On the Extreme Zeros of Jacobi Polynomials

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Abstract By applying the Euler–Rayleigh methods to a specific representation of the Jacobi polynomials as hypergeometric functions, we obtain new bounds for their largest zeros. In particular, we derive upper and lower bound for $1-x_{nn}^2(\lambda)$, with $x_{nn}(\lambda)$ being the largest zero of the n-th ultraspherical polynomial $P_n^{(\lambda)}$. For every fixed $\lambda > -1/2$, the limit of the ratio of our upper and lower bound for $1-x_{nn}^2(\lambda)$ does not exceed 1.6. This paper is a continuation of [13].

1 Introduction and Statement of the Results

The extreme zeros of the classical orthogonal polynomials of Jacobi, Laguerre and Hermite have been a subject of intensive study. We refer to Szegő's monograph [18] for earlier results, and to [2, 3, 4, 5, 6, 8, 9, 10, 11, 13, 17] for some recent developments.

Throughout this paper we use the notation

$$x_{1n}(\alpha,\beta) < x_{2n}(\alpha,\beta) < \cdots < x_{nn}(\alpha,\beta)$$

for the zeros of the n-th Jacobi polynomial $P_n^{(\alpha,\beta)}$, $\alpha,\beta > -1$, and the zeros of the n-th Gegenbauer polynomial $P_n^{(\lambda)}$, $\lambda > -1/2$ are denoted by

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$$x_{1n}(\lambda) < x_{2n}(\lambda) < \cdots < x_{nn}(\lambda)\,.$$

In the recent paper [13] we applied the Euler–Rayleigh method to the Jacobi and, in particular, the Gegenbauer polynomials, represented as hypergeometric functions, to derive new bounds for their extreme zeros. Below we state some of the bounds obtained in [13], which improve upon some results of Driver and Jordaan [5].

Theorem A. ([13, Theorem 1.4]) *For every $n \geq 3$ and $\alpha, \beta > -1$, the largest zero $x_{nn}(\alpha,\beta)$ of the Jacobi polynomial $P_n^{(\alpha,\beta)}$ satisfies*

$$1 - x_{nn}(\alpha,\beta) < \frac{2(\alpha+1)(\alpha+3)}{(n+\alpha+1)(n+\alpha+\beta+1)\big[2 - \frac{(\alpha+1)(2n+\beta-1)}{(n+\alpha+1)(n+\alpha+\beta+1)-(\alpha+1)(\alpha+2)}\big]}\,.$$

Corollary A. ([13, Corollary 1.6]) *For every $n \geq 3$ and $\lambda > -1/2$, the largest zero $x_{nn}(\lambda)$ of the Gegenbauer polynomial $P_n^{(\lambda)}$ satisfies*

$$1 - x_{nn}(\lambda) < \frac{(2\lambda+1)(2\lambda+5)}{(n+2\lambda)(2n+2\lambda+1)\big[2 - \frac{(2\lambda+1)(4n+2\lambda-3)}{2(n+2\lambda)(2n+2\lambda+1)-(2\lambda+1)(2\lambda+3)}\big]}\,.$$

Theorem B. ([13, Theorem 1.1]) *For every $n \geq 3$ and $\lambda > -1/2$, the largest zero $x_{nn}(\lambda)$ of the Gegenbauer polynomial $P_n^{(\lambda)}$ satisfies*

$$1 - x_{nn}^2(\lambda) < \frac{(2\lambda+1)(2\lambda+5)}{2n(n+2\lambda)+2\lambda+1+\frac{2(\lambda+1)(2\lambda+1)^2(2\lambda+3)}{n(n+2\lambda)+2(2\lambda+1)(2\lambda+3)}}\,.$$

The above results provide lower bounds for the largest zeros of the Jacobi and Gegenbauer polynomials. It is instructive to compare Theorem B with the following upper bound for the largest zeros of the Gegenbauer polynomials:

Theorem C. ([12, Lemma 3.5]) *For every $\lambda > -1/2$, the largest zero $x_{nn}(\lambda)$ of the Gegenbauer polynomial $P_n^{(\lambda)}$ satisfies*

$$1 - x_{nn}^2(\lambda) > \frac{(2\lambda+1)(2\lambda+9)}{4n(n+2\lambda)+(2\lambda+1)(2\lambda+5)}\,.$$

We observe that, for any fixed $\lambda > -1/2$ and large n, the ratio of the upper and the lower bound for $1 - x_{nn}^2(\lambda)$, given by Theorems B and C, does not exceed 2. With Corollary 1 below this ratio is reduced to 1.6.

In the present paper we apply the Euler–Rayleigh method to the Jacobi polynomial $P_n^{(\alpha,\beta)}$, represented as a hypergeometric function, to obtain further bounds for the largest zeros of the Jacobi and Gegenbauer polynomials. As at some points the calculations become unwieldy, we have used the assistance of Wolfram's *Mathematica*.

The following is the main result in this paper.

Theorem 1. *For every $n \geq 4$ and $\alpha,\, \beta > -1$, the largest zero $x_{nn}(\alpha,\beta)$ of the Jacobi polynomial $P_n^{(\alpha,\beta)}$ satisfies*

$$1 - x_{nn}(\alpha,\beta) < \frac{4(\alpha+1)(\alpha+2)(\alpha+4)}{(5\alpha+11)\left[n(n+\alpha+\beta+1)+\frac{1}{3}(\alpha+1)(\beta+1)\right]}\,. \tag{1}$$

Moreover, if either $n \geq \max\{4, \alpha+\beta+3\}$ or $\beta \leq 4\alpha+7$, then

$$1 - x_{nn}(\alpha,\beta) < \frac{4(\alpha+1)(\alpha+2)(\alpha+4)}{(5\alpha+11)\left[n(n+\alpha+\beta+1)+\frac{1}{2}(\alpha+1)(\beta+1)\right]}\,. \tag{2}$$

Since $P_n^{(\alpha,\beta)}(x) = (-1)^n P_n^{(\beta,\alpha)}(-x)$, Theorem 1 can be equivalently formulated as

Theorem 2. *For every $n \geq 4$ and $\alpha,\, \beta > -1$, the smallest zero $x_{1n}(\alpha,\beta)$ of the Jacobi polynomial $P_n^{(\alpha,\beta)}$ satisfies*

$$1 + x_{1n}(\alpha,\beta) < \frac{4(\beta+1)(\beta+2)(\beta+4)}{(5\beta+11)\left[n(n+\alpha+\beta+1)+\frac{1}{3}(\alpha+1)(\beta+1)\right]}\,.$$

Moreover, if either $n \geq \max\{4, \alpha+\beta+3\}$ or $\alpha \leq 4\beta+7$, then

$$1 + x_{1n}(\alpha,\beta) < \frac{4(\beta+1)(\beta+2)(\beta+4)}{(5\beta+11)\left[n(n+\alpha+\beta+1)+\frac{1}{2}(\alpha+1)(\beta+1)\right]}\,.$$

The assumption $\beta \leq 4\alpha+7$ is satisfied, in particular, when $\beta = \alpha > -1$. Therefore, as a consequence of Theorem 1, we obtain a bound for the largest zero of the ultraspherical polynomial $P_n^{(\lambda)} = c\,P_n^{(\alpha,\alpha)}$, $\alpha = \lambda - \frac{1}{2}$.

Theorem 3. *For every $n \geq 4$ and $\lambda > -1/2$, the largest zero $x_{nn}(\lambda)$ of the Gegenbauer polynomial $P_n^{(\lambda)}$ satisfies*

$$1 - x_{nn}(\lambda) < \frac{(2\lambda+1)(2\lambda+3)(2\lambda+7)}{(10\lambda+17)\left[n(n+2\lambda)+\frac{1}{8}(2\lambda+1)^2\right]}\,. \tag{3}$$

Theorem 3 and $1 - x_{nn}^2(\lambda) < 2(1 - x_{nn}(\lambda))$ imply immediately the following:

Corollary 1. *For every $n \geq 4$ and $\lambda > -1/2$, the largest zero $x_{nn}(\lambda)$ of the Gegenbauer polynomial $P_n^{(\lambda)}$ satisfies*

$$1 - x_{nn}^2(\lambda) < \frac{2(2\lambda+1)(2\lambda+3)(2\lambda+7)}{(10\lambda+17)\left[n(n+2\lambda)+\frac{1}{8}(2\lambda+1)^2\right]}\,. \tag{4}$$

Usually, comparison of the various bounds for the extreme zeros of the classical orthogonal polynomials is not an easy task due to the parameters involved. At least for large n, the bounds in Theorem 1, Theorem 3 and Corollary 1 are sharper than those in Theorem A, Corollary A and Theorem B, respectively. In fact, the actual bounds obtained with the approach here are slightly sharper but are given by rather complicated expressions; in particular, by a limit passage we reproduce a result of Gupta and Muldoon from [8] about the smallest zero of the Laguerre polynomial. These and some other observations are given in Section 4 of the paper.

The rest of the paper is organized as follows. In Section 2 we present the necessary facts about the Euler–Rayleigh method and the Newton identities. The proof of the results is given in Section 3.1 (for the reader's convenience, in Section 3.2 we include a short proof of Theorem C).

2 The Euler–Rayleigh Method

As was already mentioned, the proof of our results exploits the so-called Euler–Rayleigh method (see [10]). Since here (and in [13]) the Euler–Rayleigh method is applied to real-root polynomials, for the reader's convenience we provide some details from [13].

Let P be a monic polynomial of degree n with zeros $(x_i)_1^n$,

$$P(x) = x^n - b_1 x^{n-1} + b_2 x^{n-2} - \cdots + (-1)^n b_n = \prod_{i=1}^{n} (x - x_i)\,. \tag{5}$$

For $k \in \mathbb{N}_0$, the power sums

$$p_k = p_k(P) := \sum_{i=1}^{n} x_i^k, \qquad p_0 = n = \deg P,$$

and the coefficients $(b_i)_1^n$ of P are connected by the Newton identities (cf. [20])

$$p_r + \sum_{i=1}^{\min\{r-1,n\}} (-1)^i p_{r-i}\, b_i + (-1)^r r\, b_r = 0\,, \qquad (b_i = 0\,,\ \ i > n)\,.$$

From Newton's identities one easily obtains:

Lemma 1. *Assuming $n \ge r$, the following formulae hold for p_r, $1 \le r \le 4$:*

$$\begin{aligned} p_1(P) &= b_1\,;\\ p_2(P) &= b_1^2 - 2b_2\,;\\ p_3(P) &= b_1^3 - 3b_1 b_2 + 3b_3\,;\\ p_4(P) &= b_1^4 - 4b_1^2 b_2 + 2b_2^2 + 4b_1 3b_3 - 4b_4\,. \end{aligned}$$

Let us set

$$\ell_k(P) := \frac{p_k(P)}{p_{k-1}(P)}\,, \qquad u_k(P) := \big[p_k(P)\big]^{1/k}\,, \qquad k \in \mathbb{N}\,.$$

The following statement is Proposition 2.2 in [13], and provides a slight modification of Lemma 3.2 in [10].

Proposition 1. *Let P be as in* (5) *with positive zeros $x_1 < x_2 < \cdots < x_n$. Then the largest zero x_n of P satisfies the inequalities*

$$\ell_k(P) < x_n < u_k(P), \qquad k \in \mathbb{N}.$$

Moreover, $\{\ell_k(P)\}_{k=1}^{\infty}$ is monotonically increasing, $\{u_k(P)\}_{k=1}^{\infty}$ is monotonically decreasing, and

$$\lim_{k\to\infty} \ell_k(P) = \lim_{k\to\infty} u_k(P) = x_n .$$

3 Proof of the Results

3.1 Proof of Theorem 1

The starting point for the proof of Theorem 1 is the following representation of $P_n^{(\alpha,\beta)}$ (cf. [18, eqn. (4.21.2)]:

$$P_n^{(\alpha,\beta)}(x) = \frac{(\alpha+1)_n}{n!}\, {}_2F_1\Big(-n, n+\alpha+\beta+1; \alpha+1; \frac{1-x}{2}\Big) \tag{6}$$

(for the proof of Theorem C we used another representation of $P_n^{(\alpha,\beta)}$ as a hypergeometric function, namely, [18, eqn. (4.3.2)]). Here we use Szegő's notation for the hypergeometric ${}_2F_1$ function,

$$F(a,b;c;z) = 1 + \sum_{k=1}^{\infty} \frac{(a)_k}{k!}\frac{(b)_k}{(c)_k} z^k, \qquad (a)_k := a(a+1)\cdots(a+k-1).$$

It follows from (6) that the monic polynomial

$$P(z) = z^n + \sum_{i=1}^{n} (-1)^i b_i z^{n-i}$$

with coefficients

$$b_i = b_i(P) = \binom{n}{i} \frac{(n+\alpha+\beta+1)_i}{(\alpha+1)_i}, \qquad i = 1,\ldots,n, \tag{7}$$

has n positive zeros $z_1 < z_2 < \cdots < z_n$, connected with the zeros of $P_n^{(\alpha,\beta)}$ by the relation

$$z_i = \frac{2}{1 - x_{in}(\alpha,\beta)}, \qquad i = 1,\ldots,n.$$

According to Proposition 1, $p_{k+1}(P)/p_k(P) < z_n < \big[p_k(P)\big]^{1/k}$, $k \in \mathbb{N}$, and consequently

$$\frac{2}{\big[p_k(P)\big]^{1/k}} < 1 - x_{nn}(\alpha,\beta) < \frac{2p_k(P)}{p_{k+1}(P)}\,, \qquad k \in \mathbb{N}. \tag{8}$$

At this point, we find it suitable to substitute

$$a := \alpha + 1\,,\ b := \beta + 1\,,$$
$$t := n(n+\alpha+\beta+1)\,,$$

thus $a, b > 0$ and $t = n(n+a+b-1)$. With this notation, the first four coefficients $b_i(P)$ in (7) are given by

$$b_1(P) = \frac{t}{a}\,, \quad b_2(P) = \frac{t(t-a-b)}{2a(a+1)}\,, \quad b_3(P) = \frac{t(t-a-b)\big[t-2(a+b+1)\big]}{6a(a+1)(a+2)}\,,$$
$$b_4(P) = \frac{t(t-a-b)\big[t-2(a+b+1)\big]\big[t-3(a+b+2)\big]}{24a(a+1)(a+2)(a+3)}\,.$$

On using Lemma 1, we find $p_1(P) = b_1(P) = t/a$,

$$p_2(P) = \frac{t\big[t+a(a+b)\big]}{a^2(a+1)}\,, \tag{9}$$

$$\begin{aligned} p_3(P) &= \frac{t\,q_2(t)}{a^3(a+1)(a+2)}\,,\\ q_2(t) &= 2t^2 + a(2a+3b)t + a^2(a+b)(a+b+1)\,, \end{aligned} \tag{10}$$

$$\begin{aligned} p_4(P) &= \frac{t\,q_3(t)}{a^4(a+1)^2(a+2)(a+3)}\,,\\ q_3(t) &= (5a+6)t^3 + 2a(3a^2+5ab+4a+6b)t^2\\ &\quad + a^2(3a^3+9a^2b+6ab^2+6a^2+15ab+7b^2+2a+4b)t\\ &\quad + a^3(a+1)(a+b)(a+b+1)(a+b+2)\,. \end{aligned} \tag{11}$$

Theorem 1 follows from the right-hand inequality in (8) with $k = 3$. In order to show this, we observe that, according to (10) and (11),

$$1 - x_{nn}(\alpha,\beta) < \frac{2p_3(P)}{p_4(P)} = \frac{2a(a+1)(a+3)q_2(t)}{q_3(t)} = \frac{4(\alpha+1)(\alpha+2)(\alpha+4)}{\frac{2q_3(t)}{q_2(t)}}\,.$$

Hence, to prove the first part of Theorem 1, it suffices to show that if a, b and t are positive, then

$$\frac{2q_3(t)}{q_2(t)} \geq (5a+6)\Big(t + \frac{ab}{3}\Big) = (5\alpha+11)\Big[n(n+\alpha+\beta+1) + \frac{1}{3}(\alpha+1)(\beta+1)\Big]\,.$$

With the help of Wolfram's *Mathematica*, we find

$$2q_3(t)-(5a+6)\Big(t+\frac{ab}{3}\Big)q_2(t)=\frac{a}{3}\,r_2(t)\,,$$

where

$$\begin{aligned}r_2(t)=&(6a^2+5ab+12a+6b)t^2\\&+a(3a^3+14a^2b+6ab^2+3a^2+27ab+6b^2-6a+6b)t\\&+a^2(a+b)(a+b+1)(6a^2+ab+18a+12)\,.\end{aligned}$$

It is clear now that $r_2(t)>0$: the single negative summand in the right-hand side, $-6a^2t$, is neutralized by $6a^2t^2$, since $t=n(n+a+b-1)>n(n-1)\geq 12$ for $n\geq 4$. Consequently,

$$\frac{2q_3(t)}{q_2(t)}\geq(5a+6)\Big(t+\frac{ab}{3}\Big)\,,$$

and the first part of Theorem 1 is proved.

For the proof of the second part of Theorem 1, we need to show that

$$2q_3(t)-(5a+6)\Big(t+\frac{ab}{2}\Big)q_2(t)\geq 0 \tag{12}$$

provided either $\beta\leq 4\alpha+7$ or $n\geq\max\{4,\alpha+\beta+3\}=\max\{4,a+b+1\}$.

With the assistance of *Mathematica*, we find

$$2q_3(t)-(5a+6)\Big(t+\frac{ab}{2}\Big)q_2(t)=\frac{1}{2}a^2(a+2)\,s_2(a,b;t)\,,$$

where

$$s_2(a,b;t)=4t^2+(2a^2+6ab-b^2-2a+2b)t+a(a+b)(a+b+1)(4a-b+4)\,.$$

Assume first that $\beta\leq 4\alpha+7$, which is equivalent to $b\leq 4a+4$. Then obviously the constant term in the quadratic $s_2(a,b;\cdot)$ is non-negative. We shall prove that the sum of the other two terms is positive. Indeed, since for $n\geq 4$ we have

$$t=n(n+a+b-1)\geq 4(a+b+3)>0\,,$$

we need to show that $4t+2a^2+6ab-b^2-2a+2b>0$. The latter follows from

$$\begin{aligned}4t+2a^2+6ab-b^2-2a+2b&\geq 16(a+b+3)+2a^2+6ab-b^2-2a+2b\\&=2a^2+b(6a+18-b)+14a+48\\&\geq 2a^2+b(2a+14)+14a+48>0\,.\end{aligned}$$

Now, assume that $n\geq\max\{4,\alpha+\beta+3\}=\max\{4,a+b+1\}$. We observe that

$$t=n(n+a+b-1)\geq 2(a+b)(a+b+1)\,. \tag{13}$$

Therefore,

$$
\begin{aligned}
4t^2+(2a^2+6ab-b^2-2a+2b)t &\ge 8(a+b)(a+b+1)t+(2a^2+6ab-b^2-2a+2b)t\\
&= (10a^2+22ab+7b^2+6a+10b)t > 0\,.
\end{aligned}
$$

Using the latter inequality and applying (13) once again, we conclude that

$$
\begin{aligned}
s_2(a,b;t) &\ge (10a^2+22ab+7b^2+6a+10b)t + a(a+b)(a+b+1)(4a-b+4)\\
&\ge (a+b)(a+b+1)\Big[20a^2+44ab+14b^2+12a+20b+a(4a-b+4)\Big]\\
&= (a+b)(a+b+1)(24a^2+43ab+14b^2+16a+20b) > 0\,.
\end{aligned}
$$

Thus, (12) holds true in the case $n \ge \max\{4, \alpha+\beta+3\}$, too, which completes the proof of the second part of Theorem 1.

3.2 Proof of Theorem C

The original proof of Theorem C in [12] exploits an idea from [18, Paragraph 6.2], based on the following observation of Laguerre: if f is a real-valued polynomial of degree n having only real and distinct zeros, and $f(x_0)=0$, then

$$
3(n-2)\big[f''(x_0)\big]^2 - 4(n-1)f'(x_0)f'''(x_0) \ge 0\,. \tag{14}
$$

In [15] Uluchev and the author proved a conjecture of Foster and Krasikov [7], stating that if f is a real-valued polynomial of degree n, then for every integer m satisfying $0 \le 2m \le n$ the following inequalities hold true:

$$
\sum_{j=0}^{2m}(-1)^{m+j}\binom{2m}{j}\frac{(n-j)!(n-2m+j)!}{(n-m)!(n-2m)!}f^{(j)}(x)f^{(2m-j)}(x) \ge 0, \qquad x\in\mathbb{R}\,.
$$

It was shown in [15] that these inequalities provide a refinement of the Jensen inequalities for functions from the Laguerre-Pólya class, specialized to the subclass of real-root polynomials. In [4], (14) was deduced from the above inequalities in the special case $m=2$, and then exploited for the derivation of certain bounds for the zeros of classical orthogonal polynomials.

Let us substitute in (14) $f=P_n^{(\lambda)}$ and $x_0 = x_{nn}(\lambda)$. We make use of $f(x_0)=0$ and the second order differential equations for f and f',

$$
\begin{aligned}
&(1-x^2)f'' - (2\lambda+1)xf'(x) + n(n+2\lambda)f = 0\,,\\
&(1-x^2)f''' - (2\lambda+3)xf''(x) + (n-1)(n+2\lambda+1)f' = 0\,,
\end{aligned}
$$

to express $f'(x_0)$ and $f'''(x_0)$ in terms of $f''(x_0)$ as follows:

$$f'(x_0) = \frac{1-x_0^2}{(2\lambda+1)x_0} f''(x_0)\,,$$
$$f'''(x_0) = \Big[\frac{(2\lambda+3)x_0}{1-x_0^2} - \frac{(n-1)(n+2\lambda+1)}{(2\lambda+1)x_0}\Big] f''(x_0)\,.$$

Putting these expressions in (14), canceling out the positive factor $\big[f''(x_0)\big]^2$ and solving the resulting inequality with respect to x_0^2, we arrive at the condition

$$x_0^2 \le \frac{(n-1)(n+2\lambda+1)}{(n+\lambda)^2+3\lambda+\frac{5}{4}+3\frac{(\lambda+1/2)^2}{n-1}}\,.$$

Hence,

$$x_0^2 < \frac{(n-1)(n+2\lambda+1)}{(n+\lambda)^2+3\lambda+\frac{5}{4}} = 1 - \frac{(2\lambda+1)(2\lambda+9)}{4n(n+2\lambda)+(2\lambda+1)(2\lambda+5)}\,.$$

This accomplishes the proof of Theorem C.

4 Concluding Remarks

1. As was mentioned in the introduction, at least for large n, the bounds given in Theorem 1, Theorem 3 and Corollary 1 are sharper than those in Theorem A, Corollary A and Theorem B, respectively. For instance, for fixed $\alpha,\beta > -1$ the upper bounds for $1-x_{nn}(\alpha,\beta)$ in Theorem A and Theorem 1 are respectively

$$\frac{(\alpha+1)(\alpha+3)}{n^2} + o(n^{-2}), \qquad \frac{4(\alpha+1)(\alpha+2)(\alpha+4)}{(5\alpha+11)n^2} + o(n^{-2})\,, \qquad n\to\infty\,,$$

and

$$(\alpha+1)(\alpha+3) - \frac{4(\alpha+1)(\alpha+2)(\alpha+4)}{5\alpha+11} = \frac{(\alpha+1)^3}{5\alpha+11} > 0\,, \qquad \alpha > -1\,.$$

The same conclusion is drawn for the other two pairs of bounds when $\lambda > -1/2$ is fixed and n is large (it follows from the above consideration with $\lambda = \alpha - 1/2$).

2. Theorem 1 is deduced from the second inequality in (8) with $k = 3$. Note that (8) with $k = 2$ together with (9) and (10) implies the estimate

$$1 - x_{nn}(\alpha,\beta) < \frac{2(\alpha+1)(\alpha+3)}{2n(n+\alpha+\beta+1)+(\alpha+1)(\beta+1)}\,,$$

which however is less precise than the estimate in Theorem C, and also than the estimate of Driver and Jordaan from [5],

$$1-x_{nn}(\alpha,\beta)<\frac{2(\alpha+1)(\alpha+3)}{2n(n+\alpha+\beta+1)+(\alpha+1)(\alpha+\beta+2)}\,.$$

Of course, having found the power sums $p_i(P)$, $1\le i\le 4$, one could apply Proposition 1 for derivation of lower bounds for $1-x_{n,n}(\alpha,\beta)$ as well. For instance, the first inequality in (8) with $k=4$ yields

$$1-x_{nn}(\alpha,\beta)>\frac{2}{\big[p_4(P)\big]^{1/4}}$$

with $p_4(P)$ given by (11) and $a=\alpha+1$, $b=\beta+1$, $t=n(n+\alpha+\beta+1)$. However, the expression on the right-hand side looks rather complicated to be of any use.

3. In [8] Gupta and Muldoon proved the following upper bound for the smallest zero $x_{1n}(\alpha)$ of the n-th Laguerre polynomial $L_n^{(\alpha)}$:

$$x_{1n}(\alpha)<\frac{(\alpha+1)(\alpha+2)(\alpha+4)(2n+\alpha+1)}{(5\alpha+11)n(n+\alpha+1)+(\alpha+1)^2(\alpha+2)}\,. \tag{15}$$

Let us demonstrate how this result can be deduced from the proof of Theorem 1 and the well-known limit relation

$$x_{1n}(\alpha)=\lim_{\beta\to\infty}\frac{\beta}{2}\big(1-x_{nn}(\alpha,\beta)\big)\,.$$

Since

$$\frac{1}{2}\big(1-x_{nn}(\alpha,\beta)\big)\le\frac{p_3(P)}{p_4(P)}=\frac{a(a+1)(a+3)q_2(t)}{q_3(t)}$$

with $a=\alpha+1$, $b=\beta+1$, $t=n(n+\alpha+\beta+1)$, and $q_2(t)$, $q_3(t)$ given in (10) - (11), we have

$$x_{1n}(\alpha)=\lim_{\beta\to\infty}\frac{\beta}{2}\big(1-x_{nn}(\alpha,\beta)\big)\le a(a+1)(a+3)\lim_{b\to\infty}\frac{b\,q_2(t)}{q_3(t)}\,. \tag{16}$$

On using

$$\lim_{b\to\infty}\frac{t}{b}=n$$

and the explicit form of $q_2(t)$ and $q_3(t)$, we find

$$\begin{aligned}\lim_{b\to\infty}\frac{b\,q_2(t)}{q_3(t)}&=\lim_{b\to\infty}\frac{q_2(t)/b^2}{q_3(t)/b^3}\\&=\frac{2n^2+3a\,n+a^2}{(5a+6)n^3+2a(5a+6)n^2+a^2(6a+7)n+a^3(a+1)}\\&=\frac{2n+a+1}{(5a+6)n(n+a)+a^2(a+1)}\,.\end{aligned}$$

By substituting the latter expression in (16) and setting $a=\alpha+1$, we arrive at (15) .

4. We already mentioned in the introduction that, for every fixed $\lambda>-1/2$, the ratio $r(\lambda,n)$ of the upper and the lower bound for $1-x_{nn}^2(\lambda)$, given by Theorem 1 and Theorem C, respectively, tends to a limit which does not exceed 1.6. More precisely,

$$r(\lambda,n)=\rho(\lambda)\psi(\lambda,n)\,,$$

where

$$\rho(\lambda)=\frac{8(2\lambda+3)(2\lambda+7)}{(2\lambda+9)(10\lambda+17)}\,,\qquad \varphi(\lambda,n)=\frac{n(n+2\lambda)+(2\lambda+1)(2\lambda+5)/4}{n(n+2\lambda)+(2\lambda+1)^2/8}\,.$$

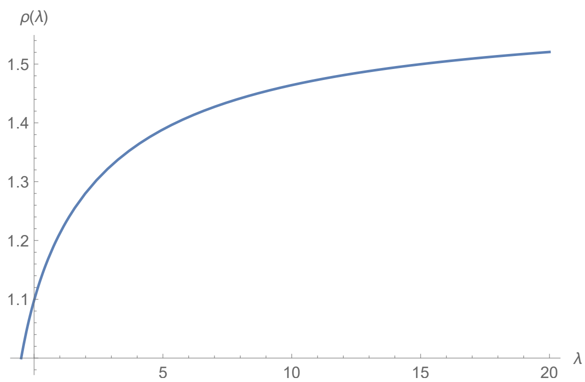

Fig. 1 The graph of $\rho(\lambda)$.

The function $\rho(\lambda)$ is monotonically increasing in the interval $(-1/2,\infty)$ assuming values between 1 and 1.6 (see Fig. 1) while, for a fixed $\lambda>-1/2$, $\lim_{n\to\infty}\varphi(\lambda,n)=1$.

5. The Euler–Rayleigh approach assisted with symbolic algebra has been applied in [17] for the derivation of bounds for the extreme zeros of the Laguerre polynomials, and in [1, 14, 16] for the estimation of the extreme zeros of some non-classical orthogonal polynomials, which are related to the sharp constants in some Markov-type inequalities in weighted L_2 norms.

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