Elliptic stochastic quantization

Sergio Albeverio, Francesco C. De Vecchi and Massimiliano Gubinelli

Hausdorff Center of Mathematics &
Institute of Applied Mathematics
University of Bonn, Germany

albeverio@iam.uni-bonn.de
francesco.devecchi@uni-bonn.de
gubinelli@iam.uni-bonn.de

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Abstract

We prove an explicit formula for the law in zero of the solution of a class of elliptic SPDE in $\mathbb{R}^2$. This formula is the simplest instance of *dimensional reduction*, discovered in the physics literature by Parisi and Sourlas (1979), which links the law of an elliptic SPDE in $d+2$ dimension with a Gibbs measure in d dimensions. This phenomenon is similar to the relation between an $\mathbb{R}^{d+1}$ dimensional parabolic SPDE and its $\mathbb{R}^d$ dimensional invariant measure. As such, dimensional reduction of elliptic SPDEs can be considered a sort of *elliptic stochastic quantisation* procedure in the sense of Nelson (1966) and Parisi and Wu (1981). Our proof uses in a fundamental way the representation of the law of the SPDE as a supersymmetric quantum field theory. Dimensional reduction for the supersymmetric theory was already established by Klein et al. (1984). We fix a subtle gap in their proof and also complete the dimensional reduction picture by providing the link between the elliptic SPDE and the supersymmetric model. Even in our $d=0$ context the arguments are non-trivial and a non-supersymmetric, elementary proof seems only to be available in the Gaussian case.

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1 Introduction

Stochastic quantisation [16, 17] broadly refers to the idea of sampling a given probability distribution by solving a stochastic differential equation (SDE). This idea is both appealing practically and theoretically since simulating or solving an SDE is sometimes simpler than sampling or studying a given distribution. If, in finite dimensions, this boils down mostly to the idea of the Monte Carlo Markov chain method (which was actually invented before stochastic quantisation), it is in infinite dimensions that the method starts to have a real theoretical appeal.

It was Nelson [37, 38, 39] and subsequently Parisi and Wu [43] who advocated the constructive use of stochastic partial differential equations (SPDEs) to realize a given Gibbs measure for the use of Euclidean quantum field theory (QFT). Indeed the original (parabolic) stochastic quantisation procedure of [43] can be understood as the equivalence

$$\mathbb{E}[F(\varphi(t))] \propto \int F(\phi) e^{-S(\phi)} \mathcal{D}\phi. \tag{1}$$

Here F belongs to a suitable space of real-valued test functions, $\mathcal{D}\phi$ is an heuristic "Lebsegue measure" on $\mathcal{S}'(\mathbb{R}^d)$, while on the left hand side the random field φ depends on $(t, x) \in \mathbb{R} \times \mathbb{R}^d$ and is a stationary solution to the parabolic SPDE

$$\partial_t \varphi(t, x) + (m^2 - \Delta)\varphi(t, x) + V'(\varphi) = \xi, \tag{2}$$

where ξ is a Gaussian white noise in $\mathbb{R}^{d+1}$, $V : \mathbb{R} \to \mathbb{R}$ a generic local potential bounded from below, m^2 a positive parameter, and $\varphi(t)$ is the fixed time marginal of φ which has a law independent of t by stationarity and on the right hand side we have the formal expression for a measure on functions on $\mathbb{R}^d$ with weight factor given by

$$S(\phi) := \int_{\mathbb{R}^d} |\nabla\phi(x)|^2 + m^2|\phi(x)|^2 + V(\phi(x))\mathrm{d}x. \tag{3}$$

Eq. (1) can be made mathematically precise and rigorous with standard tools from the theory of Markov processes [15, 36, 18], SDE/SPDEs [30, 1, 48] and Dirichlet forms [4], for example when $d = 0$, or when the equation is regularized appropriately and, in certain cases, for suitable renormalized versions of the SPDE [5, 3, 9, 11, 14, 23, 24, 25, 29, 35, 2, 28] when $d = 1, 2, 3$. Let us note for example that in the full space it is easier to make sense of eq. (2) than of the formal Gibbs measure on the right hand side of (1), see [23].

In a slightly different context, and inspired by previous perturbative computations of Imry, Ma [27] and Young [51], Parisi and Sourlas [41, 42] considered the solutions of the elliptic SPDEs

$$(m^2 - \Delta)\phi + V'(\phi) = \xi \tag{4}$$

in $\mathbb{R}^{d+2}$ where ξ is a Gaussian white noise on $\mathbb{R}^{d+2}$ and they discovered that its stationary solutions are similarly related to the same d dimensional Gibbs measure. If we take $x \in \mathbb{R}^d$ then, they claimed that, for "nice" test functions F (e.g. correlation functions) we have

$$\mathbb{E}[F(\phi(0_{\mathbb{R}^2}, \cdot))] \propto \int F(\phi) e^{-4\pi S(\phi)} D\phi. \tag{5}$$

More precisely the law of the random field $(\phi(0_{\mathbb{R}^2}, y))_{y \in \mathbb{R}^d}$, obtained by looking at the trace of ϕ on the hyperplane $\{x = (x_1, \ldots, x_{d+2}) \in \mathbb{R}^{d+2} : x_1 = x_2 = 0\} \subset \mathbb{R}^{d+2}$, should be equivalent to that of the Gibbs measure formally appearing on the right hand side of (5) and corresponding to the action functional (3). Therefore one can interpret eq. (5) as an *elliptic stochastic quantisation* prescription in the same spirit of eq. (1).

When $V = 0$ one can directly check that the formula (5) is correct. Indeed in this case the unique stationary solution ϕ to the elliptic SPDE (4) is given by a Gaussian process with covariance

$$\mathbb{E}[\phi(x)\phi(x')] = \int_{\mathbb{R}^{d+2}} \frac{e^{ik\cdot(x-x')}}{(m^2 + |k|^2)^2} \frac{\mathrm{d}k}{(2\pi)^{d+2}}, \qquad x, x' \in \mathbb{R}^{d+2}.$$

Therefore for all $y, y' \in \mathbb{R}^d$ we have

$$\mathbb{E}[\phi(0, y)\phi(0, y')] = \int_{\mathbb{R}^d} e^{ik\cdot(y-y')} \int_{\mathbb{R}^2} \frac{\mathrm{d}q}{(|q|^2 + m^2 + |k|^2)^2} \frac{\mathrm{d}k}{(2\pi)^{d+2}}$$

$$= \int_{\mathbb{R}^2} \frac{\mathrm{d}q}{(|q|^2+1)^2} \int_{\mathbb{R}^d} \frac{e^{ik\cdot(y-y')}}{m^2+|k|^2} \frac{\mathrm{d}k}{(2\pi)^{d+2}} = \frac{1}{4\pi} \int_{\mathbb{R}^d} \frac{e^{ik\cdot(y-y')}}{m^2+|k|^2} \frac{\mathrm{d}k}{(2\pi)^d}$$

where we performed a rescaling of the q integral in order to decouple the two integrations. The reader can easily check that the expression we obtained describes the covariance of the Gaussian random field formally corresponding to the right hand side of (5) for $V = 0$.

While this last argument is almost trivial, a more general justification outside the Gaussian setting is not so obvious. The equivalence (5) was derived in [41, 42] at the theoretical physics level of rigor going through a representation of the left hand side via a supersymmetric quantum field theory (QFT) involving a pair of scalar fermion fields. This is one of the instances of the *dimensional reduction* phenomenon which is conjectured in certain random systems where the randomness effectively decreases the dimensionality of the space where fluctuations take place. A crucial assumption is that the equation (4) has a unique solution, which is already a non-trivial problem for general V. Parisi and Sourlas [42] observed that non-uniqueness can lead to a breaking of the supersymmetry, in which case the relation (5) could fail. So, part of the task of clarifying the situation is to determine under which conditions *some* relations in the spirit of (5) could anyway be true.

The dimensional reduction (5) of the elliptic SPDEs (4) seems less amenable to standard probabilistic arguments than its parabolic counterpart (1). Let us remark that from the point of view of theoretical physics it is possible [17, 42] to justify also dimensional reduction in the parabolic case (2) using a supersymmetric argument much like in the elliptic setting, the difference in dimensions is only due to the different number of fermions fields needed to represent the law of the SPDE as a quantum field theory.

The only attempt we are aware of to a mathematically rigorous understanding of the relation (5) is the work of Klein, Landau and Perez [31, 32, 33] (see also the related work on the density of states of electronic systems with random potentials [34]) which however do not fully prove eq. (5) but only the equivalence between the intermediate supersymmetric theory in $d+2$ dimensions and the Gibbs measure in d dimensions. The reason for this limitation is that the problem of uniqueness of the elliptic SPDE seems to restrict unnecessarily the class of potentials for which (5) can be established and Klein et al. decided to bypass a detailed analysis of the situation by starting directly with the supersymmetric formulation. Their rigorous argument requires a cut-off, both on large momenta in d "orthogonal" dimensions and on the space variable in $d+2$ dimensions in order to obtain a well defined, finite volume problem. This regularization breaks the supersymmetry which has to be recovered by adding a suitable correction term, spoiling the final result (see below). A subtle gap in their published proof is pointed out, and closed, in Section 4.

Let us remark that, in a different context, dimensional reduction has been proven and exploited in the remarkable work of Brydges and Imbrie on branched polymers [13, 12] and more recently by Helmut [26].

In the present work we try to bridge the gap and provide a proof of *elliptic stochastic quantisation* for the SPDE (4) in the $d = 0$ case.

Fix $d = 0$ and consider the two dimensional elliptic multidimensional SPDE

$$(m^2 - \Delta)\phi(x) + f(x)\partial V(\phi(x)) = \xi(x) \qquad x \in \mathbb{R}^2 \tag{6}$$

where $\phi = (\phi^1, \ldots, \phi^n)$ takes values in $\mathbb{R}^n$, $(\xi^1, \ldots, \xi^n)$ are n independent Gaussian white noises, $V : \mathbb{R}^n \to \mathbb{R}$ a smooth potential function, $f(x) := \tilde{f}(|x|^2)$ with $\tilde{f} : \mathbb{R}_+ \to \mathbb{R}_+$ a decreasing cut-off function, such that the derivative $\tilde{f}'$ of the function $r \longmapsto \tilde{f}(r)$ is defined, tending to 0 at infinity, and $\partial V = (\partial_i V)_{i=1,\ldots,n}$ denotes the gradient of V. We will denote $f'(x) := \tilde{f}'(|x|^2)$.

Eq. (6) is the elliptic counterpart of the equilibrium Langevin reversible dynamics for finite dimensional Gibbs measures. Let us note that the elliptic dynamics is already described by an SPDE in two dimensions while in the parabolic setting one would consider a much simpler Markovian SDE [28, 2]. The question of uniqueness of solutions is however quite similar in difficulty, indeed it is non-trivial to establish uniqueness of stationary solutions to the SDE and much work in the theory of long time behavior of Markov processes is devoted precisely to this. In the elliptic context of (6) there is no (easy) Markov property helping and the question of uniqueness of weak stationary solutions seems is more open, even in the presence of the cut-off f.

What makes this $d = 0$ problem very interesting, is above all, the fact that while the statements we would like to prove are quite easy to describe (see below), to our surprise their rigorous justification is already quite involved and not quite yet complete in full generality. However we are now in a position to confirm that dimensional reduction is indeed at work for the two dimensional SPDE (6) under a subset of the following assumptions on V and on the finite volume cut-off f:

Hypothesis C. (convexity) The potential $V : \mathbb{R}^n \to \mathbb{R}$ is a positive smooth function such that

$$y \in \mathbb{R}^2 \mapsto V(y) + m^2|y|^2$$

is strictly convex and it and its first and second partial derivatives grow at most exponentially at infinity.

Hypothesis QC. (quasi convexity) The potential $V : \mathbb{R}^n \to \mathbb{R}$ is a positive smooth function, such that it and its first and second partial derivatives grow at most exponentially at infinity and such that there exists a function $H : \mathbb{R}^n \to \mathbb{R}$ with exponential growth at infinity such that we have

$$-\langle \hat{n}, \partial V(y + r\hat{n}) \rangle \leqslant H(y), \qquad \hat{n} \in \mathbb{S}^n, y \in \mathbb{R}^n \text{ and } r \in \mathbb{R}_+,$$

with $\mathbb{S}$ is the $n-1$ dimensional sphere.

Hypothesis CO. (cut-off) The function f is a real valued, has at least C^2 smoothness and in addition satisfies $f' \leqslant 0$, it decays exponentially at infinity and fulfils $\Delta(f) \leqslant b^2 f$ for $b^2 \ll m^2$ (some examples of such functions are given in [32]).

Remark 1 The following families of functions satisfy Hypothesis QC:

- Smooth convex functions (since they satisfy the stronger Hypothesis C),
- Smooth bounded functions,
- Smooth functions having the second derivative semidefinite positive outside a compact set,
- Any positive linear combinations of the previous functions.

By weak solution to equation (6) we understand a a probability measure ν on fields ϕ under which $(m^2 - \Delta)\phi + \partial V(\phi)$ is distributed like Gaussian white noise on $\mathbb{R}^2$. A strong solution ϕ to equation (6) is a measurable map $\xi \mapsto \phi = \phi(\xi)$ satisfying the equation for almost all realizations of ξ.

Define the probability measure κ on $\mathbb{R}^n$ by

$$\frac{\mathrm{d}\kappa}{\mathrm{d}y} := Z_\kappa^{-1} \exp\left[-4\pi\left(\frac{m^2}{2}|y|^2 + V(y)\right)\right], \tag{7}$$

where $Z_\kappa := \int_{\mathbb{R}^n} \exp\left[-4\pi\left(\frac{m^2}{2}|y|^2 + V(y)\right)\right] \mathrm{d}y$.

The main result of this paper is the following theorem which states that on very general conditions on V there is always a weak solution which satisfies (an approximate) elliptic stochastic quantisation relation.

Theorem 1 *Under the Hypotheses QC and CO there exists (at least) one weak solution ν to equation (6) such that for all measurable bounded functions $h : \mathbb{R}^n \to \mathbb{R}$ we have*

$$\int_{\mathcal{W}} h(\phi(0))\Upsilon_f(\phi)\nu(\mathrm{d}\phi) = Z_f \int_{\mathbb{R}^n} h(y)\mathrm{d}\kappa(y) \tag{8}$$

where $\Upsilon_f(\phi) := e^{4\int_{\mathbb{R}^2} f'(x)V(\phi(x))\mathrm{d}x}$ and $Z_f := \int_{\mathcal{W}} \Upsilon_f(\phi)\nu(\mathrm{d}\phi)$. $\mathcal{W}$ is a suitable Banach space of functions from $\mathbb{R}^2$ to $\mathbb{R}^n$ where ν is defined (see Section 2).

The drawback of this result is the lack of constructive determination of the weak solution ν for which the dimensional reduction described by eq. (8) is realized. This is of course linked with the possible non-uniqueness of the strong solution to (6). The fact that non-uniqueness is related to a possible breaking of the supersymmetry associated with (6) was already noted by Parisi and Sourlas [42]. If we are willing to assume that the potential is convex we can be more precise, as the following theorem shows.

Theorem 2 *Under Hypothesis C and CO there exists an unique strong solution $\phi = \phi(\xi)$ of eq. (6) and for all measurable bounded functions $h : \mathbb{R}^n \to \mathbb{R}$ we have*

$$\mathbb{E}[h(\phi(0))\Upsilon_f(\phi)] = Z_f \int_{\mathbb{R}^n} h(y)\mathrm{d}\kappa(y) \tag{9}$$

where $Z_f := \mathbb{E}[\Upsilon_f(\phi)]$ and where $\mathbb{E}$ denotes expectation with respect to the law of ξ.

Both theorems require the presence of a suitable cut-off $f \neq 1$ which is responsible for the weighting factor $\Upsilon_f(\phi)$ on the left hand side of the dimensional reduction statements (8) and (9). If we would be allowed to take $f = 1$ then we would have proven the $d = 0$ version of eq. (5). However, presently we are not able to do this for all QC potentials but only for those satisfying Hypothesis C (see Sect 4 for the proof). This is the first rigorous result on elliptic stochastic quantisation without any cut-off. In fact in this case the results of Klein, Landau and Perez [32], using only an integral representation of the solution to equation (6), it does not hold.

Theorem 3 *Suppose that V satisfies Hypothesis C and let ϕ be the unique strong solution in $C^0_{\exp\beta}(\mathbb{R}^2; \mathbb{R}^n)$ (see Section 5 for the definition of this space) of equation*

$$(m^2 - \Delta)\phi + \partial V(\phi) = \xi. \tag{10}$$

Then for any $x \in \mathbb{R}^2$ and any measurable and bounded function h defined on $\mathbb{R}^n$ we have

$$\mathbb{E}[h(\phi(x))] = \int_{\mathbb{R}^n} h(y)\mathrm{d}\kappa(y). \tag{11}$$

Remark 2 It is easy to generalize Theorems 1, 2 and 3 to equations of the form

$$(m^2 - \Delta)\phi^i(x) + \sum_{r=1}^{n} \gamma_r^i \gamma_r^j f(|x|^2)\partial_{\phi^j} V(\phi(x)) = \gamma_r^i \xi^r(x), \tag{12}$$

where f is as before, $\Gamma = (\gamma_j^i)_{i,j=1,\ldots,n}$ is an $n \times n$ invertible matrix and the Hypothesis C and QC generalized accordingly.

Plan. The paper is organized as follows. In Section 2 we study the strong and weak solutions of eq. (6) and also the representation of weak solutions via the theory of transformation of measures on Wiener space developed by Üstünel and Zakai in [49] whose setting and main facts needed here are summarized in Appendix A. Section 3 is devoted to the proof of our results about elliptic stochastic quantisation. The supersymmetric approach to dimensional reduction is detailed in Section 4. Finally, Section 5 discusses the proof of Theorem 3 on the cut-off removal.

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2 The elliptic SPDE

In order to study equation (6) we have to recall some definitions, notations and conventions. Fix an abstract Wiener space $(\mathcal{W}, \mathcal{H}, \mu)$ where the noise ξ is defined (for the concept of abstract Wiener space we refer e.g. to [22, 40, 49]). The Cameron-Martin space $\mathcal{H}$ is the space

$$\mathcal{H} := L^2(\mathbb{R}^2; \mathbb{R}^n),$$

with its natural scalar product and natural norm given by $\langle h, g\rangle = \sum_{i=1}^n \int_{\mathbb{R}^2} h^i(x) g^i(x)\mathrm{d}x$. Let $\mathcal{W}$ (in which $\mathcal{H}$ is densely embedded) be the space

$$\mathcal{W} = \mathcal{W}_{p,\eta} := W_\eta^{p,-1-2\epsilon}(\mathbb{R}^2; \mathbb{R}^n) \cap (1-\Delta)(C_\eta^0(\mathbb{R}^2; \mathbb{R}^n)),$$

where $p \geqslant 1$, $\eta > 0$ and $W_\eta^{p,-1-2\epsilon}(\mathbb{R}^2; \mathbb{R}^n)$ is a fractional Sobolev space with norm

$$\|g\|_{W_\eta^{p,-1-2\epsilon}} := \left(\int_{\mathbb{R}^2} (1+|x|)^{-\eta} \left|(1-\Delta)^{-\frac{1}{2}-\epsilon}(g)\right|^p \mathrm{d}x\right)^{\frac{1}{p}},$$

for some $\epsilon > 0$ small enough and $(1-\Delta)(C_\eta^0(\mathbb{R}^2; \mathbb{R}^n))$ is the space of the second order distributional derivatives of continuous functions on $\mathbb{R}^n$ growing at infinity at most as $|x|^\eta$ with norm

$$\|g\|_{(-\Delta+1)(C_\eta^0)} := \|(1+|x|)^{-\eta}((1-\Delta)^{-1}g)(x)\|_{L_x^\infty}.$$

Thus $\mathcal{W}_{p,\eta}$ is a Banach space with norm given by the sum of the norms of $W_\eta^{p,-1-2\epsilon}(\mathbb{R}^2; \mathbb{R}^n)$ and of $(1-\Delta)^{-1}(C_\eta^0(\mathbb{R}^2; \mathbb{R}^n))$. In the following we usually do not specify the indices η and p in the definition of $\mathcal{W}_{p,\eta}$ and we write only $\mathcal{W}$. We also introduce the notation $\tilde{\mathcal{W}} = (1-\Delta)^{-1}(\mathcal{W})$.

The Gaussian measure μ on $\mathcal{W}$ is the standard Gaussian measure with Fourier transform given by $e^{-\frac{1}{2}\|\cdot\|^2_{\mathcal{H}}}$. The white noise ξ is then naturally realized on $(\mathcal{H},\mathcal{W},\mu)$, in the sense that ξ is the random variable $\xi:\mathcal{W}\to\mathcal{S}'(\mathbb{R}^2;\mathbb{R}^n)$ (where $\mathcal{S}'(\mathbb{R}^2;\mathbb{R}^n)$ is the space of $\mathbb{R}^n$–valued Schwartz distributions on $\mathbb{R}^2$) defined as $\xi(w)=\mathrm{id}_{\mathcal{W}}(w)=w$. In this way the law of ξ is simply μ (or, better, it is equal to the pushforward of μ on $\mathcal{S}':=\mathcal{S}'(\mathbb{R}^2,\mathbb{R}^n)$ with respect the natural inclusion map of $\mathcal{W}$ in $\mathcal{S}'$).

Sometimes it is also useful to consider the space $\mathcal{C}^\alpha_\tau$, i.e. the space of α-Hölder continuous functions such that they and their derivatives (or Hölder norms) grow at infinity at most like $|x|^\tau$ for the real number τ (this notation is used also if τ is negative in that case the functions decrease at least like $\frac{1}{|x|^{-\tau}}$). It is important to note that $\mathcal{C}^\alpha_\eta$ can be identified with the Besov space $B^\alpha_{\infty,\infty,\eta}(\mathbb{R}^2)$ (see [8] Chapter 2 Section 2.7). It is also important to realize that $(1-\Delta)^{-1}(\mathcal{W})\subset\mathcal{C}^\alpha_\eta$ if we choose p big enough and $\alpha>0$ small enough.

We introduce now two notions of solutions for equation (6). For later convenience it is better to discuss the equation in term of the variable $\eta:=(m^2-\Delta)\phi$ for which it reads

$$\eta+f\partial V(\mathcal{I}\eta)=\eta+U(\eta)=\xi, \tag{13}$$

where

$$\mathcal{I}:=(m^2-\Delta)^{-1}$$

and where we introduced the map $U:\mathcal{W}\to\mathcal{H}$ given by

$$U(w):=f\partial V(\mathcal{I}w),\qquad w\in\mathcal{W}. \tag{14}$$

It is simple to prove that under the Hypothesis C or QC, we have indeed $U(w)\in\mathcal{H}$. Furthermore we introduce the map $T:\mathcal{W}\to\mathcal{W}$ as $T(w):=w+U(w)$. It is clear that a map $S:\mathcal{W}\to\mathcal{W}$ satisfies the equation (13), i.e. $T(S(w))=\xi(w)=w$, for (μ-)almost all $w\in\mathcal{W}$, if and only if $\mathcal{I}S(w)$ satisfies equation (6). The law ν on $\mathcal{W}$ associated to a solution of equation (13) must satisfy the relation $T_*(\nu)=\mu$. For these reasons we introduce the following definition.

Definition 1 *A measurable map $S:\mathcal{W}\to\mathcal{W}$ is a* strong solution *to equation* (13) *if $T\circ S=\mathrm{Id}_{\mathcal{W}}$ μ-almost surely. A probability measure $\nu\in\mathcal{P}(\mathcal{W})$ (where $\mathcal{P}(\mathcal{W})$ is the space of probability measures on $\mathcal{W}$) on the space $\mathcal{W}$ is a* weak solution *to equation* (13) *if $T_*(\nu)=\mu$.*

2.1 Strong solutions

In order to study the existence of strong solutions to equation (6) we introduce an equivalent version of the same equation that is simpler to study. Indeed if we write

$$\bar{\phi}=\phi-\mathcal{I}\xi,$$

and we suppose that ϕ satisfies equation (6), then the function $\bar{\phi}$ satisfies the following (random) PDE

$$(m^2-\Delta)\bar{\phi}+f\partial V(\bar{\phi}-\mathcal{I}\xi)=0. \tag{15}$$

Equation (15) can be studied pathwise for any realization of the random field $\mathcal{I}\xi$. Hereafter the symbol $\lesssim$ stands for inequality with some positive constant standing on the right hand side.

Lemma 1 *Suppose that V satisfies Hypothesis QC, and let $\bar{\phi}$ be a classical C^2 solution to the equation* (15)*, such that $\lim_{x\to\infty}\bar{\phi}(x)=0$, then for any $0<\tau<2$ and $\eta>0$ we have*

$$\|\bar{\phi}\|_\infty\ \lesssim 1+\|fe^{\alpha_1|\mathcal{I}\xi|}\|_\infty \tag{16}$$

$$\|\bar{\phi}\|_{\mathcal{C}^{2-\tau}_{-\eta}} \lesssim 1 + e^{\alpha_1\|\bar{\phi}\|_\infty}\|(|x|+1)^\eta f e^{\alpha_1|\mathcal{I}\xi|}\|_\infty, \tag{17}$$

for some positive constant α_1 and where it and the constants involved in the symbol $\lesssim$ depend only on the function H in Hypothesis QC.

Proof Putting $r_{\bar{\phi}}(x) = \sqrt{\sum_i(\bar{\phi}^i(x))^2} = |\bar{\phi}(x)|$, $x \in \mathbb{R}^2$, since the C^2 function $\bar{\phi}$ converge to zero at infinity, the function $r_{\bar{\phi}}$ must have a global maximum at some point $\bar{x} \in \mathbb{R}^2$. This means that $-\Delta(r^2_{\bar{\phi}})(\bar{x}) \geqslant 0$. On the other hand since $\bar{\phi}$ solves equation (15) we have

$$\begin{aligned} m^2 r^2_{\bar{\phi}}(\bar{x}) &\leqslant -\frac{1}{2}\Delta(r^2_{\bar{\phi}})(\bar{x}) + m^2 r^2_{\bar{\phi}}(\bar{x}) \\ &\leqslant (-\bar{\phi}\cdot\Delta\bar{\phi} - |\nabla\bar{\phi}|^2 + m^2|\bar{\phi}|^2) \\ &\leqslant -f(\bar{x}) r_{\bar{\phi}}(\bar{x})(\hat{n}_{\bar{\phi}}(\bar{x})\cdot\partial V(\mathcal{I}\xi(\bar{x}) + \hat{n}_{\bar{\phi}}(\bar{x}) r_{\bar{\phi}}(\bar{x}))) \end{aligned}$$

where $\hat{n}_{\bar{\phi}} = \frac{\bar{\phi}}{|\bar{\phi}|} \in \mathbb{R}^k$ when $r_{\bar{\phi}} \not= 0$, and 0 elsewhere. Using Hypothesis QC we obtain

$$\|r_{\bar{\phi}}\|_\infty \leqslant \frac{f(\bar{x})H(\mathcal{I}\xi(\bar{x}))}{m^2} \lesssim 1 + \|f e^{\alpha_1|\mathcal{I}\xi|}\|_\infty,$$

since H grows at most exponentially at infinity. This result implies inequality (16). The bound (17) can be obtained directly using the inclusion properties of Besov spaces. □

Remark 3 It is simple to prove that the inequalities (16) and (17) can be chosen to be uniform with respect to some rescaling of the potential of the form λV, or satisfying Hypothesis V_λ below where $\lambda \in [0,1]$.

In the following we denote by $\mathcal{F} : \mathcal{W} \to \mathcal{P}(\mathcal{C}^{2-\tau}(\mathbb{R}^2;\mathbb{R}^n))$ the set valued function which associates to a given $w \in \mathcal{W}$ the (possible empty) set of solutions to equation (15) in $\mathcal{C}^{2-\tau}(\mathbb{R}^2;\mathbb{R}^n)$, where $\tau > 0$, when $\mathcal{I}\xi$ is evaluated in w.

Theorem 4 *For any $w \in \mathcal{W}$ the set $\mathcal{F}(w)$ is non-empty and closed. Furthermore $\mathcal{F}(w) \subset C^2(\mathbb{R}^2;\mathbb{R}^n)$ and if $B \subset \mathcal{W}$ is a bounded set then $\mathcal{F}(B) = \cup_{w\in B}\mathcal{F}(w)$ is compact in $\mathcal{C}^{2-\tau}_{-\eta}(\mathbb{R}^2;\mathbb{R}^n)$ for any $\tau > 0$ and $\eta \geqslant 0$.*

Proof We introduce the map $\mathcal{C}(\mathbb{R}^2;\mathbb{R}^n) \times \mathcal{W} \ni (\bar{\phi}, w) \mapsto \mathcal{K}(\bar{\phi}, w) \in \mathcal{C}^{2-\tau}(\mathbb{R}^2;\mathbb{R}^n)$, given by

$$\mathcal{K}^i(\bar{\phi}, w) := -\mathcal{I}(f\partial V(\bar{\phi} + \mathcal{I}\xi(w))).$$

The map $\mathcal{K}$ is continuous with respect to its first argument, indeed if $\bar{\phi}, \bar{\phi}_1 \in \mathcal{C}(\mathbb{R}^2;\mathbb{R}^n)$,

$$\begin{aligned} \|\mathcal{K}^i(\bar{\phi}, w) - \mathcal{K}^i(\bar{\phi}_1, w)\|_{\mathcal{C}^{2-\tau}_{-\eta}} &\lesssim \|(|x|+1)^\eta f(\partial V(\bar{\phi}, \mathcal{I}\xi(w)) - \partial V(\bar{\phi}_1, \mathcal{I}\xi(w)))\|_\infty \\ &\lesssim \left\| \int_0^1 (|x|+1)^\eta f \partial^2 V(\bar{\phi} - t(\bar{\phi} - \bar{\phi}_1) + \mathcal{I}\xi(w)) \cdot (\bar{\phi} - \bar{\phi}_1) \mathrm{d}t \right\|_\infty \\ &\lesssim \|\bar{\phi} - \bar{\phi}_1\|_\infty \|(|x|+1)^\eta \sqrt{f}\|_\infty \left(\|\partial^2 V_B\|_\infty + e^{\alpha\|\bar{\phi} - \bar{\phi}_1\|_\infty} \|\sqrt{f} e^{\alpha|\mathcal{I}\xi|}\|_\infty \right), \end{aligned}$$

where the positive constant α depends on the exponential growth of $\partial^2 V$ at infinity. By a similar reasoning we can prove that $\mathcal{K}$ sends bounded sets of $\mathcal{C}^{2-\tau}_{-\eta}$ into bounded sets of $\mathcal{C}^{2-\tau'}_{-\eta'}$, where $\tau' < \tau$ and $\eta' > \eta$. Since the immersion $\mathcal{C}^{2-\tau'}_{-\eta'} \hookrightarrow \mathcal{C}^{2-\tau}_{-\eta}$ is compact we have that $\mathcal{K}$ is a compact map.

Since $\mathcal{I}\xi \in \mathcal{C}^{1-}_{\alpha}$, it is simple to prove, using a bootstrap argument, that if $\bar{\phi} = \mathcal{K}(\bar{\phi}, w)$ then $\bar{\phi} \in C^2(\mathbb{R})$. From this fact it follows that, using inequalities (16) and (17) of Lemma 1 and Remark 3, the solutions to the equation $\bar{\phi} = \lambda\mathcal{K}(\bar{\phi}, w)$ are uniformly bounded for $\lambda \in [0,1]$. Thanks to these properties of the map $\mathcal{K}$ we can use Schaefer's fixed-point theorem (see [21] Theorem 4 Section 9.2 Chapter 9) to prove the existence of at least one solution to the equation (15).

Finally using again Lemma 1 we have that $\mathcal{F}(B)$ is compact for any bounded set $B \subset \mathcal{W}$. □

Theorem 5 *There exists a strong solution to equation* (6) *(or equivalently of* (13)*).*

Proof For proving the existence of a strong solution to the equation (13) (in the sense of Definition 1) it is sufficient to prove that we can choose the solutions to equation (15), whose existence for any $w \in \mathcal{W}$ is guaranteed by Theorem 4, in a measurable way with respect $w \in \mathcal{W}$. More precisely we have to prove that there exists a measurable selection for the function set map $\mathcal{F}$, i.e. there exists a map $\bar{S} : \mathcal{W} \to \mathcal{C}^{2-\tau}_{-\eta}$ such that $\bar{S}(w) \in \mathcal{F}(w)$.

Fix a sequence of balls $B_1, \ldots, B_n, \ldots \subset \mathcal{W}$ of increasing radius and such that $\lim_{n\to+\infty} B_n = \mathcal{W}$, then, by Theorem 4, the map $\mathcal{F}|_{B_n \backslash B_{n-1}}$ takes values in a compact set. Since $\mathcal{K}$ (introduced in the proof of Theorem 4) is a Carathéodory map (since, as it is proven in the proof of Theorem 4, $\mathcal{K}$ is continuous in $\bar{\phi}$ and measurable in w) by Filippov's implicit function theorem (see Theorem 18.17 in [6]), there exists a (Borel) measurable function $\bar{S}_n$ defined on $B_n \backslash B_{n-1}$ such that $\bar{S}_n(w) \in \mathcal{F}(w)$. The map $\bar{S}$ defined on $B_n \backslash B_{n-1}$ by $\bar{S}|_{B_n \backslash B_{n-1}} = \bar{S}_n$ is the measurable selection that we need (since $B_n \backslash B_{n-1}$ is measurable).

A strong solution S to equation (13) is then given by $S(w) := w + (m^2 - \Delta)\bar{S}(w)$, $w \in \mathcal{W}$. □

Corollary 1 *Under the Hypothesis C there exists only one strong solution to equation* (13).

Proof Suppose that S_1, S_2 are two strong solutions to equation (13) then, putting $\phi_j(x,w) = \mathcal{I}(S_j(w)(x))$, $j = 1,2$, writing $\delta\phi(x,w) = \phi_1(x,w) - \phi_2(x,w)$ and $\delta r(x,w) = \sqrt{\sum_{i=1}^n (\delta\phi^i(x,w))^2}$, we obtain

$$(m^2 - \Delta)(\delta r^2) + 2\sum_i (|\nabla\delta\phi^i|^2) + f\delta r[\hat{n}_{\delta\phi} \cdot (\partial V(\phi_1) - \nabla V(\phi_2))] = 0.$$

By Lagrange's theorem there exists a function $g(x)$, $x \in \mathbb{R}^2$, taking values in the segment $[\phi_1(x), \phi_2(x)] \subset \mathbb{R}^n$ such that $\hat{n}_{\delta\phi} \cdot (\partial V(\phi_1) - \partial V(\phi_2)) = \delta r \partial^2 V(g(x))(\hat{n}_{\delta\phi}, \hat{n}_{\delta\phi})$. From this fact we obtain

$$(m^2 - \Delta)(\delta r^2) + (\partial^2 V(g(x))(\hat{n}_{\delta\phi}, \hat{n}_{\delta\phi}))\delta r^2 \leqslant 0.$$

Since $m^2 + \partial^2 V(g(x))(\hat{n}_{\delta\phi}, \hat{n}_{\delta\phi})$ is positive $y \mapsto V(y) + m^2|y|^2$ being convex by our Hypothesis C, and $\delta r(x)$ is positive and goes to zero as $x \to +\infty$, we have $\phi_1 = \phi_2$ and so $S_1(w) = S_2(w)$. □

2.2 Weak solutions

First of all we prove that the map U, given by (14), is a $H - C^1$ function (in the sense of [49], see Appendix A) for the abstract Wiener space $(\mathcal{W}, \mathcal{H}, \mu)$.

Proposition 1 *If V and its derivatives grow at most exponentially at infinity the map U is a $H - C^1$ function, on the abstract Wiener $(\mathcal{W}, \mathcal{H}, \mu)$ and we have*

$$\nabla U^i(w)[h] = f(x)\partial_{\phi^i\phi^j} V(\mathcal{I}w) \cdot \mathcal{I}(h^j).$$

Furthermore U is C^2 Fréchet differentiable as a map from $\mathcal{W}$ into H.

Proof The proof is essentially based on the fundamental theorem of calculus and the use of the Fourier transform. In order to give an idea of the proof we prove only the most difficult part, namely that ∇U is continuous with respect to translations by elements of $\mathcal{H}$, where continuity is understood with respect to the Hilbert-Schmidt norm for operators acting on $\mathcal{H}$.

For fixed $w \in \mathcal{W}$, $h, h' \in \mathcal{H}$ we have

$$\nabla U^i(w+h')[h] - \nabla U^i(w)[h] = f(x) \int_0^1 \partial^3_{\phi^i \phi^j \phi^r} V((m^2-\Delta)^{-1}(w+th')) \cdot \mathcal{I}(h^j) \cdot \mathcal{I}(h'^r) \mathrm{d}t, \quad (18)$$

where the sum over $j, r = 1, \ldots, n$ is implied. We recall that the Hilbert-Schmidt norm of an integral kernel is the integral of the square of the kernel. In our case the Fourier transform of the integral kernel representing the difference (18) is given by

$$\hat{K}^i_j(k,k') = \sum_{r=1}^{n} \int_{\mathbb{R}^4} \int_0^1 \frac{\hat{V}^i_{t,jr,f}(k-k_1)}{(|k_1-k_2|^2+m^2)} \cdot \frac{\hat{h}'^r(k_1-k_2)}{(|k_2-k'|^2+m^2)} \frac{\mathrm{d}k_1 \mathrm{d}k_2}{(2\pi)^4},$$

where $\hat{V}^i_{t,jk,f}(k,l)$ is the Fourier transform of $f\partial^3_{\phi^i\phi^j\phi^k} V(\mathcal{I}(w+th'))$, $t \in [0,1]$. It is simple to prove that

$$\|\nabla U(w+h')[\cdot] - \nabla U(w)[\cdot]\|_2^2 \lesssim \int_{\mathbb{R}^4} \hat{K}^i_r(k,k') \hat{K}^r_i(k',k) \mathrm{d}k \mathrm{d}k' \lesssim \|\sqrt{f} e^{\alpha|\mathcal{I}w| + \alpha|\mathcal{I}h'|}\|^2_\infty \|\sqrt{f}\|^2_{L^2} \|h'\|^2_{\mathcal{H}},$$

where α depends on the exponential growth of $\partial^3 V$. Since $\|\sqrt{f} e^{\alpha|\mathcal{I}w|+\alpha|\mathcal{I}h'|}\|_\infty$ is always finite in $\mathcal{W}$ (for η positive and small enough) we have proved the continuity of the map $h' \longmapsto \nabla U(w+h')$ with respect to the Hilbert-Schmidt norm. □

Define the measurable map $N : \mathcal{W} \to \mathbb{N} \cup \{+\infty\}$

$$N(w) := (\text{number of solutions } y \in \mathcal{W} \text{ to the equation } T(y) = w),$$

moreover let $M \subset \mathcal{W}$ be the set of zeros of $\det_2(I_{\mathcal{H}} + \nabla U(w))$.

Theorem 6 *The function N is greater or equal to 1 and it is μ-almost surely finite. Furthermore the map T is proper and N is (μ-)almost surely finite.*

Proof We define $\mathcal{T}(\hat{\phi}, w) = \hat{\phi} + U(\hat{\phi} + w)$. Obviously we have that z is a solution to the equation $T(z) = w$ if and only if $\hat{\phi} = z - w$ is a solution to the equation $\mathcal{T}(\hat{\phi}, w) = 0$. On the other hand $\hat{\phi}$ is solution to the equation $\mathcal{T}(\hat{\phi}, w) = 0$ if and only if $\bar{\phi} = \mathcal{I}(\hat{\phi})$ is a solution to equation (15). By Theorem 4, equation (15) has at least one solution for any $w \in \mathcal{W}$ and so $N(w) \geqslant 1$ for any $w \in \mathcal{W}$.

Let K be a compact set in $\mathcal{W}$ we have that $T^{-1}(K) \subset K + (m^2 - \Delta)(\mathcal{F}(K))$ (where $\mathcal{F}$ is the set valued map introduced in Theorem 4). Since K is compact, by Theorem 4, $\mathcal{F}(K)$ is compact in $\mathcal{C}^{2-}_{-\eta}$ which implies that $(m^2-\Delta)(\mathcal{F}(K))$ is compact in $\mathcal{C}^{0-}_{-\eta}$. Since the immersion $\mathcal{C}^{0-}_{-\eta} \hookrightarrow \mathcal{W}$ is compact and the sum of two compact sets is compact, we obtain that T is a proper map.

Since by Proposition 5, $\mu(T(M)) = 0$, for proving the theorem it is sufficient to prove that $N(w) < +\infty$ for $w \not\in T(M)$. If $w \not\in T(M)$ then $\mathrm{id}_H + \nabla U(w)|_{\mathcal{H}}$ is a linear invertible operator on $\mathcal{H}$ and so $\mathrm{id}_{\mathcal{W}} + \nabla U(w)$ is a linear invertible operator on $\mathcal{W}$. By the Implicit Function Theorem, we have that T is a C^1 diffeomorphism between a neighborhood B_w of w onto $T(B_w)$. This implies that the set $T^{-1}(w)$ consists of isolated points. Since the map T is proper, this means that $T^{-1}(w)$ is a compact set made only by isolated points which implies that $T^{-1}(w)$ is a finite set. □

Theorem 7 *A probability measure ν is a weak solution to equation (13) if and only if it is absolutely continuous with respect to μ and there exists a non-negative function $A \in L^{\infty}(\mu)$ such that $\sum_{y\in T^{-1}(w)} A(y) = 1$ for μ-almost all $w \in \mathcal{W}$ and $\frac{d\nu}{d\mu} = A|\Lambda_U|$ with*

$$\Lambda_U(w) := \det{}_2(I + \nabla U(w)) \exp\left(-\delta(U)(w) - \frac{1}{2}\|U(w)\|^2_{\mathcal{H}}\right),$$

where $\delta(U)$ the Skorokhod integral of the map U and where $\det_2$ denotes the regularized Fredholm determinant (see [47] Chapter 9).

Proof Recall that, by Proposition 5, $\mu(T(M)) = 0$. This implies that for any weak solution ν we have $\nu(T^{-1}(T(M)))$=0. Letting $\mathbb{W}^n := T^{-1}(N = n) \cap T^{-1}(T(M))$ we deduce that $\nu(\cup_n \mathbb{W}^n) = \sum_n \nu(\mathbb{W}^n) = 1$ and therfore if we prove that ν is absolutely continuous with respect to μ on each $\mathbb{W}^n$ we have proved that ν is absolutely continuous with respect to μ.

Using n times iteratively the Kuratowski-Ryll-Nardzewski selection theorem (Theorem 18.13 in [6]) due to the fact that $T^{-1}(x) \cap \mathbb{W}^n$ is composed by zero or n elements, we can decompose the set $\mathbb{W}^n$ into n measurable subsets $\mathbb{W}^n_1, \dots, \mathbb{W}^n_n$ where the map $T|_{\mathbb{W}^n_i}$ is invertible. This means that if $\Omega \subset \mathbb{W}^n$ we have $\nu(\Omega \cap \mathbb{W}^n_i) \leqslant \mu(T(\Omega))$. On the other hand we have that $\mu(T(\Omega)) = \int_{\Omega\cap\mathbb{W}^n_i} |\Lambda_U| d\mu$. This implies that if $\mu(\Omega) = 0$ then $\nu(\Omega \cap \mathbb{W}^n_i) \leqslant \mu(T(\Omega)) = \int_{\Omega\cap\mathbb{W}^n_i} |\Lambda_U| d\mu = 0$. As a consequence $\nu(\Omega) = \sum_i \nu(\Omega \cap \mathbb{W}^n_i) = 0$ and ν is absolutely continuous with respect to μ.

Theorem 14 below implies that for any measurable positive functions f, A we have

$$\int f \circ T(w) A(w) |\Lambda_U(w)| d\mu = \int f(w) \left(\sum_{y\in T^{-1}(w)} A(y)\right) d\mu. \tag{19}$$

Taking $f = \mathbb{I}_{T(M)}$ and $A = 1$ we deduce that $\int_{T^{-1}(T(M))} |\Lambda_U| d\mu = \mu(T(M)) = 0$. Therefore we can suppose that there exists a specific non-negative function A such that $d\nu = A|\Lambda_U| d\mu$ and since $T_*(\nu) = \mu$ we must have

$$\int f(w) d\mu = \int f \circ T(w) d\nu = \int f \circ T(w) A(w) |\Lambda_U(w)| d\mu,$$

for any bounded measurable function f. From a comparison of this formula with (19) we deduce that $\sum_{y\in T^{-1}(w)} A(y) = 1$ for (μ-)almost all $w \in \mathcal{W}$.

On the other hand, using again Theorem 14 it is simple to prove that if $d\nu = A|\Lambda_U| d\mu$ and $\left(\sum_{y\in T^{-1}(w)} A(y)\right) = 1$ then ν is a weak solution to equation (13). □

Remark 4 If S is any strong solution to equation (13) then $\nu = S_*\mu$ is a weak solution. Furthermore it is simple to prove that the weak solutions of the form $S_*\mu$, where S is some strong solution to (6), are the extremes of the convex set $\mathfrak{W} := \{\nu \text{ satisfying } T_*\nu = \mu\}$. Using a lemma (precisely Lemma 2) that we shall prove below, it follows from this that $\mathfrak{W}$ is weakly compact and thus, by Krein–Milman theorem (see Theorem 3.21 in [45]), any measure $\nu \in \mathfrak{W}$ can be written as convex combination of measures induced by strong solutions.

Corollary 2 *If V satisfies Hypothesis C there exists only one weak solution ν to equation* (13) *and we have that $\frac{\mathrm{d}\nu}{\mathrm{d}\mu} = |\Lambda_U|$ and $\nu = S_*\mu$ (where S is the only strong solution to equation* (13) *and Λ_U is as in Theorem 7).*

Proof If V satisfies Hypothesis C, by Corollary 1, T is invertible and by Theorem 7 we have that ν is unique and $\frac{\mathrm{d}\nu}{\mathrm{d}\mu} = |\Lambda_U|$. By Remark 4 we have that $S_*\mu$, where S is the unique strong solution of (13), is the unique weak solution to the same equation. □

3 Elliptic stochastic quantisation

In this section we want to prove the dimensional reduction of equation (6), namely that the law in 0 of at least a (weak) solution to equation (13), has an explicit expression in terms of the potential V.

The original idea of Parisi and Sourlas [41] for proving this relations was to transform expectations involving the solution ϕ to equation (6) (taken at the origin) into an integral of the form

$$\mathbb{E}[h(\phi(0))] = \int h(\mathcal{I}w(0)) \det(I + \nabla U(\mathcal{I}w)) e^{-\langle U(\mathcal{I}w), \mathcal{I}w\rangle - \frac{1}{2}\|U(\mathcal{I}w)\|^2_{\mathcal{H}}} \mathrm{d}\mu(w), \tag{20}$$

where U is precisely defined in equation (14). Then one can express the determinant on the right hand side of (20) as the exponential $e^{\int V(\Phi)\mathrm{d}x\mathrm{d}\theta\mathrm{d}\bar{\theta}}$ involving the superfield

$$\Phi(x, \theta, \bar{\theta}) = \varphi(x) + \psi(x)\theta + \bar{\psi}(x)\bar{\theta} + \omega(w),$$

(see Section 4.1 for a more precise description) constructed from the real Gaussian free field φ over $\mathbb{R}^2$, two additional fermionic (i.e. anticommuting) fields $\psi, \bar{\psi}$ and the complex field ω. Introducing these new anticommuting fields it can be easily proven that the integral (20) admits an invariance property with respect to supersymmetric transformations. This implies the dimensional reduction, i.e.

$$(20) = \int h(\varphi(0)) e^{-\int V(\Phi)\mathrm{d}x\mathrm{d}\theta\mathrm{d}\bar{\theta}} \mathcal{D}\Phi = \int_{\mathbb{R}^n} h(y)\mathrm{d}\kappa(y). \tag{21}$$

Unfortunately this reasoning is only heuristic since the integral on the right hand side of (20) is not well defined without a spatial cut-off, given that both the determinant and the exponential are infinite.

For polynomial potentials V, a rigorous version of this reasoning was proposed by Klein et al. [32]. More precisely Klein et al. give a rigorous proof of the relationship (21) introducing a suitable modification due to the presence of the spatial cut-off f, but they do not discuss the relationship between equation (6) and the reduction (20).

The goal of this section is to prove the following theorem.

Theorem 8 *Under the Hypotheses CO and QC there exists (at least) one weak solution ν to equation* (6) *such that for any measurable bounded function h defined on $\mathbb{R}^n$ we have*

$$\begin{aligned} \textstyle\int_{\mathcal{W}} h(\mathcal{I}w(0))\Upsilon_f(\mathcal{I}w)\mathrm{d}\nu(w) &= \textstyle\int_{\mathcal{W}} h(\mathcal{I}w(0))\Upsilon_f(\mathcal{I}w)\Lambda_U(w)\mathrm{d}\mu(w) \\ &= Z_f \textstyle\int_{\mathbb{R}^n} h(y)\mathrm{d}\kappa(y) \end{aligned} \tag{22}$$

where $Z_f = \int_{\mathcal{W}} \Upsilon_f(\mathcal{I}w)\mathrm{d}\nu(w) > 0$.

Let us first discuss the consequences of this result. The relation (22) can be expressed in the following more probabilistic way. Suppose that on a given probability space $(\Omega^\nu, \mathbb{P}^\nu)$, the map $\phi : \mathbb{R}^2 \times \Omega^\nu \to \mathbb{R}^n$ gives the weak solution ν of Theorem 8, namely that the law of the $\mathcal{W}$-random variable $(m^2 - \Delta)\phi(\cdot, \omega)$ is the measure ν. Then we have that, for any real measurable bounded function defined on $\mathbb{R}^n$,

$$\mathbb{E}_{\mathbb{P}^\nu}\left[h(\phi(0))\frac{\Upsilon_f(\phi)}{Z_f}\right] = \int_{\mathcal{W}} h(y)\mathrm{d}\kappa(y),$$

namely we have proven Theorem 1. If we assume Hypothesis C then by Corollary 1, Corollary 2 and Theorem 8 there exists a unique strong solution satisfying (22) and we have proven as a consequence Theorem 2.

The proof of Theorem 8 will be given in several step of wider degree of generality with respect to the hypothesis on the potential V. For technical reasons we need to first introduce an additional class of potentials.

Hypothesis V_λ. We have the decomposition

$$V = V_B + \lambda V_U, \qquad V_U(y) = \sum_{i=1}^{n} (y^i)^4, \qquad y = (y^1, \ldots, y^n) \in \mathbb{R}^n,$$

with $\lambda > 0$ and V_B a bounded function with all bounded derivatives on $\mathbb{R}^n$.

In Section 4 below we will exploit a supersymmetric argument for the family of potentials V satisfying the more restrictive Hypothesis V_λ to prove that in this case a cut-off version of equation (21) holds:

Theorem 9 *Under the Hypotheses CO and V_λ if h is any real measurable bounded function defined on $\mathbb{R}^n$ then we have*

$$\int_{\mathcal{W}} h(\mathcal{I}w(0))\Lambda_U(w)\Upsilon_f(\mathcal{I}w)\mathrm{d}\mu(w) = Z_f \int_{\mathbb{R}^n} h(y)\mathrm{d}\kappa(y),$$

where $Z_f = \int_{\mathcal{W}} \Lambda_U(w)\Upsilon_f(\mathcal{I}w)\mathrm{d}\mu(w) > 0$.

Proof The proof is given in Section 4 below. □

Next we prove that Theorem 9 implies the existence of a weak solution satisfying equation (22) under Hypothesis V_λ.

Theorem 10 *Under the Hypothesis V_λ we have that*

$$\int_{\mathcal{W}} g \circ T(w)\Lambda_U(w)\mathrm{d}\mu(w) = \int_{\mathcal{W}} g(w)\mathrm{d}\mu(w). \tag{23}$$

where g is any bounded measurable function defined on $\mathcal{W}$.

Proof Using the methods of Section 2 we can prove that the map T satisfies Hypotheses DEG1, DEG2, DEG3 of Appendix A. The claim then follows from Theorem 15 and Theorem 16 below, where we can choose the function g to be any bounded continuous function since $\Lambda_U \in L^1(\mu)$ under Hypothesis V_λ. □

Proposition 2 *Under the Hypotheses CO and V_λ there exists at least a weak solution ν to equation* (13) *satisfying* (22).

Proof Let $\mathcal{V} \subset L^1(|\Lambda_U|\mathrm{d}\mu)$ be the span of the two linear spaces $\mathcal{V}_1, \mathcal{V}_2 \subset L^1(|\Lambda_U|\mathrm{d}\mu)$ where $\mathcal{V}_1$ is composed by the functions of the form $g \circ T$, where g is a measurable function defined on $\mathcal{W}$ such that $g \circ T \in L^1(|\Lambda_U|\mathrm{d}\mu)$, and $\mathcal{V}_2$ is formed by the functions of the form $h(\mathcal{I}w(0))\Upsilon_f(\mathcal{I}w)$, where h is a measurable function defined on $\mathbb{R}^n$ such that $h(\mathcal{I}w(0))\Upsilon_f(\mathcal{I}w) \in L^1(|\Lambda_U|\mathrm{d}\mu)$. Note that $\mathcal{V}_1$ and $\mathcal{V}_2$, and so $\mathcal{V} = \mathrm{span}\{\mathcal{V}_1, \mathcal{V}_2\}$, are non-void since, under the Hypotheses V_λ and CO (see Lemma 15 below), $\Lambda_U \in L^p(\mu)$ and so $g \circ T, h(\mathcal{I}w(0))\Upsilon_f(\mathcal{I}w) \in L^1(\mu)$ whenever g, h are bounded. Define a positive functional $\hat{L} : \mathcal{V} \to \mathbb{R}$ by extending via linearity the relations

$$\hat{L}(h(\mathcal{I}w(0))\Upsilon_f(\mathcal{I}w)) := \int h(\mathcal{I}w(0))\Upsilon_f(\mathcal{I}w)\Lambda_U(w)\mathrm{d}\mu(w) \tag{24}$$

$$\hat{L}(g \circ T) := \int g(w)\mathrm{d}\mu(w). \tag{25}$$

to the whole $\mathcal{V}$. We have to verify that $\hat{L}$ is well defined and positive on $\mathcal{V}$. Suppose that there exist functions g and h such that $g \circ T = h(\mathcal{I}w(0))\Upsilon_f(\mathcal{I}w)$ then, by Theorem 10, we have

$$\int_{\mathcal{W}} g\mathrm{d}\mu = \int_{\mathcal{W}} g \circ T \Lambda_U \mathrm{d}\mu = \int_{\mathcal{W}} h(\mathcal{I}w(0))\Upsilon_f(\mathcal{I}w)\Lambda_U \mathrm{d}\mu. \tag{26}$$

This implies that $\hat{L}$ is well defined on $\mathcal{V}_1 \cap \mathcal{V}_2$ and so on $\mathcal{V}$. Obviously $\hat{L}$ is positive on $\mathcal{V}_2$, and, by Theorem 9 we have

$$\hat{L}(h(\mathcal{I}w(0))\Upsilon_f(\mathcal{I}w)) = \int_{\mathcal{W}} h(\mathcal{I}w(0))\Upsilon_f(\mathcal{I}w)\Lambda_U \mathrm{d}\mu = Z_f \int_{\mathbb{R}^n} h(y)\mathrm{d}\kappa(y) \geqslant 0 \tag{27}$$

whenever h, and so $h(\mathcal{I}w(0))\Upsilon_f(\mathcal{I}w)$, is positive. This means that $\hat{L}$ is positive.

For any $f = g \circ T \in \mathcal{V}_1$, by Theorem 14 and Theorem 6, we have

$$|\hat{L}(f)| = \left| \int_{\mathcal{W}} g(w)\mathrm{d}\mu(w) \right| \leqslant \int_{\mathcal{W}} |g(w)|N(w)\mathrm{d}\mu(w) = \int_{\mathcal{W}} |g \circ T(w)\Lambda_U(w)|\mathrm{d}\mu(w) = \|f\Lambda_U\|_1.$$

On the other hand, if $f \in \mathcal{V}_2$, by relation (24), $\hat{L}(f) \leqslant \|f\Lambda_U\|_1$. These two inequalities and the positivity of $\hat{L}$ imply, by Theorem 8.31 of [6] on the extension of positive functionals on Riesz spaces, that there exists at least one positive continuous linear functional L on $L^1(|\Lambda_U|\mathrm{d}\mu)$, such that $L(f) = \hat{L}(f)$ for any $f \in \mathcal{V}$. The functional L defines the weak solution to equation (13) we are looking for. Indeed, since L is a continuous positive functional on $L^1(|\Lambda_U|\mathrm{d}\mu)$ there exists a measurable positive function $B \in L^\infty(|\Lambda_U|\mathrm{d}\mu) \subset L^\infty(\mathrm{d}\mu)$ such that $L(f) = \int_{\mathcal{W}} f(w)B(w)|\Lambda_U(w)|\mathrm{d}\mu(w)$. Since $\Lambda_U \in L^p$ by Lemma 15 below, we have $1 \in \mathcal{V}_1$ and so $L(1) = \int_{\mathcal{W}} 1\mathrm{d}\mu(w) = 1$. This implies, since the function B is positive, that the σ-finite measure $\mathrm{d}\nu = B|\Lambda_U|\mathrm{d}\mu$ is a probability measure. Furthermore, since $\mathcal{V}_1$ contains all the functions $g \circ T$, where g is measurable and bounded, equality (25) implies that $T_*(\nu) = \mu$. This means that ν is a weak solution to equation (13). Finally since $\mathcal{V}_2$ contains all the functions of the form $h(\mathcal{I}w(0))\Upsilon_f(\mathcal{I}w)$ where h is measurable and bounded on $\mathbb{R}^n$ the measure ν satisfies the thesis of the theorem. □

Unfortunately we cannot repeat this reasoning for general potentials satisfying the weaker Hypothesis QC since both Theorem 9 and Proposition 2 exploit an L^p bound on Λ_U (see Lemma 15 below) that cannot be obtained for more general potentials. Thus the idea is to generalize equation (22) without passing from equation (21). Indeed it is possible to approximate any potential

V satisfying Hypothesis QC by a sequence of potentials $(V_i)_i$ satisfying Hypothesis V_λ in such a way that the sequence of weak solutions ν_i associated with V_i converges (weakly) to a weak solution associated with the potential V (see Lemma 2, Lemma 3 and Lemma 4 below). Since equation (22) involves only integrals with respect to a weak solution to equation (6), we are able to prove that equation (22) holds for any potential V approximating its weak solution ν by the sequence ν_i satisfying equation (22).

Let us now set up the approximation argument, starting with a series of lemmas about convergence of weak solutions.

Lemma 2 *Let $\{T_i\}_{i\in\mathbb{N}}$ be a sequence of continuous maps on $\mathcal{W}$ such that for any compact $K\subset\mathcal{W}$ we have that $\cup_{i\in\mathbb{N}}T_i^{-1}(K)$ is pre-compact and there exists a continuous map T such that $T_i\to T$ uniformly on the compact subsets of $\mathcal{W}$. Let $\mathbb{M}_i$ be a set of probability measures on $\mathcal{W}$ defined as follows*

$$\mathbb{M}_i := \{\nu \text{ probability measure on } \mathcal{W} \text{ such that } T_{j,*}(\nu)=\mu \text{ for some } j\geqslant i\}\,.$$

Then $\mathbb{M} := \cap_{i\in\mathbb{N}}\bar{\mathbb{M}}_i$, where the closure is taken with respect to the weak topology on the set of probability measures on $\mathcal{W}$, is non-void and

$$\mathbb{M}\subset\{\nu \text{ probability measure on } \mathcal{W} \text{ such that } T_*(\nu)=\mu\}\,.$$

Proof First of all we prove that $\mathbb{M}_i$ is pre-compact for any $i\in\mathbb{N}$. This is equivalent to prove that the measures in $\mathbb{M}_i$ are tight. Let $\tilde{K}$ be a compact set such that $\mu(\tilde{K})\geqslant 1-\epsilon$ for a fixed $0<\epsilon<1$, then $K:=\overline{\cup_{i\in\mathbb{N}}T_i^{-1}(\tilde{K})}$ is a compact set in $\mathcal{W}$. Consider $\nu\in\mathbb{M}_j$ then there exists T_k such that $T_{k,*}\nu=\mu$. This implies

$$\nu(K)\geqslant\nu(\cup_i T_i^{-1}(\tilde{K}))\geqslant\nu(T_k^{-1}(\tilde{K}))\geqslant\mu(\tilde{K})\geqslant 1-\epsilon,$$

for any $k\in\mathbb{N}$. Since $\mathbb{M}_i$ are pre-compact, $\bar{\mathbb{M}}_i$ are compact and $\bar{\mathbb{M}}_i\subset\bar{\mathbb{M}}_j$ if $i\geqslant j$. This implies that $\mathbb{M}$ is non-void. If we consider a $\nu\in\mathbb{M}$ there exists a sequence ν_k weakly converging to ν, for $k\to+\infty$, such that $T_{i_k,*}(\nu_k)=\mu$ and $i_k\to+\infty$. Proving that $T_*(\nu)=\mu$ is equivalent to prove that for any C^1 bounded function g with bounded derivatives defined on $\mathcal{W}$ taking values in $\mathbb{R}$ we have $\int g\circ T\mathrm{d}\nu=\int g\mathrm{d}\mu$. Let K the compact set defined before, then there exists a $k\in\mathbb{N}$ such that $\sup_{w\in K}\|T_{i_k}(w)-T(w)\|\leqslant\epsilon$ and that $\left|\int_{\mathcal{W}}g\circ T\mathrm{d}\nu-\int_{\mathcal{W}}g\circ T\mathrm{d}\nu_k\right|\leqslant\epsilon$, for the arbitrary $0<\epsilon<1$. This implies that

$$\begin{aligned}\left|\int_{\mathcal{W}}g\circ T\mathrm{d}\nu-\int_{\mathcal{W}}g\mathrm{d}\mu\right| &\leqslant \left|\int_{\mathcal{W}}g\circ T\mathrm{d}\nu-\int_{\mathcal{W}}g\circ T\mathrm{d}\nu_i\right|+\left|\int_K(g\circ T-g\circ T_{i_k})\mathrm{d}\nu_k\right|\\ &\quad+\|g\|_\infty\epsilon+\left|\int_{\mathcal{W}}g\circ T_{i_k}\mathrm{d}\nu_k-\int_{\mathcal{W}}g\mathrm{d}\mu\right|\\ &\leqslant \epsilon+\|\nabla g\|_\infty\epsilon+\|g\|_\infty\epsilon.\end{aligned}$$

Since ϵ is arbitrary, from this it follows that $\int_{\mathcal{W}}g\circ T\mathrm{d}\nu=\int_{\mathcal{W}}g\mathrm{d}\mu$.

□

Remark 5 The proof of Lemma 2 proves also that given any sequence of $\nu_i\in\mathbb{M}_i$ there exists a subsequence converging weakly to $\nu\in\mathbb{M}$.

Lemma 3 *Let $\{V_i\}_{i\in\mathbb{N}}$ be a sequence of potentials satisfying the Hypothesis QC converging to the potential V and such that ∂V_i converges uniformly to ∂V on compact subsets of $\mathbb{R}^n$, V_i, V,*

∂V_i and ∂V are uniformly exponentially bounded and there exists a common function H entering Hypothesis QC for $\{V_i\}_{i\in\mathbb{N}}$ and V. Let T_i, T the maps on $\mathcal{W}$ associated to V_i and V respectively. Then the sequence $\{T_i\}_{i\in\mathbb{N}}$ satisfies the hypothesis of Lemma 2.

Proof Note that the a priori estimates (16) and (17) in Lemma 1 are uniform in $i \in \mathbb{N}$ since they depend only on H and the exponential growth of $V_i, V, \partial V_i, \partial V$. From this we can deduce the pre-compactness of the set $K = \cup_{i\in\mathbb{N}} T_i^{-1}(\tilde{K})$ for any compact set $\tilde{K} \subset \mathcal{W}$ using a reasoning similar to the one proposed in Theorem 4 and Theorem 6.

Proving that T_i converges to T uniformly on the compact sets is equivalent to prove that the map $U_i(w)(x) = f(x)\partial V_i(\mathcal{I}w(x))$ converges to $U(w)(x) = f(x)\partial V(\mathcal{I}w(x))$ in L^2 uniformly on the compact subsets of $\mathcal{W}$. Let K be a compact set of $\mathcal{W}$, then there exists an $M > 0$ such that $|\mathcal{I}w(x)| \leqslant M(1 + |x|^\eta)$ (where we suppose without loss of generality that $\eta < 1$). By hypotheses we have that there exist two constants $\alpha, \beta > 0$ such that $|\partial V_i(y)|, |\partial V(y)| \leqslant e^{\alpha|y|+\beta}$, thus there exists a compact subset $\mathfrak{K}$ of $\mathbb{R}^2$ such that $\int_{\mathfrak{K}^c} (f(x))^2 \exp(2\alpha M(1 + |x|^\eta) + 2\beta)\mathrm{d}x \leqslant \epsilon$, for some $\epsilon \in (0, 1)$. Denote by B_ϵ the ball of radius $\sup_{x\in\mathfrak{K}} M(1 + |x|^\eta)$ then we have

$$\begin{aligned} \sup_{w\in K} \|U_i(w) - U(w)\|^2_{\mathcal{H}} &\leqslant 2\left|\int_{\mathfrak{K}^c} (f(x))^2 e^{2\alpha M(1+|x|^\eta)+2\beta}\mathrm{d}x\right| \\ &\quad + \sup_{w\in K}\left|\int_{\mathfrak{K}} (f(x))^2 |\partial V(\mathcal{I}w) - \partial V_i(\mathcal{I}w)|^2 \mathrm{d}x\right| \\ &\leqslant 2\epsilon + (\sup_{y\in B_\epsilon} |\partial V(y) - \partial V_i(y)|)^2 \int_{\mathfrak{K}} (f(x))^2 \mathrm{d}x \\ &\to 2\epsilon, \end{aligned}$$

as $i \to +\infty$. This means that $\lim_{i\to+\infty}(\sup_{w\in K} \|U_i(w) - U(w)\|^2_{\mathcal{H}}) \leqslant 2\epsilon$, and since ϵ is arbitrary in $(0, 1)$ the theorem is proved. □

Lemma 4 *Let V be a potential satisfying Hypothesis QC, then there exists a sequence $\{V_i\}_{i\in\mathbb{N}}$ of bounded smooth potentials converging to V and satisfying the hypothesis of Lemma 3.*

Proof Let V be a potential satisfying the Hypothesis QC and let $\tilde{H}$ the function whose existence is guaranteed by Hypothesis QC. Let, for any $N \in \mathbb{N}$, $v_N := \sup_{y\in B(0,N)} |V(y)|$ and let $\tilde{V}^N := G_{v_N} \circ V$ where

$$G_k(z) := \begin{cases} z & \text{if } |z| \leqslant k, \\ k & \text{if } |z| > k. \end{cases}$$

Let ρ be a smooth compactly supported mollifier and denote by ρ_ϵ the function $\rho_\epsilon(y) := \epsilon^{-n}\rho\left(\frac{y}{\epsilon}\right)$. We want to prove that $V^N = \tilde{V}^N * \rho_{\epsilon_N}$, for a suitable sequence $\epsilon_N \in \mathbb{R}_+$, is the approximation requested by the lemma. Without loss of generality we can suppose that $\tilde{H}$ is a positive function depending only on the radius $|y|$ and increasing as $|y| \to +\infty$. Under these conditions the request of Hypothesis QC is equivalent to say that for any unit vector $\hat{n} \in \mathbb{S}^n$ we have that for any $y \in \mathbb{R}^n$

$$\max(-\hat{n} \cdot \partial V(y + r\hat{n}), 0) \leqslant \tilde{H}(y).$$

We want to prove that $H(|y|) = \tilde{H}(|y| + \sup_N(\epsilon_N))$ is the function requested by the lemma.

Since for any unit vector $\hat{n} \in \mathbb{S}^n$ we have $|\hat{n} \cdot \partial\tilde{V}^N| \leqslant |\hat{n} \cdot \partial V|$ and since $\tilde{V}^N$ is absolutely continuous we obtain

$$-\hat{n} \cdot \partial V^N(y + r\hat{n}) = ((-\hat{n} \cdot \partial\tilde{V}^N) * \rho_{\epsilon_N})(y + r\hat{n})$$

$$\leqslant (\max(-\hat{n}\cdot\partial V(\cdot+r\hat{n}),0)*\rho_{\epsilon_N})(y) \leqslant \tilde{H}*\rho_{\epsilon_N}(y).$$

Furthermore we have that $\tilde{V}^N = V$ on $B(0,N-1)$ and so there exists a sequence $\{\epsilon_N\}_N$ such that $\epsilon_N \to 0$ and $\sup_{x\in B(0,N-1)}|\partial V^N(x) - \partial V(x)| \leqslant \frac{1}{N}$. Since V^N is smooth and bounded and

$$\tilde{H}*\rho_{\epsilon_N}(y) \leqslant \tilde{H}(|y| + \sup_N(\epsilon_N)) = H(y),$$

we conclude the claim. □

Finally we are able to prove (22) for all QC potentials, which will conclude this section.

Proof of Theorem 8 By Proposition 2 the equality (22) holds when V satisfies the Hypothesis V_λ for some $\lambda > 0$, i.e. if $V(y) = V_{\lambda,V_B}(y) = V_B(y) + \lambda\sum_{k=1}^n (y^i)^4$ for some bounded potential V_B. It is clear that if $\lambda_i \to 0$ the potentials V_{λ_i,V_B} converge to the potential V_B and the hypothesis of Lemma 3 hold. This means that if $\hat{\nu}_i$ is a sequence of probability measures such that $\hat{\nu}_i$ is a weak solution to the equation associated with V_{λ_i,V_B} satisfying the thesis of Proposition 2, by Remark 5 and Lemma 2, there exists a probability measure $\hat{\nu}$, that is a weak solution to the equation associated with V_B, such that $\hat{\nu}_i \to \nu$ in the weak sense, as $i \to \infty$ and $\lambda_i \to 0$.

We want to prove that $\hat{\nu}$ is a weak solution to the equation associated with V_B satisfying equation (22). The previous claim is equivalent to prove that

$$\int_{\mathcal{W}} g(\mathcal{I}w(0))e^{4\int f'(x)V_{\lambda_i,B}(\mathcal{I}w(x))\mathrm{d}x}\mathrm{d}\hat{\nu}^i(w) \to \int_{\mathcal{W}} g(\mathcal{I}w(0))e^{4\int f'(x)V_B(\mathcal{I}w(x))\mathrm{d}x}\mathrm{d}\hat{\nu}(w), \tag{28}$$

as $\lambda \to 0$, for any continuous bounded function g, and that $\kappa_{\lambda_i} \to \kappa_B$ weakly, where $\mathrm{d}\kappa_{\lambda_i} = \exp(-4\pi V_{\lambda_i,B})\mathrm{d}x/Z_{\lambda_i}$ and $\mathrm{d}\kappa_B = \mathrm{d}\kappa_{\lambda_i} = \exp(-4\pi V_B)\mathrm{d}x/Z_B$.

Proving relation (28) is equivalent to prove that

$$\int f'(x)V_{\lambda_i,B}(\mathcal{I}w(x))\mathrm{d}x \to \int f'(x)V_B(\mathcal{I}w(x))\mathrm{d}x$$

uniformly on compact sets of $\mathcal{W}$. This assertion can be easily proved using the methods of Lemma 3. The weak convergence of κ_{λ_i} to κ_B easily follows from Lebesgue's dominate convergence theorem.

The previous reasoning proves the theorem for any bounded potential V_B. Using Lemma 4 we can approximate any potential V satisfying Hypothesis QC by a sequence of bounded potentials $V_{B,i}$. Using Lemma 3, Remark 5, Lemma 2 and a reasoning similar to the one exploited in the first part of the proof we obtain the thesis of the theorem for a general potential satisfying Hypothesis QC. □

4 Dimensional reduction

Let

$$\Xi(h) := \int_{\mathcal{W}} h(\mathcal{I}w(0))\Lambda_U(w)\frac{\Upsilon_f(\mathcal{I}w)}{Z_f}\mathrm{d}\mu(w), \tag{29}$$

with the notations as in Section 2 (Theorem 7) and Section 3 (Theorem 8). In this section we prove Theorem 9, i.e. the identity

$$\Xi(h) = \int_{\mathbb{R}^n} h(y)\mathrm{d}\kappa(y), \tag{30}$$

using the supersymmetric representation of the integral. It is important to note that Λ_U appears without the modulus in (29).

4.1 Supersymmetric representation

Let us illustrate how to derive the supersymmetric formulation of $\Xi(h)$. Our goal is to provide a blueprint for the reader to understand the role of supersymmetry in our argument. For an introduction to the mathematical formalism of supersymmetry see e.g. [20, 7, 44, 19]. The details of the rigorous implementation of the ideas exposed here is the main goal of the paper of Klein et al. [32] and of the modifications we implement in the following subsections in order to overcome a gap in their proof.

Let us start by unfolding the definition of Λ_U and $\Upsilon_f(\mathcal{I}w)$ in (29) to get the expression

$$Z_f\Xi(h) = \int h(\mathcal{I}w(0))\,\mathrm{det}_2(I_{\mathcal{H}}+\nabla U)\exp\left(-\delta(U)-\frac{1}{2}\|U\|^2_{\mathcal{H}}+4\int V(\mathcal{I}w(x))f'(x)\mathrm{d}x\right)\mathrm{d}\mu(w).$$

In order to manipulate the regularized Fredholm determinant we approximate the right hand side by

$$Z_f^{\chi}\Xi_{\chi}(h) := \int h(\mathcal{J}_{\chi}w(0))\,\mathrm{det}_2(I_{\mathcal{H}}+\nabla U_{\chi})\times$$
$$\times\exp\left(-\delta(U_{\chi})-\frac{1}{2}\|U_{\chi}\|^2_{\mathcal{H}}+4\int V(\mathcal{J}_{\chi}w(x))f'(x)\mathrm{d}x\right)\mathrm{d}\mu(w).$$

where $\chi>0$ is a regularization parameter, $\mathcal{J}_{\chi}:=\mathcal{I}^{1+\chi}=(m^2-\Delta)^{-1-\chi}$, Z_f^{χ} is the unique positive constant such that $\Xi_{\chi}(h)=1$ and

$$U_{\chi}(w) := \frac{1}{1+2\chi}\mathcal{I}^{\chi}\partial V(\mathcal{J}_{\chi}w). \tag{31}$$

We will prove below that $\lim_{\chi\to 0}\Xi_{\chi}(h)=\Xi(h)$. When $\chi>0$, $\nabla U_{\chi}(w)=\frac{1}{1+2\chi}\mathcal{I}^{\chi}\partial V(\mathcal{J}_{\chi}w)\mathcal{J}_{\chi}$ is almost surely a trace class operator and $U_{\chi}\in\mathcal{W}^*$. This means that we can rewrite the regularized Fredholm determinant det_2 in term of the unregularized one (denoted det) (see equation (60) and the discussion in Appendix A) obtaining

$$\begin{array}{rcl} Z_f^{\chi}\Xi_{\chi}(h) & = & \int h(\mathcal{J}_{\chi}w(0))\det(I_{\mathcal{H}}+\nabla U_{\chi})\times \\ & & \times\exp\left(-\langle U_{\chi},w\rangle-\frac{1}{2}\|U_{\chi}\|^2_{\mathcal{H}}+4\int V(\mathcal{J}_{\chi}w(x))f'(x)\mathrm{d}x\right)\mathrm{d}\mu(w).\end{array} \tag{32}$$

From the expression of ∇U_{χ} and the invariance of the determinant by conjugation we deduce

$$\det(I_{\mathcal{H}}+\nabla U_{\chi})=\det(I_{\mathcal{H}}+\varpi\mathcal{I}^{\chi}f\partial^2V(\mathcal{J}_{\chi}w)\mathcal{J}_{\chi})=\det(I_{\mathcal{H}}+\varpi\mathcal{I}^{1/2+\chi}f\partial^2V(\mathcal{J}_{\chi}w)\mathcal{I}^{1/2+\chi}),$$

where $\varpi=\frac{1}{1+2\chi}$, and featuring the nicer symmetric operator $\varpi\mathcal{I}^{1/2+\chi}f\partial^2V(\mathcal{J}_{\chi}w)\mathcal{I}^{1/2+\chi}$. Let γ be the Gaussian measure given by the law of $\varphi=\mathcal{J}_{\chi}w$ under μ. The expression (32) is then equivalent to

$$\int h(\varphi(0))\det(I_{\mathcal{H}}+\varpi\mathcal{I}^{1/2+\chi}f\partial^2V(\varphi)\mathcal{I}^{1/2+\chi})\exp(-\langle\varpi f\partial V(\varphi),(m^2-\Delta)\varphi\rangle)\times$$
$$\times\exp\left(-\frac{\varpi^2}{2}\|\mathcal{I}^{\chi}f\partial V(\varphi)\|^2_{\mathcal{H}}+4\int V(\varphi(x))f'(x)\mathrm{d}x\right)\gamma(\mathrm{d}\phi).$$

At this point we introduce an auxiliary Gaussian field η distributed as the Gaussian white noise μ to write

$$\exp\left(-\frac{\varpi^2}{2}\|\mathcal{I}^{\chi}f\partial V(\phi)\|^2_{\mathcal{H}}\right)=\int\exp(-i\varpi\langle f\partial V(\phi),\mathcal{I}^{\chi}\eta\rangle)\mu(\mathrm{d}\eta).$$

We also introduce two free fermion fields $\psi, \bar{\psi}$ realized as bounded operators on a suitable Hilbert space $H_{\psi,\bar{\psi}}$ with a state $\langle\cdot\rangle_{\psi,\bar{\psi}}$ for which

$$\langle\bar{\psi}(x)\bar{\psi}(x')\rangle_{\psi,\bar{\psi}} = \langle\psi(x)\psi(x')\rangle_{\psi,\bar{\psi}} = 0, \qquad \langle\psi(x)\bar{\psi}(x')\rangle_{\psi,\bar{\psi}} = \varpi\mathcal{G}_{1+2\chi}(x-x'),$$

where $\mathcal{G}_\alpha$ is the kernel of the operator $\mathcal{I}^\alpha$. These additional fields allow to represent the determinant as

$$\det(I_{\mathcal{H}} + \varpi\mathcal{I}^{1/2+\chi} f\partial^2 V(\varphi)\mathcal{I}^{1/2+\chi}) = \left\langle \exp\left(\int \psi^i(x) f(x)\partial^2_{\phi^i\phi^j} V(\varphi(x))\bar{\psi}^j(x)\mathrm{d}x\right)\right\rangle_{\psi,\bar{\psi}}.$$

By tensorizing the fermionic Hilbert space $H_{\psi,\bar{\psi}}$ with the L^2 space of the product measure $\gamma\otimes\mu$ one can realize the fermionic free fields $\psi, \bar{\psi}$ and the Gaussian fields φ, η as operators on the same Hilbert space H_Φ with a state which we denote by $\langle\cdot\rangle_\Phi$ or $\langle\cdot\rangle_\chi$ when this does not cause ambiguity. As a consequence, we have

$$Z_f^\chi \Xi_\chi(h) = \langle h(\varphi(0))\exp(S(\varphi,\eta,\psi,\bar{\psi}))\rangle_\chi,$$

with

$$\begin{aligned} S(\varphi,\eta,\psi,\bar{\psi}) \quad := \quad & \int \psi(x) f(x)\partial^2 V(\varphi(x))\bar{\psi}(x)\mathrm{d}x + \\ & -\varpi\langle f\partial V(\varphi), (m^2-\Delta)\varphi + i\mathcal{I}^\chi\eta\rangle + 4\int V(\varphi(x)) f'(x)\mathrm{d}x. \end{aligned}$$

A "more symmetric" form for this expression involving bosonic and fermionic fields can be obtained by introducing the *superspace* $\mathfrak{S}$ and the *superfield* Φ.

The superspace Formally the superspace $\mathfrak{S}$ can be thought as the set of points $(x,\theta,\bar{\theta})$ where $x\in\mathbb{R}^2$ and $\theta,\bar{\theta}$ are two additional anticommuting coordinates. A more concrete construction is to understand $\mathfrak{S}$ via the algebra of smooth functions on it.

Let $\mathfrak{G}(\theta_1,\ldots,\theta_n)$ be the Grassmann algebra generated by symbols $\theta_1,\ldots,\theta_n$, i.e.

$$\mathfrak{G}(\theta_1,\ldots,\theta_n) = \mathrm{span}(1,\theta_i,\theta_i\theta_j,\theta_i\theta_j\theta_k,\ldots,\theta_1\cdots\theta_n)$$

with the relations $\theta_i\theta_j = -\theta_j\theta_i$.

A C^∞ function $F:\mathbb{R}^2\to\mathfrak{G}(\theta,\bar{\theta})$ is just a quadruplet $(f_\emptyset, f_\theta, f_{\bar{\theta}}, f_{\theta\bar{\theta}})\in(C^\infty(\mathbb{R}^2))^4$ the identification

$$F(x) = f_\emptyset(x) + f_\theta(x)\theta + f_{\bar{\theta}}(x)\bar{\theta} + f_{\theta\bar{\theta}}(x)\theta\bar{\theta}.$$

The function F can be considered as a function $F:\mathfrak{S}\to\mathbb{R}$ by formally writing

$$F(x,\theta,\bar{\theta}) = F(x).$$

In particular we identify $C^\infty(\mathfrak{S})$ with $C^\infty(\mathbb{R}^2;\mathfrak{G}(\theta,\bar{\theta}))$. $C^\infty(\mathfrak{S})$ is a non-commutative algebra on which we can introduce a linear functional defined as

$$F\mapsto\int F(x,\theta,\bar{\theta})\mathrm{d}x\mathrm{d}\theta\mathrm{d}\bar{\theta} := -\int_{\mathbb{R}^2} f_{\theta\bar{\theta}}(x)\mathrm{d}x,$$

induced by the standard Berezin integral on $\mathfrak{S}$ satisfying

$$\int\mathrm{d}\theta\mathrm{d}\bar{\theta} = \int\theta\mathrm{d}\theta\mathrm{d}\bar{\theta} = \int\bar{\theta}\mathrm{d}\theta\mathrm{d}\bar{\theta} = 0, \qquad \int\theta\bar{\theta}\mathrm{d}\theta\mathrm{d}\bar{\theta} = -1.$$

Remark 6 A norm on $C^\infty(\mathfrak{G})$ can be defined by

$$\|F\|_{C(\mathfrak{G})} = \sup_{x\in\mathbb{R}^2}(|f_\varnothing(x)| + |f_\theta(x)| + |f_{\bar\theta}(x)| + |f_{\theta\bar\theta}(x)|),$$

and an involution by

$$\bar F(x,\theta,\bar\theta) = \overline{f_\emptyset}(x) + \overline{f_\theta}(x)\theta + \overline{f_{\bar\theta}}(x)\bar\theta + \overline{f_{\theta\bar\theta}}(x)\theta\bar\theta,$$

where the bar on the right hand side denotes complex conjugation.

Given $r : C^1(\mathbb{R};\mathbb{R})$ we define the composition $r\circ F : \mathfrak{G}\to\mathbb{R}$ by

$$r(F(x,\theta,\bar\theta)) := r(f_\emptyset(x)) + r'(f_\emptyset(x))f_\theta(x)\theta + r'(f_\emptyset(x))f_{\bar\theta}(x)\bar\theta + r'(f_\emptyset(x))f_{\theta\bar\theta}(x)\theta\bar\theta,$$

in accordance with the same expression one would get if r were a monomial. Moreover we can define similarly the space of Schwartz superfunctions $\mathcal{S}(\mathfrak{G})$ and the Schwartz superdistributions $\mathcal{S}'(\mathfrak{G}) \approx \mathcal{S}'(\mathbb{R}^2;\mathfrak{G}(\theta,\bar\theta))$ where $T\in\mathcal{S}'(\mathfrak{G})$ can be written $T = T_\emptyset + T_\theta\theta + T_{\bar\theta}\bar\theta + T_{\theta\bar\theta}\theta\bar\theta$ with $T_\emptyset, T_\theta, T_{\bar\theta}, T_{\theta\bar\theta}\in\mathcal{S}'(\mathbb{R}^2)$ and duality pairing

$$T(f) = -T_\emptyset(f_{\theta\bar\theta}) + T_\theta(f_{\bar\theta}) - T_{\bar\theta}(f_\theta) - T_{\theta\bar\theta}(f_\emptyset), \qquad f_\emptyset, f_\theta, f_{\bar\theta}, f_{\theta\bar\theta}\in\mathcal{S}(\mathbb{R}^2).$$

The superfield We take the generators $\theta,\bar\theta$ to anticommute with the the fermion fields $\psi,\bar\psi$, and introduce the complex Gaussian field

$$\omega := -\varpi((m^2-\Delta)\varphi + i\mathcal{I}^\chi\eta)$$

and put all our fields together in a single object defining the *superfield*

$$\Phi(x,\theta,\bar\theta) := \varphi(x) + \bar\psi(x)\theta + \psi(x)\bar\theta + \omega(x)\theta\bar\theta.$$

We define also

$$\begin{aligned} V(\Phi(x,\theta,\bar\theta)) &= V(\varphi(x)) + \partial V(\varphi(x))(\bar\psi(x)\theta + \psi(x)\bar\theta) + \\ &\quad + [\partial V(\varphi(x))\omega(x) + \partial^2 V(\varphi(x))\psi(x)\bar\psi(x)]\theta\bar\theta \end{aligned}$$

and since

$$\tilde f(|x|^2 + 4\theta\bar\theta) = \tilde f(|x|^2) + 4\tilde f'(|x|^2)\theta\bar\theta$$

we observe that

$$\begin{aligned} -\int V(\Phi(x,\theta,\bar\theta))\tilde f(|x|^2+4\theta\bar\theta)\mathrm{d}x\mathrm{d}\theta\mathrm{d}\bar\theta &= \int f(x)\partial V(\varphi(x))\omega(x)\mathrm{d}x + \\ + \int [f(x)\partial^2 V(\varphi(x))\psi(x)\bar\psi(x) + 4V(\varphi(x))f'(x)]\mathrm{d}x &= S(\phi,\eta,\psi,\bar\psi). \end{aligned}$$

By introducing the superspace distribution $\theta\bar\theta\delta_0(\mathrm{d}x)$ we have also, by similar computations.

$$h(\varphi(0)) = -\int h(\Phi(x,\theta,\bar\theta))\theta\bar\theta\delta_0(\mathrm{d}x)\mathrm{d}\theta\mathrm{d}\bar\theta.$$

As a consequence we can rewrite $\Xi(h)$ as an average over the superfield Φ:

$$\begin{aligned} Z_f^\chi \Xi_\chi(h) &= \left\langle \left(-\int h(\Phi(x,\theta,\bar\theta))\theta\bar\theta\delta_0(\mathrm{d}x)\mathrm{d}\theta\mathrm{d}\bar\theta \right) \right. \\ &\quad \left. \times \exp\left(-\int V(\Phi(x,\theta,\bar\theta))\tilde f(|x|^2+4\theta\bar\theta)\mathrm{d}x\mathrm{d}\theta\mathrm{d}\bar\theta \right) \right\rangle_\chi . \end{aligned} \tag{33}$$

While all these rewriting are essentially algebraic (modulo the issue of justifying rigorously each step), the supersymmetric formulation (33) makes appear a symmetry of the expression for $\Xi_\chi(h)$ which was not clear from the original formulation. In some sense the reader can think to the superspace $(x,\theta,\bar\theta)$ and to the superfield $\Phi(x,\theta,\bar\theta)$ as a convenient bookkeeping procedure for a series of relations between the quantities one is manipulating.

A crucial observation is that the superfield Φ is a free field with mean zero, namely all its correlation functions can be expressed in terms of the two-point function $\langle\Phi(x,\theta,\bar\theta)\Phi(x,\theta',\bar\theta')\rangle_\chi$ via Wick's theorem. A direct computation of this two point function gives:

$$\begin{aligned}\langle\Phi(x,\theta,\bar\theta)\Phi(x,\theta',\bar\theta')\rangle_\chi &= \langle\varphi(x)\varphi(x')\rangle_\chi - \langle\bar\psi(x)\psi(x')\rangle_\chi\theta\bar\theta' - \langle\psi(x)\bar\psi(x')\rangle_\chi\bar\theta\theta' + \langle\varphi(x)\omega(x')\rangle_\chi\theta'\bar\theta' \\ &\quad +\langle\omega(x)\varphi(x')\rangle_\chi\theta\bar\theta + \langle\omega(x)\omega(x')\rangle_\chi\theta\bar\theta\theta'\bar\theta' \\ &= \mathcal{G}_{2+2\chi}(x-x') + \varpi\mathcal{G}_{1+2\chi}(x-x')(\theta\bar\theta'-\bar\theta\theta') - \varpi(m^2-\Delta)\mathcal{G}_{2+2\chi}(x-x')(\theta'\bar\theta'+\theta\bar\theta) \\ &\quad +((m^2-\Delta)^2\mathcal{G}_{2+2\chi}(x-x') - \mathcal{G}_{2\chi}(x-x'))\theta\bar\theta\theta'\bar\theta'.\end{aligned}$$

Upon observing that $(m^2-\Delta)\mathcal{G}_{2+2\chi} = \mathcal{G}_{1+2\chi}$, $(m^2-\Delta)^2\mathcal{G}_{2+2\chi} = \mathcal{G}_{2\chi}$ and that $-\theta\bar\theta' + \bar\theta\theta' + \theta'\bar\theta' + \theta\bar\theta = (\theta-\theta')(\bar\theta-\bar\theta')$ we conclude

$$\langle\Phi(x,\theta,\bar\theta)\Phi(x,\theta',\bar\theta')\rangle = C_\Phi(x-x',\theta-\theta',\bar\theta-\bar\theta'), \tag{34}$$

where

$$C_\Phi(x,\theta,\bar\theta) := \mathcal{G}_{2+2\chi}(x) - \varpi\mathcal{G}_{1+2\chi}(x)\theta\bar\theta.$$

Remark 7 Note that when $\chi=0$, the superfield Φ corresponds to the formal functional integral

$$e^{-\frac{1}{2}\int[\Phi(m^2-\Delta_S)\Phi]\mathrm{d}x\mathrm{d}\theta\mathrm{d}\bar\theta}\mathcal{D}\Phi$$

where $\mathcal{D}\Phi = \mathcal{D}\psi\mathcal{D}\bar\psi\mathcal{D}\varphi\mathcal{D}\eta$ and where $\Delta_S = \Delta + \partial_\theta\partial_{\bar\theta}$ is the superlaplacian. Then

$$\frac{1}{2}\int[\Phi(m^2-\Delta_S)\Phi]\mathrm{d}x\mathrm{d}\theta\mathrm{d}\bar\theta = \frac{1}{2}\int[-2\bar\psi(m^2-\Delta)\psi - \omega\omega + 2\omega(m^2-\Delta)\varphi]\mathrm{d}x$$

$$= \frac{1}{2}\int[-2\psi(m^2-\Delta)\bar\psi + ((m^2-\Delta)\varphi)^2 + \eta^2]\mathrm{d}x$$

and this indeed corresponds to the action functional appearing in the formal functional integral for $(\psi,\bar\psi,\varphi,\eta)$. This is in agreement to the fact that the two point function satisfies the equation

$$(m^2-\Delta_S)C_\Phi(x,\theta,\bar\theta) = \delta_0(x)\delta(\theta)\delta(\bar\theta),$$

where $\delta(x)\delta(\theta)\delta(\bar\theta)$ is the distribution such that

$$\int F(x,\theta,\bar\theta)\delta_0(x)\delta(\theta)\delta(\bar\theta)\mathrm{d}x\mathrm{d}\theta\mathrm{d}\bar\theta = f_\emptyset(0),$$

namely, C_Φ is the Green's function for $(m^2-\Delta_S)$.

The supersymmetry On $C^\infty(\mathfrak{S})$ one can introduce the (graded) derivations

$$Q := 2\theta\nabla + x\partial_{\bar\theta}, \qquad \bar Q := 2\bar\theta\nabla - x\partial_\theta,$$

which are such that

$$Q(|x|^2 + 4\theta\bar\theta) = \bar Q(|x|^2 + 4\theta\bar\theta) = 0,$$

namely they annihilate the quadratic form $|x|^2+4\theta\bar\theta$. Moreover if $QF = \bar QF = 0$, for F as above, then we must have

$$0 = QF(x,\theta,\bar\theta) = 2\theta\nabla f_\emptyset(x) + xf_{\bar\theta}(x) + 2\nabla f_{\bar\theta}(x)\theta\bar\theta - xf_{\theta\bar\theta}(x)\theta$$

$$0 = \bar QF(x,\theta,\bar\theta) = 2\bar\theta\nabla f_\emptyset(x) + xf_{\theta}(x) - 2\nabla f_{\theta}(x)\theta\bar\theta + xf_{\theta\bar\theta}(x)\bar\theta$$

and therefore

$$\nabla f_\emptyset(x) = \frac{x}{2} f_{\theta\bar\theta}(x) \qquad \text{and} \qquad f_\theta(x) = f_{\bar\theta}(x) = 0.$$

If we also request that F is invariant with respect to $\mathbb{R}^2$ rotations, then there exists an f such that $f(|x|^2) = f_\emptyset(x)$ from which we deduce that $2xf'(|x|^2) = \nabla f(|x|^2) = \nabla f_\emptyset(x) = \frac{x}{2} f_{\theta\bar\theta}(x)$ which implies

$$f(|x|^2 + 4\theta\bar\theta) = f(|x|^2) + 4f'(|x|^2)\theta\bar\theta = f_\emptyset(x) + f_{\theta\bar\theta}(x)\theta\bar\theta = F(x,\theta,\bar\theta).$$

Namely any function satisfying these two equations can be written in the form

$$F(x,\theta,\bar\theta) = f(|x|^2 + 4\theta\bar\theta).$$

Observe that if we introduce the linear transformations

$$\tau(b,\bar b)\begin{pmatrix} x \\ \theta \\ \bar\theta \end{pmatrix} = \begin{pmatrix} x + 2\bar b\theta\rho + 2b\bar\theta\rho \\ \theta - (x\cdot b)\rho \\ \bar\theta + (x\cdot\bar b)\rho \end{pmatrix} \in \mathfrak{G}(\theta,\bar\theta,\rho)$$

for $b,\bar b \in \mathbb{R}^2$ and where ρ is a new odd variable anticommuting with $\theta,\bar\theta$ and itself, then we have

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \tau(tb,t\bar b)F(x,\theta,\bar\theta) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} F(\tau(tb,t\bar b)(x,\theta,\bar\theta)) = (b\cdot\bar Q + \bar b\cdot Q)F(x,\theta,\bar\theta)$$

so $\tau(b,\bar b) = \exp(b\cdot\bar Q + \bar b\cdot Q)$ and $\tau(tb,t\bar b)\tau(sb,s\bar b) = \tau((t+s)b,(t+s)\bar b)$. In particular $F \in C^\infty(\mathfrak{S})$ is supersymmetric if and only if F is invariant with respect to rotations and for any $b,\bar b\in\mathbb{R}^2$ we have $\tau(b,\bar b)(F) = F$.

By duality the operators $Q,\bar Q$ and $\tau(b,\bar b)$ act also on the space $\mathcal{S}'(\mathfrak{G})$ and we say that the distribution $T \in \mathcal{S}'(\mathfrak{S})$ is supersymmetric if it is invariant with respect to rotations and $Q(T) = \bar Q(T) = 0$. For supersymmetric functions and distribution the following fundamental theorem holds.

Theorem 11 *Let $F \in \mathcal{S}(\mathfrak{S})$ and $T \in \mathcal{S}'(\mathfrak{S})$ such that T_0 is a continuous function. If both F and T are supersymmetric, then we have the reduction formula*

$$\int T(x,\theta,\bar\theta)\cdot F(x,\theta,\bar\theta)\mathrm{d}x\mathrm{d}\theta\mathrm{d}\bar\theta = 4\pi T_0(0)F_0(0). \tag{35}$$

Proof The proof can be found in [32] Lemma 4.5. □

Let us note that

$$QC_\Phi(x,\theta,\bar\theta) = \bar{Q}C_\Phi(x,\theta,\bar\theta) = 0,$$

indeed we can check that

$$\nabla\mathcal{G}_{2+2\chi}(x) = \int_{\mathbb{R}^2}\frac{\mathrm{d}k}{(2\pi)^2}\frac{(ik)e^{ik\cdot x}}{(m^2+|k|^2)^{2+2\chi}} = -\frac{i}{2(1+2\chi)}\int_{\mathbb{R}^2}\frac{\mathrm{d}k}{(2\pi)^2}e^{ik\cdot x}\nabla_k\frac{1}{(m^2+|k|^2)^{1+2\chi}}$$

$$= \frac{i}{2(1+2\chi)}\int_{\mathbb{R}^2}\frac{\mathrm{d}k}{(2\pi)^2}\frac{(ix)e^{ik\cdot x}}{(m^2+|k|^2)^{1+\delta}} = -\frac{x}{2(1+2\chi)}\mathcal{G}_{1+2\chi}(x) = -\frac{x\varpi}{2}\mathcal{G}_{1+2\chi}(x)$$

As a consequence expectation values of polynomials over the superfield Φ are invariant under the supersymmetry generated by any linear combinations of $Q,\bar{Q}$.

Remark 8 The previous discussion implies that

$$\tau(b,\bar{b})C_\Phi(x,\theta,\bar\theta) = C_\Phi(x,\theta,\bar\theta). \tag{36}$$

As a consequence, the superfield $\Phi' := \tau(b,\bar{b})\Phi$ is a Gaussian free field and has the same correlation function $C_{\Phi'}$ as Φ given by equation (34). However it is important to stress that this does not imply that Φ' has the same "law" as Φ, namely that $\langle F(\Phi')\rangle = \langle F(\Phi)\rangle$ for nice arbitrary functions. Indeed the correlation function given in equations (34) involves only the product $\langle\Phi(x,\theta,\bar\theta)\Phi(x,\theta',\bar\theta')\rangle$ of the complex superfield Φ and not also the product $\langle\Phi(x,\theta,\bar\theta)\bar\Phi(x,\theta',\bar\theta')\rangle$ of Φ with its complex conjugate $\bar\Phi$. The law of Φ would have been invariant with respect super transformations if and if only $\langle\Phi(x,\theta,\bar\theta)\Phi(x,\theta',\bar\theta')\rangle$ and $\langle\Phi(x,\theta,\bar\theta)\bar\Phi(x,\theta',\bar\theta')\rangle$ had been both supersymmetric. Unfortunately the function $\langle\Phi(x,\theta,\bar\theta)\bar\Phi(x,\theta',\bar\theta')\rangle$ is not invariant with respect to super transformations.

Expectation of supersymmetric polynomials As explained in Remark 8, the law of Φ is not supersymmetric. Nevertheless we can deduce important consequences from the supersymmetry of the correlation function C_Φ. More precisely, since Φ is a free field Wick's theorem hold and

$$\left\langle\prod_{i=1}^{2n}\Phi(x_i,\theta_i,\bar\theta_i)\right\rangle_\chi = \sum_{\{(i_k,j_k)\}_k}\prod_{k=1}^{n}C_\Phi(x_{i_k}-x_{j_k},\theta_{i_k}-\theta_{j_k},\bar\theta_{i_k}-\bar\theta_{j_k}), \tag{37}$$

$$\left\langle\prod_{i=1}^{2n+1}\Phi(x_i,\theta_i,\bar\theta_i)\right\rangle_\chi = 0. \tag{38}$$

By the supersymmetry of $C_\Phi(x-x',\theta-\bar\theta,\theta-\bar\theta')$ and of its products, we obtain that

$$\left\langle\prod_{i=1}^{2n}\tau(b,\bar{b})(\Phi)(x_i,\theta_i,\bar\theta_i)\right\rangle_\chi = \left\langle\prod_{i=1}^{2n}\Phi(x_i,\theta_i,\bar\theta_i)\right\rangle_\chi.$$

The previous equality implies that

$$\begin{aligned}\left\langle \prod_{i=1}^n \int P_i(\Phi)\cdot\tau(b,\bar b)(F^i)\mathrm{d}x\mathrm{d}\theta\mathrm{d}\bar\theta \right\rangle_\chi &= \left\langle \prod_{i=1}^n \int \tau(b,\bar b)(P_i(\Phi))\cdot F^i\mathrm{d}x\mathrm{d}\theta\mathrm{d}\bar\theta \right\rangle_\chi \\ &= \left\langle \prod_{i=1}^n \int P_i(\tau(b,\bar b)(\Phi))\cdot F^i\mathrm{d}x\mathrm{d}\theta\mathrm{d}\bar\theta \right\rangle_\chi \\ &= \left\langle \prod_{i=1}^n \int P_i(\Phi)\cdot F^i\mathrm{d}x\mathrm{d}\theta\mathrm{d}\bar\theta \right\rangle_\chi , \qquad (39)\end{aligned}$$

where $P_1,\dots,P_n$ are arbitrary polynomials and $F^1,\dots,F^n$ arbitrary functions on superspace.

Lemma 5 *Let $F^1,\dots,F^n\in\mathcal{S}(\mathfrak{S})$ be supersymmetric smooth functions and $P_1,\dots,P_n$ be n polynomials then*

$$\left\langle \prod_{i=1}^n \int P_i(\Phi)(x,\theta,\bar\theta)\cdot F^i(x,\theta,\bar\theta)\mathrm{d}x\mathrm{d}\theta\mathrm{d}\bar\theta \right\rangle_\chi = (4\pi)^n \left\langle \prod_{i=1}^n f^i_\emptyset(0)P_i(\phi(0)) \right\rangle_\chi .$$

Proof We define the distribution $\mathcal{H}^1\in\mathcal{S}'(\mathfrak{S})$ in the following way:

$$\mathcal{H}^1(G):=\left\langle \int P_1(\Phi)(x,\theta,\bar\theta)\cdot G(x,\theta,\bar\theta)\mathrm{d}x\mathrm{d}\theta\mathrm{d}\bar\theta \prod_{i=2}^n\int P_i(\Phi)(x,\theta,\bar\theta)\cdot F^i(x,\theta,\bar\theta)\mathrm{d}x\mathrm{d}\theta\mathrm{d}\bar\theta \right\rangle_\chi$$

for any $G\in\mathcal{S}(\mathfrak{S})$. Using the fact that $F^2,\dots,F^n$ are supersymmetric and relation (39) we have that

$$\begin{aligned}\mathcal{H}^1(\tau(b,\bar b)(G)) &= \left\langle \textstyle\int P_1(\Phi)\cdot\tau(b,\bar b)(G)\mathrm{d}x\mathrm{d}\theta\mathrm{d}\bar\theta\prod_{i=2}^n\int P_i(\Phi)\cdot F^i\mathrm{d}x\mathrm{d}\theta\mathrm{d}\bar\theta\right\rangle_\chi \\ &= \left\langle \textstyle\int P_1(\Phi)\cdot\tau(b,\bar b)(G)\mathrm{d}x\mathrm{d}\theta\mathrm{d}\bar\theta\prod_{i=2}^n\int P_i(\Phi)\cdot\tau(b,\bar b)(F^i)\mathrm{d}x\mathrm{d}\theta\mathrm{d}\bar\theta\right\rangle_\chi = \mathcal{H}^1(G).\end{aligned}$$

This means that $\mathcal{H}^1$ is supersymmetric and since F^1 is also supersymmetric, by Theorem 11 we conclude

$$\mathcal{H}^1(F^1)=f^1_\emptyset(0)\mathcal{H}^1_0(0)=(4\pi)\left\langle f^1_\emptyset(0)P_i(\phi(0))\prod_{i=2}^n\int F^i\cdot P_i(\Phi)\mathrm{d}x\mathrm{d}\theta\mathrm{d}\bar\theta\right\rangle_\chi = \mathcal{H}^1(V)$$

where $V:=(4\pi)\delta_0(\mathrm{d}x)\theta\bar\theta$. Setting

$$\mathcal{H}^2(G):=\left\langle \left(\int P_i(\Phi)V\mathrm{d}x\mathrm{d}\theta\mathrm{d}\bar\theta\right)\left(\int P_i(\Phi)G\mathrm{d}x\mathrm{d}\theta\mathrm{d}\bar\theta\right)\prod_{i=3}^n\int P_i(\Phi)F^i\mathrm{d}x\mathrm{d}\theta\mathrm{d}\bar\theta\right\rangle_\chi$$

and reasoning similarly we also conclude that $\mathcal{H}^2(F^2)=\mathcal{H}^2(V)$. Proceeding by transforming each subsequent factor, we can deduce that

$$\left\langle\prod_{i=1}^n\int P_i(\Phi)F^i\mathrm{d}x\mathrm{d}\theta\mathrm{d}\bar\theta\right\rangle_\chi = \left\langle\prod_{i=1}^n\int P_i(\Phi)V\mathrm{d}x\mathrm{d}\theta\mathrm{d}\bar\theta\right\rangle_\chi = (4\pi)^n\left\langle\prod_{i=1}^n f^i_\emptyset(0)P_i(\phi(0))\right\rangle_\chi .$$

□

Theorem 12 *Let $V_1, V_2, \ldots, V_n$ be smooth functions such that they and their derivatives grow at most polynomially then*

$$\left\langle \prod_{i=1}^{n} \int V_i(\Phi)(x,\theta,\bar{\theta}) \cdot F^i(x,\theta,\bar{\theta}) \mathrm{d}x\mathrm{d}\theta\mathrm{d}\bar{\theta} \right\rangle_{\chi} = (4\pi)^n \left\langle \prod_{i=1}^{n} f^i_{\emptyset}(0) V_i(\phi(0)) \right\rangle_{\chi} .$$

Proof By Lemma 5, the theorem holds for $V_i = P_i$ polynomials. Then the thesis follows by the density of polynomials in the space of two-times differentiable functions with respect to the Malliavin derivative (see [40] Corollary 1.5.1). □

Remark 9 Going back to eq. (33) the idea now would be to expand the exponential getting

$$\begin{aligned} Z_f \Xi_{\chi}(g) &= \sum_{n \geqslant 0} \frac{1}{n!} \left\langle \left(\int h(\Phi(x,\theta,\bar{\theta}))\theta\bar{\theta}\delta_0(\mathrm{d}x)\mathrm{d}\theta\mathrm{d}\bar{\theta} \right) \times \right. \\ &\quad \left. \times \left(-\int V(\Phi(x,\theta,\bar{\theta}))\tilde{f}(|x|^2 + 4\theta\bar{\theta})\mathrm{d}x\mathrm{d}\theta\mathrm{d}\bar{\theta} \right)^n \right\rangle_{\chi} \end{aligned} \qquad (40)$$

and then to use Theorem 12 to prove that each average in the right hand side is equal to

$$\langle h(\varphi(0))(-4\pi V(\varphi(0)))^n \rangle_{\chi}.$$

Since

$$\langle h(\varphi(0))(-4\pi V(\varphi(0)))^n \rangle_{\chi} = Z_f^{\chi} \int_{\mathbb{R}^n} g(y)\mathrm{d}\kappa(y)$$

equality (30) would be proved taking the limit $\chi \to 0$. Unfortunately equation (40) is not easy to prove since the series on the right hand side of (40) does not converge absolutely for a general V. For this reason we present below an indirect proof of (30).

4.2 Outline of the proof of Theorem 9

A proof of Theorem 9 requires a justification of the formal steps in the previous subsection. Much of the work is already present in [32] and indeed our result is analogous, under different hypotheses, to Theorem II in [32]. The proofs of Lemma 11, Lemma 13 and Lemma 14 below follows the same ideas of Lemma 3.1, Lemma 3.2 and Lemma 3.3 in [32]. We decided to propose a detailed proof of Theorem 9 mainly for two reasons:

1. The first reason is that the hypotheses on the potential V of Theorem 9 and of Theorem II in [32] are quite different. Indeed in [32] only polynomial potentials are considered while Hypothesis V_{λ} permits to consider polynomial of at most fourth degree perturbed by any bounded function. In order to prove the boundedness of Λ_U in $L^p(\mu)$ under these different hypotheses we need to prove Lemma 12 which is a trivial consequence of hypercontractivity when the potential V is polynomial but is based on the non-trivial inequality (47) (proven in [49]) for general potentials V.

2. The second main reason is the difference in the use of supersymmetry and of the supersymmetric representation of the integral (29). Indeed, in our opinion there is a little gap in the proof of Theorem III of [32] that cannot be fixed without developing a longer proof. More precisely putting

$$G(F) := \left\langle g(\varphi(0)) \exp\left(-\int V(\Phi) F \mathrm{d}\theta\mathrm{d}\bar{\theta}\mathrm{d}x \right) \right\rangle_{\chi},$$

in the proof of Theorem III of [32] it is tacitly assumed that the expression $G(F)$ is supersymmetric with respect to the function F, i.e. if F is a smooth function in $\mathcal{S}(\mathfrak{S})$ and $\tau(b,\bar{b})$ is a supersymmetric transformation, then we have that $G(\tau(b,\bar{b})(F)) = G(F)$. In our opinion this fact is non-trivial since the law of Φ is not supersymmetric (see Remark 8). What can be easily proven is only that the expressions

$$G^n(F) := \left\langle g(\varphi(0)) \left(\int V(\Phi) F \mathrm{d}\theta \mathrm{d}\bar{\theta} \mathrm{d}x \right)^n \right\rangle_\chi$$

are supersymmetric in F (see Theorem 12 of Section 4.1). This fact alone does not easily imply that $G(F)$ is supersymmetric. Indeed for what we say in the discussion of Remark 9, we cannot guarantee that the series (40), which is equivalent to $G(F) = \sum_{n \geqslant 0} \frac{1}{n!} G^n(F)$, converges absolutely when V growth at infinity at least as a polynomial of fourth degree (and we do not know under which conditions on V and F it converges relatively). In order to overcome this problem we propose a proof of Theorem 9 which exploits only indirectly the supersymmetric representation of the integral (29) in a way which permits to use only the supersymmetry of the expressions $G^n(F)$ and avoiding the proof of the supersymmetry of the expression $G(F)$ (see Lemma 8).

The proof is based on two steps:

1. Prove Theorem 9 under the Hypothesis V_λ, C and CO,
2. Prove Theorem 9 under the Hypothesis V_λ with λ small enough and CO.

We introduce some preliminary objects, notations and considerations. First of all, if V is a potential satisfying the Hypothesis V_λ we write

$$V_{t,\lambda} = tV_B + \lambda V_U,$$

for any $t \in \mathbb{R}$, denote by $U_{t,\lambda}$ the corresponding map from $\mathcal{W}$ into $\mathcal{H}$and use a similar notation $U^\chi_{t,\lambda}$ for the corresponding map analogous to $U_{t,\lambda}$ when U is replaced by its regularized version (31). Let $L : \mathbb{R} \to \mathbb{R}$ be a continuous bounded function. We write

$$G^{\chi,L}_\lambda(t) := \int_{\mathcal{W}} L(\mathcal{J}_\chi w(0)) \det\nolimits_2(I_{\mathcal{H}} + \nabla U^\chi_{t,\lambda}) \times$$

$$\times \exp\left(-\delta(U^\chi_{t.\lambda}) - \frac{1}{2} \| U^\chi_{t,\lambda} \|^2_{\mathcal{H}} + 4 \int_{\mathbb{R}^2} V_{t,\lambda}(\mathcal{J}_\chi w(x)) f'(x) \mathrm{d}x \right) \mathrm{d}\mu$$

and

$$H^{\chi,L}_\lambda(t) := Z^\chi_f \int_{\mathbb{R}^n} L(y) \exp\left(-4\pi \left((1+2\chi) m^{2(1+2\chi)} \frac{|y|^2}{2} + tV_B(y) + \lambda V_U(y) \right) \right) \mathrm{d}y.$$

It is evident that the thesis of Theorem 9 is equivalent to prove that

$$G^L_\lambda(1) := G^{0,L}_\lambda(1) = H^L_\lambda(1) := H^{0,L}_\lambda,$$

for any bounded potential V_B, any λ small enough and any L continuous and bounded. Since the set of bounded potentials is invariant with respect to rescaling, proving the previous equality for any bounded potential is equivalent to proving that $G^L_\lambda(t) = H^L_\lambda(t)$ for any t. Furthermore since $H^L_\lambda(t)$ is real analytic in t proving the previous equality is equivalent to proving that $\partial^k_t G^L_\lambda(0) = \partial^k_t H^L_\lambda(0)$ for any $k \in \mathbb{N}$ (or equivalently that $G^L_\lambda(t) = H^L_\lambda(t)$ in a neighborhood of 0) and $G^L_\lambda(t)$ is real analytic for any $t \in \mathbb{R}$.

4.3 Proof under the Hypothesis C

Let V_B be a bounded smooth function with first and second derivatives which are both bounded.

Lemma 6 *Let V_B be a bounded potential satisfying the Hypothesis C, then* $\exp(-t\delta(U^\chi)) \in L^1(\mu)$ *for any* $|t| \in \left[0, \frac{m^2}{2\|\partial^2 V_B\|_\infty}\right)$ *and* $\chi \in [0,1]$. *Furthermore the integral* $\int \exp(-t\delta(U^\chi))\mathrm{d}\mu$ *is uniformly bounded for* $\chi \in [0,1]$ *and t in the compact subsets of* $\left(-\frac{m^2}{2\|\partial^2 V_B\|_\infty}, \frac{m^2}{2\|\partial^2 V_B\|_\infty}\right)$.

Proof Under the Hypothesis of the lemma we have that

$$\|\nabla U^\chi_{t,0}\| \leqslant |t| \frac{\|\partial^2 V_B\|_\infty}{m^{2(1+\chi)}},$$

where $\|\cdot\|$ is the usual operator norm on $\mathcal{L}(H)$. Proposition B.8.1 of [49] states that

$$\mathbb{E}\left[\exp\left(-\frac{1}{2}\delta(K)\right)\right] \leqslant (\mathbb{E}[\exp(\|K\|^2_{\mathcal{H}})])^{\frac{1}{4}} \cdot \left(\mathbb{E}\left[\exp\left(\frac{\|\nabla K\|_2^2}{(1-\|\nabla K\|)}\right)\right]\right)^{\frac{1}{4}}$$

whenever K is a $H - C^1$ map such that $\|\nabla K\| < 1$. Taking $K = 2tU^\chi$ in the previous inequality we obtain the thesis. □

Lemma 7 *The function* $G_0^{\chi,L}(t)$ *is real analytic in* $\left(-\frac{m^2}{2\|\partial^2 V_B\|_\infty}, \frac{m^2}{2\|\partial^2 V_B\|_\infty}\right)$.

Proof First of all we have that for any $t \in \mathbb{R}$ the map $r \to \det_2(I + \nabla U^\chi_{t+r,0}) =: D_t(r)$ is holomorphic in r (see [47] Theorem 9.3). By Cauchy theorem this means that

$$|\partial_t^n(\det_2(I + \nabla U^\chi_{t,0}))| \leqslant \frac{n! \sup_{\theta \in \mathbb{S}^1} |D_t(Re^{i\theta})|}{R^n}.$$

On the other hand we have for any $\chi \in [0,1]$

$$|D_t(r)| \leqslant \exp\left(\frac{1}{2}\|\nabla U^\chi_{t,0} + \nabla U^\chi_{r,0}\|_2^2\right) \leqslant \exp(C(t^2 + |r|^2)\|\partial^2 V_B\|^2_\infty),$$

where $C \in \mathbb{R}_+$ is some positive constant depending on f but not on V_B. Thus we obtain

$$|\partial_t^n(\det_2(I + \nabla U^\chi_{t,0}))| \leqslant \frac{n! \exp(C(t^2 + |R|^2)\|\partial^2 V_B\|^2_\infty)}{R^n}.$$

With a similar reasoning we obtain a uniform bound of this kind for $\partial_t^n \exp\left(-\frac{1}{2}|U^\chi_{t,0}|^2\right)$.

Finally we note that

$$\mathbb{E}[\exp(-\delta(U^\chi_{t+r,0}))] = \sum \frac{(-1)^n r^n}{n!} \mathbb{E}[\exp(-\delta(U^\chi_{t,0}))(\delta(U^\chi_{1,0}))^n].$$

By Lemma 6, we note that

$$\begin{aligned} |\mathbb{E}[\partial_t^n \exp(-\delta(U^\chi_{t+r,0}))]| &= |\mathbb{E}[\exp(-\delta(U^\chi_{t+r,0}))(\delta(U^\chi_{1,0}))^n]| \\ &\leqslant \frac{1}{\epsilon^n}\mathbb{E}[\exp(-\delta(U^\chi_{t+\epsilon})) + \exp(-\delta(U^\chi_{t-\epsilon}))] < +\infty \end{aligned}$$

for any $t \in \left(-\frac{m^2}{2\|\partial^2 V_B\|_\infty}, \frac{m^2}{2\|\partial^2 V_B\|_\infty}\right)$ and $0 < \epsilon < \frac{m^2}{2\|\partial^2 V_B\|_\infty} - |t|$. Using the previous inequality it follows that $G_0^{\chi,L}(t)$ is real analytic in the required interval. □

Lemma 8 *For $\chi \in (0,1)$ we have that $G_0^{\chi,L}(t) = H_0^{\chi,L}(t)$ for $t \in \left(-\frac{m^2}{2\|\partial^2 V_B\|_\infty}, \frac{m^2}{2\|\partial^2 V_B\|_\infty}\right)$.*

Proof By Lemma 7 the function $G_0^{\chi,L}(t)$ is real analytic in $\left(-\frac{m^2}{2\|\partial^2 V_B\|_\infty}, \frac{m^2}{2\|\partial^2 V_B\|_\infty}\right)$. It is enough then to prove $\partial_t^n G_0^{\chi,L}(0) = \partial_t^n H_0^{\chi,L}(0)$ for any $n \in \mathbb{N}$. By the discussion in Section 4.1 we note that

$$\begin{aligned} G_0^{\chi,L}(t) &= \int L(\mathcal{J}_\chi w(0)) \det{}_2(I_{\mathcal{H}} + \nabla U_{t,0}^\chi) \exp\left(-\delta(U_{t,0}^\chi) - \frac{1}{2}\|U_{t,0}^\chi\|_{\mathcal{H}}^2\right) \times \\ &\quad \times \exp\left(4\int V_{t,0}(\mathcal{J}_\chi w(x)) f'(x) \mathrm{d}x\right) \mathrm{d}\mu \\ &= \left\langle L(\varphi(0)) \exp\left(t\int \psi^i(x) f(x) \partial^2_{\phi^i\phi^j} V(\varphi(x)) \bar{\psi}^j(x) \mathrm{d}x\mathrm{d}\theta\mathrm{d}\bar{\theta}\right) \times \right.\\ &\quad \left. \times \exp\left(t\int f(x)\partial_{\phi^i} V(\varphi(x))\omega^i(x)\mathrm{d}x + 4t\int V(\varphi(x)) f'(x)\mathrm{d}x\right)\right\rangle_\chi \\ &= \left\langle \left(-\int L(\Phi(x,\theta,\bar{\theta}))\theta\bar{\theta}\delta_0(\mathrm{d}x)\mathrm{d}\theta\mathrm{d}\bar{\theta}\right) \times \right. \\ &\quad \left. \times \exp\left(-t\int V(\Phi(x,\theta,\bar{\theta}))\tilde{f}(|x|^2 + 4\theta\bar{\theta})\mathrm{d}x\mathrm{d}\theta\mathrm{d}\bar{\theta}\right)\right\rangle_\chi. \end{aligned}$$

From the previous equality we get

$$\begin{aligned} \partial_t^n G_0^{\chi,L}(0) &= \big\langle \left(-\textstyle\int L(\Phi(x,\theta,\bar{\theta}))\theta\bar{\theta}\delta_0(\mathrm{d}x)\mathrm{d}\theta\mathrm{d}\bar{\theta}\right) \times \\ &\quad \times \left(-\textstyle\int V(\Phi(x,\theta,\bar{\theta}))\tilde{f}(|x|^2 + 4\theta\bar{\theta})\mathrm{d}x\mathrm{d}\theta\mathrm{d}\bar{\theta}\right)^n \big\rangle_\chi \end{aligned}$$

Since the distribution $f(|x|^2 + \theta\bar{\theta})$ and $\theta\bar{\theta}\delta_0(\mathrm{d}x)$ are supersymmetric, by Theorem 12 we have

$$\begin{aligned} &\left\langle \left(-\textstyle\int L(\Phi(x,\theta,\bar{\theta}))\theta\bar{\theta}\delta_0(\mathrm{d}x)\mathrm{d}\theta\mathrm{d}\bar{\theta}\right)\left(-\textstyle\int V(\Phi(x,\theta,\bar{\theta}))\tilde{f}(|x|^2 + 4\theta\bar{\theta})\mathrm{d}x\mathrm{d}\theta\mathrm{d}\bar{\theta}\right)^n\right\rangle_\chi = \\ &= (-4\pi)^n \langle L(\varphi(0))(V(\varphi(0))^n)\rangle_\chi. \end{aligned}$$

And since

$$\partial_t^n H_0^{\chi,L}(0) = (-4\pi)^n \langle L(\varphi(0))(V(\varphi(0))^n)\rangle_\chi,$$

the thesis follows. □

Proposition 3 *We have that $G_0^L(t) = H_0^L(t)$ in $\left(-\frac{m^2}{2\|\partial^2 V_B\|_\infty}, \frac{m^2}{2\|\partial^2 V_B\|_\infty}\right)$.*

Proof We need only to prove that $G_0^{\chi,L}(t) \to G_0^{\chi,L}(t)$ as $\chi \to 0$. Since $\det_2$, δ, $|\cdot|_{\mathcal{H}}$ are continuous with respect to the natural norm of $\mathcal{H}$ and the Hilbert-Schmidt norm on $\mathcal{H}\otimes\mathcal{H}$ (see [47] Theorem 9.2 for the continuity of $\det_2$ and [40] Proposition 1.5.4 for the continuity of δ), and since $\exp(-\delta(U_{t,0}^\chi))$ is bounded uniformly in L^p (for p small enough) we only have to prove that, for $\chi \to 0$, $U_{1,0}^\chi(w) \to U_{1,0}(w)$ in $\mathcal{H}$ and $\nabla U_{1,0}^\chi(w) \to \nabla U_{1,0}(w)$ in $\mathcal{H}\otimes\mathcal{H}$ for almost every $w \in \mathcal{W}$. We present only the proof of the second convergence, the proof of the first one being simpler and similar.

We have that

$$\nabla U_{1,0}^\chi(w)[h] = (f\partial^2 V_B(\mathcal{J}_\chi w)\cdot \mathcal{J}_\chi h),$$

thus proving the convergence of $\nabla U^{\chi}_{1,0}(w)$ in $\mathcal{H}\otimes\mathcal{H}$ is equivalent to proving the convergence of $(m^2-\Delta)^{-1-\chi}$ to $(m^2-\Delta)^{-1}$ in $\mathcal{H}\otimes\mathcal{H}$ and the convergence of $f\partial^2 V_B(\mathcal{J}_\chi w)$ to $f\partial^2 V_B(\mathcal{I}w)$ in $C^0(\mathbb{R}^2)$. The first convergence follows from direct computation using the Fourier transform of this operators. The second convergence follows from the fact that V_B is smooth with bounded derivatives, f decays exponentially at infinity and $\mathcal{J}_\chi w$ converges to $\mathcal{I}w$ pointwise and uniformly on compact sets since $(m^2-\Delta)^{-\chi}\to \mathrm{id}_{L^2}$, weakly as bounded operator on $L^2(\mathbb{R}^2)$ and $(m^2-\Delta)^{-1}$ is a compact operator from $L^2(\mathbb{R}^2)$ into $C^0_{\mathrm{loc}}(\mathbb{R}^2)$. □

Introduce the following equation for $\phi_t=\bar{\phi}_t+\mathcal{I}\xi$:

$$(m^2-\Delta)\bar{\phi}_t+tf\partial V_B(\bar{\phi}_t+\mathcal{I}\xi)=0. \tag{41}$$

Denote by λ_- the infimum on $y\in\mathbb{R}^n$ over the eigenvalues of the y dependent matrix $(\partial^2 V_B(y))$, and with λ_+ the supremum on $y\in\mathbb{R}^n$ over the eigenvalues of the same matrix.

For $t\in\left(-\frac{m^2}{|\lambda_+\wedge 0|},\frac{m^2}{|\lambda_-\wedge 0|}\right)$we have that equation (41) has an unique solution that, by the Implicit Function Theorem, is infinitely differentiable with respect to t when $V_B\in C^\infty(\mathbb{R}^n)$. Define the formal series

$$S_t(r):=\sum_{n\geqslant 1}\frac{\sup_{x\in\mathbb{R}^2}|\partial^n_t\bar{\phi}_t(x)|}{n!}r^n. \tag{42}$$

Lemma 9 *Suppose that V_B is a bounded function with all derivatives bounded such that*

$$\|\partial^n V_B\|_\infty\leqslant C^n n!,$$

where the norm is the one induced by the identification of $\partial^n V_B$ as a multilinear operator and for some $C\in\mathbb{R}_+$, then the r power series $S_t(r)$ is holomorphic for any $t\in\left(-\frac{m^2}{|\lambda_+\wedge 0|},\frac{m^2}{|\lambda_-\wedge 0|}\right)$. Furthermore the radius of convergence of $S_t(r)$ can be chosen uniformly for t in compact subsets of $\left(-\frac{m^2}{|\lambda_+\wedge 0|},\frac{m^2}{|\lambda_-\wedge 0|}\right)$.

Proof We define the following functions

$$\bar{V}^1(r):=\sum_{n\geqslant 0}\frac{\|\partial^{n+1}V\|_\infty}{n!}r^n,\qquad \bar{V}^2(r):=\sum_{n\geqslant 0}\frac{\|\partial^{n+2}V\|_\infty}{n!}r^n.$$

We have that the partial derivative $\partial^n_t\bar{\phi}_t$ solves the following equation

$$\begin{aligned}(m^2-\Delta)\partial^n_t\bar{\phi}+t\partial^2 V_B(\bar{\phi}_t)\cdot\partial^n_t\bar{\phi}_t &= -\partial^{n-1}_t(\partial V_B(\bar{\phi}_t)+t\partial^2 V_B(\bar{\phi}_t)\cdot\partial_t\bar{\phi}_t)+\\ &\quad +t\partial^2 V_B(\bar{\phi}_t)\cdot\partial^n_t\bar{\phi}_t\end{aligned}$$

Using a reasoning similar to the one of Lemma 1, it is easy to prove that

$$\|\partial^n_t\bar{\phi}_t\|_\infty\leqslant\frac{\|-\partial^{n-1}_t(\partial V_B(\bar{\phi}_t)+t\partial^2 V_B(\bar{\phi}_t)\cdot\partial_t\bar{\phi}_t)+t\partial^2 V_B(\bar{\phi}_t)\cdot\partial^n_t\bar{\phi}_t\|_\infty}{m^2-|t|(\lambda_{\mathrm{sign}(t)}\wedge 0)},$$

where it is important to note that the right hand side of the previous inequality depends only on the derivatives of order at most $n-1$. The previous inequality and the method of majorants (see [50]) of holomorphic functions permit to get the following differential inequality for $S_t(r)$

$$(m^2-|t|(\lambda_{\mathrm{sign}(t)}\wedge 0)-r\bar{V}^2(S_t(r)))\partial_r(S_t)(r)\leqslant\bar{V}^1(S_t(r)). \tag{43}$$

From the previous inequality we obtain that $S_t(r)$ is majorized by the holomorphic function $F_t(r)$ that is the solution of the differential equation (43) (where the symbol $\leqslant$ is replaced by $=$) depending parametrically on t with initial condition $F_t(0) = 0$. Since $F_t(r)$ is majorized by $F_k(r)$ or by $F_{-k}(r)$ if $|t| \leqslant k$ the thesis follows. $\square$

Remark 10 An example of potential satisfying the hypotheses of Lemma 9 is given by the family of trigonometric polynomials in $\mathbb{R}^n$.

Lemma 10 *Under the hypotheses of Lemma 9 and supposing that L is an entire function we have that $G_0^L(t) = H_0^L(t)$ for any $t \in \left(-\frac{m^2}{|\lambda_+ \wedge 0|}, \frac{m^2}{|\lambda_- \wedge 0|}\right)$. In other words the thesis of Theorem 9 holds if $\lambda = 0$, V_B satisfies Hypothesis C as well as the hypotheses of Lemma 9.*

Proof By Proposition 3 we need only to prove that G_0^L is real analytic in the required set. By Corollary 2 we have that

$$G_0^L(t) = \mathbb{E}\left[L(\mathcal{I}\xi(0) + \bar{\phi}_t(0))e^{4\int tV_B(\mathcal{I}\xi(x)+\bar{\phi}_t(x))f'(x)\mathrm{d}x}\right].$$

Then the thesis follows from Lemma 9 and the analyticity of L and of the exponential. $\square$

Proposition 4 *Under Hypothesis V_λ we have that $G_\lambda^L(t) = H_\lambda^L(t)$ for any $t \in \left(-\frac{m^2}{|\lambda_+ \wedge 0|}, \frac{m^2}{|\lambda_- \wedge 0|}\right)$. In other words the thesis of Theorem 9 holds if V satisfies Hypothesis C.*

Proof By Lemma 10 we know that Theorem 9 holds for any $\lambda = 0$ and for any bounded potential satisfying Hypothesis C and the hypothesis of Lemma 9. Thus if we are able to approximate any potential V satisfying Hypothesis V_λ and Hypothesis C by potentials of the form requested by Lemma 10 the thesis is proved.

We can use the methods of the proof of Lemma 4 for approximating a potential V satisfying Hypothesis V_λ by a sequence of potentials $V_{B,N}$ satisfying the hypothesis of Lemma 9. More in detail, using the notations of Lemma 4, we have that the sequence of functions V^N is composed by smooth, bounded functions and, if V satisfies Hypothesis V_λ, they are identically equal to N outside a growing sequence of squares $Q_N \subset \mathbb{R}^2$. This means that $V^{N,p}$, which is the periodic extension of V^N outside the square Q_N, is a smooth function for any $N \in \mathbb{N}$. Since $V^{N,p}$ is periodic it can be approximated with any precision we want by a trigonometric polynomial P^N. Furthermore since V satisfies Hypothesis C, also $V^{N,p}$ satisfies Hypothesis C and we can choose the trigonometric polynomial P^N satisfying Hypothesis C too. In this way we construct a sequence of potentials $V_{B,N} = P^N$ satisfying the hypotheses of Lemma 9 and converging to V uniformly on compact sets. Thus the thesis follows from Lemma 2, Lemma 3, Corollary 2 and the fact that the functions of the form $L(\mathcal{I}\xi(0) + \bar{\phi}_t(0))$, where L is an entire function, are dense in the set of measurable functions in $\mathcal{I}\xi(0) + \bar{\phi}_t(0)$ with respect to the $L^p(\mu)$ norm. $\square$

4.4 Extension

Lemma 11 *Under the Hypothesis V_λ we have $\det_2(I + \nabla U(w)) \in L^\infty(\mu)$.*

Proof We follow the same reasoning proposed in [32] for polynomials. First of all, by the invariance property of the determinant with respect to conjugation, we have that

$$\det_2(I + \nabla U(w)) = \det_2(I + O(w))$$

where $O(w)$ is the selfadjoint operator given by

$$O^i_j(w)[h] = (m^2 - \Delta)^{-\frac{1}{2}} (f \partial^2_{\phi^i \phi^j} V(\mathcal{I} w) \cdot (m^2 - \Delta)^{-\frac{1}{2}} h). \tag{44}$$

Since V satisfies the Hypothesis QC the eigenvalues of the symmetric matrix $\partial^2 V(y)$ (where $y \in \mathbb{R}^n$) are bounded from below. Furthermore we can write the matrix $\partial^2 V(y)$ as the difference of two commuting matrices $\partial^2 V(y) = V_+(y) - V_-(y)$ where $V_+(y), V_-(y)$ are symmetric, they have only eigenvalues greater or equal to zero and $\ker V_+(y) \cap \ker V_-(y) = \{0\}$. We denote by O^+, O^- the two operators defined as O in equation (44) replacing $\partial^2 V$ by V_+ and V_- respectively. Obviously O^+ and O^- are positive definite and $O = O^+ - O^-$. By Lemma 3.3 [32] we have that

$$|\det{}_2(I + O(w))| \leqslant \exp(2\|O^-(w)\|_2^2).$$

Using a reasoning similar to the one of Proposition 1 and the fact that, under the Hypothesis V_λ, the minimum eigenvalue $\lambda(y)$ of $\partial^2 V(y)$ has a finite infimum λ_- that is the same as the one for V_- we obtain

$$|\det{}_2(I + \nabla U(w))| = |\det{}_2(I + O(w))| \leqslant \exp(C\lambda_0 \|f\|_2^2)$$

for some positive constant C. In particular we have $\det_2(I + \nabla U(w)) \in L^\infty$. □

In order to prove that $\exp(-\delta(U)) \in L^p$ we split U into two pieces. First of all if $\lambda(y)$ is the minimum eigenvalue of $\partial^2 V(y)$ we recall that $\lambda_- = \inf_{y \in \mathbb{R}^n} \lambda(y)$. Moreover we shall set

$$\bar{U} := U - (\lambda_- \wedge 0) f \mathcal{I}(w),$$

and $\hat{U} := U - \bar{U}$. We also set $W := V + \frac{\lambda_-}{2}|y|^2$. We introduce a useful approximation of $\bar{U}(w)$ for proving Theorem 15. Let P_n the projection of an $L^2(\mathbb{R}^2)$ function on the momenta less then n, i.e.

$$P_n(h) = \int_{|k|<n} e^{ik\cdot x} \hat{h}(k) \mathrm{d}k,$$

where $\hat{h}$ is the Fourier transform of h defined on $\mathbb{R}^2$. We can uniquely extend the operator P_n to all tempered distributions. In this way we define $U_n(w)$ as

$$U_n(w) := P_n[f \partial V(\mathcal{I} P_n w)] \tag{45}$$

We shall denote by $\bar{U}_n$ the expression corresponding to (45) where V is replaced by W.

Lemma 12 *Under the Hypothesis V_λ there exist two positive constants C, α independent on $p \geqslant 2$ and $n \in \mathbb{N}$ such that*

$$\mathbb{E}[|\delta(\bar{U}_n - \bar{U})|^p] \leqslant C(p-1)^{2p} n^{-\alpha}. \tag{46}$$

Furthermore a similar bound holds also for $\mathbb{E}[|\|\nabla U_n\|_2^2 - \|\nabla U\|_2^2|^p]$ and $\mathbb{E}[|\|\mathcal{I}w\|_{\mathcal{H}}^2 - \|P_n(\mathcal{I}w)\|_{\mathcal{H}}^2|^p]$.

Proof First of all we write $\bar{U} = U_B + \bar{U}_U$ where $U_B = f \partial V_B(\mathcal{I}w)$, and we consider the corresponding decomposition for $\bar{U}_n$. If we prove that an inequality analogous to (46) holds for $U_B - U_{B,n}$ and $\bar{U}_U - \bar{U}_{U,n}$ separately then the inequality (46) holds.

In order to prove the lemma we use the following inequality (proven in [49] Proposition B.8.1)

$$\mathbb{E}\left[\cosh\left(\frac{\sqrt{\rho}}{2\sqrt{2}}\delta(K)\right)\right] \leqslant (\mathbb{E}[\exp(\rho\|K\|_{\mathcal{H}}^2)])^{\frac{1}{4}} \cdot \left(\mathbb{E}\left[\exp\left(\frac{\rho}{1-\rho c}\|\nabla K\|_2^2\right)\right]\right)^{\frac{1}{4}} \tag{47}$$

that holds when $\|\nabla K\|_2^2 \in L^\infty$, $\|\nabla K\| \leqslant c < 1$ and $0 \leqslant \rho < \frac{1}{2c^2}$. Putting $K = \bar{\epsilon}(U_B - U_{B,n})$ for $\bar{\epsilon}$ small enough, since $\|\nabla(U_{B,n} - U_B)\|_2^2, \|\nabla(U_{B,n} - U_B)\| \in L^\infty$ with a bound uniform in n, we have that

$$\mathbb{E}[\cosh(\epsilon\delta(U_B - U_{B,n}))] \leqslant (\mathbb{E}[\exp(\epsilon'\|U_B - U_{B,n}\|_{\mathcal{H}}^2)])^{\frac{1}{4}} \cdot (\mathbb{E}[\exp(\epsilon'\|\nabla(U_B - U_{B,n})\|_2^2)]), \quad (48)$$

for suitable $\epsilon, \epsilon' > 0$ and for all $n \in \mathbb{N}$. First of all we want to give a bound for the right hand side of (48) providing a precise convergence rate to the constant 1 of the upper bound for the right hand side as $n \to +\infty$. We first note that

$$\mathbb{E}[\exp(\epsilon'\|U_B - U_{B,n}\|_{\mathcal{H}}^2)] = \sum_{k=1}^{\infty} \frac{\epsilon'^k}{k!} \mathbb{E}[\|U_B - U_{B,n}\|_{\mathcal{H}}^{2k}]. \quad (49)$$

Using a reasoning like the one in the proof of Proposition 1 we have that

$$\|U_B - U_{B,n}\|_{\mathcal{H}}^2 \lesssim \|\partial V_B\|_\infty^2 \|Q_n(f)\|_{\mathcal{H}}^2 + \|\partial^2 V_B\|_\infty^2 \int_{\mathbb{R}^2} (f(x) Q_n(\mathcal{I}w)(x))^2 \mathrm{d}x,$$

where $Q_n = I - P_n$. From the previous inequality and the hypercontractivity of Gaussian random fields we obtain that

$$\begin{aligned} \mathbb{E}[\|U_B - U_{B,n}\|_{\mathcal{H}}^{2k}] &\lesssim k \left(\|Q_n(f)\|_{\mathcal{H}}^{2k} + \int_{\mathbb{R}^2} f(x)^k \mathbb{E}[(Q_n(\mathcal{I}w)(x))^2] \mathrm{d}x \right) \\ &\lesssim k\|Q_n(f)\|_{\mathcal{H}}^{2k} + k(2k-1)^k \|f^k\|_1 \mathbb{E}[(Q_n(\mathcal{I}w)(x))^2]^k, \end{aligned}$$

where the constants implied by the symbol $\lesssim$ do not depend on k. The right hand side converges then for $n \to +\infty$ to 1 as we have announced. Using the Fourier transform, the fact that f is a Schwartz function, and the fact that $\mathcal{I}w$ is equivalent to a white noise transformed by the operator $(m^2 - \Delta)$ it is simple to obtain that $\|Q_n(f)\|^2, \mathbb{E}[(Q_n(\mathcal{I}w)(x))^2] \lesssim \frac{1}{n^2}$. Then using the fact that $(2k-1)^k \lesssim C_1^k k!$ and inserting the previous inequality in equation (49) we obtain

$$\mathbb{E}[\exp(\epsilon'\|U_B - U_{B,n}\|^2)] \leqslant 1 + C_3 \frac{\frac{\epsilon'}{n^2}}{\left(1 - \frac{C_2\epsilon'}{n^2}\right)^2},$$

that holds when $\epsilon' > 0$ is small enough and for two positive constants C_2, C_3. Using similar methods it is possible to prove a similar estimate for $\mathbb{E}[\exp(\epsilon'\|\nabla(U_B - U_{B,n})\|_2^2)]$. Inserting these estimates in the inequality (48), we obtain

$$\mathbb{E}[\cosh(\epsilon\delta(U_B - U_{B,n}))] - 1 \lesssim \frac{\epsilon'}{n^2}, \quad (50)$$

where the constants implied by the symbol $\lesssim$ do not depend on n and on ϵ', when ϵ' is smaller than a suitable $\epsilon'_0 > 0$. Using the inequality (50) we obtain that

$$\sum_{k,n=1}^{+\infty} \frac{n^{1/2}\epsilon^{2k}}{(2k)!} \mathbb{E}[(\delta(U_B - U_{B,n}))^{2k}] = \sum_{n=1}^{+\infty} n^{\frac{1}{2}} (\mathbb{E}[\cosh(\epsilon\delta(U_B - U_{B,n}))] - 1) \lesssim \sum_{n=1}^{\infty} \frac{\epsilon'}{n^{\frac{3}{2}}} < +\infty.$$

Since the terms of an absolutely convergent series are bounded we obtain

$$\mathbb{E}[(\delta(U_B - U_{B,n}))^{2k}] \lesssim \frac{(2k)!}{\epsilon^{2k} n^{\frac{1}{2}}} \lesssim (2k-1)^{4k} n^{-\frac{1}{2}}.$$

Using Young inequality we obtain that the inequality (46) holds for any $p \geqslant 2$. The estimate for $\delta(\bar{U}_U - \bar{U}_{U,n})$ follows from the fact that $\bar{U}_U$ is a polynomial of at most third degree and from hypercontractivity estimates for polynomial expressions of Gaussian random fields.

The result for $\|\nabla U\|_2^2 - \|\nabla U_n\|_2^2$ can be proved using the same decomposition of U and U_n and following a similar reasoning. The result for $\mathbb{E}[|\|f\mathcal{I}w\|_{\mathcal{H}}^2 - \|fP_n(\mathcal{I}w)\|_{\mathcal{H}}^2|^p]$ can be proved using hypercontractivity for polynomial expressions of Gaussian random fields. □

In the following we write $c_n = \mathrm{Tr}(P_n \circ \mathcal{I})$. It is important to note that

$$c_n = \int_{|x|<n} \frac{1}{|x|^2 + m^2} \mathrm{d}x \lesssim \log(n),$$

where the integral is taken on the ball $|x| < n$ on $\mathbb{R}^2$.

Lemma 13 *There exists a $\lambda_0 > 0$ depending only on f and m^2 such that for any $0 < \lambda < \lambda_0$ and V satisfying the Hypothesis V_λ there exist some constants $\alpha, C_1, C_2 > 0$ such that*

$$\delta(\bar{U}_n) - R \int_{\mathbb{R}^2} f(P_n \mathcal{I}w)^2 \mathrm{d}x - \|\nabla U_n\|_2^2 \geqslant -C_1 - C_2 c_n^\alpha$$

for any $R \in \mathbb{R}_+$.

Proof If $\mathrm{Tr}(|\nabla K|) < +\infty$ and $K \in \mathcal{W}$ we have that $\delta(K) = \langle K, w\rangle_{\mathcal{H}} - \mathrm{Tr}(\nabla K)$. Using this relation we obtain that

$$\delta(\bar{U}_k) = \sum_{i=1}^{n} \left(\int_{\mathbb{R}^2} P_k(f\partial_{\phi^i} W(P_k\mathcal{I}w))(x) w^i(x) \mathrm{d}x - \mathrm{Tr}_{L^2}(P_k(f\partial^2_{\phi^i\phi^i} W(P_k\mathcal{I}w) \cdot P_k(m^2 - \Delta))) \right).$$

From this we obtain the lower bound

$$\begin{aligned}
\int_{\mathbb{R}^2} P_k(f\partial_{\phi^i} W(P_k\mathcal{I}w)) w^i \mathrm{d}x &= \int_{\mathbb{R}^2} f\partial_{\phi^i} W(P_k\mathcal{I}w)(m^2 - \Delta)(P_k\mathcal{I}w^i)\mathrm{d}x \\
&= \int_{\mathbb{R}^2} f\partial_{\phi^i} W(\mathcal{I}w_k)(m^2 - \Delta)(\mathcal{I}w_k^i)\mathrm{d}x \\
&= \int_{\mathbb{R}^2} f\partial_{\phi^i\phi^r} W(\mathcal{I}w_k)\nabla\mathcal{I}w_k^i \cdot \nabla\mathcal{I}w_k^r \mathrm{d}x + \\
&\quad - \int_{\mathbb{R}^2} (\Delta f) W(\mathcal{I}w_k)\mathrm{d}x + m^2 \int_{\mathbb{R}^2} f\mathcal{I}w_k^i \partial_{\phi^i} W(\mathcal{I}w_k)\mathrm{d}x \\
&\geqslant \int_{\mathbb{R}^2} f(m^2\mathcal{I}w_k^i\partial_{\phi^i} W(\mathcal{I}w_k) - b^2 W(\mathcal{I}w_k))\mathrm{d}x
\end{aligned}$$

On the other hand we have

$$\begin{aligned}
\mathrm{Tr}_{L^2}(P_k(f\partial^2_{\phi^i\phi^i} W(\mathcal{I}w_k) \cdot P_k(m^2 - \Delta))) &= c_n \int_{\mathbb{R}^2} \partial^2_{\phi^i\phi^i} W(\mathcal{I}w_k) f \mathrm{d}x \\
&\leqslant \frac{c_n^p}{p} + \frac{1}{q} \int_{\mathbb{R}^2} (\partial^2_{\phi^i\phi^i} W(\mathcal{I}w_k)(\mathcal{I}w_k))^q f \mathrm{d}x,
\end{aligned}$$

where $\frac{1}{q} + \frac{1}{p} = 1$ and $q < 2$. Furthermore we have that

$$\|\nabla U_k\|_2^2 \leqslant \int_{\mathbb{R}^2} \frac{1}{(|x|^2 + m^2)^2} \mathrm{d}x \int_{\mathbb{R}^2} (\partial^2_{\phi^i\phi^i} V(\mathcal{I}w_k))^2 f \mathrm{d}x = \ell \int_{\mathbb{R}^2} (\partial^2_{\phi^i\phi^i} V(\mathcal{I}w_k))^2 f \mathrm{d}x,$$

where $\ell = \int_{\mathbb{R}^2} \frac{1}{(|x|^2+m^2)^2} \mathrm{d}x$. Using the previous inequality we obtain that

$$\begin{aligned}
&\delta(\bar{U}_n) - R \int_{\mathbb{R}^2} f|\mathcal{I}w_k|^2 \mathrm{d}x - \|\nabla U_n\|_2^2 \\
&\geqslant -\frac{c_n^p}{p} + \int_{\mathbb{R}^2} f(m^2 \mathcal{I}w_k^i \partial_{\phi^i} W(\mathcal{I}w_k) - b^2 W(\mathcal{I}w_k)) \mathrm{d}x + \\
&\quad - \int_{\mathbb{R}^2} f \left(\frac{(\partial^2_{\phi^i\phi^i} W(\mathcal{I}w_k))^q}{q} + \ell(\partial^2_{\phi^i\phi^i}(V)(\mathcal{I}w_k))^2 + R|\mathcal{I}w_k|^2 \right) \mathrm{d}x
\end{aligned}$$

It is simple to see that there exists a $\lambda_0 > 0$ (depending only on b^2 and m^2) such that for any potential V satisfying the Hypothesis V_λ with $\lambda < \lambda_0$ the expression

$$m^2 y_k^i \partial_{\phi^i} W(y) - b^2 W(y) - \frac{(\partial^2_{\phi^i\phi^i} W(y))^q}{q} - \ell(\partial^2_{\phi^i\phi^i} V(\mathcal{I}w_k))^2 - R|y|^2$$

is bounded from below and thus the thesis of the lemma holds. □

Lemma 14 *Given a* $p \in [1, +\infty)$*there is a* $R > 0$ *big enough such that*

$$\exp\left(-\delta(\hat{U}) - R \int_{\mathbb{R}^2} f(x) |\mathcal{I}w(x)|^2 \mathrm{d}x \right) \in L^p(\mu).$$

Proof This lemma is proven in [32] Lemma 3.2. □

Lemma 15 *Suppose that* f *satisfies the Hypotheses CO, then there exists* $\lambda_0 > 0$ *depending only on* f *and* m^2 *such that for any* $\lambda < \lambda_0$ *and any* V *satisfying the Hypothesis* V_λ *we have that*

$$\exp(-\delta(U) + (1 + \|\nabla U\|_2^2)) \in L^p(\mu)$$

for any $p \in [1, +\infty)$.

Proof The thesis follows from Lemma 11, Lemma 12, Lemma 13 and Lemma 14 using a standard reasoning due to Nelson (see Lemma V.5 of [46]) due to the fact that from the previous results it follows that there exist two constants $\alpha, \beta > 0$ independent on N such that

$$\mu(\{w \in \mathcal{W} | \delta(U^N)(w) \geqslant \beta(\log(N))\}) \leqslant e^{-N^\alpha}.$$

□

Proof of Theorem 9 By Proposition 4 in order to prove the theorem it remains only to prove that $G^L_\lambda(t)$ is real analytic for any $t \in \mathbb{R}$. The proof of this fact easily follows from Lemma 15 exploiting a reasoning similar to the one used in Lemma 7. □

5 Removal of the spatial cut-off

In this section we prove Theorem 3 on the removal of the spatial cut-off in the setting of Hypothesis C. It is important to note that, differently from Theorem 8, we explicitly require that the potential V satisfies Hypothesis C and not only Hypothesis QC. This is not due to problems in proving the existence of solutions to equation (10) or in proving the convergence of the cut-offed solution to the non-cut-offed one without the Hypothesis C (see Lemma 16). The main difficulty is instead to prove the convergence of $\Upsilon_f(\phi)/Z_f$ to 1. Indeed the previous factor does not actually converge and what we can reliably expect is that

$$\lim_{f\to 1} Z_f^{-1}\mathbb{E}[\Upsilon_f(\phi_f)|\sigma(\phi_f(0))] \to 1, \tag{51}$$

where hereafter ϕ_f denotes the solution to the equation (6) with cut-off f, i.e. $\Upsilon_f(\phi_f)/Z_f$ becomes independent with respect to the σ-algebra generated by $\phi_f(0)$.

To prove (51) directly is quite difficult due to the non-linearity of the equation or equivalently to the presence of the regularized Fredholm determinant in the expressions (23) and (22) (which is a strongly non-local operator). For this reason we want to exploit a reasoning similar to the one used in Section 4. With this aim we introduce the equation

$$(m^2 - \Delta)\phi_{f,t} + tf\partial V(\phi_{f,t}) = \xi \tag{52}$$

and the functions

$$F_f^L(t) := Z_f^{-1}\mathbb{E}\left[L(\phi_{f,t}(0))e^{4t\int_{\mathbb{R}^2} f'(x)V(\phi_{f,t}(x))\mathrm{d}x}\right],$$

where t is taken such that $t\partial^2 V(y) + m^2$, and $F^L(t) = \mathbb{E}[L(\phi_t(0))]$ (where ϕ_t is the solution to (52) with $f \equiv 1$). By Lemma 9 (whose proof does not use in any point the cut-off f) $F^L(t)$ is real analytic whenever V is a trigonometric polynomial, $t\partial^2 V(y) + m^2$ is definite positive for any $y \in \mathbb{R}^n$ and L is an entire bounded function. Furthermore, by Theorem 8, $F_f^L(t) = H^L(t)$ (where $H^L(t) = \int L(y)\mathrm{d}\kappa_t(y)$, see Section 4) which is real analytic. Thus if we are able to prove that $\lim_{f\to 1} \partial_t^n F_f^L(0) = \partial_t^n F^L(0)$ we have that $H^L(t) = F^L(t)$ whenever $t\partial^2 V + m^2$ is definite positive proving that Theorem 3 when V is a trigonometric polynomial satisfying Hypothesis C. The idea, then, is to apply a generalization of Lemma 2, Lemma 3, Lemma 4 and the reasoning in the proof of Proposition 4 and in the proof of Theorem 8 in order to obtain Theorem 3.

Remark 11 Hypothesis C is required in an essential way in the proof of the holomorphy of $F^L(t)$, in particular in Lemma 9. The fact that the cutoff is removed does not allow to reason by approximation as we did in Theorem 9.

Since the proof is composed by many steps which are a straightforward generalization of the results of the previous sections of the paper, we write here only some details of the parts of the proof of Theorem 3 which largely differ from what has been obtained before.

Hereafter we denote by $\omega_\beta(x)$ the function

$$\omega_\beta(x) := \exp(-\beta\sqrt{(1 + |x|^2)})$$

and introduce the space $\mathcal{W}_\beta$ where $\beta > 0$ in the following way

$$\mathcal{W}_\beta := (-\Delta + 1)C^0_{\exp\beta}(\mathbb{R}^2; \mathbb{R}^n),$$

where $C^0_{\exp\beta}$ is the space of continuous function with respect to the weighted L^∞ norm

$$\|g\|_{\infty,\exp\beta} := \sup_{x\in\mathbb{R}^2} |\omega_\beta(x)g(x)|.$$

The triple $(\mathcal{W}_\beta, \mathcal{H}, \mu)$ is an abstract Wiener space. We introduce the obvious generalization of equation (15)

$$(m^2 - \Delta)\bar{\phi} + \partial V(\bar{\phi} + \mathcal{I}\xi) = 0, \tag{53}$$

where $\bar{\phi} = \phi - \mathcal{I}\xi$.

Now we want to prove a result that can replace Lemma 1. Indeed Lemma 1 plays a central role in the previous sections of the paper, allowing to prove the existence of strong solutions to equation (6), the characterization of weak solutions in Theorem 6 and Theorem 7 and finally allowing to show the convergence of weak solutions using the convergence of potentials in Lemma 2, Lemma 3.

Lemma 16 *Suppose that V satisfies the Hypothesis QC and suppose that $\bar{\phi}$ is a classical solution to equation (53), then there exists a β_0 depending only on m^2 such that, for any $\beta < \beta_0$*

$$\|\bar{\phi}\|_{\infty,\exp\beta} \lesssim \|\exp(\alpha|\mathcal{I}\xi|)\|_{\infty,\exp\beta}, \tag{54}$$

where $\|\exp(\alpha|\mathcal{I}\xi|)\|_{\infty,\exp\beta}$ is almost surely finite and the constants implied by the symbol $\lesssim$ depend only on H and m^2. Furthermore for any U open and bounded we have

$$\|\bar{\phi}\|_{\mathcal{C}^{2-\tau}(U)} \lesssim \|\exp(\alpha p|\mathcal{I}\xi|)\|_{U_\epsilon,\infty}\exp(\alpha p\|\bar{\phi}\|_{\infty,\exp\beta}\|\omega_\beta^{-1}\|_{U_\epsilon,\infty}) \tag{55}$$

where $U_\epsilon := \{x \in \mathbb{R}^2 | \exists y \in U, |y - x| \leqslant \epsilon\}$ and $\epsilon > 0$.

Proof The proof is very similar to the proof of Lemma 1. We report here only the passages having the main differences. For any $\epsilon > 0$ there is a $\beta_\epsilon > 0$ and for any $\beta < \beta_\epsilon$ we have

$$\left|\frac{\Delta(\omega_{2\beta}(x))}{\omega_{2\beta}(x)} - \frac{|\nabla\omega_{2\beta}(x)|^2}{\omega_{4\beta}(x)}\right| < \epsilon, \qquad x \in \mathbb{R}^2.$$

Without loss of generality (using the result of Lemma 1) we have that $\lim_{x\to\infty}|\bar{\phi}(x)|^2\omega_{2\beta}(x) = 0$ and so $x \mapsto |\bar{\phi}(x)|^2\omega_{2\beta}(x)$ has a positive maximum at $\bar{x} \in \mathbb{R}^2$. This means that $-\Delta(|\bar{\phi}|^2\omega_{2\beta})(\bar{x}) \geqslant 0$ and $\nabla\bar{\phi} = -\frac{\bar{\phi}}{2\omega_{2\beta}}\nabla\omega_{2\beta}$ we have that

$$\begin{aligned}(m^2 - \epsilon)|\bar{\phi}(\bar{x})|^2\omega_{2\beta}(\bar{x}) &\leqslant \frac{-\Delta(|\bar{\phi}|^2\omega_{2\beta})(\bar{x})}{2} + m^2|\bar{\phi}(\bar{x})|^2\omega_{2\beta}(\bar{x}) \\ &\leqslant -\omega_{2\beta}(\bar{x})(\bar{\phi}(\bar{x})\cdot\partial V(\mathcal{I}\xi(\bar{x}) + \bar{\phi}(\bar{x}))).\end{aligned}$$

Using a reasoning similar to the one of Lemma 1 the thesis follows. □

Since the bounds (54) and (55) in $C^0_{\exp\beta}$ and $\mathcal{C}^{2-\tau}_{\text{loc}}$ imply the compactness in $C^0_{\exp\beta'}$ when $\beta' < \beta$, Lemma 16 permits to prove the existence of strong solutions to equation (10), their uniqueness when V satisfies Hypothesis C and the generalization of Lemma 2, Lemma 3, Lemma 4, Proposition 4 and Theorem 8 needed in order to prove Theorem 3.

At this point the proof of Theorem 3 requires only the following additional statement.

Theorem 13 *Let V be a trigonometric polynomial, let L be a polynomial and let f_r be a sequence of cut-offs satisfying Hypothesis CO, such that $f_r \equiv 1$ on the ball of radius $r \in \mathbb{N}$ and such that $f_r'(x) \leqslant C_1\exp(-C_2(|x| - r))$ for some positive constants $C_1, C_2 \in \mathbb{R}_+$ independent on r, then*

$$\partial_t^k H^L(0) = \lim_{r\to+\infty}\partial_t^k F^L_{f_r}(0) = \partial_t^k F^L(0).$$

To make the proof easy to follow we restrict ourselves to the scalar case, i.e. the case where $n = 1$. The general case is a straightforward generalization. We will also need certain results about iterated Gaussian integrals. So let us introduce first some notations.

We denote by $\mathcal{T}$ the set of rooted trees with at least a external vertex which is not the root. We denote by τ_0 the tree with only one vertex other than the root. In this set we introduce two operations: if $\tau \in \mathcal{T}$ we denote by $[\tau]$ the tree obtained from τ by adding a new vertex at the root which becomes the new root, and if $\tau' \in \mathcal{T}$ we denote by $\tau \cdot \tau'$ the tree obtained by identifying the root of τ and τ'. It is possible to obtain any element of $\mathcal{T}$ by applying iteratively a finite number of times the previous operations to τ_0. Furthermore we define $\mathcal{I}^f_\tau(x) \in C^0(\mathbb{R}^2)$ by induction in the following way

$$\mathcal{I}^f_{\tau_0}(x) := \mathcal{I}\xi, \qquad \mathcal{I}^f_{[\tau]}(x) := \int_{\mathbb{R}^2} \mathcal{G}(x-y) f(y) \mathcal{I}^f_\tau(y) \mathrm{d}y, \qquad \mathcal{I}^f_{\tau\cdot\tau'}(x) := \mathcal{I}^f_\tau(x) \cdot \mathcal{I}^f_{\tau'}(x),$$

where $\mathcal{G}(x)$ is the Green function of the operator $\mathcal{I} = (m^2 - \Delta)^{-1}$. We need also to introduce the following notation. Suppose that $\tau, \tau' \in \mathcal{T}$ and let $\mathcal{P}_{\tau,\tau'}$ be the set of all possible pairing between the external vertices (excepted their roots) of the forest $\tau \sqcup \tau'$ and let $\mathcal{P}^{\mathrm{int}}_{\tau,\tau'} \subset \mathcal{P}_{\tau,\tau'}$ the set of all possible pairing involving separately the vertices of τ and τ'. If $\pi \in \mathcal{P}$ we write

$$\mathfrak{I}^{\pi,f}_{\tau,\tau'}(x,y) = \mathbb{E}[\widetilde{\mathcal{I}}^{\pi,f}_\tau(x) \cdot \widetilde{\mathcal{I}}^{\pi,f}_{\tau'}(y)],$$

where $\widetilde{\mathcal{I}}^{\pi,f}_\tau(x), \widetilde{\mathcal{I}}^{\pi,f}_{\tau'}(y)$ are the expression $\mathcal{I}^f_\tau(x)$ where ξ is replaced by some copies of Gaussian white noises ξ_V one for each vertex V of τ and τ' which have correlation 0 if $(V,V') \not\in \pi$ and are identically correlated otherwise.

Lemma 17 *With the notations and the hypotheses of Theorem 13 we have that for any $\tau, \tau' \in \mathcal{T}$*

$$\lim_{r\to+\infty} \left(\mathbb{E}\left[\mathcal{I}^{f_r}_\tau(0) \cdot \prod_{i=1}^p \int f'_r(x) \mathcal{I}^{f_r}_{\tau_i}(x) \mathrm{d}x \right] - \mathbb{E}[\mathcal{I}^{f_r}_\tau(0)] \cdot \mathbb{E}\left[\prod_{i=1}^p \int f'_r(x) \mathcal{I}^{f_r}_{\tau_i}(x) \mathrm{d}x \right] \right) = 0.$$

Proof We present the proof only for the case $p = 1$, since the general case is a straightforward generalization. Since $\mathcal{I}^{f_r}_\tau$ are Gaussian random variables depending polynomially with respect to the white noise ξ, using the notation previously introduced we have

$$\mathbb{E}\left[\mathcal{I}^{f_r}_\tau(0) \cdot \int f'_r(x) \mathcal{I}^{f_r}_{\tau'}(x) \mathrm{d}x \right] - \mathbb{E}[\mathcal{I}^{f_r}_\tau(0)] \cdot \mathbb{E}\left[\int f'_r(x) \mathcal{I}^{f_r}_{\tau'}(x) \mathrm{d}x \right] =$$

$$= \sum_{\pi \in \mathcal{P}_{\tau,\tau'} \setminus \mathcal{P}^{\mathrm{int}}_{\tau,\tau'}} \int_{\mathbb{R}^2} \mathfrak{I}^{\pi,f_r}_{\tau,\tau'}(0,x) f'_r(x) \mathrm{d}x.$$

Let us consider the simplest case when $\tau = \tau_k := [\ldots[\tau_0]\ldots]$ k times and $\tau' = \tau_{k'} = [\ldots[\tau_0]\ldots]$ k' times. In this case we have

$$\mathfrak{I}^{\pi,f_r}_{\tau,\tau'}(0,x) = \int \mathcal{G}(0-y_1) f_r(y_1) \ldots \mathcal{G}(y_k - x_1) \mathcal{G}(x_1 - x_2) \times$$

$$\times f_r(x_2) \ldots f_r(x_{k'}) \mathcal{G}(x_{k'} - x) \mathrm{d}y_1 \cdots \mathrm{d}y_k \mathrm{d}x_1 \cdots \mathrm{d}x_k.$$

In particular, since $\mathcal{C}(x) = \mathcal{G} * \mathcal{G}$, which is the Green function of $\mathcal{I}^2 = (m^2 - \Delta)^{-2}$, is bounded and positive, and since $\mathcal{G}$ is positive we obtain that

$$|\mathfrak{I}^{\pi,f_r}_{\tau,\tau'}(0,x)| \leqslant \underbrace{\mathcal{G} * \ldots * \mathcal{G}}_{k+k' \quad \text{times}} (0-x) = \int_{\mathbb{R}^2} \frac{e^{-il\cdot x}}{(|l|^2 + m^2)^{k+k'}} \mathrm{d}l.$$

Thus we get

$$|\mathfrak{I}^{\pi,f_r}_{\tau,\tau'}(0,x)|\cdot(|x|^2+1)\leqslant\left|\int_{\mathbb{R}^2}(-\Delta_l+1)\frac{e^{-il\cdot x}}{(|l|^2+m^2)^{k+k'}}\mathrm{d}l.\right|\leqslant C_3,$$

where $C_3\in\mathbb{R}_+$. Thus

$$\int_{\mathbb{R}^2}\mathfrak{I}^{\pi,f_r}_{\tau,\tau'}(0,x)f'_r(x)\mathrm{d}x\leqslant\int_{B_r^c}\frac{C_3}{(|x|^2+1)}C_1\exp(-C_2(|x|-r))\mathrm{d}x\lesssim_{C_1,C_2,C_3}\frac{1}{r^2+1}\to 0.$$

For the general case let us note that $\mathfrak{I}^{\pi,f_r}_{\tau,\tau'}(0,x)$ is built by taking the product and the convolution with the functions $\mathcal{G}, f_r$ and $\mathcal{C}=\mathcal{G}*\mathcal{G}$. We note that $\mathcal{C}$ appears one time for every pair of vertices $(V_1,V_2)\in\pi$. Then, since $\pi\not\in\mathcal{P}^{\text{int}}_{\tau,\tau'}$ there is at least a couple $(V,V')\in\pi$ such that V is a vertex of τ and V' is a vertex of τ'. Now we can bound the function $\mathcal{C}$ with a constant C_4 for all pairs of vertices $(V_1,V_2)\neq(V,V')$ and f_r by 1 obtaining, for any $x\in\mathbb{R}^2$, that

$$\mathfrak{I}^{\pi,f_r}_{\tau,\tau'}(0,x)\lesssim C_4^{k_1}\mathfrak{I}^{f_r}_{\tau_{k_2},\tau_{k3}}(0,x)$$

for some $k_1,k_2,k_3\in\mathbb{N}$. The thesis follows from the previous inequality and the bounds obtained on $\mathfrak{I}^{f_r}_{\tau_{k_2},\tau_{k3}}(0,x)$. □

Proof of Theorem 13 We write

$$\mathfrak{L}_{f_r}(t):=L(\phi_{f_r,t}(0))\qquad\mathfrak{E}_{f_r}(t):=\exp\left(4t\int_{\mathbb{R}^2}f'_r(x)V(\phi_{f_r,t}(x))\mathrm{d}x\right).$$

We have

$$\begin{aligned}\partial_t^kF^L_{f_r}(t) &= \sum_{0\leqslant l\leqslant k}\binom{k}{l}\mathbb{E}\left[\mathfrak{L}^{(k-l)}_{f_r}(0)\,\partial_t^l\left(\frac{\mathfrak{E}_{f_r}(t)}{\mathbb{E}[\mathfrak{E}_{f_r}(t)]}\right)\bigg|_{t=0}\right]\\ &= \mathbb{E}[\mathfrak{L}^{(k)}_{f_r}(0)]+\sum_{1\leqslant l\leqslant k}\sum_{0\leqslant p\leqslant l-1}\binom{k}{l}\binom{l}{p}(\mathbb{E}[\mathfrak{L}^{(k-l)}_{f_r}(0)\cdot\mathfrak{E}^{(l-p)}_{f_r}(0)]+\\ &\quad -\mathbb{E}[\mathfrak{L}^{(k-l)}_{f_r}(0)]\mathbb{E}[\mathfrak{E}^{(l-p)}_{f_r}(0)])\cdot\partial_t^p\left(\frac{1}{\mathbb{E}[\mathfrak{E}_{f_r}(t)]}\right)\bigg|_{t=0},\end{aligned}$$

where we used the Leibniz rule for the derivative of the product and the relation

$$\partial_t^l\left(\frac{1}{\mathbb{E}[\mathfrak{E}_{f_r}(t)]}\right)\bigg|_{t=0}=-\sum_{0\leqslant p\leqslant l-1}\binom{l}{p}\mathbb{E}[\mathfrak{E}^{(l-p)}_{f_r}(0)]\cdot\partial_t^p\left(\frac{1}{\mathbb{E}[\mathfrak{E}_{f_r}(t)]}\right)\bigg|_{t=0}.$$

Since $\partial_t^p\left(\frac{1}{\mathbb{E}[\mathfrak{E}_{f_r}(t)]}\right)\Big|_{t=0}$ is bounded from above and below when $r\to+\infty$ if we are able to prove that $\mathbb{E}[\mathfrak{L}^{(k)}_{f_r}(0)]\to\partial_t^kF^L(0)$ and $\mathbb{E}[\mathfrak{L}^{(k-l)}_{f_r}(0)\cdot\mathfrak{E}^{(l-p)}_{f_r}(0)]-\mathbb{E}[\mathfrak{L}^{(k-l)}_{f_r}(0)]\mathbb{E}[\mathfrak{E}^{(l-p)}_{f_r}(0)]\to 0$ the theorem is proven.

First of all we note that

$$(m^2-\Delta)\partial_t^k\phi_{f_r,t}|_{t=0}=kf_r\partial_t^{k-1}(V(\phi_{f_r,t}))|_{t=0}\tag{56}$$

for $k>0$ and $\phi_{f_r,0}=\mathcal{I}\xi$ for $k=0$. This means that $\mathfrak{L}^{(k-l)}_{f_r}(0),\mathfrak{E}^{(l-p)}_{f_r}(0)$ are given by a finite combination of convolutions and products between the function $\mathcal{G}$ (i.e. the Green function of $\mathcal{I}$), the functions of the form $V^{(l)}(\phi_{f_r,0})$ (where $V^{(l)}$ is the l-th derivative of V), the cut-off f_r and

f_r'. Since V is a trigonometric polynomial, by developing V and its derivative by Taylor series, we obtain the following formal expressions

$$\mathfrak{L}^{(k)}_{f_r}(0)=\sum_{\tau\in\mathcal{T}}A^k_\tau\mathcal{I}^{f_r}_\tau(0),\qquad \mathfrak{E}^{(k)}_{f_r}(0)=\sum_l\sum_{\tau_1,\dots,\tau_l\in\mathcal{T}}B^{k,l}_{(\tau_1,\dots,\tau_l)}\prod_{i=1}^l\int_{\mathbb{R}^2}f_r'(x)\mathcal{I}^{f_r}_{\tau_i}(x)\mathrm{d}x. \tag{57}$$

The previous series are not only formal but they are actually absolutely convergent series. Furthermore we can change the integral, the expectation and the limit with the series.

In order to prove this we now note that there exist two positive constants $C,\alpha>0$ such that the function V is majorized (in the meaning of the majorants method) by $C\exp(\alpha x)$ and let $\tilde{L}$ be the polynomial which majorizes the polynomial L. We now consider $\widetilde{\mathfrak{L}}_{f_r}(t)=\tilde{L}(\phi_{f_r,f}(0))$ and $\widetilde{\mathfrak{E}}_{f_r}(t)=\left(tC\int_{\mathbb{R}^2}f_r'\exp(\alpha\phi_{f_r,t}(x))\mathrm{d}x\right)$. For what we said, $\widetilde{\mathfrak{L}}^{(k)}_{f_r}(0)$ and $\widetilde{\mathfrak{E}}^{(p)}_{f_r}(0)$ are a finite combination of convolutions and products between $\mathcal{G}$, the functions of the form $V^{(l)}(\phi_{f_r,0})$ (where $V^{(l)}$ is the l-th derivative of V), the cut-off f_r and f_r'. Let $\widehat{\mathfrak{L}}^k_{f_r}$ and $\widehat{\mathfrak{E}}^k_{f_r}$ be some random variables having the same expression of $\widetilde{\mathfrak{L}}^{(k)}_{f_r}(0)$ and $\widetilde{\mathfrak{E}}^{(p)}_{f_r}(0)$ where we replace every appearance of $V(\phi_{f_r,0}(x))$ by $C\exp(\alpha|\phi_{f_r,0}(x)|)$, every appearance of $V'(\phi_{f_r,0}(x))$ with $C\alpha\exp(\alpha|\phi_{f_r,0}(x)|)$ and so on. We introduce the following functions dependent on $\tau\in\mathcal{T}$ and defined recursively as follows

$$\mathcal{J}^{f_r}_{\tau_0}(x):=|\mathcal{I}^{f_r}_{\tau_0}(x)|\qquad \mathcal{J}^{f_r}_{[\tau]}(x):=\int_{\mathbb{R}^2}\mathcal{G}(x-y)f_r(y)\mathcal{J}^{f_r}_\tau(y)\mathrm{d}y\qquad \mathcal{J}^{f_r}_{\tau\cdot\tau'}(x):=\mathcal{J}^{f_r}_\tau(x)\cdot\mathcal{J}^{f_r}_{\tau'}(x).$$

We, then, obtain that

$$\hat{\mathfrak{L}}^{(k)}_{f_r}=\sum_{\tau\in\mathcal{T}}\hat{A}^k_\tau\mathcal{J}^{f_r}_\tau(0)\qquad \hat{\mathfrak{E}}^{(k)}_{f_r}=\sum_l\sum_{\tau_1,\dots,\tau_l\in\mathcal{T}}\hat{B}^{k,l}_{(\tau_1,\dots,\tau_l)}\prod_{i=1}^l\int_{\mathbb{R}^2}f_r'(x)\mathcal{J}^{f_r}_{\tau_i}(x)\mathrm{d}x.$$

By our construction we have that $\hat{A}^k_\tau,\hat{B}^{k,l}_{\tau,i}$ are all greater or equal than zero and also the following inequalities hold $|A^k_\tau|\leqslant\hat{A}^k_\tau,|B^{k,l}_{(\tau_1,\dots,\tau_l)}|\leqslant\hat{B}^{k,l}_{(\tau_1,\dots,\tau_l)}$. Furthermore we have $|\mathcal{I}^{f_r}_\tau(x)|\leqslant\mathcal{J}^{f_r}_\tau(x)$. Finally $\mathbb{E}[|\hat{\mathfrak{L}}^{(k)}_{f_r}|^p],\mathbb{E}[|\hat{\mathfrak{G}}^{(k)}_{f_r}|^p]$ are finite for any p, since the $x_1,\dots,x_l$ function

$$\mathbb{E}\left[\exp\left(\beta\alpha\sum_{i=1}^l|\phi_{f_r,0}(x_i)|\right)\right]\leqslant+\infty,$$

for any $\beta>0$. Since $\mathcal{G}$ is positive the bounds on $\mathbb{E}[|\hat{\mathfrak{L}}^{(k)}_{f_r}|^p],\mathbb{E}[|\hat{\mathfrak{G}}^{(k)}_{f_r}|^p]$ can be chosen uniformly on r. This implies that the series (57) are absolutely convergent and by Lebesgue's dominate convergence theorem we can exchange the series with the summation and the limit. This means that

$$\begin{gathered}\lim_{r\to+\infty}\mathbb{E}[\mathfrak{L}^{(k)}_{f_r}(0)\cdot\mathfrak{E}^{(l)}_{f_r}(0)]-\mathbb{E}[\mathfrak{L}^{(k)}_{f_r}(0)]\mathbb{E}[\mathfrak{E}^{(l)}_{f_r}(0)]=\\ =\lim_{r\to+\infty}\sum_{l\in\mathbb{N}}\sum_{\tau,\tau_1,\dots,\tau_l\in\mathcal{T}}A^k_\tau B^{k,l}_{(\tau_1,\dots,\tau_l)}\left(\mathbb{E}\left[\mathcal{I}^{f_r}_\tau(0)\cdot\prod_{i=1}^l\int f_r'(x)\mathcal{I}^{f_r}_{\tau_i}(x)\mathrm{d}x\right]+\right.\\ \left.-\mathbb{E}[\mathcal{I}^{f_r}_\tau(0)]\cdot\mathbb{E}\left[\prod_{i=1}^l\int f_r'(x)\mathcal{I}^{f_r}_{\tau_i}(x)\mathrm{d}x\right]\right)=\\ =\sum_{l\in\mathbb{N}}\sum_{\tau,\tau_1,\dots,\tau_l\in\mathcal{T}}A^k_\tau B^{k,l}_{(\tau_1,\dots,\tau_l)}\lim_{r\to+\infty}\left(\mathbb{E}\left[\mathcal{I}^{f_r}_\tau(0)\cdot\prod_{i=1}^l\int f_r'(x)\mathcal{I}^{f_r}_{\tau_i}(x)\mathrm{d}x\right]+\right.\\ \left.-\mathbb{E}[\mathcal{I}^{f_r}_\tau(0)]\cdot\mathbb{E}\left[\prod_{i=1}^l\int f_r'(x)\mathcal{I}^{f_r}_{\tau_i}(x)\mathrm{d}x\right]\right)=0,\end{gathered}$$

where in the last line we used Lemma 17. In a similar way it is simple to prove that

$$\mathbb{E}[\mathfrak{L}^{(k)}_{f_r}(0)] \to \partial^k_t F^L(0),$$

and this concludes the proof. □

A Transformations in abstract Wiener spaces

This appendix summarizes the results of [49] which are used in the paper and establish the related notations. Hereafter we consider an abstract Wiener space (W, H, μ) where W is a separable Banach space, H is an Hilbert space densely and continuously embedded in W (with inclusion map denoted by $i : H \to W$) called Cameron-Martin space and μ is the Gaussian measure on W associated with the Cameron-Martin space, i.e. μ is the centered Gaussian measure on W such that for any $w^* \in W^*$ we have $\hat{\mu}(w^*) = \int_W \exp(i\langle w^*, w\rangle) \mathrm{d}\mu(w) = \exp\left(-\frac{\|i^*(w^*)\|^2_H}{2}\right)$ where $i^* : W^* \to H$ is the dual operator of i.

If $u : W \to \mathbb{R}$ is a measurable non-linear functional we denote by $\nabla u : W \to H$ the following non-linear operator

$$\nabla u(w)[h] = \langle \nabla u(w), h\rangle_H := \lim_{\epsilon\to 0} \frac{u(w + \epsilon h) - u(w)}{\epsilon}.$$

The operator ∇ is called Malliavin derivative and it is possible to prove that it is closable (with unique closure) on the set of measurable $L^p(\mu)$ functions. We denote the domain of ∇ in $L^p(\mu)$ by $\mathbb{D}_{p,1}$. The previous operation can be extended for functional $\mathfrak{u} : W \to H^{\otimes k}$ where $\nabla \mathfrak{u} : W \to H^{\otimes k+1}$ with its natural topology. Also this extension of the operator ∇ is closable.

If the measurable non-linear operator $F : W \to H$, where $|F|_H \in L^p(\mu)$, is such that $\mathbb{E}[\langle F, \nabla u\rangle_H] = \mathbb{E}[\tilde{F}u]$ for some $\tilde{F} \in L^p(\mu)$, we say that F is in the domain of the operator δ and we denote by $\delta(F) = \tilde{F} \in L^p(\mu)$ the Skorokhod integral of the measurable operator F. The following expression for $\delta(F)$ used in the following holds: suppose that $F(w) \in i^*(W)$ and that $\nabla F(w)$ is a trace class operator on H for μ almost every $w \in W$ then

$$\delta(F)(w) = \langle i^{*,-1}(F(w)), w\rangle - \mathrm{Tr}(\nabla F(w)). \tag{58}$$

We introduce a definition for studying the random transformations defined on abstract Wiener spaces.

Definition 2 *Let $U : W \to H$ be a measurable map. We say that U is a $H - C^1$ map if for μ almost every $w \in W$ the map $U_w : H \to H$, defined as $h \longmapsto U_w(h) := U(w + h)$, is a Fréchet differentiable function in H and if $\nabla U_w : H \to H^{\otimes 2}$, defined as $h \longmapsto \nabla U_w(h) := \nabla U(w + h)$ where ∇ is the Malliavin derivative, is continuous for almost every $w \in W$ and with respect to the natural (Hilbert-Schmidt) topology of $H^{\otimes 2}$.*

We introduce the shift $T : W \to W$ associated with U, i.e. the map defined as $T(w) = w + U(w)$, and the non-linear functional $\Lambda_U : W \to \mathbb{R}$ as follows

$$\Lambda_U(w) = \det{}_2(I_H + \nabla U(w)) \exp\left(-\delta(U)(w) - \frac{1}{2}|U(w)|^2_H\right), \tag{59}$$

where $\det_2(I_H + \cdot)$ is the regularized Fredholm determinant (see [47] Chapter 9) at it is well defined for any Hilbert-Schmidt operator K. In particular if K is self adjoint we have

$$\det{}_2(I + K) = \prod_{i\in\mathbb{N}} (1 + \lambda_i)e^{-\lambda_i},$$

where λ_i are the eigenvalues of the operator K.

Suppose that $U(w) \in i^*(W)$ and that $\nabla U(w)$ is a trace class operator for almost any $w \in W$, then using the expression (58) and the properties of $\det_2$ we obtain

$$\Lambda_U(w) = \det(I_H + \nabla U(w)) \exp\left(-\langle i^{*,-1}(U(w)), w\rangle_W - \frac{1}{2}|U(w)|_H^2\right), \tag{60}$$

where $\det(I_H + \cdot)$ is the standard Fredholm determinant. The functional Λ_U is closely related to the transformation of the measure μ with respect the transformation T. Indeed suppose that W is finite dimensional then we have

$$\mathrm{d}\mu = \exp\left(-\frac{1}{2}\langle w, w\rangle_H\right)\frac{\mathrm{d}x}{Z} = \exp\left(-\frac{1}{2}\langle i^{*,-1}(w), w\rangle_W\right)\frac{\mathrm{d}x}{Z},$$

where $Z \in \mathbb{R}_+$ is a suitable renormalization constant and $\mathrm{d}x$ is the Lebesgue measure on W. Thus, if T is a diffeomorphism on W, we evidently have, thanks to equation (60),

$$T_*(\mu) = \left|\det(I + \nabla U(w)) \exp\left(-\langle i^{*,-1}(U(w)), w\rangle_W - \frac{1}{2}\langle i^{*,-1}(U(w)), U(w)\rangle_W\right)\right| = |\Lambda_U(w)|.$$

The previous relation can be extended to the case where W and H are infinite dimensional and the transformation T is not a diffeomorphism but it is only a $H - C^1$ map.

First of all we need the following generalization to the abstract Wiener space context of the finite dimensional Sard Lemma.

Proposition 5 *Let T be a $H-C^1$ map and let $M \subset \mathcal{W}$ be the set of the zeros of* $\det_2(I+\nabla U(w))$ *then the μ measure of the set $T(M)$ is zero, i.e. $\mu(T(M)) = 0$.*

Proof See Theorem 4.4.1 [49]. □

The following is the change of variable theorem for (generally not invertible) $H - C^1$ maps.

Theorem 14 *Let T be an $H - C^1$ map and let be f, g two positive measurable functions then*

$$\int_{\mathcal{W}} f \circ T(w) g(w) |\Lambda_U| \mathrm{d}\mu(w) = \int_{\mathcal{W}} f(w) \left(\sum_{y \in T^{-1}(w)} g(y)\right) \mathrm{d}\mu(w). \tag{61}$$

In particular if $K \subset W$ is a measurable subset where $T|_K$ is invertible we

$$\int_K f \circ T(w) |\Lambda_U| \mathrm{d}\mu(w) = \int_{T(K)} f(w) \mathrm{d}\mu(w).$$

Proof See Theorem 4.4.1 [49]. □

In order to prove Theorem 9, and so the relationship between the weak solutions to equation (6) and the integrals with respect to the signed measure $\Lambda_U \mathrm{d}\mu$, it is not enough to consider Theorem 14 but we need a relationship analogous to (61) with $|\Lambda_U|$ replaced by Λ_U. In order to achieve this result we need some more hypotheses on the map U:

Hypothesis DEG1 The map $U : W \to H \hookrightarrow W$ is a Fréchet differentiable map from W into itself and furthermore it is C^1 with respect to the usual topology of W;

Hypothesis DEG2 The map T is proper (i.e. inverse images of compact subsets are compact) and the equation $T^{-1}(y) = w$ has a finite number of solution y for μ almost every $w \in W$.

Under the Hypothesis DEG1 and DEG2 we can define the following number

$$\mathrm{DEG}(w,T) := \sum_{y\in T^{-1}(w)} \mathrm{sign}(\det{}_2(I_W + \nabla U(y))).$$

This index is a suitable modification of the Leray-Schauder degree of a Fredholm non-linear operator described, for example, in [10] Section 5.3C, where the following definition is given: if B is a bounded set of W such that $T^{-1}(w)\cap\partial B=\emptyset$ and $\nabla T(y) \not= 0$ for $y \in T^{-1}(w)\cap B$ we have

$$\mathrm{DEG}_B(w,T) = \sum_{y\in T^{-1}(w)} (-1)^{(\text{number of egative eigenvalues of } \nabla T(y))}.$$

It is evident that under the Hypothesis DEG2 and, as a consequence of Proposition 5, we have

$$\lim_{B\to W} \mathrm{DEG}_B(w,T) = \mathrm{DEG}(w,T)$$

for almost all $w \in W$.

Theorem 15 *Under the Hypotheses DEG1 and DEG2 we have that* $\mathrm{DEG}(w,T)$ *is* μ *almost surely equal to the constant* $\mathrm{DEG}(T)\in\mathbb{Z}$ *independent on* w *and for any* f *bounded function such that* $f\circ T\cdot\Lambda_U \in L^1(\mu)$ *we have*

$$\int_W f\circ T(w)\Lambda_U(w)\mathrm{d}\mu(w) = \mathrm{DEG}(T)\cdot\int_W f(w)\mathrm{d}\mu(w).$$

Proof The proof can be found in [49] Theorem 9.4.1 and Theorem 9.4.6. □

In general is not simple to compute $\mathrm{DEG}(T)$ but this computation simplified under the following Hypothesis:

Hypothesis DEG3 The map $T_\epsilon(w) = w + \epsilon U(w)$ has bounded level set uniformly in $\epsilon \in [0,1]$, i.e. if $B \subset W$ is bounded $\cup_{\epsilon\in[0,1]} T_\epsilon^{-1}(B)$ is a bounded set in W.

Theorem 16 *Under the Hypotheses DEG1, DEG2 and DEG3 we have that, fro any* $\epsilon\in[0,1]$*:*

$$\mathrm{DEG}(T) = \mathrm{DEG}(w,T) = \mathrm{DEG}(w,T_\epsilon) = 1.$$

Proof The proposition follows from the invariance of DEG_B under homotopies of the operator T. In other words for any B such that $T_\epsilon^{-1}(w)\cap\partial B=\emptyset$ we have $\mathrm{DEG}_B(w,T_\epsilon) = \mathrm{DEG}_B(w,T)$. Under the Hypothesis DEG3 we can choose B big enough such that $\mathrm{DEG}_B(w,T_\epsilon) = \mathrm{DEG}(w,T_\epsilon)$ for any $\epsilon\in[0,1]$. Since $\mathrm{DEG}(w,T_0) = \mathrm{DEG}(w,\mathrm{id}_W) = 1$ the thesis follows. □

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