RELAXATION OF NON-LOCAL ENERGIES FOR STRUCTURED DEFORMATIONS WITH APPLICATIONS TO PLASTICITY

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ABSTRACT. An integral representation result is obtained for the asymptotics of energies including both local and non-local terms, in the context of structured deformations. Starting from an initial energy featuring a local bulk and interfacial contribution and a non-local measure of the jump discontinuities, an iterated limiting procedure is performed. First, the initial energy is relaxed to structured deformation, and then the measure of non-locality is sent to zero, with the effect of obtaining an explicit local energy in which the non-linear contribution of submacroscopic slips and separations is accounted for. Two terms, different in nature, emerge in the bulk part of the final energy: one coming from the initial bulk energy and one arising from the non-local contribution to the initial energy. This structure turns out to be particularly useful for studying mechanical phenomena such as yielding and hysteresis. Moreover, in the class of invertible structured deformations, applications to crystal plasticity are presented.

Keywords: structured deformations, relaxation, non-local energies, crystal plasticity.

CONTENTS

1. INTRODUCTION

In continuum mechanics, structured deformations [14] provide a rich tool for including the multiscale geometry of deformations. In light of the modern developments of analytical tools for the energetic formulation of mechanical phenomena, structured deformations have been cast in a variational framework in the pioneering work of Choksi and Fonseca [11]. In their setting, a (first-order) structured deformation is a pair

(g, G), where g represents the macroscopic deformation and G represents the contribution at the macrolevel of smooth submacroscopic geometrical changes; in order to allow the macroscopic deformation g to include non-smooth behavior, such as slips and separations, Choksi and Fonseca required that $g \in SBV(\Omega; \mathbb{R}^d)$, the space of special functions of bounded variations (see [2]). The matrix-valued field $G \in L^1(\Omega; \mathbb{R}^{d \times N})$ captures the contribution of the smooth submacroscopic geometrical changes to the deformation gradient ∇g, so that a relevant object in the theory of structured deformation is the disarrangement tensor $M := \nabla g - G$.

The connection between structured deformations and the actual submacroscopic geometrical changes occurring during a deformation is captured in the Approximation Theorem [14, Theorem 5.8] and [11, Theorem 2.12], stating that for each $(g, G) \in SD(\Omega; \mathbb{R}^d) := SBV(\Omega; \mathbb{R}^d) \times L^1(\Omega; \mathbb{R}^{d \times N})$ there exists a sequence $u_n \in SBV(\Omega; \mathbb{R}^d)$ such that

$$u_n \to g \quad \text{in } L^1(\Omega; \mathbb{R}^d) \qquad \text{and} \qquad \nabla u_n \overset{*}{\rightharpoonup} G \quad \text{in } \mathcal{M}(\Omega; \mathbb{R}^{d \times N}). \tag{1.1}$$

In the formula above, $\mathcal{M}(\Omega; \mathbb{R}^{d \times N})$ is the space of matrix-valued Radon measures and the symbol $\overset{*}{\rightharpoonup}$ denotes the weak-* convergence in $\mathcal{M}(\Omega; \mathbb{R}^{d \times N})$. (In the context of SBV functions, the symbol ∇ is used to denote the part of the distributional derivative which is absolutely continuous with respect to the Lebesgue measure.) The approximating functions u_n in (1.1) are interpreted as a description of both smooth and non-smooth submacroscopic geometrical changes, and we may write

$$M = \nabla \big(\lim_{n \to \infty} u_n \big) - \lim_{n \to \infty} \nabla u_n,$$

so that the disarrangement tensor emerges as a measure of the non-commutativity of the limit operation and taking the absolutely continuous part of the distributional derivative; because of this, it captures the contribution in the limit of the jump discontinuities of the u_n's. Notice that the approximating sequence u_n in (1.1) need not be unique.

The main issues that Choksi and Fonseca addressed were the assignment of an energy to a structured deformation and the establishment of an integral representation for that energy. They took an initial energy $E_L \colon SBV(\Omega; \mathbb{R}^d) \to [0, +\infty)$ featuring a bulk energy density $W \colon \mathbb{R}^{d \times N} \to [0, +\infty)$ and an interfacial energy density $\psi \colon \mathbb{R}^d \times \mathbb{S}^{N-1} \to [0, +\infty)$ in the form

$$E_L(u) := \int_\Omega W(\nabla u(x)) \, \mathrm{d}x + \int_{\Omega \cap S_u} \psi([u](x), \nu_u(x)) \, \mathrm{d}\mathcal{H}^{N-1}(x), \tag{1.2}$$

where $\mathrm{d}x$ and $\mathrm{d}\mathcal{H}^{N-1}(x)$ denote integration with respect to the N-dimensional Lebesgue measure and the $(N-1)$-dimensional Hausdorff measures, respectively, S_u is the jump set of u and $\nu_u(x)$ is the outer unit normal at $x \in S_u$.

Because of the non-uniqueness of the approximating sequence u_n, the energy $I_L(g, G)$ for a given structured deformation is defined as the most economical way, in terms of energies $E_L(u_n)$ in (1.2), to reach (g, G). From the mathematical point of view, this corresponds to a relaxation procedure, namely

$$I_L(g, G) := \inf_{\{u_n\} \subset SBV(\Omega; \mathbb{R}^d)} \Big\{ \liminf_{n \to \infty} E_L(u_n) : u_n \text{ converges to } (g, G) \text{ as in (1.1) and}$$
$$\sup_n \|\nabla u_n\|_{L^p(\Omega; \mathbb{R}^{d \times N})} < \infty \Big\} \tag{1.3}$$

where $p \geqslant 1$. The representation theorems [11, Theorems 2.16 and 2.17] state that, under suitable hypotheses on W and ψ, there exist a certain relaxed bulk energy density $H \colon \mathbb{R}^{d \times N} \times \mathbb{R}^{d \times N} \to [0, +\infty)$ and a certain relaxed interfacial energy density $h \colon \mathbb{R}^d \times \mathbb{S}^{N-1} \to [0, +\infty)$ such that

$$I_L(g, G) = \int_\Omega H(\nabla g(x), G(x)) \, \mathrm{d}x + \int_{\Omega \cap S_g} h([g](x), \nu_g(x)) \, \mathrm{d}\mathcal{H}^{N-1}(x). \tag{1.4}$$

Formulas for the relaxed energy densities H and h are obtained via the blow-up method [6, 24, 25] and involve the contributions of both W and ψ for H and of ψ alone for h. We refer the reader to Section 5.1 for the integral representation theorem providing (1.4). As a matter of fact, we will present a more general version where we allow the initial bulk and surface energy densities W and ψ to depend on the space variable x. We remark that this analysis neglects second-order effects such as curving and bending, which are instead captured by second-order structured deformations [8, 22, 36].

In [16] a one-dimensional procedure inspired by that in [11] was carried out for the notion of structured deformations introduced in [14]; there the initial energy (1.2) had the form

$$E_L(u) = \int_0^1 W(\nabla u(x)) \, \mathrm{d}x + \sum_{z \in S_u} \psi([u](z)) \tag{1.5}$$

and the resulting integral representation (1.4) was shown to be

$$I_L(g, G) = \int_0^1 \big(W(G(x)) + \lambda(\nabla g(x) - G(x)) \big) \, \mathrm{d}x + \sum_{z \in S_g} \psi([g](z)),$$

where $\lambda := \liminf_{\zeta \to 0^+} \psi(\zeta)/\zeta$. In this example, the contribution to the relaxed bulk energy density H of the initial interfacial energy density ψ has a special character: as the definition of λ shows, only arbitrarily small jumps influence the relaxed bulk response, which, in turn, is linear in the disarrangement tensor M. This result seems unduly restrictive in light of the descriptive power in the context of structured deformations of non-linear dependence of the bulk energy on $M = \nabla g - G$ as illustrated in [10]: there (and subsequently in [17, 18]) a periodic dependence upon M was shown to account for yielding, hysteresis, and hardening in single crystals undergoing two-level shears.

A proposal in [16] toward capturing a non-linear dependence on M was to modify the initial energy (1.5) as follows: for each $r \in (0, 1)$ let

$$F^r(u) := \int_0^1 W(\nabla u(x)) \, \mathrm{d}x + \sum_{z \in S_u} \psi([u](z)) + \int_0^1 \Psi \bigg(\sum_{z \in S_u \cap (x-r, x+r)} \frac{[u](z)}{2r} \bigg) \mathrm{d}x, \tag{1.6}$$

where the added, non-local term includes the bounded and uniformly continuous bulk energy density $\Psi \colon [0, +\infty) \to [0, +\infty)$ which accounts for the average of the jumps within each interval of radius r.

Relaxing the energy F^r in (1.6) to structured deformations and then taking the limit as $r \to 0^+$ yields (see [16, Proposition 2.3 and (2.21)])

$$J(g, G) = \int_0^1 \big(W(G(x)) + \lambda(\nabla g(x) - G(x)) \big) \, \mathrm{d}x + \sum_{z \in S_g} \psi([g](z)) + \int_0^1 \Psi(\nabla g(x) - G(x)) \mathrm{d}x, \tag{1.7}$$

where a second, possibly non-linear, dependence on the disarrangements appears through the density Ψ in the last integral above.

The goal of this paper is to show that an analogous procedure in higher dimensions can be carried out in the SBV framework of [11], by adding to the energy E_L in (1.2) a term analogous to the last term on the right-hand side of (1.6). Let $\Omega \subset \mathbb{R}^N$ be a bounded connected open set with Lipschitz boundary $\partial\Omega$; for a continuous function $\Psi \colon \Omega \times \mathbb{R}^{d \times N} \to [0, +\infty)$ and for $u \in SBV(\Omega; \mathbb{R}^d)$, define

$$E^{\alpha_r}(u) := \int_{\Omega_r} \Psi\big(x, (D^s u * \alpha_r)(x)\big) \, \mathrm{d}x, \tag{1.8}$$

where $D^s u$ is the jump part of the distributional derivative $Du = \nabla u \mathcal{L}^N + D^s u$; for $r > 0$, $\Omega_r := \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) > r\}$, B_r is the ball of radius r centered at the origin, and

$$\alpha_r := \frac{1}{r^N} \alpha\left(\frac{x}{r}\right), \tag{1.9a}$$

where

$$\alpha \in C_b(B_1) = \{\alpha \colon B_1 \to [0, +\infty) : \alpha \text{ is continuous and bounded}\} \quad \text{and} \quad \int_{B_1} \alpha(x) \, \mathrm{d}x = 1. \tag{1.9b}$$

The symbol $*$ denotes the convolution operation which, for a generic $\mu \in \mathcal{M}(\Omega; \mathbb{R}^d)$ and f a continuous function, is defined as (see [2, Definition 2.1])

$$(\mu * f)(x) := \int_\Omega f(x - y) \, \mathrm{d}\mu(y). \tag{1.10}$$

Notice that we have introduced an explicit dependence on x in the non-local energy density Ψ in (1.8). The need for such a dependence is motivated by some explicit applications to yielding, hysteresis, and crystal plasticity that we have in mind, and that we will discuss in Sections 6 and 7.

Putting (1.8) and (1.10) together, and using the structure theorem for the derivative of SBV functions (see formula (2.5) below), yields the following form for the averaged interfacial energy E^{α_r}

$$E^{\alpha_r}(u) = \int_{\Omega_r} \Psi\left(x, \int_{B_r(x) \cap S_u} \alpha_r(x-y)[u](y) \otimes \nu_u(y) \, d\mathcal{H}^{N-1}(y)\right) dx, \tag{1.11}$$

with $B_r(x) = x + B_r$ the ball of radius r centered at x. We notice that in the expression above the non-local character of the averaged interfacial energy emerges through the appearance of two iterated integrations, the inner surface integral with respect to $\mathcal{H}^{N-1}$ and the outer volume integral with respect to $\mathcal{L}^N$.

Remark 1.1. The definition of the energy E^{α_r} in (1.8) as the integral on Ω_r is motivated by the fact that the convolution operation is well defined on the sum of the supports. Noting that $\Omega_r + B_r = \Omega$, the energy E^{α_r} is well defined according to [2, page 41]. In principle, it can be convenient to define the non-local energy as an integral over all of Ω (thus having the dependence on the parameter r only in the integrand function). To this aim, we define

$$\widetilde{E}^{\alpha_r}(u) := \int_\Omega \Psi\big(x, (D^s u * \alpha_r)(x)\big) \, dx, \tag{1.12}$$

which, in the spirit of (1.11) can be written as

$$\widetilde{E}^{\alpha_r}(u) = \int_\Omega \Psi\left(x, \int_{B_r(x) \cap S_u} \alpha_r(x-y)[u](y) \otimes \nu_u(y) \, d\mathcal{H}^{N-1}(y)\right) dx, \tag{1.13}$$

with the understanding that the points $x \in \Omega$ such that $\mathrm{dist}(x, \partial\Omega) < r$ contribute with a different weight to the averaging process. As we shall see in Lemma 4.1, the difference between the relaxed energies given by E^{α_r} and $\widetilde{E}^{\alpha_r}$ respectively is lost in the limit as $r \to 0^+$.

Our primary results deal with the relaxation of energy (1.11) to reach a target structured deformation followed by taking the limit $r \to 0^+$. Notice that E^{α_r} in (1.8) and $\widetilde{E}^{\alpha_r}$ in (1.12) describe two different averaging processes, so that it is expected that they give rise to different energies, say I^{α_r} and $\widetilde{I}^{\alpha_r}$, respectively, when relaxing to structured deformations. Besides the difference in the definition of the energies (1.8) and (1.12), two different notions of convergence are needed to perform the relaxation to structured deformations. Yet, in view of Remark 1.1, it is useful, for the purpose of this introduction, to write the relaxation by means of a unified notation: we will denote by $\mathcal{E}^{\alpha_r}$ either E^{α_r} in (1.8) or $\widetilde{E}^{\alpha_r}$ in (1.12), and by $\mathcal{I}^{\alpha_r}$ either I^{α_r} or $\widetilde{I}^{\alpha_r}$, the relaxed functionals mentioned above.

Recalling the relaxation in (1.3), given $(g, G) \in SD(\Omega; \mathbb{R}^d)$, we first fix $r > 0$ to obtain a representation of the relaxation of $\mathcal{E}^{\alpha_r}$

$$\mathcal{I}^{\alpha_r}(g, G) := \inf_{\{u_n\} \subset SBV(\Omega; \mathbb{R}^d)} \left\{ \liminf_{n \to \infty} \mathcal{E}^{\alpha_r}(u_n) : u_n \rightsquigarrow (g, G) \text{ and } \sup_n \|\nabla u_n\|_{L^p(\Omega; \mathbb{R}^{d \times N})} < \infty. \right\} \tag{1.14}$$

The symbol $\rightsquigarrow$ above denotes the convergence in (1.1) when relaxing E^{α_r} to I^{α_r}, and the same symbol denotes the convergence in (1.1), complemented by the following condition in which $\mathcal{M}^+(\Omega)$ denotes the set of positive Radon measures on Ω:

there exist scalar fields Λ_g^a, Λ_g^s such that $\Lambda_g := \Lambda_g^a \mathcal{L}^N + \Lambda_g^s \mathcal{H}^{N-1} \llcorner S_g \in \mathcal{M}^+(\Omega)$ and

$$|D^s u_n| \overset{*}{\rightharpoonup} \Lambda_g \quad \text{in } \mathcal{M}^+(\Omega), \tag{1.15}$$

when relaxing $\widetilde{E}^{\alpha_r}$ to $\widetilde{I}^{\alpha_r}$. We will use the notation $u_n \xrightarrow{SD} (g, G)$ to denote the convergence in (1.1), whereas, when adding the condition (1.15), we adopt the symbol $u_n \xrightarrow{SD} (g, G)$.

We state now our first result concerning the representation of $\mathcal{I}^{\alpha_r}(g, G)$ in (1.14). In item (i) of the following theorem, $C_0(B_1) = \{\alpha \in C_b(B_1) : \alpha|_{\partial B_1} = 0\}$ (see Section 2.1 for the precise definition).

Theorem 1.2. *Let $\Omega \subset \mathbb{R}^N$ be a bounded Lipschitz domain, $\Psi \colon \Omega \times \mathbb{R}^{d \times N} \to [0, +\infty)$ be a continuous function, and, for $r > 0$, let α_r be as in (1.9). For any $(g, G) \in SD(\Omega; \mathbb{R}^d)$*

    (i) *if $\mathcal{E}^{\alpha_r} = E^{\alpha_r}$ as in (1.8) and $\alpha \in C_0(B_1)$, then the relaxed energy $\mathcal{I}^{\alpha_r}(g, G)$ in (1.14), taking $\rightsquigarrow$ as $\xrightarrow{SD}$, has the following integral representation*

$$\mathcal{I}^{\alpha_r}(g, G) = I^{\alpha_r}(g, G) = \int_{\Omega_r} \Psi\big(x, ((\nabla g - G)\mathcal{L}^N * \alpha_r)(x) + (D^s g * \alpha_r)(x)\big) \, dx, \tag{1.16}$$

namely, in the spirit of (1.11),

$$I^{\alpha_r}(g,G) = \int_{\Omega_r} \Psi\left(x, \int_{B_r(x)} \alpha_r(y-x)(\nabla g - G)(y)\,\mathrm{d}y \right.$$
$$\left. + \int_{B_r(x)\cap S_g} \alpha_r(y-x)[g](y) \otimes \nu_g(y)\,\mathrm{d}\mathcal{H}^{N-1}(y)\right)\mathrm{d}x; \tag{1.17}$$

(ii) *if $\mathcal{E}^{\alpha_r} = \widetilde{E}^{\alpha_r}$ as in* (1.12), *then the relaxed energy $\mathcal{I}^{\alpha_r}(g,G)$ in* (1.14), *taking $\rightsquigarrow$ as $\xrightarrow{SD}$, has the following integral representation*

$$\mathcal{I}^{\alpha_r}(g,G) = \widetilde{I}^{\alpha_r}(g,G) = \int_\Omega \Psi\big(x, ((\nabla g - G)\mathcal{L}^N * \alpha_r)(x) + (D^s g * \alpha_r)(x)\big)\,\mathrm{d}x, \tag{1.18}$$

namely, in the spirit of (1.13),

$$\widetilde{I}^{\alpha_r}(g,G) = \int_\Omega \Psi\left(x, \int_{B_r(x)} \alpha_r(y-x)(\nabla g - G)(y)\,\mathrm{d}y \right.$$
$$\left. + \int_{B_r(x)\cap S_g} \alpha_r(y-x)[g](y) \otimes \nu_g(y)\,\mathrm{d}\mathcal{H}^{N-1}(y)\right)\mathrm{d}x. \tag{1.19}$$

Remark 1.3. Recalling that the condition $\alpha \in C_b(B_1)$ is part of (1.9b), we highlight the difference in the hypotheses of cases (i) and (ii) in Theorem 1.2. In (ii), no further restrictions on the regularity of the convolution kernels α_r is assumed, whereas we need them to take zero value at the boundary in (i).

In Theorem 1.2(i) the formula (1.17) remains valid for a larger class of convolution kernels α (in particular, not necessarily vanishing at the boundary of B_1), provided that the convergence $\xrightarrow{SD}$ in (i) is replaced by $\xrightarrow{SD}$. On the other hand, if one restricts the class of convolution kernels in Theorem 1.2(ii) to $\alpha \in C_0(B_1)$, then the formula (1.19) remains valid even if we weaken the convergence $\xrightarrow{SD}$ in (ii) to $\xrightarrow{SD}$. In view of this, the theses of Theorem 1.2 can be summarized as follows

(i) $\alpha \in C_0(B_1)$, $\quad \rightsquigarrow = \xrightarrow{SD}, \quad \left\{ \begin{array}{c} \mathcal{E}^{\alpha_r} = E^{\alpha_r} \\ \mathcal{E}^{\alpha_r} = \widetilde{E}^{\alpha_r} \end{array} \right\} \Longrightarrow \left\{ \begin{array}{c} (1.17) \\ (1.19) \end{array} \right\},$

(ii) $\alpha \in C_b(B_1)$, $\quad \rightsquigarrow = \xrightarrow{SD}, \quad \left\{ \begin{array}{c} \mathcal{E}^{\alpha_r} = E^{\alpha_r} \\ \mathcal{E}^{\alpha_r} = \widetilde{E}^{\alpha_r} \end{array} \right\} \Longrightarrow \left\{ \begin{array}{c} (1.17) \\ (1.19) \end{array} \right\}.$

After proving Theorem 1.2 in Section 3 and obtaining an integral representation for $\mathcal{I}^{\alpha_r}$, we devote Section 4 to deducing an explicit formula for its limit

$$\mathcal{I}(g,G) := \lim_{r\to 0^+} \mathcal{I}^{\alpha_r}(g,G). \tag{1.20}$$

To introduce our main result concerning the limit above, we restrict our attention to functions Ψ with at most linear growth at infinity. By denoting

$$\mu := (\nabla g - G)\mathcal{L}^N + D^s g \qquad \text{and} \qquad \mu_r := (\mu * \alpha_r)\mathcal{L}^N, \tag{1.21}$$

the functionals I^{α_r} in (1.16) and $\widetilde{I}^{\alpha_r}$ in (1.18) can be seen as functionals defined on measures and open subsets of Ω_r and Ω, denoted by $\mathcal{A}(\Omega_r)$ and $\mathcal{A}(\Omega)$, respectively:

$$I^{\alpha_r}(g,G) = \mathscr{I}_r(\mu_r;\Omega_r) := \int_{\Omega_r} \Psi\big(x,(\mu*\alpha_r)(x)\big)\,\mathrm{d}x, \quad \widetilde{I}^{\alpha_r}(g,G) = \widetilde{\mathscr{I}}(\mu_r;\Omega) := \int_\Omega \Psi\big(x,(\mu*\alpha_r)(x)\big)\,\mathrm{d}x, \tag{1.22}$$

where the values of $\mathscr{I}_r \colon \mathcal{M}(\Omega;\mathbb{R}^{d\times N}) \times \mathcal{A}(\Omega_r) \to [0,+\infty)$ and $\widetilde{\mathscr{I}} \colon \mathcal{M}(\Omega;\mathbb{R}^{d\times N}) \times \mathcal{A}(\Omega) \to [0,+\infty)$ are defined to be

$$\int_A \Psi\left(x, \frac{\mathrm{d}\lambda}{\mathrm{d}\mathcal{L}^N}(x)\right)\mathrm{d}x + \int_{A\cap\mathrm{spt}(|\lambda^s|)} \Psi^\infty\left(x, \frac{\mathrm{d}\lambda}{\mathrm{d}|\lambda^s|}(x)\right)\mathrm{d}|\lambda^s|(x) \tag{1.23}$$

for $A \in \mathcal{A}(\Omega_r)$ and $A \in \mathcal{A}(\Omega)$, respectively; in (1.23), $\lambda \in \mathcal{M}(\Omega;\mathbb{R}^{d\times N})$, λ^s is the part of λ which is singular with respect to $\mathcal{L}^N$, $|\lambda^s|$ is its total variation, and Ψ^∞ denotes the recession function at infinity of Ψ (see (1.31) below).

We notice that for every $A \in \mathcal{A}(\Omega_r)$ we have

$$\widetilde{\mathscr{I}}(\mu_r;A) = \mathscr{I}_r(\mu_r;A),$$

so that we can consider the difference

$$\widetilde{\mathscr{I}}(\mu_r;\Omega) - \mathscr{I}_r(\mu_r;\Omega_r) = \int_{\Omega\setminus\Omega_r} \Psi(x,\mu_r(x))\,\mathrm{d}x = \widetilde{\mathscr{I}}(\mu_r;\Omega\setminus\overline{\Omega}_r), \qquad (1.24)$$

whenever the class of convolution kernels α and one notion of convergence $\rightsquigarrow$ in Theorem 1.2 are chosen in such a way that they both apply to parts (a) and (b) at the same time (see Remark 1.3). Since we will prove in Lemma 4.1 that $\widetilde{\mathscr{I}}(\mu_r;\Omega\setminus\overline{\Omega}_r)$ can be controlled by a quantity vanishing with r as $r \to 0^+$, we can choose

$$\mathcal{I}^{\alpha_r}(g,G) = \widetilde{I}^{\alpha_r}(g,G) = \widetilde{\mathscr{I}}(\mu_r;\Omega)$$

to compute the limit in (1.20). This analysis requires tools from measure theory to deal with the presence of concentrated limiting measures. In fact, the linear growth of Ψ at infinity allows one to detect the contribution of the singular part of these limiting measures via the application of Reshetnyak-type continuity theorems. We focus on two different classes of continuous functions $\Psi\colon \Omega \times \mathbb{R}^{d\times N} \to [0,+\infty)$, namely

(E) for all $x \in \overline{\Omega}$ and $\xi \in \mathbb{R}^{d\times N}$, there exists the limit

$$\lim_{\substack{x'\to x \\ \xi'\to\xi \\ t\to+\infty}} \frac{\Psi(x',t\xi')}{t}; \qquad (1.25)$$

(L) Ψ is Lipschitz with respect to the second variable, i.e., there exists $L_\Psi > 0$ such that

$$|\Psi(x,\xi) - \Psi(x,\xi')| \leqslant L_\Psi|\xi - \xi'|, \qquad \text{for all } x \in \Omega; \qquad (1.26)$$

there exists $C > 0$ such that

$$|\Psi(x,\xi)| \leqslant C(1+|\xi|) \qquad \text{for all } x \in \Omega, \qquad (1.27)$$

and there exists a continuous function $\omega\colon [0,+\infty) \to [0,+\infty)$, with $\omega(s) \to 0^+$ as $s \to 0^+$, such that

$$|\Psi(x,\xi) - \Psi(x',\xi)| \leqslant \omega(|x-x'|)(1+|\xi|), \qquad \text{for all } x, x' \in \Omega, \, \xi \in \mathbb{R}^{d\times N}. \qquad (1.28)$$

A detailed description of the class (E) can be found in [30, Section 2.4], where it is also pointed out that functions Ψ belonging to (E) satisfy automatically the at most linear growth at infinity condition (1.27).

Remark 1.4. The two classes (E) and (L) have a non-empty intersection, but also a non-trivial symmetric difference.

An example of a function which belongs to (E) but not to (L), with $N = d = 1$, is the function $\Psi\colon \mathbb{R} \to \mathbb{R}$ defined by $\Psi(\xi) = \sqrt{1-\xi^2}$ for $\xi \in [-1,1]$ and extended by periodicity. The limit in (1.25) exists and equals 0, but Ψ fails to be Lipschitz.

An example of a function which belongs to (L) but not to (E), again with $N = d = 1$, is given by $\Psi\colon \mathbb{R} \to \mathbb{R}$ defined in terms of the sequence $\{x_n\}_{n=1}^\infty$, defined recursively by $\xi_1 = 1$ and $\xi_{n+1} = 2n\xi_n$, for $n \in \mathbb{N}\setminus\{0\}$, and such that

$$\Psi(\xi) = \begin{cases} 0, & 0 \leqslant \xi \leqslant 1, \\ \xi - \xi_n, & \xi_n \leqslant \xi \leqslant \dfrac{\xi_n + \xi_{n+1}}{2}, & n \in \mathbb{N}\setminus\{0\}, \\ \xi_{n+1} - \xi, & \dfrac{\xi_n + \xi_{n+1}}{2} \leqslant \xi \leqslant \xi_{n+1}, & n \in \mathbb{N}\setminus\{0\}. \end{cases}$$

Then $\Psi(\xi_n)/\xi_n = 0$ for all $n \in \mathbb{N}\setminus\{0\}$ and

$$\frac{\Psi\left(\dfrac{\xi_n + \xi_{n+1}}{2}\right)}{\dfrac{\xi_n + \xi_{n+1}}{2}} = \frac{\xi_{n+1} - \xi_n}{\xi_{n+1} + \xi_n} = \frac{2n-1}{2n+1}, \qquad \text{for } n \in \mathbb{N}\setminus\{0\}.$$

Consequently,

$$\lim_{n\to\infty} \frac{\Psi(\xi_n)}{\xi_n} = 0 < 1 = \lim_{n\to\infty} \frac{\Psi\left(\dfrac{\xi_n + \xi_{n+1}}{2}\right)}{\dfrac{\xi_n + \xi_{n+1}}{2}},$$

so that the recession function for Ψ is not defined.

We are now in a position to state our result concerning the limit (1.20).

Theorem 1.5. *Let $\Omega \subset \mathbb{R}^N$ be a bounded Lipschitz domain, let $\Psi \colon \Omega \times \mathbb{R}^{d \times N} \to [0, +\infty)$ be a continuous function belonging to (E) or (L), let α_r be as in (1.9), and let $\widetilde{\mathscr{I}}$ be given by (1.22). Then for any $(g, G) \in SD(\Omega; \mathbb{R}^d)$ we have that*

$$\lim_{r \to 0^+} \widetilde{\mathscr{I}}(\mu_r; \Omega) = \widetilde{\mathscr{I}}(\mu; \Omega), \tag{1.29}$$

namely, recalling (1.21) and (1.23),

$$\widetilde{\mathscr{I}}(\mu; \Omega) = \int_\Omega \Psi\big(x, \nabla g(x) - G(x)\big) \, \mathrm{d}x + \int_{\Omega \cap S_g} \Psi^\infty\Big(x, \frac{\mathrm{d}D^s g}{\mathrm{d}|D^s g|}(x)\Big) \, \mathrm{d}|D^s g|(x), \tag{1.30}$$

with Ψ^∞ defined by

$$\Psi^\infty(x, \xi) := \limsup_{\substack{x' \to x \\ \xi' \to \xi \\ t \to +\infty}} \frac{\Psi(x', t\xi')}{t} \tag{1.31}$$

for every $x \in \overline{\Omega}$, $\xi \in \mathbb{S}^{d \times N - 1}$, and extended to $\mathbb{R}^{d \times N}$ by positive 1-homogeneity.

Remark 1.6. We observe the following:

- Theorem 1.5 together with Lemma 4.1 imply the following equalities

$$\widetilde{\mathscr{I}}(\mu; \Omega) = \lim_{r \to 0^+} \mathscr{I}_r(\mu_r; \Omega_r) = \mathcal{I}(g, G) =: I(g, G), \tag{1.32}$$

  so that the dependence upon α disappears in the limit and we can use the symbol $I(g, G)$ to denote the energy in (1.30).
- The recession function Ψ^∞ defined in (1.31) is finite whenever Ψ is in (E) or (L). Notice that it is a limit if Ψ is in (E).
- In Theorem 1.5, the resulting bulk energy density retains the character of the function Ψ that defines the initial non-local energy (1.8). Moreover, we observe that since the recession function Ψ^∞ vanishes in the case of sublinear growth at infinity the formula (1.30) reduces to

$$I(g, G) = \int_\Omega \Psi\big(x, \nabla g(x) - G(x)\big) \, \mathrm{d}x \tag{1.33}$$

  when Ψ has sublinear growth.

It is now natural to consider an initial energy that combines both a local contribution, described by the functional E_L in (1.2), and a non-local one, described either by E^{α_r} in (1.8) or by $\widetilde{E}^{\alpha_r}$ in (1.12). Recalling the notation $\mathcal{E}^{\alpha_r}$ introduced just above (1.14), we focus our attention on the relaxation to structured deformations, in the context of [11], of the energy functional

$$\mathcal{F}^{\alpha_r}(u) := E_L(u) + \mathcal{E}^{\alpha_r}(u), \tag{1.34}$$

namely, we consider

$$\mathcal{J}^{\alpha_r}(g, G) := \inf_{\{u_n\} \subset SBV(\Omega; \mathbb{R}^d)} \Big\{ \liminf_{n \to \infty} \mathcal{F}^{\alpha_r}(u_n) : u_n \rightsquigarrow (g, G) \text{ and } \sup_n \|\nabla u_n\|_{L^p(\Omega; \mathbb{R}^{d \times N})} < \infty \Big\} \tag{1.35}$$

where $p \geqslant 1$. If the convergence $\rightsquigarrow$ is $\xrightarrow{SD}$, the coupling between the local and non-local parts is immediately understood recalling the definition (1.3) of $I_L(g, G)$; on the contrary, if the convergence $\rightsquigarrow$ is $\xrightarrow{SD}$, Lemma 5.3 grants that the recovery sequences for the relaxation (1.3) satisfy property (1.15), so that they can be used as competitors for the strong convergence $\xrightarrow{SD}$. We will prove in Theorem 5.4 that

$$\mathcal{J}^{\alpha_r}(g, G) = I_L(g, G) + \mathcal{I}^{\alpha_r}(g, G), \tag{1.36}$$

so that, defining

$$J(g, G) := \lim_{r \to 0^+} \mathcal{J}^{\alpha_r}(g, G), \tag{1.37}$$

we immediately obtain, by Theorem 1.5 and (1.32)

$$J(g, G) = I_L(g, G) + I(g, G). \tag{1.38}$$

The formula above couples together the contributions to the total energy $J(g, G)$ coming from the local part $I_L(g, G)$ and from the limit $I(g, G)$ of the non-local energy. The contribution of the singularities of g enters the expression of $J(g, G)$ both through the surface term of $I_L(g, G)$ and through the surface term of $I(g, G)$,

via the recession function Ψ^∞, thus retaining the linear character at infinity of Ψ. The effect of Ψ on the disarrangements is encoded in the bulk term of $I(g, G)$.

The present approach to relaxation of non-local energies rests on two limiting processes:

(1) Starting from a submacroscopic level at which a weighted average of disarrangements within each neighborhood of a fixed size $r > 0$ determines the initial energy density, one passes to the macrolevel, permitting disarrangements to diffuse throughout each such neighborhood. This limiting process determines a structured deformation as well as the non-local dependence of the energy density on that structured deformation.

(2) Starting at the macrolevel from neighborhoods of the given size r above, one passes to neighborhoods of smaller and smaller sizes to obtain in the limit $r \to 0$ purely local bulk and interfacial energy densities for the structured deformation identified above.

Previous research on relaxation of energies for continuous bodies has rested on one or the other, but not on both, of these two limiting processes. In [7, 11, 12, 16, 37, 39] the first process is carried out for purely local energy densities, so that the parameter r does not appear, and the second process is irrelevant. The important results in [39] (that are exemplified in [7, 12, 16, 37]) show that the relaxed bulk energies obtained via the limiting process (1), when the initial energy is both purely local and purely interfacial, form a class that excludes the periodic functions used in [10] to predict yielding and hysteresis.

Peridynamics provides a context in which only the second limiting process is employed: for example, classical, local theories of elasticity and fracture are recovered in [38] and [33] from peridynamic theories under the limiting process $r \to 0$. In the case of peridynamics, the principal focus with respect to storage of energy and with respect to associated field theories is the non-local case in which the "horizon" r remains fixed, while the present approach via the two limiting processes achieves a purely local relaxed energy that, unlike the relaxations based on process (1) alone, admits periodic relaxed energy responses in the context of the field theories [20, 21, 35] for bodies undergoing structured deformations.

We note that there are particular classes of local initial energy densities W and ψ for which explicit formulas are available [7, 37, 39] for the relaxed energy $I_L(g, G)$ in (1.4). Notice that formula (1.30) does provide an explicit formula for the non-local energy $I(g, G)$, so that an explicit formula for $J(g, G)$ could then be obtained via (1.38). Moreover, in the one-dimensional case, explicit formulas for I_L are available in [12] and [16, Part II, Sections 2.1–2.5]. The distinction between the types of dependence of the relaxed bulk energy on the disarrangement tensor $M = \nabla g - G$ that we pointed out earlier in this introduction (see (1.7)) for the one-dimensional results in [16] remains apparent in the multi-dimensional cases where explicit formulas are available for both. Specifically, Remark 6.1 provides sufficient conditions in the multi-dimensional case in order that the relaxed bulk energy density be given by the sum $W(x, G(x)) + \Psi(x, M(x))$, with W the initial bulk energy response function and Ψ the initial non-local bulk energy response function.

The overall plan of this work is the following: in Section 2 we fix the notation and recall some basic results used throughout this article. In Section 3 and Section 4, we prove Theorem 1.2 and Theorem 1.5, respectively, that are the main novel contribution of this work. In Section 5, we consider an initial energy featuring both a local and a non-local term and discuss its relaxation and the limit for a vanishing measure of non-locality. The brief Section 5.3 contains some comments on the inversion of the two limiting procedures and discusses why, at least in the present context, we cannot expect a commutability result to hold. In Section 6 we show how our results provide a firm foundation for the predictions of yielding and hysteresis in earlier studies based on structured deformations; in Section 7, we turn to crystal plasticity to present an example of bulk energies of the type recovered by our relaxation of non-local energies.

2. Preliminaries

We start this section by fixing the notation used throughout this work; then we recall some results on measure theory and give a contained presentation of special functions of bounded variation, and finally we conclude by introducing structured deformations in the framework of [11].

2.1. **Notation.** We will use the following notations

- $\Omega \subset \mathbb{R}^N$ is a bounded connected open set with $\mathcal{L}^N(\partial\Omega) = 0$;
- $\mathcal{A}(\Omega)$ is the family of all open subsets of Ω;

- $\mathcal{M}(\Omega)$ and $\mathcal{M}(\Omega; \mathbb{R}^\ell)$ are the sets of (signed) finite real-valued or vector-valued Radon measures on Ω, respectively; $\mathcal{M}^+(\Omega)$ is the set of non-negative finite Radon measures on Ω;
- given $\mu \in \mathcal{M}(\Omega)$ or $\mu \in \mathcal{M}(\Omega; \mathbb{R}^\ell)$, the measure $|\mu| \in \mathcal{M}^+(\Omega)$ denotes the total variation of μ;
- $\mathcal{L}^N$ and $\mathcal{H}^{N-1}$ denote the N-dimensional Lebesgue measure and the $(N-1)$-dimensional Hausdorff measure in $\mathbb{R}^N$, respectively; the symbol $\mathrm{d}x$ will also be used to denote integration with respect to $\mathcal{L}^N$, while $\mathrm{d}\mathcal{H}^{N-1}$ will be used to denote surface integration with respect to $\mathcal{H}^{N-1}$;
- given $\mu \in \mathcal{M}(\Omega; \mathbb{R}^d)$, we denote by $\mu = m^a \mathcal{L}^N + \mu^s$ its decomposition into absolutely continuous part with respect to the Lebesgue measure and singular part; for every $A \in \mathcal{A}(\Omega)$, we define $\langle\mu\rangle(A) := \int_A \sqrt{1 + |m^a(x)|^2}\, \mathrm{d}x + |\mu^s|(A)$;
- $\mathbb{S}^{N-1}$ denotes the unit sphere in $\mathbb{R}^N$;
- for any $r > 0$, B_r denotes the open ball of $\mathbb{R}^N$ centred at the origin of radius r; for any $x \in \mathbb{R}^N$, $B_r(x) := x + rB$ denotes the open ball centred at x of radius r; $Q := (-\frac{1}{2}, \frac{1}{2})^N$ denotes the open unit cube of $\mathbb{R}^N$ centred at the origin; for any $\eta \in \mathbb{S}^{N-1}$, Q_η denotes the open unit cube in $\mathbb{R}^N$ with two faces orthogonal to η;
- for any $r > 0$ and $\Omega \subset \mathbb{R}^N$, $\Omega_r := \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) > r\}$ and $\Omega^r := \Omega + B_r$;
- C represents a generic positive constant that may change from line to line;
- $C_b(B_1) := \{\alpha \colon B_1 \to [0, +\infty) : \alpha \text{ is continuous and bounded}\}$;
  $C_c(B_1) := \{\alpha \colon B_1 \to [0, +\infty) : \alpha \text{ is continuous and has compact support in } B_1\}$;
  $C_0(B_1)$ denotes the closure of $C_c(B_1)$ in the sup norm;
- for any $p \in [1, +\infty)$, $L^p(\Omega; \mathbb{R}^\ell)$ denotes the space of vector-valued functions whose p-th power is integrable;
  $BV(\Omega; \mathbb{R}^\ell)$ denotes the space of vector-valued functions of bounded variation;
  $SBV(\Omega; \mathbb{R}^\ell)$ denotes the space of vector-valued special functions of bounded variation;
- the set of structured deformations $SD(\Omega; \mathbb{R}^d)$ is defined as $SD(\Omega; \mathbb{R}^d) := SBV(\Omega; \mathbb{R}^d) \times L^1(\Omega; \mathbb{R}^{d \times N})$.

2.2. Measure Theory.

We collect here some basic definitions and results from measure theory that will be used throughout the paper. In particular, we introduce the notions of weak-* and $\langle\cdot\rangle$-strict convergences and conclude by stating the Reshetnyak Continuity Theorem. Throughout this subsection, the symbol μ denotes a general measure.

Given $\Omega \subset \mathbb{R}^N$ a measurable set, we denote by $\mathcal{M}(\Omega; \mathbb{R}^\ell)$ the set of $\mathbb{R}^\ell$-valued Radon measures defined on Ω. The Radon-Nikodým Theorem [2, Theorem 1.28] ensures that, for any $\mu \in \mathcal{M}(\Omega; \mathbb{R}^\ell)$ there exists a unique pair of Radon measures μ^a and μ^s such that μ^a is absolutely continuous with respect to the Lebesgue measure $\mathcal{L}^N$, μ^s is singular with respect to $\mathcal{L}^N$, and $\mu = \mu^a + \mu^s$. Moreover, there exists a unique function $m^a \in L^1(\Omega; \mathbb{R}^\ell)$ such that $\mu^a = m^a \mathcal{L}^N$, so that $\mu = m^a \mathcal{L}^N + \mu^s$. The singular part μ^s of μ is supported on a set of Lebesgue measure zero.

Definition 2.1. *Let $\mu_n = m_n^a \mathcal{L}^N + \mu_n^s \in \mathcal{M}(\Omega; \mathbb{R}^\ell)$ be a sequence of measures and let $\mu = m^a \mathcal{L}^N + \mu^s \in \mathcal{M}(\Omega; \mathbb{R}^\ell)$.*

(i) *We say that μ_n converges weakly-* to μ (in symbols $\mu_n \overset{*}{\rightharpoonup} \mu$) if*

$$\lim_{n \to \infty} \int_\Omega \varphi(x)\, \mathrm{d}\mu_n(x) = \int_\Omega \varphi(x)\, \mathrm{d}\mu(x) \qquad \text{for every } \varphi \in C_0(\Omega).$$

(ii) *We say that μ_n converges $\langle\cdot\rangle$-strictly to μ if $\mu_n \overset{*}{\rightharpoonup} \mu$ and $\langle\mu_n\rangle(\Omega) \to \langle\mu\rangle(\Omega)$, where, for every $A \in \mathcal{A}(\Omega)$,*

$$\langle\mu\rangle(A) := \int_A \sqrt{1 + |m^a(x)|^2}\, \mathrm{d}x + |\mu^s|(A).$$

Proposition 2.2 ([2, Proposition 1.62(b)]). *Let $\mu_n \in \mathcal{M}(\Omega; \mathbb{R}^\ell)$ be a sequence of bounded Radon measures locally weakly-* converging to μ. Then, if $|\mu_n|$ locally weakly-* converges to Λ, then $\Lambda \geqslant |\mu|$. Moreover, if E is a relatively compact μ-measurable set such that $\Lambda(\partial E) = 0$, then $\mu_n(E) \to \mu(E)$ as $n \to \infty$. More generally,*

$$\int_{\mathbb{R}^d} u(x)\, \mathrm{d}\mu(x) = \lim_{n \to \infty} \int_{\mathbb{R}^d} u(x)\, \mathrm{d}\mu_n(x),$$

for every bounded Borel function $u \colon \mathbb{R}^d \to \mathbb{R}$ with compact support, such that the set of discontinuity points is Λ-negligible.

Theorem 2.3. *Let $\mu \in \mathcal{M}(\Omega; \mathbb{R}^\ell)$ and, for $r > 0$, let α_r be as in (1.9). Then $\mu * \alpha_r \in L^1_{\mathrm{loc}}(\Omega; \mathbb{R}^\ell)$ and*

(i) *the measures $\mu_r := (\mu * \alpha_r)\mathcal{L}^N$ locally weakly-* converge to μ as $r \to 0^+$ and, for every $E \in \Omega_r$ a Borel set,*

$$\int_E |\mu * \alpha_r|(x)\, \mathrm{d}x \leqslant |\mu|(E^r);$$

the measures $|\mu_r|$ locally weakly-* converge in Ω to $|\mu|$;

(ii) *if $|\mu|(\partial\Omega) = 0$, it follows that $\langle\mu_r\rangle(\Omega) \to \langle\mu\rangle(\Omega)$, that is μ_r converges $\langle\cdot\rangle$-strictly to μ according to Definition 2.1(ii).*

The theorem above presents some straightforward generalizations of known results in measure theory to convolutions with kernels $\alpha_r \in C_b(B_r)$ only. More precisely, part (i) generalizes [2, Theorem 2.2(b) and (c)] and part (ii) generalizes [4, Proposition 2.22(iii)], both of which are proved in the case $\alpha_r \in C_c^\infty(B_r)$.

The following theorem collects two results concerning the continuity or upper-semicontinuity of functionals defined on measures. Given $\mu = m^a \mathcal{L}^N + \mu^s \in \mathcal{M}(\Omega; \mathbb{R}^\ell)$ and $\Phi \colon \Omega \times \mathbb{R}^\ell \to [0, +\infty)$ continuous, let

$$\mathscr{I}(\mu) := \int_\Omega \Phi(x, m^a(x))\, \mathrm{d}x + \int_\Omega \Phi^\infty\left(x, \frac{\mathrm{d}\mu^s}{\mathrm{d}|\mu^s|}(x)\right) \mathrm{d}|\mu^s|(x),$$

where Φ^∞ is the *recession function* of Φ at infinity, defined by

$$\Phi^\infty(x, \xi) := \limsup_{\substack{x' \to x \\ \xi' \to \xi \\ t \to +\infty}} \frac{\Phi(x', t\xi')}{t} \tag{2.1}$$

for every $x \in \overline{\Omega}$, $\xi \in \mathbb{S}^{\ell-1}$ and extended to $\mathbb{R}^\ell$ by positive 1-homogeneity (compare with (1.31)). We say that $\Phi \in E(\Omega \times \mathbb{R}^\ell)$ if Φ belongs to the class (E) defined in the introduction. Notice that if $\Phi \in E(\Omega \times \mathbb{R}^\ell)$, then Φ^∞ is a limit, namely

$$\Phi^\infty(x, \xi) = \lim_{\substack{x' \to x \\ \xi' \to \xi \\ t \to +\infty}} \frac{\Phi(x', t\xi')}{t} \tag{2.2}$$

for every $x \in \overline{\Omega}$, $\xi \in \mathbb{S}^{\ell-1}$ and extended to $\mathbb{R}^\ell$ by positive 1-homogeneity. We point out that (i) continuous and positively 1-homogeneous functions and (ii) convex functions with linear growth are two classes of functions belonging to $E(\Omega \times \mathbb{R}^\ell)$ (see [31]). We refer the reader to [30, Section 2.4] for a detailed description of the class $E(\Omega \times \mathbb{R}^\ell)$. Motivated by the fact that Lipschitz functions do not necessarily belong to $E(\Omega \times \mathbb{R}^\ell)$ (see Remark 1.4), we made the distinction between (E) and (L) in the Introduction.

Theorem 2.4. *Let $\mu_n = m_n^a \mathcal{L}^N + \mu_n^s \in \mathcal{M}(\Omega; \mathbb{R}^\ell)$ be a sequence of measures and let $\mu = m^a \mathcal{L}^N + \mu^s \in \mathcal{M}(\Omega; \mathbb{R}^\ell)$ be such that μ_n $\langle\cdot\rangle$-strictly converges to μ. Let $\Phi \colon \Omega \times \mathbb{R}^\ell \to [0, +\infty)$ be a continuous function.*

(i) *(Reshetnyak continuity theorem, [30, Theorem 4]) if $\Phi \in E(\Omega \times \mathbb{R}^\ell)$, then Φ^∞ is given by (2.2) and*

$$\mathscr{I}(\mu_n) \to \mathscr{I}(\mu) \qquad as\ n \to \infty;$$

(ii) *(Reshetnyak upper-semicontinuity theorem, [4, Corollary 2.11]) if Φ has linear growth at infinity (see (1.27)), then Φ^∞ is given by (2.1) and*

$$\mathscr{I}(\mu) \geqslant \limsup_{n \to \infty} \mathscr{I}(\mu_n).$$

We conclude this subsection by proving the following property for functions Φ belonging to the class (L).

Lemma 2.5. *Let $\Phi \colon \Omega \times \mathbb{R}^\ell \to [0, +\infty)$ belong to the class (L). Then the recession function Φ^∞ defined in (2.1) can be computed as*

$$\Phi^\infty(x, \xi) = \limsup_{t \to +\infty} \frac{\Phi(x, t\xi)}{t}, \qquad for\ all\ x \in \Omega,\ \xi \in \mathbb{R}^\ell. \tag{2.3}$$

Proof. Fix $x \in \Omega$ and $\xi \in \mathbb{R}^\ell$. The inequality $\Phi^\infty(x, \xi) \geqslant \limsup_{t \to +\infty} t^{-1}\Phi(x, t\xi)$ is obvious from the definition of $\limsup$. The proof of the converse inequality is a matter of a computation, using the subadditivity of the $\limsup$ and keeping (1.26) and (1.28) in mind. $\qquad\square$

2.3. *BV* **and** *SBV* **functions.** We start by recalling some facts on functions of bounded variation. We refer to [2] for a detailed treatment of this subject.

A function $u \in L^1(\Omega; \mathbb{R}^d)$ is said to be of *bounded variation*, and we write $u \in BV(\Omega; \mathbb{R}^d)$, if the distributional derivative $Du \in \mathcal{M}(\Omega; \mathbb{R}^{d \times N})$, that is, it is a (signed) finite Radon measure. The space $BV(\Omega; \mathbb{R}^d)$ is a Banach space when endowed with the norm $\|u\|_{BV(\Omega; \mathbb{R}^d)} := \|u\|_{L^1(\Omega; \mathbb{R}^d)} + |Du|(\Omega)$. Since this norm is too strong for practical applications, it is customary to consider the weak* convergence in BV, which is the appropriate notion for compactness properties (see [2]). We say that a sequence $u_n \in BV(\Omega; \mathbb{R}^d)$ converges weakly-* to a function $u \in BV(\Omega; \mathbb{R}^d)$, in symbols $u_n \overset{*}{\rightharpoonup} u$, if

$$u_n \to u \quad \text{in } L^1(\Omega; \mathbb{R}^d) \qquad \text{and} \qquad Du_n \overset{*}{\rightharpoonup} Du \quad \text{in } \mathcal{M}(\Omega; \mathbb{R}^{d \times N}).$$

Since $Du \in \mathcal{M}(\Omega; \mathbb{R}^{d \times N})$, it can be split into the sum of two mutually singular measures $D^a u$ and $D^s u$. By ∇u we denote the density of $D^a u$ with respect to $\mathcal{L}^N$, so that we can write

$$Du = \nabla u \, \mathcal{L}^N + D^s u.$$

The measure $D^s u$ can be further split into the sum of two contributions, $D^j u$ measuring the discontinuities of u and $D^c u$ measuring the Cantor-like behavior of the distributional derivative. In particular, denoting by S_u the set of points $x \in \Omega$ for which there exist two vectors $a, b \in \mathbb{R}^d$ and a unit vector $\nu \in \mathbb{S}^{N-1}$, normal to S_u at x, such that $a \neq b$ and

$$\lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon^N} \int_{\{y \in x + \varepsilon Q_\nu : (y-x) \cdot \nu > 0\}} |u(y) - a| \, \mathrm{d}y = 0, \qquad \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon^N} \int_{\{y \in x + \varepsilon Q_\nu : (y-x) \cdot \nu < 0\}} |u(y) - b| \, \mathrm{d}y = 0, \quad (2.4)$$

the triple (a, b, ν) is uniquely determined by (2.4) up to permutation of a and b and a change of sign of ν and is denoted by $(u^+(x), u^-(x), \nu_u(x))$. The set S_u is called the *jump set* of u and it is $(N-1)$-rectifiable. In conclusion, the distributional derivative Du can be written as the sum of three mutually singular measures as

$$Du = \nabla u \, \mathcal{L}^N + [u] \otimes \nu_u \, \mathcal{H}^{N-1} \llcorner S_u + D^c u,$$

where $[u] := u^+ - u^-$.

The space of *special functions of bounded variation*, $SBV(\Omega; \mathbb{R}^d)$ is the space of functions $u \in BV(\Omega; \mathbb{R}^d)$ such that $D^c u = 0$; therefore, for each $u \in SBV(\Omega; \mathbb{R}^d)$

$$Du = \nabla u \mathcal{L}^N + [u] \otimes \nu_u \mathcal{H}^{N-1} \llcorner S_u. \tag{2.5}$$

2.4. **Structured deformations.** Following [11], we define the set of structured deformations as

$$SD(\Omega; \mathbb{R}^d) := SBV(\Omega; \mathbb{R}^d) \times L^1(\Omega; \mathbb{R}^{d \times N}).$$

We introduce the shorthand notation $\|(g, G)\|_{SD(\Omega; \mathbb{R}^d)} := \|g\|_{BV(\Omega; \mathbb{R}^d)} + \|G\|_{L^1(\Omega; \mathbb{R}^{d \times N})}$, which we are going to denote simply by $\|(g, G)\|_{SD}$ when no domain specification is needed.

A fundamental result in the theory of structured deformations is the Approximation Theorem [14, Theorem 5.8], a counterpart of which was recovered in [11, Theorem 2.12] in the SBV framework. Its proof is a consequence of the following two results.

Theorem 2.6 ([1, Theorem 3]). *Let $f \in L^1(\Omega; \mathbb{R}^{d \times N})$. Then there exist $u \in SBV(\Omega; \mathbb{R}^d)$, a Borel function $\beta \colon \Omega \to \mathbb{R}^{d \times N}$, and a constant $C > 0$ depending only on N such that*

$$Du = f \, \mathcal{L}^N + \beta \mathcal{H}^{N-1} \llcorner S_u, \qquad \int_{S_u \cap \Omega} |\beta(x)| \, \mathrm{d}\mathcal{H}^{N-1}(x) \leqslant C \|f\|_{L^1(\Omega; \mathbb{R}^{d \times N})}.$$

Lemma 2.7 ([11, Lemma 2.9]). *Let $u \in BV(\Omega; \mathbb{R}^d)$. Then there exist piecewise constant functions $\bar{u}_n \in SBV(\Omega; \mathbb{R}^d)$ such that $\bar{u}_n \to u$ in $L^1(\Omega; \mathbb{R}^d)$ and*

$$|Du|(\Omega) = \lim_{n \to \infty} |D\bar{u}_n|(\Omega) = \lim_{n \to \infty} \int_{S_{\bar{u}_n}} |[\bar{u}_n](x)| \, \mathrm{d}\mathcal{H}^{N-1}(x).$$

Theorem 2.8 (Approximation Theorem (see [11, Theorem 2.12])). *Let $(g, G) \in SD(\Omega; \mathbb{R}^d)$. Then there exist $u_n \in SBV(\Omega; \mathbb{R}^d)$ such that $u_n \xrightarrow{SD} (g, G)$ in the sense of (1.1) and there exists $C > 0$ such that*

$$|Du_n|(\Omega) \leqslant C \big(1 + \|(g, G)\|_{SD}\big). \tag{2.6}$$

3. Relaxation of the non-local energy $\mathcal{E}^{\alpha_r}$

We start by proving that for each structured deformation $(g, G) \in SD(\Omega; \mathbb{R}^d)$ the relaxed energy $\mathcal{I}^{\alpha_r}(g, G)$ defined in (1.14) is finite.

Lemma 3.1. *Let* $\Omega \subset \mathbb{R}^N$ *be a bounded Lipschitz domain, let* $\Psi \colon \Omega \times \mathbb{R}^{d \times N} \to [0, +\infty)$ *be a continuous function and, for* $r > 0$, *let* α_r *be as in* (1.9). *Then for any* $(g, G) \in SD(\Omega; \mathbb{R}^d)$ *the relaxed energy* $\mathcal{I}^{\alpha_r}(g, G)$ *defined in* (1.14) *is finite, namely there exists a positive constant* $\overline{C}_r$ *(given in* (3.5) *below) such that*

$$\mathcal{I}^{\alpha_r}(g, G) \leqslant \overline{C}_r. \tag{3.1}$$

In addition, if Ψ *has at most linear growth at infinity, then the estimate above is uniform in* r, *namely there exists a positive constant* $\overline{C}$ *(given in* (3.6) *below) such that*

$$\mathcal{I}^{\alpha_r}(g, G) \leqslant \overline{C}. \tag{3.2}$$

Proof. Let $(g, G) \in SD(\Omega; \mathbb{R}^d)$ and let $u_n \in SBV(\Omega; \mathbb{R}^d)$ be the sequence provided by the Approximation Theorem 2.8. Then

$$I^{\alpha_r}(g, G) \leqslant \liminf_{n \to \infty} E^{\alpha_r}(u_n) = \liminf_{n \to \infty} \int_{\Omega_r} \Psi\big(x, (D^s u_n * \alpha_r)(x)\big) \, \mathrm{d}x. \tag{3.3}$$

Recalling that for every $x \in \Omega_r$

$$(D^s u_n * \alpha_r)(x) = \int_{\Omega_r \cap B_r(x) \cap S_{u_n}} \alpha_r(x - y)[u_n](y) \otimes \nu_{u_n}(y) \, \mathrm{d}\mathcal{H}^{N-1}(y),$$

we have

$$|(D^s u_n * \alpha_r)(x)| \leqslant C_{\alpha_r} |D^s u_n|(B_r(x)) \leqslant C_{\alpha_r} |Du_n|(\Omega) \leqslant C_{\alpha_r} C\big(1 + \|(g, G)\|_{SD}\big),$$

where $C_{\alpha_r} = \sup_{B_r} |\alpha_r|$ and we have used (2.6). This implies that, for every $x \in \Omega_r$, $(D^s u_n * \alpha_r)(x)$ will take values in the closed ball of radius $C_{\alpha_r} \|(g, G)\|_{SD}$ in $\mathbb{R}^{d \times N}$. Therefore, the continuity of Ψ implies that, for every $x \in \Omega_r$,

$$\big|\Psi\big(x, (D^s u_n * \alpha_r)(x)\big)\big| \leqslant \max\Big\{\Psi(x, A) : x \in \Omega_r, |A| \leqslant C_{\alpha_r} C\big(1 + \|(g, G)\|_{SD(\Omega; \mathbb{R}^d)}\big)\Big\}. \tag{3.4}$$

In conclusion,

$$I^{\alpha_r}(g, G) \leqslant |\Omega_r| \max\Big\{\Psi(x, A) : x \in \Omega_r, |A| \leqslant C_{\alpha_r} C\big(1 + \|(g, G)\|_{SD(\Omega; \mathbb{R}^d)}\big)\Big\}, \tag{3.5}$$

which is (3.1).

Consider now a continuous Ψ with at most linear growth at infinity. Then we can bound the integral in the right-hand side of (3.3) and, reasoning as in the proof of [2, Theorem 2.2(b)], we obtain, using also (2.6),

$$\int_{\Omega_r} C_\Psi(1 + |(D^s u_n * \alpha_r)(x)|) \, \mathrm{d}x \leqslant C_\Psi(|\Omega_r| + |D^s u_n|(\Omega)) \leqslant C_\Psi C\big(|\Omega| + 1 + \|(g, G)\|_{SD(\Omega; \mathbb{R}^d)}\big), \tag{3.6}$$

which provides (3.2).

It is easy to see that the same estimates (3.5) and (3.6) also hold for $\tilde{I}^{\alpha_r}(g, G)$. $\qquad\square$

Proof of Theorem 1.2. We start by proving part (i), namely we take $\mathcal{E}^{\alpha_r} = E^{\alpha_r}$ from (1.8), $\alpha \in C_0(B_1)$, and the relaxation problem (1.14) with respect to the convergence $\xrightarrow{SD}$ (see (1.1)). Let $(g, G) \in SD(\Omega; \mathbb{R}^d)$ be fixed and let $\mu = (\nabla g - G)\mathcal{L}^N + D^s g$ (as in (1.21)). For any admissible sequence $u_n \in SBV(\Omega; \mathbb{R}^d)$ for the relaxation problem (1.14), let $\mu_n := D^s u_n$ and observe that, since u_n converges to (g, G) in the sense of (1.1), we have

$$\mu_n \overset{*}{\rightharpoonup} \mu. \tag{3.7}$$

By standard results on convolutions of measures, (see, e.g., [2, page 41]), we then have

$$\mu_n * \alpha_r \to \mu * \alpha_r \quad \text{uniformly on compact sets as } n \to \infty,$$

which, in particular, ensures that $\mu_n * \alpha_r \to \mu * \alpha_r$ and $\Psi(x, \mu_n * \alpha_r) \to \Psi(x, \mu * \alpha_r)$ $\mathcal{L}^N$-a.e. in Ω_r, as $n \to \infty$.

We now wish to prove that (1.16) hold. To this end, it is sufficient to show that

$$\lim_{n \to \infty} E^{\alpha_r}(u_n) = \int_{\Omega_r} \Psi\big(x, (\mu * \alpha_r)(x)\big) \, \mathrm{d}x. \tag{3.8}$$

We observe that the integrand in the right hand side of (3.8) is bounded $\mathcal{L}^N$-a.e., by the continuity of Ψ and the fact that $\mu * \alpha_r \in C_0(\mathbb{R}^d)$; indeed, analogously to (3.4), the following estimate holds

$$|\Psi(x, \mu * \alpha_r)(x)| \leqslant \max\left\{\Psi(x, A) : x \in \Omega_r, |A| \leqslant C_{\alpha_r}\|(g, G)\|_{SD}\right\},$$

Moreover, (3.4) guarantees that, the the sequence of functions $\Psi(\cdot, \mu_n * \alpha_r(\cdot))$ is equi-integrable, thus (3.8) follows by the Vitali-Lebesgue convergence theorem (see [23, Theorem 2.24]). This concludes the proof of part (i).

We now prove part (ii), namely we take $\mathcal{E}^{\alpha_r} = \widetilde{E}^{\alpha_r}$ from (1.12) with α_r as in (1.9) and the relaxation problem (1.14) with respect to the convergence $\overset{SD}{\longrightarrow}$ (see (1.1) and (1.15)). Let us fix a structured deformation $(g, G) \in SD(\Omega; \mathbb{R}^d)$ and let us recall that S_g is a $(N-1)$-rectifiable set, so that for every fixed $r \in (0, 1)$, the set

$$\mathcal{N}_r := \{x \in \Omega : \mathcal{H}^{N-1}(\partial B_r(x) \cap S_g) > 0\}$$

has Lebesgue measure zero.

We notice that, for any sequence $u_n \overset{SD}{\longrightarrow} (g, G)$ and for μ_n, μ as defined in the proof of part (i), (3.7) holds; moreover, by Proposition 2.2 we can infer that for every $x \in \Omega \setminus \mathcal{N}_r$

$$D^s u_n(\Omega \cap B_r(x)) \to \int_{B_r(x) \cap \Omega} (\nabla g(y) - G(y))\, \mathrm{d}y + \int_{\Omega \cap B_r(x) \cap S_g} [g](y) \otimes \nu_g(y)\, \mathrm{d}\mathcal{H}^{N-1}(y). \qquad (3.9)$$

For every $x \in \Omega \setminus \mathcal{N}_r$ (that is, for $\mathcal{L}^N$-a.e. $x \in \Omega$) we define

$$f_{r,n}(x) := \int_{\Omega \cap B_r(x) \cap S_{u_n}} \alpha_r(y - x)[u_n](y) \otimes \nu_{u_n}(y)\, \mathrm{d}\mathcal{H}^{N-1}(y),$$

$$f_r(x) := \int_{\Omega \cap B_r(x)} \alpha_r(y - x)(\nabla g(y) - G(y))\, \mathrm{d}y + \int_{\Omega \cap B_r(x) \cap S_g} \alpha_r(y - x)[g](y) \otimes \nu_g(y)\, \mathrm{d}\mathcal{H}^{N-1}(y).$$

By classical results (see, e.g., [3]), $f_{r,n}$ and f_r are $\mathcal{L}^N$-measurable in Ω if we prove that they are continuous $\mathcal{L}^N$-a.e. in Ω. Indeed, fix $x, x' \in \Omega \setminus \mathcal{N}_r$ such that $|D^s u_n|(\partial B_r(x)) = 0$ (this happens for $\mathcal{L}^N$-a.e. x); we have

$$|f_{r,n}(x) - f_{r,n}(x')| = \left|\int_{\Omega \cap (B_r(x) \triangle B_r(x')) \cap S_{u_n}} (\alpha_r(y - x) - \alpha_r(y - x'))[u_n](y) \otimes \nu_{u_n}(y)\, \mathrm{d}\mathcal{H}^{N-1}(y)\right|$$
$$\leqslant C_{\alpha_r}|D^s u_n|(\Omega \cap (B_r(x) \triangle B_r(x'))) \to 0$$

when $x' \to x$, where $C_{\alpha_r} = \sup_{B_r}|\alpha_r|$, which proves the continuity of $f_{r,n}$ for $\mathcal{L}^N$-a.e. $x \in \Omega$. The continuity of f_r can be proved in a similar way. Moreover, we have the bound

$$\int_\Omega |f_{r,n}(x)|\, \mathrm{d}x \leqslant C_{\alpha_r}|\Omega||D^s u_n|(\Omega) < +\infty, \qquad (3.10)$$

which proves that $f_{r,n} \in L^1(\Omega; \mathbb{R}^{d \times N})$; similarly we can conclude that also $f_r \in L^1(\Omega; \mathbb{R}^{d \times N})$. Finally, by (3.9), since $\alpha_r \in C_b(B_r)$ and hence in $\mathcal{E}'(\mathbb{R}^N)$ (the space of continuous linear forms on $C^\infty(\mathbb{R}^N)$), and by [26, Theorem 5.1.3] we conclude that $f_{r,n} \overset{*}{\rightharpoonup} f_r$ in the sense of distributions. It is easily seen that the bound in (3.10) also entails that the limit is also in the sense of measures.

Defining

$$\bar{f}_{r,n}(x) := \int_{\Omega \cap B_r(x) \cap S_{u_n}} \alpha_r(y - x)|[u_n](y) \otimes \nu_{u_n}(y)|\, \mathrm{d}\mathcal{H}^{N-1}(y),$$

$$\bar{f}_r(x) := \int_{\Omega \cap B_r(x)} \alpha_r(y - x)\Lambda_g^a(y)\, \mathrm{d}y + \int_{\Omega \cap B_r(x) \cap S_g} \alpha_r(y - x)\Lambda_g^s(y)\, \mathrm{d}\mathcal{H}^{N-1}(y),$$

and recalling (1.15) we obtain that $\bar{f}_{r,n} \overset{*}{\rightharpoonup} \bar{f}_r$, by applying [26, Theorem 5.1.3] once again. Again by Proposition 2.2 we get that $f_{r,n} \to f_r$ $\mathcal{L}^N$-a.e. in Ω.

The proof of

$$\lim_{n \to \infty} \int_\Omega \Psi(x, f_{r,n}(x))\, \mathrm{d}x = \int_\Omega \Psi(x, f_r(x))\, \mathrm{d}x$$

follows along the lines of the proof of (3.8), and this completes the proof of Theorem 1.2. $\qquad \square$

4. The limit as $r \to 0$

In this section we deal with the limit (1.20), that is, we find an explicit formula for the energy in the limit as the measure of non-locality r tends to zero. As mentioned in the introduction, we restrict our attention to continuous functions $\Psi \colon \Omega \times \mathbb{R}^{d \times N} \to [0, +\infty)$, belonging to the class (E) or (L). We start with the assertion that, in the limit $r \to 0^+$, the difference between functionals I^{α_r} in (1.16) and $\tilde{I}^{\alpha_r}$ in (1.18) vanishes. To do this, we rely on the formalism of the functional $\widetilde{\mathscr{I}}$ defined in (1.22), where μ_r and μ are the measures introduced in (1.21).

Lemma 4.1. *Let $\Omega \subset \mathbb{R}^N$ be a bounded Lipschitz domain, let $\Psi \colon \Omega \times \mathbb{R}^{d \times N} \to [0, +\infty)$ be a continuous function with (at most) linear growth at infinity, that is, there exists $C_\Psi > 0$ such that $\Psi(x, \xi) \leqslant C_\Psi(1 + |\xi|)$ for each $x \in \overline{\Omega}$, $\xi \in \mathbb{R}^{d \times N}$, and let $\alpha \in C_b(B_1)$ satisfy (1.9). Then for any $(g, G) \in SD(\Omega; \mathbb{R}^d)$*

$$\widetilde{\mathscr{I}}(\mu_r; \Omega \setminus \overline{\Omega}_r) \leqslant C_\Psi \big(|\Omega \setminus \Omega_r| + \|(g, G)\|_{SD(N_r; \mathbb{R}^d)} \big); \tag{4.1}$$

where $N_r = (\Omega \setminus \overline{\Omega}_r)^r$ is the r-neighborhood of $\Omega \setminus \overline{\Omega}_r$. In particular,

$$\lim_{r \to 0^+} I^{\alpha_r}(g, G) = \lim_{r \to 0^+} \tilde{I}^{\alpha_r}(g, G). \tag{4.2}$$

Proof. Let us fix $(g, G) \in SD(\Omega; \mathbb{R}^d)$. It is possible to extend g outside of Ω without adding jumps at the boundary $\partial \Omega$ (see [27], also [28, Theorem 1.4] for a simple proof). Extend G by zero outside of Ω as well, to obtain $(\tilde{g}, \tilde{G}) \in SD(\Omega + B_2; \mathbb{R}^d)$ such that

$$(\tilde{g}, \tilde{G})|_\Omega = (g, G) \qquad \text{and} \qquad \|(\tilde{g}, \tilde{G})\|_{SD(\Omega + B_2; \mathbb{R}^d)} \leqslant 2\|(g, G)\|_{SD(\Omega; \mathbb{R}^d)}. \tag{4.3}$$

Define the measure $\tilde{\mu} := (\nabla \tilde{g} - \tilde{G})\mathcal{L}^N + D^s \tilde{g}$ and observe that $\tilde{\mu}|_\Omega = \mu$, where μ is defined (1.20); define $\tilde{\mu}_r$ accordingly. Then, by the (at most) linear growth of Ψ and the properties of convolutions (see [2, Lemma 2.2(b)]), we have that

$$\widetilde{\mathscr{I}}(\mu_r; \Omega \setminus \overline{\Omega}_r) = \widetilde{\mathscr{I}}(\tilde{\mu}_r; \Omega \setminus \overline{\Omega}_r) = \int_{\Omega \setminus \overline{\Omega}_r} \Psi\big(x, ((\nabla \tilde{g} - \tilde{G})\mathcal{L}^N * \alpha_r)(x) + (D^s \tilde{g} * \alpha_r)(x)\big) \, \mathrm{d}x$$

$$\leqslant C_\Psi \int_{\Omega \setminus \overline{\Omega}_r} \big(1 + |((\nabla \tilde{g} - \tilde{G}) * \alpha_r)(x)| + |(D^s \tilde{g} * \alpha_r)(x)|\big) \, \mathrm{d}x$$

$$\leqslant C_\Psi \Big(|\Omega \setminus \Omega_r| + \|(\tilde{g}, \tilde{G})\|_{SD(N_r; \mathbb{R}^d)} \Big),$$

which is (4.1). Equality (4.2) follows immediately from (1.24) and (1.22), keeping in mind that $|D^s \tilde{g}|(\partial \Omega) = 0$. The lemma is proved. $\qquad \square$

Now we prove Theorem 1.5.

Proof of Theorem 1.5. Let us fix $(g, G) \in SD(\Omega; \mathbb{R}^d)$. Recalling the definition of μ_r from (1.21), we claim that

$$\mu_r \overset{*}{\rightharpoonup} \mu \quad \text{in } \mathcal{M}(\Omega; \mathbb{R}^{d \times N}) \tag{4.4a}$$

and

$$|\mu_r| \overset{*}{\rightharpoonup} |\mu| \quad \text{in } \mathcal{M}^+(\Omega). \tag{4.4b}$$

Indeed, starting from the extension $(\tilde{g}, \tilde{G}) \in SD(\Omega + B_2; \mathbb{R}^d)$ of the proof of Lemma 4.1, construct $\tilde{\mu}$ and $\tilde{\mu}_r$ according to (1.21) and notice that $\tilde{\mu} \llcorner \Omega = \mu$, by the first relationship in (4.3). Therefore, by Theorem 2.3(i) we get

$$|\mu_r|(\Omega) \leqslant |\tilde{\mu}_r|(\Omega) = \int_\Omega |(\tilde{\mu} * \alpha_r)(x)| \, \mathrm{d}x = \int_\Omega \left| \int_{B_r(x)} \alpha_r(x - y) \, \mathrm{d}\tilde{\mu}(y) \right| \mathrm{d}x \leqslant \int_\Omega \int_{B_r(x)} \alpha_r(x - y) \, \mathrm{d}|\tilde{\mu}|(y) \mathrm{d}x$$

$$= \int_{\Omega^r} \int_\Omega \alpha_r(x - y) \, \mathrm{d}x \, \mathrm{d}|\tilde{\mu}|(y) = |\tilde{\mu}|(\Omega^r) \leqslant |\tilde{\mu}|(\Omega + B_2) < +\infty,$$

where the last inequality follows from the second relationship in (4.3). Thus, up to a subsequence (not relabeled) of μ_r, there exists $\mu^* \in \mathcal{M}(\Omega; \mathbb{R}^{d \times N})$ such that $\mu_r \overset{*}{\rightharpoonup} \mu^*$ and there exists $\lambda \in \mathcal{M}^+(\Omega)$ such that $|\mu_r| \overset{*}{\rightharpoonup} \lambda$. Now it remains to identify μ^* and λ. Recall that convolution kernels α_r as in (1.9) converge in the sense of distributions in Ω to the Dirac delta δ_0 centred at the origin as $r \to 0^+$. Thus, μ_r converges to

μ in the sense of distributions as $r \to 0^+$ by [26, Theorem 5.1.3]. On the other hand, the uniqueness of the distributional limit and $\mu_r \overset{*}{\rightharpoonup} \mu^*$ entail that $\mu^* = \mu$. This proves (4.4a).

The nonnegativity of λ and the lower semicontinuity of the total variation with respect to the weak* convergence in the sense of measures entail that $\lambda \geqslant |\mu|$. On the other hand, since $|\mu_r| \leqslant |\mu| * \alpha_r$, by applying [26, Theorem 5.1.3] to the sequence $|\mu| * \alpha_r$, we obtain $|\mu| \geqslant \lambda$. Thus, $\lambda = |\mu|$ so that (4.4b) is proved. In particular, we have

$$|\mu_r| \overset{*}{\rightharpoonup} |\nabla g - G| \mathcal{L}^N + |D^s g|. \tag{4.5}$$

Finally, in order to prove that $\mu_r \langle \cdot \rangle$-strict converges to μ (see Definition 2.1(ii)) we are left to showing that

$$\langle \mu_r \rangle(\Omega) \to \langle \mu \rangle(\Omega).$$

To this end, it is sufficient to exploit the same argument as in [4, Lemma 2.2(iii)], replacing the sequence of standard mollifiers therein by α_r, and exploiting the convergence of $\mu_r \overset{*}{\rightharpoonup} \mu$, convergence (4.5), and the fact that $|\mu|(\partial \Omega) = 0$.

Finally, if Ψ belongs to the class (E), since the $\liminf$ in the definition of Ψ^∞ is indeed a limit (see Remark 1.6) we can apply Theorem 2.4(i), to obtain (1.29).

If Ψ belongs to the class (L), Theorem 2.4(ii) provides the upper bound

$$\limsup_{r \to 0^+} \int_\Omega \Psi\big(x, (\mu * \alpha_r)(x)\big) \, \mathrm{d}x \leqslant \int_\Omega \Psi\big(x, (\nabla g - G)(x)\big) \, \mathrm{d}x + \int_{\Omega \cap S_g} \Psi^\infty\Big(x, \frac{\mathrm{d}D^s g}{\mathrm{d}|D^s g|}(x)\Big) \, \mathrm{d}|D^s g|(x) \tag{4.6}$$

We now prove that

$$\int_\Omega \Psi\big(x, \nabla g(x) - G(x)\big) \, \mathrm{d}x + \int_{\Omega \cap S_g} \Psi^\infty\Big(x, \frac{\mathrm{d}D^s g}{\mathrm{d}|D^s g|}(x)\Big) \, \mathrm{d}|D^s g|(x) \leqslant \liminf_{r \to 0^+} \int_\Omega \Psi\big(x, (\mu * \alpha_r)(x)\big) \, \mathrm{d}x \tag{4.7}$$

To this end, set $\{\theta_r\} \subset \mathcal{M}^+(\Omega)$ by $\theta_r := \Psi(\cdot, (\mu * \alpha_r)(\cdot))\mathcal{L}^N$. Since this is a bounded sequence of Radon measures, it converges weakly-* to some positive measure θ. We obtain (4.7) if we show that

$$\frac{\mathrm{d}\theta}{\mathrm{d}\mathcal{L}^N}(x) \geqslant \Psi\big(x, (\nabla g - G)(x)\big) \quad \text{for } \mathcal{L}^N\text{-a.e. } x \in \Omega, \tag{4.8a}$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}|D^s g|}(x) \geqslant \Psi^\infty\Big(x, \frac{\mathrm{d}D^s g}{\mathrm{d}|D^s g|}(x)\Big) \quad \text{for } \mathcal{H}^{N-1}\text{-a.e. } x \in S_g. \tag{4.8b}$$

We start with (4.8a). By the linearity of the convolution operator and the definition of μ_r, we know that, as $r \to 0^+$,

$$\big((\nabla g - G)\mathcal{L}^N * \alpha_r\big)\mathcal{L}^N \overset{*}{\rightharpoonup} (\nabla g - G)\mathcal{L}^N \qquad \text{and} \qquad (D^s g * \alpha_r)\mathcal{L}^N \overset{*}{\rightharpoonup} D^s g, \tag{4.9}$$

and, by [29, Corollary 2.1.17], we have

$$(\mu * \alpha_r)(x) \to (\nabla g - G)(x) \qquad \text{for } \mathcal{L}^N\text{-a.e. } x \in \Omega. \tag{4.10}$$

Let us fix $x_0 \in \Omega \setminus S_g$ which is a Lebesgue point for $\nabla g - G$ and let us compute

$$\frac{\mathrm{d}\theta}{\mathrm{d}\mathcal{L}^N}(x_0) = \lim_{k \to \infty} \frac{\theta(Q_{\delta_k}(x_0))}{\mathcal{L}^N(Q_{\delta_k}(x_0))} = \lim_{k \to \infty} \lim_{r \to 0^+} \frac{\theta_r(Q_{\delta_k}(x_0))}{\delta_k^N} = \lim_{k \to \infty} \lim_{r \to 0^+} \frac{1}{\delta_k^N} \int_{Q_{\delta_k}(x_0)} \Psi(x, (\mu * \alpha_r)(x)) \, \mathrm{d}x$$

$$= \lim_{k \to \infty} \lim_{r \to 0^+} \frac{1}{\delta_k^N} \int_{Q_{\delta_k}(x_0)} \Psi\big(x, ((\nabla g - G)\mathcal{L}^N * \alpha_r)(x) + (D^s g * \alpha_r)(x)\big) \, \mathrm{d}x$$

$$\geqslant \lim_{k \to \infty} \lim_{r \to 0^+} \frac{1}{\delta_k^N} \int_{Q_{\delta_k}(x_0)} \Psi\big(x, ((\nabla g - G)\mathcal{L}^N * \alpha_r)(x)\big) \, \mathrm{d}x - \lim_{k \to \infty} \lim_{r \to 0^+} \frac{L_\Psi}{\delta_k^N} \int_{Q_{\delta_k}(x_0)} |(D^s g * \alpha_r)(x)| \, \mathrm{d}x,$$

where we have used (1.26). Since, by the second convergence in (4.9), the last integral is the Radon-Nikodým of $|D^s g|$ with respect to $\mathcal{L}^N$, it vanishes, so, that we have

$$\frac{\mathrm{d}\theta}{\mathrm{d}\mathcal{L}^N}(x_0) \geqslant \lim_{k \to \infty} \lim_{r \to 0^+} \frac{1}{\delta_k^N} \int_{Q_{\delta_k}(x_0)} \Psi\big(x, ((\nabla g - G)\mathcal{L}^N * \alpha_r)(x)\big) \, \mathrm{d}x \geqslant \lim_{k \to \infty} \frac{1}{\delta_k^N} \int_{Q_{\delta_k}(x_0)} \Psi\big(x, (\nabla g - G)(x)\big) \, \mathrm{d}x$$

$$\geqslant \lim_{k \to \infty} \int_Q \Psi\big(x_0 + \delta_k y, (\nabla g - G)(x_0 + \delta_k y)\big) \, \mathrm{d}y \geqslant \Psi\big(x_0, (\nabla g - G)(x_0)\big),$$

where we have used the continuity of Ψ, (4.10), and Fatou's Lemma in the first line above, a change of variables and (1.26) and (1.28) in the following estimates. This proves (4.8a).

To prove (4.8b), let us fix $x_0 \in S_g$ and let $\tau(x_0) := \dfrac{\mathrm{d} D^s g}{\mathrm{d}|D^s g|}(x_0)$. By Lemma 2.5, the recession function $\Psi^\infty(x_0, \tau(x_0))$ can be computed using formula (2.3). Let now $\{t_k\}$ be a sequence diverging to $+\infty$ as $k \to \infty$ along which the $\limsup$ in (2.3) is indeed a limit, that is,

$$\Psi^\infty(x_0, \tau(x_0)) = \lim_{k \to \infty} \frac{\Psi(x_0, t_k \tau(x_0))}{t_k}.$$

Since Ψ is Lipschitz continuous, a reasoning analogous to that of [4, Lemma 4.2] grants that the sequence $\{t_k\}$ can be chosen as

$$t_k := \frac{|D^s g|(Q_{\delta_k}(x_0))}{\delta_k^N}, \tag{4.11}$$

with $\{\delta_k\}$ a vanishing sequence such that $\theta(\partial Q_{\delta_k}(x_0)) = 0$. Then

$$\frac{\mathrm{d}\theta}{\mathrm{d}|D^s g|}(x_0) = \lim_{k \to \infty} \frac{\theta(Q_{\delta_k}(x_0))}{|D^s g|(Q_{\delta_k}(x_0))} = \lim_{k \to \infty} \lim_{r \to 0^+} \frac{\theta_r(Q_{\delta_k}(x_0))}{|D^s g|(Q_{\delta_k}(x_0))} = \lim_{k \to \infty} \lim_{r \to 0^+} \frac{\displaystyle\int_{Q_{\delta_k}(x_0)} \Psi\big(x, (\mu * \alpha_r)(x)\big)\,\mathrm{d}x}{|D^s g|(Q_{\delta_k}(x_0))}$$

$$= \lim_{k \to \infty} \lim_{r \to 0^+} \frac{1}{t_k} \int_Q \Psi\big(x_0 + \delta_k y, (\mu * \alpha_r)(x_0 + \delta_k y)\big)\,\mathrm{d}y$$

where the last equality follows by a change of variables, taking (4.11) into account. Defining

$$w_{k,r}(y) := \frac{(\mu * \alpha_r)(x_0 + \delta_k y)}{t_k},$$

we can continue the chain of equalities above as follows

$$\frac{\mathrm{d}\theta}{\mathrm{d}|D^s g|}(x_0) = \lim_{k \to \infty} \lim_{r \to 0^+} \frac{1}{t_k} \int_Q \Psi\big(x_0 + \delta_k y, (\mu * \alpha_r)(x_0 + \delta_k y)\big)\,\mathrm{d}y$$

$$= \lim_{k \to \infty} \lim_{r \to 0^+} \frac{1}{t_k} \int_Q \Psi\big(x_0 + \delta_k y, t_k w_{k,r}(y)\big)\,\mathrm{d}y$$

$$\geqslant \lim_{k \to \infty} \lim_{r \to 0^+} \bigg[\frac{1}{t_k} \int_Q \Psi\big(x_0, t_k \tau(x_0)\big)\,\mathrm{d}y - L_\Psi \int_Q \big|w_{k,r}(y) - \tau(x_0)\big|\,\mathrm{d}y$$

$$- \frac{1}{t_k} \int_Q \omega(\delta_k|y|)(1 + t_k|\tau(x_0)|)\,\mathrm{d}y\bigg] \geqslant \Psi^\infty(x_0, \tau(x_0)),$$

where we have used (1.26) and (1.28) and where the last two terms in the square bracket vanish since $\lim_{k \to \infty} \lim_{r \to 0^+} \int_Q w_{k,r}(y)\,\mathrm{d}y = \tau(x_0)$ by (4.11) and by the properties of the modulus of continuity ω. This concludes the proof of (4.8b) and, consequently, of (4.7). Combining (4.6) and (4.7) yields a chain of equalities, which is precisely (1.30). $\qquad\square$

Recalling (1.32) in Remark 1.6, the limiting energy $\widetilde{\mathscr{I}}(\mu; \Omega)$ in (1.30) can be written as

$$I(g, G) = \int_\Omega \Psi\big(x, \nabla g(x) - G(x)\big)\,\mathrm{d}x + \int_{\Omega \cap S_g} \Psi^\infty\Big(x, \frac{\mathrm{d} D^s g}{\mathrm{d}|D^s g|}(x)\Big)\,\mathrm{d}|D^s g|(x)$$

$$= \int_\Omega \Psi\big(x, \nabla g(x) - G(x)\big)\,\mathrm{d}x + \int_{\Omega \cap S_g} \Psi^\infty\big(x, [g](x) \otimes \nu_g(x)\big)\,\mathrm{d}\mathcal{H}^{N-1}(x) \tag{4.12}$$

Moreover, as a particular case of Ψ with sublinear growth, one can consider a bounded Ψ. In this case, the formula above reduces clearly to (1.33) (since $\Psi^\infty = 0$) and we observe that the proof of Theorem 1.5 can be obtained by a simple application of the Lebesgue Dominated Convergence Theorem, in view of (4.10).

5. Coupling local and non-local energies

In this section we extend the results first proved in the pioneering paper [11] to the case of x-dependent energy densities. The integral representation results [11, Theorems 2.16 and 2.17] are expected to hold with the obvious modifications, namely with the relaxed energy densities depending on x as well. This generalization is somewhat natural and can be obtained with minor modifications to the original proofs, but since it is not presented elsewhere, we highlight here the adaptation of the proofs from [11] for sake of completeness.

5.1. **Relaxation of the local energy E_L.** In this subsection we present the relaxation results for local energies, like E_L defined in (1.2), contained in the pioneering paper [11]. We start by introducing the assumptions on the bulk and interfacial energy densities W and ψ. Let $p \geqslant 1$ and let $W \colon \Omega \times \mathbb{R}^{d \times N} \to [0, +\infty[$ and $\psi \colon \Omega \times \mathbb{R}^d \times \mathbb{S}^{N-1} \to [0, +\infty[$ be continuous functions satisfying the following conditions

$(W1)_p$ there exists $C > 0$ such that, for all $x \in \Omega$ and $A, B \in \mathbb{R}^{d \times N}$,

$$|W(x, A) - W(x, B)| \leqslant C|A - B|\big(1 + |A|^{p-1} + |B|^{p-1}\big)$$

$(W2)$ there exist $C, T > 0$ and $0 < \alpha < 1$ such that, for all $x \in \Omega$ and $A \in \mathbb{R}^{d \times N}$ with $|A| = 1$,

$$\left| W^\infty(x, A) - \frac{W(x, tA)}{t} \right| \leqslant \frac{C}{t^\alpha},$$

for all $t > T$, where W^∞ denotes the *recession function* at infinity of W (with respect to A), namely

$$W^\infty(x, A) := \limsup_{t \to +\infty} \frac{W(x, tA)}{t}; \tag{5.1}$$

$(W3)$ there exists a continuous function $\omega_W \colon [0, +\infty) \to [0, +\infty)$ with $\omega_W(s) \to 0$ as $s \to 0^+$ such that, for every $x_0 \in \Omega$ and $A \in \mathbb{R}^{d \times N}$,

$$|W(x, A) - W(x_0, A)| \leqslant \omega_W(|x - x_0|)(1 + |A|^p);$$

$(\psi 1)$ there exist $c, C > 0$ such that, for all $x \in \Omega$, $\lambda \in \mathbb{R}^d$, and $\nu \in \mathbb{S}^{N-1}$,

$$c|\lambda| \leqslant \psi(x, \lambda, \nu) \leqslant C|\lambda|;$$

$(\psi 2)$ (positive 1-homogeneity) for all $x \in \Omega$, $\lambda \in \mathbb{R}^d$, $\nu \in \mathbb{S}^{N-1}$, and $t > 0$

$$\psi(x, t\lambda, \nu) = t\psi(x, \lambda, \nu),$$

$(\psi 3)$ (sub-additivity) for all $x \in \Omega$, $\lambda_1, \lambda_2 \in \mathbb{R}^d$, and $\nu \in \mathbb{S}^{N-1}$,

$$\psi(x, \lambda_1 + \lambda_2, \nu) \leqslant \psi(x, \lambda_1, \nu) + \psi(x, \lambda_2, \nu).$$

$(\psi 4)$ there exists a continuous function $\omega_\psi \colon [0, +\infty) \to [0, +\infty)$ with $\omega_\psi(s) \to 0$ as $s \to 0^+$ such that, for every $x_0 \in \Omega$, $\lambda \in \mathbb{R}^d$, and $\nu \in \mathbb{S}^{N-1}$,

$$|\psi(x, \lambda, \nu) - \psi(x_0, \lambda, \nu)| \leqslant \omega_\psi(|x - x_0|)|\lambda|.$$

Given W and ψ as above, and $u \in SBV(\Omega; \mathbb{R}^d)$, we defined the initial energy $E_L(u)$ as

$$E_L(u) := \int_\Omega W(x, \nabla u(x)) \, \mathrm{d}x + \int_{\Omega \cap S_u} \psi(x, [u](x), \nu_u(x)) \, \mathrm{d}\mathcal{H}^{N-1}(x) \tag{5.2}$$

and, given $(g, G) \in SD(\Omega)$, we defined the relaxed energies $I_p(g, G)$ as

$$I_p(g, G) := \inf_{\{u_n\} \subset SBV(\Omega; \mathbb{R}^d)} \Big\{ \liminf_{n \to \infty} E_L(u_n) : u_n \to (g, G) \text{ in the sense of (1.1)},$$
$$(1 - \delta_1(p)) \sup_n \|\nabla u_n\|_{L^p(\Omega; \mathbb{R}^{d \times N})} < \infty \Big\}. \tag{5.3}$$

In the formula above, and in the sequel, we use the symbol $\delta_1(p)$ as the Kronecker delta computed at p, namely $\delta_1(p) = 1$ if $p = 1$ and zero otherwise, and use it as a selector between the cases $p = 1$ and $p > 1$. In particular, in (5.3), the control on the L^p norm of $|\nabla u_n|$ does not appear in the formula if $p = 1$, since, in that case $1 - \delta_1(p) = 0$.

We introduce now the classes of competitors for the cell formulae for the relaxed bulk and surface energy densities. For $A, B \in \mathbb{R}^{d \times N}$ let

$$\mathcal{C}_p^{\mathrm{bulk}}(A, B) := \left\{ u \in SBV(Q; \mathbb{R}^d) : u|_{\partial Q}(x) = Ax, \int_Q \nabla u \, \mathrm{d}x = B, |\nabla u| \in L^p(Q) \right\} \tag{5.4}$$

and for $\lambda \in \mathbb{R}^d$ and $\nu \in \mathbb{S}^{N-1}$ let

$$\mathcal{C}_p^{\mathrm{surface}}(\lambda, \nu) := \left\{ u \in SBV(Q_\nu; \mathbb{R}^d) : u|_{\partial Q_\nu}(x) = u_{\lambda, \nu}(x), \delta_1(p)\mathfrak{C}_1(u) + (1 - \delta_1(p))\mathfrak{C}(u) \right\},$$

where the function $u_{\lambda,\nu}$ is defined by

$$u_{\lambda,\nu}(x) := \begin{cases} \lambda & \text{if } x \cdot \nu \geqslant 0, \\ 0 & \text{if } x \cdot \nu < 0, \end{cases}$$

and the conditions $\mathfrak{C}_1(u)$ and $\mathfrak{C}(u)$ are

$$\mathfrak{C}_1(u) \Longleftrightarrow \int_Q \nabla u \, dx = 0 \qquad \text{and} \qquad \mathfrak{C}(u) \Longleftrightarrow \nabla u(x) = 0 \text{ for } \mathcal{L}^N\text{-a.e. } x \in Q_\nu \tag{5.5}$$

We state now the integral representation theorem for the relaxed energies I_p defined in (5.3). It generalizes the results contained in [11, Theorems 2.16 and 2.17] to the inhomogeneous case considered here. For the sake of being concise, we give a unified statement through the use of the selector $\delta_1(p)$, which takes into account the different nuances between the case $p = 1$ and the case $p > 1$.

Theorem 5.1. *Let $p \geqslant 1$ and let $W \colon \Omega \times \mathbb{R}^{d \times N} \to [0, +\infty[$ and $\psi \colon \Omega \times \mathbb{R}^d \times \mathbb{S}^{N-1} \to [0, +\infty[$ be continuous functions satisfying hypotheses $(W1)_p$, $\delta_1(p)(W2)$, $(W3)$, $(\psi1)$, $(\psi2)$, $(\psi3)$, and $(\psi4)$; let $(g, G) \in SD(\Omega)$ and let $I_p(g, G)$ be given by (5.3). Then there exist $H_p \colon \Omega \times \mathbb{R}^{d \times N} \times \mathbb{R}^{d \times N} \to [0, +\infty)$ and $h_p \colon \Omega \times \mathbb{R}^d \times \mathbb{S}^{N-1} \to [0, +\infty)$ such that*

$$I_p(g, G) = \int_\Omega H_p(x, \nabla g(x), G(x)) \, dx + \int_{\Omega \cap S_g} h_p(x, [g](x), \nu_g(x)) \, d\mathcal{H}^{N-1}(x). \tag{5.6}$$

For all $x \in \Omega$ and $A, B \in \mathbb{R}^{d \times N}$,

$$H_p(x, A, B) := \inf \left\{ \int_Q W(x, \nabla u) \, dx + \int_{Q \cap S_u} \psi(x, [u], \nu_u) \, d\mathcal{H}^{N-1} : u \in \mathcal{C}_p^{\mathrm{bulk}}(A, B) \right\}; \tag{5.7}$$

for all $x \in \Omega$, $\lambda \in \mathbb{R}^d$, and $\nu \in \mathbb{S}^{N-1}$,

$$h_p(x, \lambda, \nu) := \inf \left\{ \delta_1(p) \int_{Q_\nu} W^\infty(x, \nabla u) \, dx + \int_{Q_\nu \cap S_u} \psi(x, [u], \nu_u) \, d\mathcal{H}^{N-1} : u \in \mathcal{C}_p^{\mathrm{surface}}(\lambda, \nu) \right\}, \tag{5.8}$$

with W^∞ defined in (5.1).

Remark 5.2. Theorem 5.1 collects the content of Theorems 2.16 and 2.17 in [11] in a compact form. In particular, the form of the integral representation of the relaxed energies (5.3) provided by formula (5.6) is structurally the same both for $p = 1$ and for $p > 1$: it features a bulk energy and an interfacial energy.

We make the following observations.

- The condition $|\nabla u| \in L^p(Q)$ in (5.4) is redundant if $p = 1$ (see [11, Remark 2.15]);
- If $p = 1$, hypothesis $(W2)$ is required and we notice that in formula (5.8) the recession function at infinity W^∞ defined in (5.1) appears, to account for concentration phenomena arising when taking the limit of functions in L^1.
- In (5.5), condition $\mathfrak{C}_1$ contains condition $\mathfrak{C}$, so that, for every $\lambda \in \mathbb{R}^d$ and $\nu \in \mathbb{S}^{N-1}$, we have the inclusion $\mathcal{C}_{p>1}^{\mathrm{surface}}(\lambda, \nu) \subset \mathcal{C}_1^{\mathrm{surface}}(\lambda, \nu)$.
- The cell formula (5.8) for $p > 1$ corrects formula (2.17) in [11], where the dependence on the normal ν was mistakenly omitted, as already noted in [37, Theorem 3] and [39, formula (4)].

We point out the following final remarks.

- hypothesis $(W1)_p$ could be strengthened to include coercivity (p-growth from below). Although this would be a strong restriction from the mechanical point of view, it would make the proofs easier. We refer the reader to [11, Step 1 in the proof of Proposition 2.22] for a discussion on this.
- If $p > 1$, hypotheses $(\psi1)$ and $(\psi2)$ can be relaxed. We refer the reader to [11, Remark 3.3] for a discussion on this.

These final remarks pave the way for a statement of Theorem 5.1 under the minimal set of hypotheses.

Sketch of the proof of Theorem 5.1. Formula (5.6) is obtained by using the blow-up method [6, 24, 25] to prove that the energy densities (5.7) and (5.8) provide upper and lower bound for the Radon-Nikodým derivatives of suitable measures associated with $I_p(g, G)$ with respect to $\mathcal{L}^N$ and $|[g]|\mathcal{H}^{N-1} \llcorner S_g$. The dependence on x is not involved in this process, and the existence of the moduli of continuity ω_W and ω_ψ is a strong enough assumption to estimate the error when passing from the evaluation of the energy densities at generic $x \in Q(x_0, \delta)$ to the evaluation at x_0. A similar strategy was undertaken in [8], in the spirit of [6]. $\qquad \square$

5.2. **Relaxation of the complete functional.**

Lemma 5.3. *Given $(g, G) \in SD(\Omega; \mathbb{R}^d)$, there exists a sequence of functions $u_n \in SBV(\Omega; \mathbb{R}^d)$ admissible for the relaxation process of Theorem 5.1 satisfying* (1.15). *In particular, $u_n \xrightarrow{SD} (g, G)$ as $n \to \infty$.*

Proof. The Approximation Theorem 2.8 provides sequences u_n that converge to (g, G) in the sense of (1.1). We claim that any recovery sequence underlying Theorem 5.1 is such that (1.15) holds. In fact, by the coercivity of the surface energy density ψ (see property $(\psi 1)$), $|D^s u_n|$ is bounded uniformly with respect to n and therefore it converges weakly-* to a measure Λ_g, whose singular part is concentrated on S_g by (5.6). The lemma is proved. $\qquad\square$

Theorem 5.4. *In the conditions of Theorem 1.2 and Theorem 5.1, the relaxation* (1.35) *of the initial energy* (1.34) *admits the integral representation* (1.36), *where, for any $(g, G) \in SD(\Omega; \mathbb{R}^d)$, the relaxed energy $I_L(g, G)$ of the local initial energy E_L in* (1.2) *is given by* (5.6) *and the relaxed energy $\mathcal{I}^{\alpha_r}(g, G)$ of the non-local initial energy $\mathcal{E}^{\alpha_r}$ is provided by Theorem 1.2. In particular,*

(i) *under the hypotheses of Theorem 1.2(i), using* (1.16), *we have the formula*

$$
J^{\alpha_r}(g, G) = \int_\Omega H(x, \nabla g(x), G(x)) \, \mathrm{d}x + \int_{\Omega \cap S_g} h(x, [g](x), \nu_g(x)) \, \mathrm{d}\mathcal{H}^{N-1}(x)
$$
$$
+ \int_{\Omega_r} \Psi\big(x, ((\nabla g - G) * \alpha_r)(x) + (D^s g * \alpha_r)(x)\big) \, \mathrm{d}x; \tag{5.9}
$$

(ii) *under the hypotheses of Theorem 1.2(ii), using* (1.18), *we have the formula*

$$
\widetilde{J}^{\alpha_r}(g, G) = \int_\Omega H(x, \nabla g(x), G(x)) \, \mathrm{d}x + \int_{\Omega \cap S_g} h(x, [g](x), \nu_g(x)) \, \mathrm{d}\mathcal{H}^{N-1}(x)
$$
$$
+ \int_\Omega \Psi\big(x, ((\nabla g - G) * \alpha_r)(x) + (D^s g * \alpha_r)(x)\big) \, \mathrm{d}x. \tag{5.10}
$$

Proof. In case (i), the representation formula (5.9) is an immediate consequence of Theorem 1.2(i) and Theorem 5.1 and the superadditivity properties of the $\liminf$. In case (ii), Lemma 5.3 yields that the recovery sequences for the relaxation of Theorem 5.1 indeed converge in the $\xrightarrow{SD}$ sense, therefore, they are admissible for the relaxation process of Theorem 1.2(ii). Invoking again the superadditivity properties of the $\liminf$ allows us to obtain (5.10). $\qquad\square$

Corollary 5.5. *In the conditions of Theorem 1.5 and Theorem 5.4, for any $(g, G) \in SD(\Omega; \mathbb{R}^d)$, the functional $J(g, G)$ defined in* (1.37) *admits the integral representation in* (1.38), *namely, recalling the expression* (4.12) *for $I(g, G)$ and the expression* (5.6) *for $I_L(g, G)$,*

$$
J(g, G) = \int_\Omega H(x, \nabla g(x), G(x)) \, \mathrm{d}x + \int_{\Omega \cap S_g} h(x, [g](x), \nu_g(x)) \, \mathrm{d}\mathcal{H}^{N-1}(x)
$$
$$
+ \int_\Omega \Psi\big(x, \nabla g(x) - G(x)\big) \, \mathrm{d}x + \int_{\Omega \cap S_g} \Psi^\infty\big(x, [g](x) \otimes \nu_g(x)\big) \, \mathrm{d}\mathcal{H}^{N-1}(x). \tag{5.11}
$$

Proof. The result follows immediately by Theorem 1.5 and Theorem 5.4. $\qquad\square$

5.3. **On the reverse order of the limits.** After the presentation of the iterated limiting procedure carried out in Sections 3 and 4, a legitimate question is whether the two operations commute, namely, whether we obtain the same result if we reverse the order in which the two limits are taken: first letting $r \to 0$ and then letting $n \to \infty$. The problem is a relevant one in the scientific community and a similar question was studied in [9] for a problem of dimension reduction in the context of structured deformations. In the following few lines, we will give a brief explanation of why in the present case a commutability result does not hold.

Under the hypotheses of the previous sections on W, ψ, and Ψ, let us consider the reversed iterated limiting procedure for an initial energy of the type $E_L + E^{\alpha_r}$, with E_L as in (5.2) and E^{α_r} as in (1.8). We first let the measure of non-locality tend to zero and then relax to structured deformations, namely we consider, for $u \in SBV(\Omega; \mathbb{R}^d)$

$$
I_L(u) := \lim_{r \to 0} \big(E_L(u) + E^{\alpha_r}(u)\big) \tag{5.12}
$$

and then we relax this energy as in (5.3), for $(g, G) \in SD(\Omega; \mathbb{R}^d)$:

$$I_p^{(R)}(g, G) := \inf_{\{u_n\} \subset SBV(\Omega; \mathbb{R}^d)} \left\{ \liminf_{n \to \infty} I_L(u_n) : u_n \to (g, G) \text{ in the sense of } (1.1), \right.$$

$$\left. (1 - \delta_1(p)) \sup_n \|\nabla u_n\|_{L^p(\Omega; \mathbb{R}^{d \times N})} < \infty \right\}. \tag{5.13}$$

Given that E_L is independent of r, it is easy to deal with (5.12). Since Ψ belongs to the class (E) or (L), an application of the Reshetnyak Continuity Theorem 2.4(i) with $\Phi = \Psi$ gives

$$\lim_{r \to 0} E^{\alpha_r}(u) = \lim_{r \to 0} \int_\Omega \Psi\big(x, (\alpha_r * D^s u)(x)\big) \, \mathrm{d}x = \int_\Omega \Psi(x, 0) \, \mathrm{d}x + \int_{\Omega \cap S_u} \Psi^\infty\Big(x, \frac{\mathrm{d}D^s u}{\mathrm{d}|D^s u|}(x)\Big) \, \mathrm{d}|D^s u|(x)$$

$$= \int_\Omega \Psi(x, 0) \, \mathrm{d}x + \int_{\Omega \cap S_u} \Psi^\infty\big(x, [u](x) \otimes \nu_u(x)\big) \, \mathrm{d}\mathcal{H}^{N-1}(x).$$

Therefore, in (5.12) we obtain

$$I_L(u) = E_L(u) + \int_\Omega \Psi(x, 0) \, \mathrm{d}x + \int_{\Omega \cap S_u} \Psi^\infty\big(x, [u](x) \otimes \nu_u(x)\big) \, \mathrm{d}\mathcal{H}^{N-1}(x).$$

Now, it is easy to prove that if Ψ is in the class (L), then also Ψ^∞ is in the class (L). Therefore, for either (i) Ψ belonging to (L) or (ii) Ψ belonging to (E) with Ψ^∞ Lipschitz in the second variable uniformly with respect to the first one, it is immediate to see that Ψ^∞ is a surface energy density that satisfies hypotheses $(\psi 1)$, $(\psi 2)$, and $(\psi 3)$ (see [11, Remark 3.3] and [34, Remark 3.1]). Thus, the relaxation process (5.13) is the same as that of Theorem 5.1 for a local energy of the type (5.2) whose densities are $\widetilde{W}(x, A) := W(x, A) + \Psi(x, 0)$ and $\widetilde{\psi}(x, \lambda, \nu) := \psi(x, \lambda, \nu) + \Psi^\infty(x, \lambda \otimes \nu)$. The cell formulas (5.7) and (5.8) imply that only the behavior of $\Psi(x, A)$ at $A = 0$ or as $|A| \to \infty$ can influence the relaxed energy in (5.13), whereas the presence of the third integral in (5.11) shows that all of the values of $\Psi(x, A)$ can influence the relaxed energy $J(g, G)$ in (5.11).

6. Bulk relaxed densities of the form $W(x, G(x)) + \Psi(x, \nabla g(x) - G(x))$

The representation (5.11) of the relaxed energy $J(g, G)$ established in Corollary 5.5 contains the bulk part

$$\int_\Omega \big(H(x, \nabla g(x), G(x)) + \Psi(x, \nabla g(x) - G(x))\big) \, \mathrm{d}x,$$

in which the bulk relaxed density is a sum of the contribution $H(x, \nabla g(x), G(x))$ from the initial local energy $E_L(u)$ in (5.2) and the contribution $\Psi(x, \nabla g(x) - G(x))$ from the initial non-local energy E^{α_r} in (1.11) or from $\tilde{E}^{\alpha_r}$ in (1.13). The second term $\Psi(x, \nabla g(x) - G(x))$ has the distinction of capturing a bulk energy density due to disarrangements alone through its sole dependence on the deformation due to disarrangements $M(x) = \nabla g(x) - G(x)$, while the first term $H(x, \nabla g(x), G(x))$ can be written as $H(x, G(x) + M(x), G(x))$ and so depends in general on both the deformation due to disarrangements $M(x)$ and the deformation without disarrangements $G(x)$. This situation leads naturally to the question of finding conditions on the initial local energy $E_L(u)$ that imply that the term $H(x, G(x) + M(x), G(x))$ depends on $G(x)$ alone or, more generally, that

$$H(x, G(x) + M(x), G(x)) = H_\backslash(x, G(x)) + H_d(x, M(x)), \tag{6.1}$$

in which case the bulk relaxed density becomes

$$H(x, G(x) + M(x), G(x)) + \Psi(x, M(x)) = H_\backslash(x, G(x)) + (H_d(x, M(x)) + \Psi(x, M(x))), \tag{6.2}$$

a function $H_\backslash$ of deformation without disarrangements plus a function $H_d + \Psi$ of deformation due to disarrangements.

The existence of a decomposition of the form (6.1) was raised in [10] and was shown not to be available, in general, in the study [32]. A modified form of (6.1) was established in [5]: there the x-dependence was absent, and the term $H_\backslash(x, G(x))$ in (6.2) was replaced by $H_\backslash(G(x), \nabla G(x))$. In the context of plasticity addressed in the articles [10, 13, 17], the availability of (6.1) was shown to provide a variational basis for describing and predicting the phenomena of yielding, hysteresis, and hardening observed in both single crystals and in polycrystalline materials. In this section we provide conditions on the initial local energy $E_L(u)$ that guarantee that (6.1) and (6.2) hold with H_d identically zero. This outcome provides a setting in which

the term $H_\backslash(x, G(x))$ in (6.2) arises solely from the local part $E_L(u)$ of the initial energy while the term $\Psi(x, M(x))$ arises solely from the non-local part E^{α_r} of the initial energy in (1.11).

Remark 6.1. Consider the local, purely bulk initial energy

$$E_L(u) \coloneqq \int_\Omega W(x, \nabla u(x))\, \mathrm{d}x \tag{6.3}$$

where $W \colon \mathbb{R}^{3\times 3} \to \mathbb{R}$ is a continuous function that satisfies with $p > 1$ the condition $(W1)_p$, so that the initial local energy in (5.2) is employed with the interfacial energy ψ identically zero. Consequently, the conditions $(\psi 2)$ and $(\psi 3)$ are satisfied, but not the linear lower bound in $(\psi 1)$. Nevertheless, the final point in Remark 5.2 along with the fact that the lower bound in $(\psi 1)$ was used only in proving part (ii) of Theorem 5.4 allow us to use (5.7) and (5.8) in Theorem 5.1 to obtain the following representation formula for the relaxed local energy (5.3):

$$I_p(g, G) = \int_\Omega W(x, G(x))\, \mathrm{d}x.$$

This formula rests on the fact that in (5.7) the absence of the interfacial energy in E_L causes no penalization of jumps in u, so that $H_p(x, A, B) = W(x, B)$. Therefore, under the assumption (6.3) and under the hypotheses of part (i) of Theorem 5.4, the bulk part of the relaxed energy $J(g, G)$ is given by the expression $\int_\Omega (W(x, G(x)) + \Psi(x, M(x)))\, \mathrm{d}x$.

7. Example from crystal plasticity

We turn to the subject of the mechanics of single crystals to identify an example of bulk energies of the type recovered in the volume integral in (1.30) through our relaxation of non-local energies. The example emerges within the special class of *invertible structured deformations* (g, G) in which the tensors G and $K_{(g,G)}$ in (7.2) below play the role of F^e and $(F^p)^{-1}$ in the standard treatments of crystal plasticity.

7.1. **Invertible structured deformations.** The main mechanisms of deformation in single crystals are the distortion without disarrangements of the crystalline lattice and the shearing due to disarrangements. The articles [14, 19] show that the class of invertible structured deformations is appropriate for capturing such multiscale geometrical changes. In the present setting, we can identify (g, G) as an *invertible structured deformation* when (see [14] for a broader setting for this notion)

(I1) g is a diffeomorphism of class C^1 for which ∇g and $(\nabla g)^{-1}$ are Lipschitzian,
(I2) G is continuous with invertible values,
(I3) the macroscopic volume change multiplier $\det \nabla g$ and the multiplier for volume change without disarrangements $\det G$ are equal: $\det \nabla g = \nabla G$.

For an open set $\Omega \subset \mathbb{R}^3$, we define

$$ISD(\Omega; \mathbb{R}^3) \coloneqq \big\{ (g, G) \in SD(\Omega; \mathbb{R}^3) : (I1),\, (I2),\, \text{and } (I3) \text{ hold} \big\}.$$

Invertible structured deformations turn out to be a useful setting for understanding some kinematical ingredients in continuum models of single crystals undergoing plastic deformations, partly because the relation $\det \nabla g = \det G$ reflects the fact that the disarrangements occurring in single crystals typically do not involve changes in volume, *i.e.*, arise without the formation of submacroscopic voids.

One useful mathematical property of invertible structured deformations rests on the notion of composition of invertible structured deformations: if $(g, G) \in ISD(\Omega)$ and $(h, H) \in ISD(g(\Omega))$, then the composition $(h, H) \diamond (g, G)$ is defined by

$$(h, H) \diamond (g, G) \coloneqq (h \circ g, (H \circ g)G). \tag{7.1}$$

It is easy to show that $(h, H) \diamond (g, G) \in ISD(\Omega)$ and each $(g, G) \in ISD(\Omega)$ has the factorization

$$(g, G) = (g, \nabla g) \diamond (i, K_{(g,G)}) \tag{7.2}$$

where $i \coloneqq x \mapsto x$ is the identity mapping on Ω and $K_{(g,G)} \coloneqq (\nabla g)^{-1}G$. The factor $(g, \nabla g) \in ISD(\Omega)$ carries all of the macroscopic deformation and is a classical deformation, *i.e.*, it causes no disarrangements because $M_{(g,\nabla g)} \coloneqq \nabla g - \nabla g = 0$. The factor $(i, K_{(g,G)}) \in ISD(\Omega)$ is purely submacroscopic, *i.e.*, it causes no macroscopic deformation, and carries the disarrangements

$$M_{(i, K_{(g,G)})} \coloneqq \nabla i - K_{(g,G)} = I - (\nabla g)^{-1}G = (\nabla g)^{-1}(\nabla g - G) = (\nabla g)^{-1}M_{(g,G)}. \tag{7.3}$$

Moreover, both factors in (7.2) are invertible structured deformations, because $\det K_{(g,G)} = \det G / \det \nabla g = 1 = \det \nabla i$ and, trivially, $\det \nabla g = \det \nabla g$.

7.2. Slip systems for single crystals; crystallographic structured deformations.
For a single crystal in the reference configuration Ω the crystallographic data required for the analysis of crystallographic slip consists of pairs of orthogonal unit vectors (s^a, m^a) for $a = 1, \ldots, A$, with A the number of potentially active slip systems. For crystallographic slip, the discontinuity in deformation arises only across a limited family of slip planes identified via the slip systems. The unit vector s^a provides the direction of slip, while the unit vector m^a is a normal to the slip plane for the ath slip-system (s^a, m^a). For the case of face-centered cubic crystals, the vectors m^a are chosen from the normals to the faces of a preassigned regular octahedron and the slip vectors s^a are chosen to be one of the directed edges of the face associated with m^a.

We wish next to identify a collection of invertible structured deformations for which the disarrangements arise only through the action of the slip systems of a give crystal. To this end, we recall [15] that for each structured deformation (g, G) in the sense of [14] and, hence, for each invertible structured deformation there exists a sequence of injective, piecewise smooth deformations f_n such that

$$f_n \to g \quad \text{in } L^\infty(\Omega; \mathbb{R}^3), \qquad \nabla f_n \to G \quad \text{in } L^\infty(\Omega; \mathbb{R}^{3\times3}),$$

and, for every such sequence and for every $x \in \Omega$, the disarrangement tensor $M_{(g,G)}$ is given by the identification relation

$$M_{(g,G)}(x) := \nabla g(x) - G(x) = \lim_{r \to 0} \lim_{n \to \infty} \frac{1}{V_3(r)} \int_{B_r(x) \cap S_{f_n}} [f_n](y) \otimes \nu_{f_n}(y) \, d\mathcal{H}^{N-1}(y), \tag{7.4}$$

and the deformation without disarrangements G by the identification relation

$$G(x) = \lim_{r \to 0} \lim_{n \to \infty} \frac{1}{V_3(r)} \int_{B_r(x)} \nabla f_n(y) \, dy. \tag{7.5}$$

In both (7.4) and (7.5), $V_3(r)$ denotes the volume of the three-dimensional ball of radius r. Suppose now that the approximating deformations f_n are such that the dyadic fields $[f_n] \otimes \nu_{f_n}$ are compatible with the A slip-systems of the crystal in the sense that for every $n \in \mathbb{N}$ and for every $r > 0$ there exist continuous fields $\gamma_n^a(\cdot, r) \colon \Omega \to \mathbb{R}$ for $a = 1, \ldots, A$ such that

$$\int_{B_r(x) \cap S_{f_n}} [f_n](y) \otimes \nu_{f_n}(y) \, d\mathcal{H}^{N-1}(y) = \sum_{a=1}^{A} V_3(r) \gamma_n^a(x, r) \nabla g(x) s^a \otimes m^a, \tag{7.6}$$

and such that $\lim_{r \to 0} \lim_{n \to \infty} \gamma_n^a(x, r) =: \gamma^a(x)$ exists for every $x \in \Omega$ and for $a = 1, \ldots, A$. Under these assumptions, the identification relation (7.4) becomes

$$M_{(g,G)}(x) = \sum_{a=1}^{A} \gamma^a(x) \nabla g(x) s^a \otimes m^a,$$

so that the first relation in (7.3) becomes

$$M_{(i, K_{(g,G)})}(x) = I - K_{(g,G)}(x) = \sum_{a=1}^{A} \gamma^a(x) s^a \otimes m^a \tag{7.7}$$

and

$$K_{(g,G)}(x) = I - \sum_{a=1}^{A} \gamma^a(x) s^a \otimes m^a \tag{7.8}$$

We have provided through (7.6) sufficient conditions that the disarrangement tensor field $M_{(i, K_{(g,G)})}$ for the purely submacroscopic part $(i, (\nabla g)^{-1} G)$ of (g, G) is a linear combination of the (spatially constant) crystallographic slip dyads $s^a \otimes m^a$ for $a = 1, \ldots, A$ associated with the given crystal. In this context we may say that the invertible structured deformation (g, G) generates disarrangements only in the form of crystallographic slips or, more briefly, that (g, G) is *crystallographic*. We note in passing that the article [19] provided a precise sense in which one may consider approximations by crystallographic slips of the disarrangement matrix $M_{(g,G)} = \nabla g - G$ of any invertible structured deformations. Since we here restrict our attention to those invertible structured deformations for which (7.7) holds, the approximations in [19] become exact in the present context.

For a crystallographic structured deformation (g, G) and a point $x \in \Omega$ we say that a slip-system a is active at x if $\gamma^a(x) \neq 0$, and we say that single slip occurs at x if there is only one slip-system that is active at x. If more than one slip system is active at x we say that multiple slip occurs at x. If (g, G) is crystallographic, so that (7.6), (7.7), and (7.8) hold, we may use the relations $\operatorname{tr}(s^a \otimes m^a) = s^a \cdot m^a = 0$ for all a and $\det K_{(g,G)} = \det G / \det \nabla g$ to conclude from (7.8) and the definition of invertible structured deformations that

$$\operatorname{tr} K_{(g,G)} = 3 \qquad \text{and} \qquad \det K_{(g,G)} = 1. \tag{7.9}$$

Consequently, the crystallographic structured deformations are among those for which $K_{(g,G)} = (\nabla g)^{-1} G$ satisfies (7.9). We note that a slip system a is active at x for (g, G) if and only if a is active at x for the purely submacroscopic part $(i, K_{(g,G)})$ of (g, G).

Examples of crystallographic structured deformations that undergo single slip at every point are the two-level shears $(g^a_{\mu, x_o}, G^a_\nu)$ for $a = 1, \dots, A$, for $\mu, \nu \in \mathbb{R}$ and for $x_o \in \Omega$:

$$\begin{aligned} g^a_{\mu, x_o}(x) &:= x_o + (I + \mu s^a \otimes m^a)(x - x_o) \\ G^a_\nu(x) &:= I + \nu s^a \otimes m^a, \end{aligned} \tag{7.10}$$

for which it can be verified [14] via the "deck of cards" family of approximations f_n that (7.6) is satisfied, and for which

$$\nabla g^a_\mu(x) = I + \mu s^a \otimes m^a, \tag{7.11a}$$

$$M_{(g^a_{\mu, x_o}, G^a_\nu)}(x) = (\mu - \nu) s^a \otimes m^a, \tag{7.11b}$$

and

$$K_{(g^a_{\mu, x_o}, G^a_\nu)}(x) = I + (\nu - \mu) s^a \otimes m^a = I - M_{(g^a_{\mu, x_o}, G^a_\nu)}(x). \tag{7.12}$$

The "deck of cards" approximations f_n show that each two-level shear $(g^a_{\mu, x_o}, G^a_\nu)$ is approximated for each n by smooth shears of amount ν of the crystal lattice between $n - 1$ slip planes, along with slip-discontinuities in the direction s^a across the $n - 1$ planes, each slip-discontinuity of amount $\frac{\mu - \nu}{n}$ times a reference dimension in the direction m^a. By virtue of the "deck of cards" approximations f_n and, in view of (7.11a), (7.10)$_2$, (7.11b), and the trivial relation

$$\mu = \nu + (\mu - \nu),$$

we may then call μ the macroscopic shear, ν the shear without slip, and $\mu - \nu$ the shear due to slip for the two-level shear $(g^a_{\mu, x_o}, G^a_\nu)$. Of particular interest is the case $\nu = 0$, *i.e.*, the two-level shear (g^a_{μ, x_o}, I), in which the region between slip planes undergoes no shear and the macroshear μ arises entirely from slips on slip-system a.

7.3. Slip-neutral two-level shears.
We now summarize arguments provided in [10] in a more limited setting that are based on the observation that crystallographic slip is physically activated within very thin bands, the so-called slip-bands, whose thickness is typically of the order 10^2 atomic units, while the separation of active slip-bands is typically of order 10^4 atomic units. The arguments in [10] indicate the following: for each $a = 1, \dots, A$, there is a number $p^a > 0$ such that a two-level shear $(g^a_{\mu, x_o}, G^a_\nu)$ for which the shear due to slip $\mu - \gamma$ is an integral multiple of p^a gives rise to submacroscopic slips equal to an integral number of atomic units in the direction of slip s^a. The dimensionless number p^a equals a shift of one atomic unit in the direction of slip s^a divided by the thickness in the direction m^a of the slip-band associated with system a (measured in the same atomic units). Consequently, p^a is of the order of 10^{-2}, and a two-level shear $(g^a_{\mu, x_o}, G^a_\nu)$ with

$$\mu - \gamma = n p^a, \qquad \text{with } n \in \mathbb{Z}, \tag{7.13}$$

produces a shift of n atomic units and so does not produce a misfit of the crystalline lattice across the active slip bands, no matter what the amount of shear without slip ν. Thus, when (7.13) holds, the disarrangements due to slip are not revealed by the deformed positions under the two-level shear attained by the lattice points away from the slip bands. We refer to a two-level shear $(g^a_{\mu, x_o}, G^a_\nu)$ satisfying (7.13) as slip-neutral for the slip-system a. In particular, when $\nu = 0$ we have $G^a_\nu = I$, and the two-level shear (g^a_μ, I) is slip-neutral if the macroshear $\mu = \mu - \nu$ is an integral multiple of p^a. Although a slip-neutral shear of the form $(g^a_{np^a}, I)$ causes a macroscopic shearing of the body, not only does it cause no misfit of the lattice across slip bands, it also causes no distortion of the lattice. Consequently, we call the two-level shear $(g^a_{np^a}, I)$ completely neutral for the slip-system a.

We suppose now that the given body undergoes a completely neutral two-level shear (g^a_{μ,x_o}, I) with $\mu = np^a$, starting from the region Ω, and suppose further that (g^a_{μ,x_o}, I) is then followed by a crystallographic deformation (g, G), so that we have the composition and factorization as in (7.1) and (7.2):

$$(g,G) \diamond (g^a_{\mu,x_o}, I) = (g \circ g^a_{\mu,x_o}, G \circ g^a_{\mu,x_o}) = (g \circ g^a_{\mu,x_o}, \nabla(g \circ g^a_{\mu,x_o})) \diamond (i, K_{(g \circ g^a_{\mu,x_o}, G \circ g^a_{\mu,x_o})}),$$

with $K_{(g \circ g^a_{\mu,x_o}, G \circ g^a_{\mu,x_o})}$ given by

$$\begin{aligned}
K_{(g \circ g^a_{\mu,x_o}, G \circ g^a_{\mu,x_o})} &= (\nabla(g \circ g^a_{\mu,x_o}))^{-1}(G \circ g^a_{\mu,x_o}) = ((\nabla g \circ g^a_{\mu,x_o})\nabla g^a_{\mu,x_o})^{-1}(G \circ g^a_{\mu,x_o}) \\
&= (\nabla g^a_{\mu,x_o})^{-1}(\nabla g \circ g^a_{\mu,x_o})^{-1}(G \circ g^a_{\mu,x_o}) = ((\nabla g^a_{\mu,x_o})^{-1}(\nabla g)^{-1}G) \circ g^a_{\mu,x_o} \\
&= (\nabla g^a_{\mu,x_o})^{-1} K_{(g,G)}) \circ g^a_{\mu,x_o}
\end{aligned}$$

and with $K_{(g^a_{\mu,x_o}, I)}$ given by (7.12):

$$K_{(g^a_{\mu,x_o}, I)} = I - \mu s^a \otimes m^a = (\nabla g^a_{\mu,x_o})^{-1}.$$

Therefore, we have the relation

$$K_{(g,G) \diamond (g^a_{mp^a}, I)} = K_{(g^a_{\mu,x_o}, I)}(K_{(g,G)} \circ g^a_{\mu,x_o}),$$

and the relations $M_{(i,K_{(g,G)})} = I - K_{(g,G)}$, (7.8), (7.11b), and (7.12) then yield

$$\begin{aligned}
I - M_{(i,K_{(g,G) \diamond (g^a_{mp^a}, I)})} &= K_{(g,G) \diamond (g^a_{mp^a}, I)} = K_{(g^a_{\mu,x_o}, I)}(K_{(g,G)} \circ g^a_{\mu,x_o}) \\
&= (I - M_{(g^a_{\mu,x_o}, I)})(I - M_{(i,K_{(g,G)})} \circ g^a_{\mu,x_o}) \\
&= I - M_{(i,\dot{K}_{(g,G)})} \circ g^a_{\mu,x_o} - M_{(g^a_{\mu,x_o}, I)} + M_{(g^a_{\mu,x_o}, I)} M_{(i,K_{(g,G)})} \circ g^a_{\mu,x_o} \qquad (7.14) \\
&= I - M_{(i,\dot{K}_{(g,G)})} \circ g^a_{\mu,x_o} - M_{(g^a_{\mu,x_o}, I)} + \mu(s^a \otimes m^a) M_{(i,K_{(g,G)})} \circ g^a_{\mu,x_o} \\
&= I - M_{(i,\dot{K}_{(g,G)})} \circ g^a_{\mu,x_o} - M_{(g^a_{\mu,x_o}, I)} + \mu s^a \otimes (M^T_{(i,K_{(g,G)})} \circ g^a_{\mu,x_o}) m^a.
\end{aligned}$$

When $\mu \neq 0$ the last term in (7.14) vanishes at a point x if and only if $M^T_{(i,K_{(g,G)})}(g^a_{\mu,x_o}(x)) m^a = 0$, and, because $g^a_{\mu,x_o}(x_o) = x_o$, we conclude from (7.7):

Remark 7.1. The disarrangement tensor $M_{(i,K_{(g,G) \diamond (g^a_{\mu,x_o}, I)})}(x_o)$ at $x_o \in \Omega$ for the submacroscopic part of the composition $(g,G) \diamond (g^a_{\mu,x_o}, I)$ of a crystallographic deformation (g,G) with the completely neutral two-level shear (g^a_{μ,x_o}, I), where $\mu = np^a$, is given by

$$M_{(i,K_{(g,G) \diamond (g^a_{\mu,x_o}, I)})}(x_o) = M_{(i,K_{(g,G)})}(x_o) + M_{(g^a_{\mu,x_o}, I)}(x_o) \qquad (7.15)$$

if and only if

$$\sum_{b=1}^{A} \gamma^b(x)(s^b \cdot m^a)m^b = M^T_{(i,K_{(g,G)})}(x_o) m^a = 0. \qquad (7.16)$$

The identification relation (7.4) for M shows that the vanishing of $M^T_{(i,K_{(g,G)})}(x_o) m^a$ in (7.16) is the statement that, on average, as $n \to \infty$ and $r \to 0$, the jumps in approximating deformations f_n must be parallel to the slip plane for the ath slip system. A sufficient condition on the crystallographic deformation (g, G) in order that the sum in (7.16) vanish is the following: every slip system b that is active at x_o for (g, G) satisfies $s^b \cdot m^a = 0$, i.e., the slip plane for the completely neutral two-level shear $(g^a_{np^a}, I)$ contains every slip direction s^b of every slip system b, active at x_o for (g, G). In particular, if (g, G) is a double slip at x_o with active slip systems (s^1, m^1) and (s^2, m^2), then (7.15) holds for every a such that $s^1 \cdot m^a = s^2 \cdot m^a = 0$. Such double slips (g, G) include the case of "cross slip" in which $(s^1, m^1) = (s^a, m^1)$ and $(s^2, m^2) = (s^a, m^2)$ with $m^a = m^1 \neq m^2$ in which slips in one and the same direction s^a occur in two different slip systems at x_o.

Our discussion above of the relationship between the disarrangement tensor $M_{(i,K_{(g,G) \diamond (g^a_{\mu,x_o}, I)})}$ for the purely submacroscopic part of the composition $(g,G) \diamond (g^a_{\mu,x_o}, I)$ and the disarrangement tensor $M_{(i,K_{(g,G)})}$ for the purely submacroscopic part of (g, G) is of particular interest for energetics, because (g^a_{μ,x_o}, I) was assumed to be completely neutral for the slip-system a, i.e., $\mu = np^a$ with n an integer. In that case, the lattice on which (g, G) acts when following (g^a_{μ,x_o}, I) differs from that on which (g, G) acts when *not* following (g^a_{μ,x_o}, I) only by undetectable translations of the lattice between active slip-planes for system a.

Consequently, the submacroscopic kinematical states of the crystal attained by means of the two purely sub-macroscopic structured deformations $(i, K_{(g,G) \diamond (g^a_{\mu,x_o}, I)})$ and $(i, K_{(g,G)})$ are indistinguishable. In particular, the corresponding disarrangement fields $M_{(i, K_{(g,G) \diamond (g^a_{\mu,x_o}, I)})}$ and $M_{(i, K_{(g,G)})}$ are indistinguishable, so that the validity of (7.15) would have significant implications with respect to properties of the energetic response of the crystal. We now provide specific circumstances under which the relaxed energies recovered in Corollary 5.5 would be subject to those implications, and we set the stage by highlighting the role of $M_{(i, K_{(g,G)})}$, the disarrangement tensor for the purely submacroscopic deformation $(i, K_{(g,G)})$, in providing constitutive relations for the energetic response to crystallographic deformation that are frame-indifferent (independent of observer).

7.4. Frame-indifferent energetic responses.
We noted in the text above the relation (7.3) that contains the formulas

$$M_{(i, K_{(g,G)})} = I - K_{(g,G)} = (\nabla g)^{-1} M_{(g,G)},$$

relating $M_{(g,G)}$, the disarrangement tensor for an invertible structured deformation (g, G), and $M_{(i, K_{(g,G)})}$, the disarrangement tensor for the purely submacroscopic deformation $(i, K_{(g,G)})$ in (7.2). Because ∇g and M both are premultiplied by a rotation Q under a change of observer associated with the rotation Q, the tensor field $K_{(g,G)}$ as well as the disarrangement tensor $M_{(i, K_{(g,G)})}$ are unchanged under such a change of observer. Therefore, for a function $\Psi_i : \mathbb{R}^{3 \times 3} \to \mathbb{R}$ the mapping

$$x \mapsto \Psi_i(M_{(i, K_{(g,G)})}(x)) = \Psi_i((\nabla g(x))^{-1} M_{(g,G)}(x))$$

has the property that its dependence on the structured deformation (g, G) is independent of observer. The function Ψ_i specifies the energetic response of a body from the reference cofiguration Ω to the disarrangements arising in purely submacroscopic deformations.

If we define for the given macroscopic deformation g the mapping $\Psi_g : \Omega \times \mathbb{R}^{3 \times 3} \to \mathbb{R}$ by,

$$\Psi_g(x, L) := \Psi_i((\nabla g(x))^{-1} L), \qquad \text{for all } L \in \mathbb{R}^{3 \times 3}, \tag{7.17}$$

then the mapping $x \mapsto \Psi_g(x, M_{(g,G)}(x)) = \Psi_i((\nabla g(x))^{-1} M_{(g,G)}(x))$ also has the property that its dependence on (g, G) is independent of observer. The following constitutive assumption for the dependence on invertible structured deformations (g, G) of $\psi : \Omega \to \mathbb{R}$, the free energy density due to disarrangements, namely,

$$\psi(x) = \Psi_i(M_{(i, K_{(g,G)})}(x)) = \Psi_i((\nabla g(x))^{-1} M_{(g,G)}(x)) = \Psi_g(x, M_{(g,G)}(x)), \qquad x \in \Omega, \tag{7.18}$$

then is independent of observer and carries the assumption that the free energy density due to disarrangements depends only on the disarrangements associated with the submacroscopic factor $(i, K_{(g,G)})$ in (7.2). When (g, G) is a crystallographic deformation, then the response functions Ψ_i and Ψ_g determine the free energy density due to crystallographic slip as a function of the disarrangement tensors $M_{(i, K_{(g,G)})}$ and $M_{(g,G)}$, respectively.

7.5. Periodic properties of the energetic response Ψ_i to crystallographic slip.
Let $x_o \in \Omega$, $a \in \{1, \ldots, A\}$, $\mu = np^a$ with $n \in \mathbb{Z}$, and a crystallographic structured deformation (g, G) be given. We argued above that the lattice on which (g, G) acts, when following the completely neutral two-level shear (g^a_{μ,x_o}, I), differs from that on which (g, G) acts, when *not* following (g^a_{μ,x_o}, I), only by the undetectable translations of the lattice between active slip bands for system a. Consequently, the submacroscopic kinematical states of the crystal lattice attained by means of the two purely submacroscopic structured deformations $(i, K_{(g,G) \diamond (g^a_{\mu,x_o}, I)})$ and $(i, K_{(g,G)})$ are indistinguishable. We invoke this indistinguishability to assert that the free energy density $\psi(x_o)$ due to crystallographic slip should be the same for $(i, K_{(g,G) \diamond (g^a_{\mu,x_o}, I)})$ and for $(i, K_{(g,G)})$ at the fixed point x_o of g^a_{μ,x_o}. Under the constitutive assumption (7.18) applied to the point x_o this assertion means that, for every $\mu = np^a$ with $n \in \mathbb{Z}$,

$$\Psi_i(M_{(i, K_{(g,G) \diamond (g^a_{\mu,x_o}, I)})}(x_o)) = \Psi_i(M_{(i, K_{(g,G)})}(x_o)). \tag{7.19}$$

We wish to translate (7.19) into a property of the response function $\Psi_i : \mathbb{R}^{3 \times 3} \to \mathbb{R}$ by invoking the additivity property (7.15) in Remark 7.1. This property requires that we restrict attention to matrices

$M \in \mathbb{R}^{3 \times 3}$ of the form

$$M = \sum_{b=1}^{A} \beta^b s^b \otimes m^b \tag{7.20}$$

with $\beta^1, \ldots, \beta^A \in \mathbb{R}$ such that

$$M^\top m^a = 0, \tag{7.21}$$

and such that

$$\det(I - M) = 1. \tag{7.22}$$

If we define

$$\mathbb{M}^a := \{M \in \mathbb{R}^{3 \times 3} : (7.20), (7.21), (7.22) \text{ hold}\},$$

then it is easy to show that if there exists $b \in \{1, \ldots, A\}$, $s \in \mathbb{R}^3$, $\xi \in \mathbb{R}$, such that $s \cdot m^a = 0$ and

$$M = s \otimes m^a + \xi(m^a \times m^b) \otimes m^b \tag{7.23}$$

then $M \in \mathbb{M}^a$. When $m^b = \pm m^a$ then the matrix M in (7.23) reduces to $s \otimes m^a$ and represents disarrangements arising from slips in the crystallographic plane with normal m^a, but not necessarily in one of the slip directions in the list of slip systems for the crystal. When $m^b \neq \pm m^a$, $s^a = m^a \times m^b = s$, and $\xi \neq 0$, M represents disarrangements of the previous type along with slips in the direction $m^a \times m^b$ in the crystallographic plane with normal m^b and so corresponds to the cross-slip described in Remark 6.1.

Suppose now that (g, G), a, and x_o are such that $M_{(i, K_{(g,G)})}(x_o) \in \mathbb{M}^a$. By Remark 7.1, (7.15) holds for every completely neutral two-level shear $(g^a_{np^a, x_o}, I)$, $i.e.$,

$$M_{(i, K_{(g,G)} \diamond (g^a_{np^a, x_o}, I))}(x_o) = M_{(i, K_{(g,G)})}(x_o) + M_{(g^a_{np^a, x_o}, I)}(x_o)$$

which by (7.11b) we may write in the following form

$$M_{(i, K_{(g,G)} \diamond (g^a_{\mu, x_o}, I))}(x_o) = M_{(i, K_{(g,G)})}(x_o) + np^a s^a \otimes m^a.$$

Consequently, when $M_{(i, K_{(g,G)})}(x_o) \in \mathbb{M}^a$, this formula and the constitutive restriction (7.19) on the response function Ψ_i yield the relation

$$\Psi_i\big(M_{(i, K_{(g,G)})}(x_o) + np^a s^a \otimes m^a\big) = \Psi_i\big(M_{(i, K_{(g,G)})}(x_o)\big), \qquad \text{for every } n \in \mathbb{Z}. \tag{7.24}$$

For matrices $M \in \mathbb{M}^a$ satisfying (7.23) this restriction takes the form

$$\Psi_i((s + np^a s^a) \otimes m^a + \xi(m^a \times m^b) \otimes m^b) = \Psi_i(s \otimes m^a + \xi(m^a \times m^b) \otimes m^b) \tag{7.25}$$

for every $s \in \{m^a\}^\perp$, $b \in \{1, \ldots, A\}$, $\xi \in \mathbb{R}$, and $n \in \mathbb{Z}$. In other terms, (7.25) is the assertion that for each $b \in \{1, \ldots, A\}$ and $\xi \in \mathbb{R}$ the mapping

$$s \mapsto \Psi_i(s \otimes m^a + \xi(m^a \times m^b) \otimes m^b) \tag{7.26}$$

is periodic on $\{m^a\}^\perp$ with (vector) period $p^a s^a$. Thus, the presence of completely neutral two-level shears $(g^a_{p^a}, I)$ has led via (7.24) to the identification of a family of affine subspaces

$$\mathbb{M}^a_{b,\xi} := \{s \otimes m^a + \xi(m^a \times m^b) \otimes m^b : \ s \in \{m^a\}^\perp\}$$

of $\mathbb{R}^{3 \times 3}$, each two-dimensional and on each of which the restriction of Ψ_i is periodic with corresponding period $p^a s^a$.

7.6. **Form of the relaxed energy appropriate for crystalline plasticity.** In this subsection we take the basic constitutive assumption (7.18) and the property (7.24) of Ψ_i that reflects the complete neutrality of certain two-level shears, and we identify additional properties of Ψ_i that permit the application of our principal relaxation result Theorem 1.5 when Ψ_g appears in place of Ψ in the formula (1.11) for the averaged interfacial energy. The following theorem provides conditions on Ψ_i and (g, G) sufficient for the application of Theorem 1.5 in the context of crystal plasticity. We note in advance that the fact that the macroscopic deformation g for a crystallographic structured deformation (g, G) is smooth (as is the case, more generally, for $(g, G) \in ISD(\Omega)$) means that the singular part $D^s g$ of the distributional derivative Dg is zero and, consequently, that the term in (1.30) involving the recession function Ψ_g^∞ is zero. (This would not be the case were one to use the original definition of invertible structured deformation in [14] in which g is allowed to be discontinuous.)

Theorem 7.2. *Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipshitz domain, and let $\Psi_i \colon \mathbb{R}^{3\times3} \to \mathbb{R}$ be Lipschitz continuous and satisfy for each $a, b \in \{1, \dots, A\}$ and for each $\xi \in \mathbb{R}$ the periodicity condition (7.26). Moreover, for each crystallographic structured deformation (g, G) let $\Psi_g \colon \Omega \times \mathbb{R}^{3\times3} \to \mathbb{R}$ be given in terms of Ψ_i by (7.17), and for each $u \in SBV(\Omega; \mathbb{R}^3)$ define as in (1.8) the averaged interfacial energy*

$$E_g^{\alpha_r}(u) = \int_{\Omega_r} \Psi_g(x, (D^s u * \alpha_r)(x)) \, \mathrm{d}x.$$

Then for each mollifier $\alpha \in C_0(B_1)$, for each $r > 0$, and for the convergence in (1.1), the relaxed energy $\mathcal{I}^{\alpha_r}(g, G)$ in (1.14) is given by

$$\mathcal{I}^{\alpha_r}(g, G) = \int_{\Omega_r} \Psi_g(x, (\nabla g - G) * \alpha_r)(x)) \, \mathrm{d}x, \tag{7.27}$$

and the relaxed energy $\mathcal{I}(g, G)$ in (1.20) takes the form given in (1.36) (1.33)

$$\mathcal{I}(g, G) = \lim_{r \to 0^+} \mathcal{I}^{\alpha_r}(g, G) = \int_{\Omega} \Psi_g(x, \nabla g(x) - G(x)) \, \mathrm{d}x = \int_{\Omega} \Psi_i(I - \nabla g(x)^{-1} G(x)) \, \mathrm{d}x$$
$$= \int_{\Omega} \Psi_i(I - K_{(g,G)}(x) \, \mathrm{d}x = \int_{\Omega} \Psi_i(M_{(i, K_{(g,G)})}(x)) \, \mathrm{d}x. \tag{7.28}$$

In particular, the relaxed bulk energy density retains the periodicity property (7.26).

Proof. We note that for $(g, G) \in ISD(\Omega)$ the Lipschitz continuity of $(\nabla g)^{-1}$ and the assumed Lipschitz continuity of Ψ_i imply that Ψ_g satisfies (L), so that we may invoke not only part (i) of Theorem 1.2 to obtain (7.27) but also Theorem 1.5 to obtain (7.28). $\qquad\square$

This result provides a representation formula for the relaxed energy associated with crystallographic slip only, and so does not provide information about energy stored due to distortion of the crystal lattice. To remove this shortcoming, we add to the averaged energy $E_g^{\alpha_r}(u)$ considered above the local, purely bulk initial energy

$$E_L(u) := \int_{\Omega} W(\nabla u(x)) \, \mathrm{d}x$$

where $W \colon \mathbb{R}^{3\times3} \to \mathbb{R}$ is a continuous function that satisfies with $p > 1$ the condition $(W1)_p$. Remark 6.1 then yields the formula for the relaxed local energy

$$I_p(g, G) = \int_{\Omega} W(G(x)) \, \mathrm{d}x.$$

Therefore, the energy stored due to smooth distortion of the crystal lattice emerges alone from the relaxation of the local initial energy E_L. Moreover, that remark permits us to conclude that the relaxation of the initial energy $E_L + E_g^{\alpha_r}$ followed by taking the limit as r goes to zero results in the following purely bulk energy for the crystal

$$J(g, G) = \int_{\Omega} [W(G(x)) + \Psi_i(M_{(i, K_{(g,G)})}(x))] \, \mathrm{d}x. \tag{7.29}$$

It has the distinctive property that the (bulk) energy density is the sum of a function W of the deformation without slip G and of a function Ψ_i of the deformation $M_{(i, K_{(g,G)})}$ due to slip. Moreover, the latter has the periodicity property (7.26) reflecting the ability of the crystal to undergo completely neutral crystallographic slips. Both of these properties arise in earlier variational studies of the basic phenomena of plastic behavior: yielding, hysteresis, and the existence of an elastic range. An additive decomposition of the bulk energy similar to that in (7.29) was the starting point in the study [13], and that very property along with the periodicity property of the energy due to slip was central to the energetic studies [10, 18].

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