RESOLUTIONS BY PERMUTATION MODULES

PAUL BALMER AND DAVE BENSON

Abstract. We prove that, up to adding a complement, every modular representation of a finite group admits a finite resolution by permutation modules.

Let G be a finite group and $\Bbbk$ be a field of characteristic $p > 0$ dividing the order of G. It is well-known that if G has non-cyclic Sylow p-subgroups, the $\Bbbk$-linear representation theory of G is complicated. In particular, the Krull–Schmidt abelian category, $\Bbbk G$-mod, of finite-dimensional $\Bbbk G$-modules admits *infinitely many* isomorphism classes of indecomposable objects. On the other hand, there is a much simpler class of $\Bbbk G$-modules, the *permutation modules*, *i.e.*, those isomorphic to $\Bbbk X$ for X a finite G-set. The *finite* collection $\{\Bbbk(G/H)\}_{H \leqslant G}$ additively generates all such modules.

For a $\Bbbk G$-module $M \in \Bbbk G$-mod, we want to analyze the existence of what we'll call a *permutation resolution* for short, *i.e.*, an exact sequence

$$0 \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to M \to 0 \qquad (1)$$

where all P_i are permutation modules. Up to direct summands, it is always possible:

2. **Theorem.** *Let G be a finite group and $M \in \Bbbk G$-mod. Then there exists a $\Bbbk G$-module N such that $M \oplus N$ admits a finite resolution (1) by permutation modules.*

The related problem of resolutions (1) that are not only exact but remain exact under all fixed-point functors has been recently discussed in [BSW17]. Allowing p-permutation modules P_i (that is, direct summands of permutation modules), Bouc–Stancu–Webb prove that such resolutions exist for all M if and only if G has a Sylow subgroup that is either cyclic or dihedral (for $p = 2$).

Unsurprisingly, Theorem 2 reduces to a Sylow subgroup S of G, since every M is a direct summand of $\mathrm{Ind}_S^G \mathrm{Res}_S^G(M)$ and since the functor Ind_S^G is exact and preserves permutation modules. So we focus on the case where G is a p-group.

For the proof, we shall consider a stronger property:

3. *Definition.* We say that a resolution (1) is *free up to degree* $m \geqslant 0$ if P_i is a free module for $i = 0, \ldots, m$. We say that M admits *good permutation resolutions* if for every integer $m \geqslant 0$, there exists a finite resolution (1) by permutation modules that is free up to degree m.

4. *Remark.* Let G be a p-group. A $\Bbbk G$-module M admits good permutation resolutions if and only if for all $m \geqslant 1$ the mth Heller loop $\Omega^m M$ admits a finite permutation resolution. Also, if Q is free and $M \oplus Q$ admits a permutation resolution as in (1) then the epimorphism $P_0 \twoheadrightarrow M \oplus Q \twoheadrightarrow Q$ forces Q to be a direct summand of P_0 and one can remove $0 \to Q \xrightarrow{=} Q \to 0$ from the resolution. So if $M \oplus Q$ has a permutation resolution that is free up to degree m then so does M.

Date: 2020 March 9.
First-named author supported by NSF grant DMS-1901696.

An advantage of good permutation resolutions is the *two out of three property*:

5. **Proposition.** *Let G be a p-group. Let $0 \to L \to M \to N \to 0$ be an exact sequence of $\Bbbk G$-modules. If two out of L, M and N have good permutation resolutions then so does the third.*

Proof. If $P \twoheadrightarrow N$ is a projective cover, we obtain by 'rotation' an exact sequence $0 \to \Omega^1 N \to L \oplus P \to M \to 0$. In view of Remark 4, we can rotate in this way and reduce to the case where L and M admit good permutation resolutions and then prove that N does. Let $m \geqslant 0$. Choose $P_\bullet \to M$ a permutation resolution of M that is free up to degree m. Let $\ell \geqslant m$ be such that $P_i = 0$ for all $i > \ell$. Now choose $Q_\bullet \to L$ a permutation resolution of L that is free up to degree ℓ. We have the following picture (plain part) with exact rows:

$$
\begin{array}{ccccccccccccccccc}
0 & \to & Q_n & \to & \cdots & \to & Q_{\ell+1} & \to & Q_\ell & \to & \cdots & \to & Q_0 & \to & L & \to & 0 \\
 & & \downarrow & & & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \\
0 & \to & 0 & \to & \cdots & \to & 0 & \to & P_\ell & \to & \cdots & \to & P_0 & \to & M & \to & 0
\end{array}
\tag{6}
$$

The standard lifting argument, using that Q_j is projective for $j = 0, \ldots, \ell$ shows that there exists a lift $f_\bullet\colon Q_\bullet \to P_\bullet$ of the morphism $L \to M$. Then the mapping cone complex $\mathrm{cone}(f_\bullet)$ yields a resolution of $\mathrm{coker}(L \to M) = N$ and this complex $\mathrm{cone}(f_\bullet)$ has free objects in degree $0, \ldots, m$ since $P_\bullet$ and $Q_\bullet$ do. ☐

Let us discuss an example of Theorem 2, where we can even take $N = 0$.

7. **Proposition.** *Let $E = (C_p)^{\times r} = C_p \times \cdots \times C_p$ be an elementary abelian group of rank r. Then every $\Bbbk E$-module admits good permutation resolutions.*

Proof. Consider for each $1 \leqslant i \leqslant r$ the ('coordinate-wise') subgroup

$$H_i = C_p \times \cdots \times C_p \times 1 \times C_p \times \cdots \times C_p$$

of rank $r-1$. Let $m \geqslant 0$. Inflating from $E/H_i \simeq C_p$ the usual 2-periodic resolutions $0 \to \Bbbk \to \Bbbk C_p \to \cdots \to \Bbbk C_p \to \Bbbk \to 0$ of length at least m, we obtain quasi-isomorphisms of $\Bbbk E$-modules $s_i\colon Q(i) \to \Bbbk[0]$ where the $Q(i)$ are defined as follows:

$$
\begin{array}{llccccccccccccc}
Q(i) := & 0 & \to & \Bbbk & \to & \Bbbk(E/H_i) & \to & \cdots & \to & \Bbbk(E/H_i) & \to & \Bbbk(E/H_i) & \to & 0 \\
{\scriptstyle s_i}\downarrow & & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \\
\Bbbk[0] = & 0 & \to & 0 & \to & 0 & \to & \cdots & \to & 0 & \to & \Bbbk & \to & 0
\end{array}
$$

Tensoring all the above, we obtain a quasi-isomorphism

$$s_1 \otimes \cdots \otimes s_r\colon P_\bullet := Q(1) \otimes \cdots \otimes Q(r) \to (\Bbbk[0])^{\otimes r} \cong \Bbbk[0],$$

i.e., a permutation resolution $P_\bullet$ of $\Bbbk$. In other words, we performed an 'external tensor' of all the periodic resolutions over each copy of C_p in E. Since the Mackey formula gives by induction $\Bbbk(E/H_{i_1}) \otimes \cdots \otimes \Bbbk(E/H_{i_n}) \cong \Bbbk(E/(H_{i_1} \cap \cdots \cap H_{i_n}))$, we have produced a permutation resolution $P_\bullet$ of $\Bbbk$ that is easily seen to be free up to degree m. As $m \geqslant 0$ was arbitrary, we proved that the trivial module $\Bbbk$ admits good permutation resolutions. A general module $M \in \Bbbk E\text{-mod}$ admits a filtration whose successive quotients are trivial. We therefore conclude by induction, via Proposition 5. ☐

8. *Remark.* The proof of Proposition 7 shows that the stabilisers in the permutation resolution may be taken to be products of subsets with respect to the given decomposition of E. Applying the proposition to a module and its dual shows that given

a module M we may form a finite exact complex of permutation modules with these stabilisers in such a way that the image of one of the maps is M. This should be compared with the main theorem of [BC] which shows that a finite exact sequence of permutation E-modules in which the set of stabilisers has no containment of index p necessarily splits, so that the image of every map is again a permutation module.

Proof of Theorem 2. As already mentioned, we can reduce to the case where G is a p-group. By [Car00], we know that for every $\Bbbk G$-module M, there exists a $\Bbbk G$-module N and a finite filtration $0 = L_0 \subset L_1 \subset \cdots \subset L_s = M \oplus N$ such that every L_i/L_{i-1} is induced from some elementary abelian subgroup $E_i \leqslant G$. Since the result holds for elementary abelian groups (Proposition 7) and is stable by induction, we see that all L_i/L_{i-1} admit good permutation resolutions. By Proposition 5, we conclude that so does $M \oplus N$. In particular, $M \oplus N$ has a permutation resolution. ☐

Acknowledgements: The authors are grateful to Serge Bouc, Martin Gallauer and Peter Webb for useful discussions. The authors would like to thank the Isaac Newton Institute for Mathematical Sciences, Cambridge, for support and hospitality during the programmes 'K-theory, algebraic cycles and motivic homotopy theory' and 'Groups, representations and applications: new perspectives', where work on this paper was undertaken. The Isaac Newton Institute is supported by EPSRC grant no EP/R014604/1.

References

[BC] David J. Benson and Jon F. Carlson. Bounded complexes of permutation modules. In preparation.

[BSW17] Serge Bouc, Radu Stancu, and Peter Webb. On the projective dimensions of Mackey functors. *Algebr. Represent. Theory*, 20(6):1467–1481, 2017.

[Car00] Jon F. Carlson. Cohomology and induction from elementary abelian subgroups. *Q. J. Math.*, 51(2):169–181, 2000.

Paul Balmer, Mathematics Department, UCLA, Los Angeles, CA 90095-1555, USA
Email address: balmer@math.ucla.edu
URL: http://www.math.ucla.edu/∼balmer

Dave Benson, Institute of Mathematics, University of Aberdeen, King's College, Aberdeen AB24 3UE, Scotland U.K.
Email address: d.j.benson@abdn.ac.uk
URL: https://homepages.abdn.ac.uk/d.j.benson/pages/