POTENTIAL USES OF REPRESENTATIONS OF $SL(4,\mathbb{R})$ IN PARTICLE PHYSICS

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Abstract. I investigate ways in which the representations of $SL(4,\mathbb{R})$ might describe fundamental particles, and how this representation theory might be connected to the standard model of particle physics.

In the process I discover a mathematical reason for the complexification of the Clifford algebra, and draw attention to experimental evidence that this complexification may not have been done in the best possible way.

At the same time I investigate the uses of duality between representations, and uncover experimental evidence that a duality may have been inadvertently introduced into the standard model in an inappropriate place.

1. Introduction

1.1. **The real forms of A_3.** There is a huge literature on the use of Lie groups of type A_3 in theories of fundamental physics. Famous examples are the Pati–Salam model [1, 2] based around a group $SU(4)$, and Penrose's twistor theory [3, 4] based around $SU(2,2)$. There are five distinct real forms of the Lie algebra, each of which gives rise to two different groups. In each case, one of the groups is simply-connected (that is, fermionic) and acts on a 4-dimensional space of spinors, while the other group is of adjoint (that is, bosonic) type, and acts on a 6-dimensional Lie algebra. For reference, these groups are tabulated in Table 1, together with the signature of the Killing form.

The specific feature of $SU(2,2)$ that appeals in particular to physicists is that it contains the Poincaré group, so that there are obvious and direct connections to general relativity. It is the only one of the five real forms to do so. Modern approaches to twistor theory [4] allow one to switch between different real forms, but this may have the effect of obscuring the questions rather than answering them.

The case $SL(2,\mathbb{H})$ is attractive to particle physicists, as it clearly generalises the group $SL(2,\mathbb{C})$ used in the Dirac model of relativistic quantum mechanics. Various uses of $SL(2,\mathbb{H})$ and larger related groups appear in the literature, for example [5, 6, 7]. A further attraction is that the signature $(5,10)$ is reminiscent of the structure of the Georgi–Glashow grand unified theory [8] based on $SU(5)$, although there is no obvious direct connection.

But $SL(2,\mathbb{H})$ is not the only real form that contains $SL(2,\mathbb{C})$: both $SU(2,2)$ and $SL(4,\mathbb{R})$ have this property also. Hence the particular attraction of $SU(2,2)$ as a place where unification of quantum mechanics and general relativity might take place. However, twistor theory has not in practice led to such unification. It may therefore be worth considering the merits of $SL(4,\mathbb{R})$.

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Table 1. Real forms of A_3

Signature	Fermionic	Bosonic
$(0,15)$	$SU(4)$	$SO(6)$
$(6,9)$	$SU(3,1)$	
$(8,7)$	$SU(2,2)$	$SO(4,2)$
$(5,10)$	$SL(2,\mathbb{H})$	$SO(5,1)$
$(9,6)$	$SL(4,\mathbb{R})$	$SO(3,3)$

1.2. **The split real form.** What singles this case out is that it is the only one in which the spinor representation is real. This property is certainly not obviously a requirement for unification, but it does have quite a lot of consequences for the structures of all the representations, and it is possible that the representations in this case fit better to the observed fundamental particles than do the representations in the other cases.

In [9] I analysed the automorphisms and the subgroup structure of $SL(4,\mathbb{R})$ in some detail, and drew attention to many parallels between specific structures in the group and specific parts of the standard model of particle physics. The paper [9] also contains two particular mass equations that are suggested by this investigation, that are not only in exact agreement with experiment (correct to 5 and (nearly) 4 significant figures respectively), but are also predictive.

These parallels do not of course constitute a model, but they do make some very specific suggestions for how a model might be built on this basis. Any such model must also use representations of $SL(4,\mathbb{R})$ to classify particles. It is the aim of this paper to investigate the representation theory in enough detail to show how some of this classification might arise, and how it restricts to the standard model classification in certain cases.

2. Some representations

2.1. **Tensors of rank 1, 2 and 3.** Let us begin with a 4-dimensional real vector space V, consisting of (say) column vectors, acted on in the natural way by $SL(4,\mathbb{R})$ consisting of 4×4 real matrices with determinant 1. Initially I consider V to be the space of Majorana spinors for a subgroup $SL(2,\mathbb{C})$, but other interpretations may emerge later. All irreducible (finite-dimensional) representations can be made as tensors of some rank r. If r is odd then the representation is faithful (that is, in physics language, fermionic), while if r is even the representation has -1 acting trivially, and is called bosonic.

As is well-known [10], the rank 2 tensors split into a symmetric part $S^2(V)$ of dimension 10 and an anti-symmetric part $\Lambda^2(V)$ of dimension 6. Mathematically this is expressed in the equation

$$V\otimes V = S^2(V)\oplus\Lambda^2(V), \tag{1}$$

but we might more informally write something like

$$\begin{aligned} V\otimes V &= 6+10, \text{ or}\\ 4\times 4 &= 6+10. \end{aligned} \tag{2}$$

In rank 3 there is, in addition to the symmetric and anti-symmetric tensors, also a 'mixed' or 'middle' tensor cube $M^3(V)$, that is also written as $S^{2,1}(V)$ in a notation that generalises to higher ranks. The fundamental equations are

$$\begin{array}{rcl} V\otimes\Lambda^2(V) &=& \Lambda^3(V)\oplus S^{2,1}(V)\\ V\otimes S^2(V) &=& S^3(V)\oplus S^{2,1}(V) \end{array} \tag{3}$$

so that the middle cube actually occurs twice in the full tensor cube $V\otimes V\otimes V$. Informally, we have

$$\begin{array}{rcl} 4\times 6 &=& 4'+20_m\\ 4\times 10 &=& 20_m+20_s, \end{array} \tag{4}$$

where the subscripts m and s are used to distinguish the middle cube from the symmetric cube, and $4'$ denotes the dual V' of V.

2.2. **Rank 4 tensors.** It is very important to keep a clear distinction between a representation and its dual. This is clear physically, where momentum is dual to position but must under no circumstances be confused with it! It is also clear mathematically, where $4\times 4=6+10$, but $4\times 4'=1+15$.

To get a full classification of the rank 4 tensors it is easiest to look at squares of rank 2 tensors. Informally we have

$$\begin{array}{rcl} \Lambda^2(6) &=& 15\\ S^2(6) &=& 1+20_c\\ \Lambda^2(10) &=& 45\\ S^2(10) &=& 20_c+35. \end{array} \tag{5}$$

In particular, there is yet another 20-dimensional representation, that is used in general relativity [11] for the Riemann Curvature Tensor, so I have denoted it with a subscript c. To see that we have everything in rank 4 we can also calculate the decompositions of products of distinct tensors as follows:

$$\begin{array}{rcl} 6\times 10 &=& 15+45\\ 4\times 4' &=& 1+15\\ 4\times 20_m &=& 15+20_c+45\\ 4\times 20_s &=& 45+35 \end{array} \tag{6}$$

2.3. **Classification of tensors.** Notice that the trivial (1-dimensional) representation arises in rank 4. Hence every representation that arises in rank r also arises in rank $r+4$. So the rank itself is not an invariant of an irreducible representation, but the rank modulo 4 is an invariant. Therefore there are four fundamentally different types of representation, with ranks 0, 1, 2 and 3 modulo 4. Odd rank representations are fermionic, and even rank representations are bosonic.

The fermionic representations always come in dual pairs, but duality changes the rank and so changes the physical properties. For example 20_m and 20_s have duals $20'_m$ and $20'_s$ that first appear in rank 9. The bosonic representations sometimes come in dual pairs (for example, 10 in rank 2 is paired with $10'$ which first appears in rank 6), but are sometimes self-dual (for example $6=6'$).

But recall that duality is a mathematical operation that does not correspond to anything physical. For example, it might be possible to associate V and V' with left-handed and right-handed spinors in some way, in which case the fact that left-handed and right-handed spinors behave completely differently physically would be already built in to the structure of the group theory and representation theory.

3. The structure of fermions

3.1. **Spin structure.** Let us investigate how much of the structure and properties of fundamental fermions can be modelled in ranks 1 and 3. To do this we should first restrict the fermionic representations to the subgroup $SL(2,\mathbb{C})$. There is a subtlety here in that all our representations are real, so we cannot distinguish between a given complex irreducible for $SL(2,\mathbb{C})$ and its complex conjugate. Thus we cannot distinguish *a priori* between left-handed and right-handed representations. This distinction can only be made by imposing a complex structure, so we will need to look at the available complex structures, and try to interpret them physically.

With this proviso, both V and V' restrict as spin $1/2$ representations, either of which may be either left-handed or right-handed as far as we can tell so far. The middle cube 20_m restricts as two spin $1/2$ representations, and one with spin $(1,1/2)$ or $(1/2,1)$. The symmetric cube also contains $(1,1/2)$ or $(1/2,1)$, together with a spin $3/2$ representation. Altogether we have six copies of the spin $1/2$ representation, which is enough for three Dirac spinors provided we can find appropriate ways to choose the complex structures.

Associated with the spin $1/2$ representations we also have two copies of the spin $(1,1/2)$ representation in the two copies of 20_m. One might conjecture that the spin 1 factor represents the three colours of quarks, in which case ignoring the colours would give us another three Dirac spinors. However, this is highly speculative, and we should first look at how the representations break up on restriction to $SL(3,\mathbb{R})$.

3.2. **Colour/generation structure.** Restricting the irreducible fermionic representations to $SL(3,\mathbb{R})$ we have:

$$\begin{array}{rcl} 4 & \mapsto & 1+3 \\ 6 & \mapsto & 3+3' \\ 10 & \mapsto & 1+3+6' \\ 4' & \mapsto & 1+3' \\ 20_m & \mapsto & 3+3'+6'+8 \\ 20_s & \mapsto & 1+3+6'+10. \end{array} \tag{7}$$

Some things are rather striking about these decompositions. In particular the spin $1/2$ representations (lying inside 20_m) are associated with an 8-dimensional representation that is reminiscent of the baryon octet, of spin $1/2$ baryons. Similarly the spin $3/2$ representation (inside 20_s) is associated with a 10-dimensional representation that is reminiscent of the baryon decuplet, of 10 spin $3/2$ baryons. Moreover, in the spin $1/2$ sector one sees $3+3'+6'$ which looks extraordinarily like the classification of leptons into 3 neutrinos, 3 electrons and 6 quarks.

Of course, the 3 representation and its dual $3'$ are completely different physically, so it is highly plausible, for example, to associate one of them with electrons and the other with neutrinos. In other words, one of them is associated with significant differences in mass, and the other with insignificant differences in mass. If so, then this suggests that the concepts of colour and generation as used in the standard model might benefit from a little tweaking. A more explicit association of 'generation' with 'mass difference' and of 'colour' with 'no mass difference' might conceivably simplify the model. However, such speculation is premature.

4. The structure of bosons

4.1. **Spin structure.** Restricting to $SL(2,\mathbb{C})$, the fundamental bosonic representations split as

$$\begin{array}{rcl} 6 & \mapsto & 1+1+4 \\ 10 & \mapsto & 4+6, \end{array} \tag{8}$$

where 4 denotes the representation as $SO(3,1)$ on Minkowski space, and 6 denotes the complex 3-dimensional representation as $SO(3,\mathbb{C})$. The whole 16-dimensional representation $V\otimes V$ can be given the structure of the Clifford algebra of Minkowski space, and this is how this representation is usually presented in the standard model. In fact, the same structure applies to the rank 4 version $V\otimes V'$, and in the standard model these two are effectively combined into a single complexified Clifford algebra. I will describe this process in more detail in Section 5.

For our present purposes, however, it is essential to keep these two versions of the Clifford algebra separate. To add to the confusion, there is a third version $V'\otimes V'$, and one might also want to distinguish between $V\otimes V'$ and $V'\otimes V$.

4.2. **Colour and flavour.** One way to clarify the distinctions is to use the group $SL(3,\mathbb{R})$, which was also helpful in identifying fermions. For this group we have

$$\begin{array}{rcl} 6 & \mapsto & 3+3' \\ 10 & \mapsto & 1+3+6' \\ 15 & \mapsto & 1+3+3'+8 \end{array} \tag{9}$$

so that what appears as $1+8$ in rank 4 appears as $3+6'$ in rank 2.

By analogy with the fermionic $3+3'$ which I suggested might represent electrons and neutrinos, I suggest the bosonic $3+3'$ might similarly represent the photon and the three intermediate vector bosons. In rank 4, this might be extended by 8 gluons and 1 Higgs boson, for example. In rank 2 we need a different interpretation of $1+3+6'$, which cannot involve gluons. There is an additional complication, in that this representation is not self-dual, so that there is a dual version with a completely different physical interpretation. Whatever happens, we are looking for something in $1+3+6'$ or its dual, that possibly has some relationship to $1+1+8$ in rank 4, and therefore might have some bearing on the strong force.

4.3. **Pseudoscalar mesons.** It is well-known [12] that the pions are involved in the process whereby the strong force holds the nucleus of an atom together. So it is plausible to interpret the 3 component as the three pions. This suggests following Gell-Mann [13] into including also the strange quark, so that we obtain four kaons and the eta and eta-prime mesons, making 9 in all. But if this picture is correct, then there should be 10 such mesons altogether, not 9.

Predicting new particles at this stage is unlikely to be a productive strategy, since it is pretty clear that experiment has already found everything there is to be found in this range of energies. The alternative is a re-interpretation of particles that are already known. That is, one can suggest that there is more to the identities of the kaons than the quark content. This shouldn't be too radical a proposal, since it is well-known that the quark content of baryons does not determine the baryon, even if the spin is also specified. The fact that the Λ and Σ^0 baryons both have quark content uds and spin 1/2 is enough to alert us to this possibility.

I therefore make the suggestion that the three neutral kaons that are clearly distinguished by experiment, that is the K_1, K_2, and $K_0/\bar{K}_0$, be treated as independent particles, rather than quantum superpositions of each other. In support of this suggestion, I would draw attention to the fact that the original analysis of these particles in Gell-Mann's eightfold way puts the quarks into a 3-dimensional representation of $SU(3)$, and the anti-quarks into the dual representation.

Now as I have shown with many examples, a duality usually changes a physical concept into a completely different concept. Quarks and anti-quarks are not different concepts: they differ only by changing the directions of the time coordinate and one space coordinate. They must therefore lie in the same representation, not in dual representations.

4.4. **Possible new experimental evidence.** Lest this proposal be thought to be gratuitous iconoclasm, I point out a recently discovered anomaly [14, 15] affecting kaons in the standard model, whereby the rare decays of K_1 into $\pi^0\nu\bar{\nu}$ occur ten times as often as the standard model predicts. I am certainly not qualified to suggest any connection between my proposals and this anomaly, but if a change of kaon type can happen without any change in mass, and without any change in quark content, then it seems likely that some channels of decay may have been missed in the analysis.

I point out, moreover, that on restricting from $SL(3,\mathbb{R})$ to $SO(3)$ the representations 3 and $3'$ become equivalent, therefore self-dual, so that both $V \otimes V$ and $V \otimes V'$ restrict as $1+1+3+3+3+5$, and more specifically, both $1+8$ and $3+6'$ restrict as $1+3+5$. In terms of pseudoscalar mesons, this represents 1 eta meson, 3 pions and 5 kaons. In terms of gluons, it represents 3 anti-symmetric gluons and 5 symmetric gluons. (In the standard model, the anti-symmetric gluons are multiplied by i in order to make all 8 of them Hermitian.)

5. The Clifford algebra

5.1. **Real versus complex.** The standard model makes much use of the complexified Clifford algebra of Minkowski space. As an algebra, this is just the full matrix algebra of complex 4×4 matrices, but what makes it a Clifford algebra is that it supports a representation of $SL(2,\mathbb{C})$ with graded pieces

$$1 + 4 + (3 + 3) + 4 + 1. \tag{10}$$

Mathematically, it is best to start with the real Clifford algebra, that is the real 4×4 matrices, breaking up into real representations of dimensions $1+4+6+4+1$. This is an associative algebra, and the corresponding Lie algebra is $gl(4,\mathbb{R})$. This Lie algebra arises naturally in the rank 4 tensors over a Majorana spinor representation.

But there is another copy of the same representation of $SL(2,\mathbb{C})$ in the rank 2 tensors. I propose to show how this can be identified with the imaginary part of the Clifford algebra in the standard model. To do this, we need a rank 2 tensor to play the role of multiplication by i. Since it must square to -1 it must be anti-symmetric, which would appear to give us a choice of 6 dimensions. These 6 possibilities are, up to sign,

$$i\gamma_5, i\gamma_0, \gamma_0\gamma_5, \qquad \gamma_2\gamma_3, \gamma_3\gamma_1, \gamma_1\gamma_2. \tag{11}$$

The one that is chosen in the standard model is $i\gamma_5$. More specifically, the projection $1-\gamma_5$ identifies i with $i\gamma_5$.

5.2. **A problem.** Unfortunately this choice does not take account of the representation theory. The 6-dimensional space of anti-symmetric matrices must support a representation of $SL(2,\mathbb{C})$ of type $1+1+4$. Hence only two of the six choices for i are compatible with this requirement. It is easy to calculate which two they are, and they turn out to be $i\gamma_0$ and $\gamma_0\gamma_5$. Mathematically there is no significant distinction between these two choices, but physically it is probably important to choose the right one. My instinct is to choose $i\gamma_0$, on the grounds that projection by $1-\gamma_0$ looks like a plausible way to impose conservation of energy.

However this may be, the rather astonishing conclusion is that the standard model choice of $i\gamma_5$ at this point is mathematically inconsistent. Let us look at this problem also from a physical point of view. The functions of the Weinberg angle appear in the vertex factors of the Feynman calculus in the $i\gamma_\mu$ terms and not in the $i\gamma_\mu\gamma_5$ terms. The fact that γ_5 is invariant under $SL(2,\mathbb{C})$, identified in the standard model with a double cover of the Lorentz group, therefore implies that in this model, the Weinberg angle is Lorentz-invariant. Experiment, however, tells us that this is not in fact the case. The Weinberg angle is known to run with the energy scale, and is therefore invariant only under $SU(2)$, and not under $SL(2,\mathbb{C})$.

Hence the mathematical and physical arguments both lead to the same conclusion: the projection by $1-\gamma_5$ used in the standard model is incompatible with the representation theory of $SL(2,\mathbb{C})$, and should be replaced by either $1-\gamma_0$ or $1-i\gamma_0\gamma_5$.

6. Conclusion

In a previous paper [9] I investigated the subgroup structure of $SL(4,\mathbb{R})$ with a view to showing how the various symmetry groups that are needed in the standard model of particle physics can all be found as subgroups of this larger group. In particular, there are two chiral copies of $SL(2,\mathbb{C})$ as well as a subgroup $SL(3,\mathbb{R})$. I proposed to use one copy of $SL(2,\mathbb{C})$ as the Dirac group and the other as an effective gauge group for the weak force. While these two groups do not commute with each other, their respective subgroups $SU(2)$ do. Thus it is possible to use the left-handed $SU(2)$ as a gauge group for the weak force in non-relativistic quantum mechanics, with some possibility for some running of parameters such as the Weinberg angle when one extends to the relativistic case.

I proposed also to use $SL(3,\mathbb{R})$ in place of the gauge group of the strong force. This is more controversial, not so much because of the change from compact to split real form, which is a technicality that can be easily repaired, but more because the group does not commute with the Dirac group $SL(2,\mathbb{C})$. Moreover, use of the split real form highlights the difference between the 3-dimensional natural representation and its dual, which cannot be used for colours and anti-colours, but can be used for colours and generations respectively. In order to demonstrate that this may actually be a positive development, I used it to obtain two conjectured mass equations, both of which are in exact agreement with experiment, and neither of which has any explanation within the standard model.

In the present paper, I extend the analysis from the group theory to the representation theory. In particular, there is a 20-dimensional irreducible fermionic representation that on restriction to $SL(2,\mathbb{C})$ splits as the sum of a Dirac spinor and a 6-dimensional complex irreducible, so must represent some some $1/2$ particles. On restriction to $SL(3,\mathbb{R})$, this representation breaks as $3+3'+6'+8$.

I suggest therefore that this representation unifies the 6 leptons and 6 quarks of the standard model with the spin 1/2 baryon octet. This may not be exactly the unification that one might have expected, but perhaps it might be considered to be even better than the best one could have hoped for. Is it perhaps too good to be true? Possibly, but I don't believe so.

A similar analysis of the small bosonic representations reveals that the rank 2 bosons in $6+10$ are completely different from the rank 4 bosons in $1+15$. Nothing like this occurs in the standard model, where the 8 gluons in rank 4 are matched by the meson octet in rank 2. But the representation theory of $SL(3,\mathbb{R})$ requires at least a meson nonet, and the extension to $SL(4,\mathbb{R})$ requires a meson decuplet. I resolve this issue by proposing to incorporate 3 neutral kaons in the model, as is strongly indicated by experiment, rather than the two neutral kaons with which the standard model struggles to explain the experimental results.

The underlying mathematical reason for this necessary change to the standard model is that a duality always swaps completely different concepts, such as position and momentum, or time and energy. The duality is therefore not between colours and anti-colours, but between colours and generations.

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