Spatial moments for high-dimensional critical contact process, oriented percolation and lattice trees

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October 9, 2018

Abstract

Recently, Holmes and Perkins identified conditions which ensure that for a class of critical lattice models the scaling limit of the range is the range of super-Brownian motion. One of their conditions is an estimate on a spatial moment of order higher than four, which they verified for the sixth moment for spread-out lattice trees in dimensions $d > 8$. Chen and Sakai have proved the required moment estimate for spread-out critical oriented percolation in dimensions $d + 1 > 4 + 1$. We prove estimates on all moments for the spread-out critical contact process in dimensions $d > 4$, which in particular fulfills the spatial moment condition of Holmes and Perkins. Our method of proof is relatively simple, and, as we show, it applies also to oriented percolation and lattice trees.

1 Introduction and results

1.1 Introduction

It is by now well established that super-Brownian motion arises as the scaling limit in a number of critical lattice models above the upper critical dimension, including the voter model, the contact process, oriented percolation, percolation, and lattice trees (see, e.g., [6, 8, 13, 24, 25]). There are various ways, of differing strengths, of stating such convergence results. A particularly strong statement is that the scaling limit of the range of the critical lattice model is the range of super-Brownian motion, with the convergence with respect to the Hausdorff metric on the set of compact subsets of $\mathbb{R}^d$. Recently, Holmes and Perkins [20] have identified conditions which imply this strong form of convergence. These conditions also imply an asymptotic formula for the probability of exiting a large ball, i.e., for the extrinsic one-arm exponent.

One of the substantial conditions of [20] is an estimate on a spatial moment of degree higher than four. Holmes and Perkins have proved the required bound on the sixth moment for spread-out lattice trees in dimensions $d > 8$, and Chen and Sakai have proved asymptotic formulas for

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all spatial moments for spread-out critical oriented percolation in dimensions $d+1 > 4+1$ (their case $\alpha = \infty$, see [5, p.510]). These results are sufficient to establish the spatial moment condition of Holmes and Perkins in these contexts.

In this paper, we prove, in a unified and relatively simple fashion, estimates on all spatial moments for critical spread-out models of the contact process in dimensions $d > 4$, oriented percolation in dimensions $d+1 > 4+1$, and lattice trees in dimensions $d > 8$. Our results for the contact process are new, and, together with the other conditions verified in [20], yield the conclusions of Holmes and Perkins. Our method of proof simplifies the bound of [20] on the sixth moment for lattice trees and extends it to all moments. It also provides a simpler approach to bounding the moments for oriented percolation than the method of [5] (who however did obtain stronger results). Our proof is based on the lace expansion, which has been applied to study the critical contact process (e.g., [14, 16, 23]), critical oriented percolation (e.g., [3–5, 17, 21, 22]), and lattice trees (e.g., [7, 12, 18, 19]). We emphasise the contact process throughout the paper, because it is the greatest novelty in our work, and because it is the most delicate of the three models to analyse.

Our spread-out models are formulated in terms of a probability measure D on $\mathbb{Z}^d$, which is defined as follows. Let $h : \mathbb{R}^d \to [0, \infty)$ be bounded, continuous almost everywhere, invariant under the symmetries of $\mathbb{Z}^d$, and such that

$$\int h(x)\ \mathrm{d}^d x = 1, \qquad \int |x|^n h(x)\ \mathrm{d}^d x < \infty \quad (n \in \mathbb{N}). \tag{1.1}$$

Given $L \geq 1$, we define D by

$$D(x) = \begin{cases} \dfrac{h(x/L)}{\sum_{y \in \mathbb{Z}^d \setminus \{0\}} h(y/L)} & (x \neq 0) \\ 0 & (x = 0). \end{cases} \tag{1.2}$$

It follows that, for any $n \in \mathbb{N}$,

$$\|D\|_\infty = O(L^{-d}), \qquad \sup_{x \in \mathbb{Z}^d} |x|^n D(x) = O(L^{n-d}), \qquad \sum_{x \in \mathbb{Z}^d} |x|^n D(x) = O(L^n). \tag{1.3}$$

1.2 Oriented percolation

We begin with oriented percolation, which also serves as a discretisation of the contact process. We will use this discretisation to analyse the contact process.

Spread-out oriented percolation is defined on the graph with vertex set $\mathbb{Z}^d \times \{0, 1, 2, \ldots\}$ and directed bonds $((x, n), (y, n+1))$, for $x, y \in \mathbb{Z}^d$ and $n \geq 0$. To the directed bonds $((x, n), (y, n+1))$, we associate independent random variables taking the value 1 with probability $pD(y - x)$ and 0 with probability $1 - pD(y - x)$. We say a bond is *occupied* when its random variable takes the value 1, and *vacant* when its random variable is 0. The parameter $p \in [0, \|D\|_\infty^{-1}]$ is the expected number of occupied bonds per vertex (it is *not* a probability). The joint probability distribution of the bond variables is $\mathbb{P}_p$, with expectation denoted $\mathbb{E}_p$.

We say that (x, n) is *connected* to (y, m), and write $(x, n) \longrightarrow (y, m)$, if there is an oriented path from (x, n) to (y, m) consisting of occupied bonds, or if $(y, m) = (x, n)$. This requires that $n \leq m$.

Let $C(x,n)$ denote the set of sites (y,m) such that $(x,n) \longrightarrow (y,m)$; its cardinality is denoted by $|C(x,n)|$. The *critical point* p_c is defined to be the supremum of the set of $p \in [0, \|D\|_\infty^{-1}]$ for which $\mathbb{E}_p|C(0,0)| < \infty$. It is known that $p_c = 1 + O(L^{-d})$ (and more) [15]. Let

$$\tau_n(x) = \mathbb{P}_{p_c}((0,0) \longrightarrow (x,n)). \tag{1.4}$$

Our main result for oriented percolation is the following theorem.

Theorem 1.1. *Let $d > 4$. There is an $L_0 = L_0(d)$ such that for $L \geq L_0$, and for any $s \geq 0$, there is a constant $c_s = c_s(L)$ such that*

$$\sum_{x \in \mathbb{Z}^d} |x|^s \tau_n(x) \leq c_s n^{s/2}. \tag{1.5}$$

The cases $s = 0, 2$ were proved previously in [22] and also in [17]. The general case was proved in [5], where a precise asymptotic formula for all moments was given.

1.3 Contact process

The contact process is a continuous-time Markov process with state space $\{0,1\}^{\mathbb{Z}^d}$, with $d \geq 1$. The state of the contact process is determined by a variable $\xi_x \in \{0,1\}$, for each $x \in \mathbb{Z}^d$. When $\xi_x = 0$, then x is "healthy," and when $\xi_x = 1$, then x is "infected." An infected particle spontaneously becomes healthy at rate 1, and, given $p > 0$, a healthy particle at x becomes infected at rate $p \sum_{y \in \mathbb{Z}^d} \xi_y D(x-y)$.

We assume that at time zero there is a single infected individual at the origin, with all others healthy. Let C_t denote the set of infected particles at time $t \geq 0$. The *susceptibility* is defined by

$$\chi(p) = \sum_{x \in \mathbb{Z}^d} \int_0^\infty \mathbb{P}_p(x \in C_t)\, \mathrm{d}t. \tag{1.6}$$

The *critical point* is defined by $p_c = \sup\{p : \chi(p) < \infty\}$. It is known that $p_c = 1 + O(L^{-d})$ (and more) [15]. The following theorem is our main result for the contact process.

Theorem 1.2. *Let $d > 4$. There is an $L_0 = L_0(d)$ such that for $L \geq L_0$, and for any $s \geq 0$, there is a constant $c_s = c_s(L)$ such that*

$$\sum_{x \in \mathbb{Z}^d} |x|^s \mathbb{P}_{p_c}(x \in C_t) \leq c_s \begin{cases} t^{s/2} & (t \geq 1) \\ 1 & (t < 1). \end{cases} \tag{1.7}$$

The case $s = 2$ of (1.7) is proved in [14], as is a uniform bound on the zeroth moment (in fact more is proved in [14]). The dichotomy in (1.7) does not arise for integer-time models such as oriented percolation and lattice trees.

It is well known that the contact process can be approximated by an oriented percolation model (see, e.g., [1, 2, 14, 16, 23]). For this, we replace the time interval $[0,\infty)$ by $\varepsilon\mathbb{Z}_+ \equiv \{0, \varepsilon, 2\varepsilon, 3\varepsilon, \dots\}$, and define an oriented percolation model on $\mathbb{Z}^d \times \varepsilon\mathbb{Z}_+$, as follows. Bonds have the form $((x,t),(y,t+\varepsilon))$ with $t \in \varepsilon\mathbb{Z}_+$ and $x, y \in \mathbb{Z}^d$. A bond is occupied with probability $1-\varepsilon$ if $x = y$, and with probability $\varepsilon p D(y-x)$ if $x \neq y$. Bonds with $x = y$ are called *temporal* and bonds with $x \neq y$

are called *spatial*. Let $\mathbb{P}_p^\varepsilon$ denote the probability measure for the oriented percolation model on $\mathbb{Z}^d \times \varepsilon\mathbb{Z}_+$. Then $\mathbb{P}_p^\varepsilon$ converges weakly as $\varepsilon \to 0^+$ to the contact process measure $\mathbb{P}_p$, and the critical value p_c^ε of the discretised model converges to the critical value p_c of the contact process [2, 23] (see also [26]). In particular,

$$\lim_{\varepsilon\to 0^+} \mathbb{P}_{p_c^\varepsilon}^\varepsilon\big((0,0) \longrightarrow (x,[t/\varepsilon]\varepsilon)\big) = \mathbb{P}_{p_c}(x \in C_t). \tag{1.8}$$

In order to deal with all models simultaneously, we adopt the notational convenience

$$\tau_n(x) = \mathbb{P}_{p_c^\varepsilon}^\varepsilon\big((0,0) \longrightarrow (x,n\varepsilon)\big). \tag{1.9}$$

1.4 Lattice trees

For lattice trees, we assume that $h(x) = 0$ if $\|x\|_\infty > \frac{1}{2}$, so that D of (1.2) is supported on $[-\frac{L}{2}, \frac{L}{2}]^d$. Let B be the set of bonds $\{x, y\}$ in $\mathbb{Z}^d$ with $0 < \|x - y\|_\infty \le L$. A lattice tree is a finite connected set of bonds in B with no cycles. Let $\mathcal{T}_N$ denote the set of N-bond lattice trees containing the origin 0, let $B(T)$ denote the set of bonds in $T \in \mathcal{T}_N$, and let

$$t_N^{(1)} = \sum_{T\in\mathcal{T}_N} \prod_{\{x,y\}\in B(T)} D(x-y). \tag{1.10}$$

For $x \in \mathbb{Z}^d$, let $\mathcal{T}_{N,n}(x)$ denote the set of lattice trees $T \in \mathcal{T}_N$ which contain x and for which the unique path in T connecting 0 and x consists of n bonds. Let

$$t_N^{(2)}(x;n) = \sum_{T\in\mathcal{T}_{N,n}(x)} \prod_{\{x,y\}\in B(T)} D(x-y). \tag{1.11}$$

A standard subadditivity argument implies that there exists $p_c > 0$ such that the 1-point function $g_p = \sum_{N=0}^\infty t_N^{(1)} p^N$ has radius of convergence p_c.

Critical exponents and the scaling limit of lattice trees in dimensions $d > 8$ are discussed in, e.g., [7, 18, 25]. Here, we rely on results from [18]. Our assumption that D has finite range is to conform with [18]; we expect that this restriction is actually unnecessary. It is known that $p_c = 1 + O(L^{-d})$ and $1 \le g_{p_c} \le 4$, if $d > 8$ and if L is large enough [11]. Let

$$\tau_n(x) = \sum_{N=0}^\infty t_N^{(2)}(x;n) p_c^N, \tag{1.12}$$

which is finite since $t_N^{(2)}(x;n) \le t_N^{(1)}$. The following theorem is our main result for lattice trees.

Theorem 1.3. *Let $d > 8$. There is an $L_0 = L_0(d)$ such that for $L \ge L_0$, and for any $s \ge 0$, there is a constant $c_s = c_s(L)$ such that*

$$\sum_{x\in\mathbb{Z}^d} |x|^s \tau_n(x) \le c_s n^{s/2}. \tag{1.13}$$

The cases $s = 0, 2, 4, 6$ are considered in detail in [20] (stronger results than (1.13) for $s = 0$ are proved in [12] and for $s = 2$ in [7]). Our method of proof is simpler than that of [20], and it applies to all $s \ge 0$.

2 Sufficient condition on generating function

For any one of the three models under consideration, let

$$t_z(x) = \sum_{n=0}^{\infty} \tau_n(x) z^n. \tag{2.1}$$

The Fourier transform of an absolutely summable function $f : \mathbb{Z}^d \to \mathbb{R}$ is defined by

$$\hat{f}(k) = \sum_{x \in \mathbb{Z}^d} f(x) e^{ik\cdot x} \qquad (k = (k_1, \ldots, k_d) \in [-\pi, \pi]^d). \tag{2.2}$$

We use the Fourier–Laplace transform

$$\hat{t}_z(k) = \sum_{n=0}^{\infty} \hat{\tau}_n(k) z^n. \tag{2.3}$$

Let $\Delta = \sum_{i=1}^{d} \frac{\partial^2}{\partial k_i^2}$. Then, for $|z| < 1$ and $r \in \mathbb{N}$,

$$\Delta^r \hat{t}_z(0) = \sum_{n=0}^{\infty} d_n^{(r)} z^n \quad \text{with} \quad d_n^{(r)} = (-1)^r \sum_{x \in \mathbb{Z}^d} |x|^{2r} \tau_n(x). \tag{2.4}$$

A key element of our proof is [7, Lemma 3.2(i)], which is a kind of Tauberian theorem (see also [9]). We restate it as follows.

Lemma 2.1. *If for $u \geq 1$ and $v \geq 0$ the power series $\sum_{n=0}^{\infty} a_n z^n$ obeys*

$$\left| \sum_{n=0}^{\infty} a_n z^n \right| \leq \frac{C}{|1-z|^u (1-|z|)^v} \qquad (|z| < 1), \tag{2.5}$$

then

$$|a_n| \leq \begin{cases} C' n^{u+v-1} & (u > 1) \\ C' n^v \log n & (u = 1). \end{cases} \tag{2.6}$$

We will prove the following proposition for the discretised contact process. The proposition also applies for oriented percolation for $d > 4$ and lattice trees for $d > 8$, with the parameter ε given simply by $\varepsilon = 1$.

Proposition 2.2. *Let $d > 4$, $p = p_c^{\varepsilon}$, and $r \in \mathbb{N}$. There is an ε-independent $L_0 > 0$ such that for any $L \geq L_0$ there is an ε-independent constant $C_{2r} = C_{2r}(L)$ such that*

$$|\Delta^r \hat{t}_z(0)| \leq \frac{C_{2r}\varepsilon}{|1-z|^2} \sum_{j=0}^{r-1} \frac{\varepsilon^j}{(1-|z|)^j} \qquad (|z| < 1). \tag{2.7}$$

By Lemma 2.1, Proposition 2.2 implies that there is an ε-independent constant c'_{2r} such that, for all $r, n \in \mathbb{N}$,

$$\sum_{x \in \mathbb{Z}^d} |x|^{2r} \tau_n(x) \leq c'_{2r} \sum_{j=1}^{r} (n\varepsilon)^j. \tag{2.8}$$

When $\varepsilon = 1$ this gives a bound $c_{2r} n^r$ and proves Theorems 1.1 and 1.3 for even integer powers. For the contact process, the bound becomes

$$\sum_{x \in \mathbb{Z}^d} |x|^{2r} \mathbb{P}^{\varepsilon}_{p_c^{\varepsilon}}((0,0) \to (x, n\varepsilon)) \leq c'_{2r} r \big((n\varepsilon) \vee (n\varepsilon)^r \big). \tag{2.9}$$

By (1.8), this implies the conclusion of Theorem 1.2 for even integer powers.

In fact, it suffices to prove Theorems 1.1 and 1.3 for s a non-negative even integer, since the general case then follows by Hölder's inequality (using the known results for the zeroth moment). In detail, if s is not an even integer then let r be the smallest integer such that $s < 2r$. Let $p = 2r/s > 1$ and define q by $\frac{1}{p} + \frac{1}{q} = 1$. Then $sp = 2r$ is an even integer and, for oriented percolation and lattice trees ($\varepsilon = 1$),

$$\begin{aligned}
\sum_{x \in \mathbb{Z}^d} |x|^s \tau_n(x) &= \sum_{x \in \mathbb{Z}^d} |x|^s \tau_n(x)^{1/p} \tau_n(x)^{1/q} \\
&\leq \Big(\sum_{x \in \mathbb{Z}^d} |x|^{sp} \tau_n(x) \Big)^{1/p} \Big(\sum_{x \in \mathbb{Z}^d} \tau_n(x) \Big)^{1/q} \\
&\leq (c_{2r} n^r)^{1/p} c_0^{1/q} = c_s n^{s/2}.
\end{aligned} \tag{2.10}$$

For the contact process, the above argument gives instead, by using (2.9) for the $(2r)^{\text{th}}$ moment, the upper bound

$$\big(c'_{2r} r (t \vee t^r) \big)^{1/p} c_0^{1/q} = c_s (t^{s/2r} \vee t^{s/2}). \tag{2.11}$$

For $t \leq 1$, the right-hand side is $O(1)$, consistent with (1.7). For $t \geq 1$, since $s/2r \leq s/2$ the right-hand side is $O(t^{s/2})$, which is again consistent with (1.7). Thus, for Theorem 1.2 it also suffices to consider even integer powers s.

3 Oriented percolation: proof of Theorems 1.1–1.2

We fix $d > 4$, L sufficiently large, and $p = p_c^{\varepsilon}$ throughout this section. We are generally not concerned with the L-dependence of constants, and usually allow them to depend on L. As discussed above, we can restrict to even integer powers s, and since the cases $s = 0, 2$ are already well established in previous papers, we can restrict attention to even integers $s \geq 4$.

3.1 Lace expansion for oriented percolation

The proof uses the lace expansion, which provides a formula for the coefficients $\pi_n(x)$ in the convolution equation

$$\tau_{n+1}(x) = (q * \tau_n)(x) + \sum_{m=2}^{n-1} (\tau_{n-m} * q * \pi_m)(x) + \pi_{n+1}(x) \qquad (n \geq 0), \tag{3.1}$$

with the convention that an empty sum is zero. Here $(f*g)(x)=\sum_{y\in\mathbb{Z}^d} f(y)g(x-y)$, $\pi_0(x)=\pi_1(x)=0$ for all x, and

$$q(x)=(1-\varepsilon)\delta_{0,x}+\varepsilon p_c^\varepsilon D(x). \tag{3.2}$$

By [14, (2.30) & (2.34)] (and [17, (1.12)] for $\varepsilon=1$), $p_c^\varepsilon=1+O(L^{-d})$. There are three different versions of the lace expansion for oriented percolation, which provide different representations for $\pi_n(x)$ [10,21,23]. See [25] for a discussion of these three representations. We require few details here.

The following proposition is proved in [14] (see also [17,22] for $\varepsilon=1$). We will extend Proposition 3.1 to arbitrary integers $r\geq 0$.

Proposition 3.1. [14, Propositions 2.1 & 2.3] *Let $d>4$ and $p=p_c^\varepsilon$. There is an $L_0>0$ and a finite C, both independent of ε, such that for $L\geq L_0$ and for $n\geq 0$,*

$$\sum_{x\in\mathbb{Z}^d}|x|^{2r}\tau_n(x)\leq C(L^2n\varepsilon)^r \qquad (r=0,1), \tag{3.3}$$

$$\sup_{x\in\mathbb{Z}^d}|x|^{2r}\tau_n(x)\leq (1-\varepsilon)^n+C\big(L^2(1+n\varepsilon)\big)^{r-d/2} \qquad (r=0,1), \tag{3.4}$$

$$\sum_{x\in\mathbb{Z}^d}|x|^{2r}|\pi_n(x)|\leq \varepsilon^2C\big(L^2(1+n\varepsilon)\big)^{r-d/2} \qquad (r=0,1,2). \tag{3.5}$$

We use the Fourier–Laplace transforms

$$\hat t_z(k)=\sum_{n=0}^\infty\hat\tau_n(k)z^n, \qquad \hat\Pi_z(k)=\sum_{n=2}^\infty\hat\pi_n(k)z^n. \tag{3.6}$$

By (3.3) with $r=0$, the power series $\hat t_z(k)$ converges in the open unit disk $|z|<1$ in the complex plane. The stronger result that the limit $A=\lim_{n\to\infty}\sum_{x\in\mathbb{Z}^d}\tau_n(x)$ exists (see [14,17]) implies that $\hat t_1(0)=\infty$. On the other hand, the following corollary to Proposition 3.1 shows that $\hat\Pi_z$ and some of its derivatives remain bounded on the *closed* disk $|z|\leq 1$.

The transformation converts the spatio-temporal convolutions in (3.1) into products. A linear equation for $\hat t_z(k)$ results, and its solution is

$$\hat t_z(k)=1+\hat\Pi_z(k)+\hat\Phi_z(k)\hat t_z(k)=\frac{1+\hat\Pi_z(k)}{1-\hat\Phi_z(k)}, \tag{3.7}$$

with

$$\hat\Phi_z(k)=z\hat q(k)\big(1+\hat\Pi_z(k)\big), \qquad \hat q(k)=1-\varepsilon+\varepsilon p_c\hat D(k). \tag{3.8}$$

Corollary 3.2. *Under the hypotheses of Proposition 3.1, uniformly in $|z|\leq 1$,*

$$|\hat\Pi_z(0)|\leq O(L^{-d}\varepsilon), \tag{3.9}$$

$$|\partial_z\hat\Pi_z(0)|\leq O(L^{-d}), \tag{3.10}$$

$$|\Delta\hat\Pi_z(0)|\leq O(L^{2-d}\varepsilon), \tag{3.11}$$

$$|\Delta\hat\Phi_z(0)|\leq O(L^2\varepsilon). \tag{3.12}$$

Proof. Let $|z| \le 1$. By (3.5) with $r = 0, 1$,

$$|\hat{\Pi}_z(0)| \le \sum_{n=2}^{\infty} \sum_{x\in\mathbb{Z}^d} |\pi_n(x)| \le O(L^{-d})\ \varepsilon^2 \sum_{n=2}^{\infty} (1+n\varepsilon)^{-d/2} \le O(L^{-d}\varepsilon), \tag{3.13}$$

$$|\partial_z\hat{\Pi}_z(0)| \le \sum_{n=2}^{\infty} \sum_{x\in\mathbb{Z}^d} n\,|\pi_n(x)| \le O(L^{-d})\ \varepsilon \sum_{n=2}^{\infty} (1+n\varepsilon)^{1-d/2} \le O(L^{-d}), \tag{3.14}$$

$$|\Delta\hat{\Pi}_z(0)| \le \sum_{n=2}^{\infty} \sum_{x\in\mathbb{Z}^d} |x|^2|\pi_n(x)| \le O(L^{2-d})\ \varepsilon^2 \sum_{n=2}^{\infty} (1+n\varepsilon)^{1-d/2} \le O(L^{2-d}\varepsilon). \tag{3.15}$$

Also, by (1.3),

$$|\Delta\hat{\Phi}_z(0)| = |z|\Big|\hat{q}(0)\,\Delta\hat{\Pi}_z(0) + \big(1+\hat{\Pi}_z(0)\big)\varepsilon p_c \Delta\hat{D}(0)\Big| = O(L^2\varepsilon). \tag{3.16}$$

This completes the proof. □

Since $\lim_{z\uparrow 1} \hat{t}_z(0) = \infty$ and $\hat{\Pi}_1(0) = O(L^{-d}\varepsilon)$, it follows from (3.7) that

$$\hat{\Phi}_1(0) = 1. \tag{3.17}$$

By (3.7) and (3.8), we can therefore rewrite $\hat{t}_z(0)$ as

$$\hat{t}_z(0) = \frac{1+\hat{\Pi}_z(0)}{\hat{\Phi}_1(0) - \hat{\Phi}_z(0)} = \frac{1+\hat{\Pi}_z(0)}{\hat{q}(0)\big((1-z)(1+\hat{\Pi}_1(0)) + z(\hat{\Pi}_1(0) - \hat{\Pi}_z(0))\big)}. \tag{3.18}$$

It then follows from (3.9)–(3.10) that

$$|\hat{t}_z(0)| \le \frac{O(1)}{|1-z|} \qquad (|z|<1). \tag{3.19}$$

To bound the second moment $\Delta\hat{t}_z(0)$, we differentiate (3.7) using the quotient rule and apply (3.11)–(3.12). This gives (with L-dependent constant)

$$\begin{aligned} |\Delta\hat{t}_z(0)| &= \left|\frac{1}{1-\hat{\Phi}_z(0)}\Big(\Delta\hat{\Pi}_z(0) + \hat{t}_z(0)\Delta\hat{\Phi}_z(0)\Big)\right| \\ &= \frac{O(1)}{|1-z|}\left(\varepsilon + \frac{\varepsilon}{|1-z|}\right) = \frac{O(\varepsilon)}{|1-z|^2} \qquad\qquad (|z|<1). \end{aligned} \tag{3.20}$$

Note that the bounds (3.19)–(3.20) are better than those naively obtained by multiplying $|z|^n$ on both sides of (3.3) and then summing over nonnegative integers n, which results in inverse powers of $(1-|z|)$ instead of $|1-z|$.

3.2 Induction

The proof of Proposition 2.2 is by induction on $r \in \mathbb{N}$. The case $r = 1$ is provided by (3.20). To advance the induction, we first note that, by Hölder's inequality,

$$|x|^{2r} = \left(\sum_{j=1}^d x_j^2\right)^r \le \left(\left(\sum_{j=1}^d 1\right)^{1-1/r}\left(\sum_{j=1}^d x_j^{2r}\right)^{1/r}\right)^r = d^{r-1}\sum_{j=1}^d x_j^{2r}. \tag{3.21}$$

Then, for a $\mathbb{Z}^d$-symmetric nonnegative function f on $\mathbb{Z}^d$, we have

$$\sum_{x\in\mathbb{Z}^d} x_1^{2r} f(x) \le \sum_{x\in\mathbb{Z}^d} |x|^{2r} f(x) \le d^{r-1}\sum_{x\in\mathbb{Z}^d}\sum_{j=1}^d x_j^{2r} f(x) = d^r \sum_{x\in\mathbb{Z}^d} x_1^{2r} f(x). \tag{3.22}$$

Therefore, to prove Proposition 2.2, it suffices to show (2.7) for a single component, i.e.,

$$|\partial_1^{2r}\hat{t}_z(0)| \le \frac{O(\varepsilon)}{|1-z|^2}\sum_{j=0}^{r-1}\frac{\varepsilon^j}{(1-|z|)^j} \qquad (|z|<1), \tag{3.23}$$

where ∂_1 is an abbreviation for $\frac{\partial}{\partial k_1}$.

By applying Leibniz's rule to the first equality in (3.7) and using the spatial symmetry, we obtain

$$\begin{aligned}\partial_1^{2r}\hat{t}_z(0) &= \partial_1^{2r}\hat{\Pi}_z(0) + \sum_{j=0}^r \binom{2r}{2j}\partial_1^{2j}\hat{\Phi}_z(0)\,\partial_1^{2r-2j}\hat{t}_z(0) \\ &= \frac{1}{1-\hat{\Phi}_z(0)}\left(\partial_1^{2r}\hat{\Pi}_z(0) + \sum_{j=1}^r \binom{2r}{2j}\partial_1^{2j}\hat{\Phi}_z(0)\,\partial_1^{2r-2j}\hat{t}_z(0)\right) \qquad (r\in\mathbb{N}).\end{aligned} \tag{3.24}$$

(This reproduces the first equality of (3.20) when $r = 1$.) The right-hand side of (3.24) only involves lower-order derivatives of $\hat{t}$ than the left-hand side. This opens up the possibility of an inductive proof, though we must deal with the fact that the right-hand side does involve $\partial_1^{2r}\hat{\Pi}_z$. The idea is similar to that of [5], where for oriented percolation an asymptotic formula for $\sum_{x\in\mathbb{Z}^d}|x_1|^s t_n(x)$ is derived for any $s > 0$. We only prove upper bounds in this paper, and can be less careful in dealing with the recursion relation (3.24).

The next lemma shows that ℓ_1 estimates on $|x|^q\tau_n(x)$ imply corresponding ℓ_∞ estimates.

Lemma 3.3. *Assume the same setting as Proposition 2.2, and let $q \in \mathbb{N}$. Suppose there is an ε-independent constant C such that $\sum_{x\in\mathbb{Z}^d}|x|^q\tau_n(x) \le C(1+n\varepsilon)^{q/2}$ holds for all $n \ge 0$. Then there is an ε-independent constant C' such that $\sup_{x\in\mathbb{Z}^d}|x|^q\tau_n(x) \le C'(1+n\varepsilon)^{(q-d)/2}$ also holds for all $n \ge 0$.*

Proof. We write $m = \lfloor n/2\rfloor$. Since $(a+b)^q \le 2^q(a^q+b^q)$ for any $a, b > 0$, and since $\tau_n(x) \le \sum_y \tau_m(y)\tau_{n-m}(x-y)$, we have

$$|x|^q\tau_n(x) \le 2^q\|\tau_{n-m}\|_\infty \sum_y |y|^q\tau_m(y) + 2^q\|\tau_m\|_\infty \sum_y |x-y|^q\tau_{n-m}(x-y). \tag{3.25}$$

By hypothesis, each of the two sums is bounded by $C(1+n\varepsilon)^{q/2}$. By (3.4) for $r = 0$, each ℓ_∞ norm is bounded above by a multiple of $(1+n\varepsilon)^{-d/2}$ (we use $(1-\varepsilon)^n \le e^{-n\varepsilon} \le c(1+n\varepsilon)^{-d/2}$ with c independent of ε). This completes the proof. □

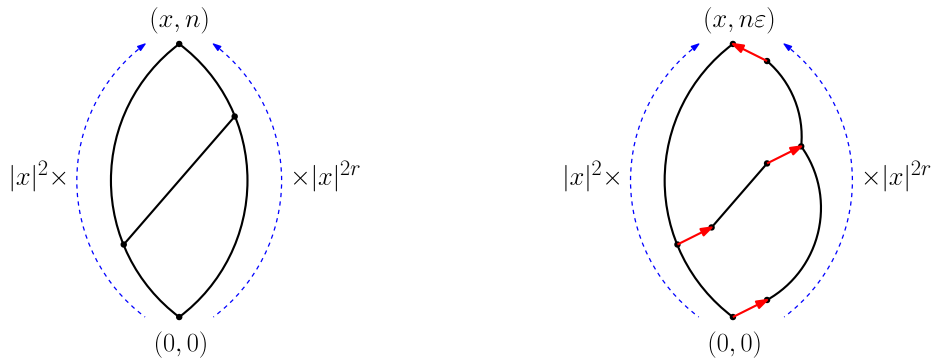

Figure 1: Allocation of $|x|^2$ and $|x|^{2r}$ to a diagram bounding $\pi_n^{(1)}(x)$ for oriented percolation (left) and the contact process (right).

The next lemma is a key step in advancing the induction. It shows that a bound on the $(2r)^{\text{th}}$ moment of τ_n implies a bound on the $(2r+2)^{\text{th}}$ moment of π_n. The proof uses standard diagrammatic estimates on π_n, which are explained in detail in [14, (4.26)] (and [17, (4.10)] for $\varepsilon = 1$). Figure 1 illustrates the diagrams for $\pi_n^{(1)}(x)$, which is one contribution to $\pi_n(x)$. The oriented percolation case $\varepsilon = 1$ (left figure) needs less care, as compared to the contract process case $\varepsilon < 1$ (right figure). For the latter, correct factors of ε must be extracted. Those factors can be obtained by noting that, at each intersection of two bond-disjoint paths, at least one must use a spatial bond (red arrows) to leave or enter that point. Summation over the temporal location of the two middle red bonds consumes their ε factors to form a Riemann sum, leaving the two ε factors from the top and bottom red bonds; these are responsible for the factor ε^2 in the bound (3.5) on $\pi_n(x)$.

Lemma 3.4. *Assume the same setting as Proposition 2.2, and let $r \in \mathbb{N}$. Suppose there is an ε-independent constant C such that $\sum_{x\in\mathbb{Z}^d} |x|^{2r}\tau_n(x) \le C(1+n\varepsilon)^r$ holds for all $n \ge 0$. Then there is an ε-independent constant C' such that $\sum_{x\in\mathbb{Z}^d} |x|^{2r+2}|\pi_n(x)| \le C'\varepsilon^2(1+n\varepsilon)^{r+1-d/2}$ also holds for all $n \ge 0$.*

Proof. We make the split $|x|^{2r+2} = |x|^2|x|^{2r}$ and multiply $|x|^2$ on the left side and $|x|^{2r}$ to the right side of the diagrams bounding π_n (see Figure 1 for an example of a diagram). Then we use the triangle inequality to decompose $|x|^2$ along the left side of the diagram and $|x|^{2r}$ along the right side of the diagram. The proof of the $r = 1$ case of (3.5) uses the ℓ_1 and ℓ_∞ estimates (3.3)–(3.4) for $r = 0, 1$ to obtain an upper bound of order $\varepsilon^2(1+n\varepsilon)^{1-d/2}$, where ε^2 arises from the occupied spatial bonds at $(0,0)$ and $(x, n\varepsilon)$. By Lemma 3.3 and the hypothesis, we also have $\sup_x |x|^{2r}\tau_n(x) \le C'(1+n\varepsilon)^{r-d/2}$. If the same bounds as in the proof of the $r = 1$ case of (3.5) are applied with one line having weight $|x|^{2r}$, then our assumption tells us that this line contributes an additional factor $(1+m\varepsilon)^r$ where $m\varepsilon$ is the temporal displacement of the line. Since $m \le n$, we obtain an upper bound of order $(1+n\varepsilon)^{r+1-d/2}$, as required. $\square$

Example 3.5. We illustrate the diagrammatic estimate underlying the proof of Lemma 3.4 with an example for $\varepsilon = 1$ and $r = 2$. Suppose that $\sum_x |x|^4\tau_n(x) \le O(n^2)$, and consider the left diagram in Figure 1 weighted with $|x|^6$ and summed over x, which is a prototype for a contribution to

$\sum_x |x|^6|\pi_n(x)|$. We write $|x|^6 = |x|^2|x|^4$ and decompose each factor along the two sides of the diagram using the triangle inequality. One contribution that results is

$$X_n = \sum_{l\le m\le n}\sum_{u,v,x} |x-u|^4|x-v|^2\tau_l(u)\tau_m(v)\tau_{m-l}(v-u)\tau_{n-l}(x-u)\tau_{n-m}(x-v). \tag{3.26}$$

By hypothesis and Lemma 3.3, $\sup_z |z|^4\tau_n(z) \le O(n^{2-d/2})$. Therefore, by (3.3),

$$\begin{aligned} X_n &\le c\sum_{l\le m\le n}\sum_u \tau_l(u)m^{-d/2}\sum_v \tau_{m-l}(v-u)(n-l)^{2-d/2}\sum_x |x-v|^2\tau_{n-m}(x-v) \\ &\le c'\sum_{l\le m\le n} m^{-d/2}(n-l)^{2-d/2}(n-m) \\ &\le c''n^3\sum_{m=1}^{n-1} m^{-d/2}\sum_{l=1}^{m}(n-l)^{-d/2}. \end{aligned} \tag{3.27}$$

The elementary verification that the sum on the right-hand side decays like $n^{-d/2}$ is carried out in detail in [17, Example 4.4]. This gives an overall bound $n^{3-d/2}$ and illustrates the origin of the bound in the conclusion of Lemma 3.4. Note that the factor n^3 in the overall bound results from the inequalities $(n-l)^2 \le n^2$ and $n-m \le n$ used above to estimate the factors arising from the spatial moments.

The next lemma promotes the bounds of Lemma 3.4 to bounds on generating functions.

Lemma 3.6. *Assume the same setting as Proposition 2.2, and let $r \in \mathbb{N}$. Suppose there is an ε-independent constant C such that $\sum_x |x|^{2r}\tau_n(x) \le C(1+n\varepsilon)^r$ holds for all $n \ge 0$. Then, uniformly in ε,*

$$|\Delta^{r+1}\hat{\Pi}_z(0)| \le \frac{O(\varepsilon^2)}{1-|z|} + \mathbb{1}_{r+1>d/2}\frac{O(\varepsilon^{r+3-d/2})}{(1-|z|)^{r+2-d/2}} \qquad (|z|<1). \tag{3.28}$$

Proof. We drop the argument "0" from $\hat{\Pi}_z(0)$. Let $a = r+1-d/2$. By hypothesis and Lemma 3.4,

$$|\Delta^{r+1}\hat{\Pi}_z| \le O(\varepsilon^2)\sum_{n=2}^{\infty}(1+n\varepsilon)^a|z|^n. \tag{3.29}$$

The case $r+1 \le d/2$ readily follows from $(1+n\varepsilon)^a \le 1$.

Suppose $r+1 > d/2$, so that $a > 0$. Since $(1+x)^a \le 2^a(1+x^a)$ and $|z| \le e^{-(1-|z|)}$, we find that

$$\begin{aligned} |\Delta^{r+1}\hat{\Pi}_z| &\le O(\varepsilon^2)\sum_{n=2}^{\infty}|z|^n + O(\varepsilon^2)\sum_{n=2}^{\infty}(n\varepsilon)^a e^{-n(1-|z|)} \\ &\le \frac{O(\varepsilon^2)}{1-|z|} + \frac{O(\varepsilon^{2+a})}{(1-|z|)^{a+1}}, \end{aligned} \tag{3.30}$$

as required. □

We note that, by (3.22), the bound (3.28) implies that

$$|\partial_1^{2r+2}\hat{\Pi}_z(0)| \leq \frac{O(\varepsilon^2)}{1-|z|} + \mathbb{1}_{r+1>d/2}\frac{O(\varepsilon^{r+3-d/2})}{(1-|z|)^{r+2-d/2}} \qquad (|z|<1). \tag{3.31}$$

Lemma 3.7. *Suppose that the bound* (3.31) *holds for all* $r+1 \leq R+1$. *Then, for* $2 \leq j \leq R+1$,

$$|\partial_1^{2j}\hat{\Phi}_z(0)| \leq O(\varepsilon) + \frac{O(\varepsilon^2)}{1-|z|} + \mathbb{1}_{j>d/2}\left(\frac{O(\varepsilon^{j+2-d/2})}{(1-|z|)^{j+1-d/2}} + \sum_{i=\lfloor d/2\rfloor+1}^{j-1}\frac{O(\varepsilon^{i+3-d/2})}{(1-|z|)^{i+1-d/2}}\right). \tag{3.32}$$

Proof. We again drop the argument "0" from the transforms. We use the fact that $\partial_1^{2j}\hat{q} = O(\epsilon)$ for $j \geq 1$ (but $\hat{q} = O(1)$). For $2 \leq j \leq d/2$,

$$\begin{aligned}
|\partial_1^{2j}\hat{\Phi}_z| &\leq |1+\hat{\Pi}_z||\partial_1^{2j}\hat{q}| + |\partial_1^{2j}\hat{\Pi}_z|\hat{q} + \sum_{i=1}^{j-1}\binom{2j}{2i}|\partial_1^{2i}\hat{\Pi}_z||\partial_1^{2j-2i}\hat{q}| \\
&= O(\varepsilon) + \frac{O(\varepsilon^2)}{1-|z|}.
\end{aligned} \tag{3.33}$$

For $j > d/2$, (3.33) needs to be modified as

$$\begin{aligned}
|\partial_1^{2j}\hat{\Phi}_z| &\leq O(\varepsilon) + \frac{O(\varepsilon^2)}{1-|z|} + \frac{O(\varepsilon^{j+2-d/2})}{(1-|z|)^{j+1-d/2}} + \sum_{i=\lfloor d/2\rfloor+1}^{j-1}\binom{2j}{2i}|\partial_1^{2i}\hat{\Pi}_z||\partial_1^{2j-2i}\hat{q}| \\
&= O(\varepsilon) + \frac{O(\varepsilon^2)}{1-|z|} + \frac{O(\varepsilon^{j+2-d/2})}{(1-|z|)^{j+1-d/2}} + \sum_{i=\lfloor d/2\rfloor+1}^{j-1}\frac{O(\varepsilon^{i+3-d/2})}{(1-|z|)^{i+1-d/2}}.
\end{aligned} \tag{3.34}$$

This completes the proof. □

Proof of Proposition 2.2. We again drop the argument "0." We use induction on $r \in \mathbb{N}$ to prove (3.23), which asserts that

$$|\partial_1^{2r}\hat{t}_z| \leq \frac{O(\varepsilon)}{|1-z|^2}\sum_{j=0}^{r-1}\frac{\varepsilon^j}{(1-|z|)^j} \qquad (r \geq 1). \tag{3.35}$$

By (3.20), the initial case $r=1$ is already confirmed.

Suppose that (3.35) holds for all positive integers $r \leq R$. By Lemma 2.1, there are ε-independent constants C_r such that, for all nonnegative integers n,

$$\sum_x |x|^{2r}\tau_n(x) \leq C_r(1+n\varepsilon)^r. \tag{3.36}$$

By Lemma 3.6, this provides the estimate (3.31) on $\partial_1^{2r+2}\hat{\Pi}_z$ for all $r \leq R$.

Suppose first that $R+1 \leq d/2$. By (3.24) with $r = R+1$,

$$\partial_1^{2R+2}\hat{t}_z = \frac{1}{1-\hat{\Phi}_z}\left(\partial_1^{2R+2}\hat{\Pi}_z + \sum_{j=1}^{R+1}\binom{2R+2}{2j}\partial_1^{2j}\hat{\Phi}_z\,\partial_1^{2R+2-2j}\hat{t}_z\right). \tag{3.37}$$

By Lemmas 3.6–3.7, followed by application of the induction hypothesis,

$$\begin{aligned}|\partial_1^{2R+2}\hat{t}_z| &\le \frac{O(1)}{|1-z|}\Big(|\partial_1^{2R+2}\hat{\Pi}_z| + |\partial_1^2\hat{\Phi}_z|\,|\partial_1^{2R}\hat{t}_z| + \sum_{j=2}^{R}|\partial_1^{2j}\hat{\Phi}_z|\,|\partial_1^{2R+2-2j}\hat{t}_z| + |\partial_1^{2R+2}\hat{\Phi}_z|\,|\hat{t}_z|\Big)\\ &\le \frac{O(1)}{|1-z|}\Bigg(\frac{\varepsilon^2}{1-|z|} + \frac{\varepsilon^2}{|1-z|^2}\sum_{j=0}^{R-1}\frac{\varepsilon^j}{(1-|z|)^j}\\ &\qquad + \left(\varepsilon + \frac{\varepsilon^2}{1-|z|}\right)\frac{\varepsilon}{|1-z|^2}\sum_{j=2}^{R}\sum_{i=0}^{R-j}\frac{\varepsilon^i}{(1-|z|)^i} + \left(\varepsilon + \frac{\varepsilon^2}{1-|z|}\right)\frac{1}{|1-z|}\Bigg). \end{aligned}\tag{3.38}$$

It is then an exercise in bookkeeping to verify that this implies that, as required,

$$|\partial_1^{2R+2}\hat{t}_z| \le \frac{O(\varepsilon)}{|1-z|^2}\sum_{j=0}^{R}\frac{\varepsilon^j}{(1-|z|)^j}. \tag{3.39}$$

Suppose finally that $R+1 > d/2$. Then (3.38) is modified due to the extra terms in (3.31)–(3.32). The contribution due to the extra term in $\partial_1^{2R+2}\hat{\Pi}_z$ can be estimated by

$$\frac{O(1)}{|1-z|}\frac{\varepsilon^{R+3-d/2}}{(1-|z|)^{R+2-d/2}} = \frac{O(\varepsilon)}{|1-z|}\frac{\varepsilon}{1-|z|}\frac{\varepsilon^{R+1-d/2}}{(1-|z|)^{R+1-d/2}}. \tag{3.40}$$

The above is of the correct form to advance the induction if $d > 4$ is even. If instead d is odd (in which case $R+1 \ge (d+1)/2$), then we use

$$\frac{\varepsilon^{R+1-d/2}}{(1-|z|)^{R+1-d/2}} \le \frac{\varepsilon^{R+1-(d+1)/2}}{(1-|z|)^{R+1-(d+1)/2}} + \frac{\varepsilon^{R+1-(d-1)/2}}{(1-|z|)^{R+1-(d-1)/2}} \tag{3.41}$$

to obtain a result of the correct form to advance the induction also in this case. We write the absolute value of the additional term in (3.32) as X_j. It contributes to the last two terms of the first line of (3.38) an amount

$$\frac{O(1)}{|1-z|}\left(\sum_{j=\lfloor d/2\rfloor+1}^{R} X_j|\partial_1^{2R+2-2j}\hat{t}_z| + X_{R+1}|\hat{t}_z|\right). \tag{3.42}$$

The X_{R+1} term is bounded by

$$\frac{O(\varepsilon)}{|1-z|^2}\left(\frac{\varepsilon^{R+2-d/2}}{(1-|z|)^{R+2-d/2}} + \sum_{i=\lfloor d/2\rfloor+1}^{R}\frac{\varepsilon^{i+2-d/2}}{(1-|z|)^{i+1-d/2}}\right). \tag{3.43}$$

If $d > 4$ is even, then this is bounded by the right-hand side of (3.39). If d is odd, then we again apply (3.41) to obtain an estimate that is appropriate to advance the induction. Finally, for the sum in (3.42), we use the induction hypothesis to obtain an upper bound

$$\frac{O(\varepsilon)}{|1-z|^3}\sum_{j=\lfloor d/2\rfloor+1}^{R}\left(\frac{\varepsilon^{j+2-d/2}}{(1-|z|)^{j+1-d/2}} + \sum_{i=\lfloor d/2\rfloor+1}^{j-1}\frac{\varepsilon^{i+3-d/2}}{(1-|z|)^{i+1-d/2}}\right)\sum_{i=0}^{R-j}\frac{\varepsilon^i}{(1-|z|)^i}. \tag{3.44}$$

This again has the correct form to advance the induction, again with the distinction between even and odd $d > 4$. This completes the proof. □

4 Lattice trees: proof of Theorem 1.3

We sketch the proof, and only point out where it differs from the proof of Theorem 1.1. We assume henceforth that $d > 8$ and L is sufficiently large.

Recall the definitions of Section 1.4, and, as in (2.1), let

$$t_z(x) = \sum_{n=0}^{\infty} \tau_n(x) z^n \qquad (|z| < 1). \tag{4.1}$$

The lace expansion gives (see, e.g., [7, (4.7)–(4.9)])

$$\hat{t}_z(k) = \frac{\hat{h}_z(k)}{1 - \hat{\Phi}_z(k)}, \tag{4.2}$$

with

$$\hat{h}_z(k) = g_{p_c} + \hat{\Pi}_z(k), \qquad \hat{\Phi}_z(k) = z p_c \hat{D}(k) \hat{h}_z(k). \tag{4.3}$$

The critical 1-point function g_{z_c} is a constant in the interval $[1, 4]$ [11], and the coefficients of the power series $\Pi_z(x)$ are given by

$$\Pi_z(x) = \sum_{n=0}^{\infty} \pi_n(x) z^n. \tag{4.4}$$

The following proposition is a small modification of [18, Proposition 5.1 & Lemma 5.4]. It plays the role of Proposition 3.1; note that the power $n^{r-d/2}$ of (3.5) is replaced by $n^{r-(d-4)/2}$ for lattice trees. This reflects the increase in the upper critical dimension from 4 for oriented percolation to 8 for lattice trees.

Proposition 4.1. *Let $d > 8$ and fix any $\delta > 0$. There is an $L_0 > 0$ and a finite C such that for $L \geq L_0$ and for $n \geq 1$,*

$$\sum_{x \in \mathbb{Z}^d} |x|^{2r} \tau_n(x) \leq C L^{2r} n^r, \qquad \sup_{x \in \mathbb{Z}^d} |x|^{2r} \tau_n(x) \leq C L^{2r-d} n^{r-d/2} \qquad (r = 0, 1), \tag{4.5}$$

$$\sum_{x \in \mathbb{Z}^d} |x|^{2r} |\pi_n(x)| \leq C L^{2r-d+\delta} n^{r-(d-4)/2} \quad (r = 0, 1, 2). \tag{4.6}$$

By (4.6), $|\hat{\Pi}_z(k)|$ and $|\Delta \hat{\Pi}_z(k)|$ are uniformly bounded in $|z| \leq 1$. As in (3.19), we obtain

$$|\hat{t}_z(0)| \leq \frac{C}{|1 - z|} \qquad (|z| < 1), \tag{4.7}$$

in fact much more is known (see [7, (2.5)]). Lemma 3.3 applies equally well to lattice trees. Lemma 3.4 is replaced by the following lemma, whose proof we discuss below.

Lemma 4.2. *Assume the same setting as Proposition 4.1, and let $r \in \mathbb{N}$. Suppose there is a finite C such that $\sum_{x \in \mathbb{Z}^d} |x|^{2r} \tau_n(x) \leq C n^r$ holds for all $n \geq 1$. Then there is a finite C' such that $\sum_{x \in \mathbb{Z}^d} |x|^{2r+2} |\pi_n(x)| \leq C' n^{r+1-(d-4)/2}$ holds for all $n \geq 1$.*

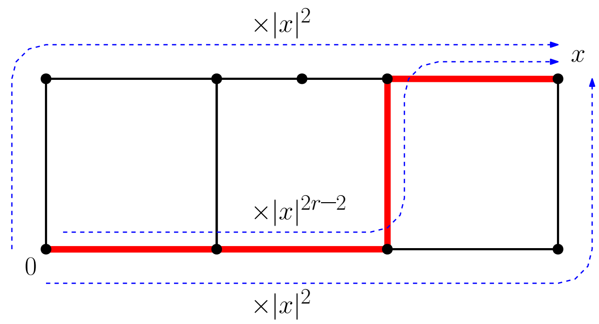

Figure 2: Allocation of $|x|^2$ and $|x|^{2r-2}$ to a diagram bounding $\pi_n^{(3)}(x)$ for lattice trees. The backbone is represented by the sequence of bold lines (in red).

The conclusion of Lemma 3.6 is then replaced by

$$\sum_{n=1}^{\infty} \sum_{x \in \mathbb{Z}^d} |x|^{2r+2} |\pi_n(x)| \, |z|^n \leq O\big((1-|z|)^{-(r-(d-8)/2)\vee 1}\big) \qquad (|z|<1). \tag{4.8}$$

The proof of Proposition 2.2 is essentially the same but is simpler because now we can set $\varepsilon = 1$, and it applies also to prove Theorem 1.3. It remains only to discuss the proof of Lemma 4.2. This is carried out in detail for $r = 1$ in the proof of [18, Proposition 5.1], and for $r = 2$ in the proof of [20, (9.32)]. The same method applies more generally to handle higher values of r. Briefly, the proof goes as follows.

Proof of Lemma 4.2. The diagrammatic estimate in the proof of Lemma 3.4 must be replaced by an estimate for the diagrams that arise for lattice trees (the diagrams are discussed in [20, Section 9.2] — see, in particular, [20, Figure 2]). We divide $|x|^{2r+2}$ as $|x|^2|x|^{2r-2}|x|^2$, distribute one $|x|^2$ factor along the top of a diagram via the triangle inequality, and distribute the other $|x|^2$ factor along the bottom of the diagram (see Figure 2 for an example of a 3-loop diagram). This leads to terms with one line on the top of the diagram weighted with the displacement squared, and one line on the bottom similarly weighted. The factor $|x|^{2r-2}$ is distributed along the backbone, which includes lines on top and bottom of the diagram, which may or may not be already weighted with the displacement squared. Thus, altogether, we have one line weighted with $|y|^2$ and a different line (which must lie on the backbone) weighted with $|y|^2|y|^{2r-2} = |y|^{2r}$, or we have two lines weighted with $|y|^2$ and a third line (which must lie on the backbone) weighted with $|y|^{2r-2}$. The case $r = 1$ is handled in [18, Proposition 5.1] by using the bounds (4.5) on the backbone lines. For $r > 1$, we first apply Hölder's inequality (as in (2.10)) to see that the hypothesis on the $(2r)^{\text{th}}$ moment of τ_n implies $\sum_{x\in\mathbb{Z}^d} |x|^{2r-2}\tau_n(x) = O(n^{r-1})$. With Lemma 3.3, these bounds on the $(2r)^{\text{th}}$ and $(2r-2)^{\text{th}}$ moments of τ_n imply corresponding ℓ_∞ bounds. Together, these imply that the estimate for the $(2r+2)^{\text{th}}$ moment of π_n will be at most n^{r-1} times larger than the fourth moment estimate of (4.6), i.e.,

$$\sum_{x \in \mathbb{Z}^d} |x|^{2r+2} |\pi_n(x)| \leq n^{r-1} O(n^{2-(d-4)/2}) = O(n^{r+1-(d-4)/2}), \tag{4.9}$$

as required. $\square$

Acknowledgements

We thank Mark Holmes and Ed Perkins for bringing this problem to our attention and for several helpful conversations. The work of AS was supported by JSPS KAKENHI Grant Number 18K03406. The work of GS was supported in part by NSERC of Canada and by a grant from the Simons Foundation. GS would like to thank the Isaac Newton Institute for Mathematical Sciences for support and hospitality during the programme "Scaling limits, rough paths, quantum field theory" when work on this paper was undertaken; this work was supported by EPSRC Grant Numbers EP/K032208/1 and EP/R014604/1.

References

[1] D.J. Barsky and C.C. Wu. Critical exponents for the contact process under the triangle condition. *J. Stat. Phys.*, **91**:95–124, (1998).

[2] C. Bezuidenhout and G. Grimmett. Exponential decay for subcritical contact and percolation processes. *Ann. Probab.*, **19**:984–1009, (1991).

[3] L.-C. Chen and A. Sakai. Critical behavior and the limit distribution for long-range oriented percolation. I. *Probab. Theory Related Fields*, **142**:151–188, (2008).

[4] L.-C. Chen and A. Sakai. Critical behavior and the limit distribution for long-range oriented percolation. II: Spatial correlation. *Probab. Theory Related Fields*, **145**:435–458, (2009).

[5] L.-C. Chen and A. Sakai. Asymptotic behavior of the gyration radius for long-range self-avoiding walk and long-range oriented percolation. *Ann. Probab.*, **39**:507–548, (2011).

[6] J.T. Cox, R. Durrett, and E.A. Perkins. Rescaled voter models converge to super-Brownian motion. *Ann. Probab.*, **28**:185–234, (2000).

[7] E. Derbez and G. Slade. The scaling limit of lattice trees in high dimensions. *Commun. Math. Phys.*, **193**:69–104, (1998).

[8] R. Durrett and E.A. Perkins. Rescaled contact processes converge to super-Brownian motion in two or more dimensions. *Probab. Theory Related Fields*, **114**:309–399, (1999).

[9] P. Flajolet and A. Odlyzko. Singularity analysis of generating functions. *SIAM J. Disc. Math.*, **3**:216–240, (1990).

[10] T. Hara and G. Slade. Mean-field critical behaviour for percolation in high dimensions. *Commun. Math. Phys.*, **128**:333–391, (1990).

[11] T. Hara and G. Slade. On the upper critical dimension of lattice trees and lattice animals. *J. Stat. Phys.*, **59**:1469–1510, (1990).

[12] T. Hara and G. Slade. The number and size of branched polymers in high dimensions. *J. Stat. Phys.*, **67**:1009–1038, (1992).

[13] M. Heydenreich and R. van der Hofstad. *Progress in High-Dimensional Percolation and Random Graphs*. Springer International Publishing Switzerland, (2017).

[14] R. van der Hofstad and A. Sakai. Gaussian scaling for the critical spread-out contact process above the upper critical dimension. *Electr. J. Probab.*, **9**:710–769, (2004).

[15] R. van der Hofstad and A. Sakai. Critical points for spread-out self-avoiding walk, percolation and the contact process. *Probab. Theory Related Fields*, **132**:438–470, (2005).

[16] R. van der Hofstad and A. Sakai. Convergence of the critical finite-range contact process to super-Brownian motion above the upper critical dimension. *Electr. J. Probab.*, **15**:801–894, (2010).

[17] R. van der Hofstad and G. Slade. Convergence of critical oriented percolation to super-Brownian motion above 4+1 dimensions. *Ann. Inst. H. Poincaré Probab. Statist.*, **39**:413–485, (2003).

[18] M. Holmes. Convergence of lattice trees to super-Brownian motion above the critical dimension. *Electr. J. Probab.*, **13**:671–755, (2008).

[19] M. Holmes. Backbone scaling for critical lattice trees in high dimensions. *J. Phys. A: Math. Theor.*, **49**:314001, (2016).

[20] M. Holmes and E. Perkins. On the range of lattice models in high dimensions - extended version. Preprint, `https://arxiv.org/abs/1806.08497`, (2018).

[21] B.G. Nguyen and W-S. Yang. Triangle condition for oriented percolation in high dimensions. *Ann. Probab.*, **21**:1809–1844, (1993).

[22] B.G. Nguyen and W-S. Yang. Gaussian limit for critical oriented percolation in high dimensions. *J. Stat. Phys.*, **78**:841–876, (1995).

[23] A. Sakai. Mean-field critical behavior for the contact process. *J. Stat. Phys.*, **104**:111–143, (2001).

[24] G. Slade. Scaling limits and super-Brownian motion. *Not. Am. Math. Soc.*, **49**(9):1056–1067, (2002).

[25] G. Slade. *The Lace Expansion and its Applications.* Springer, Berlin, (2006). Lecture Notes in Mathematics Vol. 1879. Ecole d'Eté de Probabilités de Saint–Flour XXXIV–2004.

[26] C.C. Wu. The contact process on a tree: Behavior near the first phase transition. *Stoch. Proc. Appl.*, **57**:99–112, (1995).