Existence results for the Helmholtz equation in periodic wave-guides with energy methods

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Abstract: The Helmholtz equation $-\nabla \cdot (a\nabla u) - \omega^2 u = f$ is considered in an unbounded wave-guide $\Omega := \mathbb{R} \times S \subset \mathbb{R}^d$, where $S \subset \mathbb{R}^{d-1}$ is a bounded domain. The coefficient a is strictly elliptic and (locally) periodic in the unbounded direction $x_1 \in \mathbb{R}$. For non-singular frequencies ω, we show the existence of a solution u. While previous proofs of such results were based on operator theory, our proof uses only energy methods.

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1 Introduction

We investigate the existence of solutions to the Helmholtz equation

$$-\nabla \cdot (a\nabla u) - \omega^2 u = f \tag{1.1}$$

in an infinite wave-guide $\Omega := \mathbb{R} \times S$. The cross-section S is given by a bounded Lipschitz domain $S \subset \mathbb{R}^{d-1}$, the right hand side $f \in H^{-1}(\Omega)$ has compact support, the frequency $\omega > 0$ is assumed to be non-singular. The differential operator $Au := -\nabla \cdot (a\nabla u)$ is given by coefficients $a : \Omega \to \mathbb{R}^{d\times d}$ of class $L^\infty(\Omega)$ with $a(x)$ symmetric and positive, satisfying $\lambda|\xi|^2 \le \xi \cdot a(x)\xi \le \Lambda|\xi|^2$ for some $0 < \lambda < \Lambda < \infty$ and all $\xi \in \mathbb{R}^d$, $x \in \Omega$. The coefficient is assumed to be 1-periodic in the direction of x_1, $a(x + e_1) = a(x)$ for every $x \in \Omega$, but we also treat coefficients that are only locally periodic. We impose Neumann conditions on $\partial\Omega$; Dirichlet conditions can be treated in the same way. For an illustration of the geometry see Figure 1.

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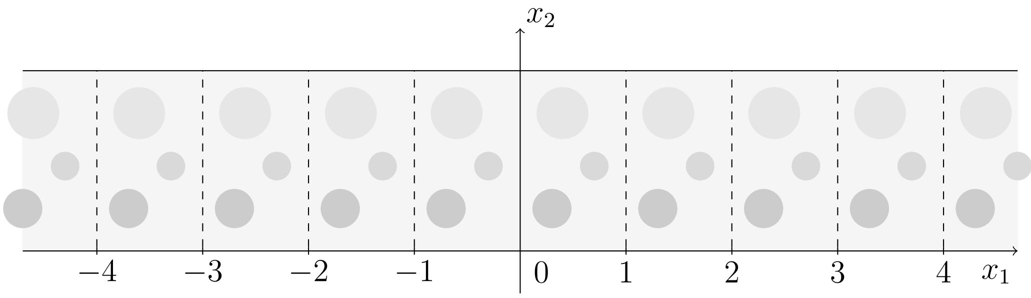

Figure 1: The wave-guide geometry in two dimensions. The coefficient a is indicated by different levels of gray. It is 1-periodic in x_1-direction.

We say that a function $u \in H^1_{\text{loc}}(\Omega)$ solves the *radiation problem* if the following three conditions are met:

(i) u solves (1.1) in Ω in the sense of distributions.
(ii) $\sup_{r\in\mathbb{Z}} \|u\|_{L^2((r,r+1)\times S)} < \infty$.
(iii) the radiation condition of Definition 2.4 is satisfied.

One of our main results is the following existence statement.

Theorem 1.1 (Existence and uniqueness result for periodic media)**.** *Let the data Ω, f, ω, and a be as described above. Let ω be non-singular in the sense of Assumption 2.3 below. Let a be 1-periodic in the first direction, $a(x+e_1) = a(x)$ for every $x \in \Omega$. Then there exists one and only one solution u to the radiation problem (i)–(iii).*

The statement of Theorem 1.1 is not new. It is contained, e.g., in [8]. The decisive difference between existing literature and the paper at hand regards the method of proof. The proof in [8] uses operator theory (just as the proofs of similar results in e.g. [9] and [16]): One constructs families of operators in subsets of the complex plane, sketches specific curves in the complex plane and evaluates corresponding line integrals of operators. The constructions provide bounded families of operators and thus, as a result, an inverse to the Helmholtz operator. The proofs rely on analyticity properties and exploit the operator perturbation theory of Kato.

By contrast, our proof uses only energy methods and is self-contained.

Our approach has the character of a Fredholm alternative. The assumption that the frequency ω is non-singular implies that the homogeneous problem has only the trivial solution. From this uniqueness property, we obtain the existence result. In order to obtain the existence, we introduce an approximate problem which is easy to solve. If the approximate solutions are bounded, then any limit is the desired solution to the original problem. If the approximate solutions are unbounded, we normalize them and obtain, in the limit, a non-trivial solution to the homogeneous problem — in contradiction to the uniqueness property.

From the above description of the proof it is clear that the approach is very direct. The two difficulties are 1) the construction of a useful approximate problem and 2) the verification of the radiation condition for limits. Our choice is inspired by constructions of [4] and [15]: We work with radiation boxes and demand that approximate solutions look like outgoing waves in the radiation boxes. All proofs then rely on the flux equality for solutions: In every cross-section of the wave-guide the solution has the same energy flux.

Our methods are very flexible and provide also other results. As an example, we give the following result for media that are periodic in two half-spaces. For such piecewise periodic media, even for non-singular frequencies ω, the number ω^2 can be an eigenvalue of the Helmholtz operator. We therefore have to assume the uniqueness property for the homogeneous problem. Our result has therefore very much the character of a Fredholm alternative: If solutions are unique, then they exist for arbitrary f.

Theorem 1.2 (Piecewise periodic media)**.** *Let $\Omega = \mathbb{R} \times S$ be the wave-guide, S and a as above. Let a be 1-periodic at the far right and at the far left: There exists $R_0 > 0$ such that $a(x + e_1) = a(x)$ holds for every $x \in \Omega$ with $|x_1| > R_0$. Let $\omega > 0$ be a non-singular frequency in the sense of Assumption 2.3 below for the two periodic media at the far right and at the far left. If the radiation problem (i)–(iii) with $f = 0$ possesses only the trivial solution, then the radiation problem (i)–(iii) has a unique solution u for arbitrary $f \in H^{-1}(\Omega)$ with compact support.*

Regarding literature we mention [12] for classical methods. In [10], a uniqueness result is obtained for a small perturbation of a periodic medium. In general, uniqueness does not hold in the situation of Theorem 1.2, see [1, 5, 6]. The work [3] treats a similar problem and makes a connection to a Lippmann-Schwinger equation, uniqueness is obtained there from a positive absorption parameter.

We mention that the Fredholm alternative for a limiting absorption principle was also exploited in [18] in order improve the existence statement of [4] with a vanishing absorption principle. The analysis of guided modes in a wave guide with purely harmonic dependence in the unbounded direction was treated in [2]. In the work [7], the solution to half-space problems is used for the computation of guided modes, which is further exploited in [11].

2 Preliminaries

In this section we discuss various properties of the system and specify the setting for our results. We start with the conservation of fluxes. This is a fundamental property of the Helmholtz equation and our existence result is built on it. We recall the concept of propagating modes and introduce the non-singularity assumption on ω, which allows also to introduce a useful radiation condition in Definition 2.4. We furthermore show some results on approximate orthogonality and the equivalence of our radiation condition with a more standard formulation.

2.1 Conservation of fluxes and the form Q

During the entire approach, we will work with a number $l \in \mathbb{N}$ that gives the width of a so called “radiation box”. The number is fixed throughout this article, but it must be chosen sufficiently large. Below, we will be specific on the choice of l.

Given $l > 0$, we consider the domain $W_0 := (0, l) \times S$ and, for arbitrary $r \in \mathbb{Z}$, the shifted domains $W_r := (r, r + l) \times S$. Later on, we will identify a function $u : W_r \to \mathbb{C}$ with the function $\tilde{u} : W_0 \to \mathbb{C}$, which is obtained with a shift, $\tilde{u}(x) = u(x + re_1)$.

Of crucial importance in our approach will be the following sesquilinear form Q. For $u \in H^1(W_0)$ and $v \in L^2(W_0)$, we define

$$Q(u, v) := \frac{1}{l} \int_{W_0} a \nabla u \cdot e_1 \bar{v}\,, \qquad \text{and} \qquad \mathcal{Q}(u) := Q(u, u)\,, \tag{2.1}$$

where the overbar denotes complex conjugation. The forms Q and $\mathcal{Q}$ can be used to measure the energy flux of solutions. We also consider at one point a symmetrized variant of Q, namely

$$Q^s(u, v) := \frac{1}{2} \left(Q(u, v) - \overline{Q(v, u)} \right) . \tag{2.2}$$

The symmetrized variant satisfies $Q^s(u, v) = -\overline{Q^s(v, u)}$ and $Q^s(u, u) = i \operatorname{Im} Q(u, u) = i \operatorname{Im} \mathcal{Q}(u)$, hence also $\operatorname{Im} Q^s(u, u) = \operatorname{Im} Q(u, u) = \operatorname{Im} \mathcal{Q}(u)$.

We will repeatedly use the piecewise affine cutoff-functions ϑ that are 1 in an interior interval and 0 outside a larger interval. More precisely, given four consecutive points $(\rho, \rho + l, r, r + l)$, we set: $\vartheta(s) = 0$ for $s \leq \rho$ and for $s \geq r + l$, $\vartheta(s) = 1$ for $\rho + l \leq s \leq r$, and ϑ affine linear in the two remaining intervals, compare Figure 2. By slight abuse of notation, we identify ϑ with a cutoff-function on Ω by setting $\vartheta(x) := \vartheta(x_1)$.

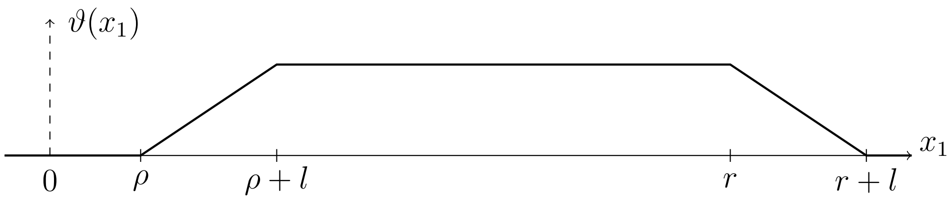

Figure 2: The cutoff function ϑ.

The basis for our approach is the energy flux equality. In its simplest form, it states: For a homogeneous solution ϕ, the energy flux quantity $\operatorname{Im} Q(\phi|_{W_r}, \phi|_{W_r}) = \operatorname{Im} \mathcal{Q}(\phi|_{W_r})$ is independent of the position r.

Lemma 2.1 (Simple flux equality)**.** *Let $\phi \in H^1_{\text{loc}}(\Omega)$ be a solution to $A\phi = \omega^2 \phi$ on Ω. Then, for arbitrary $\rho, r \in \mathbb{R}$ with $\rho + l \leq r$, there holds the flux equality*

$$\operatorname{Im} \mathcal{Q}(\phi|_{W_\rho}) = \operatorname{Im} \mathcal{Q}(\phi|_{W_r})\,. \tag{2.3}$$

Proof. We use the piecewise affine cutoff-function ϑ corresponding to the four points $(\rho, \rho + l, r, r + l)$. We furthermore set $\Omega_{r_1, r_2} := (r_1, r_2) \times S$.

We multiply the equation $A\phi = \omega^2\phi$ with $\vartheta\bar{\phi}$, we recall that the overbar stands for complex conjugation. An integration over Ω yields

$$
\begin{aligned}
0 &= \int_{\Omega_{\rho,r+l}} a\nabla\phi \cdot \nabla(\bar{\phi}\vartheta) - \int_{\Omega_{\rho,r+l}} \omega^2\, \phi\, \bar{\phi}\, \vartheta \\
&= \int_{\Omega_{\rho,r+l}} a\nabla\phi \cdot \nabla\bar{\phi}\, \vartheta - \int_{\Omega_{\rho,r+l}} \omega^2\, \phi\, \bar{\phi}\, \vartheta - \int_{W_r} a\nabla\phi\, \bar{\phi} \cdot \frac{1}{l} e_1 + \int_{W_\rho} a\nabla\phi\, \bar{\phi} \cdot \frac{1}{l} e_1 \\
&= \int_{\Omega_{\rho,r+l}} a\nabla\phi \cdot \nabla\bar{\phi}\, \vartheta - \int_{\Omega_{\rho,r+l}} \omega^2\, |\phi|^2\, \vartheta - Q(\phi|_{W_r}, \phi|_{W_r}) + Q(\phi|_{W_\rho}, \phi|_{W_\rho})\,.
\end{aligned}
$$

Since the first two integrals are real, taking the imaginary part, we find (2.3). □

Remark 2.2 (The flux through an interface)**.** *Let ϕ be a solution of $A\phi = \omega^2\phi$ on Ω. Multiplication with $\bar{\phi}$, integration over $(\rho, r) \times S \subset \Omega$, and taking the imaginary part yields*

$$
\operatorname{Im} \int_{\{\rho\}\times S} a\nabla\phi \cdot e_1\ \bar{\phi} = \operatorname{Im} \int_{\{r\}\times S} a\nabla\phi \cdot e_1\ \bar{\phi}\,. \tag{2.4}
$$

This shows that the expression on the right does not depend on the position r.

The fact that the surface integral is indepent of r implies that every volume integral $\operatorname{Im} \mathcal{Q}(\phi|_{W_r})$ *of* (2.3) *actually coincides with the expression in* (2.4)*.*

2.2 Propagating modes and radiation condition

We next study solutions to the radiation problem. We cannot expect that solutions decay at infinity. On the other hand, in general, we cannot expect that solutions are locally of class L^∞. We therefore introduce a new norm to measure functions. For $u : \Omega \to \mathbb{C}$ we set

$$
\|u\|_{sL} := \sup_{r\in\mathbb{Z}} \|u\|_{L^2((r,r+1)\times S)}\,, \tag{2.5}
$$

the symbol sL is chosen since we take a supremum over L^2-norms. We study the following subspace of $H^1(W_0)$:

$$
X := \left\{ u|_{W_0} \,\middle|\, u \in H^1_{\mathrm{loc}}(\Omega),\ \|u\|_{sL} < \infty,\ Au = \omega^2 u \text{ in } \Omega \right\}\,. \tag{2.6}
$$

We will assume that the space X is finite-dimensional. This is indeed the case for all but a countable number of frequencies ω.

Let us briefly recall some spectral analysis facts, for details see [8, 13, 14, 17]: With a Floquet-Bloch transform the problem $Au = \omega^2 u$ can be decomposed in a familiy of problems $A(\xi)u = \omega^2 u$, where $\xi \in [0, 2\pi)$ is the quasimoment and $A(\xi)$ is the operator A, restricted to ξ-quasiperiodic functions f, i.e. to functions with $f(x + e_1) = e^{i\xi} f(x) \forall x$. Every operator $A(\xi)$ has a compact resolvent and hence a pure point spectrum. Since the eigenvalues depend continuously on ξ (and have increasing lower bounds), for generic values of ω, the number ω^2 coincides with eigenvalues of $A(\xi)$ for finitely many values of ξ. The corresponding eigenfunctions then form a basis for the space X of (2.6).

In the following, we assume that ω^2 is an eigenvalue of $A(\xi)$ for only finitely many ξ_j and that the imaginary part of the sesquilinear form Q does not vanish in the basis functions. Our definition of non-singular frequencies coincides with the one in [8].

Assumption 2.3 (Non-singular frequency). *We assume that $\omega > 0$ is a non-singular frequency in the following sense:*

(a) The space X of (2.6) *is finite-dimensional. There exists a basis $(\phi_1^+, ..., \phi_{N_+}^+, \phi_1^-, ..., \phi_{N_-}^-)$ with corresponding quasimoments $\xi_j^\pm \in [0, 2\pi)$ such that the quasiperiodic extensions with $\phi_j^\pm(x + e_1) = e^{i\xi_j^\pm}\phi_j^\pm(x)$ are solutions of $A\phi = \omega^2\phi$ in Ω.*

(b) The basis functions have the property that, for every j,

$$\operatorname{Im} \mathcal{Q}(\phi_j^+) > 0 \quad \text{and} \quad \operatorname{Im} \mathcal{Q}(\phi_j^-) < 0 \,. \tag{2.7}$$

For a non-singular frequency ω and basis functions as above we define the following two subspaces of $H^1(W_0)$,

$$X_+ := \operatorname{span}\{\phi_j^+ \,|\, 1 \le j \le N_+\}\,, \quad X_- := \operatorname{span}\{\phi_j^- \,|\, 1 \le j \le N_-\}\,. \tag{2.8}$$

Every function $\phi^+ \in X_+$ can be extended to a solution of the homogeneous problem. Indeed, $\phi^+ \in X_+$ implies $\phi^+(x) := \sum_j \alpha_j \phi_j^+(x)$ for some coefficients $\alpha_j \in \mathbb{C}$. The extension $\phi^+(x + me_1) = \sum_j \alpha_j e^{im\xi_j^+}\phi_j^+(x)$ for every $m \in \mathbb{Z}$ is a solution to the homogenenous problem $A\phi = \omega^2\phi$ in Ω. Analogously, an extension operator can be defined for $\phi^- \in X_-$.

Since the basis functions are linearly independent, there holds $X = X_+ \oplus X_-$. Corresponding to this decomposition, there are two projections, $\Pi_{X,+} : X \to X$ onto X_+ and $\Pi_{X,-} : X \to X$ onto X_-. An arbitrary element $u \in X$ can be uniquely written as $u = \sum_{j=1}^{N_+} \alpha_j \phi_j^+ + \sum_{j=1}^{N_-} \beta_j \phi_j^-$, and the projections of this element are $\Pi_{X,+}(u) = \sum_{j=1}^{N_+} \alpha_j \phi_j^+$ and $\Pi_{X,-}(u) = \sum_{j=1}^{N_-} \beta_j \phi_j^-$. We emphasize that the basis function $\phi_j^\pm$ are, in general, not L^2-orthogonal.

The $L^2(W_0)$-orthogonal projection onto the subspace X is denoted as $\Pi_X : L^2(W_0) \to L^2(W_0)$. With the help of Π_X we define the two projections $\Pi_+ := \Pi_{X,+} \circ \Pi_X : L^2(W_0) \to L^2(W_0)$ onto X_+ and $\Pi_- := \Pi_{X,-} \circ \Pi_X : L^2(W_0) \to L^2(W_0)$ onto X_-.

The projections allow to introduce the radiation condition that is used in this work. We remark that the equivalence with a more standard radiation condition is established in Lemma 2.8. The norm $\|u\|_{sL}$ was introduced in (2.5).

Definition 2.4 (Radiation condition). *Let ω be non-singular in the sense of Assumption 2.3. Let $l > 0$ be a fixed width of the radiation boxes $W_{\pm r}$ and let $\Pi_\pm$ be the above projections. We say that $u : \Omega \to \mathbb{C}$ with $\|u\|_{sL} < \infty$ satisfies the radiation condition if*

$$\Pi_-(u|_{W_r}) \to 0 \quad \text{and} \quad \Pi_+(u|_{W_{-r}}) \to 0 \quad \text{as } r \to +\infty\,. \tag{2.9}$$

In this formula, we identify a function on W_r with a function on W_0 via a shift, the convergence is that of X.

2.3 Approximate orthogonality

We have to deal with the fact that the spaces X_+ and X_- are not orthogonal, neither in the sense of $L^2(W_0)$ nor in the sense of the form $\operatorname{Im} Q(.,.)$. In order to have at least an approximate orthogonality, which will be sufficient for our approach, we choose the width l of the radiation boxes W_r very large. The choice of $l \in \mathbb{N}$ is made in this subsection.

We always normalize the basis functions such that

$$\frac{1}{l}\int_{W_0} |\phi_j^\pm|^2 = 1\,. \tag{2.10}$$

A normalized basis function remains normalized when l is changed; this is a consequence of quasiperiodicity.

In the case that several basis functions, say $\phi_l^+, ..., \phi_m^+$, have the same quasimoment ξ, we diagonalize the functions with respect to the form Q^s,

$$Q^s(\phi_i^+, \phi_j^+) = 0 \quad \text{for all} \quad l \le i, j \le m,\ i \ne j\,. \tag{2.11}$$

This is possible by a standard diagonalization procedure; we exploit $Q^s(\phi_j^+, \phi_j^+) \neq 0$ for all j and the symmetry $Q^s(u,v) = -\overline{Q^s(v,u)}$. The orthogonalization is performed for all basis functions $\phi_j^\pm$ that have a common quasimoment.

Lemma 2.5 (Approximate orthogonality)**.** *For every $\eta_0 > 0$ there exists $l_0 \in \mathbb{N}$ such that, for every $l_0 \le l \in \mathbb{N}$, there holds: Every two different elements u and v of the set $\{\phi_1^+, ..., \phi_{N_+}^+, \phi_1^-, ..., \phi_{N_-}^-\}$ satisfy*

$$|Q^s(u,v)| \le \eta_0\,. \tag{2.12}$$

Proof. In the case that u and v have the same quasimoment ξ, there holds $Q^s(u,v) = 0$ by the orthogonalization of the basis functions, see (2.11).

Let u have the quasimoment ξ and let v have the quasimoment $\zeta \neq \xi$. We want to calculate the expression

$$\begin{aligned} Q^s(u,v) &= \frac{1}{2}\left(Q(u,v) - \overline{Q(v,u)}\right) \\ &= \frac{1}{2l}\int_{W_0} a\nabla u\cdot e_1\bar v - \frac{1}{2l}\int_{W_0} a\nabla\bar v\cdot e_1 u\,. \end{aligned}$$

We perform the argument for the first term and use once more the notation $\Omega_{r_1,r_2} := (r_1, r_2)\times S$. We can calculate

$$\begin{aligned} \frac{1}{2l}\int_{W_0} a\nabla u\cdot e_1\bar v &= \frac{1}{2l}\sum_{k=0}^{l-1}\int_{\Omega_{k,k+1}} a\nabla u\cdot e_1\bar v \\ &= \frac{1}{2l}\sum_{k=0}^{l-1}\int_{\Omega_{0,1}} e^{ik\xi} a\nabla u\cdot e_1\, e^{-ik\zeta}\bar v = \frac{1}{2l}\sum_{k=0}^{l-1} e^{ik(\xi-\zeta)}\int_{\Omega_{0,1}} a\nabla u\cdot e_1\,\bar v\,. \end{aligned}$$

Because of $\xi \neq \zeta$, the factor $\frac{1}{l}\sum_{k=0}^{l-1} e^{ik(\xi-\zeta)}$ is small for all large values of l. This allows to choose l_0 large in order to have the expression in absolute value bounded by $\eta_0/2$. The other term is treated in the same way and we obtain (2.12). $\square$

Lemma 2.6 (Sign of the sesquilinear form)**.** *For a sufficiently large width $l > 0$ there holds, for some $\gamma > 0$,*

$$\pm \mathrm{Im}\, Q(u,u) \geq \frac{\gamma}{l} \|u\|^2_{L^2(W_0)} \quad \forall\, u \in X_\pm\,. \tag{2.13}$$

Proof. Let $u \in X_+$ be arbitrary, $u = \sum_{j=1}^N \alpha_j \phi_j^+$. With the two numbers $\gamma_0 := \min_j \mathrm{Im}\, Q(\phi_j^+, \phi_j^+) > 0$ and $\eta_0 := \max_{k\neq j} |Q^s(\phi_k^+, \phi_j^+)|$ we find

$$\begin{aligned} \mathrm{Im}\, Q(u,u) = \mathrm{Im}\, Q^s(u,u) &= \sum_{j=1}^N |\alpha_j|^2\, \mathrm{Im}\, Q^s(\phi_j^+, \phi_j^+) + \sum_{k\neq j}^N \mathrm{Im}\, \left(\alpha_k \overline{\alpha_j}\, Q^s(\phi_k^+, \phi_j^+)\right) \\ &\geq \gamma_0 \sum_{j=1}^N |\alpha_j|^2 - \eta_0 \sum_{k\neq j}^N |\alpha_j|\,|\alpha_k|\,. \end{aligned}$$

Lemma 2.5 implies that, choosing l large, we can achieve that η_0 is arbitrarily small. With η_0 sufficiently small (depending on γ_0 and N) we obtain the strict positivity

$$\mathrm{Im}\, Q(u,u) \geq \frac{\gamma_0}{2} \sum_{j=1}^N |\alpha_j|^2\,.$$

At this point we have exploited that the normalization (2.10) for a function $\phi_j^\pm$ remains valid when l is varied. The immediate inequality $\|u\|^2_{L^2(W_0)} \leq Cl \sum_{j=1}^N |\alpha_j|^2$ for a constant $C > 0$ provides the claim for some $\gamma > 0$.

The argument for X_- is analogous. □

By definition of the space X, restrictions of homogeneous solutions u on Ω are contained in X. We now turn to a more quantitative version of this fact: If u is a homogeneous solutions on a large subdomain, then its restriction is close to an element of X.

Lemma 2.7 (Solutions are close to X)**.** *Let $l \in \mathbb{N}$ be fixed and let $\eta > 0$ be an arbitrary error quantifier. There exists a large number $r_0 \in \mathbb{N}$ such that, for every $\mathbb{N} \ni r > r_0$, there holds: Every function $u_r \in H^1_{\mathrm{loc}}(\Omega)$ with the properties*

$$A u_r = \omega^2 u_r \text{ in } \Omega_{-r,r} \quad \text{and} \quad \|u_r\|_{sL} \leq 1 \tag{2.14}$$

satisfies

$$\|u_r|_{W_0} - \Pi_X(u_r|_{W_0})\|_{H^1} \leq \eta\,. \tag{2.15}$$

We will later use repeatedly the following immediate consequence of (2.15), which exploits $\Pi_X = \Pi_+ + \Pi_-$:

$$\|u_r|_{W_0} - \Pi_+(u_r|_{W_0}) - \Pi_-(u_r|_{W_0})\|_{L^2} \leq \eta\,. \tag{2.16}$$

Proof. The aim is to show that $u_r|_{W_0}$ is near to an element of X.

For a contradiction argument we assume that, with l and η fixed, there exists a sequence of functions u_r with $r \to \infty$, which satisfy (2.14), but not (2.15).

The boundedness of (2.14) allows to select a subsequence and to find a limit function u such that $u_r \to u$ locally weakly in $H^1(\Omega)$. The limit u also satisfies both properties of (2.14), the solution property and the boundedness. Locally, the sequence u_r converges even strongly in H^1, as can be shown easily by testing the equation for $u_r - u$ with $(u_r - u)\,\theta$, where θ is a cutoff function. The strong convergence $u_r \to u$ in $H^1(W_0)$ implies that the limit u satisfies the same inequality as the approximate functions:

$$\|u|_{W_0} - \Pi_X(u|_{W_0})\|_{H^1} \geq \eta\,. \tag{2.17}$$

This provides a contradiction: $u|_{W_0} \in X$ holds by definition of X in (2.6), so the left hand side of (2.17) vanishes. □

We will later also exploit the H^1-regularity of the elements $\phi \in X$: For some constant C, there holds

$$\|\phi\|_{H^1(W_0)} \leq C\|\phi\|_{L^2(W_0)} \tag{2.18}$$

for all elements $\phi \in X$. The constant $C = C(\omega, \lambda)$ depends only on the frequency ω and on the ellipticity constant λ of the coefficients. The property (2.18) can be obtained by testing the equation with the solution and a cutoff function, but it can be concluded for general C also immediately from the fact that the basis functions are of class $H^1(W_0)$ and that the space X is finite dimensional.

2.4 An equivalent radiation condition

Lemma 2.8 (Equivalent radiation condition)**.** *Let the coefficient a be as in Theorem 1.2, let $\omega > 0$ be a non-singular frequency for both periodic media, let $u \in H^1_{\mathrm{loc}}(\Omega)$ be a function that satisfies (i) and (ii) of our solution concept, i.e.: u solves* (1.1) *in the sense of distributions and* $\sup_{r\in\mathbb{Z}} \|u\|_{L^2((r,r+1)\times S)} < \infty$.

The function u satisfies the radiation condition of Definition 2.4 if and only if the following holds: There exist $\phi_+ \in X_+$ and $\phi_- \in X_-$, which we identify with their extensions to solutions on all of Ω (quasiperiodic extensions of every basis function), such that, as $r \to \infty$,

$$\|(u - \phi_+)|_{W_r}\|_{L^2(W_r)} \to 0 \quad \textit{and} \quad \|(u - \phi_-)|_{W_{-r}}\|_{L^2(W_{-r})} \to 0\,. \tag{2.19}$$

The proof has much similarity to the proofs of existence and uniqueness. Since our focus is on the main results, we perform those proofs in greater detail. The reader might therefore find it helpful to read Sections 3 and 4 first.

Proof. The "if"-part. Let u satisfy (2.19) with $\phi_+ \in X_+$ and $\phi_- \in X_-$. Using a triangle inequality, boundedness of projections, and $\Pi_-(\phi_+|_{W_r}) = 0$, we find

$$\begin{aligned}\|\Pi_-(u|_{W_r})\|_{L^2(W_r)} &\leq \|\Pi_-((u-\phi_+)|_{W_r})\|_{L^2(W_r)} + \|\Pi_-(\phi_+|_{W_r})\|_{L^2(W_r)}\\ &\leq C\|(u-\phi_+)|_{W_r}\|_{L^2(W_r)} \to 0\end{aligned}$$

by (2.19). This shows one part of (2.9), the calculation for $\Pi_+(u|_{W_{-r}})$ is analogous.

The "only-if"-part. Vice versa, let u satisfy the radiation condition (2.9). We consider the right boundary. For any sequence $R \to \infty$, the sequences $u|_{W_R}$ are bounded in $L^2(W_0)$ by the assumption on the boundedness of the sL-norm. We we want to subtract the right-going part in W_R: We consider $\phi^R \in X_+$, extended as a solution to all of Ω, with $\Pi_+((u - \phi^R)|_{W_R}) = 0$.

The sequence of functions $\phi^R|_{W_0}$ is bounded in $H^1(W_0)$. We select a subsequence $R \to \infty$ (not relabelled) and a limit function ϕ_+ with $\phi^R|_{W_0} \to \phi_+$, weakly in $H^1(W_0)$ and strongly in $L^2(W_0)$. The space X_+ is closed, hence $\phi_+ \in X_+$. As usual, we identify ϕ_+ with its extension as a homogeneous solution. In particular, $\phi^R \to \phi_+$ holds locally on all of Ω. Our aim is to show, for an arbitrary sequence $r \to \infty$, that $(u - \phi_+)|_{W_r} \to 0$ holds in L^2.

We fix an arbitrary error quantifier $\eta > 0$. A large number $r \in \mathbb{N}$ is chosen in dependence of η, the choice is specified below.

We define ϕ^r corresponding to $u|_{W_r}$ as above, with $\Pi_+((u - \phi^r)|_{W_r}) = 0$, and consider $w := u - \phi^r$. We assume that the number r is sufficiently large such that the support of f is to the left of re_1. Then w solves the homogeneous problem on $\Omega \cap \{x_1 > r\}$ and hence satisfies the flux equality

$$\operatorname{Im} \mathcal{Q}(w|_{W_R}) = \operatorname{Im} \mathcal{Q}(w|_{W_r}) \tag{2.20}$$

for every number $R \in \mathbb{N}$, $R > r + l$.

Calculation of the right hand side of (2.20). The difference $w|_{W_r} - \Pi_-(u|_{W_r}) = u|_{W_r} - \phi^r|_{W_r} - \Pi_-(u|_{W_r}) = u|_{W_r} - \Pi(u|_{W_r})$ is small by Lemma 2.7; more precisely, we acchieve $\|w|_{W_r} - \Pi_-(u|_{W_r})\|_{H^1} \le \eta$ when r is sufficiently large (large distance to the support of f). This allows calculate the right hand side of (2.20) as

$$\begin{aligned} \operatorname{Im} \mathcal{Q}(w|_{W_r}) &= \operatorname{Im} \mathcal{Q}(\Pi_-(u|_{W_r}) + [w|_{W_r} - \Pi_-(u|_{W_r})]) \\ &\le \operatorname{Im} \mathcal{Q}(\Pi_-(u|_{W_r})) + C\eta \le C\eta \,, \end{aligned} \tag{2.21}$$

where we used $\operatorname{Im} \mathcal{Q}(\Pi_-(u|_{W_r})) \le 0$ of Lemma 2.6 in the last step.

Calculation of the left hand side of (2.20). We exploit that, for large R, the function $w|_{W_R}$ is close to an element of X_+, which follows from the radiation condition. Let us make this fact precise: Inserting the definition of w and using $\Pi_+(\phi^r|_{W_R}) = \phi^r|_{W_R}$, we find

$$\begin{aligned} w|_{W_R} - \Pi_+(w|_{W_R}) &= u|_{W_R} - \phi^r|_{W_R} - \Pi_+(u|_{W_R}) + \Pi_+(\phi^r|_{W_R}) \\ &= (u|_{W_R} - \Pi_X(u|_{W_R})) + \Pi_-(u|_{W_R}) \to 0 \end{aligned}$$

as $R \to \infty$. The smallness of the first bracket follows from Lemma 2.7, the smallness of the last term from the radiation condition (2.9). The convergence is in L^2, but the solution property allows once more to lift the regularity order and we obtain convergence also in H^1. This can be used to calculate the left hand side of (2.20):

$$\begin{aligned} \operatorname{Im} \mathcal{Q}(w|_{W_R}) &= \operatorname{Im} \mathcal{Q}(\Pi_+(w|_{W_R}) + [w|_{W_R} - \Pi_+(w|_{W_R})]) \\ &\ge \operatorname{Im} \mathcal{Q}(\Pi_+(w|_{W_R})) + o(1) \ge \frac{\gamma}{l} \|\Pi_+(w|_{W_R})\|_{L^2}^2 + o(1) \end{aligned}$$

as $R \to \infty$, where we used the quantitative estimate (2.13) in the last step.

Combining the calculations for the two sides of (2.20), we have obtained the smallness result

$$\|\Pi_+(w|_{W_R})\|_{L^2}^2 \le \frac{Cl}{\gamma}\eta + o(1) \tag{2.22}$$

as $R \to \infty$. We evaluate the left hand side:

$$\Pi_+(w|_{W_R}) = \Pi_+((u - \phi^r)|_{W_R}) = (\phi^R - \phi^r)|_{W_R}\,.$$

Since both ϕ^R and ϕ^r are extensions of elements in X_+, the norm of the difference can be measured in any subdomain W_ρ with equivalent results. Relation (2.22) hence also implies, as $R \to \infty$,

$$\|(\phi_+ - \phi^r)|_{W_r}\|_{L^2}^2 \leftarrow \|(\phi^R - \phi^r)|_{W_r}\|_{L^2}^2 \le C\|(\phi^R - \phi^r)|_{W_R})\|_{L^2}^2 \le \frac{Cl}{\gamma}\eta + o(1)\,.$$

It remains to exploit once more the radiation condition (2.9) and Lemma 2.7 to find $\|(u - \phi^r)|_{W_r})\|_{L^2}^2 = \|u|_{W_r} - \Pi_+(u|_{W_r})\|_{L^2}^2 \le \eta$ for sufficiently large r. Since we have shown that ϕ^r is close to ϕ_+, and since $\eta > 0$ was arbitrary, we have shown the desired result $(u - \phi_+)|_{W_r} \to 0$.

The condition on the left follows in the same way. □

3 The truncated problem

We use truncated domains of the form $\Omega_{-L,R} := (-L, R) \times S$ with two natural numbers $R, L > 0$. With the four consecutive points on the real line, $(-L - l, -L, R, R + l)$, we define (as before) a cutoff-function $\vartheta : \mathbb{R} \to \mathbb{R}$ as the piecewise affine function which vanishes to the left of the first and to the right of the fourth point, and which is 1 between the second and the third point.

The length parameter $l \in \mathbb{N}$ is suppressed in everything that follows. We will study solutions u to $Au = \omega^2 u$ on different subdomains W_r. In this process, we exploit the following fact: For every function $\phi \in X_+$ and its extension to a solution of $A\phi = \omega^2\phi$, the restriction $\phi|_{W_r}$ is again in X_+ for every $r \in \mathbb{Z}$. The same is true for X_-. This follows immediately from the extension process, see the text after (2.8).

Definition 3.1 (Function space and sesquilinear form)**.** *For $R, L \in \mathbb{N}$ we use the function space*

$$V_{L,R} := \left\{ u \in H^1(\Omega_{-L-l,R+l}) \,\middle|\, u|_{W_R} \in X_+\,,\ u|_{W_{-L-l}} \in X_- \right\}\,. \tag{3.1}$$

With ϑ as above, corresponding to the points $(-L - l, -L, R, R + l)$, we introduce the sesquilinear form

$$\begin{aligned}\beta(u, v) := \int_{\Omega_{-L-l,R+l}} a\nabla u \nabla \bar{v}\vartheta - \int_{\Omega_{-L-l,R+l}} \omega^2\, u\bar{v}\vartheta \\ - Q(u|_{W_R}, v|_{W_R}) + Q(u|_{W_{-L-l}}, v|_{W_{-L-l}})\,.\end{aligned} \tag{3.2}$$

We define the following approximate problem.

Definition 3.2 (Truncated problem)**.** *Given $f \in H^{-1}(\Omega)$ with support in $\Omega_{-M,M}$ for some $M > 0$, we say that a function u solves the truncated problem for $\mathbb{N} \ni R, L > M$, if*

$$u \in V_{L,R} \quad \text{and} \quad \beta(u,v) = \langle f, v\rangle \quad \forall v \in V_{L,R}\,. \tag{3.3}$$

Remark 3.3 (Non-uniqueness in the truncated problem)**.** *Let $(L,R) \in \mathbb{N}^2$ be a pair of parameters such that the truncated problem has a nontrivial solution to $f = 0$. Then there exists a function u that satisfies $Au = \omega^2 u$ in $\Omega_{-L,R}$ and homogeneous Dirichlet conditions on $\{-L\} \times S$ and $\{R\} \times S$.*

Proof. Let $u = u_{L,R} \neq 0$ be a solution to the truncated problem with $f = 0$. We use the test-function $v := u \in V_{L,R}$ in the sesquilinear form β. With ϑ as in Definition 3.1, we find

$$\begin{aligned}0 &= \beta(u,u)\\ &= \int_{\Omega_{-L-l,R+l}} a\nabla u \cdot \nabla u\,\vartheta - \int_{\Omega_{-L-l,R+l}} \omega^2\,|u|^2\,\vartheta - \mathcal{Q}(u|_{W_R}) + \mathcal{Q}(u|_{W_{-L-l}})\,.\end{aligned}$$

Taking the imaginary part yields the flux equality

$$\operatorname{Im}\mathcal{Q}(u|_{W_R}) = \operatorname{Im}\mathcal{Q}(u|_{W_{-L-l}})\,.$$

By definition of $V_{L,R}$, there holds $u|_{W_R} \in X_+$ and $u|_{W_{-L-l}} \in X_-$. The sign property (2.13) can be used to conclude that both flux terms vanish, and, moreover, that $u|_{W_R} = 0$ and $u|_{W_{-L-l}} = 0$.

By the $H^1(\Omega_{-L-l,R+l})$-property of $V_{L,R}$ in (3.1) we see that u is a solution on $\Omega_{-L,R}$ satisfying homogeneous Dirichlet conditions on $\{-L\} \times S$ and $\{R\} \times S$. $\square$

Remark 3.3 shows that non-uniqueness in the truncated problem occurs only if ω^2 is an eigenvalue of A to Dirichlet boundary conditions at the lateral boundaries. Independent of the subsequent lemma, we can therefore expect that uniqueness holds for many choices of L and R.

Lemma 3.4 (Uniqueness for the truncated problem)**.** *Let $\omega > 0$ be a non-singular frequency in the sense of Assumption 2.3, $a : \Omega \to \mathbb{R}$ periodic in e_1-direction. Let $\mathbb{N} \ni R_k, L_k \to \infty$ be two sequences. Then there exists $k_0 \in \mathbb{N}$ such that for every pair $(L,R) = (L_k, R_k)$ with $k \geq k_0$ there exists at most one solution to the truncated problem of Definition 3.2.*

Proof. Let us assume that, along a subsequence, the truncated problems for parameters (L_k, R_k) possess a nontrivial solution u_k to $f = 0$. We normalize u_k to have $\sup_\rho \|u_k|_{W_\rho}\|_{L^2(W_\rho)} = 1$ as in Lemma 5.1. We can extract a further subsequence and a limit function u such that $u_k \to u$ locally in H^1. Lemma 5.1 provides that the limit function u is a nontrivial radiating solution to $Au = \omega^2 u$. On the other hand, Lemma 4.1 yields that the radiation problem on Ω for $f = 0$ has only the trivial solution. This provides the desired contradiction. $\square$

We note that Lemma 3.4 improves the observation of Remark 3.3. Let us consider the trivial example of $A = -\Delta$ in dimension $d = 1$. For every $\omega > 0$ Assumption 2.3 is satisfied: With $m := \max\{n \in \mathbb{N} | 2\pi n \leq \omega\}$, the quasimoments are $\xi_1^+ = \omega - 2\pi m$ and $\underline{\xi_1^-} = -\xi_1^+$, the basis functions are $\phi_1^\pm = e^{\pm i\omega x}$. We check that $\mathcal{Q}(\phi_1^+) = \frac{1}{l}\int_0^l \partial_x \phi_1^+ \overline{\phi_1^+} = i\omega$, and accordingly $\mathcal{Q}(\phi_1^+) = -i\omega$.

In the setting of this example, we study the linear combination $v(x) := \phi_1^+(x) + \phi_1^-(x) = 2\cos(\omega x)$. For $R = L \in (\pi/\omega)(\mathbb{N} + \frac{1}{2})$, the function v solves the Dirichlet problem that was obtained in Remark 3.3. The example shows that the statement of Lemma 3.4 is stronger: For large numbers R, L, independent of their resonance properties, the solution to the truncated problem is unique.

We now turn to existence properties. As in a Fredholm alternative, the uniqueness property can imply an existence result. We use a limiting absorption principle to derive this fact.

Lemma 3.5 (Existence for the truncated problem)**.** *We consider the situation of Theorem 1.1 with $\Omega = \mathbb{R} \times S$, $f \in H^{-1}(\Omega)$ with compact support, the coefficient $a : \Omega \to \mathbb{R}$ periodic in e_1-direction. Let $\omega > 0$ be a non-singular frequency. We consider parameters $\mathbb{N} \ni R, L \geq M$ and assume that the truncated problem of Definition 3.2 has at most one solution. Then there exists a unique solution to the truncated problem of Definition 3.2.*

Proof. We use a limiting absorption principle. For every $\delta > 0$, we define a modified sesquilinear form by setting

$$\begin{aligned} \beta_\delta(u, v) := \int_{\Omega_{-L-l,R+l}} a\nabla u \nabla \bar{v} \vartheta - \int_{\Omega_{-L-l,R+l}} (\omega^2 + i\delta)\, u\bar{v}\vartheta \\ - Q(u|_{W_R}, v|_{W_R}) + Q(u|_{W_{-L-l}}, v|_{W_{-L-l}})\,. \end{aligned} \tag{3.4}$$

Step 1: Solution for $\delta > 0$. We claim that the sesquilinear form β_δ is coercive on $V_{L,R}$. For $u \in V_{L,R}$ we first calculate

$$\begin{aligned} \operatorname{Im} \beta_\delta(u, u) &= \operatorname{Im} \int_{\Omega_{-L-l,R+l}} a\nabla u \cdot \nabla u \vartheta - \operatorname{Im} \int_{\Omega_{-L-l,R+l}} (\omega^2 + i\delta)\, |u|^2 \vartheta \\ &\quad - \operatorname{Im} \mathcal{Q}(u|_{W_R}) + \operatorname{Im} \mathcal{Q}(u|_{W_{-L-l}}) \\ &\leq - \int_{\Omega_{-L-l,R+l}} \delta\, |u|^2 \vartheta - \frac{\gamma}{l} \|u|_{W_R}\|_{L^2}^2 - \frac{\gamma}{l} \|u|_{W_{-L-l}}\|_{L^2}^2\,, \end{aligned}$$

where we have used (2.13). This shows the coercivity inequality

$$-\operatorname{Im} \beta_\delta(u, u) \geq \gamma(\delta) \|u\|_{L^2(\Omega_{-L-l,R+l})}^2 \tag{3.5}$$

for $\gamma(\delta) := \min\{\delta, \gamma_0/l\} > 0$. Considering the real part of $\beta_\delta(u, u)$, we obtain

$$\operatorname{Re} \beta_\delta(u, u) \geq \lambda \|\nabla u\|_{L^2(\Omega_{-L,R})}^2 - C_0 \|u\|_{L^2(\Omega_{-L-l,R+l})}^2\,, \tag{3.6}$$

where we used the regularity property $\|\phi|_{W_0}\|_{H^1} \leq C \|\phi|_{W_0}\|_{L^2}$ for $\phi \in X$ of (2.18).

We next calculate, using first (3.6) and the Poincaré inequality and a constant $C_1 = C_1(R, L, C_0)$, then once more the regularity property of (2.18), and then (3.5):

$$
\begin{aligned}
\lambda\|u\|^2_{H^1(\Omega_{-L-l,R+l})} \\
&\le C_1 \left(\operatorname{Re}\beta_\delta(u,u) + \|u\|^2_{L^2(\Omega_{-L-l,R+l})} + \|u|_{W_R}\|^2_{H^1} + \|u|_{W_{-L-l}}\|^2_{H^1} \right) \\
&\le C_2 \left(\operatorname{Re}\beta_\delta(u,u) + \|u\|^2_{L^2(\Omega_{-L-l,R+l})} \right) \\
&\le C_2 \operatorname{Re}\beta_\delta(u,u) - \frac{C_2}{\gamma(\delta)} \operatorname{Im}\beta_\delta(u,u) \\
&= \operatorname{Re}\left[C_2 \left(1 + i\gamma(\delta)^{-1}\right) \beta_\delta(u,u) \right] .
\end{aligned}
$$

This inquality ensures coercivity of β_δ on $V_{L,R}$. We can apply the Lax-Milgram lemma and obtain that the equation $\beta_\delta(u_\delta, .) = \langle f, .\rangle$ can be solved with $u_\delta \in V_{L,R}$ for every $\delta > 0$. The solution satisfies

$$
\|u_\delta\|_{H^1(\Omega_{-L-l,R+l})} \le C(\delta)\|f\|_{H^{-1}(\Omega)} . \tag{3.7}
$$

We note that this estimate is not helpful for the limit process $\delta \to 0$ since $C(\delta) \sim 1/\gamma(\delta) \to \infty$ for $\delta \to 0$.

Step 2: Limit $\delta \to 0$. In order to perform the limit, we distinguish two cases. The distinction regards the numbers

$$
N_\delta := \|u_\delta\|_{L^2(W_{-L-l,R+l})} . \tag{3.8}
$$

Case 1: N_δ bounded along a subsequence. If N_δ is bounded along a subsequence, then we choose this subsequence $\delta \to 0$. The sequence u_δ is not only bounded in L^2, but also in H^1; this can be concluded by taking the real part of $\beta_\delta(u_\delta, u_\delta) = \langle f, u_\delta\rangle$, see (3.6), and using (2.18). We therefore find a limit function u and a further subsequence $\delta \to 0$ such that $u_\delta \rightharpoonup u$ weakly in $H^1(W_{-L-l,R+l})$.

The properties $u|_{W_R} \in X_+$ and $u|_{W_{-L-l}} \in X_-$ are satisfied, since all u_δ satisfy these properties. This shows $u \in V_{L,R}$. The weak convergence $u_\delta \rightharpoonup u$ is sufficient to take the limit $\delta \to 0$ in the relation $\beta_\delta(u_\delta, \varphi) = \langle f, \varphi\rangle$, and we obtain $\beta(u, \varphi) = \langle f, \varphi\rangle$. This shows that u is a solution of the truncated problem and the existence statement is shown.

Case 2: $N_\delta \to \infty$. In this case we study the normalized functions $v_\delta := N_\delta^{-1} u_\delta$. The sequence v_δ has all the properties of u_δ in the first case: The boundedness implies the existence of a limit function v (weak limit in H^1 and strong limit in L^2). Since v_δ solves the truncated problem with $f_\delta = N_\delta^{-1} f$, the limit v solves $\beta(v, \varphi) = \langle 0, \varphi\rangle$. Uniqueness for the truncated problem implies $v = 0$. We find a contradiction since v_δ has L^2-norm 1 and converges strongly to $v = 0$. Case 2 cannot occur. □

4 Proof of Theorem 1.1

As mentioned in the introduction, we show uniqueness and existence with energy methods, using the conservation of fluxes. The essential proofs rely on a simple trick that we want to describe here in loose terms. We explain the trick for the uniqueness proof, but it is very similar in the existence proof.

Let u be a solution to the radiation problem with $f = 0$, our aim is to show that u vanishes. We use a contradiction argument and assume that for a position $\rho \in \mathbb{N}$ the function $u|_{W_\rho}$ does not vanish. The radiation condition yields that for a large number $r \in \mathbb{N}$ the function $u|_{W_r}$ is close to a right-going wave.

If we use the flux equality for u, we conclude that the flux of u in W_ρ coincides with the flux in W_r — but this information does not help, since $u|_{W_\rho}$ can consist of right-going and left-going waves.

The trick is to consider the following: Let ϕ be the projection of $u|_{W_\rho}$ to right-going waves. We extend ϕ to all of Ω and set $w := u - \phi$. The properties of w are the following: (a) w is a solution, since u and ϕ are. (b) w is (approximately) right-going in W_r, since u and ϕ are. (c) w is left-going in W_ρ, since we subtracted the right-going part from u. The flux equality for w yields that the fluxes in W_ρ and W_r coincide. This is a valuable information, since the two fluxes have opposite sign (up to small errors). We conclude that all fluxes are small, which implies that w is small in W_ρ, which implies that u has a small left-going component in W_ρ. In the same way, choosing r to the left of ρ, one concludes that u has a small right-going component in W_ρ. This yields that u is small in W_ρ, in contradiction to the choice of ρ.

We now turn to the rigorous proofs and make the above ideas precise.

4.1 Uniqueness

We recall the norm of (2.5) for $u : \Omega \to \mathbb{C}$, $\|u\|_{sL} := \sup_{r\in\mathbb{Z}} \|u\|_{L^2((r,r+1)\times S)}$.

Lemma 4.1 (Uniqueness for the original problem)**.** *For non-singular frequencies $\omega > 0$, the problem of Theorem 1.1 has at most one solution. More precisely: Every solution $u \in H^1_{\mathrm{loc}}(\Omega)$ of $Au = \omega^2 u$ with $\|u\|_{sL} < \infty$ that satisfies the radiation conditions of Definition 2.4 vanishes identically.*

Proof. Let us assume that u is a non-vanishing solution to the homogeneous problem. Our aim is to arrive at a contradiction.

Step 1: Preparations. We normalize u with the condition $\sup_{r\in\mathbb{Z}} \|u|_{W_r}\|_{L^2(W_r)} = 1$. Let $\rho \in \mathbb{Z}$ be a number with $\|u|_{W_\rho}\|_{L^2(W_\rho)} \geq 1/2$.

We choose a small quantifier $1 \geq \varepsilon > 0$, the choice will be specified below after inequality (4.5). The radiation condition (2.9) allows to choose $r \in \mathbb{N}$, $r \geq |\rho|$ large, so that the smallness $\|\Pi_-(u|_{W_r})\|_{L^2} + \|\Pi_+(u|_{W_{-r}})\|_{L^2} \leq \varepsilon$ is satisfied (and remains satisfied for every larger r). Using the H^1-regularity property $\|\phi_\pm\|_{H^1(W_0)} \leq C\|\phi_\pm\|_{L^2(W_0)}$ of (2.18) we can improve the regularity to

$$\|\Pi_-(u|_{W_r})\|_{H^1} + \|\Pi_+(u|_{W_{-r}})\|_{H^1} \leq C\varepsilon\,. \tag{4.1}$$

We consider $\phi(x) := \sum_j \alpha_j \phi_j^+(x)$ with $\Pi_+((u-\phi)|_{W_\rho}) = 0$ and set $w = u - \phi$. There holds

$$\Pi_-(w|_{W_r}) = \Pi_-(u|_{W_r}) - \Pi_-(\phi|_{W_r}) = \Pi_-(u|_{W_r})\,,$$

hence $\|\Pi_-(w|_{W_r})\|_{L^2} \le \varepsilon$ and $\|\Pi_-(w|_{W_r})\|_{H^1} \le C\varepsilon$. This quantifies the fact that w is approximately right-going in W_r.

Regarding boundedness, we observe that $\sup_{r\in\mathbb{Z}} \|\phi\|_{L^2(W_r)} \le C$ holds, since ϕ is obtained by a projection of u. As a difference, w satisfies $\sup_{r\in\mathbb{Z}} \|w\|_{L^2(W_r)} \le 1+C$.

Step 2: Flux equality. We use a cutoff-function which is similar to that of Figure 2: We choose ϑ_ρ corresponding to the four points $(\rho, \rho+l, r, r+l)$. Multiplication of $Aw = \omega^2 w$ with $\bar{w}\,\vartheta_\rho$ yields

$$0 = \int_{\Omega_{\rho,r+l}} a\nabla w \cdot \nabla\bar{w}\vartheta_\rho - \int_{\Omega_{\rho,r+l}} \omega^2\, w\bar{w}\vartheta_\rho - \mathcal{Q}(w|_{W_r}) + \mathcal{Q}(w|_{W_\rho})\,.$$

Taking the imaginary part provides the flux equality

$$\operatorname{Im}\mathcal{Q}(w|_{W_\rho}) = \operatorname{Im}\mathcal{Q}(w|_{W_r})\,. \tag{4.2}$$

Step 3: Conclusion. The fact that w is a solution on Ω implies that $w|_{W_r}$ is an element of X, we can write $w|_{W_r} = \Pi_+(w|_{W_r}) + \Pi_-(w|_{W_r})$. The smallness of $\Pi_-(w|_{W_r})$ therefore yields

$$\|w|_{W_r} - \Pi_+(w|_{W_r})\|_{H^1} \le C\varepsilon\,. \tag{4.3}$$

This allows to calculate the quadratic form on the right hand side of (4.2) as

$$\operatorname{Im}\mathcal{Q}(w|_{W_r}) = \operatorname{Im}\mathcal{Q}\left(\Pi_+(w|_{W_r}) + [w|_{W_r} - \Pi_+(w|_{W_r})]\right)\,.$$

Inserting in the definition of $\mathcal{Q}$, using $\|w\|_{L^2(W_r)} \le 1 + C$ and $\|w\|_{H^1(W_r)} \le C_1$, we find

$$\operatorname{Im}\mathcal{Q}(w|_{W_r}) \ge \operatorname{Im}\mathcal{Q}(\Pi_+(w|_{W_r})) - \frac{C_2\Lambda\varepsilon}{l} \ge -\frac{C_2\Lambda\varepsilon}{l}\,, \tag{4.4}$$

where we used the positivity of $\mathcal{Q}$ on X_+ of (2.13). The flux equality (4.2) transfers this lower bound to the domain W_ρ.

Since also $u|_{W_\rho}$ is an element of X, there holds $w|_{W_\rho} = u|_{W_\rho} - \phi|_{W_\rho} = u|_{W_\rho} - \Pi_+(u|_{W_\rho}) = \Pi_-(u|_{W_\rho}) \in X_-$. We calculate with (4.4), (4.2), and (2.13) to conclude

$$\frac{C_2\Lambda\varepsilon}{l} \ge -\operatorname{Im}\mathcal{Q}(w|_{W_r}) = -\operatorname{Im}\mathcal{Q}(w|_{W_\rho}) \ge \frac{\gamma}{l}\|\Pi_-(u|_{W_\rho})\|_{L^2}^2\,. \tag{4.5}$$

Choosing $\varepsilon > 0$ so small that $\sqrt{C_2\Lambda\varepsilon/\gamma} \le 1/6$ holds, we find $\|\Pi_-(u|_{W_\rho})\|_{L^2} \le 1/6$.

The argument can be repeated with the left-going wave $\phi_- = \Pi_-(u|_{W_\rho})$, which yields the same estimate for $\Pi_+(u|_{W_\rho})$. Together, we obtain

$$\|u|_{W_\rho}\|_{L^2} \le \|\Pi_-(u|_{W_\rho})\|_{L^2} + \|\Pi_+(u|_{W_\rho})\|_{L^2} \le \frac{1}{3}\,, \tag{4.6}$$

in contradiction to the choice of ρ. $\square$

4.2 Flux equality

The flux equality is the central tool of our approach to existence results. It was formulated in Lemma 2.1 for solutions, but it actually holds also for solutions to the truncated problem. Here, we will show something more general: The difference of a solution and a solution of the truncated problem also satisfies the flux equality.

Lemma 4.2 (Flux equality). *Let u be a solution to the truncated problem of Definition 3.2 to parameters $R, L \in \mathbb{N}$. Let $\phi \in X_+$ be extended to a quasiperiodic solution of $A\phi = \omega^2\phi$ on Ω. Then, for every $\rho \in \mathbb{N}$, $-L \le \rho \le R - l$, the difference $w = u - \phi$ satisfies the flux equality*

$$\operatorname{Im} \mathcal{Q}(w|_{W_\rho}) = \operatorname{Im} \mathcal{Q}(w|_{W_R})\,. \tag{4.7}$$

Proof. As a solution of the truncated problem, the function $u \in V_{L,R}$ satisfies, with the cutoff-function ϑ corresponding to the four points $(-L-l, -L, R, R+l)$,

$$\begin{aligned} 0 = \beta(u, v) = \int_{\Omega_{-L-l,R+l}} a\nabla u \cdot \nabla\bar{v}\vartheta - \int_{\Omega_{-L-l,R+l}} \omega^2\, u\bar{v}\vartheta \\ - \mathcal{Q}(u|_{W_R}, v|_{W_R}) + \mathcal{Q}(u|_{W_{-L-l}}, v|_{W_{-L-l}}) \end{aligned}$$

for every $v \in V_{L,R}$.

We introduce the cutoff-function θ corresponding to the points $(\rho, \rho + l, R + l, R + 2l)$. We claim that $v := w\theta \in V_{L,R}$. Indeed, both u and ϕ are in X_+ on the right radiation box W_R, hence also w is ($\theta = 1$ in W_R). In the left radiation box W_{-L-l}, the function θ vanishes, hence $v := w\theta$ is trivially in X_-.

Due to these considerations, we can use $v = w\theta$ as a test-function. We note that the product of cutoff-functions provides a new cutoff-function: $\vartheta_\rho := \theta\vartheta$ is the piecewise affine cutoff-function which corresponds to the four points $(\rho, \rho+l, R, R+l)$. Inserting $v = w\theta$ above yields

$$\begin{aligned} 0 &= \int_{\Omega_{\rho,R+l}} a\nabla u \cdot \nabla\bar{w}\vartheta_\rho + \frac{1}{l}\int_{W_\rho} a\nabla u \cdot e_1\bar{w} - \int_{\Omega_{\rho,R+l}} \omega^2\, u\bar{w}\vartheta_\rho - \mathcal{Q}(u|_{W_R}, w|_{W_R}) \\ &= \int_{\Omega_{\rho,R+l}} a\nabla u \cdot \nabla\bar{w}\vartheta_\rho - \int_{\Omega_{\rho,R+l}} \omega^2\, u\bar{w}\vartheta_\rho - \mathcal{Q}(u|_{W_R}, w|_{W_R}) + \mathcal{Q}(u|_{W_\rho}, w|_{W_\rho})\,. \end{aligned}$$

Regarding the solution ϕ of $A\phi = \omega^2\phi$ we can proceed as in Lemma 2.1. The equation for ϕ is multiplied with $\bar{v} = \vartheta_\rho\bar{w}$ and integrated. We find essentially the same expressions as above,

$$0 = \int_{\Omega_{\rho,R+l}} a\nabla\phi \cdot \nabla\bar{w}\vartheta_\rho - \int_{\Omega_{\rho,R+l}} \omega^2\, \phi\bar{w}\vartheta_\rho - \mathcal{Q}(\phi|_{W_R}, w|_{W_R}) + \mathcal{Q}(\phi|_{W_\rho}, w|_{W_\rho})\,.$$

We can now subtract the relation for ϕ from the relation for u and obtain

$$0 = \int_{\Omega_{\rho,R+l}} a\nabla w \cdot \nabla\bar{w}\vartheta_\rho - \int_{\Omega_{\rho,R+l}} \omega^2\, w\bar{w}\vartheta_\rho - \mathcal{Q}(w|_{W_R}, w|_{W_R}) + \mathcal{Q}(w|_{W_\rho}, w|_{W_\rho})\,.$$

Taking the imaginary part, we find the flux equality (4.7). □

4.3 Existence

Lemma 4.3 (Radiation conditions for limits)**.** *For sequences $R_k, L_k \to \infty$, let u_k be a sequence of solutions to the truncated problems with right hand side f, we assume that the sequence* $\sup\{\|u_k|_{W_r}\|_{L^2(W_r)} \,|\, r \in \mathbb{Z}, -L_k - l \le r \le R_k\}$ *is bounded. Let $u \in H^1_{\text{loc}}(\Omega)$ be locally the weak H^1-limit of the solutions u_k. Then u satisfies the radiation conditions.*

Proof. We suppress the subscript k in the following and write R and L instead of R_k and L_k. As solutions to the truncated problems, the functions u_k satisfy $u_k \in V_{L,R}$, in particular $u_k \in H^1(\Omega_{-L-l,R+l})$, and $\beta(u_k, v) = \langle f, v\rangle$ for every $v \in V_{L,R}$. It is clear that the local limit u solves the Helmholtz equation with source term f. Our aim is verify the radiation condition.

A crucial step will be to derive the following property. Let $(r_k)_k$ be a sequence in $\mathbb{N}$ such that $r_k \to \infty$ and $R_k - r_k \to \infty$. We will suppress the subscript k also in the sequence r_k and claim that there holds, in $L^2(W_r)$,

$$\Pi_-(u_k|_{W_r}) \to 0\,. \tag{4.8}$$

Step 1: Verification of (4.8). We choose an error quantifier $\eta > 0$.

As in other proofs, we use $\phi(x) := \sum_j \alpha_j \phi_j^+(x)$ with $\Pi_+((u-\phi)|_{W_r}) = 0$ (ϕ is the projection onto the right-going part). We subtract this function from u_k and consider in the following $w_k := u_k - \phi$. The flux equality of Lemma 4.2 together with the positivity of (2.13) provide

$$\operatorname{Im} \mathcal{Q}(w_k|_{W_r}) = \operatorname{Im} \mathcal{Q}(w_k|_{W_R}) \ge 0\,. \tag{4.9}$$

We used that both u_k and ϕ (and hence w) are right-going waves in W_R.

We now study $w_k|_{W_r}$ and the left hand side of (4.9). Because of $r \to \infty$ and $R - r \to \infty$, the function w_k is a solution of the homogeneous problem on a large domain with center in r. This allows to use inequality (2.15) with the result that $\|w_k|_{W_r} - \Pi_+(w_k|_{W_r}) - \Pi_-(w_k|_{W_r})\|_{H^1} \le \eta$ for all $k \ge k_0(\eta)$. We observe that $\Pi_+(w|_{W_r}) = \Pi_+(u|_{W_r}) - \Pi_+(\phi|_{W_r}) = 0$ vanishes. We are therefore in the situation that $\|w_k|_{W_r} - \Pi_-(w_k|_{W_r})\|_{H^1} \le \eta$ is small, $w_k|_{W_r}$ is close to a left-going wave.

We exploit this fact in a calculation of the quadratic form, starting from (4.9) and using (2.13) in the last inequality,

$$\begin{aligned} 0 \le \operatorname{Im} \mathcal{Q}(w_k|_{W_r}) &= \operatorname{Im} \mathcal{Q}(\Pi_-(w_k|_{W_r}) + [w_k|_{W_r} - \Pi_-(w_k|_{W_r})]) \\ &\le \operatorname{Im} \mathcal{Q}(\Pi_-(w_k|_{W_r})) + \frac{C\Lambda}{l}\eta \le -\frac{\gamma}{l}\|\Pi_-(w|_{W_r})\|_{L^2}^2 + \frac{C\Lambda}{l}\eta\,, \end{aligned}$$

where we assumed in the second line $\eta \le 1$ in order to absorb the quadratic term into the linear term. The constant depends, among others, on the bound for u_k. We obtain the smallness of

$$\|\Pi_-(u|_{W_r})\|_{L^2}^2 = \|\Pi_-(w|_{W_r})\|_{L^2}^2 \le \frac{C\Lambda}{\gamma}\eta\,.$$

Since $\eta > 0$ was arbitrary, this provides the claim of (4.8).

Step 2: The radiation condition for u. We fix a sequence $r_k \to \infty$. Given this sequence, we can choose subsequences (not relabeled) $L_k, R_k \to \infty$ such that $R_k - r_k \to \infty$ and, moreover, such that $u_k - u$ is small in L^2 on the domain $W_r = W_{r_k}$ (we exploit here the local convergence $u_k \to u$ as $L, R \to \infty$).

The triangle inequality provides

$$\|\Pi_-(u|_{W_r})\|_{L^2(W_r)} \le \|\Pi_-(u_k|_{W_r})\|_{L^2(W_r)} + \|\Pi_-(u|_{W_r}) - \Pi_-(u_k|_{W_r})\|_{L^2(W_r)} .$$

The first term vanishes by (4.8) as $k \to \infty$. The second term is small by choice of the subsequence $(R_k)_k$ and $(L_k)_k$. This shows the smallness of the left hand side for large k and thus the radiation condition for u. □

Lemma 4.4 (Existence for the original problem)**.** *There exists a solution u to the radiation problem posed in Theorem 1.1.*

Proof. We use a sequence of solutions u_k to the truncated problems with $L_k, R_k \to \infty$, which exist by Lemma 3.5. We consider the sequence of real numbers

$$N_k := \sup \left\{ \|u_k\|_{L^2(W_\rho)} \middle| \rho \in \mathbb{Z}, -L_k - l \le \rho \le R_k \right\} . \tag{4.10}$$

We distinguish two cases.

Case 1: The sequence N_k is bounded. In this case, the sequence u_k is locally bounded in H^1. It therefore possesses (up to choosing a subsequence) locally a weak limit u in H^1. As a local limit of solutions, u satisfies $Au = \omega^2 u + f$ in Ω. With the local limit u, the sequence u_k satisfies all assumptions of Lemma 4.3, which yields that u satisfies the radiation condition. The function u is the desired solution and the existence assertion is shown.

Case 2: Along a subsequence, there holds $N_k \to \infty$. We choose such a subsequence and assume from now on $N_k \to \infty$. Our aim is to arrive at a contradiction. We study the normalized functions $v_k := N_k^{-1} u_k$. The sequence v_k has all the properties of u_k of Case 1: The local boundedness implies the existence of a local limit function v. Since v_k solves $Av_k = \omega^2 v_k + N_k^{-1} f$ in the sense of the truncated problem, the limit solves $Av = \omega^2 v$ in Ω. Lemma 4.3 implies that v satisfies the radiation condition. Uniqueness for this problem was shown in Lemma 4.1, we therefore obtain $v = 0$.

Another property of v_k is the following. Using v_k as a test function in the equation for v_k (with right hand side $N_k^{-1} f$), taking the imaginary part and exploiting that v_k is locally bounded, we find

$$\left| \operatorname{Im} \mathcal{Q}(v_k|_{W_{-L_k - l}}) - \operatorname{Im} \mathcal{Q}(v_k|_{v_{R_k}}) \right| \le C_0 N_k^{-1} .$$

Since $v_k|_{W_{R_k}}$ is in X_+ and $v_k|_{W_{-L_k-l}}$ is in X_-, the two flux expressions have opposite sign. Moreover, they provide a bound for the two arguments. This yields

$$\left\| v_k|_{W_{-L_k - l}} \right\|_{L^2}^2 + \left\| v_k|_{W_{R_k}} \right\|_{L^2}^2 \le C_1 N_k^{-1} . \tag{4.11}$$

The definition of N_k implies that there is a position $\rho = \rho(k) \in \mathbb{Z}$ with $\|v_k\|_{L^2(W_\rho)} \ge 1/2$. We observe that there holds $|\rho(k)| \to \infty$. Indeed, in the opposite case, we find a number $\rho_0 \in \mathbb{Z}$ and a constant subsequence, $\rho(k) = \rho_0$ along the subsequence. This is in contradiction with the local convergence $v_k|_{W_{\rho_0}} \to v|_{W_{\rho_0}} = 0$. We distinguish once more two cases.

Case 2a: Interior points. The first case is that a sequence $\rho = \rho(k)$ can be found with $R_k - \rho(k) \to \infty$ and $\rho(k) - L_k \to \infty$.

We argue as in Lemma 4.3. We fix an error quantifier $\eta > 0$. From the function v_k we want to subtract the right-going part in W_ρ: We consider $\phi \in X_+$, extended as a solution to all of Ω, with $\Pi_+((v_k - \phi)|_{W_\rho}) = 0$ (loosely speaking, $\phi = \Pi_+(v_k|_{W_\rho})$). We study the difference $w_k := v_k - \phi$. The function w_k satisfies $Aw_k = \omega^2 w_k + N_k^{-1} f$, and hence a flux equality as in Lemma 4.2, now with an error term introduced by the right hand side $N_k^{-1} f$. The flux inequality is obtained by testing the equation for w_k with the (locally bounded) function w_k, we hence find

$$\left|\mathrm{Im}\, \mathcal{Q}(w_k|_{W_\rho}) - \mathrm{Im}\, \mathcal{Q}(w_k|_{W_{R_k}})\right| \le C N_k^{-1} \,. \tag{4.12}$$

We note that $\mathrm{Im}\, \mathcal{Q}(w_k|_{W_{R_k}}) \ge 0$ holds since $w_k|_{W_{R_k}}$ is in X_+.

Regarding the first flux term, we note that $w_k|_{W_\rho} - \Pi_-(w_k|_{W_\rho}) = w_k|_{W_\rho} - \Pi(w_k|_{W_\rho})$ is small because of the fact that $w_k|_{W_\rho}$ is close to the subspace X by Lemma 2.7 (we exploit here $|\rho(k)| \to \infty$ and $R_k - \rho(k) \to \infty$ and $\rho(k) - L_k \to \infty$).

This allows to evaluate (up to small error) the first term in (4.12). For arbitrary $\eta > 0$, exploiting the sign property of Lemma 2.6, we obtain

$$\begin{aligned} -C N_k^{-1} &\le \mathrm{Im}\, \mathcal{Q}(w_k|_{W_\rho}) \le \mathrm{Im}\, \mathcal{Q}(\Pi_-(w_k|_{W_\rho})) + \eta \le -\frac{\gamma}{l} \|\Pi_-(w_k|_{W_\rho})\|_{L^2}^2 + \eta \\ &= -\frac{\gamma}{l} \|\Pi_-(v_k|_{W_\rho})\|_{L^2}^2 + \eta \end{aligned}$$

for k sufficiently large. We therefore have the smallness

$$\|\Pi_-(v_k|_{W_\rho})\|_{L^2}^2 \le \frac{l}{\gamma} (C N_k^{-1} + \eta) \tag{4.13}$$

for k sufficiently large. With the same arguments, exchanging Π_+ with Π_-, we find the same estimate for $\Pi_+(v_k|_{W_\rho})$. Invoking Lemma 2.7 once more (which is possible since $\rho(k)$ has an increasing distance to boundary points), we know that $\|v_k|_{W_\rho} - \Pi(v_k|_{W_\rho})\|_{L^2}$ is small. This is in contradiction with the normalization $\|v_k\|_{L^2(W_\rho)} \ge 1/2$. We conclude that Case 2a cannot occur.

Case 2b: Large values near boundaries. It remains to treat the case that we cannot find points $\rho = \rho(k)$ with $R_k - \rho(k) \to \infty$ and $\rho(k) - L_k \to \infty$ satisfying $\|v_k\|_{L^2(W_\rho)} \ge 1/2$. In this case, along the sequence, the distance of $\rho(k)$ to a boundary point remains bounded. Without loss of generality, let this be R_k, hence $R_k - \rho(k) \ge 0$ remains bounded. We can select a constant subsequence: Without loss of generality we can assume for $D \in \mathbb{N}$ that $R_k - \rho(k) = D$ for all k.

We consider shifted versions of the sequence v_k, defined by $\tilde{v}_k(x) = v_k(x + R_k e_1)$. The functions $\tilde{v}_k$ are defined on domains $\Omega_{-L_k - R_k - l, l}$ and have the following properties. In the subdomain W_0 the solution $\tilde{v}_k$ is outgoing, but even more is true: By (4.11), $\tilde{v}_k|_{W_0}$ is vanishing in L^2 as $k \to \infty$. In contrast, the L^2-norm in the subdomain W_{-D} is bounded from below by a positive number by the choice of ρ. For arbitrary $L_0 \in \mathbb{Z}$, in the domains $W_{-R_k+L_0}$, the solutions $\tilde{v}_k$ converge to 0 by local convergence of v_k to $v = 0$. As in Case 2a regarding interior points, we can conclude that v_k is small on any sequence of domains W_σ with $\sigma = \sigma(k)$ satisfying $\mathbb{Z} \ni \sigma(k) \to -\infty$ and $\sigma(k) + R_k \to \infty$.

The local boundedness of the functions $\tilde{v}_k$ allows to find a local limit $\tilde{v}$. The limit function solves the homogeneous problem $A\tilde{v} = \omega^2 \tilde{v}$ in $\Omega_{-\infty,0}$ and has vanishing Dirichlet data on $\{0\} \times S$. We can extend $\tilde{v}$ as an odd function to all of Ω. This provides a solution to the homogeneous problem in all of Ω, vanishing at $x_1 \to \pm\infty$, but different from 0 on W_{-D}. This is a contradiction to (a) of Assumption 2.3. □

5 Piecewise periodic media

The aim of this section is to prove Theorem 1.2 about piecewise periodic media. We use the same methods as in the problem with periodic media. We start with a lemma that will be used to show the uniqueness for the truncated problems. Uniqueness for the truncated problems will, in turn, provide the existence of a solution to the fullspace problem.

Lemma 5.1 (Limits of normalized solutions)**.** *Let $\Omega = \mathbb{R} \times S$ be a wave-guide and let $a : \Omega \to \mathbb{R}$ satisfy the general assumptions. We assume that a is piecewise periodic as in Theorem 1.2: There exists $R_0 > 0$ such that $a(x+e_1) = a(x)$ holds for every $x \in \Omega$ with $|x_1| > R_0$. Let $\omega > 0$ be a non-singular frequency in the sense of Assumption 2.3 for both periodic media. Let $u_k : \Omega \to \mathbb{C}$ be a normalized sequence of solutions to the truncated problems with L_k, R_k and $f = 0$ that converges locally,*

$$\sup_\rho \|u_k|_{W_\rho}\|_{L^2(W_\rho)} = 1 , \qquad u_k \to u \text{ locally in } H^1 , \tag{5.1}$$

the supremum is taken over all integers ρ with $-L_k - l \le \rho \le R_k$. Then the limit is a radiating solution to $Au = \omega^2 u$ with does not vanish, $u \neq 0$.

Proof. The proof has much similarity with that of Lemma 4.4. Because of the local convergence, there holds $\|u\|_{sL} \le 1$. Since locally u_k are solutions, also u is a distributional solution to $Au = \omega^2 u$. Lemma 4.3 provides that u solves the radiation condition. The important information of Lemma 5.1 is $u \neq 0$. We argue by contradiction and assume $u = 0$. We furthermore select a sequence $\rho = \rho(k)$ with $\|u_k|_{W_\rho}\|_{L^2(W_\rho)} = 1$. Once more, we have to distinguish three cases.

Case 1: $\rho(k)$ bounded. If there exists a bounded subsequence $\rho(k)$, then a further subsequence is constant: We find $\rho_0 \in \mathbb{Z}$ such that $\rho(k) = \rho_0$ for all k. Since $u_k \to u$ strongly in L^2 locally, the limit function u satisfies $\|u|_{W_{\rho_0}}\|_{L^2} = 1$ and we find a contradiction to $u = 0$.

Case 2: $\rho(k)$ unbounded. Since Case 1 is excluded, we know $|\rho(k)| \to \infty$. Without loss of generality, we assume that $\rho(k) \to \infty$.

To prepare the further arguments, we note that u_k can be used as a test-function for the truncated problem. Because of $f = 0$ there holds $\beta(u_k, u_k) = \langle f, u_k \rangle = 0$.

Taking the imaginary part and exploiting the sign properties of $\mathcal{Q}$ yields $u_k|_{W_{R_k}} = u_k|_{W_{-L_k-l}} = 0$.

Case 2a: Interior points. We assume that $R_k - \rho(k) \to \infty$, which means that the critical point $\rho(k)$ has a large distance to both 0 and R_k.

As in earlier proofs, we consider $\phi \in X_+$, extended as a solution to all of Ω, with $\Pi_+((u_k - \phi)|_{W_\rho}) = 0$. The difference $w_k := u_k - \phi$ satisfies the flux equality of Lemma 4.2,

$$\operatorname{Im} \mathcal{Q}(w_k|_{W_\rho}) = \operatorname{Im} \mathcal{Q}(w_k|_{W_{R_k}}) . \tag{5.2}$$

The right hand side is positive because of $u_k|_{W_{R_k}}, \phi|_{W_{R_k}} \in X_+$. Actually, $u_k|_{W_{R_k}}$ even vanishes (see the arguments of Case 2), hence only $\phi|_{W_{R_k}}$ remains in the argument. For the left hand side of (5.2) we use that, by choice of ϕ, there holds $\Pi_+(w_k|_{W_\rho}) = 0$; this implies the smallness of $w_k|_{W_\rho} - \Pi_-(w_k|_{W_\rho})$ by Lemma 2.7. The left hand side of (5.2) therefore can be calculated to satisfy $\operatorname{Im} \mathcal{Q}(w_k|_{W_\rho}) \leq \operatorname{Im} \mathcal{Q}(\Pi_-(w_k|_{W_\rho})) + \eta$ for any small error quantifier $\eta > 0$. Equation (5.2) yields

$$\operatorname{Im} \mathcal{Q}(\phi|_{W_{R_k}}) - \operatorname{Im} \mathcal{Q}(\Pi_-(w_k|_{W_\rho})) \leq \eta .$$

We obtain that both $\phi|_{W_{R_k}}$ (and hence $\Pi_+(u_k|_{W_\rho})$) and $\Pi_-(w_k|_{W_\rho}) = \Pi_-(u_k|_{W_\rho})$ are small. Using Lemma 2.7 again, we find a contradiction to $\|u_k|_{W_\rho}\|_{L^2(W_\rho)} = 1$.

Case 2b: Large values close to boundaries. This case is very much like Case 2b of Lemma 4.4. For a subsequence, the shifted sequence $\tilde{u}_k$ (shifted by R_k) consists of solutions to the homogeneous problem with vanishing Dirichlet data on $\{0\} \times S$, but not vanishing in some subdomain, $\|\tilde{u}_k|_{W_{-D}}\|_{L^2(W_{-D})} = 1$ for all k and for some $D \in \mathbb{N}$. We find a local limit function $\tilde{u}$. The limit shares the above properties of $\tilde{u}_k$.

For every L_0, the solutions $\tilde{u}_k|_{W_{-R_k+L_0}}$ converge to 0 by assumption (local convergence of u_k to $u = 0$). The usual argument (subtracting the right going part ϕ from $\tilde{u}_k$) we find that $\tilde{u}_k|_{W_{r_k}}$ converges to 0 for any sequence r_k with $r_k \to \infty$ and $R_k - r_k \to \infty$.

This implies that the limit $\tilde{u}$ vanishes as $x_1 \to -\infty$, but is different from 0 on W_{-D}. With an odd extension of $\tilde{u}$, we see that this is in contradiction to (a) of Assumption 2.3. □

We can now conclude the proof of our main result on non-periodic media.

Proof of Theorem 1.2. We select sequences $L_k, R_k \to \infty$ and want to solve the truncated problems. We claim that we can select the sequences $L_k, R_k \to \infty$ in such a way that the truncated problems have a uniqueness property.

Indeed, let us assume that for $L_k, R_k \to \infty$ a sequence of nontrivial solutions for the truncated problems with $f = 0$ exists. After normalization of these, they converge locally to a nontrivial solution to the full problem by Lemma 5.1. Since we have excluded the existence of such a function in the assumptions of Theorem 1.2, there cannot exist sequences $L_k, R_k \to \infty$ with non-uniqueness.

From now on, we fix $L_k, R_k \to \infty$ such that uniqueness holds along the sequence. With the proof of Lemma 3.5, we conclude that the truncated problems possess solutions. This provides a sequence u_k.

The convergence of u_k to a solution u of the original problem (along a subsequence) is shown literally as in Section 4.3: Lemma 4.3 yields that every limit of sL-bounded sequences satisfies the radiation condition. The arguments of Lemma 4.4 provide that the approximate solutions are necessarily sL-bounded. This yields that a local limit u exists. As a limit, it also satisfies $Au = \omega^2 u + f$ in Ω. $\square$

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