CONSTRUCTING THE DETERMINANT SPHERE USING A TATE TWIST

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Abstract. Following an idea of Hopkins, we construct a model of the determinant sphere $S\langle\det\rangle$ in the category of $K(n)$-local spectra. To do this, we build a spectrum which we call the Tate sphere $S(1)$. This is a p-complete sphere with a natural continuous action of $\mathbb{Z}_p^\times$. The Tate sphere inherits an action of $\mathbb{G}_n$ via the determinant and smashing Morava E-theory with $S(1)$ has the effect of twisting the action of $\mathbb{G}_n$. A large part of this paper consists of analyzing continuous $\mathbb{G}_n$-actions and their homotopy fixed points in the setup of Devinatz and Hopkins.

1. Introduction

Let p be a prime and $n > 0$ an integer; these will be fixed throughout and we will always suppress p and mostly suppress n from the notation. Let $\mathbf{E} = E_n$ denote the Lubin–Tate spectrum associated to the Honda formal group law of height n over $\mathbb{F}_{p^n}$, and let $\mathbf{K} = K(n)$ be the corresponding Morava K-theory at height n at the prime p. As is the usual convention, given any spectrum X, we write

$$\mathbf{E}_*X = \pi_*L_{\mathbf{K}}(\mathbf{E} \wedge X)$$

where $L_{\mathbf{K}}$ denotes $\mathbf{K}$-localization.

We are interested in the $\mathbf{K}$-local category and, in particular, one very interesting spectrum therein which arises from comparing two dualities. The first of these duality functors is Spanier–Whitehead duality, sending X to $D_nX = F(X, L_{\mathbf{K}}S^0)$. If X is a dualizable spectrum – for example if X is a finite spectrum – then $\mathbf{E}_*D_nX \cong \mathbf{E}^{-*}X$ and can be computed by a universal coefficient spectral sequence. The second is Gross–Hopkins duality, sending X to $I_nX = F(M_nX, I_{\mathbb{Q}/\mathbb{Z}})$, the Brown–Comenetz dual of its monochromatic layer. Specifically, M_nX is the fiber of $L_nX \to L_{n-1}X$ and $I_{\mathbb{Q}/\mathbb{Z}}$ is the spectrum representing the cohomology theory $I^*_{\mathbb{Q}/\mathbb{Z}}(X) = \mathrm{Hom}_{\mathbb{Z}}(\pi_*X, \mathbb{Q}/\mathbb{Z})$. It is a consequence of the work of Gross and Hopkins that the dual I_n of the sphere $L_{\mathbf{K}}S^0$ is invertible in the $\mathbf{K}$-local category and, hence, we have for any spectrum X a natural equivalence

$$I_nX \simeq D_nX \wedge I_n.$$

This material is based upon work supported by the National Science Foundation under Grant No. DMS-1606479 and Grant No. DMS-1725563. TB was partially supported by the DNRF92 and the European Unions Horizon 2020 research and innovation programme under the Marie Sklodowska-Curie grant agreement No. 751794. The authors would like to thank the Isaac Newton Institute for Mathematical Sciences for support and hospitality during the program Homotopy Harnessing Higher Structures when work on this paper was undertaken. This work was supported by EPSRC Grant Number EP/R014604/1.

At this point, information about the homotopy type of I_n becomes vital, and one gets a handle on it using that the spectrum $\mathbf{E}$ has an action by the Morava stabilizer group $\mathbb{G} = \mathbb{G}_n$. Consequently, the graded $\mathbf{E}_*$-module $\mathbf{E}_*X$ has a continuous action by $\mathbb{G}$, giving it the structure of a Morava module.

The key to the invertibility of I_n is the calculation of the Morava module $\mathbf{E}_*I_n$. The group $\mathbb{G}$ is a semidirect product $\mathbb{S} \rtimes Gal(\mathbb{F}_{p^n}/\mathbb{F}_p)$, where $\mathbb{S} = \mathbb{S}_n$ is the automorphism group of the formal group law of $\mathbf{K}$. The group $\mathbb{S}$ can be identified with a subgroup of the general linear group $\mathrm{Gl}_n(\mathbb{W})$, where $\mathbb{W}$ denotes the Witt vectors on the finite field $\mathbb{F}_{p^n}$. The group $\mathbb{S}$ has enough symmetry that the determinant $\mathrm{Gl}_n(\mathbb{W}) \to \mathbb{W}^\times$ restricts to a homomorphism

$$\det\colon \mathbb{S} \longrightarrow \mathbb{Z}_p^\times,$$

which can be extended to $\mathbb{G}$ as the composite

$$\det\colon \mathbb{G} = \mathbb{S} \rtimes Gal(\mathbb{F}_{p^n}/\mathbb{F}_p) \xrightarrow{\det \times \mathrm{id}} \mathbb{Z}_p^\times \times Gal(\mathbb{F}_{p^n}/\mathbb{F}_p) \xrightarrow{proj_1} \mathbb{Z}_p^\times.$$

This gives a $\mathbb{G}$-action on $\mathbb{Z}_p$, and we write the corresponding representation as $\mathbb{Z}_p\langle\det\rangle$. If M is a Morava module, we can define a new Morava module by $M\langle\det\rangle = M \otimes_{\mathbb{Z}_p} \mathbb{Z}_p\langle\det\rangle$ with the diagonal $\mathbb{G}$-action. Then we have by [HG94] and [Str00] an isomorphism of Morava modules

$$\mathbf{E}_*I_n = \mathbf{E}_*(S^{n^2-n})\langle\det\rangle.$$

If the prime is large ($2p > \max\{n^2+1, 2n+2\}$) this determines the homotopy type of I_n. If the prime is not large, then we would like a fixed model $S\langle\det\rangle$ of an invertible spectrum in the $\mathbf{K}$-local category equipped with an isomorphism

$$\mathbf{E}_*S\langle\det\rangle \cong \mathbf{E}_*\langle\det\rangle.$$

Then we have a $\mathbf{K}$-local equivalence

$$I_n \simeq S^{n^2-n} \wedge S\langle\det\rangle \wedge P_n,$$

where P_n is an invertible $\mathbf{K}$-local spectrum with $\mathbf{E}_*P_n \cong \mathbf{E}_*S^0$ as Morava modules. Attention then turns to identifying P_n. In the known cases this comes down to calculating the homotopy groups of I_nX for X a particularly nice type n complex. See [GHMR15] for analysis of P_n at $n = 2 = p-1$; the case $n = 1 = p-1$ was done by [HMS94] and also appears in [HS14, GHMR15].

The point of this note is to give a construction of a model of $S\langle\det\rangle$ valid at all primes p and all $n > 0$. We actually give two constructions of $S\langle\det\rangle$, one using homotopy fixed points, following an idea of Mike Hopkins, and another, more naive and direct one, following ideas from [GHMR15], fixing the typos therein and extending the construction to the prime 2.

The first model will evidently have the property that $L_{\mathbf{K}}(\mathbf{E}^{h\mathbb{K}} \wedge S\langle\det\rangle) = \mathbf{E}^{h\mathbb{K}}$ for all closed subgroups $\mathbb{K}$ in the kernel of the determinant. The key to this construction is to introduce a spectrum $S(1)$ with a continuous $\mathbb{G}$–action, non-equivariantly equivalent to the p-complete sphere spectrum $S^0 = S^0_p$, and such that smashing with it naturally twists $\mathbb{G}$–actions by the determinant representation. Then we define

$$S\langle\det\rangle = (\mathbf{E} \wedge S(1))^{h\mathbb{G}},$$

the action on the right-hand side being diagonal. The following is our main result.

Theorem 3.10. *There is a canonical* $\mathbb{G}$*-equivariant equivalence* $f\colon \mathbf{E} \wedge S\langle \det \rangle \to \mathbf{E} \wedge S(1)$*, where the action of* $\mathbb{G}$ *on the source is via the action on* $\mathbf{E}$*, while on the target it is diagonal. This induces an isomorphism of Morava modules* $\mathbf{E}_* S\langle \det \rangle \cong \mathbf{E}_* \langle \det \rangle$.

If $\mathbb{K}$ is a closed subgroup of $\mathbb{G}$ in the kernel of the determinant, taking $\mathbb{K}$-homotopy fixed points in this equivalence gives the desired result (Corollary 3.11)

$$\mathbf{E}^{h\mathbb{K}} \wedge S\langle \det \rangle \simeq (\mathbf{E} \wedge S(1))^{h\mathbb{K}} \simeq \mathbf{E}^{h\mathbb{K}}.$$

This project gives a chance to revisit and give an encomium on the amazing paper of Devinatz and Hopkins on fixed point spectra in the $\mathbf{K}$-local category [DH04]. Distilled down we have the following question: let X be a spectrum with a continuous action of the Morava stabilizer group $\mathbb{G}$. We can then form the $\mathbb{G}$-spectrum $Z = \mathbf{E} \wedge X$ with diagonal $\mathbb{G}$-action and discuss the homotopy type of $Z^{h\mathbb{G}} = (\mathbf{E} \wedge X)^{h\mathbb{G}}$. Note that $\mathbf{E}_* Z = \pi_* L_{\mathbf{K}}(\mathbf{E} \wedge Z)$ has two $\mathbb{G}$-actions: the Morava module action on $\mathbf{E}$ and the action on Z. A consequence of our results is that if X is dualizable in the $\mathbf{K}$-local category, then

$$\mathbf{E}_* \left(Z^{h\mathbb{G}} \right) = \mathbf{E}_* (\mathbf{E} \wedge X)^{h\mathbb{G}} \cong \mathbf{E}_* X \tag{1.1}$$

and the Morava module action on $\mathbf{E}_* (\mathbf{E} \wedge X)^{h\mathbb{G}}$ corresponds to the diagonal action on

$$\mathbf{E}_* X = \pi_* L_{\mathbf{K}} (\mathbf{E} \wedge X).$$

An analogue of this result for arbitrary spectra X with *trivial* $\mathbb{G}$-action was proven by Davis and Torii [DT12]. The equivalence (1.1) is not hard to prove once we have come to terms with the notion of a continuous $\mathbb{G}$-action. Since we are making a homology calculation we need cosimplicial techniques, and this is exactly what Devinatz and Hopkins supply.

To end this introduction, we remark that in [Wes17], Westerland considers a different variant of the determinant map on $\mathbb{G}$ and defines a corresponding determinant sphere by using a fiber sequence as in Proposition 4.2. Presumably that determinant sphere could also be constructed by the methods in our Section 3, but we have not checked the details.

2. Continuous $\mathbb{G}$ actions and their homotopy fixed points

As is perhaps apparent from the introduction, we will assume our readership has access to the standard framework of $\mathbf{K}$-local homotopy theory. The usual source for an in-depth study of the technicalities is Hovey and Strickland [HS99] and basic introductions can be found in almost any paper on chromatic homotopy theory. We were especially thorough in [BGH17, §2].

Less familiar is the analysis of point-set properties of the action of Morava stabilizer group $\mathbb{G}$ on the spectrum $\mathbf{E}$. We will need to use an explicit construction of the homotopy fixed points. For our purposes the original definition by Devinatz and Hopkins [DH04] will do. The reader interested in extensions and variations of the original notion may want to consult the work of Davis and collaborators, notably Behrens and Quick, e.g., [BD10, Qui13, DQ16].

We will also not access the full power and structure of equivariant stable homotopy theory. Our G-spectra will simply be G-objects in some suitable category of spectra; when G is profinite, we will also use a simple notion of continuity (see Definition 2.5).

We start with some algebra. Recall that $\mathbf{E}_* = \mathbb{W}[\![u_1, \ldots, u_{n-1}]\!][u^{\pm 1}]$ where the power series ring is in degree zero and the degree of u is -2 and let $\mathfrak{m} \subset \mathbf{E}_0$ be the maximal ideal.

Remark 2.1. Before we proceed further, we need to establish some more notation. Using the periodicity results of Hopkins and Smith [HS98], Hovey and Strickland produce a sequence of ideals $J(i) \subseteq \mathfrak{m} \subseteq \mathbf{E}_0$ and finite type n spectra $\mathrm{M}_{J(i)}$ with the following properties:

(1) $J(i+1) \subseteq J(i)$ and $\bigcap_i J(i) = 0$;
(2) $\mathbf{E}_0/J(i)$ is finite;
(3) $\mathbf{E}_0(\mathrm{M}_{J(i)}) \cong \mathbf{E}_0/J(i)$ and there are spectrum maps $q\colon \mathrm{M}_{J(i+1)} \to \mathrm{M}_{J(i)}$ realizing the quotient $\mathbf{E}_0/J(i+1) \to \mathbf{E}_0/J(i)$;
(4) There are maps $\eta = \eta_i\colon S^0 \to \mathrm{M}_{J(i)}$ inducing the quotient map $\mathbf{E}_0 \to \mathbf{E}_0/J(i)$ and $q\eta_{i+1} = \eta_i\colon S^0 \to \mathrm{M}_{J(i)}$;
(5) If X is a finite type n spectrum, then the map $X \to \mathrm{holim}_i(X \wedge \mathrm{M}_{J(i)})$ induced by the maps η is an equivalence.
(6) If X is any L_n-local spectrum then by [HS99] we have $L_{\mathbf{K}}X \simeq \mathrm{holim}_i\, X \wedge \mathrm{M}_{J(i)}$. In particular we have $\mathbf{E} \simeq \mathrm{holim}_i\, \mathbf{E} \wedge \mathrm{M}_{J(i)}$.

Most of this is proved in [HS99, § 4], and (6) is proved in [HS99, Prop. 7.10]. Hovey and Strickland also prove that items (1)-(5) characterize the tower $\{\mathrm{M}_{J(i)}\}$ up to equivalence in the pro-category of towers under S^0. See Proposition 4.22 of [HS99]. Note that the sequence $\{J(i)\}$ of ideals defines the same topology on $\mathbf{E}_0$ as the $\mathfrak{m}$-adic topology and that $\mathbb{G}$ acts on $\mathbf{E}_0/J(i)$ through a finite quotient.

For profinite sets $T = \lim_j T_j$ and $A = \lim_i A_i$, recall that the set of continuous maps from T to A is defined as

$$\mathrm{Map}^c(T, A) = \lim_i \mathrm{colim}_j \,\mathrm{Map}(T_j, A_i).$$

Let M be a Morava module and always assume M is $\mathfrak{m}$-complete. An important example of the previous construction is the Morava module of continuous maps

$$\mathrm{Map}^c(\mathbb{G}, M) = \lim_i \mathrm{Map}^c(\mathbb{G}, M/\mathfrak{m}^i) = \lim_i \mathrm{colim}_j \,\mathrm{Map}(\mathbb{G}/U_j, M/\mathfrak{m}^i)$$

where $U_{j+1} \subseteq U_j \subseteq \mathbb{G}$ is a nested sequence of open normal subgroups so that $\cap\, U_j = \{e\}$; then $\mathbb{G} = \lim_j \mathbb{G}/U_j$.

We now begin to make these constructions topological by giving a definition of a spectrum of continuous maps in the $\mathbf{K}$-local category.

Definition 2.2. Suppose $T = \lim_j T_j$ is a profinite set, and $A \simeq \mathrm{holim}_i\, A \wedge \mathrm{M}_{J(i)}$ is a $\mathbf{K}$-local spectrum. Define

$$F_c(T_+, A) = \mathrm{holim}_i \,\mathrm{hocolim}_j \,F({T_j}_+, A \wedge \mathrm{M}_{J(i)}).$$

In applications T will be $\mathbb{G}$ or $\mathbb{G}/\mathbb{K} \times \mathbb{G}^s$ with $s \geq 0$ and $\mathbb{K} \subseteq \mathbb{G}$ a closed subgroup, or $\mathbb{G} = \mathbb{Z}_p^\times$.

We now calculate $\pi_* F_c(T_+, A)$, at least for some A. For later applications, we will need a slightly more general result about $\pi_* F(Z, F_c(T_+, A))$ with Z arbitrary. If Z is any spectrum we may write $Z \simeq \mathrm{hocolim}\, Z^\alpha$ for some filtered collection of finite spectra. If $A \simeq \mathrm{holim}_i\, A \wedge \mathrm{M}_{J(i)}$ is a $\mathbf{K}$-local spectrum, then we have a topology on $\pi_t F(Z, A) = A^{-t}(Z)$ defined by the open system of neighborhoods of zero given by the kernels of the map

$$\pi_t F(Z, A) \longrightarrow \pi_t F(Z^\alpha, A \wedge \mathrm{M}_{J(i)}).$$

This is the *natural topology* of [HS99, Section 11]. The groups $\pi_* F(Z, A)$ are complete in this topology if

$$\pi_* F(Z, A) \cong \lim_{\alpha, i} \pi_* F(Z^\alpha, A \wedge \mathrm{M}_{J(i)}).$$

In applying the following result our main example will be $A = \mathbf{E} \wedge X$ with X dualizable in the $\mathbf{K}$-local category.

Lemma 2.3. *Suppose Z is any spectrum, $T = \lim_j T_j$ is a profinite set, and $A \simeq \mathrm{holim}_i\, A \wedge \mathrm{M}_{J(i)}$ is a $\mathbf{K}$-local spectrum. Further suppose $\pi_t(A \wedge \mathrm{M}_{J(i)})$ is finite for all i and t. We then have an isomorphism*

$$\pi_* F(Z, F_c(T_+, A)) \cong \mathrm{Map}^c(T, A^{-*}Z)$$

where $A^{-*}Z$ is equipped with the natural topology.

Proof. Let $Z \simeq \mathrm{hocolim}_\alpha\, Z^\alpha$ be some cellular filtration on Z by finite spectra. Our finiteness hypothesis on A implies

$$A^{-*}Z = \pi_* F(Z, A) \cong \lim_{\alpha, i} \pi_* F(Z^\alpha, A \wedge \mathrm{M}_{J(i)}).$$

Now we have that

$$F(Z, F_c(T_+, A)) \simeq \mathrm{holim}_\alpha\, \mathrm{holim}_i\, F(Z^\alpha, \mathrm{hocolim}_j\, F(T_{j+}, A \wedge \mathrm{M}_{J(i)}))$$

is equivalent to

$$\mathrm{holim}_\alpha\, \mathrm{holim}_i\, \mathrm{hocolim}_j\, F(Z^\alpha, F(T_{j+}, A \wedge \mathrm{M}_{J(i)})),$$

since Z^α is dualizable. The homotopy groups of

$$F(Z^\alpha, F(T_{j+}, A \wedge \mathrm{M}_{J(i)})) \simeq F(T_{j+}, F(Z^\alpha, A \wedge \mathrm{M}_{J(i)}))$$

are $\mathrm{Map}(T_j, \pi_* F(Z^\alpha, A \wedge \mathrm{M}_{J(i)}))$ and the claim follows using the Milnor sequence and our finiteness hypotheses for the vanishing of the $\lim^1$ term. □

Remark 2.4. For a $\mathbf{K}$-local spectrum $X \simeq \mathrm{holim}\, X \wedge \mathrm{M}_{J(i)}$, we can give

$$F((\mathbb{G}/U_j)_+, X \wedge \mathrm{M}_{J(i)})$$

a left $\mathbb{G} = \lim_i \mathbb{G}/U_i$ action by operating on the right on the source. (Note that the subgroups U_j are normal.) This assembles into an action on $F_c(\mathbb{G}_+, X)$. If the homotopy groups $\pi_t(X \wedge \mathrm{M}_{J(i)})$ are finite, Lemma 2.3 gives an isomorphism of continuous $\mathbb{G}$-modules

$$\pi_* F_c(\mathbb{G}_+, X) \cong \mathrm{Map}^c(\mathbb{G}, \pi_* X) \tag{2.1}$$

where again $\mathbb{G}$ acts on the source.

Writing $\mathbb{G}^s = \lim(\mathbb{G}/U_i)^s$ we define $F_c(\mathbb{G}^s_+, X)$ for $s \geq 1$ as in Definition 2.2. We have that

$$F_c(\mathbb{G}^s_+, F_c(\mathbb{G}^t_+, X)) \simeq F_c(\mathbb{G}^{s+t}_+, X).$$

The equation $F_c(\mathbb{G}^{s+1}_+, X) \simeq F_c(\mathbb{G}_+, F_c(\mathbb{G}^s_+, X))$ defines an action of $\mathbb{G}$ on $F_c(\mathbb{G}^{s+1}_+, X)$ using the right action on the first factor of $\mathbb{G}^{s+1}$.

Evaluation defines a map $\mathbb{G}_+ \wedge F((\mathbb{G}/U_j)_+, X \wedge \mathrm{M}_{J(i)}) \to X \wedge \mathrm{M}_{J(i)}$. Here $\mathbb{G}$ is simply regarded as a set, with no topology. These fit together to give a map

$$\mathbb{G}_+ \wedge F_c(\mathbb{G}_+, X) \longrightarrow X.$$

Recall that if G is a discrete group, a G-action on a spectrum X consists of the data of a map of spaces $BG \to B\mathrm{haut}(X)$, where $\mathrm{haut}(X)$ is the space of self-homotopy equivalences of X. Unpacking this data, with the bar resolution of G giving a simplicial structure on BG, this amounts to

(1) a map $\eta = \eta_X \colon X \to F(G_+, X)$, whose adjoint, explicitly given as the composition

$$G_+ \wedge X \xrightarrow{G_+ \wedge \eta} G_+ \wedge F(G_+, X) \xrightarrow{ev} X,$$

is our action map $G_+ \wedge X \to X$, such that

(2) the resulting diagram

$$X \to F(G_+^{\bullet+1}, X)$$

is an augmented cosimplicial spectrum. This diagram starts as

$$X \xrightarrow{\eta} F(G_+, X) \underset{F(G_+,\eta)}{\overset{F(m_+,X)}{\rightrightarrows}} F(G_+^2, X) \cdots$$

and continues so that η gives the last of the coface maps at each stage, while the rest of the maps come from the simplicial structure on $G^{\bullet+1}$. In particular, m here denotes the multiplication $G \times G \to G$.

Our definition of a continuous $\mathbb{G}$–spectrum will mimic this one; it is for this paper only and is not meant to replace any of the more sophisticated definitions provided by others; for example, in work of Davis and Quick (e.g., [Dav06, DQ16]).

Definition 2.5. Let X be a $\mathbf{K}$-local spectrum. The data of a *continuous* $\mathbb{G}$–action on X consists of:

(1) A map $\eta = \eta_X \colon X \to F_c(\mathbb{G}_+, X)$, and

(2) An extension of η into an augmented cosimplicial diagram

$$X \longrightarrow F_c(\mathbb{G}_+^{\bullet+1}, X), \tag{2.2}$$

in which the coface maps come from the simplicial structure on $\mathbb{G}^{\bullet+1}$ together with η.

If X is a continuous $\mathbb{G}$–spectrum and $\mathbb{K} \subseteq \mathbb{G}$ is a closed subgroup, we define $F_c(\mathbb{G}_+, X)^{\mathbb{K}} = F_c(\mathbb{G}/\mathbb{K}_+, X)$ and

$$\begin{aligned} X^{h\mathbb{K}} &= \mathrm{holim}_\Delta\, F_c(\mathbb{G}_+^{\bullet+1}, X)^{\mathbb{K}} \\ &\simeq \mathrm{holim}_\Delta\, F_c(\mathbb{G}/\mathbb{K}_+ \times \mathbb{G}_+^{\bullet}, X). \end{aligned} \tag{2.3}$$

When presented with a cosimplicial $\mathbb{G}$-spectrum of the type (2.2) we will say we have an augmented cosimplicial $\mathbb{G}$-spectrum so that the augmentation refines the action map, as in Part (1). In practice we may simply produce a cosimplicial diagram as in Part (2), checking that the coface maps are as required.

A map of continuous $\mathbb{G}$–spectra consists of a map of the respective augmented cosimplicial diagrams.

Remark 2.6. Composing with the natural map $F_c(\mathbb{G}_+^{\bullet+1}, X) \to F(\mathbb{G}_+^{\bullet+1}, X)$, we get that a continuous $\mathbb{G}$–spectrum is in particular a $\mathbb{G}$–spectrum in the discrete sense described above. Often the logic is flipped: one starts with a discrete action, and attempts to lift it to a continuous one, i.e., one tries to find the dashed lifts

$$\begin{array}{ccc} & & F_c(\mathbb{G}_+, X) \\ & \nearrow & \downarrow \\ X & \longrightarrow & F(\mathbb{G}_+, X), \end{array} \qquad \begin{array}{ccc} & & F_c(\mathbb{G}_+^{\bullet+1}, X) \\ & \nearrow & \downarrow \\ X & \longrightarrow & F(\mathbb{G}_+^{\bullet+1}, X), \end{array}$$

where the horizontal maps are already given. When such lifts exist, we say that the given discrete action is continuous.

This is what Devinatz–Hopkins [DH04] accomplish, where the discrete $\mathbb{G}$–action on $\mathbf{E}$ was already given by the Goerss–Hopkins–Miller theorem. We discuss this example further in Remark 2.12 below.

Example 2.7. A much easier example of Remark 2.6 is the following: For any $\mathbf{K}$-local spectrum X, the trivial action of $\mathbb{G}$ on X is continuous. Here we start with $\eta\colon X \to F(\mathbb{G}_+, X)$ adjoint to the "projection" map $\mathbb{G}_+ \wedge X \to X$.

Remark 2.8. Suppose further that $\mathbb{K} \subseteq \mathbb{G}$ is a closed subgroup and that $X \simeq \operatorname{holim}_i X \wedge \mathrm{M}_{J(i)}$ is a $\mathbf{K}$-local spectrum such that $\pi_t(X \wedge \mathrm{M}_{J(i)})$ is finite for all i and t. Using Lemma 2.3, one sees that these definitions are designed so that the Bousfield–Kan spectral sequence associated to (2.3) is the homotopy fixed point spectral sequence

$$E_2^{s,t} \cong H_c^s(\mathbb{K}, \pi_t X) \Longrightarrow \pi_{t-s} X^{h\mathbb{K}}$$

with E_2-term given by the continuous group cohomology.

Remark 2.9. There is an obvious generalization of this definition to other settings, for example the group may be any profinite group. Likewise, the spectrum X may live in another category where analogues of the generalized Moore spectra $\mathrm{M}_{J(i)}$ play a similar role. For example X may be a p-complete spectrum, so $X \simeq \operatorname{holim}_i X \wedge S/p^i$. While we will in effect construct a continuous p-complete $\mathbb{Z}_p^\times$ spectrum in this sense in Section 3, we refrain from setting up a general theory, as this would be beyond the scope of this paper.

The following is an easy but useful property, which we record as a lemma for convenient future reference.

Lemma 2.10. *Let X be a continuous $\mathbb{G}$–spectrum. If $X^{h\mathbb{G}}$ is given the trivial $\mathbb{G}$–action, the "inclusion of fixed points" map $X^{h\mathbb{G}} \to X$ is $\mathbb{G}$-equivariant.*

Proof. The map in question is $\operatorname{holim}_\Delta$ of the cosimplicial map

$$F_c(\mathbb{G}_+^{\bullet}, X) \simeq F_c(\mathbb{G}_+^{\bullet+1}, X)^{\mathbb{G}} \longrightarrow F_c(\mathbb{G}_+^{\bullet+1}, X),$$

given by the inclusion of fixed points, which by construction has the required properties. ∎

One way to summarize the results of Devinatz and Hopkins [DH04] is as follows. The phrase "essentially unique" means the space of choices is contractible.

Theorem 2.11. *The $\mathbb{G}$-spectrum $\mathbf{E}$ has an essentially unique structure as a continuous $\mathbb{G}$-spectrum with the property that if $\mathbb{K} \subseteq \mathbb{G}$ is closed, then the map of Morava modules $\mathbf{E}_*\mathbf{E}^{h\mathbb{K}} \to \mathbf{E}_*\mathbf{E}$ is naturally isomorphic to the inclusion*

$$\mathrm{Map}^c(\mathbb{G}/\mathbb{K}, \mathbf{E}_*) \longrightarrow \mathrm{Map}^c(\mathbb{G}, \mathbf{E}_*).$$

The Morava modules $\mathbf{E}_*\mathbf{E}^{h\mathbb{K}}$ and $\mathbf{E}_*\mathbf{E}$ are discussed in more details immediately after Remark 2.12.

Remark 2.12. The statement of Theorem 2.11 at once disguises quite a bit of difficult work and obscures the logic of the Devinatz–Hopkins argument; thus, it is surely worth going into a bit of detail.

Suppose for a moment that we knew that Theorem 2.11 was true. As above, choose a nested sequence of open normal subgroups $U_{j+1} \subseteq U_j \subseteq \mathbb{G}$ with $\cap\, U_j = \{e\}$. Then we would have a sequence of spectra

$$\cdots \longrightarrow \mathbf{E}^{hU_j} \longrightarrow \mathbf{E}^{hU_{j+1}} \longrightarrow \cdots \longrightarrow \mathbf{E} \tag{2.4}$$

with the following properties

1. $\mathbf{E}^{hU_j}$ is a $\mathbb{G}/U_j$ spectrum and all the maps of (2.4) are $\mathbb{G}$-equivariant;
2. the map $\mathbf{E}_*\mathbf{E}^{hU_j} \longrightarrow \mathbf{E}_*\mathbf{E}$ of Morava modules is isomorphic to the inclusion
$$\mathrm{Map}^c(\mathbb{G}/U_j, \mathbf{E}_*) \longrightarrow \mathrm{Map}^c(\mathbb{G}, \mathbf{E}_*);$$
3. the induced map $\mathrm{hocolim}_j\, \mathbf{E}^{hU_j} \to \mathbf{E}$ is a $\mathbf{K}$-local equivalence.

Let us give some detail on Part (3). By Remark 2.1, Part (6) we have that if X is L_n-local then $L_{\mathbf{K}}X = \mathrm{holim}\, X \wedge \mathrm{M}_{J(i)}$. The spectra $\mathbf{E}^{hU_j}$ are $\mathbf{K}$-local and, hence L_n-local. Since L_n is smashing the homotopy colimit is L_n-local, so Part (3) is equivalent to the statement that

$$\mathrm{hocolim}_j\, \mathbf{E}^{hU_j} \wedge \mathrm{M}_{J(i)} \longrightarrow \mathbf{E} \wedge \mathrm{M}_{J(i)}$$

is an equivalence for all i. This follows from (2) and the fact that $\cap\, U_j = \{e\}$.

Next observe that since $\mathbb{G}/U_j$ is finite, $\mathbf{E}_*\mathbf{E}^{hU_j}$ is finitely generated as an $\mathbf{E}_*$-module, hence $\mathbf{E}^{hU_j}$ is dualizable in the $\mathbf{K}$-local category, by [HS99, Thm 8.6]. Putting all this together – and still assuming we know Theorem 2.11 – we would have the following diagram of cosimplicial spectra, with the vertical maps being $\mathbf{K}$-local equivalences

$$\begin{array}{ccc} \mathrm{hocolim}_j\, \mathbf{E}^{hU_j} & \longrightarrow & \mathrm{hocolim}_j\, F((\mathbb{G}/U_j)_+^{\bullet+1}, \mathbf{E}^{hU_j}) \\ \simeq\downarrow & & \downarrow\simeq \\ \mathbf{E} & \longrightarrow & F_c(\mathbb{G}_+^{\bullet+1}, \mathbf{E}). \end{array} \tag{2.5}$$

Devinatz and Hopkins prove Theorem 2.11 by reversing the logical order of this discussion: Recall that the Goerss–Hopkins–Miller theorem provides $\mathbf{E}$ with an essentially unique structure as an E_∞-ring spectrum, the space $\mathrm{map}_{E_\infty}(\mathbf{E}, \mathbf{E})$ has contractible components, and $\pi_0\, \mathrm{map}_{E_\infty}(\mathbf{E}, \mathbf{E}) \cong \mathbb{G}$. This gives $\mathbf{E}$ an essentially unique structure as $\mathbb{G}$-spectrum, with the action through E_∞-ring maps.

Using the Goerss–Hopkins–Miller Theorem, Devinatz and Hopkins define a sequence of spectra which they call $\mathbf{E}^{hU_j}$ and maps as in (2.4) satisfying Parts (1)–(3) above. They then define the continuous $\mathbb{G}$-structure on $\mathbf{E}$ using the diagram of (2.5). Then they must justify the notation $\mathbf{E}^{hU_j}$; that is, they must show the spectra defined this way agree, up to equivalence, with the fixed points as defined in (2.3). Finally, they must calculate $\mathbf{E}_*\mathbf{E}^{h\mathbb{K}}$. For this they use the remarkable Proposition 2.15 below.

We further unpack the statement of Theorem 2.11 and generalize it (Proposition 2.16 and Corollary 2.17). For any X,

$$\mathbf{E}_*(\mathbf{E} \wedge X) = \pi_* L_{\mathbf{K}}(\mathbf{E} \wedge \mathbf{E} \wedge X)$$

is a Morava module, using the action of $\mathbb{G}$ on the left factor $\mathbf{E}$. Now, suppose X itself has a $\mathbb{G}$-action so that the diagonal action on $\mathbf{E} \wedge X$ is continuous. If $h \in \mathbb{G}$ and $x \in \mathbf{E}_*X$, then we write $h *_d x$ for this action. The adjoint of the diagonal action of $\mathbb{G}$ on $\mathbf{E} \wedge X$ gives rise to a map

$$\eta\colon \mathbf{E}_*(\mathbf{E} \wedge X) \longrightarrow \operatorname{Map}^c(\mathbb{G}, \mathbf{E}_*X). \tag{2.6}$$

Explicitly, if $x\colon S^t \to \mathbf{E} \wedge \mathbf{E} \wedge X$ and $g \in \mathbb{G}$, then $\eta_x(g)$ is the composite

$$S^t \xrightarrow{x} \mathbf{E} \wedge \mathbf{E} \wedge X \xrightarrow{1 \wedge g \wedge g} \mathbf{E} \wedge \mathbf{E} \wedge X \xrightarrow{\mu \wedge 1} \mathbf{E} \wedge X,$$

where μ is multiplication and we have suppressed the $\mathbf{K}$-localizations.

When X is S^0 with the trivial action and, then η gives the identification $\mathbf{E}_*\mathbf{E} \cong \operatorname{Map}^c(\mathbb{G}, \mathbf{E}_*)$ which appeared in Theorem 2.11. The following result covers every other case that arises in this note.

Lemma 2.13. *Suppose $X = Y \wedge Z$ where $\mathbf{K}_*Y$ is zero in odd degrees and Z is a $\mathbf{K}$-locally dualizable spectrum. Then the map η in (2.6) is an isomorphism.*

Proof. As in the proof of [GHMR05, Prop. 2.4], it suffices to show that the natural map

$$\mathbf{E}_*(\mathbf{E} \wedge X) \longrightarrow \lim_i \mathbf{E}_*(\mathbf{E} \wedge \mathrm{M}_{J(i)} \wedge X)$$

occurring in the Milnor sequence associated to $\operatorname{holim}_i \mathbf{E} \wedge \mathbf{E} \wedge X \wedge \mathrm{M}_{J(i)}$ is an isomorphism, i.e., that $\lim^1_i \mathbf{E}_*(\mathbf{E} \wedge \mathrm{M}_{J(i)} \wedge X) = 0$. The assumption on Y implies that $\mathbf{E}_*(Y)$ is a flat $\mathbf{E}_*$-module, so there is an isomorphism

$$\mathbf{E}_*(\mathbf{E} \wedge \mathrm{M}_{J(i)} \wedge X) \cong \mathbf{E}_*\mathbf{E} \otimes_{\mathbf{E}_*} \mathbf{E}_*(Y) \otimes_{\mathbf{E}_*} \mathbf{E}_*(\mathrm{M}_{J(i)} \wedge Z).$$

Since Z is dualizable, $\mathrm{M}_{J(i)} \wedge Z$ is $\mathbf{K}$-locally compact, hence $\mathbf{E}_*(\mathrm{M}_{J(i)} \wedge Z)$ is finite. This shows that the tower $(\mathbf{E}_*(\mathrm{M}_{J(i)} \wedge Z))_i$ is Mittag-Leffler, which implies that the required $\lim^1$ vanishes. □

Remark 2.14. We now have (at least) two actions to keep straight.

(1) For the Morava module structure on $\mathbf{E}_*(\mathbf{E} \wedge X)$ the isomorphism η becomes $\mathbb{G}$-equivariant if we give the module of functions the conjugation action

$$(h\phi)(g) = h *_d \phi(h^{-1}g).$$

(2) The diagonal action on $\mathbf{E} \wedge X$ gives an action of $\mathbb{G}$ on $\mathbf{E}_*(\mathbf{E} \wedge X)$; this involves the right factor of $\mathbf{E}$. With respect to this action η becomes $\mathbb{G}$-equivariant if we give the module of functions the action

$$(h \star \phi)(g) = \phi(gh).$$

Note that the two actions commute.

At this point, we need the following remarkable result due to Devinatz and Hopkins.

Proposition 2.15. *Let $W^\bullet$ be a cosimplicial spectrum. Suppose there exists an integer N and a finite type 0 spectrum Y so that for all spectra Z the Bousfield–Kan spectral sequence*

$$\pi^s\pi_t F(Z, Y\wedge W^\bullet) \Longrightarrow \pi_{t-s}F(Z, \operatorname{holim}_\Delta(Y\wedge W^\bullet))$$

has a horizontal vanishing line of intercept $s=N$ at the E_∞-page. Then for any spectra A and F and maps $f\colon A\to A$, there is an equivalence

$$f^{-1}L_F(A\wedge \operatorname{holim}_\Delta W^\bullet)\simeq \operatorname{holim}_\Delta(f^{-1}L_F(A\wedge W^\bullet)).$$

Proof. This is all contained in [DH04, §5], even if it is not explicitly stated this way. More specifically, we combine the material before their Lemma 5.11, Lemma 5.12, and the argument using Y given in the proof of their Theorem 5.3. □

Proposition 2.16. *Let X be a $\mathbb{G}$-spectrum, which is ($\mathbf{K}$-locally) dualizable, and such that the diagonal action of $\mathbb{G}$ on $\mathbf{E}\wedge X$ is continuous. Then for a closed subgroup $\mathbb{K}$ of $\mathbb{G}$, there is an equivalence*

$$\mathbf{E}\wedge(\mathbf{E}\wedge X)^{h\mathbb{K}}\simeq(\mathbf{E}\wedge\mathbf{E}\wedge X)^{h\mathbb{K}},$$

where on the right-hand side, $\mathbb{K}$ is acting trivially on the first factor.

Proof. We will prove this by applying Proposition 2.15 (with $A=\mathbf{E}$, $F=\mathbf{K}$, and $f=\mathrm{id}$) to the cosimplicial spectrum which computes the homotopy fixed points $(\mathbf{E}\wedge X)^{h\mathbb{K}}$. Specifically, $(\mathbf{E}\wedge X)^{h\mathbb{K}}\simeq \operatorname{holim}_\Delta W^\bullet$, with

$$W^s = F_c(\mathbb{G}_+^{s+1}, \mathbf{E}\wedge X)^{\mathbb{K}} = F_c(\mathbb{G}/\mathbb{K}_+\wedge\mathbb{G}_+^s, \mathbf{E}\wedge X).$$

We need to check that the conditions of Proposition 2.15 are satisfied; then the result follows. The argument we give exactly mirrors that of [DH04, Theorem 5.3].

We choose Y to be a finite type 0 spectrum so that $\mathbf{E}_0Y$ is free as a C-module for every cyclic subgroup $C\subseteq\mathbb{G}$ of order p and so that $\mathbf{E}_1Y=0$. Moreover, $\mathbf{E}_*Y$ is free as an $\mathbf{E}_*$-module. Such a spectrum Y is constructed by Jeff Smith; see [Rav92, §6.4, 8.3, 8.4].

Since both X and Y are dualizable, Lemma 2.3 gives us that, for any spectrum Z, there is an isomorphism

$$\pi_tF(Z, Y\wedge W^s)\cong \operatorname{Map}^c(\mathbb{G}^{s+1}, \pi_tF(Z, \mathbf{E}\wedge X\wedge Y))^{\mathbb{K}}.$$

Using again that X and Y are dualizable as well as that $\mathbf{E}_*Y$ is in even degrees and free over $\mathbf{E}_*$,

$$\pi_tF(Z,\mathbf{E}\wedge X\wedge Y)\cong \mathbf{E}^{-t}(Z\wedge DX)\otimes_{\mathbf{E}_0}\mathbf{E}_0(Y).$$

Now $\mathbf{E}_0(Y)$ is free as a C-module for every cyclic subgroup $C\subseteq\mathbb{G}$ of order p, so the same is true for $\pi_tF(Z,\mathbf{E}\wedge X\wedge Y)$, and that fact implies that

$$\pi^s\pi_tF(Z,Y\wedge W^\bullet)\cong H^s(\mathbb{K}, \mathbf{E}^{-t}(Z\wedge DX)\otimes_{\mathbf{E}_0}\mathbf{E}_0(Y))$$

is zero for $s>n^2$ [Rav92, Lemma 8.3.5].[1] In particular, this gives a horizontal vanishing line at the E_2-page, and Proposition 2.15 applies to give the claim. □

[1]The quoted results only claims the vanishing for $s>N$ where N depends only on n and p. To get $N=n^2$ would require reworking the proof and using that $\mathbb{G}$ has virtual Poincaré duality of dimension n^2.

Corollary 2.17. *If X and $\mathbb{K}$ are as in Proposition 2.16, then there is an isomorphism of Morava modules*

$$\mathbf{E}_*((\mathbf{E} \wedge X)^{h\mathbb{K}}) \cong \mathrm{Map}^c(\mathbb{G}/\mathbb{K}, \mathbf{E}_*X)$$

where the Morava module structure on the right-hand side is the conjugation action described in Remark 2.14.

Proof. Proposition 2.16 implies that $\mathbf{E}_*((\mathbf{E} \wedge X)^{h\mathbb{K}}) \cong \pi_*(\mathbf{E} \wedge \mathbf{E} \wedge X)^{h\mathbb{K}}$. We will use the homotopy fixed point spectral sequence computing $\pi_*(\mathbf{E} \wedge \mathbf{E} \wedge X)^{h\mathbb{K}}$. As was discussed in Remark 2.14, there is a $\mathbb{K}$-equivariant isomorphism

$$\mathbf{E}_*(\mathbf{E} \wedge X) \cong \mathrm{Map}^c(\mathbb{G}, \mathbf{E}_*X)$$

with the $\mathbb{K}$-action on $\mathbf{E}_*(\mathbf{E} \wedge X) = \pi_*(\mathbf{E} \wedge \mathbf{E} \wedge X)$ the diagonal action on the right two factors and the $\mathbb{K}$-action on $\mathrm{Map}^c(\mathbb{G}, \mathbf{E}_*X)$ is right multiplication on the source. It follows that the E_2-term of the homotopy fixed point spectral sequence is

$$H^*(\mathbb{K}, \pi_*(\mathbf{E} \wedge \mathbf{E} \wedge X)) \cong H^*(\mathbb{K}, \mathrm{Map}^c(\mathbb{G}, \mathbf{E}_*X)).$$

Since $H^*(\mathbb{K}, \mathrm{Map}^c(\mathbb{G}, \mathbf{E}_*X)) \cong \mathrm{Map}^c(\mathbb{G}, \mathbf{E}_*X)^{\mathbb{K}}$, the homotopy fixed point spectral sequence collapses and the edge homomorphism gives an isomorphism of Morava modules

$$\mathbf{E}_*((\mathbf{E} \wedge X)^{h\mathbb{K}}) \xrightarrow{\cong} (\mathbf{E}_*(\mathbf{E} \wedge X))^{\mathbb{K}} \cong \mathrm{Map}^c(\mathbb{G}, \mathbf{E}_*X)^{\mathbb{K}} \cong \mathrm{Map}^c(\mathbb{G}/\mathbb{K}, \mathbf{E}_*X).$$

The actions were sorted out in Remark 2.14. □

3. The Tate sphere and the determinant sphere

In order to define the determinant sphere, we need a spectrum-level construction which twists actions. This is accomplished by a sphere spectrum we suggestively denote by $S(1)$, to be indicative of a Tate twist. Namely, $S(1)$ is the p-completed sphere spectrum S^0 with a continuous action of $\mathbb{Z}_p^\times$ coming from its action as automorphisms on $\pi_0 S^0$, to be constructed below.

We can also consider $S(1)$ as a spectrum with a $\mathbb{G}$-action, where $\mathbb{G}$ acts through the determinant homomorphism

$$\det\colon \mathbb{G} \longrightarrow \mathbb{Z}_p^\times,$$

defined as in [GHMR05, Section 1.3]. The determinant is a surjection and we let $\mathbb{SG}$ denote its kernel, so that there is an exact sequence

$$1 \longrightarrow \mathbb{SG} \longrightarrow \mathbb{G} \longrightarrow \mathbb{Z}_p^\times \longrightarrow 1.$$

We will then define $S\langle\det\rangle$ as the homotopy fixed points of a particular $\mathbb{G}$-spectrum in the $\mathbf{K}$-local category.

We now begin the construction of $S(1)$; we will start by constructing a discrete action of a dense subgroup of $\mathbb{Z}_p^\times$. If $p > 2$, we have a decomposition

$$(1 + p\mathbb{Z}_p) \times \mu \cong \mathbb{Z}_p^\times$$

where $\mu = \mathbb{F}_p^\times$ is the cyclic group of order $p-1$ given by the Teichmüller lifts. Let $C \subseteq 1 + p\mathbb{Z}_p$ be the infinite cyclic subgroup generated by $\tau = 1 + p \in 1 + p\mathbb{Z}_p$.

If $p = 2$, we have a slightly different decomposition

$$(1 + 4\mathbb{Z}_2) \times \mu \cong \mathbb{Z}_2^\times$$

where now $\mu = \{\pm 1\}$. Let C be generated by $\tau = 1 + 4 = 5 \in 1 + 4\mathbb{Z}_2$.

With this setup, we write $G = C \times \mu$ for all primes. Note that G is a dense subgroup of $\mathbb{Z}_p^\times$, and τ is a generator of the torsion-free subgroup $C \cong \mathbb{Z}$. If $p > 2$ the inclusion $C \to 1 + p\mathbb{Z}_p$ completes to an isomorphism $\mathbb{Z}_p \cong 1 + p\mathbb{Z}_p$. At $p = 2$ we get a similar isomorphism $\mathbb{Z}_2 \cong 1 + 4\mathbb{Z}_2$.

Proposition 3.1. *The inclusion $G \to \mathbb{Z}_p^\times = \pi_1 B\mathrm{haut}(S^0)$ can be canonically realized by a map*

$$BG \longrightarrow B\mathrm{haut}(S^0).$$

Proof. Since $B\mathrm{haut}(S^0)$ is an infinite loop space we need only realize separately the maps $C \to \mathbb{Z}_p^\times$ and $\mu \to \mathbb{Z}_p^\times$ as maps $BC \to B\mathrm{haut}(S^0)$ and $B\mu \to B\mathrm{haut}(S^0)$. The map we want will then be the composite

$$BG \simeq BC \times B\mu \longrightarrow B\mathrm{haut}(S^0) \times B\mathrm{haut}(S^0) \longrightarrow B\mathrm{haut}(S^0)$$

where the second map is the loop space multiplication.

At all primes $BC \simeq B\mathbb{Z} \simeq S^1$ and the choice of τ defines the required map $S^1 \to B\mathrm{haut}(S^0)$.

If $p = 2$, then $B\mu \simeq B\mathbb{Z}/2 \simeq BO(1)$ and the map we need is defined by the composition

$$BO(1) \longrightarrow BO \longrightarrow B\mathrm{haut}(S^0).$$

Suppose $p > 2$ and let A be some 2-skeleton of $B\mu$. The inclusion $\mu \subseteq \mathbb{Z}_p^\times$ defines a map $A \to B\mathrm{haut}(S^0)$ by extending a generator of $\mu \subset \pi_1 B\mathrm{haut}(S^0)$ to A. Since $\pi_i B\mathrm{haut}(S^0) \cong \pi_{i-1} S^0$ is p-complete for $i \geq 2$ and μ has order prime to p, the map out of A extends uniquely to a map $B\mu \to B\mathrm{haut}(S^0)$. □

Let $k \geq 1$ and let $G_k \subseteq G$ be the kernel of the composition

$$G \xrightarrow{\subseteq} \mathbb{Z}_p^\times \longrightarrow (\mathbb{Z}/p^k)^\times.$$

If $p > 2$, then $G_1 = C$ and G_k is infinite cyclic generated by $\tau^{p^{k-1}}$. If $p = 2$, then $G_2 = C$ and for $k > 1$ the group G_k is infinite cyclic generated by $\tau^{p^{k-2}}$. We have that the intersection $\cap G_k$ is trivial, and $\lim_k G/G_k \cong \mathbb{Z}_p^\times$; thus, the subgroups G_k define the usual topology on $\mathbb{Z}_p^\times$.

Proposition 3.2. *Let $\widetilde{S(1)}$ be the p-complete sphere spectrum with the discrete action of G constructed above. If p is odd let $k \geq 1$ and if $p = 2$ let $k > 1$. Then there is an equivalence*

$$S/p^k \simeq EG_+ \wedge_{G_k} \widetilde{S(1)}$$

and the residual action of $G/G_k \cong (\mathbb{Z}/p^k)^\times$ realizes the standard action of $(\mathbb{Z}/p^k)^\times$ on $\mathbb{Z}/p^k \cong \pi_0 S/p^k$.

Proof. The homotopy orbit spectrum $EG_+ \wedge_{G_k} \widetilde{S(1)}$ is a connected spectrum and we have a homotopy orbit spectral sequence for $H_*(-) = H_*(-, \mathbb{Z})$:

$$E^2_{p,q} \cong H_p(G_k, H_q\widetilde{S(1)}) \Longrightarrow H_{p+q}(EG_+ \wedge_{G_k} \widetilde{S(1)}).$$

Let $p > 2$. The group G_k is infinite cyclic generated by $\tau^{p^{k-1}}$ where $\tau = 1+p$. Since $\tau^{p^{k-1}} \equiv 1 + p^k$ modulo p^{k+1} we have $E^2_{p,q} = 0$ unless $(p,q) = (0,0)$ and there is a surjection of G-modules

$$\mathbb{Z}_p \cong H_0(\widetilde{S(1)}) \longrightarrow H_0(G_k, H_0(\widetilde{S(1)})) \cong \mathbb{Z}/p^k.$$

It follows that $EG_+ \wedge_{G_k} \widetilde{S(1)}$ must be a Moore spectrum for $\mathbb{Z}/p^k$ with the standard action of $\mathbb{Z}/p^k$ on $\pi_0 S/p^k$. The proof at the prime 2 is completely analagous. □

Recall that continuous actions were discussed in Section 2. See in particular Definition 2.5 and Remark 2.9.

Proposition 3.3. *The G-action on $\widetilde{S(1)}$ extends to a continuous action of the profinite group $\mathbb{Z}_p^\times$, in the sense that we have an augmented cosimplicial spectrum*

$$\widetilde{S(1)} \longrightarrow F_c((\mathbb{Z}_p^\times)_+^{\bullet+1}, \widetilde{S(1)}),$$

so that the augmentation refines the $\mathbb{Z}_p^\times$–action.

Proof. Write $S/p^k(1)$ for $EG_+ \wedge_{G_k} \widetilde{S(1)}$ with its $G/G_k \cong (\mathbb{Z}/p^k)^\times$-action. Then the augmented cosimplicial spectra

$$S/p^k(1) \to F((G/G_k)_+^{\bullet+1}, S/p^k(1))$$

assemble to give a map

$$\begin{aligned}\widetilde{S(1)} \simeq \operatorname{holim}_k S/p^k(1) &\longrightarrow \operatorname{holim}_k \operatorname{hocolim}_j F((G/G_j)_+^{\bullet+1}, S/p^k(1)) \\ &= F_c((\mathbb{Z}_p^\times)_+^{\bullet+1}, \widetilde{S(1)})\end{aligned}$$

as needed. □

Definition 3.4. We will write $S(1)$ for the p-complete sphere S^0 with the continuous $\mathbb{Z}_p^\times$-action of Proposition 3.3. The same construction gives $S(1)$ as a continuous p-complete $\mathbb{G}$–spectrum, where $\mathbb{G}$ acts through the determinant surjection $\det\colon \mathbb{G} \to \mathbb{Z}_p^\times$.

We refer to this equivariant sphere as the *Tate sphere*.

Now we take the Morava E-theory spectrum $\mathbf{E}$ and give $\mathbf{E} \wedge S(1)$ the diagonal $\mathbb{G}$-action. The next result indicates that this is an interesting construction.

Proposition 3.5. *There is an isomorphism of Morava modules*

$$\mathbf{E}_*\langle \det \rangle \cong \pi_*(\mathbf{E} \wedge S(1)) = \mathbf{E}_* S(1).$$

Proof. The edge map of the Tor spectral sequence

$$\mathbf{E}_*\langle \det \rangle = \mathbf{E}_* \otimes_{\pi_0 S^0} \pi_0 S(1) \longrightarrow \pi_*(\mathbf{E} \wedge S(1))$$

is an isomorphism, and respects the $\mathbb{G}$-action by the naturality of the spectral sequence. □

The following technical result is the key to our calculations.

Proposition 3.6. *The $\mathbb{G}$-spectrum $\mathbf{E} \wedge S(1)$ has the structure of a $\mathbf{K}$-local continuous $\mathbb{G}$-spectrum.*

Proof. As in (2.2) we need to construct an augmented cosimplicial $\mathbb{G}$-spectrum

$$\mathbf{E} \wedge S(1) \longrightarrow F_c(\mathbb{G}_+^{\bullet+1}, \mathbf{E} \wedge S(1))$$

so that the augmentation refines the $\mathbb{G}$-action on $\mathbf{E} \wedge S(1)$.

As above, we continue writing $S/p^k(1)$ for $EG_+ \wedge_{G_k} \widetilde{S(1)}$ with its $G/G_k \cong (\mathbb{Z}/p^k)^\times$ action. Let us also write S/p^k for the Moore spectrum when we do not need to refer to the action.

Since $\mathrm{M}_{J(i)}$ and S/p^k are finite spectra we have

$$\begin{aligned} F_c(\mathbb{G}_+^s, \mathbf{E} \wedge S(1)) &= \operatorname{holim}_i \operatorname{hocolim}_j F((\mathbb{G}/U_j)_+^s, \mathbf{E} \wedge S(1) \wedge \mathrm{M}_{J(i)}) \\ &\xrightarrow{\simeq} \operatorname{holim}_k \operatorname{holim}_i \operatorname{hocolim}_j F((\mathbb{G}/U_j)_+^s, \mathbf{E} \wedge S/p^k(1) \wedge \mathrm{M}_{J(i)}); \end{aligned}$$

indeed, both sides of the last equivalence are p-complete and the natural map between them is an equivalence after smashing with S/p. For all j so that U_j is in the kernel of

$$\mathbb{G} \xrightarrow{\det} \mathbb{Z}_p^\times \longrightarrow (\mathbb{Z}/p^k)^\times,$$

the diagonal action of $\mathbb{G}/U_j$ on $\mathbf{E}^{hU_j} \wedge S/p^k(1) \wedge \mathrm{M}_{J(i)}$ defines an augmented cosimplicial $\mathbb{G}$-spectrum

$$\mathbf{E}^{hU_j} \wedge S/p^k(1) \wedge \mathrm{M}_{J(i)} \longrightarrow F((\mathbb{G}/U_j)_+^{\bullet+1}, \mathbf{E}^{hU_j} \wedge S/p^k(1) \wedge \mathrm{M}_{J(i)}).$$

Since $\operatorname{hocolim}_j \mathbf{E}^{hU_j} \simeq \mathbf{E}$, these assemble into the cosimplicial spectrum we need. □

We can now make our central definition.

Definition 3.7. The determinant sphere is the spectrum

$$S\langle\det\rangle = (\mathbf{E} \wedge S(1))^{h\mathbb{G}} = \operatorname{holim}_\Delta F_c(\mathbb{G}_+^{\bullet+1}, \mathbf{E} \wedge S(1))^{\mathbb{G}}.$$

Remark 3.8. If $\mathbb{K} \subseteq \mathbb{G}$ is closed we define

$$(\mathbf{E} \wedge S(1))^{h\mathbb{K}} \longrightarrow \operatorname{holim}_\Delta F_c(\mathbb{G}_+^{\bullet+1}, \mathbf{E} \wedge S(1))^{\mathbb{K}}.$$

Therefore, using Proposition 3.5 and Remark 2.8, we have a homotopy fixed point spectral sequence

$$H_c^s(\mathbb{K}, \mathbf{E}_*\langle\det\rangle) \Longrightarrow \pi_{t-s}(\mathbf{E} \wedge S(1))^{h\mathbb{K}}.$$

We now must show that there is an isomorphism of Morava modules $\mathbf{E}_*S\langle\det\rangle \cong \mathbf{E}_*\langle\det\rangle$. The key results are Proposition 3.5 and Proposition 2.16 used as its Corollary 2.17.

Proposition 3.9. *There is an isomorphism of Morava modules*

$$\mathbf{E}_*S\langle\det\rangle \cong \mathbf{E}_*\langle\det\rangle.$$

Proof. Combine Corollary 2.17 and Proposition 3.5. □

We now extend this map to an equivalence of spectra. Let $\iota\colon S\langle\det\rangle = (\mathbf{E} \wedge S(1))^{h\mathbb{G}} \to \mathbf{E} \wedge S(1)$ be the inclusion of the fixed points from Lemma 2.10, and let $\mu\colon \mathbf{E} \wedge \mathbf{E} \to \mathbf{E}$ be the multiplication. Define

$$f\colon \mathbf{E} \wedge S\langle\det\rangle \longrightarrow \mathbf{E} \wedge S(1)$$

to be the composition

$$\mathbf{E}\wedge S\langle \det\rangle \xrightarrow{1\wedge\iota} \mathbf{E}\wedge\mathbf{E}\wedge S(1) \xrightarrow{\mu\wedge 1} \mathbf{E}\wedge S(1). \tag{3.1}$$

This map is $\mathbb{G}$-equivariant if we use the action on $\mathbf{E}$ on the source and the diagonal action on the target.

Theorem 3.10. *The map $f\colon \mathbf{E}\wedge S\langle\det\rangle \to \mathbf{E}\wedge S(1)$ of (3.1) is a $\mathbb{G}$-equivariant equivalence and induces the isomorphism of Morava modules*

$$\mathbf{E}_*S\langle\det\rangle \cong \mathbf{E}_*\langle\det\rangle.$$

of Proposition 3.9.

Proof. To check that f is an equivalence we need only check that it induces the indicated map on Morava modules. Applying $\pi_*(-)$ to (3.1) gives

$$\begin{array}{ccccc} \mathbf{E}_*S\langle\det\rangle & \longrightarrow & \mathbf{E}_*(\mathbf{E}\wedge S(1)) & \xrightarrow{\mu\wedge 1} & \mathbf{E}_*S(1) \\ \downarrow\cong & & \downarrow\cong & & \downarrow = \\ \mathrm{Map}^c(\mathbb{G},\mathbf{E}_*S(1))^{\mathbb{G}} & \longrightarrow & \mathrm{Map}^c(\mathbb{G},\mathbf{E}_*S(1)) & \longrightarrow & \mathbf{E}_*S(1). \end{array} \tag{3.2}$$

The first vertical isomorphism is from Corollary 2.17, whereas the second is the isomorphism of Lemma 2.13. In the bottom row, the first map is the inclusion of fixed points and the second map is evaluation at the unit $e\in\mathbb{G}$. The fixed points on the bottom left are exactly the constant functions, so the composite is an isomorphism as claimed. □

This yields the following practical invariance result.

Corollary 3.11. *If $\mathbb{K}$ is a closed subgroup of $\mathbb{G}$ which is in the kernel of the determinant, then $\mathbf{E}^{h\mathbb{K}}\wedge S\langle\det\rangle \simeq \mathbf{E}^{h\mathbb{K}}$.*

Proof. We use Theorem 3.10. When we restrict the $\mathbb{G}$-action on the Tate sphere $S(1)$ to $\mathbb{K}$, we get that $\mathbb{K}$ acts trivially, so $S(1)$ is $\mathbb{K}$-equivariantly equivalent to S^0. We have

$$\mathbf{E}^{h\mathbb{K}}\wedge S\langle\det\rangle \simeq (\mathbf{E}\wedge S\langle\det\rangle)^{h\mathbb{K}} \simeq (\mathbf{E}\wedge S(1))^{h\mathbb{K}} \simeq \mathbf{E}^{h\mathbb{K}},$$

where the first equivalence follows since $S\langle\det\rangle$ is a $\mathbf{K}$-locally dualizable spectrum with trivial $\mathbb{K}$-action. □

4. Deconstructing the determinant sphere

Let $S\mathbb{G}\subseteq\mathbb{G}$ be the kernel of the determinant. Then we can form the fixed point spectrum $\mathbf{E}^{hS\mathbb{G}}$. This will have a residual action of $\mathbb{G}/S\mathbb{G}\cong\mathbb{Z}_p^\times$. (See the paragraph before Theorem 4 in [DH04].) Furthermore

$$(\mathbf{E}\wedge S(1))^{hS\mathbb{G}} \simeq \mathbf{E}^{hS\mathbb{G}}\wedge S(1),$$

where the right hand side has a diagonal $\mathbb{Z}_p^\times$-action.

At odd primes we get a simple description of $S\langle\det\rangle$ directly from Devinatz–Hopkins fixed point theory.

Proposition 4.1. *Let $p > 2$ and let $\phi \in \mathbb{G}$ be any element so that* $\det(\phi)$ *topologically generates $\mathbb{Z}_p^\times$. Then there is a fiber sequence*

$$S\langle \det \rangle \longrightarrow \mathbf{E}^{hS\mathbb{G}} \xrightarrow{\det(\phi)\phi - 1} \mathbf{E}^{hS\mathbb{G}}.$$

Proof. Using that $S(1)$ is non-equivariantly the sphere S^0, we have a commutative diagram

$$\begin{array}{ccc} \mathbf{E}^{hS\mathbb{G}} \wedge S(1) & \xrightarrow{\phi \wedge \det(\phi) - 1} & \mathbf{E}^{hS\mathbb{G}} \wedge S(1) \\ \simeq \downarrow & & \downarrow \simeq \\ \mathbf{E}^{hS\mathbb{G}} & \xrightarrow{\det(\phi)\phi - 1} & \mathbf{E}^{hS\mathbb{G}}. \end{array}$$

Let F be fiber of the bottom map. The composition

$$S\langle \det \rangle = (\mathbf{E} \wedge S(1))^{h\mathbb{G}} \longrightarrow \mathbf{E}^{hS\mathbb{G}} \wedge S(1) \xrightarrow{\phi \wedge \det(\phi) - 1} \mathbf{E}^{hS\mathbb{G}} \wedge S(1)$$

is null-homotopic, so we get a map $f \colon S\langle \det \rangle \to F$. Using the fact that

$$\mathbf{E}_*\mathbf{E}^{hS\mathbb{G}} \cong \mathrm{Map}^c(\mathbb{G}/S\mathbb{G}, \mathbf{E}_*) \cong \mathrm{Map}^c(\mathbb{Z}_p^\times, \mathbf{E}_*)$$

we compute that f induces an isomorphism of Morava modules. □

We can refine the fiber sequence of Proposition 4.1. We still have $p > 2$ and we have a splitting

$$\mu \times (1 + p\mathbb{Z}_p) \cong \mathbb{Z}_p^\times.$$

The group $\mu \cong \mathbb{F}_p^\times$ is cyclic of order $p - 1$ and $(1 + p\mathbb{Z}_p)$ is isomorphic to $\mathbb{Z}_p$ itself.

Let $\alpha \in \mathbb{W}^\times \subseteq \mathbb{G}$ be a $(p^n - 1)$st root of unity; then $\det(\alpha) \in \mu$ is a generator. The group $\mu \subseteq \mathbb{Z}_p^\times$ acts on $\mathbf{E}^{hS\mathbb{G}}$ and, since this group is abstractly isomorphic to C_{p-1}, the spectrum $\mathbf{E}^{hS\mathbb{G}}$ splits as a wedge of the eingenspectra for this action. Let $\mathbf{E}_\chi^{hS\mathbb{G}}$ be the summand defined by the equations

$$\pi_*\mathbf{E}_\chi^{hS\mathbb{G}} = \{\ x \in \pi_*\mathbf{E}^{hS\mathbb{G}} \mid \alpha_* x = \det(\alpha)^{-1} x\ \}.$$

Note that the spectrum $\mathbf{E}_\chi^{hS\mathbb{G}}$ corresponds to $(\mathbf{E}^{hS\mathbb{G}} \wedge S(1))^{h\mu}$. Indeed, forgetting the μ-action and remembering that the underlying spectrum of $S(1)$ is the p-complete sphere, the map which sends $x \in \pi_*(\mathbf{E}^{hS\mathbb{G}})$ to $x \wedge 1 \in \pi_*(\mathbf{E}^{hS\mathbb{G}} \wedge S(1))$ is a non-equivariant isomorphism. Now note that if $\alpha_*(x) = \det(\alpha)^{-1}x$ in $\pi_*\mathbf{E}^{hS\mathbb{G}}$ then $\alpha_*(x \wedge 1) = \alpha_*(x) \wedge \det(\alpha) = x \wedge 1$ in $\pi_*(\mathbf{E}^{hS\mathbb{G}} \wedge S(1))$ so that

$$x \wedge 1 \in (\pi_*(\mathbf{E}^{hS\mathbb{G}} \wedge S(1)))^{\mu} \cong \pi_*(\mathbf{E}^{hS\mathbb{G}} \wedge S(1))^{h\mu}.$$

Proposition 4.2. *Let $p > 2$ and let $\psi \in \mathbb{G}$ be any element so that* $\det(\psi)$ *topologically generates $1 + p\mathbb{Z}_p \subseteq \mathbb{Z}_p^\times$. Then there is a fiber sequence*

$$S\langle \det \rangle \longrightarrow \mathbf{E}_\chi^{hS\mathbb{G}} \xrightarrow{\det(\psi)\psi - 1} \mathbf{E}_\chi^{hS\mathbb{G}}.$$

The proof is very similar to that of Proposition 4.1. This fiber sequence appears in [GHMR15, Rem. 2.5] although there is a typo there: the factor of $\det(\psi^{p+1})$ should be replaced by $\det(\psi)^{-(p+1)}$ in Equation (2.6).

At the prime 2 we have $\mathbb{Z}_2^\times \cong \mu \times (1 + 4\mathbb{Z}_p)$ for $\mu = \{\pm 1\}$ and the decomposition gets a little more subtle. In particular, $\mathbf{E}^{hS\mathbb{G}_2}$ does not decompose as a wedge of μ-eigenspectra, where μ acts on $\mathbf{E}^{h\mathbb{S}\mathbb{G}_2}$ through $\mathbb{Z}_2^\times \cong \mathbb{G}_2/\mathbb{S}\mathbb{G}_2$. The following construction expands on ideas of Hans-Werner Henn.

Define a spectrum $\mathbf{E}_-^{h\mathbb{S}\mathbb{G}}$ as the cofiber of the inclusion of the fixed points

$$(\mathbf{E}^{h\mathbb{S}\mathbb{G}})^{h\mu} \longrightarrow \mathbf{E}^{h\mathbb{S}\mathbb{G}} \longrightarrow \mathbf{E}_-^{h\mathbb{S}\mathbb{G}};$$

this will end up modeling $(\mathbf{E}^{h\mathbb{S}\mathbb{G}} \wedge S(1))^{h\mu}$.

Proposition 4.3. *Let $p = 2$ and let $\psi \in \mathbb{G}$ be any element so that $\det(\psi)$ topologically generates $1 + 4\mathbb{Z}_2 \subseteq \mathbb{Z}_2^\times$. Then there is an extension of the map $\psi\colon \mathbf{E}^{h\mathbb{S}\mathbb{G}} \to \mathbf{E}^{h\mathbb{S}\mathbb{G}}$ to a commutative diagram*

$$\begin{array}{ccc} \mathbf{E}^{h\mathbb{S}\mathbb{G}} & \longrightarrow & \mathbf{E}_-^{h\mathbb{S}\mathbb{G}} \\ {\scriptstyle\psi}\downarrow & & \downarrow{\scriptstyle\psi} \\ \mathbf{E}^{h\mathbb{S}\mathbb{G}} & \longrightarrow & \mathbf{E}_-^{h\mathbb{S}\mathbb{G}} \end{array}$$

so that there is a fiber sequence

$$S\langle \det \rangle \longrightarrow \mathbf{E}_-^{h\mathbb{S}\mathbb{G}} \xrightarrow{\det(\psi)\psi - 1} \mathbf{E}_-^{h\mathbb{S}\mathbb{G}}.$$

Proof. There is a cofiber sequence of μ-spectra

$$S^0 \longrightarrow \Sigma_+^\infty \mu \longrightarrow S(1).$$

Smashing it with $\mathbf{E}^{h\mathbb{S}\mathbb{G}}$ we obtain a cofiber sequence of μ–spectra

$$\mathbf{E}^{h\mathbb{S}\mathbb{G}} \longrightarrow \mathbf{E}^{h\mathbb{S}\mathbb{G}} \wedge \mu_+ \longrightarrow \mathbf{E}^{h\mathbb{S}\mathbb{G}} \wedge S(1).$$

Now taking μ homotopy fixed points gives a cofiber sequence

$$(\mathbf{E}^{h\mathbb{S}\mathbb{G}})^{h\mu} \longrightarrow \mathbf{E}^{h\mathbb{S}\mathbb{G}} \longrightarrow (\mathbf{E}^{h\mathbb{S}\mathbb{G}} \wedge S(1))^{h\mu}.$$

Thus we have $(\mathbf{E}^{h\mathbb{S}\mathbb{G}} \wedge S(1))^{h\mu} \simeq \mathbf{E}_-^{h\mathbb{S}\mathbb{G}}$, and the result follows as in the proof of Proposition 4.1. □

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