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A three-dimensional autonomous nonlinear dynamical system modelling equatorial ocean flows

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Abstract

We investigate a nonlinear three-dimensional model for equatorial flows, finding exact solutions that capture the most relevant geophysical features: depth-dependent currents, poleward or equatorial surface drift and a vertical mixture of upward and downward motions.

Keywords: Euler equation; Nonlinear three-dimensional solutions; Equatorial ocean flows

1. Introduction

The nonlinear ocean model presented in this paper is based on the Euler equation together with the condition of incompressibility and suitable boundary conditions at a free surface that follows the curvature of the Earth and at the impermeable bottom. The application of the model is restricted to regions close to the Equator with the contribution from the Coriolis effect at the level of equatorial β-plane approximation which leads to a three-dimensional structure of the flow field. The equations are written in a coordinate system associated with the tangent plane on the Equator but still capturing the effects of the curvature of the Earth surface. We are interested in a mathematical approach that enables us to capture strong depth variations of the flow only, the density stratification and thermocline play no role in this paper—the depth of the thermocline (about 200 m for the Equatorial Undercurrent (EUC)) is short as compared with the average to-

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tal depth of the ocean (about 4 km for the Pacific). The approach that we will use building on the recent considerations made by Constantin & Johnson in [7], where the problem in the presence of a thermocline with very small change in density across it, is treated. By following a similar route here, we limit ourselves to the steady problem and, after introducing a suitable non-dimensionalisation, we make some important simplifications by virtue of the slow evolution in the azimuthal direction along the Equator. We mention that some exact, steady solutions, representing purely azimuthal flows that do not vary in the azimuthal direction were presented and explored by Constantin and Johnson in [5]; see also the very recent review by Johnson [19]. Our problem is reduced to a manageable level, but retains nonlinear features and elements of three-dimensionality, and admits exact solutions that correspond, in their general form, to the types of flows that are observed. We show that two components of the velocity field for our problem can be expressed as nonlinear functionals of the azimuthal velocity component u along the Equator, and this velocity component is restricted only by the boundary conditions and some conditions which prevent the appearance of singularities. Several choices are available for the azimuthal velocity profile and we analyse in detail some examples of these profiles: linear, parabolic and cubic. We show that the cubic profile is the simplest that satisfies all the requirements and produces a three-dimensional non purely azimuthal flow. Depending on how far from the Equator in the meridional direction we are, the sign of the non azimuthal components of the velocity, denoted by v and w, can change one or two times from the free surface downwards. Close to the free surface, at a fixed x, that is, the azimuthal (equatorial) coordinate, the sign of these two components depends on the sign of the derivative with respect to x of the function which is the leading coefficient in the cubic azimuthal component. A positive meridional component of the velocity v close to the free surface means a poleward surface drift, a negative ones meaning an equatorial surface drift, while a positive vertical component of the velocity w indicates upwelling and a negative one indicates downwelling. Thus, the presented nonlinear three-dimensional model is able to capture simultaneously the main observed features of the equatorial flows: depth-dependent currents, poleward or equatorial surface drift, and a mixture of upward and downward motions. For more informations on the physical oceanographic elements see [5,7,19] and the references therein.

We comment that previous to the studies [5] and [7], other exact solutions have been found that relate to similar, but slightly different, physical scenarios. Two-dimensional flows (in which the meridional velocity component and meridional variations are neglected) in the f-plane were discussed in Constantin and Johnson [4]. Based on a Lagrangian approach, exact nonlinear Gerstner-like three-dimensional solutions were obtained: for example, by Pollard [20], solutions in the f-plane approximation, without any restriction on the latitude, by Constantin [1,2] and Henry [12], solutions in the β-plane approximation, valid for ocean regions within a meridional distance of 200 km of the Equator, by Constantin and Monismith [8], solutions which extend Pollard's solution to include a depth invariant mean current; see the recent survey Henry [13]. However, these exact solutions fail to capture strong depth-variations of the flows. It is also very interesting that, for Gerstner-like geophysical flows one can identify short-wavelength perturbations in certain directions as a source of instabilities with an exponentially growing amplitude, the growth rate of the instabilities depending on the steepness of the travelling wave profile—the critical steepness is very close to $\frac{1}{3}$—see the papers [3,15,10,14,16]. On the other hand, for certain physically realistic velocity profiles, steady flows moving only in the azimuthal direction, with no variation in this direction, are locally stable to the short-wavelength perturbations—see [18]. Very recently, Ionescu-Kruse surveys in [17] some work on the stability/instability of some intricate geophysical flows.

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In order to have a sense of how the field of geophysical fluid dynamics developed, to understand better the ideas and the concepts of this field, to see how the theory can help improve complex models and guide observations and how it can work together with these and with numerical methods to give a fundamental understanding of the processes involved, we recommend the papers [6] and [21].

2. Governing equations

We consider a one-layer incompressible, inviscid rotating fluid close to the equatorial region. The governing equations of the steady fluid motion in the β-plane approximation are the following:

$$\begin{aligned} uu_x + vu_y + wu_z + 2\omega\left(w - \frac{y}{\mathcal{R}}v\right) &= -\frac{p_x}{\rho_0}, \\ uv_x + vv_y + wv_z + 2\omega\frac{y}{\mathcal{R}}u &= -\frac{p_y}{\rho_0}, \\ uw_x + vw_y + ww_z - 2\omega yu &= -\frac{p_z}{\rho_0} - g, \end{aligned} \tag{1}$$

$$u_x + v_y + w_z = 0. \tag{2}$$

Here (x, y, z) is a coordinate system that rotates with the Earth, with x pointing due east, y due north, and z vertically upwards, perpendicular to the Earth surface. This Cartesian coordinate system is the familiar one associated with the β-plane approximation, an approximation which is adequate within about 5° latitude (see, for example, [9], [11]). (u, v, w) are the components of the fluid velocity, p denotes the pressure and $g = 9.8\ \mathrm{m\,s^{-2}}$ is the constant gravitational acceleration at the Earth's surface. We assume that the fluid (water) has a constant density $\rho_0 = 1027\ \mathrm{kg\,m^{-3}}$, and the Earth is a perfect sphere of radius $\mathcal{R} = 6371$ km which rotates with a constant rotational speed $\omega = 73 \times 10^{-6}$ rad/s round the polar axis toward the east.

We describe the free surface of the water by

$$\eta(x, y) = \frac{y^2}{2\mathcal{R}} + h(x, y), \tag{3}$$

which incorporates the curvature in the y-direction as measured relative to the tangent plane, $h(x, y)$ being therefore the (local) disturbance from the surface of the sphere. The boundary conditions at the free surface are the kinematic and the dynamic conditions:

$$w = uh_x + vh_y + v\frac{y}{\mathcal{R}} \qquad \text{on the free surface} \quad z = \frac{y^2}{2\mathcal{R}} + h(x, y), \tag{4}$$

$$p = P_s(x, y) \qquad \text{on the free surface} \quad z = \frac{y^2}{2\mathcal{R}} + h(x, y), \tag{5}$$

P_s being the pressure prescribed at the free surface. Also, there is no flow through the flat, horizontal bottom $z = \frac{y^2}{2\mathcal{R}} - d$ which follows the curvature of the Earth, that is,

$$w = 0 \qquad \text{on} \quad z = \frac{y^2}{2\mathcal{R}} - d. \tag{6}$$

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In our problem we consider the situation that the flow essentially extends throughout the depth of the ocean and a suitable velocity profile defines the flow. The depth of the thermocline (about 200 m for the EUC) is short as compared to the average total depth of the ocean d (about 4 km for the Pacific), therefore, the presence of the thermocline is neglected.

3. Non-dimensionalisation and scaling

We now introduce a suitable non-dimensionalisation of the variables for our geophysical problem (1)–(6). For this purpose we use L as the azimuthal scale, l as the meridional scale, the average total depth of the ocean d as the vertical scale and a the typical amplitude of the surface wave:

$$x \mapsto L\,x, \quad y \mapsto l\,y, \quad z \mapsto d\,z, \quad h \mapsto a\,h, \tag{7}$$

where, to avoid new notations, we have used the same symbols for the non dimensional variables x, y, z, η on the right-hand side. We will suppose that the flows are predominantly in the azimuthal direction and these flows will involve motion in only the vertical/azimuthal plane (x, z) or in the vertical/meridional plane (y, z). The velocity components are non-dimensionalised in a form that guarantees the existence of a suitable stream function (see [7]):

$$u \mapsto U\,u, \quad v \mapsto U\frac{l}{L}\,v, \quad w \mapsto U\frac{d}{L}\,w, \tag{8}$$

where U is an appropriate speed scale and in order to avoid new notations, we have used the same symbols for the non dimensional variables u, v, w on the right-hand side. The suitable non-dimensionalisation for the pressure is (see [7]):

$$p \mapsto \rho_0 U^2 p - \rho_0 g d\,z, \tag{9}$$

and it is also natural to non-dimensionalise the constant rotational speed of the Earth by:

$$\omega \mapsto \frac{U}{d}\omega, \tag{10}$$

on the right-hand side, p and ω, respectively, are the non dimensional variables. Therefore, in the non dimensional variable (7)–(10), our geophysical boundary value problem (1)–(6) becomes

$$\begin{aligned} uu_x + vu_y + wu_z + 2\omega\left(w - \frac{l^2}{d\mathcal{R}}yv\right) &= -p_x, \\ \left(\frac{l}{L}\right)^2 \left(uv_x + vv_y + wv_z\right) + 2\omega\frac{l^2}{d\mathcal{R}}yu &= -p_y, \\ \left(\frac{d}{L}\right)^2 \left(uw_x + vw_y + ww_z\right) - 2\omega yu &= -p_z, \end{aligned} \tag{11}$$

$$u_x + v_y + w_z = 0, \tag{12}$$

with the boundary conditions

$$w = \frac{a}{d}\left(uh_x + vh_y\right) + \frac{l^2}{d\mathcal{R}} yv \qquad \text{on } z = \frac{l^2}{d\mathcal{R}}\frac{y^2}{2} + \frac{a}{d}h(x,y), \tag{13}$$

$$p = P_0(x,y) + \kappa\left[\frac{l^2}{d\mathcal{R}}\frac{y^2}{2} + \frac{a}{d}h(x,y)\right] \qquad \text{on } z = \frac{l^2}{d\mathcal{R}}\frac{y^2}{2} + \frac{a}{d}h(x,y), \tag{14}$$

$$w = 0 \qquad \text{on} \quad z = \frac{l^2}{d\mathcal{R}}\frac{y^2}{2} - 1, \tag{15}$$

where

$$P_0(x,y) := \frac{1}{\rho_0 U^2} P_s(x,y), \qquad \kappa := \frac{gd}{U^2}. \tag{16}$$

In the azimuthal direction, flows typically extend to many thousands of kilometres, e.g. 13 000 km for the flow along the Pacific Equator, and the average depth of the ocean is about 4 km. We make these choices for the length scales L and d, respectively. For the length scale l, a convenient choice for our problem, which measures how far one can go away from the Equator in the meridional direction, is

$$l = \sqrt{d\mathcal{R}}, \tag{17}$$

that is, l is about 160 km, which is reasonable for the observed width of the EUC. We then write our problem (11)–(15) as

$$\begin{aligned} uu_x + vu_y + wu_z + 2\omega\,(w - yv) &= -p_x, \\ \delta\frac{\mathcal{R}}{L}\left(uv_x + vv_y + wv_z\right) + 2\omega yu &= -p_y, \\ \delta^2\left(uw_x + vw_y + ww_z\right) - 2\omega yu &= -p_z, \end{aligned} \tag{18}$$

$$u_x + v_y + w_z = 0, \tag{19}$$

$$w = \epsilon\left(uh_x + vh_y\right) + yv \qquad \text{on } z = \frac{y^2}{2} + \epsilon h(x,y), \tag{20}$$

$$p = P_0(x,y) + \kappa\left[\frac{y^2}{2} + \epsilon h(x,y)\right] \qquad \text{on } z = \frac{y^2}{2} + \epsilon h(x,y), \tag{21}$$

$$w = 0 \qquad \text{on} \quad z = \frac{y^2}{2} - 1, \tag{22}$$

where we have denoted by

$$\delta = \frac{d}{L}, \qquad \epsilon = \frac{a}{d}. \tag{23}$$

For the chosen values of the length scales we have $\delta \approx 3 \times 10^{-4}$, $\frac{\mathcal{R}}{L} \approx 5 \times 10^{-1}$. In the context of the EUC, it is a property of the Pacific that the level of the ocean from east to west (relative to the curvature of the Earth) rises about 0.5 m because the action of the trade winds (see [7]). If we take this value for a, we get $\epsilon \approx 13 \times 10^{-5}$. We observe that in this case $\delta\frac{\mathcal{R}}{L} \approx \epsilon$.

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In what follows we investigate the problem in the limit

$$\delta \to 0, \quad \epsilon \to 0, \tag{24}$$

that is, we are looking for flows which evolve slowly in the azimuthal direction along the Equator and for which the shape of the free surface has very small and unimportant variations from the surface of the sphere. The leading-order problem is

$$\begin{aligned} uu_x + vu_y + wu_z + 2\omega(w - yv) &= -p_x, \\ 2\omega yu &= -p_y, \\ -2\omega yu &= -p_z, \end{aligned} \tag{25}$$

$$u_x + v_y + w_z = 0, \tag{26}$$

$$w = yv \qquad \text{on} \quad z = \frac{y^2}{2}, \tag{27}$$

$$p = P_0(x, y) + \kappa \frac{y^2}{2} \qquad \text{on} \quad z = \frac{y^2}{2}, \tag{28}$$

$$w = 0 \qquad \text{on} \quad z = \frac{y^2}{2} - 1. \tag{29}$$

4. Three-dimensional, nonlinear exact solutions

The system (25)–(29) can be solved completely. We are looking for a solution of the form

$$u = u(x, \zeta), \; v = v(x, y, \zeta), \; w = w(x, y, \zeta), \quad \zeta := z - \frac{y^2}{2}. \tag{30}$$

Let us introduce the function $\phi(x, \zeta)$, such that

$$u = \phi_\zeta(x, \zeta), \tag{31}$$

and

$$\phi(x, 0) = 0. \tag{32}$$

From $(25)_2$ and $(25)_3$ we get

$$p = 2\omega\phi(x, \zeta) + f(x), \tag{33}$$

with $a(x)$ an arbitrary function. By (28) and (32) we find

$$P_0(x, y) = -\kappa \frac{y^2}{2} + f(x) \quad \text{on} \quad z = \frac{y^2}{2}, \tag{34}$$

and if consider the case that the pressure is constant along $y = 0$ we get

$$f(x) = \text{const.} \tag{35}$$

Our problem reduces to

$$uu_x + vu_y + wu_z + 2\omega(w - yv) = -2\omega\phi_x \tag{36}$$

$$u_x + v_y + w_z = 0 \tag{37}$$

$$w = yv \quad \text{on} \quad z = \frac{y^2}{2} \tag{38}$$

$$w = 0 \qquad \text{on} \quad z = \frac{y^2}{2} - 1. \tag{39}$$

Taking into account (31), (33) and (35), the equation (36) becomes

$$uu_x + (w - yv)(u_\zeta + 2\omega) = -2\omega\phi_x. \tag{40}$$

If

$$u_\zeta + 2\omega \neq 0, \tag{41}$$

then

$$w = yv - \frac{uu_x + 2\omega\phi_x}{u_\zeta + 2\omega}. \tag{42}$$

If (41) is not satisfied, then, on the line of singularities we certainly must have

$$uu_x + 2\omega\phi_x = 0 \quad \text{on} \ u_\zeta + 2\omega = 0. \tag{43}$$

By denoting with $\hat{\zeta}(x)$ the solutions of the equation $u_\zeta + 2\omega = 0$, from (43), we get that the points along the line of singularities has to satisfy the equation

$$u^2(x, \hat{\zeta}(x)) + 4\omega\phi(x, \hat{\zeta}(x)) = \text{const.} \tag{44}$$

For the solution (42), the boundary condition (38) is satisfied if

$$uu_x + 2\omega\phi_x = 0 \quad \text{on} \ z = \frac{y^2}{2}, \tag{45}$$

which, with (32) in view, implies that the speed at the free surface is constant

$$u(x, 0) = \text{const} := -U_0, \tag{46}$$

where U_0 is the constant speed at the surface.

From the equation (37), we obtain

$$v_y + w_z = -u_x. \tag{47}$$

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Taking into account (30), the above equation can be written into the form

$$(w - yv)_\zeta = -u_x - v_y. \tag{48}$$

We now take the derivative in (40) with respect to ζ, and by (48), (42) and (31), we get, under the claim (41),

$$v_y = \frac{uu_{x\zeta}(u_\zeta + 2\omega) - (uu_x + 2\omega\phi_x)u_{\zeta\zeta}}{(u_\zeta + 2\omega)^2}. \tag{49}$$

Thus,

$$v \;=\; \frac{uu_{x\zeta}(u_\zeta + 2\omega) \;-\; (uu_x + 2\omega\phi_x)u_{\zeta\zeta}}{(u_\zeta + 2\omega)^2}\, y, \tag{50}$$

where we took 0 as integration constant, in order to ensure that $v = 0$ on $y = 0$ as required by symmetry about the Equator. Where $u_\zeta + 2\omega = 0$ we require that

$$\lim_{\zeta \to \hat{\zeta}} \frac{uu_{x\zeta}(u_\zeta + 2\omega) - (uu_x + 2\omega\phi_x)u_{\zeta\zeta}}{(u_\zeta + 2\omega)^2} \quad \text{exists.} \tag{51}$$

This limit does not introduce an additional constraint if (see [7])

$$u_{\zeta\zeta} \neq 0 \quad \text{where} \quad u_\zeta + 2\omega = 0. \tag{52}$$

Summing up, we expressed the components v and w of the velocity field for our reduced problem in the limit $\delta \to 0$, $\epsilon \to 0$, as highly nonlinear functionals of u, the azimuthal velocity component along the Equator, that is,

$$u(x, \zeta) = \phi_\zeta(x, \zeta) \tag{53}$$

$$v(x, y, \zeta) = \frac{uu_{x\zeta}(u_\zeta + 2\omega) - (uu_x + 2\omega\phi_x)u_{\zeta\zeta}}{(u_\zeta + 2\omega)^2}\, y \tag{54}$$

$$w(x, y, \zeta) = yv - \frac{uu_x + 2\omega\phi_x}{u_\zeta + 2\omega}, \tag{55}$$

restricted only by the boundary conditions:

$$\phi(x, 0) = 0, \quad u(x, 0) = -U_0, \quad w\,(x, y, -1) = 0. \tag{56}$$

The conditions (43) and (52) are sufficient to prevent the appearance of singularities.

We observe that the component v of the velocity is an odd function

$$v(x, y, \zeta) = -v(x, -y, \zeta) \tag{57}$$

and is zero along the Equator

$$v(x, 0, \zeta) = 0. \tag{58}$$

In what follows, we will construct some specific solutions, by choosing different azimuthal velocity profiles. We will consider the case that the flow is stationary at the bed, so, beside the last condition in (56) we also have

$$u(x, -1) = 0, \quad v(x, y, -1) = 0, \tag{59}$$

which, in view of (54) and (55), gives the equation

$$\phi_x(x, -1) = 0. \tag{60}$$

4.1. Linear azimuthal velocity profile

The first example that we consider is a linear azimuthal component u:

$$u(x, \zeta) = A(x)(\zeta + 1), \tag{61}$$

with $A(x)$ an arbitrary function. From the second condition in (56) we get that

$$A(x) = -U_0. \tag{62}$$

By (31) and (32)

$$\phi(x, \zeta) = -\frac{U_0}{2}\zeta^2 - U_0\zeta, \tag{63}$$

and we get for (53)–(56) the purely azimuthal flow:

$$u(x, \zeta) = -U_0(\zeta + 1), \quad v(x, y, \zeta) = 0, \quad w(x, y, \zeta) = 0. \tag{64}$$

4.2. Parabolic azimuthal velocity profile

We now take a parabolic azimuthal component u:

$$u(x, \zeta) = A(x)(\zeta + 1)^2 + B(x)(\zeta + 1), \tag{65}$$

with $A(x)$ and $B(x)$ arbitrary functions. In fact, by the second condition in (56), $A(x)$ and $B(x)$ are dependent

$$A(x) + B(x) = -U_0, \tag{66}$$

and by (31) and (32)

$$\phi(x, \zeta) = \frac{A(x)}{3}\zeta^3 + \frac{A(x) - U_0}{2}\zeta^2 - U_0\zeta. \tag{67}$$

The condition (60) yields

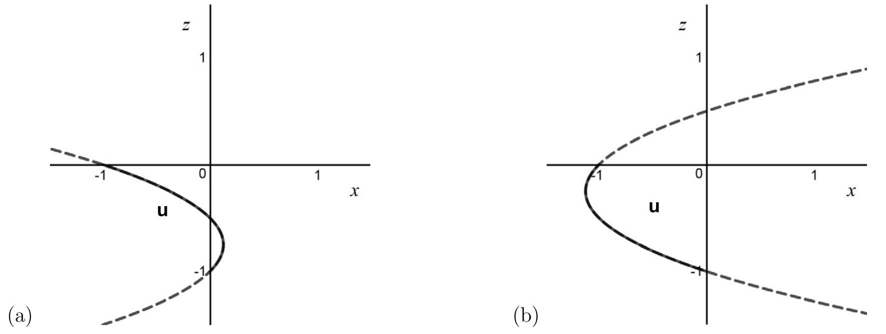

Fig. 1. The graph of the function u from (69) for (a) $U_0 = 1$, $A_0 = -2$; (b) $U_0 = 1$, $A_0 = 2$.

$$A'(x) = 0 \implies A(x) = \text{const} := A_0. \tag{68}$$

Thus, in this case, we obtain again a purely azimuthal flow:

$$u(x,\zeta) = A_0(\zeta+1)^2 - (A_0+U_0)(\zeta+1) \tag{69}$$

$$v(x,y,\zeta) = 0 \tag{70}$$

$$w(x,y,\zeta) = 0. \tag{71}$$

With the choice $U \approx 0.5\,\mathrm{m\,s^{-1}}$ for the speed scale, that is, a typical speed at the surface, we take $U_0 = 1$. In Fig. 1 we draw the graph of the function u for two values of the constant A_0. The solid line represents the profile of u from the free surface down to the bottom.

4.3. Cubic azimuthal velocity profile

For the cubic azimuthal component u:

$$u(x,\zeta) = A(x)(\zeta+1)^3 + B(x)(\zeta+1)^2 + C(x)(\zeta+1), \tag{72}$$

with $A(x)$, $B(x)$ and $C(x)$ arbitrary functions, by (56), (31) and (32), we have

$$A(x) + B(x) + C(x) = -U_0, \tag{73}$$

and

$$\phi(x,\zeta) = \frac{A(x)}{4}\zeta^4 + \frac{3A(x)+B(x)}{3}\zeta^3 + \frac{3A(x)+2B(x)+C(x)}{2}\zeta^2 - U_0\zeta. \tag{74}$$

The condition (60), which insures the vanishing of the components v and w on the bed, yields now the expression

$$9A'(x) + 8B'(x) + 6C'(x) = 0. \tag{75}$$

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On the other hand, if we derive the equation (73), we get

$$A'(x) + B'(x) + C'(x) = 0. \tag{76}$$

From the last two equations and the equation (73), we can express $B(x)$ and $C(x)$ as function of $A(x)$:

$$B(x) = -\frac{3}{2}A(x) + k_1, \quad C(x) = \frac{A(x)}{2} - k_1 - U_0, \tag{77}$$

where k_1 is a constant, and thus, the cubic azimuthal component becomes

$$\begin{aligned} u(x,\zeta) = A(x)(\zeta+1)^3 &+ \left(-\frac{3}{2}A(x) + k_1\right)(\zeta+1)^2 \\ &+ \left(\frac{A(x)}{2} - k_1 - U_0\right)(\zeta+1), \end{aligned} \tag{78}$$

with $A(x)$ an arbitrary function and k_1 an arbitrary constant.

If $u_\zeta + 2\omega \neq 0$, that is,

$$3A(x)\zeta^2 + [3A(x) + 2k_1]\zeta + \frac{A(x)}{2} + k_1 - U_0 + 2\omega \neq 0, \tag{79}$$

then

$$A(x) > 0 \implies u_\zeta + 2\omega > 0 \tag{80}$$

$$A(x) < 0 \implies u_\zeta + 2\omega < 0. \tag{81}$$

Let us now analyse the quadratic equation

$$3A(x)\zeta^2 + [3A(x) + 2k_1]\zeta + \frac{A(x)}{2} + k_1 - U_0 + 2\omega = 0. \tag{82}$$

The discriminate of this equation is

$$\Delta \;=\; 3A^2(x) + 12[U_0 - 2\omega]A(x) + 4k_1^2. \tag{83}$$

If

$$36[U_0 - 2\omega]^2 - 12k_1^2 < 0, \tag{84}$$

then we always have

$$\Delta > 0 \tag{85}$$

and if

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$$36[U_0 - 2\omega]^2 - 12k_1^2 > 0, \tag{86}$$

then

$$\begin{aligned}\Delta > 0 \iff A(x) \in &\left(-\infty, -2(U_0 - 2\omega) - \sqrt{4[U_0 - 2\omega]^2 - \frac{4}{3}k_1^2}\right) \\ &\cup \left(-2(U_0 - 2\omega) + \sqrt{4[U_0 - 2\omega]^2 - \frac{4}{3}k_1^2}, \infty\right)\end{aligned} \tag{87}$$

$$\begin{aligned}\Delta < 0 \iff A(x) \in &\left(-2(U_0 - 2\omega) - \sqrt{4[U_0 - 2\omega]^2 - \frac{4}{3}k_1^2},\right. \\ &\left. -2(U_0 - 2\omega) + \sqrt{4[U_0 - 2\omega]^2 - \frac{4}{3}k_1^2}\right)\end{aligned} \tag{88}$$

With the choice $U \approx 0.5\ \mathrm{m\,s^{-1}}$ for the speed scale, we find from (10) that the nondimensional ω has the approximate value $\omega \approx 0.6$, and that $U_0 = 1$. Hence,

$$U_0 - 2\omega \approx -0.2. \tag{89}$$

4.3.1. The case $\Delta < 0$

If we make choices for the constant k_1 that satisfy (86), that is,

$$k_1^2 < 0.12, \tag{90}$$

and for the arbitrary function $A(x)$ choices that satisfy (88), that is,

$$A(x) \in \left(0.4 - \sqrt{0.16 - \frac{4}{3}k_1^2}, 0.4 + \sqrt{0.16 - \frac{4}{3}k_1^2}\right), \tag{91}$$

then, we avoid the singularities, and we get a three-dimensional, nonlinear solution for our problem which is *non purely azimuthal*. The azimuthal component u is given by (78), and the expressions for the non-zero components v and w are provided by (54) and (55), with

$$\begin{aligned}\phi(x,\zeta) &= \frac{A(x)}{4}\zeta^4 + \left(\frac{A(x)}{2} + \frac{k_1}{3}\right)\zeta^3 \\ &\quad + \left(\frac{A(x)}{4} + \frac{k_1}{2} - \frac{U_0}{2}\right)\zeta^2 - U_0\zeta.\end{aligned} \tag{92}$$

With (78) and (92) in view, we calculate

$$u_x = A(x)'\zeta(\zeta+1)\left(\zeta + \frac{1}{2}\right), \tag{93}$$

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$$\phi_x = \frac{A'(x)}{4}\zeta^2(\zeta+1)^2, \tag{94}$$

$$uu_x + 2\omega\phi_x = A'(x)\zeta(\zeta+1)^2\Big[A(x)\zeta^3 + (A(x)+k_1)\zeta^2 + \left(\frac{A(x)}{4}+\frac{k_1}{2}-U_0+\frac{\omega}{2}\right)\zeta - \frac{U_0}{2}\Big], \tag{95}$$

$$u_{x\zeta} = A'(x)\left(3\zeta^2+3\zeta+\frac{1}{2}\right), \tag{96}$$

$$u_{\zeta\zeta} = 6A(x)\zeta + 3A(x) + 2k_1. \tag{97}$$

We get

$$v = \frac{A'(x)y}{(u_\zeta+2\omega)^2}\Big\{u\left(3\zeta^2+3\zeta+\frac{1}{2}\right)(u_\zeta+2\omega) - \zeta(\zeta+1)^2u_{\zeta\zeta}\Big[A(x)\zeta^3+(A(x)+k_1)\zeta^2 + \left(\frac{A(x)}{4}+\frac{k_1}{2}-U_0+\frac{\omega}{2}\right)\zeta-\frac{U_0}{2}\Big]\Big\}. \tag{98}$$

$$w = yv - \frac{A'(x)\zeta(\zeta+1)^2}{u_\zeta+2\omega}\Big[A(x)\zeta^3+(A(x)+k_1)\zeta^2 + \left(\frac{A(x)}{4}+\frac{k_1}{2}-U_0+\frac{\omega}{2}\right)\zeta-\frac{U_0}{2}\Big]. \tag{99}$$

In this case, at a fixed x, with $U_0 = 1$ and with (90) and (91) in view, the azimuthal component u of the velocity is negative from the free surface down to the bottom, but the sign of the components v and w depends on the sign of the derivative $A'(x)$, at the fixed x, and on the region where y is situated. The sign of the components v and w can also change from the free surface down to the bottom. For example, for $k_1 = 0$ we get from (91) that $A(x) \in (0, 0.8)$. In Fig. 2 we draw the graph of the functions u, v and w from (78), (98) and (99), respectively, for the values $k_1 = 0$, $U_0 = 1$, $A(x) = 0.2$ at a fixed x, $A'(x) = -1$ or $A'(x) = 1$, at a fixed x, and along the Equator or at $y = 1$. The solid line represents the profiles from the free surface down to the bottom. We observe that along the Equator, $v = 0$ and w changes the sign only once—for example, if $A'(x) = -1 < 0$ at a fixed x, w is positive close to the free surface and negative very close to the bottom—but at $y = 1$, v and w change the sign twice—for example, if $A'(x) = -1 < 0$ at a fixed x, v and w have positive sign close to the free surface, then there is a region where they are negative and very close to the bottom they become again positive, and the other way around if $A'(x) = 1 > 0$, at a fixed x.

4.3.2. The case $\Delta > 0$

This case happens if the constant k_1 satisfies (84) or if the constant k_1 satisfies (86) and the function $A(x)$ satisfies (87). Then, the equation (82) has the solutions

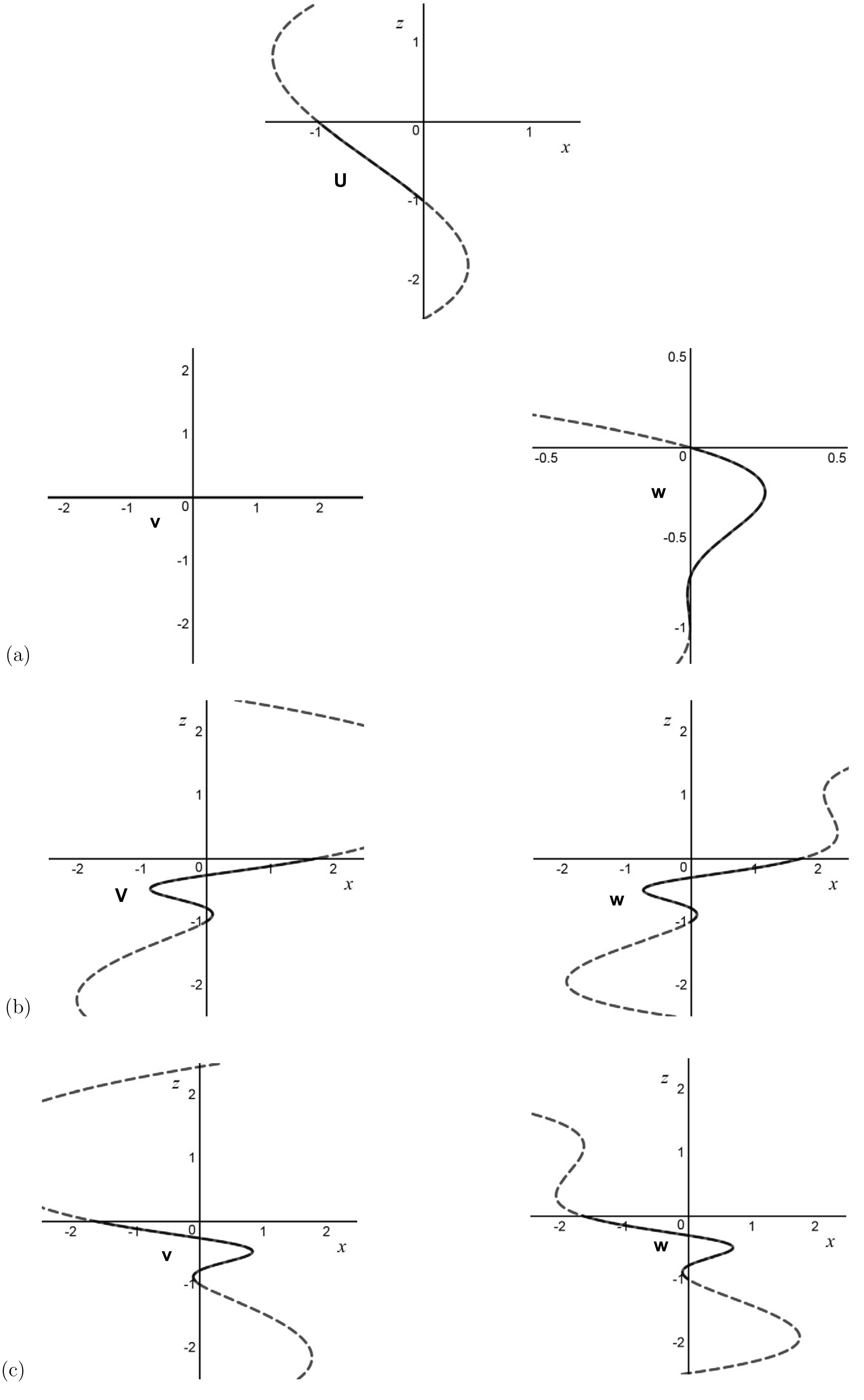

Fig. 2. The graph of the functions u, v and w from (78), (98) and (99), respectively, for the values $k_1 = 0$, $U_0 = 1$, $A(x) = 0.2$, at a fixed x and: (a) $A'(x) = -1$, at a fixed x and $y = 0$, (b) $A'(x) = -1$, at a fixed x and $y = 1$, (c) $A'(x) = 1$, at a fixed x and $y = 1$.

$$\begin{aligned} \zeta_1 &= -\tfrac{1}{2} - \tfrac{k_1}{3A(x)} + \sqrt{\tfrac{1}{12} + \tfrac{k_1^2}{9A^2(x)} + \tfrac{U_0-2\omega}{3A(x)}} \\ \zeta_2 &= -\tfrac{1}{2} - \tfrac{k_1}{3A(x)} - \sqrt{\tfrac{1}{12} + \tfrac{k_1^2}{9A^2(x)} + \tfrac{U_0-2\omega}{3A(x)}}, \end{aligned} \tag{100}$$

with

$$\zeta_1 + \zeta_2 = -1 - \frac{2k_1}{3A(x)}, \quad \zeta_1 \cdot \zeta_2 = \frac{1}{6} + \frac{k_1 - (U_0 - 2\omega)}{3A(x)}. \tag{101}$$

For our problem

$$-1 \le \zeta \le 0. \tag{102}$$

4.3.2.1. ζ_1 and ζ_2 are both in the interval $(-1, 0)$

We show that in this case the solution reduces to a purely azimuthal flow. The constraints (43) and (52) ensure the removal of the singularities (100). With $\Delta \neq 0$, the constrain (52) is immediately satisfied. In order to satisfy (43), the solutions (100) have to be also solutions of the following equation from (95):

$$A(x)\zeta^3 + [A(x) + k_1]\zeta^2 + \left(\frac{A(x)}{4} + \frac{k_1}{2} - U_0 + \frac{\omega}{2}\right)\zeta - \frac{U_0}{2} = 0. \tag{103}$$

Vieta's formulas applied to the equation (103) yield

$$\zeta_1 + \zeta_2 + \zeta_3 = -1 - \frac{k_1}{A(x)}, \quad \zeta_1 \cdot \zeta_2 \cdot \zeta_3 = \frac{U_0}{2A(x)} \tag{104}$$

Taking into account (101) we get

$$\zeta_3 = -\frac{k_1}{3A(x)} \text{ and } \zeta_3 = \frac{3U_0}{A(x) + 2k_1 - 2(U_0 - 2\omega)}, \tag{105}$$

therefore, $A(x)$ has to be a constant:

$$A(x) = \text{const} := \mathcal{A}_0. \tag{106}$$

Because the velocity components v and w have $A'(x)$ as factor, in this situation, we obtain the purely azimuthal flow:

$$\begin{aligned} u(x,\zeta) &= \mathcal{A}_0(\zeta+1)^3 + \left(-\frac{3}{2}\mathcal{A}_0 + k_1\right)(\zeta+1)^2 \\ &\quad + \left(\frac{\mathcal{A}_0}{2} - k_1 - U_0\right)(\zeta+1) \end{aligned} \tag{107}$$

$$v(x, y, \zeta) = 0 \tag{108}$$

$$w(x, y, \zeta) = 0. \tag{109}$$

From (100), ζ_1 and ζ_2 are both in the interval $(-1, 0)$ iff the following inequalities are satisfied

$$\left|\frac{k_1}{A(x)}\right| < \frac{3}{2} \quad \& \quad \frac{k_1 - (U_0 - 2\omega)}{A(x)} > -\frac{1}{2} \quad \& \quad \frac{k_1 + (U_0 - 2\omega)}{A(x)} < \frac{1}{2}. \tag{110}$$

With (89), (84) or (86)–(87), and (110) in view, we draw in Fig. 3 (a)–(b) the graph of the function u from (107) for $k_1 = 1$ and two values of the constant $\mathcal{A}_0$: $\mathcal{A}_0 = 3$ and $\mathcal{A}_0 = -3$.

4.3.2.2. $[\zeta_1 > 0$ *and* $\zeta_2 \in (-1, 0)]$ *or* $[\zeta_1 \in (-1, 0)$ *and* $\zeta_2 < -1]$

In this case, if $A(x)$ is constant we get the purely azimuthal flow in the form (107)–(109). If we consider a function $A(x)$ which is not constant, to write the non-azimuthal components (98) and (99) we must remove the singularity in the interval $(-1, 0)$. By (43), we get the following nonlinear equations for $A(x)$:

$$\mathfrak{f}_2(A(x)) := A(x)\zeta_2^3 + [A(x) + k_1]\zeta_2^2 + \left(\frac{A(x)}{4} + \frac{k_1}{2} - U_0 + \frac{\omega}{2}\right)\zeta_2 - \frac{U_0}{2} = 0,$$

or

$$\mathfrak{f}_1(A(x)) := A(x)\zeta_1^3 + [A(x) + k_1]\zeta_1^2 + \left(\frac{A(x)}{4} + \frac{k_1}{2} - U_0 + \frac{\omega}{2}\right)\zeta_1 - \frac{U_0}{2} = 0,$$

respectively, with ζ_1, ζ_2 from (100), k_1, ω and U_0 constants. If $\mathfrak{f}_2(A(x)) = 0$ or $\mathfrak{f}_1(A(x)) = 0$, respectively, does not have solutions, then we can not remove the singularity, but if $\mathfrak{f}_2(A(x)) = 0$ or $\mathfrak{f}_1(A(x)) = 0$, respectively, has solutions, then we get that $A(x)$ is expressed as function of k_1, ω and U_0, thus, we end up with a constant function $A(x)$ which yields again the purely azimuthal flow in the form (107)–(109).

Taking into account (100), $\zeta_1 > 0$ and $\zeta_2 \in (-1, 0)$ iff the following inequalities are satisfied

$$\frac{k_1 - (U_0 - 2\omega)}{A(x)} < -\frac{1}{2} \quad \& \quad \frac{k_1}{A(x)} < \frac{3}{2} \quad \& \quad \frac{k_1 + (U_0 - 2\omega)}{A(x)} < \frac{1}{2}, \tag{111}$$

and $\zeta_1 \in (-1, 0)$ and $\zeta_2 < -1$ iff the following inequalities are satisfied

$$\frac{k_1}{A(x)} > -\frac{3}{2} \quad \& \quad \frac{k_1 - (U_0 - 2\omega)}{A(x)} > -\frac{1}{2} \quad \& \quad \frac{k_1 + (U_0 - 2\omega)}{A(x)} > \frac{1}{2}. \tag{112}$$

With (89), (84) or (86)–(87), and (111) or (112) in view, we draw in Fig. 3 (c)–(f), the graph of the function u from (107) for: (c) $k_1 = 2$, $U_0 = 1$, $\mathcal{A}_0 = -2$; (d) $k_1 = -2$, $U_0 = 1$, $\mathcal{A}_0 = 2$; (e) $k_1 = 2$, $U_0 = 1$, $\mathcal{A}_0 = 1$; (f) $k_1 = -1$, $U_0 = 1$, $\mathcal{A}_0 = -1$.

4.3.2.3. $\zeta_1 > 0$ *and* $\zeta_2 < -1$

In this case, the singularities are outside the interval $[-1, 0]$ and we get a three-dimensional, nonlinear, *non purely azimuthal* solution in the form (78), (98) and (99). Taking into account (100), $\zeta_1 > 0$ and $\zeta_2 < -1$ iff the following inequalities are satisfied

$$\frac{k_1 - (U_0 - 2\omega)}{A(x)} < -\frac{1}{2} \quad \& \quad \frac{k_1 + (U_0 - 2\omega)}{A(x)} > \frac{1}{2}. \tag{113}$$

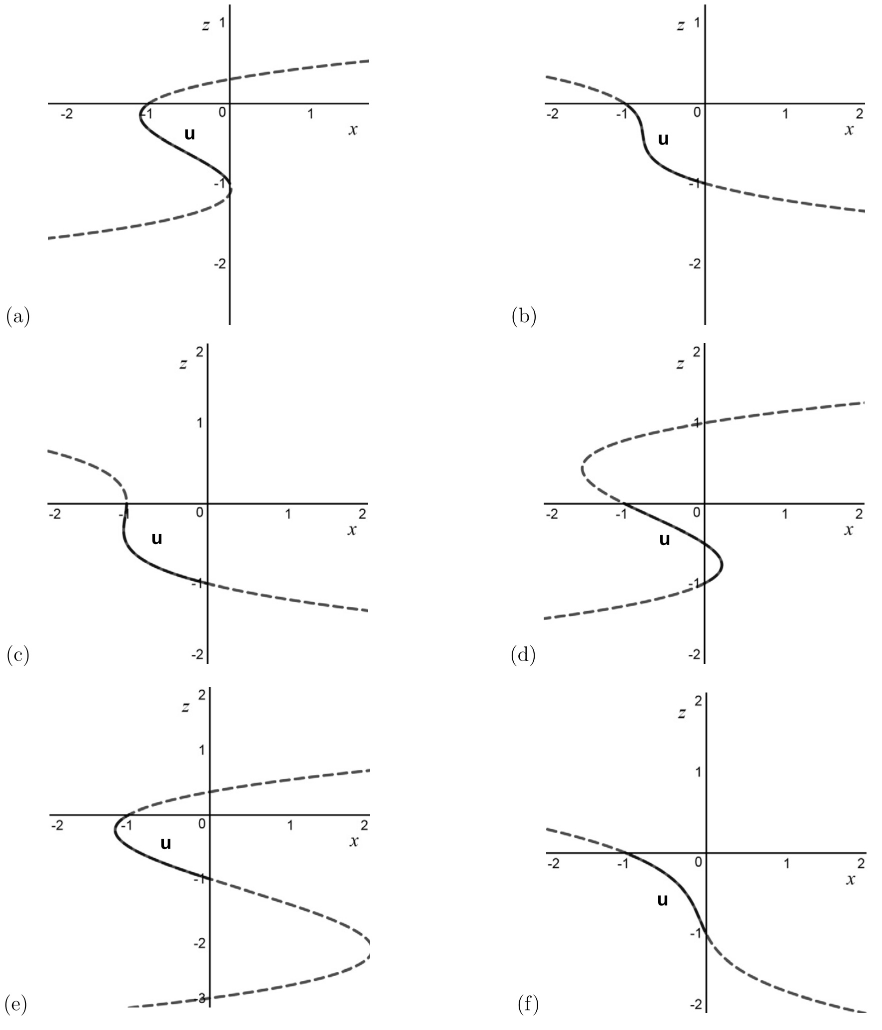

Fig. 3. The graph of the function u from (107) for (a) $k_1 = 1$, $U_0 = 1$, $\mathcal{A}_0 = 3$; (b) $k_1 = 1$, $U_0 = 1$, $\mathcal{A}_0 = -3$; (c) $k_1 = 2$, $U_0 = 1$, $\mathcal{A}_0 = -2$; (d) $k_1 = -2$, $U_0 = 1$, $\mathcal{A}_0 = 2$; (e) $k_1 = 2$, $U_0 = 1$, $\mathcal{A}_0 = 1$; (f) $k_1 = -1$, $U_0 = 1$, $\mathcal{A}_0 = -1$.

With (89), (84) or (86)–(87), and (113) in view, we draw in Fig. 4 the graph of the functions u, v and w from (78), (98) and (99), respectively, for the values $k_1 = 0.1$, $U_0 = 1$, $A(x) = -0.1$ and $A'(x) = -1$ at a fixed x, along the Equator at $y = 0$ in (a) and at $y = 1$ in (b). We observe in this case the same comportment as in Fig. 2: from the free surface down to the bottom, w changes the sign only one time along the Equator, but at $y = 1$, v and w change two times the sign.

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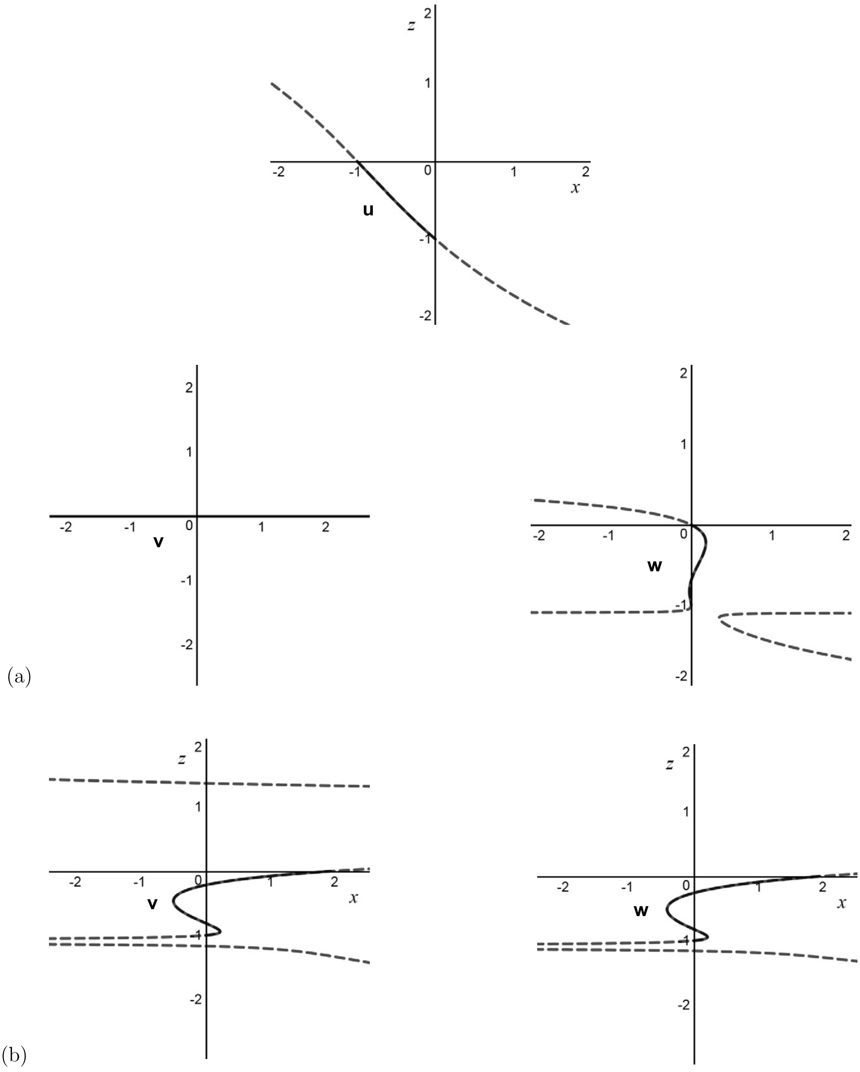

Fig. 4. The graph of the functions u, v and w from (78), (98) and (99), respectively, for the values for the values $k_1 = 0.1$, $U_0 = 1$, $A(x) = -0.1$ and $A'(x) = -1$, at a fixed x, and at: (a) $y = 0$, (b) $y = 1$.

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