm{Fold}\left(\widetilde{\mathcal{G}^j}\right)$ *satisfies the* $Gr'_*(\alpha)$*–small cancellation condition over* $\langle a_1\rangle * \ldots * \langle a_m\rangle$ *with probability*

$$P^*_{sc} \geqslant 1 - (jd)^4 d^{2\,\mathrm{diam}(\mathcal{G}) + 2\,\mathrm{girth}(\mathcal{G})}(2m)^{-(1-\eta(m))2\,\mathrm{girth}(\mathcal{G})j\alpha(2(1-\eta(m))-\beta)}.$$

Proof. Let $g := \mathrm{girth}(\mathcal{G})$. First observe that by the quasi-isometry assumption

$$\bar{g} := \min|\text{labels of non-trivial cycles in } \mathrm{Fold}(\widetilde{\mathcal{G}^j})|_* \geqslant (2(1-\eta(m)) - \beta)gj.$$

Let $\alpha < (1-\eta(m)) - \beta$. It suffices to estimate the probability that there are no α-pieces, that is, pieces p in $\mathrm{Fold}(\widetilde{\mathcal{G}^j})$ such that $|\ell(p)|_* = \alpha\bar{g}$. Let q, q' be immersed paths in $\widetilde{\mathcal{G}^j}$ whose folding equals p. For the word length $|\ell(q)|, |\ell(q')| \leqslant \frac{gj}{2}$. Indeed, otherwise, $|\ell(p)|_* > ((1-\eta(m)) - \beta)gj$ by the quasi-isometry assumption, a contradiction. On the other hand, $|\ell(q)|, |\ell(q')| \geqslant |\ell(p)| \geqslant |\ell(p)|_* = \alpha\bar{g}$.

Therefore, suppose that q and q' are $\alpha\bar{g}$ pieces in $\widetilde{\mathcal{G}^j}$. We now apply the following.

Lemma A.12 ([OW07, Prop. 7.11]). *Let q, q' be two immersed paths in a graph $\mathcal{G}$ of girth g. Suppose that q and q' have length l and l' respectively, with l and l' at most $g/2$. Endow $\mathcal{G}$ with a uniform random labeling. Suppose that after folding the graph, the paths q and q' are mapped to distinct paths. Then the probability that q and q' are labeled by two freely equal words is at most*

$$C_{l,l'}(2m)^{-(1-\eta(m))(l+l')},$$

where $C_{l,l'}$ is a term growing subexponentially in $l + l'$.

Hence, the probability that two paths q, q' in $\widetilde{\mathcal{G}^j}$ are $\alpha\bar{g}$ pieces in $\widetilde{\mathcal{G}^j}$ is at most

$$C_{gj}(2m^{-(1-\eta(m))2gj\alpha(2(1-\eta(m))-\beta)}),$$

where C_{gj} is a sub-exponential term in g. The probability that there are two such paths q, q' in $\widetilde{\mathcal{G}^j}$ is at most

$$j^4 v^{2\,\mathrm{diam}(\mathcal{G})+2g} C_{gj}(2m^{-(1-\eta(m))2gj\alpha(2(1-\eta(m))-\beta)}).$$

□

Theorem A.13. *For all $m > 64$, $d \in \mathbb{N}$, there is $j_0 > 0$ such that for all positive numbers $j > j_0$, the labeling of* $\mathrm{Fold}\left(\widetilde{\mathcal{S}_i^j}\right)$ *satisfies both*

- *the $Gr'_*(\alpha)$–small cancellation condition over $\langle a_1\rangle * \ldots * \langle a_m\rangle$, and*
- *the $Gr'(\alpha)$–small cancellation condition with respect to the word length metric in the free group on $a_1, \ldots, a_m$,*

with probability tending to 1 exponentially as $i \to \infty$.

Observe that $\mathcal{S}$ does not need to satisfy condition (2) above.

Proof. Note that $1 - \eta(m) > 1/3$, choose $\beta = 1/4$ so that $1 - \eta(m) > \beta$, and $\frac{1-\eta(m)}{2\eta(m)} - \beta > 0$. If $m \geqslant 64$, then, by an estimate as in the proof of Proposition A.10,

$$2^{\frac{1}{2}}\left(\frac{\sqrt{2}(2m)^{\eta(m)}}{2m}\right)^{1/4} < 1.$$

Choose $\alpha > 0$ such that

$$\alpha < \frac{1-\eta}{2\eta} - 1/4 \text{ and } \alpha < (1-\eta) - 1/4.$$

The probability that $\mathrm{Fold}\left(\widetilde{\mathcal{S}_i^j}\right)$ does not satisfy the required small cancellation conditions is at most

$$1 - P_{sc}(m,d,\alpha,1/4,j)(i)P_{qi}(m,d,\alpha,1/4,j)(i) + 1 - P^*_{sc}(m,d,\alpha,1/4,j)(i)P^*_{qi}(m,d,\alpha,1/4,j)(i).$$

There exists j_0 such that for all $j > j_0$ we have that

- $d^{2(C+1)}(2m)^{-(1-\eta(m))1/4j} < 1$,
- $d^{2(C+1)}(2m)^{-(1-\eta(m))2j\alpha\left(\frac{1-\eta(m)}{\eta(m)}-1/4\right)} < 1$,

- $d^{(C+1)}\left(2^{\frac{1}{2}}\left(\frac{\sqrt{2}(2m)^{\eta(m)}}{2m}\right)^{1/4}\right)^{j} < 1$, and
- $(jd)^4 d^{2(C+1)}(2m)^{-(1-\eta(m))2j\alpha(2(1-\eta(m))-1/4)} < 1$.

Then

$$P_{sc}(m,d,\alpha,1/4,j)(i)P_{qi}(m,d,\alpha,1/4,j)(i)$$

and simultaneously

$$P^*_{sc}(m,d,\alpha,1/4,j)(i)P^*_{qi}(m,d,\alpha,1/4,j)(i)$$

tend to 1 exponentially as $i \to \infty$.

□

Theorems A.1 and A.3 now follow as Theorem A.2.

REFERENCES

[Ago13] I. Agol, *The virtual Haken conjecture*, Doc. Math. **18** (2013). With an appendix by Agol, D. Groves, and J. Manning.

[BMS94] G. Baumslag, C. F. Miller III, and H. Short, *Unsolvable problems about small cancellation and word hyperbolic groups*, Bull. London Math. Soc. **26** (1994), no. 1, 97–101.

[BO08] I. Belegradek and D. Osin, *Rips construction and Kazhdan property (T)*, Groups Geom. Dyn. **2** (2008), no. 1, 1–12.

[Car14] W. Carter, *New examples of torsion-free non unique product groups*, J. Group Theory (2014), 445–464.

[CCJ+01] P.-A. Cherix, M. Cowling, P. Jolissaint, P. Julg, and A. Valette, *Groups with the Haagerup property (Gromov's a-T-menability)*, Progress in Mathematics, vol. 197, Birkhäuser Verlag, Basel, 2001.

[Del97] T. Delzant, *Sur l'anneau d'un groupe hyperbolique*, C. R. Acad. Sci. Paris Sér. I Math. **324** (1997), no. 4, 381–384.

[Gru15] D. Gruber, *Groups with graphical $C(6)$ and $C(7)$ small cancellation presentations*, Trans. Amer. Math. Soc. **367** (2015), no. 3, 2051–2078.

[GMS15] D. Gruber, A. Martin, and M. Steenbock, *Finite index subgroups without unique product in graphical small cancellation groups*, Bull. Lond. Math. Soc. **47** (2015), no. 4, 631–638.

[Gro03] M. Gromov, *Random walk in random groups*, Geom. Funct. Anal. **13** (2003), no. 1, 73–146.

[Kap57] I. Kaplansky, *Problems in the theory of rings. Report of a conference on linear algebras, June, 1956, pp. 1-3*, 1957, pp. v+60.

[Kap70] ———, *"Problems in the theory of rings" revisited*, Amer. Math. Monthly **77** (1970), 445–454.

[Kes59] H. Kesten, *Symmetric random walks on groups*, Trans. Amer. Math. Soc. **92** (1959), 335–354.

[LOS12] P. Linnell, B. Okun, and T. Schick, *The strong Atiyah conjecture for right-angled Artin and Coxeter groups*, Geom. Dedicata **158** (2012), 261–266.

[Lub94] A. Lubotzky, *Discrete groups, expanding graphs and invariant measures*, Progress in Mathematics, vol. 125, Birkhäuser Verlag, Basel, 1994. With an appendix by Jonathan D. Rogawski.

[LS77] R. C. Lyndon and P. E. Schupp, *Combinatorial group theory*, Springer-Verlag, Berlin, 1977. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 89.

[Oll04] Y. Ollivier, *Sharp phase transition theorems for hyperbolicity of random groups*, Geom. Funct. Anal. **14** (2004), no. 3, 595–679.

[OW07] Y. Ollivier and D. T. Wise, *Kazhdan groups with infinite outer automorphism group*, Trans. Amer. Math. Soc. **359** (2007), no. 5, 1959–1976.

[Ol′93] A. Ol′shanskiĭ, *On residualing homomorphisms and G-subgroups of hyperbolic groups*, Internat. J. Algebra Comput. **3** (1993), no. 4, 365–409.

[Pan99] A. E. Pankrat′ev, *Hyperbolic products of groups*, Vestnik Moskov. Univ. Ser. I Mat. Mekh. **2** (1999), 9–13, 72; English transl., Moscow Univ. Math. Bull. **54** (1999), no. 2, 9–12.

[Pro88] S. D. Promislow, *A simple example of a torsion-free, nonunique product group*, Bull. London Math. Soc. **20** (1988), no. 4, 302–304.

[Rip82] E. Rips, *Subgroups of small cancellation groups*, Bull. London Math. Soc. **14** (1982), no. 1, 45–47.

[RS87] E. Rips and Y. Segev, *Torsion-free group without unique product property*, J. Algebra **108** (1987), no. 1, 116–126.

[Sch14] K. Schreve, *The Strong Atiyah conjecture for virtually cocompact special groups*, Math. Ann. (2014), 1–8.

[Sil03] L. Silberman, *Addendum to: "Random walk in random groups" [Geom. Funct. Anal.* **13** *(2003), no. 1, 73–146] by M. Gromov*, Geom. Funct. Anal. **13** (2003), no. 1, 147–177.

[Ste15] M. Steenbock, *Rips-Segev torsion-free groups without the unique product property*, J. Algebra **438** (2015), 337–378.

UNIVERSITÄT WIEN, FAKULTÄT FÜR MATHEMATIK, OSKAR-MORGENSTERN-PLATZ 1, 1090 WIEN, AUSTRIA
E-mail address: goulnara.arzhantseva@univie.ac.at

UNIVERSITÉ DE STRASBOURG, CNRS, IRMA UMR 7501, F-67000 STRASBOURG, FRANCE
E-mail address: steenbock@math.unistra.fr