NEW INCOMPRESSIBLE SYMMETRIC TENSOR CATEGORIES IN POSITIVE CHARACTERISTIC

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CONTENTS

1. INTRODUCTION

A tensor category $\mathcal{C}$ is said to have *moderate growth* if for any object $X \in \mathcal{C}$ the length of $X^{\otimes n}$ grows at most exponentially with n. In characteristic zero, a remarkable theorem of Deligne ([15]) states that every symmetric tensor category of moderate growth is *super-Tannakian*, i.e., fibers over the category of supervector spaces. In other words, it is the category of representations of an affine supergroup scheme. However, in any characteristic $p > 0$ this theorem fails. Namely, counterexamples in characteristic $p \geq 5$ are given in [25, 27], in characteristic 2 in [34, 5], and in characteristic 3 for the first time in the present paper. In fact, in [5], the first two authors constructed a nested sequence of finite symmetric tensor categories in characteristic two,

$$\mathrm{Vec} = \mathcal{C}_0 \subset \mathcal{C}_1 \subset \mathcal{C}_2 \subset \cdots$$

which are *incompressible*, i.e., do not fiber over anything smaller. The main purpose of this paper is to construct similar incompressible symmetric tensor categories in characteristic $p > 2$. Unfortunately, the approach of [5] does not work in odd characteristic, so we propose a completely different construction. This construction also works in characteristic two, and recovers the categories defined in [5], even though there are some differences in the behavior at the prime two.

Our categories are closely connected with the non-symmetric Verlinde tensor categories in characteristic zero which categorify the Verlinde fusion rules for SL_2 ([22], 8.18), and because of this connection, we name them Ver_{p^n} $(n \geq 1)$. When $p > 2$, the category Ver_{p^n} decomposes as a tensor product $\mathrm{Ver}_{p^n}^+ \boxtimes \mathrm{sVec}$ of the "even part" and the category of supervector spaces.

The underlying abelian category of Ver_{p^n} is easy to construct. We start with the natural two dimensional module V for the algebraic group $SL_2(\mathbf{k})$, and take its tensor powers. Each $V^{\otimes m}$ has exactly one isomorphism class of indecomposable direct summand that does not appear in any previous tensor power, and this is the tilting module T_m. For $m < p$, T_m is isomorphic to the symmetric power $S^m V$, and for $m = p^k - 1$ it is the Steinberg module $\mathrm{St}_k = S^{p^k-1}V$. We form the direct sum $\bigoplus_{m=p^{n-1}-1}^{p^n-2} T_m$ and form its endomorphism ring A. Then Ver_{p^n} is the category A-mod of finite dimensional A-modules.

Constructing the symmetric tensor structure on Ver_{p^n} is more subtle. The essential point is to prove a splitting property of tilting modules.

Proposition 1.1. *(Proposition 3.2) Every morphism between tilting modules T_m with $m < p^k - 1$ is split by tensoring with St_{k-1}.*

In order to use this fact for putting an appropriate symmetric tensor structure on Ver_{p^n}, we set up a general categorical machinery of splitting ideals and abelian envelopes, which is of independent interest and likely to have other applications. Namely, given a Karoubian

rigid monoidal category $\mathcal{T}$, we say that $Q \in \mathcal{T}$ is a *splitting object* if tensoring with Q splits every morphism in $\mathcal{T}$; such objects form a thick ideal $\mathcal{S} \subset \mathcal{T}$. We say that $\mathcal{T}$ is *separated* if a morphism in $\mathcal{T}$ annihilated by tensoring with any splitting object is zero. Then we prove the following result.

Theorem 1.2. *(Theorems 2.41,2.42) If $\mathcal{T}$ is separated and $\mathcal{S}$ is finitely generated as a left ideal, then there is a multitensor (in particular, abelian) category $\mathcal{C}(\mathcal{T})$ together with a faithful monoidal functor $F : \mathcal{T} \to \mathcal{C}(\mathcal{T})$ which identifies $\mathcal{S}$ with the ideal of projectives in $\mathcal{C}(\mathcal{T})$. Moreover, if this functor is full then $\mathcal{C}(\mathcal{T})$ is the abelian envelope of $\mathcal{T}$ (i.e., faithful monoidal functors from $\mathcal{T}$ to any multitensor category $\mathcal{D}$ correspond to tensor functors $\mathcal{C}(\mathcal{T}) \to \mathcal{D}$).*

We then apply this construction to the category $\mathcal{T} = \mathcal{T}_{n,p}$ of tilting modules for $SL_2(\mathbf{k})$ modulo the tensor ideal generated by St_n. This is enabled by Proposition 3.2, which implies that the category $\mathcal{T}_{n,p}$ is separated. This yields an abelian equivalence $\mathcal{C}(\mathcal{T}_{n,p}) \cong \mathrm{Ver}_{p^n}$, which equips Ver_{p^n} with a structure of a symmetric tensor category.

Our main theorem is as follows.

Theorem 1.3. *Let $\mathbf{k}$ be an algebraically closed field of prime characteristic p. Then there are nested sequences of symmetric tensor categories $\mathrm{Ver}_{p^n}^+ = \mathrm{Ver}_{p^n}^+(\mathbf{k}) \subset \mathrm{Ver}_{p^n} = \mathrm{Ver}_{p^n}(\mathbf{k})$ over $\mathbf{k}$,*

$$\mathrm{Ver}_p^+ \subset \mathrm{Ver}_{p^2}^+ \subset \mathrm{Ver}_{p^3}^+ \subset \cdots$$

and

$$\mathrm{Ver}_p \subset \mathrm{Ver}_{p^2} \subset \mathrm{Ver}_{p^3} \subset \cdots$$

(Theorem 4.35) with the following properties.[1]

(1) *The category Ver_p is the semisimplification of the module category for $\mathbb{Z}/p$. For $n \geq 2$, the categories $\mathrm{Ver}_{p^n}, \mathrm{Ver}_{p^n}^+$ are not semisimple (Theorem 4.5(i),(ii)).*

(2) *For $p > 2$, the category Ver_{p^n} decomposes as a tensor product $\mathrm{Ver}_{p^n}^+ \boxtimes \mathrm{sVec}$ (Corollary 4.12).*

(3) *We have $\mathrm{Ver}_{2^n} = \mathcal{C}_{2n-2}$ and $\mathrm{Ver}_{2^n}^+ = \mathcal{C}_{2n-3}$, the symmetric tensor categories constructed in [5] (Theorem 4.5(iii)).*

(4) *If $p^n > 2$, the Grothendieck ring of $\mathrm{Ver}_{p^n}^+$ is isomorphic to the ring of integers in the real part of the cyclotomic field of p^n-th roots of unity, namely $\mathbb{Z}[2\cos(2\pi/p^n)]$. In other words, $\mathrm{Ver}_{p^n}^+$ gives an abelian categorification of the ring $\mathbb{Z}[2\cos(2\pi/p^n)]$. The Grothendieck ring of Ver_{p^n} is isomorphic to the group ring of $\mathbb{Z}/2$ over $\mathbb{Z}[2\cos(2\pi/p^n)]$ if $p > 2$ and to $\mathbb{Z}[2\cos(2\pi/2^{n+1})]$ if $p = 2$ (Theorem 4.5(iv)).*

(5) *The Frobenius-Perron dimension of Ver_{p^n} is $\frac{p^n}{2\sin^2(\pi/p^n)}$ (Proposition 4.11).*

(6) *The categories $\mathrm{Ver}_{p^n}^+$, $p > 2$ and Ver_{2^n} are incompressible, i.e., any (not necessarily braided) tensor functor out of them is a fully faithful embedding. In particular, these categories do not admit a tensor functor to a category of smaller Frobenius-Perron dimension. Likewise, the categories Ver_{p^n} and $\mathrm{Ver}_{2^n}^+$ are incompressible as symmetric tensor categories (Theorem 4.70).*

(7) *The category Ver_{p^n} admits a semisimple braided lift to characteristic zero, producing the semisimplification $\mathrm{Ver}_{p^n}(K)$ of the category of tilting modules for quantum SL_2 at the root of unity of order p^n (Subsection 4.4).*

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[1] We do not define the category Ver_2^+.

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(8) *(The Steinberg tensor product theorem for* Ver_{p^n}*) There exist simple objects* $\mathbb{T}_j^{[k]}$ *of* Ver_{p^n}*,* $j = 0, ..., p-1$*,* $k = 1, ..., n$*, and* $j \neq p-1$ *if* $k = 1$*, such that any simple object of* Ver_{p^n} *can be uniquely written in the form* $L_i = \mathbb{T}_{i_1}^{[1]} \otimes ... \otimes \mathbb{T}_{i_n}^{[n]}$*, where* $0 \leq i = \overline{i_1 ... i_n} \leq p^{n-1}(p-1) - 1$ *is an integer written in base* p*. The simple objects of* $\mathrm{Ver}_{p^n}^+$ *are the* L_i *with even* i *(Theorem 4.43).*

(9) *The category* Ver_{p^n} *has* $p-1$ *blocks of sizes* $1, p-1, p(p-1), ..., p^{n-2}(p-1)$*, so a total of* $n(p-1)$ *blocks. Moreover, all blocks of the same size are equivalent, even for different* n*. For instance, the blocks of size* 1 *are semisimple (i.e., generated by a simple projective object), and the blocks of size* $p-1$ *are equivalent to the unique non-semisimple block of representations of the symmetric group* S_p *over* **k** *(Proposition 4.55, Proposition 5.1).*

(10) *The Grothendieck ring of the stable category of* $\mathrm{Ver}_{p^n}^+$ *is isomorphic to* $\mathbb{F}_p[z]/z^{\frac{p^{n-1}-1}{2}}$ *for* $p > 2$ *and* $\mathbb{F}_2[z]/z^{2^{n-2}}$ *for* $p = 2$*. The Grothendieck ring of the stable category of* Ver_{2^n} *is* $\mathbb{F}_p[z]/z^{2^{n-1}-1}$ *(Proposition 4.67).*

(11) *The determinant of the Cartan matrix of a block of size* $p^r(p-1)$ *in* Ver_{p^n} *is* p^{p^r} *for* $r \geq 0$*, and its entries are* 0 *or powers of* 2 *(Proposition 4.69, Corollary 4.27).*

(12) *If* $p > 2$ *then* Ver_{p^n} *is a Serre subcategory in* $\mathrm{Ver}_{p^{n+k}}$ *(Proposition 4.58).*

Theorem 1.3 implies that we can define tensor categories $\mathrm{Ver}_{p^\infty} = \bigcup_{n \geq 1} \mathrm{Ver}_{p^n}$. Motivated by the main result of [32], we make the following (perhaps imprudently) bold conjecture:

Conjecture 1.4. Any symmetric tensor category of moderate growth over an algebraically closed field **k** of characteristic $p > 0$ admits a fiber functor to Ver_{p^∞}.

Conjecture 1.4 is open even for finite tensor categories. In particular, this conjecture would imply that Deligne's theorem holds for *integral* finite symmetric tensor categories for $p > 2$ (for example, for representation categories of triangular Hopf algebras); in other words, that such categories are representation categories of finite supergroup schemes. However, even this specialization of the conjecture is currently wide open.

In any case, the construction of the category Ver_{p^∞} opens the door for defining and studying many new interesting symmetric tensor categories, which may be constructed as representation categories of affine group schemes in Ver_{p^∞}. Examples of such group schemes are the general linear group $GL(X)$ for $X \in \mathrm{Ver}_{p^\infty}$, as well as the orthogonal group $O(X)$ and the symplectic group $Sp(X)$ when X is equipped with a symmetric, respectively skew-symmetric isomorphism $X \to X^*$. In particular, we see that in characteristic p there are infinitely many versions of general linear groups of a given "rank" r, parametrized by various objects X of length r.

The results of this paper have applications to modular representation theory of elementary abelian p-groups which are discussed in [5], Subsection 4.4. Namely, let V be the tautological 2-dimensional **k**-representation of the additive group of the field $\mathbb{F}_{p^n}$ regarded as the unipotent subgroup of $SL_2(\mathbb{F}_{p^n})$, and let $\mathcal{D}_{n,p}$ be the quotient of the Karoubian tensor category generated by V (i.e., with indecomposables being direct summands of tensor powers of V) by the ideal of projectives, i.e., the corresponding stable category. Then it is shown in [5], Proposition 4.5 that $\mathcal{D}_{n,p}$ is equivalent to $\mathcal{T}_{n,p}$. This implies that Ver_{p^n} is the abelian envelope of $\mathcal{D}_{n,p}$, which greatly clarifies the structure of $\mathcal{D}_{n,p}$.

The organization of the paper is as follows. In Section 2 we develop the technology of splitting ideals and abelian envelopes. In Section 3 we recall and prove a number of auxiliary results about tilting modules for $SL_2(\mathbf{k})$. In Section 4 we apply the technology of Sections 2, 3 to constructing and studying the categories Ver_{p^n}. In particular, we prove the properties of Ver_{p^n} listed in Theorem 1.3, and also compute the Cartan and decomposition matrices of their blocks, the fusion rules, and Ext^1 between simple objects. Finally, in Section 5 we compute explicitly the examples Ver_{p^2} and Ver_{3^3}.

Remark 1.5. The categories Ver_{p^n} were also constructed simultaneously and independently by Kevin Coulembier in [13]. This is an application of his general theory of abelian envelopes alternative to the one we propose in Section 2.

2. Splitting ideals and abelian envelopes

In this section we propose a general construction of abelian envelopes $\mathcal{C}(\mathcal{T})$ of a certain class of Karoubian rigid monoidal categories $\mathcal{T}$ which we call *separated and complete*. Under a certain finiteness assumption on $\mathcal{T}$, the resulting category $\mathcal{C}(\mathcal{T})$ is a multitensor category with enough projectives. This construction is designed to build the categories Ver_{p^n} but is interesting in its own right, and we expect it to have applications beyond the main goals of this paper. Therefore we developed this theory in a greater generality than actually needed here. In particular, a number of statements in this section, specifically Lemma 2.4, Propositions 2.8, 2.9, Corollary 2.10, Propositions 2.31, 2.32, Corollary 2.39, Theorem 2.43, are not used in the subsequent sections of this paper.

2.1. **Conventions.** Throughout the paper, $\mathbf{k}$ will denote an algebraically closed field. We will freely use the theory of tensor categories and refer the reader to [22] for the basics of this theory. In particular, we will use the conventions of this book.

Namely, by an *artinian category* over $\mathbf{k}$ we mean a $\mathbf{k}$-linear abelian category in which objects have finite length and morphism spaces are finite dimensional, and call such a category *finite* if it has finitely many (isomorphism classes of) simple objects and enough projectives. By a *(multi)tensor category* we will mean the notion defined in [22], Definition 4.1.1; in particular, such categories are artinian (hence abelian). We will denote the category of finite dimensional vector spaces by $\mathrm{Vec}_{\mathbf{k}}$ or shortly by Vec and for $\mathrm{char}(\mathbf{k}) \neq 2$ the category of finite dimensional supervector spaces by $\mathrm{sVec}_{\mathbf{k}}$ or shortly sVec. If X is an object of a tensor category with finitely many simple objects, then we denote by $\mathrm{FPdim}(X)$ the Frobenius-Perron dimension of X ([22], Subsections 3.3, 4.5). We will also consider categories with

tensor product which are not abelian but only Karoubian; in this case we will use the term *Karoubian monoidal category*. Similarly, by a *tensor functor* we will mean an exact monoidal functor between tensor categories, see [22], Definition 4.2.5 (note that such a functor is automatically faithful, see [22], Remark 4.3.10). In more general situations we will use the term *monoidal functor*; all monoidal functors we consider will be additive. Also, a symmetric tensor functor out of a symmetric tensor category into another, usually quite concrete one (such as $\text{Vec}_{\mathbf{k}}$, $\text{sVec}_{\mathbf{k}}$ or Ver_{p^n} defined below) will be called a *fiber functor*.

Finally, we will need the notions of *surjective and injective tensor functors*, [22], Subsection 6.3. Namely, a tensor functor $F : \mathcal{C} \to \mathcal{D}$ is surjective if every object $Y \in \mathcal{D}$ is a subquotient of $F(X)$ for some $X \in \mathcal{C}$, and F is injective if it is fully faithful (i.e., embedding of a tensor subcategory).

2.2. Splitting objects and splitting ideals.
Let $\mathbf{k}$ be an algebraically closed field. Throughout the paper, $\mathcal{T}$ will be a $\mathbf{k}$-linear rigid monoidal Karoubian category with bilinear tensor product and finite dimensional Hom spaces (in particular, it is Krull-Schmidt).

Recall that a *(thick) ideal* $\mathcal{P} \subset \mathcal{T}$ is a full Karoubian subcategory closed under tensoring on both sides with $\mathcal{T}$ (note that then it is also closed under left and right duals since Q is a direct summand in $Q \otimes Q^* \otimes Q$ and $Q \otimes^* Q \otimes Q$). Thus, $\mathcal{P}$ is determined by its indecomposable objects, which form a subset of indecomposable objects of $\mathcal{T}$. For a thick ideal $\mathcal{P} \subset \mathcal{T}$, the *tensor ideal* $\langle \mathcal{P} \rangle$ *generated by* $\mathcal{P}$ is the collection of morphisms which factor through objects of $\mathcal{P}$. We can form the quotient monoidal category $\mathcal{T}/\langle \mathcal{P} \rangle$, which we will sometimes also denote by $\mathcal{T}/\mathcal{P}$.

Recall also that an ideal $\mathcal{P}$ is said to be *finitely generated* (as a left ideal) if there exists an object $P \in \mathcal{P}$ such that every indecomposable object $Q \in \mathcal{P}$ is a direct summand in $X \otimes P$ for some $X \in \mathcal{T}$ (note that this is a stronger condition than finite generation as a two-sided ideal, see Example 2.44(2)). In this case by the rigidity of $\mathcal{T}$ we may (and will) choose P so that every indecomposable object $Q \in \mathcal{P}$ is also a direct summand in $P \otimes Y$ for some $Y \in \mathcal{T}$. Such an object P will be called *a generator* of $\mathcal{P}$. For example, if $\mathcal{P}$ has finitely many isomorphism classes of indecomposables, then the direct sum P of representatives of these classes is a generator.

Recall that a morphism $f : X \to Y$ in $\mathcal{T}$ is called *split* if it is a direct sum of a zero morphism and an isomorphism. In other words, f is split if it is a composition $\iota \circ \pi$ where π is a split epimorphism (projection to a direct summand) and ι is a split monomorphism (inclusion of a direct summand). It is easy to see that f is split if and only if there exists $g : Y \to X$ such that

$$(2.1) \qquad\qquad f \circ g \circ f = f, \ g \circ f \circ g = g.$$

In fact, for this it suffices that there exist $h : Y \to X$ such that $f \circ h \circ f = f$; then $g := h \circ f \circ h$ satisfies (2.1).

The following lemma is well known but we provide a proof for reader's convenience.

Lemma 2.1. *The direct sum of two morphisms is split if and only if each summand is split.*

Proof. The "if" direction is obvious, so let us prove the "only if" direction. If $f_i : X_i \to Y_i$, $i = 1, 2$, and $f = f_1 \oplus f_2$ is split then there is $h : Y_1 \oplus Y_2 \to X_1 \oplus X_2$ satisfying $f \circ h \circ f = f$. Let h_{ij} be the components of h. Then we have $f_1 \circ h_{11} \circ f_1 = f_1$, i.e., f_1 is split. $\qquad\square$

Definition 2.2. We will say that an ideal $\mathcal{P} \subset \mathcal{T}$ is a *splitting ideal* if for any $Q_1, Q_2, R \in \mathcal{P}$ and any morphism $f : Q_1 \to Q_2$ the morphism $1_R \otimes f : R \otimes Q_1 \to R \otimes Q_2$ is split.

Note that in this case it follows by taking duals that the morphism $f \otimes 1_R : Q_1 \otimes R \to Q_2 \otimes R$ is also split.

Definition 2.3. We will say that an object $R \in \mathcal{T}$ is a *splitting object* if for any $Q_1, Q_2 \in \mathcal{T}$ and a morphism $f : Q_1 \to Q_2$ the morphisms $1_R \otimes f : R \otimes Q_1 \to R \otimes Q_2$ and $f \otimes 1_R : Q_1 \otimes R \to Q_2 \otimes R$ are split.

It is clear that splitting objects form a splitting ideal, which we denote by $\mathcal{S} = \mathcal{S}(\mathcal{T})$.

Lemma 2.4. *Every splitting ideal consists of splitting objects.*

Proof. Let $\mathcal{P} \subset \mathcal{T}$ be a splitting ideal, $R \in \mathcal{P}$ and $f : Q_1 \to Q_2$ be a morphism in $\mathcal{T}$. Then $g := 1_R \otimes f : R \otimes Q_1 \to R \otimes Q_2$ is a morphism in $\mathcal{P}$ (as $\mathcal{P}$ is an ideal). Thus the morphism

$$1_{R \otimes R^*} \otimes g : R \otimes R^* \otimes R \otimes Q_1 \to R \otimes R^* \otimes R \otimes Q_2$$

is split. But R is a direct summand in $R \otimes R^* \otimes R$ since R is rigid. This implies that $1_{R \otimes R^*} \otimes g = g \oplus h$, so g itself is split by Lemma 2.1. Similarly, $f \otimes 1_R$ is split, as claimed. $\square$

Lemma 2.4 implies that $\mathcal{S}$ is the unique maximal splitting ideal in $\mathcal{T}$, containing all other splitting ideals.

A trivial example of an ideal is $\mathcal{P} = 0$ (it is clearly splitting and finitely generated with $P = 0$). Here are some more interesting examples.

Example 2.5. Suppose that $\mathcal{T}$ is a multitensor category ([22], Definition 4.1.1) with enough projectives. Then the ideal $\mathcal{P} = \mathrm{Pr}(\mathcal{T})$ of projectives in $\mathcal{T}$ coincides with $\mathcal{S}(\mathcal{T})$ and is a finitely generated splitting ideal. A generator P is the projective cover of $\mathbb{1}$. Indeed, it is clear that every projective object is splitting. Conversely, if Q splits every morphism between projective objects then in particular it splits a morphism $\xi : P_1 \to P_0$ whose cokernel is $\mathbb{1}$, which implies that Q is projective, since Q is the cokernel of the split morphism $\xi \otimes 1_Q$.

Example 2.6. Let $\mathcal{T}$ be the rigid monoidal category of rigid finite dimensional bimodules over a finite dimensional non-semisimple Frobenius algebra B (i.e., those projective as a left and as a right module); note that the functor of double dual in this category is given by the Nakayama automorphism. Then the subcategory $\mathcal{P}$ of projective bimodules is a finitely generated ideal (with generator $P = B^e := B \otimes B^{\mathrm{op}}$), but is *not* a splitting ideal. Indeed, there are non-split endomorphisms of the bimodule B^e (for example, nonzero elements of the radical of B^e), and they are not split by tensoring with any rigid bimodule. In fact, in this case there are no nonzero splitting ideals at all, since for any bimodule Q the bimodule $B^e \otimes_B Q \otimes_B B^e = B \otimes_{\mathbf{k}} Q \otimes_{\mathbf{k}} B$ is free.

Let us now describe all splitting ideals in $\mathcal{T}$.

Lemma 2.7. *(a) Let $R \in \mathcal{T}$ be an indecomposable splitting object. Then the following conditions on an object $Q \in \mathcal{T}$ are equivalent:*
(i) there exists $X \in \mathcal{T}$ such that $\mathrm{Hom}(X \otimes Q, R) \neq 0$;
(ii) there exists $X \in \mathcal{T}$ such that $\mathrm{Hom}(R, X \otimes Q) \neq 0$;
(iii) there exists $X \in \mathcal{T}$ such that R is a direct summand of $X \otimes Q$.

(b) Let $Q, R \in \mathcal{T}$ be indecomposable splitting objects. Then there exists $X \in \mathcal{T}$ such that R is a direct summand in $X \otimes Q$ if and only if there exists $X \in \mathcal{T}$ such that Q is a direct summand of $X \otimes R$.

Proof. (a) It is clear that (iii) implies (i) and (ii).

Suppose $\mathrm{Hom}(X \otimes Q, R) \neq 0$. Then $\mathrm{Hom}(X, R \otimes Q^*) \neq 0$. Thus $R \otimes Q^* \neq 0$. Hence the natural map $1 \otimes \mathrm{ev} : R \otimes Q^* \otimes Q \to R$ is not zero (as it corresponds to the identity morphism of $R \otimes Q^*$). Also this morphism is split since R is a splitting object. Since R is indecomposable, $1 \otimes \mathrm{ev}$ is a split epimorphism. Hence R is a direct summand of $Y \otimes Q$, where $Y = R \otimes Q^*$. Thus (i) implies (iii).

Suppose $\mathrm{Hom}(R, X \otimes Q) \neq 0$. Then $\mathrm{Hom}(R \otimes {}^*Q, X) \neq 0$. Thus $R \otimes {}^*Q \neq 0$. Hence the natural map $1 \otimes \mathrm{coev} : R \to R \otimes {}^*Q \otimes Q$ is not zero (as it corresponds to the identity morphism of $R \otimes {}^*Q$). Also this morphism is split since R is a splitting object. Since R is indecomposable, $1 \otimes \mathrm{coev}$ is a split monomorphism. Thus R is a direct summand of $Y \otimes Q$, where $Y = R \otimes {}^*Q$. Thus (ii) implies (iii).

(b) Suppose R is a direct summand of $X \otimes Q$. Then $\mathrm{Hom}(R, X \otimes Q) \neq 0$. It follows that $\mathrm{Hom}(X^* \otimes R, Q) \neq 0$. So by (a) there is $Y \in \mathcal{T}$ such that Q is a direct summand of $Y \otimes R$. □

Now consider the set $S = S(\mathcal{T})$ of isomorphism classes of indecomposable splitting objects in $\mathcal{T}$. Then all splitting objects are direct sums of objects from S. Define a relation on S by the condition that $Q \sim R$ if there exist $X, Y \in \mathcal{T}$ such that R is a direct summand in $X \otimes Q \otimes Y$. Clearly, this relation is reflexive and transitive, and it is also symmetric by Lemma 2.7(b), so it is an equivalence relation. Note that by Lemma 2.7(a) $Q \sim R$ iff there exist $X, Y \in \mathcal{T}$ such that $\mathrm{Hom}(Q, X \otimes R \otimes Y) \neq 0$ iff there exist $X, Y \in \mathcal{T}$ such that $\mathrm{Hom}(X \otimes R \otimes Y, Q) \neq 0$. Note also that $Q^* \sim Q$ since Q is a direct summand in $Q \otimes Q^* \otimes Q$.

Let $\overline{S} := S/\sim$. Every class $c \in \overline{S}$ generates a minimal splitting ideal $\mathcal{P}(c)$. It is easy to see that if $c_1 \neq c_2$ then $\mathcal{P}(c_1) \otimes \mathcal{P}(c_2) = 0$ (as these ideals are disjoint).

Thus we have the following proposition.

Proposition 2.8. *Splitting ideals of $\mathcal{T}$ are in 1-1 correspondence with subsets $K \subset \overline{S}$ defined by the formula $\mathcal{P}_K := \bigoplus_{c \in K} \mathcal{P}(c)$. Among them, finitely generated splitting ideals correspond to such sums that are finite with $\mathcal{P}(c)$ finitely generated for each $c \in K$. In particular, $\mathcal{S} = \mathcal{P}_{\overline{S}}$.*

Proof. This follows from Lemma 2.4. □

We can also define the equivalence relations $\sim_\ell, \sim_r$ as follows: $Q \sim_\ell R$ if there is $X \in \mathcal{T}$ such that Q is a direct summand in $X \otimes R$, and $Q \sim_r R$ if there is $X \in \mathcal{T}$ such that Q is a direct summand in $R \otimes X$ (the symmetry of these relations follows from Lemma 2.7(b)). Note that by Lemma 2.7(a) $Q \sim_\ell R$ iff there is $X \in \mathcal{T}$ such that $\mathrm{Hom}(Q, X \otimes R) \neq 0$ and $Q \sim_r R$ iff there is $X \in \mathcal{T}$ such that $\mathrm{Hom}(Q, R \otimes X) \neq 0$. The relation $\sim_\ell$ splits an equivalence class $c \in \overline{S}$ into the disjoint union of classes $c_{\bullet j}$, $j \in I_c$, and the relation $\sim_\ell$ splits c into the disjoint union of classes $c_{i\bullet}$, $i \in I_c$. We can define $c_{ij} := c_{i\bullet} \cap c_{\bullet j}$, with $c_{ij}^* = c_{ji}$, then c is a disjoint union of c_{ij}. Let $\mathcal{P}(c)_{i\bullet}, \mathcal{P}(c)_{\bullet j}, \mathcal{P}(c)_{ij}$ be the subcategories of $\mathcal{P}$ additively generated by $c_{i\bullet}, c_{\bullet j}, c_{ij}$, respectively; clearly, $\mathcal{P}_{\bullet j}$ is a left ideal and $\mathcal{P}_{i\bullet}$ is a right ideal. Thus we have decompositions

$$\mathcal{P}(c) = \bigoplus_{i \in I_c} \mathcal{P}_{i\bullet} = \bigoplus_{j \in I_c} \mathcal{P}_{\bullet j} = \bigoplus_{i,j \in I_c} \mathcal{P}(c)_{ij}.$$

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2.3. A finiteness property of splitting objects.

Proposition 2.9. *Let Q, R be indecomposable splitting objects in $\mathcal{T}$. If $Hom(R, Q) \neq 0$ then R is a direct summand in $Q \otimes Q^* \otimes Q$.*

Proof. **Step 1.** Let $Y \in \mathcal{T}$ be indecomposable. Let $\mathfrak{m}_Y$ be the radical of the local algebra $End(Y)$. For each indecomposable object $X \in \mathcal{T}$ set $U(X, Y) := X \otimes Hom(X, Y)^*$ if X is not isomorphic to Y, and $U(Y, Y) := Y \otimes \mathfrak{m}_Y^*$, and extend this definition by additivity to all objects $X \in \mathcal{T}$. Then $U(X, Y)$ is equipped with a canonical morphism $\xi : U(X, Y) \to Y$ which is not a split epimorphism, and any morphism $g : X \to Y$ which is not a split epimorphism factors through ξ.

Step 2. Now let R, Q be indecomposable splitting objects of $\mathcal{T}$. Let

$$U := U(Q \otimes Q^* \otimes Q \oplus R \otimes Q^* \otimes Q, R).$$

Then the map $\xi \otimes 1_{Q^*} : U \otimes Q^* \to R \otimes Q^*$ is split. Thus we can write $U \otimes Q^* \cong N \oplus K$, $R \otimes Q^* \cong N \oplus L$, so that $\xi \otimes 1 = 1_N \oplus 0_{K \to L}$.

Step 3. Now assume that $f : R \to Q$ is a morphism. Then the map

$$f \otimes 1_{Q^*} : R \otimes Q^* \to Q \otimes Q^*$$

is split. So we can write $Q \otimes Q^* \cong M \oplus C$, and we have a split epimorphism

$$g : R \otimes Q^* = N \oplus L \to M.$$

Step 4. Also, the map

$$f \circ \xi \otimes 1_{Q^*} : U \otimes Q^* \cong N \oplus K \to Q \otimes Q^* \cong M \oplus C$$

is split. It is a direct sum of a map $\eta : N \to M$ and $0_{K \to C}$, so the map η is split. So we can write $N \cong W \oplus V$, $M \cong W \oplus Y$, so that $\eta = 1_W \oplus 0_{V \to Y}$. Thus the map $g : N \oplus L \to M$ can be written as $g : W \oplus V \oplus L \to W \oplus Y$. Since g is a split epimorphism and $g|_V = 0$, we obtain a split epimorphism $\bar{g} : W \oplus L \to W \oplus Y$, such that $\bar{g}|_W = 1_W$. Let $h : L \to W$ be the corresponding component of $\bar{g}$. Then composing $\bar{g}$ with the automorphism $\begin{pmatrix} 1 & -h \\ 0 & 1 \end{pmatrix}$ of $W \oplus L$, we obtain a split epimorphism $\bar{g}' = 1_W \oplus \tilde{g}$, where $\tilde{g} : L \to Y$ is a split epimorphism.

Step 5. Now assume additionally that $f \neq 0$. Then we claim that the map $\xi \otimes 1_{Q^*}$ is not a split epimorphism, i.e., $L \neq 0$. Indeed, the morphism

$$\circ \xi : Hom(\mathbf{1}, U \otimes R^*) \cong Hom(R, U) \to Hom(\mathbf{1}, R \otimes R^*) \cong Hom(R, R)$$

is not surjective, as ξ is not split (so 1_R can't be in the image). Hence the morphism $\xi \otimes 1_{R^*} : U \otimes R^* \to R \otimes R^*$ is not a split epimorphism. But R, Q belong to the same $\mathcal{P}(c)_{ij}$, so there is $X \in \mathcal{T}$ such that R^* is a direct summand of $Q^* \otimes X$. Thus $\xi \otimes 1_{Q^* \otimes X} = \xi \otimes 1_{Q^*} \otimes 1_X$ is not a split epimorphism either (indeed, the direct sum of two morphisms can only be a split epimorphism if both summands are). Hence $\xi \otimes 1_{Q^*}$ is not a split epimorphism, as claimed.

Step 6. Moreover, we claim that $Y \neq 0$. Assume that $Y = 0$. Consider the map $\theta : L \to R \otimes Q^* = W \oplus V \oplus L$ given by $\theta = (\theta_1, \theta_2, \theta_3)$, where $\theta_1 = -h$, $\theta_2 = 0$, $\theta_3 = 1_L$. It is easy to see that the map $(f \otimes 1_{Q^*}) \circ \theta : L \to Q \otimes Q^*$ is zero. Let ζ be the element of $Hom(L \otimes Q, R)$ corresponding to θ. The composition $f \circ \zeta \in Hom(L \otimes Q, Q)$ is zero. Thus the map ζ is not a split epimorphism, as $f \neq 0$. Also $L \otimes Q$ is a direct summand in $R \otimes Q^* \otimes Q$, since L is a direct summand in $R \otimes Q^*$. Thus by the definition of U, ξ, there

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exists an element $\widetilde{\zeta} \in \mathrm{Hom}(L \otimes Q, U)$ which maps to ζ. Let $\widetilde{\theta} \in \mathrm{Hom}(L, U \otimes Q^*)$ correspond to $\widetilde{\zeta}$. Then $\widetilde{\theta}$ maps to θ. However, the image of any element of $\mathrm{Hom}(L, U \otimes Q^*)$ factors through N, so cannot equal θ (as $L \neq 0$), which contradicts the assumption that $Y = 0$. Thus, $Y \neq 0$.

Step 7. Since Y is a direct summand in both L and $Q \otimes Q^*$, this implies that

$$\mathrm{Hom}(Q \otimes Q^*, L) \neq 0.$$

Thus the map

$$\circ(\xi \otimes 1_{Q^*}) : \mathrm{Hom}(Q \otimes Q^*, U \otimes Q^*) \to \mathrm{Hom}(Q \otimes Q^*, R \otimes Q^*)$$

is not surjective. Hence the map

$$\circ\xi : \mathrm{Hom}(Q \otimes Q^* \otimes Q, U) \to \mathrm{Hom}(Q \otimes Q^* \otimes Q, R)$$

is not surjective either. So there exists a morphism $\pi : Q \otimes Q^* \otimes Q \to R$ that does not factor through ξ. By the definition of U, ξ, this implies that π is a split epimorphism, hence R is a direct summand in $Q \otimes Q^* \otimes Q$, as claimed. $\qquad\square$

Corollary 2.10. *For every indecomposable splitting object Q there are finitely many indecomposable splitting objects R such that $\mathrm{Hom}(R, Q) \neq 0$; namely, at most the number of distinct indecomposable direct summands in $Q \otimes Q^* \otimes Q$.*

Proof. This follows immediately from Proposition 2.9. $\qquad\square$

Remark 2.11. Suppose that in Proposition 2.9, there is a monoidal functor $F : \mathcal{T} \to \mathcal{C}$ from $\mathcal{T}$ to a multitensor category $\mathcal{C}$ with enough projectives which is fully faithful on projectives, and the objects Q, R are projective in $\mathcal{C}$ (we will show below that this is so under mild assumptions, but the proof uses Proposition 2.9). Then there is a much more direct proof of Proposition 2.9. Namely, suppose that R is the projective cover of a simple object $X \in \mathcal{C}$. Then, $\mathrm{Hom}(R, Q) \neq 0$ implies that X is a composition factor in Q. Thus $X \otimes Q^* \neq 0$ (as $X \otimes X^*$ is a subquotient of $X \otimes Q^*$), and $X \otimes Q^* \otimes Q$ is a direct summand in $Q \otimes Q^* \otimes Q$. But we have a morphism $1 \otimes \mathrm{ev} : X \otimes Q^* \otimes Q \to X$, which is nonzero since it corresponds to the identity endomorphism of the nonzero object $X \otimes Q^*$. Since $X \otimes Q^* \otimes Q$ is projective, it follows that it contains R as a direct summand.

In fact, the proof of Proposition 2.9 is just an adaptation of this argument which does not refer to the category $\mathcal{C}$.

2.4. **The abelian category attached to an ideal and its derived category.** Let $\mathcal{P} \subset \mathcal{T}$ be an ideal. Let $\mathbf{I}$ be the set of isomorphism classes of indecomposables in $\mathcal{P}$ with representatives $\mathbf{P}_i$, $i \in \mathbf{I}$. Let us say that $\mathcal{P}$ has *the finiteness property* if for any $Q \in \mathcal{P}$ there are finitely many $i \in \mathbf{I}$ such that $\mathrm{Hom}(\mathbf{P}_i, Q) \neq 0$. For example, by Corollary 2.10, the ideal $\mathcal{S}$ (and, in fact, any splitting ideal) automatically has the finiteness property.

From now on we will assume that $\mathcal{P}$ has the finiteness property. Let $C := \bigoplus_{i,j \in \mathbf{I}} \mathrm{Hom}(\mathbf{P}_i, \mathbf{P}_j)^*$. This space is naturally a coalgebra whose dual is the algebra $A := C^* = \prod_{i,j \in \mathbf{I}} \mathrm{Hom}(\mathbf{P}_i, \mathbf{P}_j)$ with operation $ab = b \circ a$ (note that this product is well defined and does not involve infinite summations because of the finiteness property of $\mathcal{P}$). For example, if $\mathbf{I}$ is finite then $A = \mathrm{End}(P)^{\mathrm{op}}$, where $P := \bigoplus_{i \in \mathbf{I}} \mathbf{P}_i$.

Let $\mathcal{C} = \mathcal{C}(\mathcal{T}, \mathcal{P}) := C\text{-comod}$ be the category of finite dimensional C-comodules (or, equivalently, C^*-modules when $\mathbf{I}$ is finite). Equivalently, $\mathcal{C}$ is the category of additive functors

$\mathcal{P}^{\mathrm{op}} \to \mathrm{Vec}$ which are of finite length, i.e., vanish on almost all $\mathbf{P}_i$. Then $\mathcal{C}$ is an artinian category with enough projectives and simple objects $\mathbf{L}_i, i \in \mathbf{I}$. Namely, we have a natural inclusion $\iota : \mathcal{P} \hookrightarrow \mathcal{C}$ as a full subcategory given by $\iota(Q) := \bigoplus_{i \in \mathbf{I}} \mathrm{Hom}(\mathbf{P}_i, Q)$, whose image is the ideal $\mathrm{Pr}(\mathcal{C})$ of projective objects in $\mathcal{C}$ (note that $\dim \iota(Q) < \infty$ since $\mathrm{Hom}(\mathbf{P}_i, Q) = 0$ for almost all i). We will identify $\mathcal{P}$ with $\mathrm{Pr}(\mathcal{C})$ using ι and, abusing notation, will often denote the image of $Q \in \mathbf{P}$ under ι also by Q. Then $\mathbf{P}_i$ are projective covers of $\mathbf{L}_i$.

Consider now the bounded above derived category $D^-(\mathcal{C})$. By a standard theorem of homological algebra ([26], III.5.21), we can realize $D^-(\mathcal{C})$ as the bounded above homotopy category of projectives $K^-(\mathrm{Pr}(\mathcal{C})) \cong K^-(\mathcal{P})$. This allows us to realize the objects of $D^-(\mathcal{C})$ explicitly as complexes

$$... \to P_{n+2} \to P_{n+1} \to P_n,$$

where $P_j \in \mathrm{Pr}(\mathcal{C})$. Moreover, the subcategory $\mathcal{C} \subset D^-(\mathcal{C})$ (the heart of the tautological t-structure) is realized as the subcategory of such complexes which are exact in nonzero degrees, i.e., the subcategory of resolutions

$$... \to P_2 \to P_1 \to P_0.$$

2.5. **The semigroup category structure on $\mathcal{C}$.** Note that $\mathcal{P}$ is a semigroup category ([22], p.25; i.e., it has a tensor product with associativity isomorphism satisfying the pentagon axiom, but has no unit in general). This implies

Proposition 2.12. *The category $K^-(\mathcal{P}) \cong D^-(\mathcal{C})$ has a natural structure of a semigroup category, given by tensoring bounded above complexes of objects from $\mathcal{P}$. Therefore, the category $\mathcal{C}$ is equipped with the induced semigroup category structure (given by taking the zeroth cohomology of the tensor product), which extends the tensor product on $\mathcal{P}$.*

Proof. The proof is standard. Namely, given $X, Y \in \mathcal{C}$, fix projective resolutions $P_\bullet$ of X and $Q_\bullet$ of Y. We define the derived tensor product $X \otimes^L Y$ to be the object of $D^-(\mathcal{C})$ represented by the complex $P_\bullet \otimes Q_\bullet$. In particular, we set $X \otimes Y := H^0(X \otimes^L Y) \in \mathcal{C}$.

Let us see why $X, Y \mapsto X \otimes^L Y$ is a well defined bifunctor. Let $P'_\bullet, Q'_\bullet$ be other resolutions of X, Y. Then there exist morphisms of resolutions $f : P_\bullet \to P'_\bullet$, $g : Q_\bullet \to Q'_\bullet$ which extend the identity morphisms 1_X, 1_Y, and they are defined uniquely up to homotopy. Similarly, we have $f' : P'_\bullet \to P_\bullet$, $g' : Q'_\bullet \to Q_\bullet$, and $f \circ f', f' \circ f, g \circ g', g' \circ g$ are homotopic to the identity morphisms. Now consider the morphism $f \otimes g : P_\bullet \otimes Q_\bullet \to P'_\bullet \otimes Q'_\bullet$. This is well defined up to homotopy, and the inverse is $f' \otimes g'$ up to homotopy. Thus $f \otimes g$ defines an isomorphism $X \otimes^L Y \to (X \otimes^L Y)'$ of the derived tensor products defined using these two pairs of resolutions, and it is easy to show that this isomorphism is independent of the choice of f, g. Moreover, if we have three resolutions for X and for Y then it is shown in a standard way that the composition around the corresponding triangle is the identity. Thus $X \otimes^L Y$ and hence $X \otimes Y$ is well defined. $\square$

Remark 2.13. In fact, to define $X \otimes Y$, one does not need the full resolutions of X and Y by objects of $\mathcal{P}$. All one needs is *presentations* of X, Y, i.e., their representations as cokernels of morphisms $f : P_1 \to P_0$ and $g : Q_1 \to Q_0$ between objects from $\mathcal{P}$. In this case we have a morphism

$$f \otimes 1 + 1 \otimes g : P_1 \otimes Q_0 \oplus P_0 \otimes Q_1 \to P_0 \otimes Q_0,$$

and $X \otimes Y$ is just the cokernel of this morphism. We refer the reader to [30, Section 3] for more details.

Example 2.14. 1. If $\mathcal{P} = 0$ then $\mathcal{C} = 0$.

2. In the setting of Example 2.5 we have $\mathcal{C} = \mathcal{T}$, and the semigroup structure on $\mathcal{C}$ constructed above recovers the original monoidal structure on $\mathcal{T}$.

3. Similarly, in Example 2.6, $\mathcal{C} \supset \mathcal{T}$ is the category of all finite dimensional B-bimodules with tensor product over B, and $X \otimes^L Y$ is the usual derived tensor product over B.

2.6. **Properties of the tensor product on $\mathcal{C}$.** Let us now study the properties of the tensor product on $\mathcal{C}$ defined in Proposition 2.12.

Lemma 2.15. *(i) Let $Q_\bullet$ be a projective resolution of an object $Y \in \mathcal{C}$. Then for each $T \in \mathcal{T}$ the complexes $T \otimes Q_\bullet$ and $Q_\bullet \otimes T$ in $K^-(\mathcal{P})$ are acyclic in strictly negative degrees.*

(ii) Let $X \in \mathcal{P}$. Then for any $Y \in \mathcal{C}$ we have $H^{-i}(X \otimes^L Y) = H^{-i}(Y \otimes^L X) = 0$ for all $i > 0$.

Proof. (i) Let $P_1 \to P_2 \to P_3$ be an exact sequence of projective objects of $\mathcal{C}$. This means that the complex

$$\mathrm{Hom}(Q, P_1) \to \mathrm{Hom}(Q, P_2) \to \mathrm{Hom}(Q, P_3)$$

is acyclic for any $Q \in \mathcal{P}$. Now consider the complex

(2.2) $$\mathrm{Hom}(Q, T \otimes P_1) \to \mathrm{Hom}(Q, T \otimes P_2) \to \mathrm{Hom}(Q, T \otimes P_3).$$

It can be written as

$$\mathrm{Hom}(T^* \otimes Q, P_1) \to \mathrm{Hom}(T^* \otimes Q, P_2) \to \mathrm{Hom}(T^* \otimes Q, P_3).$$

But $T^* \otimes Q \in \mathcal{P}$. Thus the complex (2.2) is acyclic. This implies that the complex

$$T \otimes P_1 \to T \otimes P_2 \to T \otimes P_3$$

is acyclic, which implies the claim for $T \otimes Q_\bullet$. The claim for $Q_\bullet \otimes T$ is proved similarly.

(ii) follows from (i). $\qquad \square$

Corollary 2.16. *The bifunctor $(X, Y) \mapsto X \otimes Y$ on $\mathcal{C}$ is right exact in both arguments, and*

$$X \otimes^{L_i} Y = H^{-i}(X \otimes^L Y),$$

where $\otimes^{L_i}$ is the i-th left derived functor of the tensor product. Moreover, if X is projective then the functors $X \otimes ?, ? \otimes X$ on $\mathcal{C}$ are exact.

Proof. This follows immediately from Lemma 2.15(ii). $\qquad \square$

Corollary 2.17. *The monoidal category $\mathcal{T}$ acts on $\mathcal{C}$ on the left by exact functors $T_l(X) := T \otimes X$, $T \in \mathcal{T}, X \in \mathcal{C}$, commuting with the action of $\mathcal{C}$ on itself by right multiplication. There is also a similar action on the right given by $T_r(X) := X \otimes T$ which commutes with the action of $\mathcal{C}$ on itself by left multiplication. Also, these actions map projective objects to projective ones.*

Proof. This follows from Lemma 2.15(i): the left action is given by $T_l(X) = H^0(T \otimes Q_\bullet)$, where $Q_\bullet$ is a projective resolution of X. It is clear that the functors T_l are well defined and right exact, and $L_j T_l(X) = H^{-j}(T \otimes Q_\bullet) = 0$ for $j > 0$. Thus the functors T_l are exact. The same applies to the functors T_r. $\qquad \square$

2.7. The unit object of $\mathcal{C}$. Now assume that $\mathcal{P}$ is a finitely generated ideal, and $P \in \mathcal{P}$ a generator. Let $\mathbf{1}$ be the unit object of $\mathcal{T}$. Since P is rigid, we have the evaluation morphism $\mathrm{ev} : P^* \otimes P \to \mathbf{1}$. Consider the morphism $\tau : P^* \otimes P \otimes P^* \otimes P \to P^* \otimes P$ given by the formula

$$\tau = \mathrm{ev} \otimes 1 \otimes 1 - 1 \otimes 1 \otimes \mathrm{ev}.$$

We define the object $\mathbb{1}$ in the category $\mathcal{C}$ to be the cokernel of τ (note that we use a different font to distinguish it from $\mathbf{1}$).

Proposition 2.18. *For $Q \in \mathcal{P} \subset \mathcal{T}$ there are isomorphisms*

$$Q \cong Q \otimes \mathbb{1}, \ Q \cong \mathbb{1} \otimes Q$$

which are functorial in Q.

Proof. By Corollary 2.16, $Q \otimes \mathbb{1}$ is the cokernel of the morphism

$$d_2 := 1_Q \otimes \tau : Q \otimes P^* \otimes P \otimes P^* \otimes P \to Q \otimes P^* \otimes P.$$

Let us extend this morphism to the "bar complex"

$$(2.3) \qquad\qquad Q \otimes P^* \otimes P \otimes P^* \otimes P \to Q \otimes P^* \otimes P \to Q \to 0,$$

where the last map is $d_1 := 1_Q \otimes \mathrm{ev}$ (it is easy to see that $d_1 d_2 = 0$). To construct a functorial isomorphism $Q \otimes \mathbb{1} \cong Q$, it suffices to show that this complex is exact. Since P is a generator, every Q is a direct summand in $X \otimes P$ for some $X \in \mathcal{C}$, so it suffices to show that (2.3) is exact for $Q = X \otimes P$. Hence, it suffices to show that (2.3) is split for $Q = P$.

Since P is rigid, (2.3) for $Q = P$ is split in the third term, with splitting $h_1 : P \to P \otimes P^* \otimes P$ given by $h_1 := \mathrm{coev} \otimes 1_P$, where $\mathrm{coev} : \mathbf{1} \to P \otimes P^*$ is the coevaluation morphism. So it remains to show that (2.3) is split in the second term as well. To this end, we define a homotopy

$$h_2 : P \otimes P^* \otimes P \to P \otimes P^* \otimes P \otimes P^* \otimes P$$

by

$$h_2 := \mathrm{coev} \otimes 1^{\otimes 3} = h_1 \otimes 1 \otimes 1.$$

It is easy to check that $h_1 d_1 + d_2 h_2 = 1$, which implies the statement.

This proves that there is a canonical isomorphism $Q \to Q \otimes \mathbb{1}$. The isomorphism $Q \to \mathbb{1} \otimes Q$ is constructed similarly. $\qquad\square$

Corollary 2.19. *The semigroup category $\mathcal{C}$ is actually a monoidal category with unit $\mathbb{1}$. Moreover, if $\mathcal{T}$ is braided or symmetric then so is $\mathcal{C}$.*

Proof. Since $\mathcal{C}$ is the category of resolutions in $\mathcal{P}$, the functorial isomorphism $\mathbb{1} \otimes Q \cong Q$ for $Q \in \mathcal{P}$ from Proposition 2.18 gives rise to an isomorphism $\mathbb{1} \otimes \cong \mathrm{Id}_{\mathcal{C}}$. Similarly, we have an isomorphism $\otimes \mathbb{1} \cong \mathrm{Id}_{\mathcal{C}}$. Thus, $\mathbb{1}$ is a unit in $\mathcal{C}$, hence $\mathcal{C}$ is monoidal. The second statement is clear. $\qquad\square$

Example 2.20. The following example shows that the assumption that $\mathcal{P}$ is finitely generated cannot be dropped. Let $\mathrm{char}(\mathbf{k}) = 2$ and $Q = \mathbf{k}\mathbb{Z}/2$ be the regular representation of $\mathbb{Z}/2$ over $\mathbf{k}$. For a set M let $(\mathbb{Z}/2)^M$ be the group of maps $M \to \mathbb{Z}/2$, and for $m \in M$ let Q_m be the representation of $\mathrm{Fun}(M, \mathbb{Z}/2)$ on the vector space Q given by $\rho_{Q_m}(g) = \rho_Q(g(m))$, $g \in (\mathbb{Z}/2)^M$, where ρ_Q defines the action of $\mathbb{Z}/2$ on Q. Let $\mathcal{T}' = \mathcal{T}'(M)$ be the full subcategory of $\mathrm{Rep}_{\mathbf{k}}((\mathbb{Z}/2)^M)$ spanned by the tensor products $Q_N := \bigotimes_{m \in N} Q_m$

for finite subsets $N \subset M$. Let $\mathcal{T} = \mathcal{T}(M)$ be the quotient of $\mathcal{T}'$ by the tensor ideal generated by $Q_{ij} = Q_i \otimes Q_j$, where $i, j \in M$ are distinct (i.e., we set all morphisms factoring through a direct sum of Q_{ij} to zero). Then $\mathcal{T}$ has indecomposables $\mathbf{1}$ and $Q_m, m \in M$, with $\text{Hom}(\mathbf{1}, \mathbf{1}) = \text{Hom}(\mathbf{1}, Q_i) = \text{Hom}(Q_i, \mathbf{1}) = \mathbf{k}$, $\text{Hom}(Q_i, Q_i) = \mathbf{k}^2$, but $\text{Hom}(Q_i, Q_j) = 0$ for $i \neq j$ (since any morphism of representations $Q_i \to Q_j$ factors through Q_{ij}). It is easy to show that for any subset $N \subset M$ the ideal $\mathcal{P}_N = \langle Q_i, i \in N \rangle$ is a splitting ideal in $\mathcal{T}$, which is finitely generated if and only if N is finite. The abelian semigroup category $\mathcal{C}(\mathcal{T}, \mathcal{P}_N)$ is $\mathcal{C}_N := \bigoplus_{m \in N} \text{Rep}_{\mathbf{k}}(\mathbb{Z}/2)$, which is monoidal (i.e., has a unit object) if and only if N is finite.

2.8. **The monoidal functor F.** We continue to assume that $\mathcal{P} \subset \mathcal{T}$ is a finitely generated ideal, and $\mathcal{C} = \mathcal{C}(\mathcal{T}, \mathcal{P})$.

Corollary 2.21. *There is a monoidal functor $F : \mathcal{T} \to \mathcal{C}$ such that $F(T) = T_l(\mathbf{1}) = T_r(\mathbf{1})$ (so $F(\mathbf{1}) = \mathbf{1}$). Moreover, if $\mathcal{T}$ is braided or symmetric then so is F.*

Proof. This follows from Corollary 2.19 and Corollary 2.17. Namely, we have a monoidal functor $F_l : \mathcal{T} \to \text{End}(\mathcal{C}_{\mathcal{C}}) \cong \mathcal{C}$ given by $F_l(T) := T_l(\mathbf{1})$ and a monoidal functor $F_r : \mathcal{T} \to \text{End}(_{\mathcal{C}}\mathcal{C})^{\text{op}} \cong \mathcal{C}$ given by $F_r(T) := T_r(\mathbf{1})$. Moreover, if $Q \in \mathcal{P}$ then $F_l(Q)$ and $F_r(Q)$ are canonically isomorphic to Q. However, for any $T \in \mathcal{T}$ we have that $F_l(T)$ is the cokernel of the natural morphism

$$F_l(1 \otimes \tau) : F_l(T \otimes P^* \otimes P \otimes P^* \otimes P) \to F_l(T \otimes P^* \otimes P),$$

while $F_r(T)$ is the cokernel of the natural morphism

$$F_r(1 \otimes \tau) : F_r(T \otimes P^* \otimes P \otimes P^* \otimes P) \to F_r(T \otimes P^* \otimes P).$$

This implies that $F_l \cong F_r$, which gives a construction of the functor $F \cong F_l \cong F_r$ with the claimed properties. $\square$

Corollary 2.22. *Every projective object in $\mathcal{C}$ is rigid.*

Proof. Indeed, by Corollary 2.21, every projective object in $\mathcal{C}$ is of the form $F(Q)$, where $Q \in \mathcal{P}$, hence rigid. But a monoidal functor maps rigid objects to rigid ones. $\square$

Corollary 2.23. *The functors $P \otimes ?$ and $? \otimes P$ on $\mathcal{C}$ are faithful.*

Proof. Let $f : X \to Y$ be a morphism in $\mathcal{C}$. Then the image of $1_P \otimes f$ is $P \otimes \text{Im}(f)$. Now, $\text{Hom}(X, P \otimes \text{Im}(f)) = \text{Hom}(P^* \otimes X, \text{Im}(f))$. But P^* is a generator of $\mathcal{P}$. Thus if $P \otimes \text{Im}(f) = 0$ then $\text{Hom}(Q, \text{Im}(f)) = 0$ for all projectives Q, hence $\text{Im}(f) = 0$ and $f = 0$, as claimed. $\square$

Proposition 2.24. *The map $F_{X,Y} : \text{Hom}(X, Y) \to \text{Hom}(F(X), F(Y))$ is surjective if either X or Y is in $\mathcal{P}$. Thus, the tensor ideal $\langle \text{Pr} \rangle$ in $\mathcal{C}$ of morphisms factoring through projectives is contained in the image of F.*

Proof. By definition for $Q, R \in \mathcal{P}$ the map $F_{Q,R} : \text{Hom}(Q, R) \to \text{Hom}(F(Q), F(R))$ is an isomorphism. Now let $X \in \mathcal{T}$ and $f : F(Q) \to F(X)$ be a morphism. We would like to show that $f = F(g)$ for some morphism $g : Q \to X$.

Recall that $F(X)$ is a quotient of $F(X \otimes P^* \otimes P)$. Since $F(Q)$ is projective, the morphism f admits a lift $f' : F(Q) \to F(X \otimes P^* \otimes P)$. Since $F(X \otimes P^* \otimes P)$ also is projective, there exists $g' : Q \to X \otimes P^* \otimes P$ such that $F(g') = f'$. Let g be the composition of g' with the morphism $1_X \otimes \text{ev} : X \otimes P^* \otimes P \to X$. Then $F(g) = f$, as claimed.

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Thus, the map $\mathrm{Hom}(Q, X) \to \mathrm{Hom}(F(Q), F(X))$ is surjective. Now, to show that the map $\mathrm{Hom}(X, Q) \to \mathrm{Hom}(F(X), F(Q))$ is also surjective, write it as the map $\mathrm{Hom}(Q^* \otimes X, \mathbf{1}) \to \mathrm{Hom}(F(Q^* \otimes X), F(\mathbf{1}))$, which is surjective since $Q^* \otimes X \in \mathcal{P}$. This proves the proposition. □

Let $J_{\mathcal{P}}$ be the kernel of F, i.e., the collection of morphisms f in $\mathcal{T}$ such that $F(f) = 0$. Clearly, $J_{\mathcal{P}}$ is a tensor ideal. The following proposition describes $J_{\mathcal{P}}$ explicitly.

Proposition 2.25. *Let $f : X \to Y$ be a morphism in $\mathcal{T}$. The following conditions are equivalent:*

(a) $f \in J_{\mathcal{P}}$;
(b1) $f \otimes 1_P = 0$ in $\mathcal{T}$;
(b2) $1_P \otimes f = 0$ in $\mathcal{T}$;
(c1) For any object $Q \in \mathcal{P}$ we have $f \otimes 1_Q = 0$ in $\mathcal{T}$;
(c2) For any object $Q \in \mathcal{P}$ we have $1_Q \otimes f = 0$ in $\mathcal{T}$.

Proof. If $f \in J_{\mathcal{P}}$ then $F(f) = 0$, so $F(f) \otimes 1_P = F(f \otimes 1_P) = 0$. But $X \otimes P, Y \otimes P$ are projective and F is fully faithful on projectives. Thus $f \otimes 1_P = 0$. Similarly $1_P \otimes f = 0$. Thus, (a) implies (b1) and (b2).

If $f \otimes 1_P = 0$ then $f \otimes 1_{P \otimes X} = f \otimes 1_P \otimes 1_X = 0$ for all $X \in \mathcal{C}$, so for any object $Q \in \mathcal{P}$ we have $f \otimes 1_Q = 0$. Thus, (b1) implies (c1). Similarly, (b2) implies (c2).

If $f \otimes 1_Q = 0$ then $f \otimes 1_Z = 0$ for any $Z \in \mathcal{C}$, i.e., $F(f) = 0$, hence $f \in J_{\mathcal{P}}$. Thus (c1) implies (a). Similarly, (c2) implies (a). □

Definition 2.26. If $J_{\mathcal{P}} = 0$, i.e., F is faithful, we will say that the ideal $\mathcal{P}$ is *faithful*.

Example 2.27. The ideals $\mathcal{P}$ in Examples 2.5 and Example 2.6 are faithful. On the other hand, if $\mathcal{T} = \mathcal{T}_1 \oplus \mathcal{T}_2$ with $\mathcal{T}_1 \neq 0$ and $\mathcal{P} \subset \mathcal{T}_2$ then $\mathcal{P}$ is not faithful, as $\langle \mathcal{T}_1 \rangle \subset J_{\mathcal{P}}$. Also, in Example 2.20 with M finite, $\mathcal{P}_N$ is faithful if and only if $N = M$, i.e., $\mathcal{P}_N = \mathcal{S}$ is the ideal of all splitting objects. In general, if $\mathcal{P} = \mathcal{P}_N$ then $J_{\mathcal{P}} = \langle \mathcal{P}_{M \setminus N} \rangle$.

Definition 2.28. Let us say that $\mathcal{T}$ is *of finite type* if the ideal $\mathcal{S} = \mathcal{S}(\mathcal{T})$ is finitely generated. If so, denote the ideal $J_{\mathcal{S}}$ by $J = J(\mathcal{T})$. Let us say that $\mathcal{T}$ is *separated* if $J = 0$, i.e., $\mathcal{S}$ is faithful.

In other words, $\mathcal{T}$ is separated if it has "enough splitting objects", i.e., splitting objects separate morphisms (Proposition 2.25(c1,c2)), which motivates the terminology.

Remark 2.29. It is clear that $\mathcal{T}$ is of finite type if and only if the sets $\overline{S}$ and $I_c, c \in \overline{S}$ are finite.

Example 2.30. A multitensor category with enough projectives is separated. On the other hand, the category of projective bimodules from Example 2.6 is not separated, since it has no nonzero splitting objects (so J is the collection of all morphisms in $\mathcal{T}$). Similarly, if G is a semisimple algebraic group in characteristic p then $\mathrm{Rep}(G)$ is not separated, for the same reason.

Note that if $X \in \mathcal{S}$ and $1_X \in J$ then $X \otimes X^* = 0$, hence $X = 0$ (as X is a direct summand in $X \otimes X^* \otimes X$). Thus, faithfulness is not a very restrictive condition: if $\mathcal{T}$ is of finite type then the category $\mathcal{T}/J$ is separated (with the same indecomposable objects as $\mathcal{T}$), and $\mathcal{C}(\mathcal{T}, \mathcal{S}) = \mathcal{C}(\mathcal{T}/J, \mathcal{S})$. Also we obtain the following proposition (which is not used below).

Proposition 2.31. *Let $\mathcal{T}$ be of finite type. Let $\mathcal{P} = \mathcal{P}_K$ for a subset $K \subset \overline{S}$ (as in Proposition 2.8). Then $J_{\mathcal{P}}$ contains $\langle \mathcal{P}_{S \setminus K} \rangle$. In particular, $\mathcal{P}$ is faithful if and only if $\mathcal{P} = \mathcal{S}$ and $\mathcal{T}$ is separated.*

Proof. This follows immediately since $\mathcal{P}_K \otimes \mathcal{P}_{S \setminus K} = 0$. $\qquad\qquad\square$

Let us say that $\mathcal{S}$ is *indecomposable* if $\mathcal{S} \neq 0$ and $Q \sim_\ell R$ for all indecomposable $Q, R \in \mathcal{S}$. In this case, we get

Proposition 2.32. *If $\mathcal{T}$ is separated and $\mathcal{S}$ is indecomposable then it is the unique minimal (nonzero) thick ideal in $\mathcal{T}$.*

Proof. Lemma 2.7 implies that $\mathcal{S}$ is minimal. Let $\mathcal{I} \neq 0$ be any thick ideal in $\mathcal{T}$ and $0 \neq X \in \mathcal{I}$. Since $\mathcal{T}$ is separated, there is $Q \in \mathcal{S}$ such that $Q \otimes X \neq 0$. Then $Q \otimes X$ generates $\mathcal{S}$, so $\mathcal{S} \subset \mathcal{I}$. $\qquad\qquad\square$

Let $\mathcal{P} \subset \mathcal{T}$ be a finitely generated ideal and $\widehat{\mathcal{T}}_{\mathcal{P}}$ be the full Karoubian completion of $F(\mathcal{T}) = \mathcal{T}/J$ inside $\mathcal{C} = \mathcal{C}(\mathcal{T}, \mathcal{P})$ (i.e., the Karoubian completion of the full subcategory with the same objects as $F(\mathcal{T})$). Then $\widehat{\mathcal{T}}_{\mathcal{P}}$ is a full Karoubian monoidal subcategory of $\mathcal{C}$, and the monoidal abelian category $\mathcal{C}(\widehat{\mathcal{T}}_{\mathcal{P}}, \mathcal{P})$ coincides with $\mathcal{C}$. We will call $\widehat{\mathcal{T}}_{\mathcal{P}}$ the *completion* of $\mathcal{T}$ with respect to $\mathcal{P}$.

Definition 2.33. We will say that $\mathcal{T}$ is *complete* with respect to $\mathcal{P}$ if the natural inclusion $\mathcal{T}/J \hookrightarrow \widehat{\mathcal{T}}_{\mathcal{P}}$ is an equivalence.

It is clear that $\mathcal{T}$ is complete with respect to $\mathcal{P}$ if and only if for any $T \in \mathcal{T}$, the natural map $\mathrm{Hom}(\mathbf{1}, T) \to \mathrm{Hom}(\mathbb{1}, F(T))$ is surjective.

If $\mathcal{T}$ is of finite type and $\mathcal{P} = \mathcal{S}$, we will denote $\widehat{\mathcal{T}}_{\mathcal{P}}$ simply by $\widehat{\mathcal{T}}$, and say that $\mathcal{T}$ is *complete* if it is complete with respect to $\mathcal{S}$. In this case we will also write $\mathcal{C}(\mathcal{T})$ for $\mathcal{C}(\mathcal{T}, \mathcal{S})$. Thus, $\mathcal{T}$ is separated iff $F : \mathcal{T} \to \mathcal{C}(\mathcal{T})$ is faithful and complete iff F is full. In particular, the category $\widehat{\mathcal{T}}$ is separated and complete, and $\mathcal{T}$ is separated and complete iff the natural functor $\mathcal{T} \to \widehat{\mathcal{T}}$ is an equivalence. In this case the functor F is a fully faithful embedding.

2.9. **Properties of the tensor product on $\mathcal{C}$ in the case of a splitting ideal.** Now assume that $\mathcal{P} \subset \mathcal{T}$ is a finitely generated *splitting* ideal, and as before let $\mathcal{C} = \mathcal{C}(\mathcal{T}, \mathcal{P})$.

Proposition 2.34. *(i) For any $Y \in \mathcal{C}$, $Q \in \mathcal{P}$ the tensor products $Q \otimes Y$ and $Y \otimes Q$ are projective.*
(ii) Tensoring with any object $Q \in \mathcal{P}$ on either side splits every morphism in $\mathcal{C}$.

Proof. (i) Let $f : Q_1 \to Q_0$ be a presentation of Y. By Corollary 2.16, the cokernel of the morphism $1_Q \otimes f : Q \otimes Q_1 \to Q \otimes Q_0$ is $Q \otimes Y$. But since $\mathcal{P}$ is a splitting ideal, this morphism is split. Hence $Q \otimes Y$ is a direct summand in $Q \otimes Q_0$, hence projective.

(ii) Let $f : X \to Y$ be a morphism in $\mathcal{C}$, and K, K' be the kernel and cokernel of f, respectively. Then by Corollary 2.16, $Q \otimes K$, $Q \otimes K'$ are the kernel and cokernel of $1_Q \otimes f$, respectively. But by (i) these objects are projective. This implies that the morphism $1_Q \otimes f$ is split. Similarly, the morphism $f \otimes 1_Q$ is split, as claimed. $\qquad\square$

Recall that a *multiring category* over a field $\mathbf{k}$ is a $\mathbf{k}$-linear artinian monoidal category with biexact tensor product, [22], Definition 4.2.3.

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Proposition 2.35. *The bifunctor* $(X, Y) \mapsto X \otimes Y$ *on* $\mathcal{C}$ *is exact in both arguments. Thus* $\mathcal{C}$ *is a multiring category.*

Proof. Let $Q_\bullet$ be a projective resolution of $Y \in \mathcal{C}$. Then by Corollary 2.16 $H^{-i}(X \otimes^L Y)$ is the $-i$-th cohomology of the complex $X \otimes Q_\bullet$. So it suffices to show that if P_1, P_2, P_3 are projective and the complex $P_1 \to P_2 \to P_3$ is acyclic then the complex

$$X \otimes P_1 \to X \otimes P_2 \to X \otimes P_3$$

is also acyclic. That is, we need to show that the complex

$$\mathrm{Hom}(R, X \otimes P_1) \to \mathrm{Hom}(R, X \otimes P_2) \to \mathrm{Hom}(R, X \otimes P_3)$$

is acyclic for every projective R.

Let Q be projective. Then by Proposition 2.34, $Q \otimes X$ is projective, so by Corollary 2.16 the complex

$$\mathrm{Hom}(R, Q \otimes X \otimes P_1) \to \mathrm{Hom}(R, Q \otimes X \otimes P_2) \to \mathrm{Hom}(R, Q \otimes X \otimes P_3)$$

is acyclic. Since $X \otimes P_i$ are projective (again by Proposition 2.34), this can be written as

$$\mathrm{Hom}(Q^* \otimes R, X \otimes P_1) \to \mathrm{Hom}(Q^* \otimes R, X \otimes P_2) \to \mathrm{Hom}(Q^* \otimes R, X \otimes P_3).$$

Taking $Q = R \otimes {}^*R$, we have $Q^* \otimes R = R \otimes R^* \otimes R$. Thus, using that R is a direct summand in $R \otimes R^* \otimes R$ (as R is rigid), we get that the complex

$$\mathrm{Hom}(R, X \otimes P_1) \to \mathrm{Hom}(R, X \otimes P_2) \to \mathrm{Hom}(R, X \otimes P_3)$$

is acyclic, implying that the functor $X \otimes ?$ is exact. Similarly, the functor $? \otimes X$ is exact, as claimed. $\square$

Proposition 2.36. *Let* $\mathcal{D}$ *be a multiring category, and* $X \in \mathcal{D}$ *be the cokernel of a morphism* $f : Q_1 \to Q_0$ *such that the objects* Q_0, Q_1 *are rigid. Then* X *is also rigid. Namely,* X^* *is the kernel of* $f^* : Q_0^* \to Q_1^*$ *and* ${}^*X$ *is the kernel of* ${}^*f : {}^*Q_0 \to {}^*Q_1$.

Proof. Define X^* to be the kernel of the map $f^* : Q_0^* \to Q_1^*$. Let $i : X^* \to Q_0^*, j : f(Q_1) \to Q_0$ be the natural inclusions, and $\bar{f} : Q_1 \to f(Q_1)$ the natural projection. We claim that the map $\mathrm{ev}_{Q_0} \circ (i \otimes j) : X^* \otimes f(Q_1) \to \mathbb{1}$ is zero. To show this, it suffices to show that the map $\mathrm{ev}_{Q_0} \circ (i \otimes f) : X^* \otimes Q_1 \to \mathbb{1}$ is zero (indeed, tensoring with X^* is right exact, so the map $1 \otimes \bar{f} : X^* \otimes Q_1 \to X^* \otimes f(Q_1)$ is surjective, and $f = j \circ \bar{f}$). But we have $\mathrm{ev}_{Q_0} \circ (i \otimes f) = \mathrm{ev}_{Q_1} \circ (f^* \circ i \otimes 1)$, and by definition $f^* \circ i = 0$, which implies the claim.

This means that the map $\mathrm{ev}_{Q_0} \circ (i \otimes 1) : X^* \otimes Q_0 \to \mathbb{1}$ gives rise to a map $\mathrm{ev}_X : X^* \otimes X \to \mathbb{1}$.

Similarly, let $p : Q_0 \to X$ be the natural projection and consider the coevaluation map $\mathrm{coev}_{Q_0} : \mathbb{1} \to Q_0 \otimes Q_0^*$. It gives rise to a map $(p \otimes 1) \circ \mathrm{coev}_{Q_0} : \mathbb{1} \to X \otimes Q_0^*$. Note that since the tensor product by X is left exact, $X \otimes X^* \subset X \otimes Q_0^*$.

We claim that $(p \otimes 1) \circ \mathrm{coev}_{Q_0}$ lands in $X \otimes X^*$, i.e., that the composition

$$(p \otimes f^*) \circ \mathrm{coev}_{Q_0} : \mathbb{1} \to X \otimes Q_1^*$$

is zero. We have $= (p \otimes f^*) \circ \mathrm{coev}_{Q_0} = (p \circ f \otimes 1) \circ \mathrm{coev}_{Q_1}$. But by definition $p \circ f = 0$, as claimed.

Thus, the map $(p \otimes 1) \circ \mathrm{coev}_{Q_0}$ gives rise to a map $\mathrm{coev}_X : \mathbb{1} \to X \otimes X^*$.

Now the axioms of the left dual for X^* ([22], Subsection 2.10) are easily deduced from those for Q_0^*. For example, let us check that the composition $X \to X \otimes X^* \otimes X \to X$ is the identity (the other axiom is proved similarly). Indeed, let $x \in X$ and let $y \in X$ be the

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image of x under this composition. Fix a lift $x_1 \in Q_0$ of x and let x_2 be the image of x_1 in $Q_0 \otimes Q_0^* \otimes Q_0$, and x_3 the projection of x_2 to $X \otimes Q_0^* \otimes Q_0$. Then x_3 has a unique preimage x_4 in $X \otimes X^* \otimes Q_0$. Let x_5 be the image of x_4 in $X \otimes X^* \otimes X$. By the definition of coev_X we have $(1 \otimes \mathrm{ev}_X)(x_5) = y$. By the definition of ev_X, it follows that $(1 \otimes \mathrm{ev}_{Q_0})(x_3) = y$. Thus it follows from the rigidity of Q_0 that y is the image of x_1 in X, i.e., $y = x$, as desired.

Similarly, X admits a right dual $^*X$, which is the kernel of the map $^*f : {^*Q_0} \to {^*Q_1}$. Thus, X is rigid and the proposition is proved. $\qquad\square$

Proposition 2.37. *(i) The monoidal category $\mathcal{C} = \mathcal{C}(\mathcal{T}, \mathcal{P})$ is rigid, i.e., it is a multitensor category.*

(ii) Let $\mathcal{P} = \mathcal{S}$, so that $\mathcal{C} = \mathcal{C}(\mathcal{T})$. Then we have $\mathcal{C} = \bigoplus_{c \in \overline{S}} \mathcal{C}(c)$, where $\mathcal{C}(c)$ are indecomposable multitensor subcategories. Namely, $\mathcal{C}(c)$ is the subcategory of objects generated by c. In particular, $\mathbb{1} = \bigoplus_{c \in \overline{S}} \mathbb{1}_c$.

(iii) We have $\mathcal{C}(c) = \bigoplus_{i,j \in I_c} \mathcal{C}(c)_{ij}$, where $\mathcal{C}(c)_{ij}$ are component categories of $\mathcal{C}(c)$ ([22], Definition 4.3.5). In particular, $\mathbb{1}_c = \bigoplus_{i \in I_c} \mathbb{1}_{c,i}$ where $\mathbb{1}_{c,i}$ are simple, and $\mathcal{C}(c)_{ij} = \mathbb{1}_{c,i}\mathcal{C}(c)\mathbb{1}_{c,j}$.

(iv) $\mathcal{S}$ is indecomposable if and only if $\mathbb{1} \in \mathcal{C}$ is indecomposable, i.e., $\mathcal{C}$ is a tensor category.

Proof. (i) Any $X \in \mathcal{C}$ can be represented as a cokernel of a morphism $f : Q_1 \to Q_0$ between projective objects. By Corollary 2.22, the objects Q_0 and Q_1 are rigid. Thus by Proposition 2.36 and Proposition 2.35, X is rigid, as claimed.

The proof of (ii), (iii) and (iv) is straightforward. $\qquad\square$

Remark 2.38. Example 2.6 shows that the splitting assumption for $\mathcal{P}$ cannot be dropped, as the category of finite dimensional bimodules over a finite dimensional non-semisimple Frobenius algebra is not rigid.

As a by-product we obtain the following corollary (which is not used below).

Corollary 2.39. *If $\mathcal{T}$ is separated then the algebra $\mathrm{End}(\mathbb{1})$ is semisimple.*

Proof. This follows since $\mathrm{End}(\mathbb{1})$ is semisimple and commutative and $F_{\mathbf{1},\mathbf{1}} : \mathrm{End}(\mathbb{1}) \to \mathrm{End}(\mathbb{1})$ is injective (for $\mathcal{P} = \mathcal{S}$). $\qquad\square$

Example 2.40. Let $\mathrm{char}(\mathbf{k}) = 2$ and $\mathcal{T}_0 := \mathrm{Rep}_{\mathbf{k}}(\mathbb{Z}/2)$, with the usual monoidal structure. Then $\mathcal{T}_0$ has two indecomposable objects, $\mathbf{1}$ and P (the regular representation). Let $\mathcal{T}$ be the following modification of $\mathcal{T}_0$: we set $\mathrm{End}(\mathbf{1}) := \mathbf{k}[\varepsilon]/\varepsilon^2$ instead of $\mathbf{k}$. The monoidal structure is modified in an obvious way (namely, the tensor product of ε with any morphism $X \to Y$ where $X = P$ or $Y = P$ is zero). Then $\mathcal{S}$ is generated by P, so $J = \mathbf{k}\varepsilon$, and $\mathcal{T}/J = \mathcal{C}(\mathcal{T}/J) = \mathrm{Rep}_{\mathbf{k}}(\mathbb{Z}/2)$ (i.e., $\mathcal{T}$ is not separated but complete). This example shows that J does not have to be a thick ideal, and that the separatedness assumption in Corollary 2.39 cannot be dropped.

2.10. The main theorems.

Theorem 2.41. *$\mathcal{T}$ admits a faithful monoidal functor E into a multitensor category $\mathcal{D}$ with enough projectives which is full on projectives (i.e., any morphism between projectives is in the image) if and only if $\mathcal{T}$ is separated. Moreover, if such a functor exists, it is unique up to an isomorphism, with $\mathcal{D} \cong \mathcal{C}(\mathcal{T})$ and $E \cong F$. The same holds for braided and symmetric categories.*

Proof. Suppose $\mathcal{T}$ is separated. Then by Proposition 2.37, there is a monoidal functor $F : \mathcal{T} \to \mathcal{C}(\mathcal{T})$ which satisfies the conditions of the theorem.

Conversely, let $E : \mathcal{T} \to \mathcal{D}$ be a faithful monoidal functor into a multitensor category with enough projectives which is full on projectives. Then, since E is full on projectives, there exists $Q \in \mathcal{T}$ such that $E(Q) = U$. Moreover, since U splits every morphism in $\mathcal{D}$ and E is faithful, we have $Q \in \mathcal{S}$.

Now let $f : X \to Y$ be a morphism in J. Then $1_Q \otimes f = 0$, hence $1_{E(Q)} \otimes E(f) = 0$, i.e., $1_U \otimes E(f) = 0$. Thus $U \otimes \mathrm{Im}(E(f)) = \mathrm{Im}(1_U \otimes E(f)) = 0$ in $\mathcal{D}$. This holds for every projective U, which implies $\mathrm{Im}(E(f)) = 0$, hence $E(f) = 0$. Thus $f = 0$ by the faithfulness of E. Thus, $\mathcal{T}$ is separated (the ideal of projectives is generated as a left ideal by the projective cover of $\mathbb{1}$, hence is finitely generated).

Also, in this case $\mathcal{S}$ coincides with the ideal of projectives $\mathrm{Pr} \subset \mathcal{D}$, so we get that $\mathcal{D} \cong \mathcal{C}(\mathcal{T})$ and $E \cong F$. $\qquad\square$

Recall ([20], Section 9) that a multitensor category $\mathcal{C} \supset \mathcal{T}$ is called the *abelian envelope* of $\mathcal{T}$ if it has the following universal property: for any multitensor category $\mathcal{D}$ there is an equivalence between the category of tensor functors $E : \mathcal{C} \to \mathcal{D}$ and the category of faithful monoidal functors $\overline{E} : \mathcal{T} \to \mathcal{D}$, defined by the formula $\overline{E} = E|_{\mathcal{T}}$. The same definition is made for braided and symmetric categories (in which case the functors have to be braided).

Theorem 2.42. *(i) $\mathcal{T}$ admits a fully faithful monoidal functor $E : \mathcal{T} \to \mathcal{D}$ into a multitensor category $\mathcal{D}$ with enough projectives if and only if it is separated and complete. Moreover, in this case there exists a tensor embedding $\widetilde{E} : \mathcal{C}(\mathcal{T}) \hookrightarrow \mathcal{D}$ such that $E \cong \widetilde{E} \circ F$. The same applies to braided and symmetric categories.*

(ii) In this case $\mathcal{C}(\mathcal{T})$ is the abelian envelope of $\mathcal{T}$.

Proof. (i) If $\mathcal{T}$ is separated and complete then by Proposition 2.37, we have a fully faithful monoidal functor $F : \mathcal{T} \to \mathcal{C}(\mathcal{T})$ satisfying the conditions in the theorem.

Conversely, let $E : \mathcal{T} \to \mathcal{D}$ be a fully faithful monoidal functor into a multitensor category $\mathcal{D}$ with enough projectives. Let $\mathcal{E} \subset \mathcal{D}$ be the tensor subcategory generated by $E(\mathcal{T})$, [22], Definition 4.11.1 (full and closed under subquotients). Then $\mathcal{E}$ is a multitensor category with enough projectives, and we have a fully faithful monoidal functor $E : \mathcal{T} \to \mathcal{E}$. Thus we may assume without loss of generality that $\mathcal{E} = \mathcal{D}$ (i.e., E is surjective). In this case, let U be a projective object of $\mathcal{D}$. Since E is surjective, there exists $T \in \mathcal{T}$ such that U is a subquotient of $E(T)$. Since U is both projective and injective, it is in fact a direct summand of $E(T)$. Since F is full, this means that $U = E(Q)$ for some direct summand Q of T. Thus, F is full on projectives. By Theorem 2.41 this implies that $\mathcal{T}$ is separated and $\mathcal{D} \cong \mathcal{C}(\mathcal{T})$, $E \cong F$. Finally, since F is full, $\mathcal{T}$ is complete.

(ii) This follows from [20], Theorem 9.2.1. More precisely, this theorem applies to symmetric categories, but the proof extends without significant changes to the case of braided categories and categories without a fixed braided structure. $\qquad\square$

Let $\mathcal{D}$ be a multitensor category with enough projectives. Recall that the *stable category* $\mathrm{Stab}(\mathcal{D})$ is the quotient $\mathcal{D}/\mathrm{Pr}$, where Pr is the ideal of projectives. The indecomposables of $\mathrm{Stab}(\mathcal{D})$ are the non-projective indecomposables of $\mathcal{D}$, so an object of $\mathrm{Stab}(\mathcal{D})$ may be realized as an object in $\mathcal{D}$ with no nonzero projective direct summands. There is a natural monoidal functor $\mathrm{Stab} : \mathcal{D} \to \mathrm{Stab}(\mathcal{D})$.

Suppose $\mathcal{E}$ is a rigid monoidal Karoubian category and $E : \mathcal{E} \to \mathrm{Stab}(\mathcal{D})$ a faithful monoidal functor. Consider the fiber product $\mathcal{A} := \mathcal{E} \times_{\mathrm{Stab}(\mathcal{D})} \mathcal{D}$, whose objects are triples (X, Y, f), where $X \in \mathcal{E}$, $Y \in \mathcal{D}$, and $f : E(X) \to \mathrm{Stab}(Y)$ is an isomorphism, and morphisms $(X_1, Y_1, f_1) \to (X_2, Y_2, f_2)$ are pairs (g, h), $g : X_1 \to X_2$, $h : Y_1 \to Y_2$ such that $f \circ E(g) = \mathrm{Stab}(h) \circ f$. This is naturally a rigid monoidal Karoubian category. If (X, Y, f) and (X, Y, f') are objects of $\mathcal{A}$ then there exists an automorphism $g : Y \to Y$ such that $\mathrm{Stab}(g) = f' \circ f^{-1}$ and it defines an isomorphism $(X, Y, f) \cong (X, Y, f')$. Thus isomorphism classes of objects in $\mathcal{A}$ are isomorphism classes of pairs (X, Y) such that $E(X) \cong \mathrm{Stab}(Y)$, i.e., isomorphism classes of pairs (X, Q) where Q is a projective object of $\mathcal{D}$; namely, $(X, Q) = X \oplus Q$. Thus the indecomposables of $\mathcal{A}$ are the indecomposables of $\mathcal{E}$ and the indecomposable projectives of $\mathcal{D}$. Moreover, for $X, X' \in \mathcal{E}$ and projective $Q, Q' \in \mathcal{D}$ we have

$$\mathrm{Hom}_{\mathcal{A}}(X, X') = \mathrm{Hom}_{\mathcal{E}}(X, X'), \ \ \mathrm{Hom}_{\mathcal{A}}(Q, Q') = \mathrm{Hom}_{\mathcal{D}}(Q, Q'),$$

$$\mathrm{Hom}_{\mathcal{A}}(Q, X) = \mathrm{Hom}_{\mathcal{D}}(Q, E(X)), \mathrm{Hom}_{\mathcal{A}}(X, Q) = \mathrm{Hom}_{\mathcal{D}}(E(X), Q).$$

The category $\mathcal{A}$ is equipped with a natural faithful monoidal functor $\widetilde{E} : \mathcal{A} \to \mathcal{D}$.

Note that if E is pseudomonic, i.e., full on isomorphisms (in particular, if it is fully faithful) then $\mathcal{A}$ is naturally equivalent to a rigid monoidal isomorphism-closed subcategory of $\mathcal{D}$ containing the tensor ideal generated by projective objects.

Theorem 2.43. *There is a 1-1 correspondence between (equivalence classes of):*
(1) Separated categories $\mathcal{T}$ such that $\mathcal{C}(\mathcal{T}) \cong \mathcal{D}$;
(2) Rigid monoidal Karoubian categories $\overline{\mathcal{T}}$ together with a faithful monoidal functor $\overline{F} : \overline{\mathcal{T}} \to \mathrm{Stab}(\mathcal{D})$;
which sends complete categories $\mathcal{T}$ to full subcategories $\overline{\mathcal{T}} \subset \mathrm{Stab}(\mathcal{D})$, and vice versa. Namely, this correspondence is defined by the formulas

$$\overline{\mathcal{T}} = \mathcal{T}/\mathrm{Pr}, \ \mathcal{T} = \overline{\mathcal{T}} \times_{\mathrm{Stab}(\mathcal{D})} \mathcal{D}.$$

Proof. Given $\mathcal{T} \subset \mathcal{D}$, define $\overline{\mathcal{T}} := \mathcal{T}/\mathrm{Pr}$. Then $\overline{\mathcal{T}}$ is a rigid monoidal Karoubian category with a faithful monoidal functor $\overline{F} : \overline{\mathcal{T}} \to \mathrm{Stab}(\mathcal{D})$. Conversely, suppose $\mathcal{E}$ is a rigid monoidal Karoubian category with a faithful monoidal functor $E : \mathcal{E} \to \mathrm{Stab}(\mathcal{D})$ and set $\widetilde{\mathcal{E}} := \mathcal{E} \times_{\mathrm{Stab}(\mathcal{D})} \mathcal{D}$. Then $\widetilde{\mathcal{E}}$ is a separated category, and it is clear that $\mathcal{C}(\widetilde{\mathcal{E}}) \cong \mathcal{D}$.

Now, it is clear that $\widetilde{\overline{\mathcal{E}}} = \mathcal{E}$, and the equality $\overline{\widetilde{\mathcal{T}}} = \mathcal{T}$ follows from Proposition 2.24. $\square$

Example 2.44. 1. Suppose $\mathcal{D}$ is semisimple. Then $\mathrm{Stab}(\mathcal{D}) = 0$, so Theorem 2.43 says that a separable category $\mathcal{T}$ such that $\mathcal{C}(\mathcal{T}) = \mathcal{D}$ must be complete and coincide with $\mathcal{D}$ (which is easy).

2. Let $\mathcal{C}$ be a finite multitensor category and $\mathcal{T} = \mathrm{Stab}^{-1}(\mathrm{Vec})$ the preimage of Vec under the projection $\mathrm{Stab} : \mathcal{C} \to \mathrm{Stab}(\mathcal{C})$. The indecomposable objects of $\mathcal{T}$ are $\mathbb{1}$ and the indecomposable projectives of $\mathcal{C}$ (note that $\mathbb{1}$ may be decomposable in $\mathcal{C}$ but not in $\mathcal{T}$). Then the ideal $\mathcal{P} \subset \mathcal{T}$ of projectives coincides with $\mathcal{S}$, $\mathcal{T}$ is separated and $\mathcal{C}(\mathcal{T}) = \mathcal{C}$. But if $\mathbb{1}$ is decomposable and has length $m \geq 2$ then the map $\mathbf{k} = \mathrm{Hom}_{\mathcal{T}}(\mathbb{1}, \mathbb{1}) \to \mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, \mathbb{1}) = \mathbf{k}^m$ is not surjective, hence F is not full and $\mathcal{T}$ is not complete.

A simple example of this is the case when $\mathcal{C} = \mathrm{Mat}_N(\mathrm{Rep}_{\mathbf{k}}(\mathbb{Z}/2))$, $\mathrm{char}(\mathbf{k}) = 2$, where N is a finite set (i.e., the objects of $\mathcal{C}$ are matrices of objects of $\mathrm{Rep}_{\mathbf{k}}(\mathbb{Z}/2)$ with rows and columns labeled by N and tensor product given by matrix multiplication using the usual tensor product of $\mathrm{Rep}_{\mathbf{k}}(\mathbb{Z}/2)$). In this case the indecomposable objects of $\mathcal{T}$ are $\mathbf{1}$ and also

R_{ij}, $i, j \in N$, where R_{ij} is the regular representation of $\mathbb{Z}/2$ standing in the (i,j)-th position. Note that this example also makes sense when N is an infinite set, in which case the ideal $\mathcal{S}$ (generated by all the R_{ij}) is not finitely generated as a left ideal, but is finitely generated as a two-sided ideal (in fact, by any one of the R_{ij}).

3. Let $\mathcal{C} = \mathrm{Rep}_{\mathbf{k}}(\mathbb{Z}/p)$ where $\mathrm{char}(\mathbf{k}) = p > 2$. We have a faithful symmetric monoidal functor $E : \mathrm{sVec} \hookrightarrow \mathrm{Stab}(\mathcal{C})$ sending the non-trivial simple object to the indecomposable module $L = \Omega(\mathbf{1})$ of size $p - 1$ (here Ω is the Heller shift). Let $\mathcal{T} = \mathrm{Stab}^{-1}(\mathrm{Im}(E))$. Then $\mathcal{T}$ has three indecomposables: $\mathbf{1}, L, P$, with $L \otimes P = P^{p-1}$, $P \otimes P = P^p$ and $L \otimes L = P^{p-2} \oplus \mathbf{1}$, and $\mathrm{Hom}(\mathbf{1}, P) = \mathrm{Hom}(P, \mathbf{1}) = \mathbf{k}$, $\mathrm{Hom}(P, P) = \mathbf{k}^p$, $\mathrm{Hom}(P, L) = \mathrm{Hom}(L, P) = \mathbf{k}^{p-1}$, $\mathrm{Hom}(L, L) = \mathbf{k}[z]/z^{p-1}$, $\mathrm{Hom}(\mathbf{1}, L) = \mathrm{Hom}(L, \mathbf{1}) = 0$. In this case $\mathcal{S} := \langle P \rangle$ is a finitely generated ideal and $\mathcal{T}$ is separated, but the functor F is not full, since the morphism $0 = \mathrm{Hom}_{\mathcal{T}}(\mathbf{1}, L) \to \mathrm{Hom}(\mathbb{1}, F(L)) = \mathbf{k}$ is not injective. Thus $\mathcal{T}$ is not complete.

On the other hand, in this example we may replace sVec by the *full* monoidal subcategory of $\mathrm{Stab}(\mathcal{C})$ generated by $\mathbf{1}$ and L (i.e., $\mathrm{Hom}(\mathbf{1}, L) = \mathrm{Hom}(L, \mathbf{1}) = \mathbf{k}$ rather than zero), and the functor E with the tautological embedding. In this case $\mathcal{T} = \mathrm{Stab}^{-1}(\mathrm{Im}(E))$ is both separated and complete; it is the full monoidal subcategory of $\mathcal{C}$ spanned by $\mathbf{1}, L, P$. We have $\mathcal{T} = \mathcal{C}$ for $p = 3$, but for $p > 3$ it is smaller than $\mathcal{C}$ (and not abelian). But for any $p > 2$ the category $\mathcal{C}$ is the abelian envelope of $\mathcal{T}$.

4. Let $\mathrm{char}(\mathbf{k}) = 2$ and $\mathcal{C} = \mathrm{Rep}_{\mathbf{k}}(\mathbb{Z}/2)$. Then $\mathrm{Stab}(\mathcal{C}) = \mathrm{Vec}$ is the category of finite dimensional vector spaces. Let H be a finite dimensional Hopf algebra over $\mathbf{k}$ and $\mathcal{E} := \mathrm{Rep}(H)$. Let $E : \mathcal{E} \to \mathrm{Vec} = \mathrm{Stab}(\mathcal{C})$ be the forgetful functor. Let $\mathcal{T} = \mathcal{E} \times_{\mathrm{Stab}(\mathcal{C})} \mathcal{C}$. Then $\mathcal{T}$ is separated and $\mathcal{C}(\mathcal{T}) = \mathcal{C}$. Note that E is not full on isomorphisms in general, so $\mathcal{T}$ is not equivalent to an isomorphism-closed subcategory of $\mathcal{C}$, but it admits a faithful monoidal functor $F : \mathcal{T} \to \mathcal{C}$.

Example 2.45. Here is a prototypical example of a separated and complete category $\mathcal{T}$. Let $\mathcal{C}$ be a multitensor category with enough projectives which is tensor generated by an object $X \in \mathcal{C}$. This means that any object of $\mathcal{C}$ is a subquotient of a direct sum Y of tensor products of objects $X^{(n)}$, where $X^{(n)}$ is the n-th dual of X. Let $\mathcal{T}$ be the full rigid monoidal *Karoubian* subcategory of $\mathcal{C}$ generated by X, i.e., its objects are *direct summands* (rather than subquotients) of direct sums Y as above. Then by Theorem 2.42, $\mathcal{S} \subset \mathcal{T}$ is the ideal of projective objects in $\mathcal{C}$, and $\mathcal{C} = \mathcal{C}(\mathcal{T})$.

This example becomes interesting when the category $\mathcal{T}$ has an independent (often diagrammatic) description not involving $\mathcal{C}$, and the corresponding universal property. This happens in a number of concrete examples, notably the following.

1. Let $\mathcal{C} = \mathrm{Rep}^{\mathrm{ab}}(S_t)$ is the abelian envelope of the Deligne category $\mathrm{Rep}(S_t)$ for $t \in \mathbb{Z}_+$, $\mathrm{char}(\mathbf{k}) = 0$ ([16, 9]). Let X be the "tautological" object (the interpolation of the permutation representation of S_n). Then $\mathcal{T}$ is the Karoubian Deligne category $\mathrm{Rep}(S_t)$ ([16]). The indecomposable objects of $\mathcal{T}$ are X_λ labeled by partitions λ, with $X_0 = \mathbf{1}$. The blocks in this category are described in [9]; there are semisimple blocks corresponding to a single isolated partition and also semi-infinite chains of partitions, whose structure depends on t. The ideal $\mathcal{S}$ is then spanned by X_λ for the isolated λ and also all other λ except the first element of each chain; these exceptions are exactly the partitions for which $t \geq \lambda_1 + |\lambda|$ (they are in bijection with partitions of t defined by $\lambda \mapsto (t - \lambda_1, \lambda_1, ..., \lambda_n)$). Indeed, it is precisely these X_λ that become projective in $\mathcal{C}$ ([19], 4.4).

2. Let $\mathcal{C} = \mathrm{Rep}(GL(m|n))$, $\mathrm{char}(\mathbf{k}) = 0$, and assume for simplicity that $m \geq n$. Let $X = \mathbf{k}^{m|n}$ be the defining representation. Then the category $\mathcal{T}$ is the category of modules that occur as direct summands in $X^{\otimes i} \otimes X^{*\otimes j}$. Let $t = m - n$ and $\mathrm{Rep}(GL_t)$ be the Karoubian Deligne category for parameter t. It is shown in [11, 8] that the only nontrivial tensor ideals in $\mathrm{Rep}(GL_t)$ are $J_0 \supset J_1 \supset J_2...$, such that J_i is the kernel of the natural functor $\mathrm{Rep}(GL_t) \to \mathrm{Rep}(GL(t+i|i))$. Then $\mathcal{T} = \mathrm{Rep}(GL_t)/J_n$ (see [8, 10, 20]).

3. Let G be a simply connected simple algebraic group and q a root of unity of sufficiently large order. Let $\mathcal{C} = \mathrm{Rep}(G_q)$ be the category of finite dimensional representations of the corresponding quantum group ($\mathrm{char}(\mathbf{k}) = 0$), and $\mathcal{T} = \mathrm{Tilt}(G_q) \subset \mathrm{Rep}(G_q)$ be the category of tilting modules; it is obtained by the above construction from the direct sum of the fundamental representations $X = \bigoplus_i L_{\omega_i}$ (note that all projectives in $\mathcal{C}$ are tiltings, so $\mathcal{T}$ is separated and complete and $\mathcal{C} = \mathcal{C}(\mathcal{T})$). If $G = SL_n$, the category $\mathcal{T}$ is the category of modules occurring as direct summands in tensor powers of the tautological representation V, and it is expected to have a nice diagrammatic description as a braided category using webs (a root of unity analog of [7]). In particular, we see that $\mathcal{C}$ is the abelian envelope of $\mathcal{T}$, which is a quantum analog of [14], Theorem 3.3.1.

4. Let G be as above, p a prime, $\mathbf{k}$ a field of characteristic p, $\mathcal{C} = \mathrm{Rep}(G)$ be the category of finite dimensional representations of G over $\mathbf{k}$, and $\mathcal{T} \subset \mathcal{C}$ be the category of tilting modules. Then $\mathcal{C}$ is the abelian envelope of $\mathcal{T}$ ([14], Theorem 3.3.1). Since the category $\mathcal{C}$ has no nonzero projective objects, this is not quite an instance of the above theory. However, it is a limit of such instances. More precisely, let G_n be the n-th Frobenius kernel of G. Then we have the restriction functor $\mathrm{Rep}(G) \to \mathrm{Rep}(G_n)$. Denote by $\mathcal{T}(n)$ the image of $\mathcal{T}$ in $\mathrm{Rep}(G_n)$. Then $\mathrm{Rep}(G_n) = \mathcal{C}(\mathcal{T}(n))$. This follows from the fact that the n-th Steinberg representation St_n of G is projective when restricted to G_n.

5. Let $\mathcal{T}_p$ be the category of tilting modules for the group $SL_2(\mathbf{k})$ ($\mathrm{char}(\mathbf{k}) = p$), with indecomposable tilting objects T_i, $i \geq 0$, with highest weight i. Then $\mathcal{T}_p$ has a thick ideal $\mathcal{I}_n$ generated by the n-th Steinberg representation $\mathrm{St}_n := T_{p^n-1}$, which is spanned by the objects T_i with $i \geq p^n - 1$. Let $\mathcal{T}_{n,p} := \mathcal{T}_p/\mathcal{I}_n$ ([5]). It is shown in [5], Theorem 4.14 that for $n \geq 0$ the category $\mathcal{T}_{n+1,2}$ has an abelian envelope $\mathcal{C}_{2n}$, a finite symmetric tensor category in which it sits as a full monoidal generating subcategory. Thus, $\mathcal{T}_{n+1,2}$ is separated and complete, and $\mathcal{C}_{2n} = \mathcal{C}(\mathcal{T}_{n+1,2})$.

However, while these are nice examples, the true power of the above theory is seen when the category $\mathcal{C}(\mathcal{T})$ is not known a priori, and this theory provides its first construction. This is what happens for $\mathcal{T}_{n,p}$ in the case $p > 2$, and the main goal of this paper is to define and study the corresponding categories $\mathcal{C}(\mathcal{T}_{n,p})$.

2.11. **Splitting objects in monoidal categories over discrete valuation rings.** For applications, we will need to generalize the above theory of splitting ideals and associated abelian envelopes to the deformation-theoretic setting, when the ground ring is not a field but rather a complete discrete valuation ring (DVR), which in applications will be of mixed characteristic.

Let R be a complete DVR with maximal ideal $\mathfrak{p}$ and uniformizer ε. Let $\mathbf{k} := R/\mathfrak{p}$ be its residue field, and K be the fraction field of R.

Definition 2.46. By a *flat monoidal Karoubian category* over R we will mean an R-linear monoidal Karoubian category $\mathcal{T}$ for which $\mathrm{Hom}(X, Y)$ is a free finite rank R-module for any $X, Y \in \mathcal{T}$.

If $\mathcal{T}$ is a flat monoidal Karoubian category over R then we have the monoidal Karoubian category $\mathcal{T}_{\mathbf{k}} := \mathcal{T}/\mathfrak{p}$ over the residue field $\mathbf{k}$. We also have the category $\mathcal{T}_K$, the Karoubian closure of $K \otimes_R \mathcal{T}$ (which may not be Karoubian by itself), a monoidal Karoubian category over K.[2] Moreover, it is easy to show that if $\mathcal{T}_{\mathbf{k}}$ is rigid then so are $\mathcal{T}, \mathcal{T}_K$.

We have a natural bijection between objects of $\mathcal{T}$ and $\mathcal{T}_{\mathbf{k}}$ given by $X \in \mathcal{T} \mapsto X_{\mathbf{k}} \in \mathcal{T}_{\mathbf{k}}$, $Y \in \mathcal{T}_{\mathbf{k}} \mapsto \widetilde{Y} \in \mathcal{T}$, such that $\mathrm{Hom}(X_{\mathbf{k}}, Z_{\mathbf{k}}) = \mathbf{k} \otimes_R \mathrm{Hom}(X, Z)$. Also, any object $X \in \mathcal{T}$ defines an object $X_K \in \mathcal{T}_K$, with $\mathrm{Hom}(X_K, Z_K) = K \otimes_R \mathrm{Hom}(X, Z)$. Thus any object $Y \in \mathcal{T}_{\mathbf{k}}$ uniquely lifts to an object of $\widetilde{Y}_K \in \mathcal{T}_K$ (although the lift of an indecomposable in $\mathcal{T}$ may be decomposable in $\mathcal{T}_K$).

Note that the assignments $X \mapsto X_{\mathbf{k}}$ and $X \mapsto X_K$ are symmetric monoidal functors $\mathcal{T} \to \mathcal{T}_{\mathbf{k}}$ and $\mathcal{T} \to \mathcal{T}_K$. Moreover, if $f : X \to Y$ is a split surjection or injection in $\mathcal{T}$ then so are $f_{\mathbf{k}} : X_{\mathbf{k}} \to Y_{\mathbf{k}}$ and $f_K : X_K \to Y_K$.

A flat monoidal category $\mathcal{T}$ over R cannot have nonzero splitting objects in the sense of Definition 2.3, since any element $x \in \mathfrak{p}$, $x \neq 0$ defines a morphism $x_{Q^*} : Q^* \to Q^*$ which is not split by tensoring with Q. However we can make a less restrictive definition.

Definition 2.47. 1. A morphism $f : Q_1 \to Q_0$ in $\mathcal{T}$ is *R-split* if for any $X \in \mathcal{T}$ the induced morphism of free R-modules $f_X : \mathrm{Hom}(X, Q_1) \to \mathrm{Hom}(X, Q_0)$ is split (i.e., its cokernel is a free R-module).

2. An object $Q \in \mathcal{T}$ is *a splitting object* if tensoring with Q on either side splits any R-split morphism.

Example 2.48. Any split morphism in $\mathcal{T}$ is automatically R-split. Indeed, let $f : X \to Y$ be a split morphism in $\mathcal{T}$. Then $f = 0 \oplus g$, where g is an isomorphism. Since 0 and g are R-split, so is f.

Lemma 2.49. *Let $f : X \to Y$ be an R-split morphism in $\mathcal{T}$.*
(i) If f is nonzero then the corresponding morphism $f_{\mathbf{k}} : X_{\mathbf{k}} \to Y_{\mathbf{k}}$ is nonzero.
(ii) If $f_{\mathbf{k}}$ is split then f is split.
(iii) If $Q \in \mathcal{T}$ then the morphisms $1_Q \otimes f$ and $f \otimes 1_Q$ are R-split.
(iv) If $Q_{\mathbf{k}}$ is a splitting object of $\mathcal{T}_{\mathbf{k}}$ then Q is a splitting object of $\mathcal{T}$.

Proof. (i) Suppose $f_{\mathbf{k}} = 0$. For an object $Z \in \mathcal{T}$ consider the map $f \circ : \mathrm{Hom}(Z, X) \to \mathrm{Hom}(Z, Y)$. The map $f_{\mathbf{k}} \circ : \mathrm{Hom}(Z_{\mathbf{k}}, X_{\mathbf{k}}) \to \mathrm{Hom}(Z_{\mathbf{k}}, Y_{\mathbf{k}})$ is zero. Thus $f \circ = \varepsilon g$ for some map $g : \mathrm{Hom}(Z, X) \to \mathrm{Hom}(Z, Y)$. But $f \circ$ is R-split, which implies that $g = 0$, i.e., $f \circ = 0$. Thus $f = 0$.

(ii) Since $f_{\mathbf{k}}$ is split, up to composing with isomorphisms on both sides it can be written as a direct sum of the zero morphism $X' \to Y'$ and the identity morphism $Z \to Z$, i.e., as a block matrix

$$f_{\mathbf{k}} \sim \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

This means that f can be similarly written as a block matrix

$$f \sim \begin{pmatrix} \varepsilon a & \varepsilon b \\ \varepsilon c & 1 + \varepsilon d \end{pmatrix}.$$

Thus up to composing with isomorphisms

$$f \sim \begin{pmatrix} \varepsilon(a - \varepsilon b(1 + \varepsilon d)^{-1}c) & 0 \\ 0 & 1 + \varepsilon d \end{pmatrix}.$$

Since f is R-split, $\varepsilon(a - \varepsilon b(1 + \varepsilon d)^{-1}c)$ must be R-split, hence it is zero by (i). Thus

$$f \sim \begin{pmatrix} 0 & 0 \\ 0 & 1 + \varepsilon d \end{pmatrix},$$

i.e., f is a direct sum of zero and an isomorphism. Thus, f is split.

(iii) The morphism $1_Q \otimes f$ is R-split since the corresponding map

$$(1_Q \otimes f)\circ : \mathrm{Hom}(Z, Q \otimes X) \to \mathrm{Hom}(Z, Q \otimes Y)$$

coincides with the map $f\circ : \mathrm{Hom}(Q^* \otimes Z, X) \to \mathrm{Hom}(Q^* \otimes Z, Y)$, which is split by definition. Similarly, $f \otimes 1_Q$ is R-split.

(iv) Assume that $Q_{\mathbf{k}}$ is splitting. Let $f : X \to Y$ be a morphism in $\mathcal{T}$. Then the morphism $1_{Q_{\mathbf{k}}} \otimes f_{\mathbf{k}} = (1_Q \otimes f)_{\mathbf{k}}$ is split. Also the morphism $1_Q \otimes f$ is R-split by (iii). Thus by (ii) the morphism $1_Q \otimes f$ is split. Similarly, the morphism $f \otimes 1_Q$ is split. Thus, Q is a splitting object. $\qquad\square$

It follows from Lemma 2.49(iii) that splitting objects form a thick ideal $\mathcal{S} \subset \mathcal{T}$. Thus, if Q is a splitting object then so is Q^*. Hence, if f is an R-split morphism and Q is a splitting object then Q splits the morphism f^* (as Q^* splits f).

From now on assume that the category $\mathcal{T}_{\mathbf{k}}$ is separated and its ideal of splitting objects $\mathcal{S}(\mathcal{T}_{\mathbf{k}})$ is finitely generated as a left ideal. Thus we can define the multitensor category $\mathcal{C}(\mathcal{T}_{\mathbf{k}})$ with enough projectives and a faithful monoidal functor $F : \mathcal{T}_{\mathbf{k}} \to \mathcal{C}(\mathcal{T}_{\mathbf{k}})$.

Lemma 2.50. *(i) Let P be a generator of $\mathcal{S}(\mathcal{T}_{\mathbf{k}})$. Then the morphism $\mathrm{coev}_{\widetilde{P}} : \mathbf{1} \to \widetilde{P} \otimes \widetilde{P}^*$ is R-split.*

(ii) If Q is a splitting object of $\mathcal{T}$ then $Q_{\mathbf{k}}$ is a splitting object of $\mathcal{T}_{\mathbf{k}}$. In other words, $\mathcal{S}(\mathcal{T}_{\mathbf{k}}) = \mathcal{S}_{\mathbf{k}}$.

Proof. (i) By applying the faithful monoidal functor $F : \mathcal{T}_{\mathbf{k}} \to \mathcal{C}(\mathcal{T}_{\mathbf{k}})$ and using Proposition 2.24 we get that for any $X \in \mathcal{T}_{\mathbf{k}}$ the map $\mathrm{coev}_P\circ : \mathrm{Hom}(X, \mathbf{1}_{\mathbf{k}}) \to \mathrm{Hom}(X, P \otimes P^*)$ is injective. This implies that the map

$$(\mathrm{coev}_{\widetilde{P}}\circ)_K : K \otimes_R \mathrm{Hom}(\widetilde{X}, \mathbf{1}) \to K \otimes_R \mathrm{Hom}(\widetilde{X}, \widetilde{P} \otimes \widetilde{P}^*)$$

is also injective, and thus has the same rank as coev_P (the dimension of the source). Thus $\mathrm{coev}_{\widetilde{P}}$ is split as a morphism of R-modules, as claimed.

(ii) Let Q be an indecomposable splitting object of $\mathcal{T}$, and P a generator of $\mathcal{S}(\mathcal{T}_{\mathbf{k}})$. Then $Q_{\mathbf{k}} \otimes P \neq 0$ as $\mathcal{T}_{\mathbf{k}}$ is separated. Consider the morphism $f = 1_Q \otimes \mathrm{coev}_{\widetilde{P}} : Q \to Q \otimes \widetilde{P} \otimes \widetilde{P}^*$. Since $Q_{\mathbf{k}} \otimes P \neq 0$, we have $f_{\mathbf{k}} \neq 0$, hence $f \neq 0$. On the other hand, the morphism $\mathrm{coev}_{\widetilde{P}}$ is R-split by (i), so the morphism f is split. Since Q is indecomposable, this implies that f is a split injection. Thus $f_{\mathbf{k}} : Q_{\mathbf{k}} \to Q_{\mathbf{k}} \otimes P \otimes P^*$ is also a split injection, i.e., $Q_{\mathbf{k}}$ is a direct summand in $Q_{\mathbf{k}} \otimes P \otimes P^*$, hence split, as claimed. $\qquad\square$

For simplicity assume that $\mathcal{T}$ has finitely many indecomposables. Then we can define the category $\mathcal{C}(\mathcal{T})$ in the same way as we did over a field. Namely, let P be the direct sum of all indecomposable splitting objects of $\mathcal{T}_\mathbf{k}$, $\widetilde{P}$ the corresponding object of $\mathcal{T}$, $A = \mathrm{End}(\widetilde{P})^\mathrm{op}$, and $\mathcal{C}(\mathcal{T}) = \mathrm{Rep}(A)$ be the category of finitely generated A-modules. Then $\mathcal{C}(\mathcal{T})$ can be realized as a category of *resolutions*

$$... \to Q_2 \to Q_1 \to Q_0,$$

where $Q_i \in \mathcal{T}$ are splitting objects (i.e., complexes $Q^\bullet$ of splitting objects such that the complex $\mathrm{Hom}(\widetilde{P}, Q^\bullet)$ is exact). As before, this is an R-linear abelian monoidal category with right exact tensor product. Moreover, we have a faithful monoidal functor $F : \mathcal{T} \to \mathcal{C}(\mathcal{T})$ given by $F(X) = \mathrm{Hom}(\widetilde{P}, X)$.

Of course, we cannot hope that the tensor product in $\mathcal{C}(\mathcal{T})$ is exact, since for example $\mathcal{C}(\mathcal{T})$ contains the category of finitely generated R-modules with tensor product over R (quotients of a multiple of $\mathbb{1}_i$ for some i). However, it turns out that it is exact on a certain class of objects we call *flat*.

Definition 2.51. An object $X \in \mathcal{C}(\mathcal{T})$ is *flat* if it is presented by an R-split morphism $f : Q_1 \to Q_0$, $Q_1, Q_0 \in \mathcal{S}$; i.e., $X = \mathrm{Coker}(F(f))$.

For instance, every splitting object is automatically flat.

Lemma 2.52. *The following conditions on $X \in \mathcal{C}(\mathcal{T})$ are equivalent:*
(i) X is flat;
(ii) X is a free R-module;
(iii) Any morphism $f : Q_0 \to Q_1$ with $Q_0, Q_1 \in \mathcal{S}$ presenting X is R-split.

Proof. Clearly (iii) implies (i).

Suppose X is flat, and presented by an R-split morphism $f : Q_1 \to Q_0$. Then X is the cokernel of the map $f\circ : \mathrm{Hom}(\widetilde{P}, Q_1) \to \mathrm{Hom}(\widetilde{P}, Q_0)$. By definition, this map is R-split, hence X is a free R-module. Thus (i) implies (ii).

Suppose that X is a free R-module, and presented by a morphism $f : Q_1 \to Q_0$. Then by Proposition 2.24 for $Z \in \mathcal{T}$ the natural map $\mathrm{Hom}(Z, Q_i) \to \mathrm{Hom}(F(Z), F(Q_i))$ is an isomorphism. Thus we have an exact sequence

$$\mathrm{Hom}(Z, Q_1) \to \mathrm{Hom}(Z, Q_0) \to \mathrm{Hom}(F(Z), X).$$

Thus the cokernel of the map $f\circ : \mathrm{Hom}(Z, Q_1) \to \mathrm{Hom}(Z, Q_0)$ is an R-submodule of $\mathrm{Hom}(F(Z), X)$ which is, in turn, contained in $\mathrm{Hom}_R(F(Z), X)$. But $\mathrm{Hom}_R(F(Z), X)$ is a free R-module since X is free as an R-module. Thus, $\mathrm{Coker}(f\circ)$ is a free R-module and f is R-split. $\square$

Corollary 2.53. *Let $Q_0, Q_1 \in \mathcal{S}$. A morphism $f : Q_1 \to Q_0$ is R-split if and only if $F(f) : F(Q_1) \to F(Q_0)$ is split as a morphism of R-modules.*

Proof. Suppose f is R-split. Then $F(f) = f\circ : \mathrm{Hom}(\widetilde{P}, Q_1) \to \mathrm{Hom}(\widetilde{P}, Q_0)$ is split as a morphism of R-modules. Conversely, if $F(f)$ is split then f presents the object $\mathrm{Coker}(F(f))$, which is free as an R-module, so by Lemma 2.52 f is R-split. $\square$

Corollary 2.54. *For every $Z \in \mathcal{T}$ the object $F(Z)$ is flat.*

Proof. This follows from Lemma 2.52 since $\mathrm{Hom}(\widetilde{P}, Z) = F(Z)$ is a free R-module. $\square$

Proposition 2.55. $X \in \mathcal{C}(\mathcal{T})$ *is flat if and only if it is rigid, and in this case the functors* $X \otimes ?$ *and* $? \otimes X$ *are exact.*

Proof. We start with the following lemma.

Lemma 2.56. *For any flat* $Y \in \mathcal{C}(\mathcal{T})$ *and projective* $Q \in \mathcal{C}(\mathcal{T})$ *the tensor products* $Q \otimes Y$ *and* $Y \otimes Q$ *are projective.*

Proof. Let $f : Q_1 \to Q_0$ be a presentation of Y by an R-split morphism. By a generalization of Corollary 2.16 to R-linear categories, the cokernel of the morphism

$$1_Q \otimes F(f) : Q \otimes F(Q_1) \to Q \otimes F(Q_0)$$

is $Q \otimes Y$. Since f is R-split and $Q = F(Q')$ for some splitting object Q', the morphism $1_Q \otimes F(f) = F(1_{Q'} \otimes f)$ is split (as so is $1_{Q'} \otimes f$). Hence $Q \otimes Y$ is a direct summand in $Q \otimes F(Q_0) = F(Q' \otimes Q_0)$, hence projective. $\square$

Lemma 2.57. *The following conditions on* $X \in \mathcal{C}(\mathcal{T})$ *are equivalent:*
  (i) X *is flat;*
  (ii) *the functor* $X \otimes$ *is exact;*
  (iii) *the functor* $\otimes X$ *is exact.*

Proof. The proof that (i) implies (ii), (iii) is the same as that of Proposition 2.35, using Lemma 2.56 instead of Proposition 2.34.

To prove that (ii) implies (i), we may assume that $X \in \mathcal{C}(\mathcal{T})_{ij}$ for some i, j. If the functor $X \otimes$ is exact then in particular it is exact on the subcategory of finitely generated R-modules, R-mod $\subset \mathcal{C}(\mathcal{T})_{jj}$ (quotients of a multiple of $\mathbb{1}_j$). If $M \in R$-mod then $X \otimes M$ is just the usual tensor product $X \otimes_R M$. Since this functor (of M) is exact, X is flat as an R-module, i.e., free (as it is finitely generated). Hence it is flat as an object of $\mathcal{C}(\mathcal{T})$ by Lemma 2.52. $\square$

Lemma 2.58. *Let* $X \in \mathcal{C}(\mathcal{T})$ *be a flat object which is the cokernel of an R-split morphism* $f : Q_1 \to Q_0$. *Then X is rigid. Namely, X^* is the kernel of $f^* : Q_0^* \to Q_1^*$ and $^*X$ is the kernel of $^*f : {^*Q_0} \to {^*Q_1}$.*

Proof. The proof is the same as that of Proposition 2.36, using Lemma 2.57 instead of Proposition 2.35. $\square$

Lemma 2.58 shows that any flat object is rigid. Also, for every rigid X the functor $X \otimes$ is exact, which implies that X is flat by Lemma 2.57. This completes the proof of Proposition 2.55. $\square$

Thus, in terms of the realization of $\mathcal{C}(\mathcal{T})$ as the category of finitely generated left A-modules, the tensor product looks as follows: we have an $(A, A \otimes_R A)$-bimodule T, and

$$X \otimes Y = T \otimes_{A \otimes_R A} (X \otimes_R Y).$$

The exactness of this functor on flat objects proved in Proposition 2.55 is equivalent to the condition that T is projective as a right module (in particular, it is free over R).

We see that the algebra A has a structure of a *pseudo-Hopf algebra* over R ([24], 4.2). Thus $A_{\mathbf{k}}$ is a pseudo-Hopf algebra over $\mathbf{k}$ and $A_K := K \otimes_R A$ is one over K. We have $\mathcal{C}(\mathcal{T}_{\mathbf{k}}) = \mathrm{Rep}(A_{\mathbf{k}})$. Also it follows that the category $\mathcal{T}_K$ is separated (namely, for any splitting object $Q \in \mathcal{T}$ the object Q_K is splitting in $\mathcal{T}_K$) and $\mathcal{C}(\mathcal{T}_K) = \mathrm{Rep}(A_K)$.

Remark 2.59. If $X \in \mathcal{C}(\mathcal{T})$ is a flat object then X^*, $^*X$ are both the dual R-module of X with the action of $a \in A$ given by $a_{X^*} = S(a)_X^*$, $a_{*X} = S^{-1}(a)_X^*$ where $S : A \to A$ is the antipode.

If there exists a pseudo-Hopf algebra A over R which is free of finite rank as an R-module and $\mathcal{C}_{\mathbf{k}} = \text{Rep}(A_{\mathbf{k}})$, $\mathcal{C}_K = \text{Rep}(A_K)$, then we will say that $\mathcal{C}_K$ is *a lift* of $\mathcal{C}_{\mathbf{k}}$ over K. Note that a specific object of $\mathcal{C}_{\mathbf{k}}$ may admit more than one lift over K, or no such lifts at all.

For example, let H be a (quasi-)Hopf algebra over R which is free of finite rank as an R-module (for example, the group algebra RG of a finite group G). Then the tensor category $\text{Rep}(H_K)$ is a lift of $\text{Rep}(H_{\mathbf{k}})$ over K.

We see that in the above setting $\mathcal{C}(\mathcal{T}_K)$ is a lift of $\mathcal{C}(\mathcal{T}_{\mathbf{k}})$ over K.

From now on assume that the algebra A_K is split over K, or, equivalently, the category $\mathcal{C}_K$ is split (note that this can always be achieved by a finite extension of R). For brevity denote $\mathcal{C}(\mathcal{T}), \mathcal{C}(\mathcal{T})_{\mathbf{k}}, \mathcal{C}(\mathcal{T})_K$ by $\mathcal{C}, \mathcal{C}_{\mathbf{k}}, \mathcal{C}_K$. Given a simple object $Y \in \mathcal{C}_K$, it is well known that one can choose (non-uniquely) a flat object $\overline{Y} \in \mathcal{C}$ such that $\overline{Y}_K = Y$. For a simple object $X \in \mathcal{C}_{\mathbf{k}}$ let $[Y : X]$ be the multiplicity of X in the Jordan-Hölder series of the reduction $\overline{Y}_{\mathbf{k}} \in \mathcal{C}_{\mathbf{k}}$. This notation is justified because this multiplicity is, in fact, independent on the choice of $\overline{Y}$ (see [21]). The *decomposition matrix* D of $\mathcal{C}$ is the matrix D with entries $D_{XY} = [Y : X]$.

Proposition 2.60. *(i) (CDE triangle) Let $C_{\mathbf{k}}, C_K$ be the Cartan matrices of $\mathcal{C}_{\mathbf{k}}, \mathcal{C}_K$, and D be the decomposition matrix of $\mathcal{C}$. Then the decomposition matrix for lifts of projectives in $\mathcal{C}_{\mathbf{k}}$ into projectives in $\mathcal{C}_K$ is D^T, and we have $C_{\mathbf{k}} = DC_KD^T$.*
(ii) $\text{FPdim}(\mathcal{C}_{\mathbf{k}}) = \text{FPdim}(\mathcal{C}_K)$;
(iii) If $\mathcal{T}_K$ is complete then so is $\mathcal{T}_{\mathbf{k}}$.

Proof. (i) This holds for any flat deformation of finite dimensional algebras over a DVR, [21].

(ii) Let $\mathbf{d}_{\mathbf{k}}, \mathbf{d}_K$ be the vectors of Frobenius-Perron dimensions in $\mathcal{C}_{\mathbf{k}}, \mathcal{C}_K$ ([22], Subsections 3.3, 4.5). By (i) we have

$$\text{FPdim}(\mathcal{C}_{\mathbf{k}}) = \mathbf{d}_{\mathbf{k}}^T C_{\mathbf{k}} \mathbf{d}_{\mathbf{k}} = \mathbf{d}_{\mathbf{k}}^T DC_KD^T\mathbf{d}_{\mathbf{k}} = \mathbf{d}_K^T C_K\mathbf{d}_K = \text{FPdim}(\mathcal{C}_K).$$

(iii) We need to show that the functor $F_{\mathbf{k}} : \mathcal{T}_{\mathbf{k}} \to \mathcal{C}_{\mathbf{k}}$ is full. Since it is faithful, it suffices to show that $\dim \text{Hom}_{\mathcal{T}_{\mathbf{k}}}(T_i, T_j) \geq \dim \text{Hom}_{\mathcal{C}_{\mathbf{k}}}(F_{\mathbf{k}}(T_i), F_{\mathbf{k}}(T_j))$. But

$$\dim \text{Hom}_{\mathcal{T}_{\mathbf{k}}}(T_i, T_j) = \dim \text{Hom}_{\mathcal{T}_K}(T_i, T_j) = \dim \text{Hom}(F_K(T_i), F_K(T_j)),$$

so it suffices to show that $\dim \text{Hom}(F_{\mathbf{k}}(T_i), F_{\mathbf{k}}(T_j)) \geq \dim \text{Hom}(F_K(T_i), F_K(T_j))$. But this follows since $F_K(T_i)$ are lifts of $F_{\mathbf{k}}(T_i)$ over K (as A-modules). $\square$

Corollary 2.61. *If $\mathcal{T}_K$ is semisimple then $\mathcal{T}_{\mathbf{k}}$ is complete and the Cartan matrix of $\mathcal{C}_{\mathbf{k}}$ is given by $C = DD^T$.*

Proof. This follows immediately from Proposition 2.60, since the Cartan matrix of $\mathcal{C}_K = \mathcal{T}_K$ is the identity matrix. $\square$

3. Properties of tilting modules for $SL_2(\mathbf{k})$ and splitting objects of $\mathcal{T}_{n,p}$

3.1. **Splitting objects in $\mathcal{T}_{n,p}$.** Let $\mathcal{T} = \mathcal{T}_p$ be the category of tilting $SL_2(\mathbf{k})-$modules, $\text{char}(\mathbf{k}) = p > 0$. Its indecomposable object of highest weight $m \in \mathbb{Z}_{\geq 0}$ will be denoted T_m. For $r \in \mathbb{Z}_{\geq 0}$ let $\text{St}_r = T_{p^r-1}$ be the r-th Steinberg module; we will also denote St_1 as St. For $k \in \mathbb{Z}_{\geq 1}$ let $\mathcal{T}_{<p^k-1}$ be the additive subcategory of $\mathcal{T}$ generated by T_m with $m < p^k - 1$.

Let L_m be the simple $SL_2(\mathbf{k})-$module of highest weight $m \in \mathbb{Z}_{\geq 0}$. Let $M \mapsto M^{(1)}$ be the Frobenius twist. We first recall the following standard facts.

Proposition 3.1. *(i) ([29, E.1]) $L_a = T_a$ for $0 \leq a \leq p - 1$.*

(ii) (Steinberg tensor product theorem, [29, 3.17]) $L_{a+bp} = L_a \otimes L_b^{(1)}$, $0 \leq a \leq p-1, b \in \mathbb{Z}_{\geq 0}$.

(iii) ([29, E.9]) One has

$$(3.1) \qquad T_{a+pb} = T_a \otimes T_b^{(1)} \text{ for } p - 1 \leq a \leq 2p - 2, \ b \in \mathbb{Z}_{\geq 0}.$$

In particular $\mathrm{St}_r = \mathrm{St} \otimes \mathrm{St}_{r-1}^{(1)}$ *for* $r \in \mathbb{Z}_{\geq 1}$.

An important role below will be played by the following proposition.

Proposition 3.2. *Let $k \in \mathbb{Z}_{\geq 1}$ and $X, Y \in \mathcal{T}_{<p^k-1}$. Then for any morphism $f : X \to Y$ the morphism $f \otimes 1 : X \otimes \mathrm{St}_{k-1} \to Y \otimes \mathrm{St}_{k-1}$ is split.*

Proof. We will use the following lemma.

Lemma 3.3. *Assume that $k \in \mathbb{Z}_{\geq 1}$ and $m \leq p^k - 1$. Then $L_m \otimes \mathrm{St}_{k-1}$ is tilting.*

Proof. We use induction in k. In the base case $k = 1$, $\mathrm{St}_{k-1} = \mathrm{St}_0 = \mathbf{1}$ and the result follows from Proposition 3.1(i).

Now assume that Lemma 3.3 holds for some k. Let $m \leq p^{k+1} - 1$. We write $m = a + bp$ where $0 \leq a \leq p - 1$ and $0 \leq b \leq p^k - 1$, so $L_m = L_{a+bp} = L_a \otimes L_b^{(1)}$ by Proposition 3.1(ii). We have

$$L_m \otimes \mathrm{St}_k = L_a \otimes L_b^{(1)} \otimes \mathrm{St} \otimes \mathrm{St}_{k-1}^{(1)} = T_a \otimes \mathrm{St} \otimes (L_b \otimes \mathrm{St}_{k-1})^{(1)}.$$

By the induction assumption $L_b \otimes \mathrm{St}_{k-1}$ is tilting, so $\mathrm{St} \otimes (L_b \otimes \mathrm{St}_{k-1})^{(1)}$ is also tilting by Proposition 3.1(iii). Thus $T_a \otimes \mathrm{St} \otimes (L_b \otimes \mathrm{St}_{k-1})^{(1)}$ is also tilting as a product of two tilting modules. We have thus proved the induction step and the Lemma follows. $\square$

Now consider the exact sequences

$$0 \to \mathrm{Ker}\,(f) \to X \to \mathrm{Im}(f) \to 0,$$

$$0 \to \mathrm{Im}(f) \to Y \to \mathrm{Coker}(f) \to 0.$$

Let M be one of $\mathrm{Ker}\,(f), \mathrm{Im}(f), \mathrm{Coker}(f)$. The Jordan-Holder factors of M are of the form $L_m, 0 \leq m < p^k - 1$. Thus $M \otimes \mathrm{St}_{k-1}$ has a filtration with subquotients of the form $L_m \otimes \mathrm{St}_{k-1}, 0 \leq m < p^k - 1$ which are tilting by Lemma 3.3. Since all Exts between tilting modules vanish (see [29, Corollary E.2 (b)]) this filtration splits and $M \otimes \mathrm{St}_{k-1}$ is tilting. Thus the sequences

$$0 \to \mathrm{Ker}\,(f) \otimes \mathrm{St}_{k-1} \to X \otimes \mathrm{St}_{k-1} \to \mathrm{Im}(f) \otimes \mathrm{St}_{k-1} \to 0,$$

$$0 \to \mathrm{Im}(f) \otimes \mathrm{St}_{k-1} \to Y \otimes \mathrm{St}_{k-1} \to \mathrm{Coker}(f) \otimes \mathrm{St}_{k-1} \to 0$$

also split and the result follows. $\square$

Remark 3.4. The inequality for m in Lemma 3.3 is the best possible: the module $L_{p^k} \otimes \mathrm{St}_{k-1}$ is not tilting. Namely this module has highest weight $p^k + p^{k-1} - 1$ and dimension $2p^{k-1}$ (since $L_{p^k} = L_1^{(k)}$); this dimension is less than the dimension of the Weyl module $W_{p^k+p^{k-1}-1}$, which equals $p^k + p^{k-1}$.

3.2. The functor $\mathcal{T}_{<p^n-1} \to \mathcal{T}_{n,p}$ is fully faithful.

Proposition 3.5. *The composite functor $\mathcal{T}_{<p^n-1} \hookrightarrow \mathcal{T}_p \to \mathcal{T}_{n,p}$ is fully faithful.*

Proof. This functor is full by definition. To prove its faithfulness we need to show that no nonzero morphism in $\mathcal{T}_{<p^n-1}$ factors through $\mathcal{I}_n = \mathcal{T}_{\geq p^n-1}$. Assume that this is not the case and let $\widetilde{f} \in \mathrm{Hom}(T_m, T_k)$ be a morphism in $\langle \mathcal{I}_n \rangle$ with $0 \leq m, k < p^n - 1$. Then the morphism $f \in \mathrm{Hom}(\mathbf{1}, T_m^* \otimes T_k)$ which is the image of $\widetilde{f}$ under the canonical isomorphism $\mathrm{Hom}(T_m, T_k) \simeq \mathrm{Hom}(\mathbf{1}, T_m^* \otimes T_k)$ is also in $\langle \mathcal{I}_n \rangle$. Thus there exists an object T_s with $s \geq p^n - 1$ and morphisms $\mathbf{1} \to T_s$ and $T_s \to T_m^* \otimes T_k$ such that their composition is not zero.

Lemma 3.6. *(see for example [11], Lemma 5.3.3 and references therein) Assume $\mathrm{Hom}(\mathbf{1}, T_s) \neq 0$ in $\mathcal{T}_p$. Then $s = 2p^l - 2$ for some $l \in \mathbb{Z}_{\geq 0}$ and the socle of T_s is one-dimensional.*

Proof. For $0 \leq s \leq 2p - 2$ the lemma follows from the description of tilting modules in [29, E.1]. Now assume that $s > 2p-2$ and write $s = a+bp$ where $p-1 \leq a \leq 2p-2$ and $b \in \mathbb{Z}_{\geq 0}$. Then by [29, Lemma E.9] we have $T_s = T_a \otimes T_b^{(1)}$. Thus $\mathrm{Hom}(\mathbf{1}, T_s) = \mathrm{Hom}(T_a^*, T_b^{(1)})$. Thus T_a^* must have a quotient on which the first Frobenius kernel of $SL_2(\mathbf{k})$ acts trivially. Again by the description in [29, E.1] we see that $a = 2p - 2$ and the quotient is the object $\mathbf{1} = L_0$. Thus $\mathrm{Hom}(\mathbf{1}, T_b^{(1)}) \neq 0$, so $\mathrm{Hom}(\mathbf{1}, T_b) \neq 0$. By induction we get that $b = 2p^{l-1} - 2$, so $s = 2p^l - 2$ as required.

The module T_{2p^l-2} is a lift to $SL_2(\mathbf{k})$ of the injective hull of the trivial module over the l-th Frobenius kernel, see [29, E.9]; this implies the statement about the socle. $\square$

Lemma 3.6 implies that the map $T_s \to T_m^* \otimes T_k$ is injective and $s = 2p^l - 2$ with $l \geq n$. This is impossible as all weights of $T_m^* \otimes T_k$ have absolute value $\leq m + k < 2(p^n - 1)$. $\square$

3.3. Tensor ideals in $\mathcal{T}_p$. Lemma 3.6 can be used to establish the classification of tensor ideals in $\mathcal{T}$.

Proposition 3.7. *([11], Theorem 5.3.1) The only nonzero tensor ideals in $\mathcal{T}_p$ are the ideals $\langle \mathcal{I}_k \rangle$, $k \geq 0$.*

Proof. Let I be a nonzero tensor ideal in $\mathcal{T}_p$. Let $f \in I$ be a nonzero morphism, $f : X \to Y$. Then f defines a nonzero element $\widetilde{f} \in \mathrm{Hom}(\mathbf{1}, X \otimes Y^*)$ which belongs to I. Thus there exists a nonzero morphism $g : \mathbf{1} \to T_m$ for some $m \geq 0$ which belongs to I. Let m be the smallest such number. By Lemma 3.6, $m = 2p^k - 2$ for some $k \geq 0$, and I contains $\mathrm{Hom}(\mathbf{1}, T_{2p^k-2})$. Since the map $\mathrm{Hom}(\mathbf{1}, T_{2p^l-2}) \to \mathrm{Hom}(\mathbf{1}, \mathrm{St}_l \otimes \mathrm{St}_l^*)$ induced by the inclusion of the direct summand $T_{2p^l-2} \hookrightarrow \mathrm{St}_l \otimes \mathrm{St}_l^*$ is an isomorphism, we see that 1_{St_k} is contained in I, i.e., $\langle \mathcal{I}_k \rangle \subset I$. This implies that 1_{St_l} is contained in I for all $l \geq k$. Thus, $\mathrm{Hom}(\mathbf{1}, T_{2p^l-2})$ is contained in I if and only if $l \geq k$, i.e., if and only if it is contained in $\langle \mathcal{I}_k \rangle$. Hence $I = \langle \mathcal{I}_k \rangle$, since a tensor ideal is uniquely determined by its components in $\mathrm{Hom}(\mathbf{1}, T)$ where T runs over indecomposable objects. $\square$

3.4. Separatedness of $\mathcal{T}_{n,p}$ and its ideal of splitting objects. Now for $n \geq 0$ let $\mathcal{I}_n = \mathcal{I}_{n,p}$ be the thick ideal in $\mathcal{T} = \mathcal{T}_p$ generated by St_n. We have $\mathcal{I}_n = \mathcal{T}_{\geq p^n-1}$, the additive subcategory of $\mathcal{T}_p$ spanned by the T_m with $m \geq p^n - 1$. Thus the tensor ideal $\langle \mathcal{I}_n \rangle$ consists of morphisms which factor through a direct sum of T_m, $m \geq p^n - 1$.

For $n \geq 1$ consider the category $\mathcal{T}_{n,p} := \mathcal{T}_p/\mathcal{I}_n$. This is a rigid monoidal Karoubian category with indecomposables T_i, $0 \leq i \leq p^n - 2$.

Corollary 3.8. *The category $\mathcal{T}_{n,p}$ is separated.*

Proof. By Proposition 3.2, $\mathrm{St}_{n-1} = T_{p^{n-1}-1}$ is a splitting object in $\mathcal{T}_{n,p}$. Also it is clear that if $\mathrm{St}_{n-1} \otimes X = 0$ for an object X then $X = 0$, which implies the statement by Proposition 3.7. $\qquad\square$

Let $\mathcal{S}_{n,p}$ be the ideal of splitting objects of $\mathcal{T}_{n,p}$.

Proposition 3.9. *We have $\mathcal{S}_{n,p} = \mathcal{I}_{n-1}/\mathcal{I}_n$.*

Proof. By Proposition 3.7 and Corollary 3.8, $\mathcal{S}_{n,p} = \mathcal{I}_m/\mathcal{I}_n$ for some $m < n$. Thus $\mathcal{S}_{n,p}$ is indecomposable. Hence the claim follows from Corollary 3.8 and Proposition 2.32. $\qquad\square$

3.5. **Donkin's recursive formula and auxiliary properties of tilting modules.** We recall Donkin's recursive formula for the tilting $SL_2(\mathbf{k})-$modules, see [17, Example 2, p. 47].

Let W_a denote the standard (i.e., Weyl) $SL_2(\mathbf{k})$–module with highest weight a. Recall that the modules $T_i = W_i = L_i$ are simple for $0 \le i \le p - 1$, and $T_0 = \mathbf{1}$. Let $V := L_1$. Recall the following result.

Lemma 3.10. *([29, E.1]) (i) One has*

$$(3.2) \qquad\qquad V \otimes T_i = T_{i-1} \oplus T_{i+1} \ \text{if} \ 1 \le i \le p - 2.$$

(ii) For $p \le a \le 2p - 2$ the module T_a has a submodule W_a such that $T_a/W_a = W_{2p-2-a}$; thus its composition series is $T_a = [L_{2p-2-a}, V^{(1)} \otimes L_{a-p}, L_{2p-2-a}]$.

Using Proposition 3.1(iii), this implies

Proposition 3.11. *(i) for $p > 2$ we have*

$$(3.3) \qquad V \otimes T_{a+pb} = \begin{cases} T_p \otimes T_b^{(1)} = T_{p+pb} & \text{if} & a = p - 1, \\ 2T_{p-1+pb} \oplus T_{p+1+pb} & \text{if} & a = p, \\ T_{a-1+pb} \oplus T_{a+1+pb} & \text{if} \ p < a < 2p - 2, \\ T_{2p-3+pb} \oplus T_{p-1} \otimes (V \otimes T_b)^{(1)} & \text{if} & a = 2p - 2. \end{cases}$$

(ii) For $p = 2$ we have

$$(3.4) \qquad V \otimes T_{a+2b} = \begin{cases} T_2 \otimes T_b^{(1)} = T_{2+2b}, & a = 1 \\ 2T_{1+2b} \oplus V \otimes (V \otimes T_b)^{(1)}, & a = 2. \end{cases}$$

We also have the following result describing tensor products of tilting modules with small highest weights.

Proposition 3.12. *The tensor product $T_m \otimes T_r$ for $1 \le m, r \le p - 1$ is given by the formula*

$$T_m \otimes T_r = \bigoplus_{|m-r| \le k \le 2(p-2)-m-r} T_k \oplus \bigoplus_{p-1 \le k \le m+r} T_k,$$

if $m + r \ge p$, and

$$T_m \otimes T_r = \bigoplus_{|m-r| \le k \le m+r} T_k$$

if $m + r < p$, where the summation is over integers k such that $m + r - k$ is even.

Proof. This follows immediately from Lemma 3.10 and Equations (3.3),(3.4) $\qquad\square$

Proposition 3.13. *Assume that $0 \le i \le p-1$ and $m = pr + p - 2$ for some $r \in \mathbb{Z}_{\ge 0}$. Then $T_i \otimes T_k$ contains T_m as a summand if and only if $k = m - i$, in which case T_m appears with multiplicity 1.*

Proof. For $p = 2$ the statement is clear from (3.4), so we may assume that $p > 2$. It is clear that $T_i \otimes T_{m-i}$ contains T_m with multiplicity 1, so we only need to prove the "only if" statement. The case $i = 0$ is clear as $T_0 = \mathbf{1}$. For $i \geq 1$ we proceed by induction in i. The base case $i = 1$ is clear by inspection of (3.3) and (3.2). Suppose $1 < i \leq p - 1$ and the statement is known for $i-1$, and let us prove it for i. So by (3.2), $T_{i-1} \otimes V \otimes T_k$ contains T_m. By the induction assumption, $V \otimes T_k$ must contain T_{m-i+1}. But $m - i + 1 \equiv -i - 1 \not\equiv -1 \pmod{p}$. Thus by (3.3) we have either $k = m - i$ or $k = m - i + 2$ and $V \otimes T_k$ contains T_{m-i+1} with multiplicity 1. For the sake of contradiction assume that $k = m - i + 2$. Then $T_{i-1} \otimes V \otimes T_k = (T_{i-2} \oplus T_i) \otimes T_k$ contains T_m with multiplicity 1. But $T_{i-2} \otimes T_k$ already contains $T_{i-2+k} = T_m$, so $T_i \otimes T_k$ cannot contain T_m (as the multiplicity is 1). Thus this case is impossible and $k = m - i$, as desired. $\square$

4. THE VERLINDE CATEGORIES Ver_{p^n}

4.1. **The definition of the Verlinde categories** Ver_{p^n}, $\mathrm{Ver}_{p^n}^+$. Let p be a prime. Recall that Ver_p denotes the Verlinde category, which is the semisimplification of the category of modular representations $\mathrm{Rep}_{\mathbf{k}}(\mathbb{Z}/p)$ ([25, 27, 32]).

Let $n \geq 1$ be an integer. By Corollary 3.8, the category $\mathcal{T}_{n,p}$ is separated, and by Proposition 3.9 its ideal of splitting objects $\mathcal{S}_{n,p} = \mathcal{I}_{n-1}/\mathcal{I}_n$ is spanned by the objects T_i, $p^{n-1} - 1 \leq i \leq p^n - 2$ and is indecomposable. Thus, using the main construction of Section 2, we can define finite symmetric tensor categories

$$\mathrm{Ver}_{p^n} = \mathrm{Ver}_{p^n}(\mathbf{k}) := \mathcal{C}(\mathcal{T}_{n,p}).$$

If $p^n > 2$, we can also consider subcategories $\mathcal{T}_{n,p}^+ \subset \mathcal{T}_{n,p}$ spanned by T_i with even i, and the corresponding subcategories $\mathrm{Ver}_{p^n}^+ := \mathcal{C}(\mathcal{T}_{n,p}^+) \subset \mathrm{Ver}_{p^n}$. This makes Ver_{p^n} into faithfully $\mathbb{Z}/2$-graded tensor categories with trivial components $\mathrm{Ver}_{p^n}^+$ ([22], Subsection 4.14).

Definition 4.1. We will call the categories Ver_{p^n}, $\mathrm{Ver}_{p^n}^+$ the **Verlinde categories**.

The motivation for this terminology will be clarified in Subsection 4.4.

4.2. $\mathcal{T}_{n,p}$ **is complete.**

Theorem 4.2. *(i) The category $\mathcal{T}_{n,p}$ is complete, so it is a full rigid monoidal Karoubian subcategory of $\mathrm{Ver}_{p^n} = \mathcal{C}(\mathcal{T}_{n,p})$.*
(ii) Ver_{p^n} is the abelian envelope of $\mathcal{T}_{n,p}$.

Proof. (i) For $p = 2$ the result follows from [5], so we may assume that $p \geq 3$.

We need to show that for any $X, Y \in \mathcal{T}_{n,p}$, the map $\mathrm{Hom}(X, Y) \to \mathrm{Hom}(F(X), F(Y))$ is an isomorphism, or, equivalently, that the map $\mathrm{Hom}(\mathbf{1}, Y \otimes X^*) \to \mathrm{Hom}(\mathbb{1}, F(Y) \otimes F(X)^*) = \mathrm{Hom}(\mathbb{1}, F(Y \otimes X^*))$ is an isomorphism. Thus, it suffices to show that for any $i \in [0, p^n - 2]$ the map $\mathrm{Hom}(\mathbf{1}, T_i) \to \mathrm{Hom}(\mathbb{1}, F(T_i))$ is an isomorphism. For $i \geq p^{n-1} - 1$ this follows from Proposition 2.24, so it suffices to treat the case $i \leq p^{n-1} - 2$.

Let $S = T_{p^{n-1}-1}$. To compute $\mathrm{Hom}(\mathbb{1}, F(T_i))$, we will use the presentation of $\mathbb{1}$ as the cokernel of the map

$$\tau : S^* \otimes S \otimes S^* \otimes S \to S^* \otimes S.$$

Note that $S \cong S^*$. Hence we get that

$$\mathrm{Hom}(\mathbb{1}, F(T_i)) = \mathrm{Ker}(\mathrm{Hom}(S^{\otimes 2}, T_i) \to \mathrm{Hom}(S^{\otimes 4}, T_i)).$$

Thus it suffices to check that the sequence

$$\operatorname{Hom}_{\mathcal{T}_{n,p}}(\mathbf{1}, T_i) \to \operatorname{Hom}_{\mathcal{T}_{n,p}}(S^{\otimes 2}, T_i) \to \operatorname{Hom}_{\mathcal{T}_{n,p}}(S^{\otimes 4}, T_i)$$

is exact. This sequence can be rewritten as

$$(4.1) \qquad \operatorname{Hom}_{\mathcal{T}_{n,p}}(\mathbf{1}, T_i) \to \operatorname{Hom}_{\mathcal{T}_{n,p}}(S^{\otimes 2}, T_i) \to \operatorname{Hom}_{\mathcal{T}_{n,p}}(S^{\otimes 2}, T_i \otimes S^{\otimes 2}).$$

Let us now consider the same sequence in $\mathcal{T}_p$:

$$(4.2) \qquad \operatorname{Hom}_{\mathcal{T}_p}(\mathbf{1}, T_i) \to \operatorname{Hom}_{\mathcal{T}_p}(S^{\otimes 2}, T_i) \to \operatorname{Hom}_{\mathcal{T}_p}(S^{\otimes 2}, T_i \otimes S^{\otimes 2}),$$

which is clearly exact. But since $i \leq p^{n-1} - 2$ and $p \geq 3$, the sequence (4.2) actually belongs to $\mathcal{T}_{<p^n-1}$, hence is isomorphic to (4.1) by Proposition 3.5. Thus (4.1) is exact, as required.[3]
(ii) now follows from Theorem 2.42. □

Another proof of Theorem 4.2(i) is given in Remark 4.10 below.

Theorem 4.2 allows one to easily compute the dimension of the space of invariants in $V^{\otimes r}$, where $V := F(T_1) \in \operatorname{Ver}_{p^n}$. It is clear that if r is odd then this space is zero, so it suffices to compute the invariants in $V^{\otimes 2m}$. Let $d_{mn} = \dim \operatorname{Hom}(\mathbf{1}, V^{\otimes 2m})$.

Proposition 4.3. *Let* $f_n(z) = \sum_{m \geq 0} d_{mn} z^m$. *Then*

$$(4.3) \qquad f_n(z) = \frac{(t + t^{-1})(t^{p^n-1} - t^{-p^n+1})}{t^{p^n} - t^{-p^n}},$$

where $z = (t + t^{-1})^{-2}$.

Proof. By Theorem 4.2, $d_{mn} = \dim \operatorname{Hom}(\mathbf{1}, T_1^{\otimes 2m})$. The rest of the argument is the same as in [5], proof of Proposition 3.9. □

Remark 4.4. Proposition 4.3 generalizes [5], Proposition 3.3. Note that there is a misprint in [5], Proposition 3.3 and the proof of Proposition 3.9: 2^n should be replaced by 2^{n+1}.

4.3. **Basic properties of** Ver_{p^n}**.**

Theorem 4.5. *(i) If* $n = 1$ *then* $\operatorname{Ver}_{p^n} = \operatorname{Ver}_p$, $\operatorname{Ver}_{p^n}^+ = \operatorname{Ver}_p^+$.
(ii) We have a fully faithful monoidal functor $F : \mathcal{T}_{n,p} \to \operatorname{Ver}_{p^n}$ *which defines an equivalence between* $\mathcal{S}_{n,p}$ *and the category of projectives in* Ver_{p^n}. *In particular,* Ver_{p^n} *has* $p^{n-1}(p-1)$ *simple objects whose projective covers are* $F(T_i)$, $p^{n-1} - 1 \leq i \leq p^n - 2$, *and all simple objects in* Ver_{p^n} *are self-dual.*
(iii) We have $\operatorname{Ver}_{2^n} = \mathcal{C}_{2n-2}$ *and if* $n \geq 2$ *then* $\operatorname{Ver}_{2^n}^+ = \mathcal{C}_{2n-3}$, *where* $\mathcal{C}_i$ *are the categories defined in* [5].
(iv) For $p^n > 2$ *let* $\mathcal{O}_{n,p}$ *be the ring of integers in the field* $\mathbb{Q}(2\cos(2\pi/p^n)) = \mathbb{Q}(q^2 + q^{-2})$, *where* $q = q_{n,p} := \exp(\pi i/p^n)$; *that is,* $\mathcal{O}_{n,p} = \mathbb{Z}[2\cos(2\pi/p^n)]$ *(*[35]*, p.16, Proposition 2.16). Then the Frobenius-Perron dimension* FPdim *defines an isomorphism of the Grothendieck ring* $\operatorname{Gr}(\operatorname{Ver}_{p^n}^+)$ *onto* $\mathcal{O}_{n,p}$.
(v) There are no tensor functors $\operatorname{Ver}_{p^n}^+ \to \operatorname{Ver}_{p^m}^+$ *for* $m < n$.

Proof. Parts (i), (ii) are immediate from the previous results. Part (iii) follows from (ii) and [5], Theorem 4.14.

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[3]This method of proof was suggested by K. Coulembier.

Let us prove (iv). Note that the split Grothendieck ring of $\mathcal{T}_{n,p}$ is a $\mathbb{Z}_+$-ring of finite rank ([22], Section 3), so for $T \in \mathcal{T}_{n,p}$ we can define its Frobenius-Perron dimension $\mathrm{FPdim}(T)$. Let

$$(4.4) \qquad Q_{n,p}(2\cos x) := \frac{\sin(p^n x)}{\sin x}$$

be the Chebyshev polynomial of the second kind. By [5], Corollary 3.5,

$$Q_{n,p}(\mathrm{FPdim}(T_1)) = 0,$$

which implies that $\mathrm{FPdim}(T_1) = \mathrm{FPdim}(F(T_1)) = 2\cos(\pi/p^n)$. Thus we have $\mathrm{FPdim}(T_2) = 2\cos(2\pi/p^n) + 1$ for $p > 2$ and $\mathrm{FPdim}(T_2) = 2\cos(2\pi/p^n) + 2$ for $p = 2$. Hence the image of FPdim contains $\mathcal{O}_{n,p}$. On the other hand, since the rank of $\mathrm{Ver}_{p^n}^+$ is $p^{n-1}(p-1)/2$, which equals the degree of $\mathbb{Q}(2\cos(2\pi/p^n))$ over $\mathbb{Q}$, we see that FPdim lands in $\mathbb{Q}(2\cos(2\pi/p^n))$, hence in $\mathcal{O}_{n,p}$, since the Frobenius-Perron dimension of any object is an algebraic integer ([22], Proposition 3.3.4). Thus $\mathrm{FPdim} : \mathrm{Gr}(\mathrm{Ver}_{p^n}^+) \to \mathcal{O}_{n,p}$ is an isomorphism.

(v) follows from (iv). $\qquad\square$

Remark 4.6. For $p = 2$ Theorem 4.5 is proved in [5], Corollary 2.2 (the ring $\mathcal{O}_{n,2}$ is denoted there by $\mathcal{O}_{n-2}$).

Let $Q_{n,p} = Q_{n,p}^+ - Q_{n,p}^-$ be the splitting of the Chebyshev polynomial of the second kind $Q_{n,p}$ (see (4.4)) into the parts with positive and negative coefficients.

Corollary 4.7. *(The universal property of Ver_{p^n}) Let $\mathcal{D}$ be a symmetric tensor category over* **k**. *Then the category of* **k**-*linear symmetric monoidal functors $E\colon \mathrm{Ver}_{p^n} \to \mathcal{D}$ is equivalent to the category of objects $X \in \mathcal{D}$ such that $\wedge^3 X = 0$ and $Q_{n,p}^+(X) \cong Q_{n,p}^-(X)$, equipped with an isomorphism $\wedge^2 X \cong \mathbb{1}$. This equivalence is given by $E \mapsto E(V)$, where $V := F(T_1)$. In particular, the group of tensor automorphisms of the identity functor of Ver_{p^n} is $\mathbb{Z}/2$ for $p > 2$ and trivial for $p = 2$.*

Proof. This follows from Theorem 4.2 and [5], Corollary 3.5. The last statement follows since the group of automorphisms of V preserving the identification $\wedge^2 V \cong \mathbb{1}$ is $\mathbb{Z}/2$ for $p > 2$ and trivial for $p = 2$. $\qquad\square$

Remark 4.8. The condition that $\dim X = 2$ in [5], Corollary 3.5 is redundant since this follows in any characteristic from the conditions $\wedge^2 X \cong \mathbb{1}$, $\wedge^3 X = 0$. Indeed, let $d = \dim X$. Then for $p > 3$ we have $\frac{d(d-1)}{2} = 1$, $\frac{d(d-1)(d-2)}{6} = 0$, so $d = 2$. For $p = 3$ we have $\frac{d(d-1)}{2} = 1$, so $d^2 - d - 2 = 0$, which in characteristic 3 can be written as $(d-2)^2 = 0$, hence again $d = 2$. Finally, for $p = 2$ we have $\dim(\wedge^3 X) = \dim(\wedge^2 X)d$ (see for example [23]), which implies that $d = 0 = 2$.

4.4. **The lift of Ver_{p^n} to characteristic zero.** Let $W(\mathbf{k})$ be the ring of Witt vectors of **k**, K_0 its fraction field, and $\overline{K_0}$ the algebraic closure of K_0. Fix n and let $\zeta = \zeta_{n,p} \in \overline{K_0}$ be a primitive p^n-th root of unity. Consider the discrete valuation ring $R_{n,p} := W(\mathbf{k})[\zeta]$ with uniformizing element $\zeta - 1$. Let K be the fraction field of $R_{n,p}$ (a ramified extension of K_0 of degree $p^{n-1}(p-1)$). In this subsection we will show that the category $\mathrm{Ver}_{p^n}(\mathbf{k})$ admits a flat deformation to a braided category over $R_{n,p}$, whose generic fiber is the semisimple Verlinde category $\mathrm{Ver}_{p^n}(K)$ associated to the quantum group SL_2 at a p^n-th root of unity. To this end, we will use the theory of splitting objects for categories defined over a complete DVR developed in Subsection 2.11.

Let $TL(\zeta)$ be the lopsided version of the Temperley-Lieb category with parameter $\delta = \zeta^{1/2} + \zeta^{-1/2}$, see [31], Subsection 2.5; it is defined over $\mathbb{Z}[\zeta]$, hence over $R_{n,p}$, and we'll use the $R_{n,p}$-linear version. Then $TL(\zeta)$ is a flat braided monoidal Karoubian category over $R_{n,p}$ with indecomposables $\widetilde{T}_i$, $i \geq 0$. Also, $TL(\zeta)_{\mathbf{k}} = \mathcal{T}_p$ is the symmetric category of tilting modules for $SL_2(\mathbf{k})$, while $TL(\zeta)_K$ is (the lopsided version of) the braided category of tilting modules for the quantum group $SL_2^{\zeta^{1/2}}$ over K. Moreover, $TL(\zeta)$ has a tensor ideal $\mathcal{I}(\zeta)$ generated by the Steinberg module $\mathbf{St} = \widetilde{T}_{p^n-1}$ over the quantum group, which specializes to $\mathcal{I}_n$ over $\mathbf{k}$, and localizes to $\mathcal{I}(\zeta)_K := K \otimes_{R_{n,p}} \mathcal{I}(\zeta)$ over K (note that other tensor ideals $\mathcal{I}_k$, $k \neq n$, do not lift to tensor ideals in $TL(\zeta)$). Thus we can define the flat braided monoidal category $\mathcal{T}(\zeta) := TL(\zeta)/\mathcal{I}(\zeta)$ over $R_{n,p}$ with indecomposables $\widetilde{T}_i$, $0 \leq i \leq p^n - 2$, whose reduction $\mathcal{T}(\zeta)_{\mathbf{k}}$ is the symmetric category $\mathcal{T}_{n,p}$ with indecomposables T_i, $0 \leq i \leq p^n - 2$, while the localization $\mathcal{T}_K$ is the split semisimple braided Verlinde category $\mathrm{Ver}_{p^n}(K)$ (the semisimplification of $TL(\zeta)_K$, see [3], Subsection 3.3 and references therein), with simple objects $T_i^\zeta = W_i^\zeta = L_i^\zeta$, $0 \leq i \leq p^n - 2$, the Weyl modules over the quantum group (which are also tilting and simple modules). Thus the objects $\widetilde{T}_i$, $p^{n-1} - 1 \leq i \leq p^n - 2$ are splitting objects of $\mathcal{T}(\zeta)$.

Remark 4.9. Note that for $p > 2$, since ζ is a root of unity of odd order, the category $TL(\zeta)_K$ is **not** modular (unlike the situation of [3], Subsection 3.3); rather, its symmetric (Müger) center is sVec_K generated by $T_0^\zeta = \mathbf{1}$ and $T_{p^n-2}^\zeta = \psi$. In fact, we have $\mathrm{Ver}_{p^n}(K) = \mathrm{Ver}_{p^n}^+(K) \boxtimes \mathrm{sVec}_K$ as braided categories, where $\mathrm{Ver}_{p^n}^+(K)$ is the (modular) category generated by the objects of $\mathrm{Ver}_{p^n}(K)$ with even highest weight.

Remark 4.10. The construction of this subsection gives another proof of Theorem 4.2(i) which works for any p and does not use [5]. Namely, Theorem 4.2(i) follows from Corollary 2.61, since the category $\mathrm{Ver}_{p^n}(K)$ is semisimple.

4.5. **The Frobenius-Perron dimension and the Grothendieck ring of** Ver_{p^n}.

Proposition 4.11. *(i) We have*

$$\mathrm{FPdim}(\mathrm{Ver}_{p^n}) = \frac{p^n}{2\sin^2(\frac{\pi}{p^n})}.$$

(ii) The Cartan matrix of Ver_{p^n} *is given by* $C = DD^T$, *where* D *is its decomposition matrix (with respect to its lift to characteristic zero).*

Proof. (i) This follows from Proposition 2.60(i) since

$$\mathrm{FPdim}(\mathrm{Ver}_N(K)) = \sum_{k=1}^{N-1} \frac{\sin^2(\frac{\pi k}{N})}{\sin^2(\frac{\pi}{N})} = \frac{N}{2\sin^2(\frac{\pi}{N})}.$$

(ii) This follows from Corollary 2.61, as the category $\mathrm{Ver}_{p^n}(K)$ is semisimple. $\square$

Corollary 4.12. *If* $p > 2$ *then we have* $\mathrm{Ver}_{p^n} = \mathrm{Ver}_{p^n}^+ \boxtimes \mathrm{sVec}$. *Thus the Grothendieck ring of* Ver_{p^n} *is given by* $\mathrm{Gr}(\mathrm{Ver}_{p^n}) = \mathcal{O}_{n,p}\mathbb{Z}/2 = \mathcal{O}_{n,p}[g]/(g^2 - 1)$.

Proof. The first statement holds because it holds for the category $\mathrm{Ver}_{p^n}(K)$ (cf. Remark 4.9). Namely, the nontrivial invertible object of $\mathrm{Ver}_{p^n}(K)$ has to descend to an invertible object over $\mathbf{k}$ (as it has FPdim $= 1$). The second statement then follows from Theorem 4.5(iv). $\square$

4.6. The Cartan matrix and the decomposition matrix of Ver_{p^n}**.** Now we can compute the decomposition matrix D (and hence the Cartan matrix C) of Ver_{p^n}. Recall that the indecomposable projectives of Ver_{p^n} are $\mathbf{P}_i = F(T_i)$ for $p^{n-1} - 1 \leq i \leq p^n - 2$. So we may label the simple objects by the same integers and denote them by $\mathbf{L}_i$; thus $\mathbf{P}_i$ is the projective cover of $\mathbf{L}_i$. Also we have the *extended decomposition matrix* $\widetilde{D}$, a square matrix of size $p^n - 1$ with entries d_{ij}, $0 \leq i, j \leq p^n - 2$, where $d_{ij} := [(\widetilde{T}_i)_K : W_j^\zeta]$ is the multiplicity of W_j^ζ in the reduction of $\widetilde{T}_i$ to K. Then D is the submatrix of $\widetilde{D}$ consisting of the rows labeled by $p^{n-1} - 1 \leq i \leq p^n - 2$.

Note that the characters of W_j^ζ are $[j+1]_x = \frac{x^{j+1} - x^{-j-1}}{x - x^{-1}}$. Thus if $\chi_i(x)$ is the character of T_i as an $SL_2(\mathbf{k})$-module then the entries d_{ij} of $\widetilde{D}$ (hence D) are determined from the formula

$$(4.5) \qquad\qquad \chi_i(x) = \sum_j d_{ij} \frac{x^{j+1} - x^{-j-1}}{x - x^{-1}}.$$

Proposition 4.13. *(i) We have*

$$\sum_j d_{ij} x^{j+1} = (\chi_i(x)(x - x^{-1}))_+,$$

where the subscript $+$ denotes the part with positive powers of x.

(ii) The decomposition matrix D has maximal rank $p^{n-1}(p - 1)$;

(iii) The entries of the Cartan matrix C are $c_{ij} = (\chi_i, \chi_j)$ (the inner product of SL_2-characters).

(iv) The Cartan matrix C of Ver_{p^n} is positive definite, hence nondegenerate.

Proof. (i) is immediate from formula (4.5). (ii) follows since $d_{ij} = 0$ if $j > i$. (iii) follows from (i) and Proposition 4.11(ii). Finally, (iv) follows from (ii), (iii). $\qquad \square$

Corollary 4.14. *The Frobenius-Perron dimensions $d_i = \mathrm{FPdim}(\mathbf{L}_i)$ of simple objects of Ver_{p^n} are determined uniquely by the system of equations*

$$C\mathbf{d} = \mathbf{p},$$

where $\mathbf{d} = (d_i)$, and $\mathbf{p} = (p_i)$ is the vector of Frobenius-Perron dimensions p_i of $\mathbf{P}_i$, $p^{n-1} - 1 \leq i \leq p^n - 2$. That is,

$$\mathbf{d} = C^{-1}\mathbf{p}.$$

Proof. This follows from Proposition 4.13(iv). $\qquad \square$

The vector $\mathbf{p}$ can be computed from the formula $\mathbf{p} = D\mathbf{w}$, where $w_j = [j+1]_q = \frac{\sin(\pi(j+1)/p^n)}{\sin(\pi/p^n)}$ are the Frobenius-Perron dimensions of the Weyl modules W_j^ζ in $\mathrm{Ver}_{p^n}(K)$. Thus we get

$$\mathbf{d} = C^{-1}D\mathbf{w} = (DD^T)^{-1}D\mathbf{w}$$

So we obtain

Proposition 4.15. *We have $\mathbf{d} = (DD^T)^{-1}D\mathbf{w}$, where $w_j = [j+1]_q$ for $0 \leq j \leq p^n - 2$ and d_{ij} are the coefficients of $\chi_i(x)(x - x^{-1})$ for $p^{n-1} - 1 \leq i \leq p^n - 2$.*

This determines the Grothendieck ring of Ver_{p^n} with $\mathbb{Z}_+$-basis since the Frobenius-Perron dimension defines an isomorphism of this ring with $\mathcal{O}_{n,p}$.

Remark 4.16. We indicate here an alternative approach to the results of Subsections 4.4, 4.6 which avoids using the lifting to characteristic zero. Namely one can compute the entries of the Cartan matrix directly as

$$c_{ij} = \dim \mathrm{Hom}_{\mathrm{Ver}_{p^n}}(\mathbf{P}_i, \mathbf{P}_j) = \dim \mathrm{Hom}_{\mathcal{T}_p}(T_i, T_j)$$

using Proposition 3.5 and Theorem 4.2. However, it is well known that $\dim \mathrm{Hom}_{\mathcal{T}_p}(T_i, T_j) = (\chi_i, \chi_j)$, see for example [2, Section 3A], so we get the formula $C = DD^T$ as in Proposition 4.11 (ii) where D is *defined* by equation (4.5). This implies that Propositions 4.13 and 4.15 hold and an easy computation gives the formula from Proposition 4.11 (i).

The matrix $\widetilde{D}$ (hence D) can now be easily computed via Donkin's explicit algorithm, see Proposition 3.1(iii) and [29, E.9]. However, there is an even more explicit formula for $\widetilde{D}$, given in [33]. Namely, given a number $a \in [p^{n-1}, p^n - 1]$ written in base p as $\overline{a_1 \ldots a_n}$ (so $a_1 \neq 0$), we say that b is a *descendant* of a if $b = a_1 p^{n-1} \pm a_2 p^{n-2} \pm \ldots \pm a_n$ for some choice of signs (clearly, in this case $b \in [1, p^n - 1]$). Thus, every number a has 2^ℓ distinct descendants, where ℓ is the number of nonzero digits among $a_2, \ldots, a_n$.

Proposition 4.17. [33] *One has $d_{ij} = 1$ if $j + 1$ is a descendant of $i + 1$, otherwise $d_{ij} = 0$.*

This yields a very explicit description of the Cartan matrix of Ver_{p^n}.

Corollary 4.18. *The entry c_{ij} of the Cartan matrix is the number of common descendants of $i + 1$ and $j + 1$.*

Proof. This follows from Proposition 4.11(ii) and Proposition 4.17. $\qquad\square$

4.7. **An explicit formula for the Cartan matrix of Ver_{p^n}.** Let us now derive an explicit formula for the Cartan matrix $C = C_{n,p}$ of the category Ver_{p^n}. We first consider the case $p = 2$. In this case let $C_{n,2}^+$ be the Cartan matrix of $\mathrm{Ver}_{2^n}^+$. Then it is shown in [5], Proposition 4.2 that

$$C_{n+1,2}^+ = \begin{pmatrix} 2C_{n,2}^+ & C_{n,2}^+ \\ C_{n,2}^+ & 2C_{n-1,2}^+ \end{pmatrix}$$

and

$$C_{n,2} = C_{n,2}^+ \oplus C_{n-1,2},$$

for an appropriate labeling of rows and columns, where $C_{1,2} = 1, C_{2,2}^+ = 2$. Hence

$$\begin{pmatrix} C_{n+1,2}^+ \\ C_{n,2} \end{pmatrix} = \left(\begin{array}{cc|cc} 2 & 1 & 0 & 0 \\ 1 & 0 & 0 & 2 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right) \otimes \begin{pmatrix} C_{n,2}^+ \\ C_{n-1,2} \end{pmatrix}.$$

Thus we get

Proposition 4.19. *One has*

$$\begin{pmatrix} C_{n+1,2}^+ \\ C_{n,2} \end{pmatrix} = \left(\begin{array}{cc|cc} 2 & 1 & 0 & 0 \\ 1 & 0 & 0 & 2 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right)^{\otimes n-1} \otimes \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

This can also be derived from Corollary 4.18.

Now let us pass to the case $p > 2$, and compute $C_{n,p}$ in a somewhat different way. For two numbers a, b with exactly $m + 1$ digits in the base p expansion let $X_{ab}(m)$ be the number of common descendants of a, b. By Corollary 4.18, $C_{n+1,p} = X(n)$ (where we label simple objects of Ver_{p^n} by highest weight of the projective cover). Let $Y_{ab}(m)$ be the number of descendants $\widetilde{a}$ of a such that $\widetilde{b} := \widetilde{a} - 1$ is a descendant of b. We will often write $X_{ab}(m)$, $Y_{ab}(m)$ as X_{ab}, Y_{ab} respectively, using that m can be recovered as the number of digits of a and b. It is clear that for $m = 0$ we have $X_{ab} = \delta_{ab}$ and $Y_{ab} = \delta_{a,b+1}$.

Let $a = \overline{a_m...a_1a_0}$, $b = \overline{b_m...b_1b_0}$ where $a_m, b_m \neq 0$, and let $a' = \overline{a_m...a_1}, b' = \overline{b_m...b_1}$.

Proposition 4.20. (i) If $a_0 = b_0 = 0$ then $X_{ab} = X_{a'b'}$.
(ii) If $a_0 = b_0 \neq 0$ then $X_{ab} = 2X_{a'b'}$.
(iii) If $a_0 + b_0 = p$ then $X_{ab} = Y_{a'b'} + Y_{b'a'}$.
(iv) Otherwise $X_{ab} = 0$.

Proof. (i), (ii), (iv) are easy, let us prove (iii). Having a common descendant means that $\widetilde{a}'p \pm a_0 = \widetilde{b}'p \pm b_0$ for some choice of signs. The possible choices are $+, -$, which gives $\widetilde{a}' - \widetilde{b}' = -1$ and $-, +$, which gives $\widetilde{a}' - \widetilde{b}' = 1$, as desired. □

Proposition 4.21. (i) If $a_0 - b_0 = \pm 1$ then $Y_{ab} = X_{a'b'}$.
(ii) If $a_0 - b_0 = p - 1$ then $Y_{ab} = Y_{a'b'}$.
(iii) If $a_0 + b_0 = p + 1$ then $Y_{ab} = Y_{b'a'}$.
(iv) Otherwise $Y_{ab} = 0$.

Proof. Similar to Proposition 4.20. □

Let $Z = Y + Y^T$.

Corollary 4.22. (i) If $a_0 = b_0 = 0$ then $X_{ab} = X_{a'b'}$.
(ii) If $a_0 = b_0 \neq 0$ then $X_{ab} = 2X_{a'b'}$.
(iii) If $a_0 + b_0 = p$ then $X_{ab} = Z_{a'b'}$.
(iv) Otherwise $X_{ab} = 0$.

Corollary 4.23. (i) If $a_0 - b_0 = \pm 1$ then $Z_{ab} = 2X_{a'b'}$.
(ii) If $a_0 + b_0 = p - 1$ then $Z_{ab} = Z_{a'b'}$.
(iii) If $a_0 + b_0 = p + 1$ then $Z_{ab} = Z_{a'b'}$.
(iv) Otherwise $Z_{ab} = 0$.

Define the following $p \times p$ matrices A, B, S, D with rows and columns labeled by $0, ..., p-1$. Let A be defined by $A_{ij} = \delta_{i,j-1} + \delta_{i,j+1}$, B be defined by $B_{ij} = \delta_{i+j,p-1} + \delta_{i+j,p+1}$, S be the matrix with $S_{ij} = \delta_{i+j,p}$, and $D := \mathrm{diag}(1, 2, 2, ..., 2)$.

Corollary 4.24. We have $X(0) = \mathrm{Id}_{p-1}$, $Z(0) = \overline{A}$, where $\overline{A}$ is obtained from A by deleting the 0-th row and column, and for $m \geq 1$

$$X(m) = D \otimes X(m-1) + S \otimes Z(m-1), \quad Z(m) = 2A \otimes X(m-1) + B \otimes Z(m-1).$$

Thus we get

$$\begin{pmatrix} X(m) \\ Z(m) \end{pmatrix} = \begin{pmatrix} D & S \\ 2A & B \end{pmatrix} \otimes \begin{pmatrix} X(m-1) \\ Z(m-1) \end{pmatrix}.$$

So we get

Proposition 4.25. *One has*

$$\begin{pmatrix} C_{n,p} \\ Z(n-1) \end{pmatrix} = \begin{pmatrix} D & S \\ 2A & B \end{pmatrix}^{\otimes n-1} \otimes \begin{pmatrix} \mathrm{Id}_{p-1} \\ A \end{pmatrix}$$

Example 4.26. For $p = 3$ we have

$$\begin{pmatrix} C_{n,3} \\ Z(n-1) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 & 0 & 1 \\ 2 & 0 & 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 & 0 & 1 \end{pmatrix}^{\otimes n-1} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Thus, for $n = 2$ we get

$$C_{2,3} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 & 0 & 2 \end{pmatrix}$$

which after permuting rows and columns falls into a direct sum of the Cartan matrices for the four blocks of Ver_{3^2}, two copies of 1 and two copies of $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$.

Corollary 4.27. *All entries of the Cartan matrix $C_{n,p}$ are either 0 or 2^m with $0 \le m \le n-1$.*

Proof. It follows from Propositions 4.19, 4.25 that while computing entries of $C_{n,p}$, we never have to add nonzero numbers and only have to multiply them (for example, for $p > 2$ this is so because the matrices D and S don't have nonzero entries in the same position, and A, B don't either). This implies the statement. $\square$

Example 4.28. The Cartan matrix entry $c_{\mathbb{1},\mathbb{1}}$ of $C = C_{n,p}$ equals 2^{n-1}.

Remark 4.29. 1. For $p = 2$ Corollary 4.27 is proved in [5], Proposition 4.2.

2. Here is another proof of Corollary 4.27. Namely, by Proposition 3.1(iii), it suffices to show that for $p - 1 \le a, c \le 2p - 2$ the space

$$\mathrm{Hom}(T_a \otimes T_b^{(1)}, T_c \otimes T_d^{(1)}) = \mathrm{Hom}(T_a \otimes T_c, (T_b \otimes T_d)^{(1)})$$

has dimension 0 or 2^m, $0 \le m \le n - 1$. But among the direct summands of $T_a \otimes T_c$ only T_{2p-2} and T_{3p-2} have quotients on which the Frobenius kernel acts trivially (note that T_{3p-2} appears only if a and c have different parities and the corresponding quotient is $T_1^{(1)}$). Thus we have the following recursive formulas:

$$(4.6) \qquad \dim \mathrm{Hom}(T_a \otimes T_b^{(1)}, T_c \otimes T_d^{(1)}) = \begin{cases} 2 \dim \mathrm{Hom}(T_b, T_d) & \text{if} \quad a = c \ne p - 1, \\ \dim \mathrm{Hom}(T_b, T_d) & \text{if} \quad a = c = p - 1, \\ \dim \mathrm{Hom}(T_b, V \otimes T_d) & \text{if} \quad a + c = 3p - 2. \\ 0 \text{ otherwise.} \end{cases}$$

We can combine formula (4.6) with (3.3) and (3.2). One observes that in all cases the number $\dim \mathrm{Hom}(T_b, V \otimes T_d)$ equals to $\dim \mathrm{Hom}(T_b, T_{d\pm1})$ or $\dim \mathrm{Hom}(T_{b\pm1}, T_d)$ as the summands in

(3.3) are typically in different linkage classes. This implies that all nonzero entries of the Cartan matrix are powers of 2, and the power is $\leq n - 1$.

4.8. The objects $\mathbb{T}_i = F(T_i)$. Let us now discuss the properties of the objects $\mathbb{T}_i := F(T_i)$. Recall that for $p^{n-1} - 1 \leq i \leq p^n - 2$ the objects $\mathbb{T}_i = \mathbf{P}_i$ are precisely the indecomposable projectives of the category Ver_{p^n}.

Next consider the case $i < p$. Note that for such i we have $\mathbb{T}_i = S^i \mathbb{T}_1$.

Lemma 4.30. *For $i < p$ the objects $\mathbb{T}_i$ are simple. The projective cover of $\mathbb{T}_i$ is $\mathbb{T}_{2p^{n-1}-2-i}$.*

Proof. We claim that if, for $j \in [p^{n-1} - 1, p^n - 2]$, we have $\text{Hom}(T_j, T_i) \neq 0$ then it is 1-dimensional and $j = 2p^{n-1} - 2 - i$. Indeed,

$$\text{Hom}(T_j, T_i) = \text{Hom}(\mathbf{1}, T_i \otimes T_j^*) = \text{Hom}(\mathbf{1}, T_i \otimes T_j).$$

Thus by Lemma 3.6 $\text{Hom}(T_j, T_i) \neq 0$ if and only if $T_i \otimes T_j$ contains $T_{2p^{n-1}-2}$ as a direct summand. Now the claim follows from Proposition 3.13.

It follows that for any $\mathbb{T}_i$ there exists a unique indecomposable projective P such that $\text{Hom}(P, \mathbb{T}_i) \neq 0$ and this Hom–space is one dimensional. This implies the statement. $\qquad\square$

Lemma 4.31. *The object $\mathbb{T}_p$ has length 3 and composition series $[\mathbb{T}_{p-2}, U, \mathbb{T}_{p-2}]$ for some simple U.*

Proof. We have $\mathbb{T}_p = \mathbb{T}_1 \otimes \mathbb{T}_{p-1}$. Thus $\text{FPdim}(\mathbb{T}_p) = \frac{(q^2 - q^{-2})(q^p - q^{-p})}{(q - q^{-1})^2}$. But we have nonzero morphisms $\mathbb{T}_p \to \mathbb{T}_{p-2}$ and $\mathbb{T}_{p-2} \to \mathbb{T}_p$, so by Lemma 4.30, $\mathbb{T}_p$ has composition series as claimed with some object U. It remains to show that U is simple. To this end, note that

$$\text{FPdim}(U) = \text{FPdim}(\mathbb{T}_p) - 2\,\text{FPdim}(\mathbb{T}_{p-2}) =$$

$$= \frac{(q^2 - q^{-2})(q^p - q^{-p}) - 2(q - q^{-1})(q^{p-1} - q^{-p+1})}{(q - q^{-1})^2} = q^p + q^{-p}.$$

Thus $\text{FPdim}(U) < 2$, so U is simple. $\qquad\square$

4.9. Some properties of the Frobenius functor. Consider the Frobenius functor

$$\mathbb{F} : \text{Ver}_{p^n} \to \text{Ver}_{p^n}^{(1)} \boxtimes \text{Ver}_p$$

defined in [24] (note that $\text{Ver}_{p^n}^{(1)} \cong \text{Ver}_{p^n}$ since the categories $\mathcal{T}_{n,p}$ and Ver_{p^n} are defined over the prime field).

Proposition 4.32. *For $n \geq 2$ the functor $\mathbb{F}$ is not exact, i.e., the category Ver_{p^n} is not Frobenius exact in the sense of [24]. Equivalently, Ver_{p^n} is not locally semisimple in the sense of [12] (see [24], Remark 7.2).*

Proof. Since there are no tensor functors $\text{Ver}_{p^n} \to \text{Ver}_p$ (Theorem 4.5(v)), by [24, Section 8] the category Ver_{p^n} is not Frobenius exact. $\qquad\square$

In this section we discuss some properties of $\mathbb{F}$ restricted to $F(\mathcal{T}_{n,p}) \subset \text{Ver}_{p^n}$. We assume that $n > 1$; see [32, Example 3.8] for the case $n = 1$.

Corollary 4.33. *We have $\mathbb{F}(\mathbb{T}_1) = U \boxtimes \mathbb{1} \in \text{Ver}_{p^n} \boxtimes \text{Ver}_p$, where U is the object from Lemma 4.31.*

Proof. We have a decomposition $\mathbb{T}_1^{\otimes p} = \bigoplus_{i<p-2}\mathbb{T}_i \otimes \pi_i \oplus M$, where π_i are free $\mathbb{Z}/p$-modules and

$$M = (p-2)\mathbb{T}_{p-2} \oplus \mathbb{T}_p = (p-2)\mathbb{T}_{p-2} \oplus [\mathbb{T}_{p-2}, U, \mathbb{T}_{p-2}],$$

where the last equality comes from Lemma 4.31. This implies the statement. □

Corollary 4.33 implies that for $n > 1$ the functor $\mathbb{F}$ restricted to $F(\mathcal{T}_{n,p})$ lands in $\mathrm{Ver}_{p^n}\boxtimes\mathbb{1} \subset \mathrm{Ver}_{p^n} \boxtimes \mathrm{Ver}_p$. In what follows we identify $\mathrm{Ver}_{p^n} \boxtimes \mathbb{1}$ with Ver_{p^n}, so $\mathbb{F} : F(\mathcal{T}_{n,p}) \to \mathrm{Ver}_{p^n}$.

The following result should be compared with (3.1).

Proposition 4.34. *Let $n > 1$. For $p - 1 \leq a \leq 2p - 2$ and $b \in \mathbb{Z}_{\geq 0}$ we have*

$$(4.7) \qquad\qquad \mathbb{T}_a \otimes \mathbb{F}(\mathbb{T}_b) = \mathbb{T}_{a+pb}.$$

Proof. The morphisms $\mathbb{T}_{p-2} \to \mathbb{T}_p$ and $\mathbb{T}_p \to \mathbb{T}_{p-2}$ from the proof of Lemma 4.31 split after tensoring with $\mathrm{St} = \mathbb{T}_{p-1}$ by Proposition 3.2, so we get $\mathrm{St} \otimes U = \mathbb{T}_{2p-1}$ by an easy computation (where we denote $F(\mathrm{St})$ just by St for brevity).

Let $\mathcal{T}_{n,p}^1 = \mathcal{I}_1/\mathcal{I}_n \subset \mathcal{T}_{n,p}$ be the thick ideal generated by St. We prove now that $\mathrm{St} \otimes \mathbb{F}(\mathbb{T}_i) \in F(\mathcal{T}_{n,p}^1)$ by induction in i. The base case $i = 0$ is clear as $\mathbb{T}_0 = \mathbb{1}$. Assume that $\mathrm{St} \otimes \mathbb{F}(\mathbb{T}_i) \in F(\mathcal{T}_{n,p}^1)$ holds for some i. Then $\mathbb{T}_{i+1}$ is a direct summand of $\mathbb{T}_i \otimes \mathbb{T}_1$, so $\mathrm{St}\otimes\mathbb{F}(\mathbb{T}_{i+1})$ is a direct summand of $\mathrm{St} \otimes\mathbb{F}(\mathbb{T}_i)\otimes\mathbb{F}(\mathbb{T}_1) = \mathrm{St}\otimes U \otimes\mathbb{F}(\mathbb{T}_i) = \mathbb{T}_{2p-1}\otimes\mathbb{F}(\mathbb{T}_i)$. Since $\mathbb{T}_{2p-1}$ is a direct summand of $\mathbb{T}_p \otimes \mathrm{St}$, we see that $\mathrm{St} \otimes \mathbb{F}(\mathbb{T}_{i+1})$ is a direct summand of $\mathbb{T}_p \otimes \mathrm{St} \otimes \mathbb{F}(\mathbb{T}_i)$ and thus $\mathrm{St} \otimes \mathbb{F}(\mathbb{T}_{i+1}) \in F(\mathcal{T}_{n,p}^1)$ since the category $\mathcal{T}_{n,p}^1$ is closed under tensor products and direct summands.

For $a \geq p-1$ the object $\mathbb{T}_a$ is a direct summand of $\mathbb{T}_{a-(p-1)} \otimes \mathrm{St}$, so $\mathbb{T}_a \otimes \mathbb{F}(\mathbb{T}_b)$ is a direct summand of $\mathbb{T}_{a-(p-1)} \otimes \mathrm{St} \otimes \mathbb{F}(\mathbb{T}_b)$, so $\mathbb{T}_a \otimes \mathbb{F}(\mathbb{T}_b) \in F(\mathcal{T}_{n,p}^1)$ by the preceding paragraph.

We now prove the proposition by induction in b with the base case $b = 0$ being clear. Thus assume that (4.7) holds for all $b \leq m$. Let $S = S(m)$ be the multi-subset of $\mathbb{Z}_{\geq 0}$ such that

$$T_1 \otimes T_m = T_{m+1} \oplus \bigoplus_{s\in S}T_s.$$

It is clear that $s < m$ for any $s \in S$, so (4.7) holds for $b = s \in S$. We have

$$\mathrm{St}\otimes\mathbb{T}_a\otimes\mathbb{F}\Big(\mathbb{T}_{m+1}\oplus\bigoplus_{s\in S}\mathbb{T}_s\Big) = \mathrm{St}\otimes\mathbb{T}_a\otimes\mathbb{F}(\mathbb{T}_1\otimes\mathbb{T}_m) = \mathrm{St}\otimes U\otimes\mathbb{T}_a\otimes\mathbb{F}(\mathbb{T}_m) = \mathbb{T}_{2p-1}\otimes\mathbb{T}_{a+pm} =$$

$$F(T_{2p-1} \otimes T_{a+pm}) = F(T_{p-1} \otimes T_1^{(1)} \otimes T_a \otimes T_m^{(1)}) = \mathrm{St} \otimes F(T_a \otimes (T_1 \otimes T_m)^{(1)}) =$$

$$\mathrm{St} \otimes F\Big(T_a \otimes (T_{m+1} \oplus \bigoplus_{s\in S}T_s)^{(1)}\Big) = \mathrm{St} \otimes F(T_{a+p(m+1)} \oplus \bigoplus_{s\in S}T_{a+ps}) =$$

$$= \mathrm{St} \otimes (\mathbb{T}_{a+p(m+1)} \oplus \bigoplus_{s\in S}\mathbb{T}_{a+ps}).$$

Thus using the induction assumption for $b = s \in S$ we get

$$\mathrm{St} \otimes \mathbb{T}_a \otimes \mathbb{F}(\mathbb{T}_{m+1}) = \mathrm{St} \otimes \mathbb{T}_{a+p(m+1)},$$

that is (4.7) holds for $b = m + 1$ after tensoring with St.

Luckily we can cancel St: in the split Grothendieck ring $K(\mathcal{T}_{n,p})$ we have $K(\mathcal{T}_{n,p}^1) = [\mathrm{St}]K(\mathcal{T}_{n,p})$ and we get the cancellation by applying the following result to the ring $K(\mathcal{T}_{n,p})$ and $a = [\mathrm{St}]$: in a commutative semisimple ring the equality $a^2b_1 = a^2b_2$ implies $ab_1 = ab_2$. □

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4.10. The inclusion $\mathrm{Ver}_{p^{n-1}} \hookrightarrow \mathrm{Ver}_{p^n}$**.** In this section we use (4.7) to prove the following theorem.

Theorem 4.35. *For $n \geq 2$ we have a natural inclusion $H : \mathrm{Ver}_{p^{n-1}} \hookrightarrow \mathrm{Ver}_{p^n}$ as a tensor subcategory.*

Proof. Recall that we have a tensor functor $\mathbb{F} : F(\mathcal{T}_{n,p}) \to \mathrm{Ver}_{p^n}$.

Corollary 4.36. $\mathbb{F}(F(\mathrm{St}_{n-1})) = 0$ *and* $\mathbb{F}(F(\mathrm{St}_{n-2})) \neq 0$.

Proof. By (4.7) we have

$$\mathbb{F}(\mathbb{T}_{p^{n-1}-1}) \otimes \mathbb{T}_{p-1} = \mathbb{T}_{p-1+p(p^{n-1}-1)} = \mathrm{St}_n = 0$$

and

$$\mathbb{F}(\mathbb{T}_{p^{n-2}-1}) \otimes \mathbb{T}_{p-1} = \mathbb{T}_{p-1+p(p^{n-2}-1)} = \mathrm{St}_{n-1} \neq 0. \quad \square$$

Thus the functor $\mathbb{F} \circ F$ induces a faithful monoidal functor $\widetilde{H} : \mathcal{T}_{n-1,p} \to \mathrm{Ver}_{p^n}$, see Proposition 3.7 or [5], Lemma 3.7. But $\mathcal{T}_{n-1,p}$ has an abelian envelope $\mathrm{Ver}_{p^{n-1}}$ by Theorem 4.2. Thus the functor $\widetilde{H}$ extends to an (exact) tensor functor $H : \mathrm{Ver}_{p^{n-1}} \to \mathrm{Ver}_{p^n}$. This functor is automatically faithful, see [22, Remark 4.3.10]. We claim that it is also full, i.e., H is an embedding, see [22, 6.3].

Corollary 4.37. *The functor $\widetilde{H}$ is full. In particular the functor H is full on projective objects of $\mathrm{Ver}_{p^{n-1}}$.*

Proof. It is sufficient to show that the injective map $\mathrm{Hom}(\mathbb{1}, \mathbb{T}_b) \to \mathrm{Hom}(\mathbb{F}(\mathbb{1}), \mathbb{F}(\mathbb{T}_b))$ is surjective for any b. Thus by Lemma 3.6 we need to show that $\mathrm{Hom}(\mathbb{1}, \mathbb{F}(\mathbb{T}_b))$ is one dimensional for $b = 2p^l - 2, 0 \leq l \leq n - 1$ and zero otherwise. But since $\mathbb{1}$ is a quotient of $\mathbb{T}_{2p-2}^*$, we have

$$\dim \mathrm{Hom}(\mathbb{1}, \mathbb{F}(\mathbb{T}_b)) \leq \dim \mathrm{Hom}(\mathbb{T}_{2p-2}^*, \mathbb{F}(\mathbb{T}_b)) =$$
$$= \dim \mathrm{Hom}(\mathbb{1}, \mathbb{T}_{2p-2} \otimes \mathbb{F}(\mathbb{T}_b)) = \dim \mathrm{Hom}(\mathbb{1}, \mathbb{T}_{2p-2+pb}),$$

and the result follows from Lemma 3.6. $\qquad \square$

It follows that the functor H is full which proves the theorem. $\qquad \square$

Remark 4.38. Here is an alternative approach to the existence of the functor H. By Proposition 4.32 the category Ver_{p^n} for $n \geq 2$ is not Frobenius exact, so by [24], Proposition 7.6, $\mathbb{F}(Q) = 0$ for every projective $Q \in \mathrm{Ver}_{p^n}$. This means that the monoidal functor $\mathbb{F} \circ F : \mathcal{T}_{n,p} \to \mathrm{Ver}_{p^n} \boxtimes \mathrm{Ver}_p$ factors through $\mathcal{T}_{n-k,p}$ for some $k > 0$. But $\mathcal{T}_{n-k,p}$ has an abelian envelope $\mathrm{Ver}_{p^{n-k}}$. Thus the functor $\mathbb{F} \circ F$ factors through a tensor functor $\widetilde{H} : \mathrm{Ver}_{p^{n-k}} \to \mathrm{Ver}_{p^n} \boxtimes \mathrm{Ver}_p$. By Corollary 4.33, this functor lands in Ver_{p^n} (as $\mathbb{T}_1$ generates $\mathrm{Ver}_{p^{n-k}}$). Recall that the Frobenius-Perron dimensions of the objects of $\mathrm{Ver}_{p^{n-k}}$ are contained in the ring $\mathcal{O}_{n-k,p}$, see Theorem 4.5 (iv). However by Corollary 4.33 the image of the functor $\widetilde{H}$ contains an object $U \boxtimes \mathbb{1}$ such that its Frobenius-Perron dimension generates the ring $\mathcal{O}_{n-1,p}$, see the proof of Lemma 4.31. We conclude that $k = 1$.

In what follows we consider $\mathrm{Ver}_{p^{n-1}}$ as a subcategory of Ver_{p^n} via the functor H.

Corollary 4.39. *Assume $n \geq 2$. Let L be a simple object of $\mathrm{Ver}_{p^{n-1}}$ with projective cover $\mathbb{T}_r$ where $r \in [p^{n-2} - 1, p^{n-1} - 2]$. Then projective cover of L considered as an object of Ver_{p^n} is $\mathbb{T}_{2p-2+pr}$.*

Proof. By (4.7) we have $\mathbb{T}_{2p-2+pr} = \mathbb{F}(\mathbb{T}_r) \otimes \mathbb{T}_{2p-2}$, so $H(\mathbb{T}_r) = \mathbb{F}(\mathbb{T}_r)$ is a quotient of $\mathbb{T}_{2p-2+pr}$ as $\mathbb{T}_{2p-2}$ surjects onto $\mathbb{1}$. $\qquad\square$

Proposition 4.40. *The functor $X \mapsto H(X) \otimes \mathbb{T}_{p-1}$ is an (additive) equivalence of $\mathrm{Ver}_{p^{n-1}}$ with a direct summand of the category Ver_{p^n}.*

Proof. It is clear that this functor is exact and faithful. We claim that it sends projective objects to projective objects and that it is full on projective objects. The first assertion is immediate from (4.7) where we substitute $a = p - 1$. The second assertion is an immediate consequence of (4.6). The proposition is proved. $\qquad\square$

4.11. **The tensor product theorem for** Ver_{p^n}. Here is a generalization of Lemma 4.30.

Proposition 4.41. *Assume $n \geq 2$. Let L be a simple object of $\mathrm{Ver}_{p^{n-1}}$ with projective cover $\mathbb{T}_r$, $r \in [p^{n-2} - 1, p^{n-1} - 2]$. Then for any $i \in [0, p-1]$ the object $\mathbb{T}_i \otimes L$ of Ver_{p^n} is simple with projective cover $\mathbb{T}_{2p-2-i+pr}$. Any simple object of Ver_{p^n} is of this form.*

Proof. We claim that there is a unique $j \in [p^{n-1} - 1, p^n - 2]$ such that $\mathrm{Hom}(\mathbb{T}_j, \mathbb{T}_i \otimes L) \neq 0$ and this Hom–space is one-dimensional. But

$$\mathrm{Hom}(\mathbb{T}_j, \mathbb{T}_i \otimes L) = \mathrm{Hom}(\mathbb{T}_j \otimes \mathbb{T}_i^*, L) = \mathrm{Hom}(\mathbb{T}_j \otimes \mathbb{T}_i, L)$$

and by Corollary 4.39 this space is nonzero if and only if $\mathbb{T}_j \otimes \mathbb{T}_i$ contains $\mathbb{T}_{2p-2+pr}$ as a direct summand. The first statement now follows from Proposition 3.13. The second statement is a consequence of the count of the simple modules. $\qquad\square$

By iterating Theorem 4.35 we get a sequence of embeddings $\mathrm{Ver}_p \subset \mathrm{Ver}_{p^2} \subset \ldots \subset \mathrm{Ver}_{p^n}$. For any $r \geq 2$ and $i \in [0, p-1]$ let us denote by $\mathbb{T}_i^{[r]}$ the simple object of the category Ver_{p^r} as in Lemma 4.30 possibly considered as an object of Ver_{p^s} with $s \geq r$. Similarly, for $i \in [0, p-2]$ let $\mathbb{T}_i^{[1]}$ denote the simple object of Ver_p possibly considered as an object of Ver_{p^s} with $s \geq 1$. Then Theorem 4.35 implies

Proposition 4.42. *For $0 \leq i \leq p-1$ and $m \geq 2$, we have $\mathbb{F}(\mathbb{T}_i^{[m]}) = \mathbb{T}_i^{[m-1]}$ except if $m = 2$ and $i = p - 1$, in which case $\mathbb{F}(\mathbb{T}_i^{[m]}) = 0$.*

For a sequence of base p digits $a_1, a_2, \ldots, a_n \in [0, p-1]$ let

$$\overline{a_1 a_2 \ldots a_n} = a_1 p^{n-1} + \ldots + a_{n-1} p + a_n$$

be the corresponding integer in base p. For a base p digit a let $a^* := p - 1 - a$. By iterating Proposition 4.41 we get the following result (cf. Steinberg's tensor product theorem [29, 3.17] and [5, Theorem 2.1 (ix)]):

Theorem 4.43. *Let $i_1 \in [0, p-2]$, $i_2, \ldots, i_n \in [0, p-1]$, and $i = \overline{i_1 \ldots i_n} \in [0, p^{n-1}(p-1) - 1]$ the corresponding integer written in base p. Then the object $L_i := \mathbb{T}_{i_1}^{[1]} \otimes \mathbb{T}_{i_2}^{[2]} \otimes \ldots \otimes \mathbb{T}_{i_n}^{[n]} \in \mathrm{Ver}_{p^n}$ is simple, and any simple object of Ver_{p^n} can be uniquely written in this form.[4] The projective cover of L_i is $P_i := \mathbf{P}_s = \mathbb{T}_s$ with*

$$s = s(i) := p^{n-1} - 1 + \overline{i_1 i_2^* \ldots i_n^*};$$

in other words, $L_i = \mathbf{L}_s$. Moreover, the simple objects of $\mathrm{Ver}_{p^n}^+$ are the L_i for even i.

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[4]The object L_i should not be mixed up with the simple module over $SL_2(\mathbf{k})$ with highest weight i which we also denoted by L_i. We hope this will not lead to confusion since the simple $SL_2(\mathbf{k})$-modules will not be used from this point on.

Thus, Theorem 4.43 gives a new way of labeling the simple objects of Ver_{p^n}, and the transition between the two labelings is given by the formula $L_i = \mathbf{L}_{s(i)}$. For example, for $p = 2$ we have $i_1 = 0$, so $s(i) = 2^n - 2 - i$, so $L_i = \mathbf{L}_{2^n-2-i}$ for $0 \leq i \leq 2^{n-1} - 1$. Also, $L_{p^{n-1}(p-2)} = \mathbf{L}_{p^n-2}$ is the generator of $\mathrm{sVec} \subset \mathrm{Ver}_{p^n}$.

Remark 4.44. For $p = 2$, the simple objects $L_1^{[m]}$ were denoted in [5] by X_{m-1}, $2 \leq m \leq n$. Thus for a subset $S \in [1, n-1]$, the object $X_S \in \mathrm{Ver}_{2^n} = \mathcal{C}_{2n-2}$ from [5] is L_i where $i = \sum_{k \in S} 2^{n-1-k}$.

Corollary 4.45. *(i) The categorical dimensions of simple objects of Ver_{p^n} are given by the formula*

$$\dim(L_i) = \prod_{k=1}^{n}(i_k + 1) \in \mathbf{k}.$$

(ii) The Frobenius-Perron dimensions of simple objects of Ver_{p^n} are given by the formula

$$\mathrm{FPdim}(L_i) = \prod_{k=1}^{n}[i_k + 1]_{q^{p^{n-k}}}.$$

Thus $\mathrm{Ver}_{p^n}^+$ is a categorification of the ring $\mathcal{O}_{n,p}$ with basis $b_i = \prod_{k=1}^{n}[i_k + 1]_{q^{p^{n-k}}}$ for even $i \in [0, p^{n-1}(p-1) - 1]$.

Proof. This follows since $\dim(T_j) = j + 1$ and $\mathrm{FPdim}(\mathbb{T}_j) = [j+1]_q$ in Ver_{p^n}. $\square$

Corollary 4.46. *The Frobenius functor acts on the simple objects of Ver_{p^n} as follows. Let $0 \leq r \leq p - 2$ and $0 \leq b \leq p^{n-1} - 1$. Then if $b \geq p^{n-1} - p^{n-2}$, we have $\mathbb{F}(L_{rp^{n-1}+b}) = 0$. Otherwise $\mathbb{F}(L_{rp^{n-1}+b}) = L_b^{[n-1]} \boxtimes L_r$ if r is even and $\mathbb{F}(L_{rp^{n-1}+b}) = (L_{p-2}^{[1]} \otimes L_b^{[n-1]}) \boxtimes L_{p-2-r}$ if r is odd.*

Proof. This follows from Proposition 4.42, Theorem 4.43, [32, Example 3.8] and the fact that $\mathbb{F}$ is a monoidal functor, see [24]. $\square$

Remark 4.47. The following remark is a generalization to odd characteristic of [5], Remark 3.14. The category Ver_{p^n} is filtered by Serre subcategories $\mathrm{Ver}_{p^n}^r$, $r \geq 0$, whose simple objects are those occurring in $V^{\otimes j}$, $j \leq r$ (where we recall that $V = \mathbb{T}_1$). These categories are not closed under tensor product, but we have partial tensor products $\mathrm{Ver}_{p^n}^j \times \mathrm{Ver}_{p^n}^{r-j} \to \mathrm{Ver}_{p^n}^r$ with associativity isomorphisms satisfying the pentagon relation. Moreover, we claim that the Frobenius functor $\mathbb{F}$ maps $\mathrm{Ver}_{p^n}^r$ to $\mathrm{Ver}_{p^{n-1}}^r$ when n is large enough compared to r, preserving this partial tensor product. Indeed, for simple objects this follows from Corollary 4.46, and for arbitrary objects it follows since for a filtered object Y any composition factor of $\mathbb{F}(Y)$ is also one of $\mathbb{F}(\mathrm{gr}(Y))$ (see [24], Proposition 3.6).

Example 4.48. Let $p \leq a \leq 2p - 2$. Then by Theorem 4.43 the projective cover of $\mathbb{T}_1^{[n-1]} \otimes \mathbb{T}_{a-p}^{[n]}$ is $\mathbb{T}_{2p^{n-1}-2-a}$. We have a nonzero morphism $\mathbb{T}_{2p^{n-1}-2-a} \otimes \mathbb{T}_a \to \mathbb{T}_{2p^{n-1}-2} \to \mathbb{1}$, hence a nonzero morphism $\mathbb{T}_{2p^{n-1}-2-a} \to \mathbb{T}_a$. Hence $\mathbb{T}_1^{[n-1]} \otimes \mathbb{T}_{a-p}^{[n]}$ is a simple constituent of $\mathbb{T}_a$. Similarly, using the nonzero morphism $\mathbb{T}_a \otimes \mathbb{T}_{2p-2-a} \to \mathbb{T}_{2p-2} \to \mathbb{1}$, we see that the simple object $\mathbb{T}_{2p-2-a}$ (see Lemma 4.30) is both a subobject and a quotient object of $\mathbb{T}_a$. By comparing the Frobenius-Perron dimensions we deduce that the object $\mathbb{T}_a$ has composition series

$$\mathbb{T}_a = [\mathbb{T}_{2p-2-a}, \mathbb{T}_1^{[n-1]} \otimes \mathbb{T}_{a-p}, \mathbb{T}_{2p-2-a}],$$

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generalizing Lemma 4.31.

Corollary 4.49. *The only braided tensor functor $E : \mathrm{Ver}_{p^n} \to \mathrm{Ver}_{p^n}$ is isomorphic to the identity. In particular the group of braided autoequivalences of the category Ver_{p^n} is trivial.*

Proof. By Corollary 4.7, braided (= symmetric) tensor functors $E : \mathrm{Ver}_{p^n} \to \mathrm{Ver}_{p^n}$ correspond to objects $X \in \mathrm{Ver}_{p^n}$ with $\wedge^2 X \cong \mathbb{1}$, $\wedge^3 X = 0$ and $Q_{n,p}^+(X) \cong Q_{n,p}^-(X)$. Such an object X must have Frobenius-Perron dimension $q + q^{-1}$, in particular, it is simple, since this number is less than 2. Thus it follows from Corollary 4.45(ii) that $X \cong L_1$. Thus there is a unique braided tensor functor E up to an isomorphism, which must therefore be isomorphic to the identity. $\square$

4.12. **Tensor product of simple objects.** Let us now compute the tensor product of simple objects of Ver_{p^n} (in its Grothendieck ring). We start with the following result about the tensor product $\mathbb{T}_m \otimes \mathbb{T}_r$ for $0 \le m, r \le p - 1$.

Proposition 4.50. *The tensor product $\mathbb{T}_m \otimes \mathbb{T}_r$ for $1 \le m, r \le p-1$ is given by the formula*

$$\mathbb{T}_m \otimes \mathbb{T}_r = \bigoplus_{|m-r| \le k \le 2(p-2)-m-r} \mathbb{T}_k \oplus \bigoplus_{p-1 \le k \le m+r} \mathbb{T}_k,$$

if $m + r \ge p$, and

$$\mathbb{T}_m \otimes \mathbb{T}_r = \bigoplus_{|m-r| \le k \le m+r} \mathbb{T}_k$$

if $m + r < p$, where the summation is over integers k such that $m + r - k$ is even.

Proof. This follows immediately from Lemma 3.10. $\square$

Corollary 4.51. *The multiplication of simple objects in the Grothendieck ring $\mathrm{Gr}(\mathrm{Ver}_{p^n})$ is given by the following recursive rule. Let $n \ge 2$, $a = a'p + m$, $b = b'p + r$, where $0 \le m, r \le p - 1$ and $0 \le a', b' \le p^{n-2}(p-1) - 1$. Then for $m + r \ge p$ we have*

$$L_a L_b =$$

$$\left(\sum_{|m-r| \le k \le 2(p-2)-m-r} L_k + \sum_{2(p-1)-m-r \le k \le p-1} (2 - \delta_{k,p-1}) L_k \right) (L_{a'} L_{b'})^{[n-1]} +$$

$$+ \sum_{p \le k \le m+r} L_{k-p} (V L_{a'} L_{b'})^{[n-1]}.$$

and for $m + r < p$ we have

$$L_a L_b = \left(\sum_{|m-r| \le k \le m+r} L_k \right) (L_{a'} L_{b'})^{[n-1]},$$

where $V := \mathbb{T}_1$ and the summation is over integers k such that $m + r - k$ is even.

Proof. This follows from Proposition 4.50 and Example 4.48. $\square$

This allows one to quickly compute products in $\mathrm{Gr}(\mathrm{Ver}_{p^n})$, since the products $L_{a'} L_{b'}$ and $V L_{a'} L_{b'}$ are computed in the smaller category $\mathrm{Ver}_{p^{n-1}}$.

Here is a somewhat more elegant way to reformulate Corollary 4.51. Let A be a unital commutative $\mathbb{Z}_+$-ring ([22], Subsection 3.1). By a *unital $\mathbb{Z}_+$-ring over A* we will mean a ring $B \supset A$ with a free A-basis $b_i, i \in I$ (where $b_0 = 1$ for some element $0 \in I$), such that A is

central in B and $b_i b_j = \sum_k N_{ij}^k b_k$, where N_{ij}^k are *nonnegative* elements of A. In this case B is a unital $\mathbb{Z}_+$-ring in the sense of [22], Subsection 3.1, with basis $a_i b_j$, where a_i is the basis of A.

Corollary 4.52. *For $n > 1$ the ring $\mathrm{Gr}(\mathrm{Ver}_{p^n})$ is a unital $\mathbb{Z}_+$-ring over $\mathrm{Gr}(\mathrm{Ver}_{p^{n-1}})$ with basis L_m, $0 \le m \le p-1$, and multiplication given by*

$$L_m L_r = \sum_{|m-r| \le k \le 2(p-2)-m-r} L_k + \sum_{2(p-1)-m-r \le k \le p-1} (2 - \delta_{k,p-1}) L_k + \sum_{p \le k \le m+r} V L_{k-p}$$

for $m + r \ge p$ and

$$L_m L_r = \sum_{|m-r| \le k \le m+r} L_k,$$

for $m + r < p$, where $V = \mathbb{T}_1^{[n-1]} \in \mathrm{Gr}(\mathrm{Ver}_{p^{n-1}})$ and the summation is over integers k such that $m + r - k$ is even.

Example 4.53. Since $\mathrm{Ver}_3 = \mathrm{sVec}$, the ring $\mathrm{Gr}(\mathrm{Ver}_3)$ is $\mathbb{Z}[V]/(V^2-1)$ with basis $1, V$. Thus by Corollary 4.52 the ring $\mathrm{Gr}(\mathrm{Ver}_9)$ has basis $1, L_1, L_2$ over $\mathrm{Gr}(\mathrm{Ver}_3)$ with multiplication given by

$$L_1^2 = 1 + L_2, \ \ L_1 L_2 = 2L_1 + V, \ \ L_2^2 = 2 + L_2 + V L_1.$$

So the basis of $\mathrm{Gr}(\mathrm{Ver}_9^+)$ is $1, M_1 = V L_1, M_2 = L_2$ with multiplication given by

$$M_1^2 = 1 + M_2, \ \ M_1 M_2 = 1 + 2M_1, \ \ M_2^2 = 1 + M_1 + M_2.$$

This agrees with the data in Subsection 5.1.

4.13. **The blocks of Ver_{p^n}.**

Proposition 4.54. *The objects $\mathbb{T}_i$ and $\mathbb{T}_j$ are in the same block if and only if the following holds:*
  (a) i and j have the same parity;
  (b) base p expansions of $i + 1$ and $j + 1$ have the same number of zeros at the end;
  (c) the last nonzero base p digits of $i + 1$ and $j + 1$ are either equal or sum up to p.

Proof. This follows from Proposition 4.17. □

It now follows from Proposition 4.54 that Ver_{p^n} has the following block structure.

Proposition 4.55. *The category Ver_{p^n} has $n(p-1)$ blocks. More specifically, it has $p-1$ blocks of size 1 formed by projective (simple) objects $\mathbb{T}_i$ with $i + 1$ divisible by p^{n-1}, $p-1$ blocks of size $p-1$ formed by projective objects $\mathbb{T}_i$ with $i + 1$ divisible by p^{n-2} but not p^{n-1}, $p-1$ blocks of size $p^2 - p$ formed by projective objects $\mathbb{T}_i$ with $i + 1$ divisible by p^{n-3} but not p^{n-2}, etc. In particular it has $p-1$ blocks of maximal size $p^{n-1} - p^{n-2}$ formed by projective objects $\mathbb{T}_i$ with $i + 1$ not divisible by p; all these blocks are equivalent to each other by the translation principle, see [29].*

Observe that by Proposition 4.40 the functor $T \mapsto \mathrm{St} \otimes T^{(1)}$ from $\mathcal{T}_{\le p^{n-1}-1}$ to $\mathcal{T}_{\le p^n - 1}$ is fully faithful; its image is generated by the objects T_i such that $i + 1$ is divisible by p, i.e., by the objects lying in the blocks not of maximal size. Thus we get the following statement:

Proposition 4.56. *The blocks of the same size in the categories Ver_{p^n} (where n is allowed to vary) are all equivalent to each other; in particular they have the same Cartan matrices.*

Remark 4.57. Let $p = 2$ and let $\mathrm{Ver}_{2^n}^- \subset \mathrm{Ver}_{2^n}$ be the additive subcategory generated by $\mathbb{T}_i$ with odd i. Then $\mathrm{Ver}_{2^n}^+$ acts on $\mathrm{Ver}_{2^n}^-$ by left multiplication. At the same time the category $\mathrm{Ver}_{2^{n-1}}$ acts on $\mathrm{Ver}_{2^n}^-$ by right multiplication: $X, Y \mapsto X \otimes H(Y)$; clearly these two actions commute. It follows from the discussion above that $\mathrm{Ver}_{2^n}^-$ considered as a module category over $\mathrm{Ver}_{2^{n-1}}$ is equivalent to the regular module (i.e., to $\mathrm{Ver}_{2^{n-1}}$ considered as a module over itself). This implies that there exists a tensor (but not braided) functor $\mathrm{Ver}_{2^n}^+ \to \mathrm{Ver}_{2^{n-1}}$; thus the category $\mathrm{Ver}_{2^n}^+$ can be described as the representation category of some triangular Hopf algebra in $\mathrm{Ver}_{2^{n-1}}$. This can be considered as an a posteriori explanation of the construction in [5].

4.14. Ver_{p^n} is a Serre subcategory in $\mathrm{Ver}_{p^{n+k}}$.

Proposition 4.58. *Let $p > 2$. Then*
(i) Any tensor subcategory $\mathcal{C} \subset \mathrm{Ver}_{p^{n+1}}$ containing Ver_{p^n} as a full subcategory coincides with Ver_{p^n} or with $\mathrm{Ver}_{p^{n+1}}$.
(ii) The category Ver_{p^n} is a Serre subcategory in $\mathrm{Ver}_{p^{n+k}}$.

Proof. (i) Let $X = \mathbb{T}_{p^{n-1}-1} \in \mathrm{Ver}_{p^n}$. This is a simple projective object. On the other hand, in $\mathrm{Ver}_{p^{n+1}}$ it is not projective and has projective cover P_X.

Lemma 4.59. *We have $[P_X : X] = 2$.*

Proof. By Corollary 4.39 we have $P_X = \mathbb{T}_{p^n+p-2}$. Now $[P_X : X] = \dim \mathrm{Hom}(P_X, P_X) = 2$ by Proposition 4.17. $\qquad\square$

Now suppose $\mathrm{Ver}_{p^n} \subset \mathcal{C} \subset \mathrm{Ver}_{p^{n+1}}$. Let Q_X be the projective cover of X in $\mathcal{C}$. Then we have a diagram of surjective morphisms $P_X \to Q_X \to X$. It is clear that Q_X is self-dual, so by Lemma 4.59 either $Q_X = P_X$ or $Q_X = X$. In the first case every indecomposable projective in $\mathcal{C}$, being a direct summand in $Y \otimes Q_X$ for some simple $Y \in \mathcal{C}$, is projective in $\mathrm{Ver}_{p^{n+1}}$, which implies that $\mathcal{C} = \mathrm{Ver}_{p^{n+1}}$ (using the multiplication rule for simple objects from Corollary 4.52). On the other hand, if $Q_X = X$ then similarly all projectives of Ver_{p^n} are projective in $\mathcal{C}$, so we get $\mathcal{C} = \mathrm{Ver}_{p^n}$.

(ii) It suffices to prove the statement for $k = 1$, in which case it follows from (i). $\qquad\square$

Corollary 4.60. *The only tensor subcategories of Ver_{p^n} are Ver_{p^m} and $\mathrm{Ver}_{p^m}^+$ for $m \leq n$.*

Proof. For $p = 2$ this is proved in [5], Corollary 2.6, so let us consider the case $p > 2$. It suffices to show that the only tensor subcategories of $\mathrm{Ver}_{p^n}^+$ are $\mathrm{Ver}_{p^m}^+$ for $m \leq n$. Let $\mathcal{C} \subset \mathrm{Ver}_{p^n}^+$ be a tensor subcategory. Let m be the largest integer such that $\mathcal{C}$ contains all simple objects of Ver_m^+. Then by Corollary 4.51, $\mathcal{C}$ cannot contain any other simple objects. Thus $\mathcal{C}$ is contained in the Serre closure of $\mathrm{Ver}_{p^m}^+$ inside $\mathrm{Ver}_{p^n}^+$. So it follows from Proposition 4.58 that $\mathcal{C} = \mathrm{Ver}_{p^m}^+$. $\qquad\square$

4.15. Ext^1 between simple objects. Let us describe a recursive procedure for computing Ext^1 between simple objects of Ver_{p^n}. We will focus on the case $p > 2$ since the case $p = 2$ is addressed in [5], Subsection 4.5. Recall that by Theorem 4.43 the simple objects of Ver_{p^n} look like $X \otimes \mathbb{T}_i$ where $X \in \mathrm{Ver}_{p^{n-1}}$ and $0 \leq i \leq p - 1$. Also by block considerations for $X, Y \in \mathrm{Ver}_{p^{n-1}}$ we have

$$\mathrm{Ext}^m_{\mathrm{Ver}_{p^n}}(X \otimes \mathbb{T}_i, Y \otimes \mathbb{T}_j) = 0, \ m \geq 0$$

unless $i = j$ or $i = p - 2 - j$. Consider first the case $i = j$.

Lemma 4.61. *For each m we have a canonical isomorphism* $\mathrm{Ext}^m_{\mathrm{Ver}_{p^n}}(X \otimes \mathbb{T}_{p-1}, Y \otimes \mathbb{T}_{p-1}) \cong$ $\mathrm{Ext}^m_{\mathrm{Ver}_{p^{n-1}}}(X, Y)$.

Proof. This is an immediate consequence of Proposition 4.40. □

Lemma 4.62. *(i) If $0 \le i \le p - 2$ then we have a canonical isomorphism*
$$\mathrm{Ext}^m_{\mathrm{Ver}_{p^n}}(X \otimes \mathbb{T}_i, Y \otimes \mathbb{T}_i) \cong \mathrm{Ext}^m_{\mathrm{Ver}_{p^n}}(X, Y).$$

(ii) If $p > 2$ and $0 \le i \le p - 2$ then we have a canonical isomorphism
$$\mathrm{Ext}^1_{\mathrm{Ver}_{p^n}}(X \otimes \mathbb{T}_i, Y \otimes \mathbb{T}_i) \cong \mathrm{Ext}^1_{\mathrm{Ver}_{p^{n-1}}}(X, Y).$$

Proof. (i) We have
$$\mathrm{Ext}^m_{\mathrm{Ver}_{p^n}}(X \otimes \mathbb{T}_i, Y \otimes \mathbb{T}_i) = \mathrm{Ext}^m_{\mathrm{Ver}_{p^n}}(X, Y \otimes \mathbb{T}_i \otimes \mathbb{T}_i).$$
Now, $\mathbb{T}_i \otimes \mathbb{T}_i$ can be written as a direct sum of indecomposables, and by block considerations the only direct summand which contributes to $\mathrm{Ext}^\bullet$ is $\mathbb{1}$. This implies the statement.
(ii) This follows from (i) and Proposition 4.58(ii). □

Now consider the case $j = p - 2 - i$. In this case we can again write
$$\mathrm{Ext}^m_{\mathrm{Ver}_{p^n}}(X \otimes \mathbb{T}_i, Y \otimes \mathbb{T}_j) = \mathrm{Ext}^m_{\mathrm{Ver}_{p^n}}(X, Y \otimes \mathbb{T}_i \otimes \mathbb{T}_j),$$
and by block considerations the only indecomposable direct summand in $\mathbb{T}_i \otimes \mathbb{T}_j$ that contributes to $\mathrm{Ext}^\bullet$ is $\mathbb{T}_{p-2}$. Thus, it suffices to compute
$$\mathrm{Ext}^1_{\mathrm{Ver}_{p^n}}(X, Y \otimes \mathbb{T}_{p-2}).$$
To this end, note that $\mathbb{T}_1 \otimes \mathbb{T}_{p-1} = [\mathbb{T}_{p-2}, \mathbb{T}_1^{[n-1]}, \mathbb{T}_{p-2}]$ (Lemma 4.31). Thus,
$$\mathrm{Ext}^m_{\mathrm{Ver}_{p^n}}(X, Y \otimes [\mathbb{T}_{p-2}, \mathbb{T}_1^{[n-1]}, \mathbb{T}_{p-2}]) =$$
$$= \mathrm{Ext}^m_{\mathrm{Ver}_{p^n}}(X, Y \otimes \mathbb{T}_1 \otimes \mathbb{T}_{p-1}) = \mathrm{Ext}^m_{\mathrm{Ver}_{p^n}}(X \otimes \mathbb{T}_{p-1}, Y \otimes \mathbb{T}_1) = 0$$
by block considerations. From the long exact sequence it then follows that
$$\mathrm{Ext}^1_{\mathrm{Ver}_{p^n}}(X, Y \otimes \mathbb{T}_{p-2}) \cong \mathrm{Hom}_{\mathrm{Ver}_{p^n}}(X, Y \otimes [\mathbb{T}_{p-2}, \mathbb{T}_1^{[n-1]}]) \cong \mathrm{Hom}_{\mathrm{Ver}_{p^{n-1}}}(X, Y \otimes V).$$
Finally, this Hom can be computed using the following proposition.

Proposition 4.63. *Let $X, Y \in \mathrm{Ver}_{p^{n-1}}$ and $0 \le i, j \le p-2$. Then $\mathrm{Hom}(X \otimes L_i, Y \otimes L_j \otimes V) = 0$ unless $|i - j| = 1$. In the latter case, $\mathrm{Hom}(X \otimes L_i, Y \otimes L_j \otimes V) = \mathrm{Hom}(X, Y)$.*

Proof. We have
$$\mathrm{Hom}(X \otimes L_i, Y \otimes L_j \otimes V) = \mathrm{Hom}(L_i \otimes L_j \otimes V, Y \otimes X^*).$$
Now, the product $L_i \otimes L_j \otimes V$ decomposes as a direct sum of $\mathbb{T}_j$ for $0 \le j \le 2p - 1$. But by block considerations for every object $Z \in \mathrm{Ver}_{p^{n-1}} \subset \mathrm{Ver}_{p^n}$, we have $\mathrm{Hom}(\mathbb{T}_j, Z) = 0$ unless $j = 0$ or $j = 2p - 2$, and
$$\mathrm{Hom}(\mathbb{T}_{2p-2}, Z) = \mathrm{Hom}(\mathbb{1}, Z).$$
This implies the statement. □

Thus, we obtain the following proposition.

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Proposition 4.64. *Let $n \geq 2$, $p > 2$, $a = a'p + m$, $b = b'p + r$, where $0 \leq m, r \leq p - 1$ and $0 \leq a', b' \leq p^{n-2}(p-1) - 1$. Then the following hold for Ext^1 between simple objects of Ver_{p^n}.*

(i) $\mathrm{Ext}^1_{\mathrm{Ver}_{p^n}}(L_a, L_b) = 0$ unless $r = m$ or $r = p - 2 - m$.

(ii) If $r = m$ then $\mathrm{Ext}^1_{\mathrm{Ver}_{p^n}}(L_a, L_b) = \mathrm{Ext}^1_{\mathrm{Ver}_{p^{n-1}}}(L_{a'}, L_{b'})$.

(iii) If $r = p - 2 - m$ then $\mathrm{Ext}^1_{\mathrm{Ver}_{p^n}}(L_a, L_b) = 0$ unless the last digit of a' is $< p - 1$ and $b' = a' + 1$ or the last digit of a' is > 0 and $b' = a' - 1$. In the latter two cases, $\mathrm{Ext}^1_{\mathrm{Ver}_{p^n}}(L_a, L_b) = \mathbf{k}$.

In other words, $\mathrm{Ext}^1_{\mathrm{Ver}_{p^n}}(L_a, L_b) \neq 0$ if and only if a and b differ only in two consecutive digits, of which the first ones differ by 1 and the second ones add up to $p - 2$, and in this case $\mathrm{Ext}^1_{\mathrm{Ver}_{p^n}}(L_a, L_b) = \mathbf{k}$.

Remark 4.65. For $p = 2$ the fact that for any a, b, $\mathrm{Ext}^1_{\mathrm{Ver}_{p^n}}(L_a, L_b)$ is either 0 or $\mathbf{k}$ is proved in [5], Proposition 4.12.

Example 4.66. 1. If $p > 2$ then $\mathrm{Ext}^1_{\mathrm{Ver}_{p^n}}(L_a, L_a) = 0$ for all a.

2. Let $n = 2$, $p > 2$. Then $\mathrm{Ext}^1_{\mathrm{Ver}_{p^n}}(L_a, L_b) = 0$ unless $a = kp + c - 1$ and $b = (k+2)p - c - 1$ or $b = kp + c - 1$ and $a = (k+2)p - c - 1$ or $a = kp + c - 1$ and $b = kp - c - 1$ or $b = kp + c - 1$ and $a = kp - c - 1$ for some $1 \leq c \leq p - 1$. In these cases, $\mathrm{Ext}^1_{\mathrm{Ver}_{p^n}}(L_a, L_b) = \mathbf{k}$. Note that this agrees with the data in Subsection 5.1.

4.16. **The Grothendieck ring of the stable category.** Given a finite tensor category $\mathcal{C}$, one may consider the Grothendieck ring $\mathrm{GrStab}(\mathcal{C})$ of the stable category $\mathrm{Stab}(\mathcal{C})$, which by definition is the quotient of $\mathrm{Gr}(\mathcal{C})$ by the ideal of classes of projective objects. It is clear that as an abelian group, $\mathrm{GrStab}(\mathcal{C})$ is the cokernel of the Cartan matrix C of $\mathcal{C}$. Thus if C is invertible then this ring is finite of order $|\det(C)|$.

In this section we compute the ring $\mathrm{GrStab}(\mathcal{C})$ for $\mathcal{C} = \mathrm{Ver}_{p^n}$ and $\mathcal{C} = \mathrm{Ver}_{p^n}^+$ for $n \geq 2$.[5] Note that for $p > 2$ we have $\mathrm{GrStab}(\mathrm{Ver}_{p^n}) = \mathrm{GrStab}(\mathrm{Ver}_{p^n}^+)[u]/(u^2 - 1)$, so it suffices to compute $\mathrm{GrStab}(\mathrm{Ver}_{p^n}^+)$.

Recall that $q = e^{\pi i/p^n}$ and $[a]_q = \frac{q^a - q^{-a}}{q - q^{-1}}$.

Proposition 4.67. *(i) For $p > 2$ the ring $\mathrm{GrStab}(\mathrm{Ver}_{p^n}^+)$ is isomorphic to $\mathcal{O}_{n,p}/([p^{n-1}]_q)$.*

(ii) The ring $\mathrm{GrStab}(\mathrm{Ver}_{2^n})$ is isomorphic to $\mathcal{O}_{n+1,2}/([2^{n-1}]_q)$.

(iii) The ring $\mathrm{GrStab}(\mathrm{Ver}_{2^n}^+)$ is isomorphic to $\mathcal{O}_{n,2}/((q + q^{-1})[2^{n-1}]_q)$.

Proof. (i) Recall that $\mathrm{Gr}(\mathrm{Ver}_{p^n}^+) \cong \mathcal{O}_{n,p}$. Also, since

$$\mathbb{T}_1^{\otimes m} \otimes \mathbb{T}_{p^{n-1}-1} = \mathbb{T}_{p^{n-1}+m-1} \oplus \bigoplus_{i=0}^{m-1} b_m \mathbb{T}_{p^{n-1}-1+i},$$

the ideal of projectives is generated by $\mathbb{T}_{p^{n-1}-1}$. Therefore, the corresponding ideal in $\mathcal{O}_{n,p}$ is generated by the Frobenius-Perron dimension of $\mathbb{T}_{p^{n-1}-1}$, which equals $[p^{n-1}]_q$. This implies the statement.

(ii) The proof is the same as (i), using that $\mathrm{Gr}(\mathrm{Ver}_{2^n}) \cong \mathcal{O}_{n+1,2}$.

(iii) The proof is the same as (ii), using that $\mathrm{Gr}(\mathrm{Ver}_{2^n}^+) \cong \mathcal{O}_{n,2}$ and replacing $\mathbb{T}_{2^{n-1}-1}$ by $\mathbb{T}_{2^{n-1}} = \mathbb{T}_1 \otimes \mathbb{T}_{2^{n-1}-1}$ (as $\mathbb{T}_{2^{n-1}-1} \notin \mathrm{Ver}_{2^n}^+$). $\qquad\square$

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[5]We thank R. Rouquier for asking this question.

Now we would like to compute these rings more explicitly, which is now a problem in elementary number theory. First of all, note that $[p^{n-1}]_q = \frac{q^{p^{n-1}} - q^{-p^{n-1}}}{q - q^{-1}}$ divides p, since already $q^{p^{n-1}} - q^{-p^{n-1}} = e^{\pi i/p} - e^{-\pi i/p}$ divides p. Indeed, for $p > 2$ the norm of $e^{\pi i/p} - e^{-\pi i/p}$ equals $-p$, while $e^{\pi i/2} - e^{-\pi i/2} = 2i$ obviously divides 2.

Moreover, for $p = 2$ the number $(q + q^{-1})[2^{n-1}]_q$ divides 2. Indeed, $\frac{2}{(q+q^{-1})[2^{n-1}]_q} = -i\frac{q^2-1}{q^2+1}$. But the number $x = \frac{q^2-1}{q^2+1}$ is a unit, since $q^2 = \frac{1+x}{1-x}$, thus $(1 + x)^{2^{n-1}} = -(1 - x)^{2^{n-1}}$, i.e., $P(x) := \frac{1}{2}((1 + x)^{2^{n-1}} + (1 - x)^{2^{n-1}}) = 0$, while P is a monic polynomial with integer coefficients and constant term 1.

Thus, the above rings are $\mathbb{F}_p$-algebras. So in Proposition 4.67(i) the ring R in question is the quotient of $\mathbb{F}_p[q^2 + q^{-2}]$ by the relations

$$(q - q^{-1})^{p^{n-1}-1} = 0, \ q^{-(p-1)p^{n-1}}(1 + q^{2p^{n-1}} + ... + q^{2(p-1)p^{n-1}}) = 0.$$

The second relation can be rewritten in characteristic p as $(q - q^{-1})^{(p-1)p^{n-1}} = 0$, hence it follows from the first relation. Thus

$$R = \mathbb{F}_p[z]/z^{\frac{p^{n-1}-1}{2}}$$

with $z := (q - q^{-1})^2$.

In Proposition 4.67(ii) we similarly get

$$R = \mathbb{F}_p[w]/(w^{2^{n-1}-1}, \ w^{2^{n-1}}) = \mathbb{F}_p[w]/w^{2^{n-1}-1},$$

where $w = q + q^{-1}$.

Finally, in Proposition 4.67(iii) we get

$$R = \mathbb{F}_p[z]/(z^{2^{n-1}}, z^{2^{n-2}}) = \mathbb{F}_p[z]/z^{2^{n-2}}$$

with $z := (q - q^{-1})^2 = q^2 + q^{-2}$.

Thus we obtain

Proposition 4.68. *(i) For $p > 2$ the ring $\mathrm{GrStab}(\mathrm{Ver}_{p^n}^+)$ is isomorphic to $\mathbb{F}_p[z]/z^{\frac{p^{n-1}-1}{2}}$ and the ring $\mathrm{GrStab}(\mathrm{Ver}_{p^n})$ is isomorphic to $\mathbb{F}_p[z, g]/(z^{\frac{p^{n-1}-1}{2}}, g^2 - 1)$.*
(ii) The ring $\mathrm{GrStab}(\mathrm{Ver}_{2^n})$ is isomorphic to $\mathbb{F}_p[z]/z^{2^{n-1}-1}$.
(iii) The ring $\mathrm{GrStab}(\mathrm{Ver}_{2^n}^+)$ is isomorphic to $\mathbb{F}_p[z]/z^{2^{n-2}}$.

Proposition 4.69. *The determinant of the Cartan matrix of any block of Ver_{p^n} of size $p^{m-1}(p - 1)$ equals $p^{p^{m-1}}$.*

Proof. By Proposition 4.56, all blocks of the same size have the same Cartan matrices. So the determinant in question depends only on m; let us denote it by D_m. Then Proposition 4.68 and Proposition 4.55 imply that

$$(\prod_{m=1}^{n-1} D_m)^{p-1} = p^{p^{n-1}-1}.$$

Thus $D_m^{p-1} = p^{p^{m-1}(p-1)}$. Also since $C_{n,p}$ is positive definite (Proposition 4.13(iv)), D_m is a positive integer. Thus $D_m = p^{p^{m-1}}$, as claimed. $\qquad\square$

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4.17. Incompressibility. Let $\mathcal{D}$ be a (not necessarily braided) tensor category.

Theorem 4.70. *(i) Assume $p > 2$. Let $E : \mathrm{Ver}_{p^n}^+ \to \mathcal{D}$ be a (not necessarily braided) tensor functor. Then E is an embedding (i.e., it is fully faithful).*

(ii) Let $\mathcal{D}$ be braided and $E : \mathrm{Ver}_{p^n} \to \mathcal{D}$ be a braided tensor functor. Then E is an embedding.

Remark 4.71. (i) In the language of [5, Definition 4.3] Theorem 4.70 says that the category $\mathrm{Ver}_{p^n}^+$ is *incompressible* as a tensor category, and Ver_{p^n} is incompressible as a symmetric (or braided) category.

(ii) Theorem 4.70(i) is not true if $\mathrm{Ver}_{p^n}^+$ is replaced by Ver_{p^n}: since there exists a tensor functor $\mathrm{sVec} \to \mathrm{Vec}$, we have a non-injective tensor functor $\mathrm{Ver}_{p^n} = \mathrm{Ver}_{p^n}^+ \boxtimes \mathrm{sVec} \to \mathrm{Ver}_{p^n}^+$, see Corollary 4.12.

(iii) See [5, Theorem 4.4] for the case $p = 2$. Note that in this case Ver_{2^n} is incompressible as a tensor category while $\mathrm{Ver}_{2^n}^+$ is not. However, both are incompressible as symmetric (or braided) categories.

Proof. (ii) easily follows from (i), so it suffices to prove (i). By replacing $\mathcal{D}$ with the image of E we can assume that $\mathcal{D}$ is finite and E is surjective. Moreover by Corollary 4.12 we can extend E to a tensor functor $R : \mathrm{Ver}_{p^n} \to \mathcal{D} \boxtimes \mathrm{sVec}$.

Lemma 4.72. *(i) Assume $i \in [0, p-1]$. Then $R(\mathbb{T}_i)$ is simple and for $i > 0$ it is not isomorphic to an object from $R(\mathrm{Ver}_{p^{n-1}})$.*

(ii) Let $a \in [p-1, 2p-2]$ and let L be a simple object from $R(\mathrm{Ver}_{p^{n-1}}) \subset \mathcal{D}$. Assume $\mathrm{Hom}(R(\mathbb{T}_a), L) \neq 0$. Then $a = 2p-2$, $L = \mathbb{1}$ and $\mathrm{Hom}(R(\mathbb{T}_a), L)$ is one-dimensional.

Proof. (i) We employ induction in i. The base case $i = 0$ is clear. Assume $R(\mathbb{T}_{i+1})$ is not simple. Thus it has a nontrivial subobject N; clearly either $\mathrm{FPdim}(N) \leq \frac{1}{2}\mathrm{FPdim}(\mathbb{T}_{i+1})$ or $\mathrm{FPdim}(R(\mathbb{T}_{i+1})/N) \leq \frac{1}{2}\mathrm{FPdim}(\mathbb{T}_{i+1})$; since $R(\mathbb{T}_{i+1})$ is self-dual, in either case we will have a quotient M with $\mathrm{FPdim}(M) \leq \frac{1}{2}\mathrm{FPdim}(\mathbb{T}_{i+1})$. By (3.2) we have a nonzero morphism $R(\mathbb{T}_1) \otimes R(\mathbb{T}_i) \to M$, so we have a nonzero morphism $R(\mathbb{T}_i) \to R(\mathbb{T}_1) \otimes M$. This must be an embedding as $R(\mathbb{T}_i)$ is simple by the induction assumption. We have

$$\mathrm{FPdim}(R(\mathbb{T}_1) \otimes M/R(\mathbb{T}_i)) =$$

$$= [2]_q \mathrm{FPdim}(M) - [i+1]_q \leq \frac{[2]_q[i+2]_q}{2} - [i+1]_q = \frac{[i+3]_q - [i+1]_q}{2} < 1,$$

so it must vanish and $\mathrm{FPdim}(M) = \frac{[i+1]_q}{[2]_q}$. But we claim that the number $\alpha := \frac{[i+1]_q}{[2]_q} = \frac{q^{m+2}-q^{-m-2}}{q^2-q^{-2}}$ cannot be the Frobenius-Perron dimension of any object. Indeed, let g be the Galois group element such that $g(q^4) = q^2$ (it exists since the order of q^2 is a power of p). Then $g(q^{2m+4}) = (-1)^m q^{m+2}$, so we have

$$|g(\alpha)| = \left|\frac{(-1)^m q^{m+2} - 1}{q^2 - 1}\right| = |\alpha| \cdot \left|\frac{q^2 + 1}{(-1)^m q^{m+2} + 1}\right| > |\alpha|,$$

as desired. Thus $R(\mathbb{T}_{i+1})$ must be simple and (i) is proved.

(ii) We already know that the statement holds for $a = p - 1$ by (i). For $a \in [p, 2p-2]$ the object $\mathbb{T}_a$ has 3 simple components: $\mathbb{T}_{2p-2-a}$ with multiplicity 2 and $\mathbb{T}_1^{[n-1]} \otimes \mathbb{T}_{a-p}$, see Example 4.48. If $L \neq \mathbb{1}$ or $a \neq 2p-2$ then $\mathrm{Hom}(R(\mathbb{T}_{2p-2-a}), L) = 0$, so we must have

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$\mathrm{Hom}(R(\mathbb{T}_1^{[n-1]} \otimes \mathbb{T}_{a-p}), L) \neq 0$ which implies $\mathrm{Hom}(R(\mathbb{T}_{a-p}), R(\mathbb{T}_1^{[n-1]}) \otimes L) \neq 0$ which is impossible for $a \neq p$ by (i) as $R(\mathbb{T}_1^{[n-1]}) \otimes L \in R(\mathrm{Ver}_{p^{n-1}})$.

It remains to show that $\mathrm{Hom}(R(\mathbb{T}_p), R(\mathbb{T}_1^{[n-1]})) = 0$ and $\dim \mathrm{Hom}(R(\mathbb{T}_{2p-2}), \mathbb{1}) = 1$. For the former we observe that $\mathbb{T}_p = \mathbb{T}_1 \otimes \mathbb{T}_{p-1}$ by (3.3), so $\mathrm{Hom}(R(\mathbb{T}_p), R(\mathbb{T}_1^{[n-1]})) \neq 0$ would imply that $R(\mathbb{T}_{p-1}) \subset R(\mathbb{T}_1 \otimes \mathbb{T}_1^{[n-1]})$. This is impossible for $p \geq 5$ since

$$\mathrm{FPdim}(\mathbb{T}_{p-1}) > \mathrm{FPdim}(\mathbb{T}_1 \otimes \mathbb{T}_1^{[n-1]});$$

it is also impossible for $p = 3$ since $0 < \mathrm{FPdim}(\mathbb{T}_1 \otimes \mathbb{T}_1^{[n-1]}) - \mathrm{FPdim}(\mathbb{T}_{p-1}) < 1$ in this case. Finally, $\mathrm{Hom}(R(\mathbb{T}_{2p-2}), \mathbb{1})$ is a direct summand of $\mathrm{Hom}(R(\mathbb{T}_{p-1}) \otimes R(\mathbb{T}_{p-1})^*, \mathbb{1}) = \mathrm{Hom}(R(\mathbb{T}_{p-1}), R(\mathbb{T}_{p-1}))$, so it is one-dimensional by (i). $\qquad\square$

Lemma 4.72(i) implies that the restriction of R to $\mathrm{Ver}_p \subset \mathrm{Ver}_{p^n}$ is an embedding. Thus Theorem 4.70 is implied by the following result.

Proposition 4.73. *Let $R : \mathrm{Ver}_{p^n} \to \mathcal{D}$ be a surjective tensor functor such that its restriction to $\mathrm{Ver}_p \subset \mathrm{Ver}_{p^n}$ is an embedding. Then R is an equivalence.*

Proof. We proceed by induction in n. The base case $n = 1$ is immediate. By applying the inductive assumption to the image of the restriction of R to $\mathrm{Ver}_{p^{n-1}}$ we can assume that R restricted to $\mathrm{Ver}_{p^{n-1}}$ is an embedding.

It is known that the functor R sends projective objects to projective objects, see [22, Theorem 6.1.16]. To prove the proposition it remains to show that R is fully faithful on projectives. Note that it is faithful automatically by [22, Remark 4.3.10], so let us show that it is full on projectives. Since $\mathrm{Hom}(Q_1, Q_2) = \mathrm{Hom}(Q_1 \otimes Q_2^*, \mathbb{1})$, it is sufficient to show that for any projective $P \in \mathrm{Ver}_{p^n}$ the map $\mathrm{Hom}(P, \mathbb{1}) \to \mathrm{Hom}(R(P), \mathbb{1})$ is surjective. We can assume that P is indecomposable. Thus it is enough to show that for $p^{n-1} - 1 \leq r \leq p^n - 2$ we have $\mathrm{Hom}(R(\mathbb{T}_r), \mathbb{1}) = 0$ except for the case $r = 2p^{n-1} - 2$ when $\mathrm{Hom}(R(\mathbb{T}_r), \mathbb{1})$ should be one-dimensional, see Lemma 3.6. Let us write $r = a + pb$ where $p - 1 \leq 2p - 2$ and $p^{n-2} - 1 \leq b \leq p^{n-1} - 2$. By (4.7) we have $\mathrm{Hom}(R(\mathbb{T}_r), \mathbb{1}) = \mathrm{Hom}(R(\mathbb{T}_a), R(\mathbb{F}(\mathbb{T}_b))^*)$ where $R(\mathbb{F}(\mathbb{T}_b)) \in R(\mathrm{Ver}_{p^{n-1}})$ by definition of the functor H, see Theorem 4.35. Thus by Lemma 4.72 $\mathrm{Hom}(R(\mathbb{T}_a), R(\mathbb{F}(\mathbb{T}_b))^*) \neq 0$ implies that $a = 2p - 2$ and any nonzero morphism factorizes through $\mathbb{1}$. Using the inductive assumption we see that $b = 2p^{n-2} - 2$ and the nonzero Hom space is one-dimensional. Thus the functor R is full on projectives, which proves Proposition 4.73. $\qquad\square$

Theorem 4.70 is proved. $\qquad\square$

5. EXAMPLES

5.1. **The category** Ver_{p^2}**.** The category Ver_p, as already mentioned, is semisimple. The simple objects are $L_0, \ldots, L_{p-2}$, where $L_0 = \mathbb{1}$ is the tensor identity.[6] The tensor products are determined by the formula $L_1 \otimes L_m = \begin{cases} L_1 & m = 0 \\ L_{m-1} \oplus L_{m+1} & 0 < m < p - 2. \\ L_{p-3} & m = p - 2 \end{cases}$

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[6]Note that this labeling differs by a shift from the one in some other papers, for example [24], where the simple objects of Ver_p are denoted $L_1 = \mathbb{1}, L_2, ..., L_{p-1}$.

The Frobenius-Perron dimension of L_m is $\frac{\sin(\pi(m+1)/p)}{\sin(\pi/p)}$.

The category Ver_{p^2} is not semisimple, but has finite representation type. There are $p-1$ semisimple blocks, corresponding to the projective simple modules $L_{p-1}, L_{2p-1}, \ldots, L_{p^2-1}$, and $p-1$ non-semisimple blocks, each of whose basic algebra is a Brauer tree algebra with inertial index $p-1$, so that there is no exceptional vertex. The tree is a straight line with p vertices. Thus we obtain

Proposition 5.1. *The non-semisimple blocks of Ver_{p^2} are equivalent to the non-semisimple block of the group algebra of the symmetric group of degree p.*

These results can be obtained from Proposition 3.1 and the results of Subsection 3.5.

Here is the structure of the tilting modules for $p = 3$ and $p = 5$, with $n = 2$. The ones we need to use are T_m with $p^{n-1} - 1 \le m \le p^n - 2$. These are the ones below the double line. In these diagrams, we denote by $[m]$ the simple module with highest weight m. This information can be read off from the paper of Doty and Henke [18].

$p = 3$:

|	$T_0 = [0]$	$T_1 = [1]$
$T_2 = [2]$	$T_3 = \begin{matrix}[1]\\[3]\\[1]\end{matrix}$	$T_4 = \begin{matrix}[0]\\[4]\\[0]\end{matrix}$
$T_5 = [5]$	$T_6 = \begin{matrix}[4]\\[0]\,[6]\\[4]\end{matrix}$	$T_7 = \begin{matrix}[3]\\[1]\,[7]\\[3]\end{matrix}$

$p = 5$:

	$T_0 = [0]$	$T_1 = [1]$	$T_2 = [2]$	$T_3 = [3]$
$T_4 = [4]$	$T_5 = \begin{matrix}[3]\\[5]\\[3]\end{matrix}$	$T_6 = \begin{matrix}[2]\\[6]\\[2]\end{matrix}$	$T_7 = \begin{matrix}[1]\\[7]\\[1]\end{matrix}$	$T_8 = \begin{matrix}[0]\\[8]\\[0]\end{matrix}$
$T_9 = [9]$	$T_{10} = \begin{matrix}[8]\\[0]\,[10]\\[8]\end{matrix}$	$T_{11} = \begin{matrix}[7]\\[1]\,[11]\\[7]\end{matrix}$	$T_{12} = \begin{matrix}[6]\\[2]\,[12]\\[6]\end{matrix}$	$T_{13} = \begin{matrix}[5]\\[3]\,[13]\\[5]\end{matrix}$
$T_{14} = [14]$	$T_{15} = \begin{matrix}[13]\\[5]\,[15]\\[13]\end{matrix}$	$T_{16} = \begin{matrix}[12]\\[6]\,[16]\\[12]\end{matrix}$	$T_{17} = \begin{matrix}[11]\\[7]\,[17]\\[11]\end{matrix}$	$T_{18} = \begin{matrix}[10]\\[8]\,[18]\\[10]\end{matrix}$
$T_{19} = [19]$	$T_{20} = \begin{matrix}[18]\\[10]\,[20]\\[18]\end{matrix}$	$T_{21} = \begin{matrix}[17]\\[11]\,[21]\\[17]\end{matrix}$	$T_{22} = \begin{matrix}[16]\\[12]\,[22]\\[16]\end{matrix}$	$T_{23} = \begin{matrix}[15]\\[13]\,[23]\\[15]\end{matrix}$

In general if p is odd, for $p-1 \le ap+b-1 \le p^2-2$, with $1 \le a \le p-1$ and $0 \le b \le p-1$, the structure is

$$T_{ap+b-1} = \begin{matrix}[ap-b-1]\\[(a-2)p+b-1]\quad[ap+b-1]\\[ap-b-1]\end{matrix}$$

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unless $b = 0$, in which case $T_{ap+b-1} = [ap + b - 1]$. Also, if $a = 1$ then $(a - 2)p + b - 1 < 0$, and we omit this composition factor so that we are left with a uniserial module.

The simple modules for $A = \mathrm{End}_{SL_2(\mathbf{k})}\left(\bigoplus_{m=p-1}^{p^2-2} T_m\right)$ correspond to the indecomposable tilting modules T_m with $p - 1 \leq m \leq p^2 - 2$. We label each simple module with the isomorphism type of the top composition factor of the corresponding projective object $\mathbb{T}_m := F(T_m)$. So the simple corresponding to $\mathbb{T}_{ap+b-1}$ is labelled L_{ap-b-1}. This has the effect that the simples are L_i with $0 \leq i \leq p^2 - p - 1$ (this agrees with the labeling in Subsection 4.11). The structure of their projective covers is as follows. If $i = ap - 1$ then L_i is a projective simple in a block of its own. The block containing L_{p-b-1} with $1 \leq b \leq p - 1$ also contains the modules $L_{(2a+1)p\pm b-1}$, which are the ones whose highest weights are in the same orbit of the affine Weyl group under the dot action. The Brauer tree is as follows:
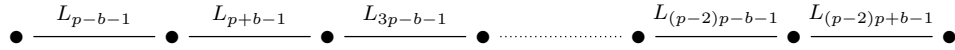
Thus the structure of the projective indecomposables is:
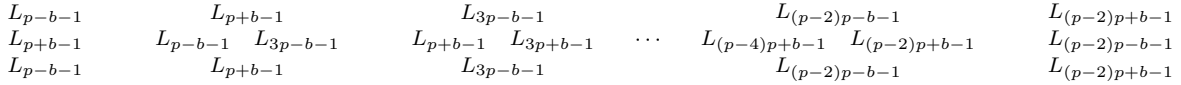
Projective resolutions in this category are formed in the usual way by walking around the Brauer tree, see Green [28]. In particular, we have

Proposition 5.2. *If p is odd then the algebra $\mathrm{Ext}^{\bullet}_{\mathrm{Ver}_{p^2}}(\mathbb{1}, \mathbb{1})$ is isomorphic to $\mathbf{k}[x, \xi]$ where x is an even generator of degree $2p - 2$ and ξ is an odd generator of degree $2p - 3$ (thus $\xi^2 = 0$). In particular, the Hilbert series of this algebra is $\frac{1+t^{2p-3}}{1-t^{2p-2}}$.*

For $p = 3$, the tensor products in Ver_{p^2} are given by the following table, in which P_n denotes the projective cover of the simple module L_n.

	L_0	L_1	L_2	L_3	L_4	L_5
L_1	$L_0 \oplus L_2$	P_3	L_4	$L_3 \oplus L_5$	P_4	
L_2	P_3	$L_2 \oplus P_0$	L_5	P_4	$L_5 \oplus P_3$	
L_3	L_4	L_5	L_0	L_1	L_2	
L_4	$L_3 \oplus L_5$	P_4	L_1	$L_0 \oplus L_2$	P_3	
L_5	P_4	$L_5 \oplus P_3$	L_2	P_3	$L_2 \oplus P_0$	

Note that L_3 satisfies $\Lambda^2(L_3) = L_0$ and $S^2(L_3) = 0$, so it generates $\mathrm{sVec} \subset \mathrm{Ver}_9$, and tensoring with L_3 acts as a parity change (recall that in general the generator of $\mathrm{sVec} \subset \mathrm{Ver}_{p^n}$ is $L_{p^{n-1}(p-2)}$).

For $p = 5$, we only list the tensor products of simples in $\mathrm{Ver}_{p^2}^+$, together with their tensor products with the parity change module L_{15}, which satisfies $\Lambda^2(L_{15}) = L_0$, $S^2(L_{15}) = 0$.

L_0	L_2	L_4	L_6	L_8	L_{10}	L_{12}	L_{14}	L_{16}	L_{18}
L_2	$L_0 \oplus L_2 \oplus L_4$	$L_4 \oplus P_2$	$L_6 \oplus L_8$	$L_6 \oplus P_8$	L_{12}	$L_{10} \oplus L_{12} \oplus L_{14}$	$L_{14} \oplus P_{12}$	$L_{16} \oplus L_{18}$	$L_{16} \oplus P_{18}$
L_4	$L_4 \oplus P_2$	$L_4 \oplus P_0 \oplus P_2$	P_8	$P_6 \oplus P_8$	L_{14}	$L_{14} \oplus P_{12}$	$L_{14} \oplus P_{10} \oplus P_{12}$	P_{18}	$P_{16} \oplus P_{18}$
L_6	$L_6 \oplus L_8$	P_8	$L_0 \oplus L_2 \oplus L_{10} \oplus L_{12}$	$L_2 \oplus L_4 \oplus L_{12} \oplus L_{14}$	$L_6 \oplus L_{16}$	$L_6 \oplus L_8 \oplus L_{16} \oplus L_{18}$	$P_8 \oplus P_{18}$	$L_{10} \oplus L_{12}$	$L_{12} \oplus L_{14}$
L_8	$L_6 \oplus P_8$	$P_6 \oplus P_8$	$L_2 \oplus L_4 \oplus L_{12} \oplus L_{14}$	$L_0 \oplus L_4 \oplus L_{10} \oplus L_{14} \oplus P_2 \oplus P_{12}$	$L_8 \oplus L_{18}$	$L_6 \oplus L_{10} \oplus P_8 \oplus P_{18}$	$P_6 \oplus P_8 \oplus P_{16} \oplus P_{18}$	$L_{12} \oplus L_{14}$	$L_{10} \oplus L_{14} \oplus P_{12}$
L_{10}	L_{12}	L_{14}	$L_6 \oplus L_{16}$	$L_8 \oplus L_{18}$	$L_0 \oplus L_{10}$	$L_2 \oplus L_{12}$	$L_4 \oplus L_{14}$	L_6	L_8
L_{12}	$L_{10} \oplus L_{12} \oplus L_{14}$	$L_{14} \oplus P_{12}$	$L_6 \oplus L_8 \oplus L_{16} \oplus L_{18}$	$L_6 \oplus L_{10} \oplus P_8 \oplus P_{18}$	$L_2 \oplus L_{12}$	$L_0 \oplus L_2 \oplus L_4 \oplus L_{10} \oplus L_{12} \oplus L_{14}$	$L_4 \oplus L_{14} \oplus P_2 \oplus P_{12}$	$L_6 \oplus L_8$	$L_6 \oplus P_8$
L_{14}	$L_{14} \oplus P_{12}$	$L_{14} \oplus P_{10} \oplus P_{12}$	$P_8 \oplus P_{18}$	$P_6 \oplus P_8 \oplus P_{16} \oplus P_{18}$	$L_4 \oplus L_{14}$	$L_4 \oplus L_{14} \oplus P_2 \oplus P_{12}$	$L_4 \oplus L_{14} \oplus P_0 \oplus P_2 \oplus P_{10} \oplus P_{12}$	P_8	$P_6 \oplus P_8$
L_{16}	$L_{16} \oplus L_{18}$	P_{18}	$L_{10} \oplus L_{12}$	$L_{12} \oplus L_{14}$	L_6	$L_6 \oplus L_8$	P_8	$L_0 \oplus L_2$	$L_2 \oplus L_4$
L_{18}	$L_{16} \oplus P_{18}$	$P_{16} \oplus P_{18}$	$L_{12} \oplus L_{14}$	$L_{10} \oplus L_{14} \oplus P_{12}$	L_8	$L_6 \oplus P_8$	$P_6 \oplus P_8$	$L_2 \oplus L_4$	$L_0 \oplus L_4 \oplus P_2$
L_{15}	L_{17}	L_{19}	L_{11}	L_{13}	L_5	L_7	L_9	L_1	L_3

5.2. The category $\mathrm{Ver}_{p^3}^+$ in the case $p = 3$. We now describe the category $\mathrm{Ver}_{p^3}^+$ in the case $p = 3$. The relevant even indexed tilting modules have the following Loewy series.

$$T_8 = [8] \qquad T_{10} = \begin{matrix}[6]\\[4]\\[10]\\[4]\\[6]\end{matrix} \qquad T_{12} = \begin{matrix}[4]\\[10]\,[0]\,[6]\\[4]\,[12]\,[4]\\[6]\,[0]\,[10]\\[4]\end{matrix} \qquad T_{14} = \begin{matrix}[2]\\[14]\\[2]\end{matrix} \qquad T_{16} = \begin{matrix}[0]\\[4]\,[12]\\[0]\,[10]\,[16]\,[0]\\[12]\,[4]\\[0]\end{matrix}$$

$$T_{18} = \begin{matrix}[16]\\[12]\\[0]\,[18]\\[12]\\[16]\end{matrix} \qquad T_{20} = \begin{matrix}[14]\\[2]\,[20]\\[14]\end{matrix} \qquad T_{22} = \begin{matrix}[12]\\[0]\,[18]\,[10]\,[16]\\[4]\,[12]\,[22]\,[12]\\[0]\,[18]\,[10]\,[16]\\[12]\end{matrix} \qquad T_{24} = \begin{matrix}[10]\\[4]\,[12]\,[22]\\[0]\,[6]\,[10]\,[18]\,[24]\,[10]\\[4]\,[12]\,[22]\\[10]\end{matrix}$$

Again we label the simples for A by the top composition factors of the tilting modules. So here is a table of the correspondence:

L_0	L_2	L_4	L_6	L_8	L_{10}	L_{12}	L_{14}	L_{16}
T_{16}	T_{14}	T_{12}	T_{10}	T_8	T_{24}	T_{22}	T_{20}	T_{18}

These fall into three blocks. One is a semisimple block containing the projective simple L_8, one is a block of finite representation type containing L_2 and L_{14}, and the remaining six simples, L_0, L_4, L_6, L_{10}, L_{12}, L_{16} lie in a single block of wild representation type. The Cartan matrix is

|	L_0	L_4	L_6	L_{10}	L_{12}	L_{16}	L_2	L_{14}	L_8
L_0	4	2	0	1	2	1			
L_4	2	4	2	2	1	0			
L_6	0	2	2	1	0	0			
L_{10}	1	2	1	4	2	0			
L_{12}	2	1	0	2	4	2			
L_{16}	1	0	0	0	2	2			
L_2							2	1	
L_{14}							1	2	
L_8									1

This can be easily obtained from the results of Section 4.

The structure of the projective A-modules (obtained using a computer calculation) is as follows (where the diagrams for the non-uniserial modules are in the sense of Alperin [1] or Benson and Carlson [4]):
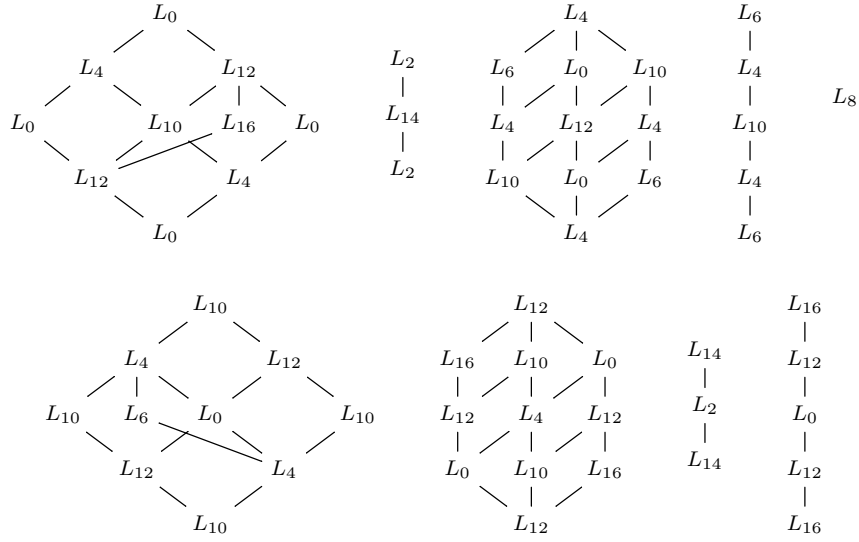
For the wild block, the basic algebra is given by the following quiver and relations:
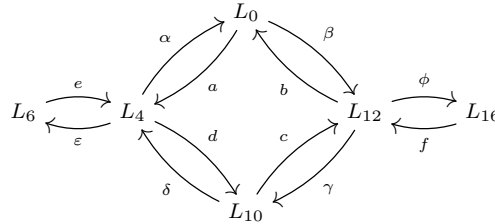
Relations: $\alpha e = 0$, $\gamma f = 0$, $\varepsilon e = 0$, $\varepsilon a = 0$, $\phi c = 0$, $\phi f = 0$, $d\delta d = 0$, $\delta d\delta = 0$, $b\beta b = 0$, $\beta b\beta = 0$, $\alpha\delta = bc$, $\beta\alpha = cd$, $\gamma\beta = da$, $\delta\gamma = ab$, $e\varepsilon = a\alpha$, $f\phi = (1 + \beta b)c\gamma$.

Note that it follows from this presentation that $\beta b f\phi = \beta bc\gamma + \beta b\beta bc\gamma = \beta bc\gamma$, so $f\phi = c\gamma + \beta b f\phi$. So the last relation can be replaced by $c\gamma = (1 - \beta b)f\phi$. Moreover,

$$\beta bc\gamma = \beta\alpha\delta\gamma = cdab = c\gamma\beta b$$

so $c\gamma$ and hence also $f\phi$ commute with βb.

Table 1 describes the tensor products of the simples in $\mathrm{Ver}_{3^3}^+$.

L_0	L_2	L_4	L_6	L_8	L_{10}	L_{12}	L_{14}	L_{16}
L_2	L_0 $L_2 \oplus L_4$ L_0	L_4 $L_0\ L_6$ L_4	L_8	$P_6 \oplus L_8$	L_{10} L_{12} L_{10}	L_{14}	L_{12} $L_{14} \oplus L_{10}\ L_{16}$ L_{12}	P_{16}
L_4	L_4 $L_0\ L_6$ L_4	$L_0 \oplus L_2$ $\oplus L_6 \oplus L_8$	L_4 L_{10} L_4	P_4	$L_{12} \oplus L_{14}$	$L_{10} \oplus L_{16}$	L_{10} $P_{16} \oplus L_{12}$ L_{10}	L_{12} $P_{14} \oplus\ L_0$ L_{12}
L_6	L_8	L_4 L_{10} L_4	L_0 $L_6 \oplus L_{12}$ L_0	$P_2 \oplus L_8$	L_{16}	L_{12} L_0 L_{12}	P_{14}	L_{10} $L_{16} \oplus\ L_4$ L_{10}
L_8	$P_6 \oplus L_8$	P_4	$P_2 \oplus L_8$	$P_0 \oplus P_2$ $\oplus P_6 \oplus L_8$	P_{16}	P_{14}	$P_{12} \oplus P_{14}$	$P_{10} \oplus P_{16}$
L_{10}	L_{10} L_{12} L_{10}	$L_{12} \oplus L_{14}$	L_{16}	P_{16}	$L_0 \oplus L_2$	L_4	L_4 $L_0\ L_6$ L_4	$L_6 \oplus L_8$
L_{12}	L_{14}	$L_{10} \oplus L_{16}$	L_{12} L_0 L_{12}	P_{14}	L_4	$L_0 \oplus L_6$	$L_2 \oplus L_8$	L_4 L_{10} L_4
L_{14}	L_{12} $L_{14} \oplus L_{10}\ L_{16}$ L_{12}	L_{10} $P_{16} \oplus L_{12}$ L_{10}	P_{14}	$P_{12} \oplus P_{14}$	L_4 $L_0\ L_6$ L_4	$L_2 \oplus L_8$	L_0 $L_2 \oplus L_8 \oplus P_6 \oplus\ L_4$ L_0	P_4
L_{16}	P_{16}	L_{12} $P_{14} \oplus\ L_0$ L_{12}	L_{10} $L_{16} \oplus\ L_4$ L_{10}	$P_{10} \oplus P_{16}$	$L_6 \oplus L_8$	L_4 L_{10} L_4	P_4	L_{10} $L_{16} \oplus P_{10} \oplus P_{16} \oplus\ L_4$ L_{10}

TABLE 1. Tensor products in $\mathrm{Ver}_{3^3}^+$

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