ON THE BV STRUCTURE OF THE HOCHSCHILD COHOMOLOGY OF FINITE GROUP ALGEBRAS

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Abstract. We give a simple algebraic recipe for calculating the components of the BV operator Δ on the Hochschild cohomology of a finite group algebra with respect to the centraliser decomposition. We use this to investigate the properties of Δ and to make some computations for some particular finite groups.

1. Introduction

The Hochschild cohomology of a finite group is a Batalin–Vilkovisky (BV) algebra. This follows for example by observations of Tradler [23] from the fact that the group algebra over a commutative ring k is a symmetric algebra. So there is a BV operator $\Delta\colon HH^n(kG) \to HH^{n-1}(kG)$, which is related to the Gerstenhaber bracket by the formula

$$[x, y] = (-1)^{|x|}\Delta(xy) - (-1)^{|x|}\Delta(x)y - x\Delta(y).$$

Thus, if the cup product and the BV operator are known, so is the Gerstenhaber bracket.

The BV operator on the Hochschild cohomology $HH^*(kG)$ of a finite group G over k coming from the standard symmetrising form on kG preserves the centraliser decomposition

$$HH^*(kG) \cong \bigoplus_g H^*(C_G(g), k)$$

under which Δ is the sum of degree -1 operators Δ_g on $H^*(C_G(g), k)$ (see Section 3). Here g runs over a set of representatives of the conjugacy classes in G. An individual component Δ_g depends only on g and $C_G(g)$ but not on G itself, and hence in order to describe Δ_g we may assume that g is central in G. The following theorem gives a description of Δ_g in this situation, and some of its properties.[1]

Theorem 1.1. *Let k be a commutative ring of coefficients. Let g be a central element in a finite group G, and let $z_g\colon \mathbb{Z} \times G \to G$ be the map which sends (m, h) to $g^m h$. Then*

$$(z_g)^*\colon H^n(G, k) \to H^n(\mathbb{Z} \times G, k) \cong H^n(G, k) \oplus H^{n-1}(G, k)$$

has the form $(1, \Delta_g)$, where $\Delta_g\colon H^n(G, k) \to H^{n-1}(G, k)$ is the component of the BV operator indexed by g. The following properties hold for the map Δ_g.

(i) *Δ_g is a k-linear derivation with respect to multiplication in $H^*(G, k)$: for $x, y \in H^*(G, k)$ we have*

$$\Delta_g(x \cdot y) = \Delta_g(x) \cdot y + (-1)^{|x|} x \cdot \Delta_g(y).$$

In particular, Δ_g is determined by its values on a set of generators for $H^*(G, k)$.

[1] We have to be careful to distinguish the cup product in Hochschild cohomology from cup product in ordinary cohomology $H^*(C_G(g), k)$ in a component of the centraliser decomposition. We use juxtaposition xy to denote Hochschild cup product, and $x \cdot y$ for cup product inside $H^*(C_G(g), k)$.

(ii) *If $g \in Z(G)$ and $g' \in Z(G')$, then the following diagram commutes.*

$$\begin{array}{ccc} H^i(G,k)\otimes_k H^j(G',k) & \longrightarrow & H^{i+j}(G\times G',k) \\ \downarrow{\scriptstyle (\Delta_g\otimes 1,1\otimes\Delta_{g'})} & & \downarrow{\scriptstyle \Delta_{(g,g')}} \\ (H^{i-1}(G,k)\otimes_k H^j(G',k))\oplus(H^i(G,k)\otimes_k H^{j-1}(G',k)) & \longrightarrow & H^{i+j-1}(G\times G',k) \end{array}$$

where the horizontal maps are the Künneth maps and

$$(\Delta_g\otimes 1)(x\otimes y)=\Delta_g(x)\otimes y, \qquad (1\otimes\Delta_{g'})(x\otimes y)=(-1)^{|x|}x\otimes\Delta_{g'}(y).$$

If k is a field, then $\Delta_{(g,g')}$ is determined by Δ_g and $\Delta_{g'}$ by combining this with the Künneth formula.

(iii) *If $\phi\colon G\to G'$ is a group homomorphism sending $g\in Z(G)$ to $g'\in Z(G')$ then the following diagram commutes.*

$$\begin{array}{ccc} H^n(G',k) & \xrightarrow{\phi^*} & H^n(G,k) \\ \downarrow{\scriptstyle \Delta_{g'}} & & \downarrow{\scriptstyle \Delta_g} \\ H^{n-1}(G',k) & \xrightarrow{\phi^*} & H^{n-1}(G,k) \end{array}$$

(iv) *If $\rho\colon k\to k'$ is a homomorphism of commutative rings, then the following diagram commutes.*

$$\begin{array}{ccc} H^n(G,k) & \xrightarrow{\rho_*} & H^n(G,k') \\ \downarrow{\scriptstyle \Delta_g} & & \downarrow{\scriptstyle \Delta_g} \\ H^{n-1}(G,k) & \xrightarrow{\rho_*} & H^{n-1}(G,k') \end{array}$$

(v) *If g and g' are elements of $Z(G)$ then we have $\Delta_{gg'}=\Delta_g+\Delta_{g'}$.*

(vi) *If H is a subgroup of G containing $g\in Z(G)$ then Δ_g commutes with the transfer map: if $x\in H^n(H,k)$ then*

$$\mathrm{Tr}_{H,G}(\Delta_g(x))=\Delta_g(\mathrm{Tr}_{H,G}(x))\in H^n(G,k).$$

(vii) *If $g\in Z(G)$ and k is a field of prime characteristic p, then the map Δ_g is determined by its action on the cohomology of a Sylow p-group of G, as follows. Let $g=g_pg_{p'}=g_{p'}g_p$ with g_p a p-element and $g_{p'}$ a p'-element, and let P be a Sylow p-subgroup of G. Then $g_p\in Z(P)$ and the following diagram commutes.*

$$\begin{array}{ccc} H^n(G,k) & \xrightarrow{\mathrm{Res}_{G,P}} & H^n(P,k) \\ \downarrow{\scriptstyle \Delta_g} & & \downarrow{\scriptstyle \Delta_{g_p}} \\ H^{n-1}(G,k) & \xrightarrow{\mathrm{Res}_{G,P}} & H^{n-1}(P,k) \end{array}$$

(viii) *In the case $k=\mathbb{F}_p$, the map Δ_g commutes with the action of Steenrod operations on $H^*(G,\mathbb{F}_p)$, and with the Bockstein homomorphism.*

(ix) *If $x\in H^1(G,k)$, we can regard x as an element of $\mathsf{Hom}(G,k)$. With this identification, we have $\Delta_g(x)=x(g)$.*

(x) *If $x\in H^2(G,k)$ is in the image of $H^2(G,\mathbb{Z})\to H^2(G,k)$ then $\Delta_g(x)=0$.*

(xi) *If $x \in H^2(G,k)$ corresponds to a central extension*

$$1 \to k^+ \to K \to G \to 1,$$

then for $h \in G$ we choose any inverse image $\hat{h} \in K$. Then we have

$$\Delta_g(x)(h) = [\hat{h}, \hat{g}] \in k^+.$$

The proof of Theorem 1.1 appears below in the following places.The first assertion is proved in Theorem 4.2. The statements (i), (iii), (iv), (vi), (viii) are proved in Corollary 4.3, statement (ii) is in Proposition 6.6, the statements (v), (vii) are in Proposition 6.7, statement (ix) is Proposition 4.4, statement (x) is Proposition 4.6, and statement (xi) is Theorem 8.1.

Part (vii) of Theorem 1.1 says that if k is a field of prime characteristic p we may as well suppose that G is a p-group for the purpose of computation, and hence we give in Section 9 a number of examples where we compute Δ_g in the case where $g \in Z(G)$ and G is a finite p-group. Our first examples are cyclic groups, and more generally, abelian groups. Then we deal with dihedral, quaternion, and semidihedral 2-groups.

Following Tradler [23], the operator Δ_g is defined as the dual of the Connes operator $B \circ I$ in Hochschild homology. By work of Burghelea [5], the Connes exact sequence preserves the centraliser decomposition, and the corresponding components of the Connes exact sequence can be used to describe Δ_g; see Theorems 2.4 and 3.4 below for details. For our analysis of Δ_g, we first provide a description of the Connes operator in the context of a general discrete group G. Again, it only depends on g and $C_G(g)$, and we give the following description for g central in G. Consider the map $z_g \colon \mathbb{Z} \times G \to G$ as in the theorem above. Then

$$(z_g)_* \colon H_n(\mathbb{Z} \times G, k) \cong H_n(G,k) \oplus H_{n-1}(G,k) \to H_n(G,k)$$

is equal to $(1, B \circ I)$. Dualising this statement to give a statement about Hochschild cohomology requires the use of a symmetrising form on kG, and we make use of the standard one, whose value on (g,h) is one if $gh = 1$ and zero otherwise.

We use Theorem 1.1 to give an explicit description of the BV operator Δ in the Hochschild cohomology of finite groups over an arbitrary commutative ring k in terms of ordinary cohomology, based on a simple general principle using homotopies in order to construct degree -1 operators in cohomology. This general principle, and its relation to Δ_g, is expressed in the following theorem, which summarises the main parts of the Theorems 5.1 and 6.1 below. By a homotopy on a chain complex (resp. cochain complex) we mean a graded map of degree 1 (resp. -1).

Theorem 1.2. *Let A be an algebra over a commutative ring k and let U, V be A-modules. Let $z \in Z(A)$ such that z annihilates U and V. Let (P, δ) be a projective resolution of U.*

There is a homotopy s on P such that the chain endomorphism $s \circ \delta + \delta \circ s$ of P is equal to multiplication by z on the components of P. For any such homotopy, the induced homotopy $s^\vee$ on the cochain complex $\mathsf{Hom}_A(P,V)$ obtained from applying the functor $\mathsf{Hom}_A(-,V)$ to s is a cochain map $\mathsf{Hom}_A(P,V) \to \mathsf{Hom}_A(P[1],V)$. In particular, upon taking cohomology, $s^\vee$ induces a degree -1 operator, denoted D_z, on $\mathsf{Ext}^*_A(U,V)$, and then D_z is independent of the choice of s.

If G is a finite group, $A = kG$, $U = V = k$ and $g \in Z(G)$, then $\Delta_g = D_{g-1}$.

See the Remark 5.2 below for sign conventions for the differentials of the complexes arising in this theorem. While the first part of Theorem 1.2 is a routine verification (see the proof of Theorem 5.1), the identification $\Delta_g = D_{g^{-1}}$ in the case $A = kG$ in Theorem 6.1 requires Theorem 4.2, the proof of which is based on the Connes exact sequence relating relating Hochschild and cyclic cohomology. We give explicit homotopies for the bar resolution in Theorem 7.1. We use this description in Section 8 to give a short proof of the formula in Theorem 1.1 (xi), restated as Theorem 8.1 below, for the components of the BV operator in degree 2, in terms of central extensions corresponding to degree 2 elements in group cohomology.

The theme of deciding when $HH^1(kG)$ is a soluble Lie algebra has been recently investigated by a number of authors [8, 16, 26]. As an application of Theorem 1.1, we add to these the following results, which are proved as Theorems 10.9 and 11.6.

Theorem 1.3. *If G is a finite p-group such that $|Z(G) : Z(G) \cap \Phi(G)| \geqslant 3$ then the Lie algebra $HH^1(kG)$ is not soluble.*

Theorem 1.4. *If G is an extraspecial p-group then the Lie algebra $HH^1(kG)$ is soluble.*

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2. Hochschild and cyclic homology of kG

Background material for this section may be found in Benson [4, §§2.11–2.15]. Other references include Loday [18, Chapter 7], Burghelea [5], and Karoubi and Villamayor [14].

Let G be a discrete group, and k a commutative ring of coefficients. Hochschild homology of kG has a centraliser decomposition

$$HH_*(kG) \cong \bigoplus_g H_*(C_G(g), k)$$

where g runs over a set of representatives of the conjugacy classes of G. This decomposition is unique up to unique isomorphism. A similar description of cyclic homology appears in [5, 14], and was reinterpreted in [4] in terms of extended centralisers, as follows. If $g \in G$, we define the extended centraliser $\hat{C}_G(g)$ to be the quotient $(\mathbb{R} \times C_G(g))/\mathbb{Z}$, where $\mathbb{Z}$ is embedded in $\mathbb{R} \times C_G(g)$ via the group homomorphism sending 1 to $(1, g)$. Recall that for a discrete group G we have $H_*(G, k) \cong H_*(BG; k)$, where BG is the classifying space of G. On the other hand, we regard $\hat{C}_G(g)$ as a one dimensional Lie group, so we need to use classifying space homology, and we have

$$HC_*(kG) \cong \bigoplus_g H_*(B\hat{C}_G(g); k).$$

The Connes exact sequence (Connes [7] §II.4, Loday and Quillen [19]) connecting Hochschild and cyclic homology

$$\cdots \to HH_{n+2}(kG) \xrightarrow{I} HC_{n+2}(kG) \xrightarrow{S} HC_n(kG) \xrightarrow{B} HH_{n+1}(kG) \to \cdots \tag{2.1}$$

respects the centraliser decomposition (Burghelea [5]), and may be described as follows. Applying the classifying space construction to the short exact sequence of Lie groups

$$1 \to \mathbb{Z} \to \mathbb{R} \times C_G(g) \to \hat{C}_G(g) \to 1$$

we obtain a fibration sequence

$$S^1 \to BC_G(g) \to B\hat{C}_G(g), \tag{2.2}$$

where $\mathbb{R}/\mathbb{Z} = S^1 = B\mathbb{Z}$. The (co-)homology of S^1 is $\mathbb{Z}$ in degree 0 and 1 and vanishes in all other degrees. In particular, $H_1(S^1;\mathbb{Z}) = H_1(B\mathbb{Z};\mathbb{Z}) \cong \mathbb{Z}$, and we write ν for the element of $H_1(S^1;\mathbb{Z})$ corresponding to $1 \in \mathbb{Z}$. Dually, we write μ for the element of $H^1(S^1;\mathbb{Z}) = H^1(B\mathbb{Z};\mathbb{Z}) \cong \mathsf{Hom}(\mathbb{Z},\mathbb{Z}) \cong \mathbb{Z}$ representing the identity element. We use the same letters μ and ν for their images in $H^1(\mathbb{Z},k)$ and $H_1(\mathbb{Z},k)$.

The Künneth formula yields a canonical identification

$$H_n(\mathbb{Z} \times G, k) = H_0(\mathbb{Z},k) \otimes_k H_n(G,k) \;\oplus\; H_1(\mathbb{Z},k) \otimes_k H_{n-1}(G,k)$$

(see Weibel [25], Theorem 3.6.1), where the summand $H_0(\mathbb{Z},k) \otimes_k H_n(G,k)$ is equal to the image of $H_n(G,k)$ under the map induced by the canonical inclusion $G \to \mathbb{Z} \times G$. Using ν yields an identification

$$H_n(\mathbb{Z} \times G, k) = H_n(G,k) \oplus H_{n-1}(G,k).$$

Similarly, using μ yields an identification

$$H^n(\mathbb{Z} \times G, k) = H^n(G,k) \oplus H^{n-1}(G,k)\ ,$$

where $H^n(G,k)$ is indentified to its image via the map induced by the canonical projection $\mathbb{Z} \times G \to G$. See Proposition 6.4 for more details on this identification.

The Serre spectral sequence of the fibration (2.2) has two non-vanishing rows, and therefore induces a long exact sequence

$$\begin{aligned}\cdots \to H_{n+2}(BC_G(g);k) &\xrightarrow{I} H_{n+2}(B\hat{C}_G(g);k) \\ &\xrightarrow{S} H_n(B\hat{C}_G(g);k) \xrightarrow{B} H_{n+1}(BC_G(g);k) \to \cdots\end{aligned} \tag{2.3}$$

where we have used ν to identify $E^2_{*,1}$ with $H_*(B\hat{C}_G(g);k)$. Note that the map I is induced by the inclusion $BC_G(g) \to B\hat{C}_G(g)$.

It was observed by Burghelea [5] that the Connes sequence is the direct sum of these sequences. Note that the maps in these sequences depend only on g and $C_G(g)$, but not on G itself. So in the following theorem, we assume that g is central in G, and we identify $H_n(\mathbb{Z}\times G,k)$ with $H_n(G,k)\oplus H_{n-1}(G,k)$ as described above, using the element $\nu \in H_1(\mathbb{Z},k)$.

Theorem 2.4. *Let G be a discrete group with a central element $g \in Z(G)$. Consider the group homomorphism $z_g\colon \mathbb{Z} \times G \to G$ which sends (m,h) to $g^m h$. The induced map*

$$(z_g)_*\colon H_n(\mathbb{Z} \times G, k) = H_n(G,k) \oplus H_{n-1}(G,k) \to H_n(G,k)$$

has the form $\binom{1}{\psi}$ *where* ψ *is equal to the composite*

$$H_{n-1}(G,k) \xrightarrow{I} H_{n-1}(\hat{G},k) \xrightarrow{B} H_n(G,k).$$

Proof. The map $G \to \mathbb{Z} \times G$ sending h to $(0,h)$ is a section both of z_g and of the canonical projection $\mathbb{Z} \times G \to G$, so $(z_g)_*$ has the form $\binom{1}{\psi}$.

Consider the diagram of groups

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z} \times G & \longrightarrow & G & \longrightarrow & 1 \\ & & \| & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{R} \times G & \longrightarrow & \hat{G} & \longrightarrow & 1 \end{array}$$

Here, the upper sequence is just the direct product sequence, and the lower sequence is the defining sequence for $\hat{G}$. The middle vertical map sends (n,h) to $(n, z_g(h)) = (n, g^n h)$, while the right hand vertical map is the inclusion. It is easy to check that the two squares commute.

Taking classifying spaces, we have a map of fibrations

$$\begin{array}{ccccc} S^1 & \longrightarrow & S^1 \times BG & \longrightarrow & BG \\ \| & & \downarrow{\scriptstyle (z_g)_*} & & \downarrow \\ S^1 & \longrightarrow & BG & \longrightarrow & B\hat{G} \end{array}$$

and hence a map of long exact sequences

$$\begin{array}{ccccccccc} \cdots & \xrightarrow{0} & H_{n-1}(BG;k) & \xrightarrow{\binom{0}{1}} & H_n(BG;k) \oplus H_{n-1}(BG;k) & \xrightarrow{(1,0)} & H_n(BG;k) & \xrightarrow{0} & \cdots \\ & & \downarrow{\scriptstyle I} & & \downarrow{\scriptstyle (1,\psi)} & & \downarrow{\scriptstyle I} & & \\ \cdots & \longrightarrow & H_{n-1}(B\hat{G};k) & \xrightarrow{B} & H_n(BG;k) & \xrightarrow{I} & H_n(B\hat{G};k) & \longrightarrow & \cdots \end{array}$$

The commutativity of the left hand square proves the theorem. □

3. The BV operator Δ on $HH^*(kG)$

For any algebra Λ which is projective as a module over the coefficient ring k, to set up a duality between Hochschild homology and cohomology, we need to use coefficients Λ in homology and $\Lambda^\vee = \mathsf{Hom}_k(\Lambda, k)$ in cohomology. This is because if P_* is a projective resolution of Λ as a Λ-Λ-bimodule, then

$$\mathsf{Hom}_{\Lambda^e}(P_*, \Lambda^\vee) \cong \mathsf{Hom}_k(\Lambda \otimes_{\Lambda^e} P_*, k).$$

Thus if k is a field, then $HH^n(\Lambda, \Lambda^\vee) \cong (HH_n(\Lambda,\Lambda))^\vee$, but for example for $k = \mathbb{Z}$ we have a universal coefficient sequence

$$0 \to \mathsf{Ext}^1_{\mathbb{Z}}(HH_{n-1}(\Lambda,\Lambda), \mathbb{Z}) \to HH^n(\Lambda, \Lambda^\vee) \to \mathsf{Hom}_{\mathbb{Z}}(HH_n(\Lambda,\Lambda),\mathbb{Z}) \to 0.$$

Similarly, cyclic cochains on Λ are described in terms of the Hochschild cochain complex for Λ with coefficients in $\Lambda^\vee$, and are dual to cyclic chains described in terms of the Hochschild chain complex for Λ with coefficients in Λ. The Connes sequence for cyclic cohomology takes the form

$$\cdots \to HH^{n+1}(\Lambda, \Lambda^\vee) \xrightarrow{B} HC^n(\Lambda) \xrightarrow{S} HC^{n+2}(\Lambda) \xrightarrow{I} HH^n(\Lambda, \Lambda^\vee) \to \cdots . \tag{3.1}$$

For g an element in a discrete group G, the Serre spectral sequence in cohomology of the fibration (2.2) induces a long exact sequence

$$\cdots \to H^{n+1}(BC_G(g);k) \xrightarrow{B} H^n(B\hat{C}_G(g);k) \xrightarrow{S} H^{n+2}(B\hat{C}_G(g);k) \xrightarrow{I} H^{n+2}(BC_G(g);k) \to \cdots$$

where we have used μ to identify $E_2^{*,1}$ with $H^*(B\hat{C}_G(g);k)$. Note that the map I is induced by $BC_G(g) \to B\hat{C}_G(g)$.

Theorem 3.2. *Let G be a discrete group and k a commutative ring of coefficients. Then we have a centraliser decomposition*

$$HH^*(kG, kG^\vee) \cong \prod_g H^*(C_G(g), k),$$

where g runs over a set of representatives of the conjugacy classes in G. The Connes sequence is a direct product of the sequences

$$\cdots \to H^{n+1}(BC_G(g);k) \xrightarrow{B} H^n(B\hat{C}_G(g);k) \xrightarrow{S} H^{n+2}(B\hat{C}_G(g);k) \xrightarrow{I} H^{n+2}(BC_G(g);k) \to \cdots$$

Proof. For the centraliser decomposition see [4, Theorem 2.11.2]. The statement on the Connes sequence is dual to Burghelea [5], with essentially the same proof. See also Sections 2.11 to 2.15 of [4]. □

If Λ is a symmetric algebra over k, finitely generated and projective as a k-module, then $\Lambda^\vee$ is isomorphic to Λ as a Λ-Λ-bimodule, but the isomorphism depends on the choice of symmetrising form.

From now on, we assume that G is a finite group. Then for kG, there is a canonical symmetrising form. This is the bilinear pairing $kG \otimes_k kG \to k$ which sends $g \otimes g'$ to $1 \in k$ if $gg' = 1$ and zero otherwise. Using this bilinear pairing, we obtain an isomorphism between $kG^\vee$ and kG, and hence between $HH^*(kG, kG^\vee)$ and $HH^*(kG, kG)$. From now on, we write $HH^*(kG)$ for $HH^*(kG, kG)$. This is a graded commutative ring, whose product structure was elucidated by Siegel and Witherspoon [22].

Definition 3.3. For G finite, we define the operator $\Delta\colon HH^n(kG) \to HH^{n-1}(kG)$ to be the map induced by $I \circ B$ under the isomorphism between $HH^*(kG)$ and $HH^*(kG, kG^\vee)$ given by the symmetrising form on kG described above.

Theorem 3.4. *For G a finite group, we have a centraliser decomposition*

$$HH^*(kG) \cong \bigoplus_g H^*(C_G(g), k).$$

The map Δ preserves the centraliser decomposition of $HH^*(kG)$, and is the sum of the maps $\Delta_g = I \circ B\colon H^*(C_G(g), k) \to H^{*-1}(C_G(g), k)$.

Proof. The centraliser decomposition in the finite case comes from Theorem 3.2, see also Siegel and Witherspoon [22]. Also by Theorem 3.2, the Connes sequence is a direct sum of the Connes sequences for each centraliser, and hence the components Δ_g of Δ are as stated. □

Definition 3.5. A *Gerstenhaber algebra* is a graded k-algebra R which is associative and graded commutative:

$$yx = (-1)^{|x||y|}xy$$

and has a Lie bracket $[x, y]$ of degree -1 which satisfies

(i) Anticommutativity:

$$[y,x] = -(-1)^{(|x|-1)(|y|-1)}[x,y]$$

(ii) The graded Jacobi identity holds:

$$[[x,y],z]] = [x,[y,z]] - (-1)^{(|x|-1)(|y|-1)}[y,[x,z]]$$

(iii) The bracket is a derivation with respect to the product (Leibniz identity):

$$[x,yz] = [x,y]z + (-1)^{(|x|-1)|y|}y[x,z]$$

Definition 3.6. A *Batalin–Vilkovisky algebra* (or BV algebra) is a Gerstenhaber algebra together with an operator Δ of degree -1 satisfying $\Delta \circ \Delta = 0$ and

$$[x,y] = (-1)^{|x|}\Delta(xy) - (-1)^{|x|}(\Delta x)y - x(\Delta y).$$

Thus the multiplication and the BV operator Δ determine the Lie bracket.

For background on BV algebra structures in Hochschild cohomology and relationship with loop space topology, see Abbaspour [1].

Proposition 3.7 (Tradler [23], Menichi [20])**.** *The operator Δ of Definition 3.3 defines a BV operator on $HH^*(kG)$ making it a BV algebra in which the Lie bracket $[-,-]$ is the Gerstenhaber bracket in Hochschild cohomology.*

Remark 3.8. Tradler gives the following formula for Δ at the level of Hochschild cochains, on an algebra Λ with a symmetric, invariant, non-degenerate bilinear form $\langle -,- \rangle \colon \Lambda \times \Lambda \to k$. For $f \in C^n(\Lambda,\Lambda)$, define $\Delta f \in C^{n-1}(\Lambda,\Lambda)$ by

$$\langle \Delta f(a_1,\dots,a_{n-1}),a_n\rangle = \sum_{i=1}^{n}(-1)^{i(n-1)}\langle f(a_i,\dots,a_{n-1},a_n,a_1,\dots,a_{i-1}),1\rangle.$$

Explicit calculations of the BV structure on Hochschild cohomology of finite groups have been made in a number of different cases, see the references in Section 9. Also relevant are the papers of Le and Zhou [15], and Volkov [24]. For k a field, Liu and Zhou [17] have given an explicit description of the BV operator on $HH^*(kG)$ in terms of Hochschild cochains, combining the centraliser decomposition and Tradler's description of the BV operator in [23].

4. The map Δ_g

In this section, we dualise the construction in Theorem 2.4 and use it to describe the components

$$\Delta_g \colon H^n(G,k) \to H^{n-1}(G,k)$$

of the BV operator Δ, where G is a finite group. As mentioned previously, for $g \in Z(G)$, the long exact sequence in cohomology for the fibration

$$S^1 \to BG \to B\hat{G}$$

has the form

$$\cdots \to H^{n+1}(BG;k) \xrightarrow{B} H^n(B\hat{G};k) \xrightarrow{S} H^{n+2}(B\hat{G};k) \xrightarrow{I} H^{n+2}(BG;k) \to \cdots \tag{4.1}$$

where we have used μ to identify $E_2^{*,1}$ with $H^*(B\hat{G};k)$. Again, the map I is induced by $BG \to B\hat{G}$. The following theorem describes the map $\Delta_g\colon H^n(G,k) \to H^{n-1}(G,k)$ coming from the element $g \in Z(G)$.

Theorem 4.2. *Let G be a finite group and let $g \in Z(G)$. Consider the group homomorphism $z_g\colon \mathbb{Z} \times G \to G$ which sends (m,h) to $g^m h$. The induced map*

$$(z_g)^*\colon H^n(G,k) \to H^n(\mathbb{Z} \times G, k) = H^n(G,k) \oplus H^{n-1}(G,k)$$

has the form $(1, \Delta_g)$, where $\Delta_g = I \circ B$ is equal to the composite

$$H^n(G,k) \xrightarrow{B} H^{n-1}(\hat{G},k) \xrightarrow{I} H^{n-1}(G,k).$$

Proof. The proof is dual to the proof of Theorem 2.4. □

The following corollary proves the properties (i), (iii), (iv), (vi), and (viii) in Theorem 1.1.

Corollary 4.3. *Let G be a finite group and let $g \in Z(G)$. Then*

(i) *The map*

$$\Delta_g\colon H^*(G,k) \to H^{*-1}(G,k)$$

is a derivation with respect to the ordinary cohomology cup product. In particular, Δ_g is determined by its values on a generating set of $H^*(G,k)$.

(ii) *The map Δ_g is natural with respect to group homomorphisms, and with respect to homomorphisms of coefficient rings.*

(iii) *If H is a subgroup of G containing g then Δ_g commutes with the transfer map and the restriction map.*

(iv) *In the case $k = \mathbb{F}_p$, the map Δ_g commutes with the Steenrod operations and with the Bockstein homomorphism.*

Proof. The map

$$(z_g)^*\colon H^*(BG,k) \to H^*(S^1 \times BG, k) = H^*(S^1,k) \otimes_k H^*(BG,k)$$

is a k-algebra homomorphism. Let μ be the generator for $H^1(S^1,k)$ chosen so that for $x \in H^*(BG,k)$ we have

$$(z_g)^*(x) = 1 \otimes x + \mu \otimes \Delta_g(x).$$

Then

$$\begin{aligned}(z_g)^*(x \cdot y) &= (1 \otimes x + \mu \otimes \Delta_g(x)) \cdot (1 \otimes y + \mu \otimes \Delta_g(y)) \\ &= 1 \otimes x \cdot y + \mu \otimes (\Delta_g(x) \cdot y + (-1)^{|x|} x \cdot \Delta_g(y)).\end{aligned}$$

Examining the coefficient of μ, we see that

$$\Delta_g(x \cdot y) = \Delta_g(x) \cdot y + (-1)^{|x|} x \cdot \Delta_g(y).$$

This proves (i). The naturality statements (ii)–(iv) follow directly from the fact that $(z_g)^* = (1, \Delta_g)$. □

For degree one elements, it is easy to describe Δ_g; this is statement (ix) in Theorem 1.1.

Proposition 4.4. *Let G be a finite group, and let $g \in Z(G)$. Then identifying $H^1(G,k)$ with $\mathsf{Hom}(G,k)$, we have*

$$\Delta_g(x) = x(g)$$

for any $x \in H^1(G,k)$.

Proof. The composite $\mathbb{Z} \times G \xrightarrow{z_g} G \xrightarrow{x} k$ is equal to $1 \otimes x + \mu \otimes x(g)$. □

Remark 4.5. It follows from Proposition 4.4 that if g is in the derived subgroup of G, then Δ_g is identically zero on $H^1(G, k)$.

The next result is statement (x) in Theorem 1.1.

Proposition 4.6. *Let G be a finite group and let $g \in Z(G)$. Then the map Δ_g vanishes on the image of*

$$H^2(G, \mathbb{Z}) \to H^2(G, k).$$

Proof. Consider the commutative diagram

$$\begin{array}{ccc} H^2(G,\mathbb{Z}) & \xrightarrow{\Delta_g} & H^1(G,\mathbb{Z}) \\ \downarrow & & \downarrow \\ H^2(G,k) & \xrightarrow{\Delta_g} & H^1(G,k) \end{array}$$

The proposition now follows from the fact that since G is finite, we have $H^1(G, \mathbb{Z}) = \mathsf{Hom}(G, \mathbb{Z}) = 0$. □

5. Degree -1 operators on $\mathsf{Ext}^*_A(U, V)$

We describe an elementary construction principle for degree -1 operators on $\mathsf{Ext}^*_A(U, V)$ determined by a central element in an algebra A which annihilates both modules U and V. We use this in the next section to interpret the BV operator in terms of this construction principle. Let k be a commutative ring.

Theorem 5.1. *Let A be a k-algebra, let $z \in Z(A)$, and let U, V be A-modules. Suppose that z annihilates both U and V. Let $P = (P_n)_{n\geqslant 0}$ together with a surjective A-homomorphism $\pi\colon P_0 \to U$ be a projective resolution of U, with differential $\delta = (\delta_n\colon P_n \to P_{n-1})_{n\geqslant 1}$. For notational convenience, set $P_i = 0$ for $i < 0$ and $\delta_i = 0$ for $i \leqslant 0$. Then the following hold.*

(i) *There is a graded A-homomorphism $s\colon P \to P$ of degree 1 such that the chain endomorphism $\delta \circ s + s \circ \delta$ of P is equal to multiplication by z on P.*

(ii) *The graded k-linear map*

$$s^\vee = \mathsf{Hom}_A(s, V)\colon \mathsf{Hom}_A(P, V) \to \mathsf{Hom}_A(P[1], V)$$

sending $f \in \mathsf{Hom}_A(P_n, V)$ to $f \circ s \in \mathsf{Hom}_A(P_{n-1}, V)$ for all $n \geqslant 0$ is a homomorphism of cochain complexes. In particular, $s^\vee$ induces a graded k-linear map of degree -1

$$D^A_z = H^*(s^\vee)\colon \mathsf{Ext}^*_A(U, V) \to \mathsf{Ext}^{*-1}_A(U, V)$$

(iii) *The graded map D^A_z is independent of the choice of the projective resolution P and of the choice of the the homotopy s satisfying (i). In particular, we have $D^A_0 = 0$.*

Remark 5.2. If A is obvious from the context, we write D_z instead of D^A_z. Note that D_z is a graded map which depends on U and V. With the notation of the Theorem, we use the following sign conventions. For i an integer, the shifted complex $P[i]$ is equal, in degree n, to P_{n-i}, with differential $(-1)^i\delta$. The cochain complex $\mathsf{Hom}_A(P, V)$ has differential in degree n given by precomposing with $(-1)^{n+1}\delta_{n+1}$. (This is consistent with the standard sign conventions, as described in [3, Section 2.7], for total complexes of double complexes of the form $\mathsf{Hom}_A(P,Q)$, where Q is another chain complex, modulo regarding $\mathsf{Hom}_A(P,V)$ as a chain complex.) Combining the above sign conventions for shifts and total complexes, we get that the chain complex $P[1]$ has differential $-\delta$ and the cochain complex $\mathsf{Hom}_A(P[1],V)$ has in degree n the differential sending $f \in \mathsf{Hom}_A(P_{n-1},V)$ to $-(-1)^{n+1} f \circ \delta_n = (-1)^n f \circ \delta_n \in \mathsf{Hom}_A(P_n,V)$. The sign convention for cochain complexes of the form $\mathsf{Hom}_A(P,V)$ has no impact on the definition of the operators D_z, but it does matter for the signs of Bockstein homomorphisms. Had we chosen the differential of $\mathsf{Hom}_A(P,V)$ simply being given by precomposing with δ, then the Bockstein homomorphisms in Proposition 5.9 below would anticommute with the operators D_z.

Proof of Theorem 5.1. Multiplication by z annihilates U. Thus the chain endomorphism of P induced by multiplication with z is homotopic to zero. This proves the existence of a homotopy s satisfying (i). Let n be a nonnegative integer. In order to show that $s^\vee$ is a cochain map, we need to show that the following diagram of k-linear maps

$$\begin{array}{ccccccc}
\cdots \longrightarrow & \mathsf{Hom}_A(P_n,V) & \xrightarrow{(-1)^{n+1}\delta_{n+1}^\vee} & \mathsf{Hom}_A(P_{n+1},V) & \longrightarrow \cdots \\
& \downarrow{\scriptstyle s_{n-1}^\vee} & & \downarrow{\scriptstyle s_n^\vee} & \\
\cdots \longrightarrow & \mathsf{Hom}_A(P_{n-1},V) & \xrightarrow[(-1)^n\delta_n^\vee]{} & \mathsf{Hom}_A(P_n,V) & \longrightarrow \cdots
\end{array}$$

is commutative. The commutativity of this diagram is equivalent to

$$(-1)^{n+1} f \circ \delta_{n+1} \circ s_n = (-1)^n f \circ s_{n-1} \circ \delta_n$$

hence to

$$f \circ (\delta_{n+1} \circ s_n + s_{n-1} \circ \delta_n) = 0\ ,$$

for all $f \in \mathsf{Hom}_A(P_n,V)$. By the choice of the homotopy s, the left side is equal to $f \circ \zeta$, where $\zeta\colon P_n \to P_n$ is equal to multiplication by z. Since $f(zP_n) = zf(P_n) \subseteq zV = \{0\}$, it follows that $f \circ \zeta = 0$. This shows that $s^\vee$ is a cochain map. Taking cohomology, $s^\vee$ induces a degree -1 map D_z as stated, whence (ii).

Let P' be a projective resolution of U, with differential δ' and quasi-isomorphism $P' \to U$ given by a map $\pi' : P'_0 \to U$. Let s' be a homotopy on P' with the property that the chain endomorphism $\delta' \circ s' + s' \circ \delta'$ of P' is equal to multiplication by z on P'. Let $a : P \to P'$ be a chain homotopy equivalence lifting the identity on U, via the maps π and π'. We need to show that the homotopies $a \circ s$ and $s' \circ a$ from $P[1]$ to P' induce the same map upon applying $\mathsf{Hom}_A(-,V)$. Set $t = a \circ s - s' \circ a$. We will use the same letter ζ for the chain endomorphisms of P and P' given by multiplication with z. Using that a commutes with the differentials of P and P', we have

$$\delta' \circ t + t \circ \delta = a \circ \delta \circ s - \delta' \circ s' \circ a + a \circ s \circ \delta - s' \circ \delta' \circ a = a \circ \zeta - \zeta \circ a = 0\ .$$

Taking into account that the differential of $P[1]$ is $-\delta$, this implies that t is in fact a chain map from $P[1]$ to P', or equivalently, from P to $P'[-1]$. The homotopy class of such a chain map represents an element in $\mathsf{Ext}_A^{-1}(U,U) = \{0\}$, and hence the chain map t is homotopic to zero. That is, there is a graded map $u : P[1] \to P'$ of degree 1 such that $t = \delta' \circ u - u \circ \delta$, where as before the sign comes from the fact that the differential of $P[1]$ is $-\delta$. Since t is a chain map, it follows that $t^\vee = \mathsf{Hom}_A(t,V)\colon \mathsf{Hom}_A(P',V) \to \mathsf{Hom}_A(P[1],V)$ is a cochain map. The functor $\mathsf{Hom}_A(-,V)$ sends the homotopy u to a homotopy $u^\vee$ satisfying $t^\vee =$

$u^\vee \circ \delta^\vee - \delta^\vee \circ u^\vee$. We need to adjust $u^\vee$ with the signs needed to compensate for the signs in the differentials of $\mathsf{Hom}_A(P', V)$ and $\mathsf{Hom}_A(P[1], V)$ according to the sign convention in Remark 5.2. More precisely, one checks that $((-1)^{n+1}u_n^\vee)$ is the homotopy which is needed to show that the cochain map $t^\vee$ is homotopic to zero. Thus $t^\vee$ induces the zero map on cohomology. This shows that the maps $s^\vee$ and $(s')^\vee$ induce the same map D_z upon taking cohomology, hence in particular $D_0 = 0$. This proves (iii). □

Let A be an algebra over a commutative ring k. Any element $z \in Z(A)$ induces a chain endomorphism on a projective resolution P of an A-module U, and hence a graded linear endomorphism on $\mathsf{Ext}_A^*(U, V)$, for any two A-modules U, V. In this way, the space of graded k-linear endomorphisms of $\mathsf{Ext}_A^*(U, V)$ of any fixed degree becomes a module over $Z(A)$. Since multiplication by z induces an element in the centre of the module category of A, it follows easily that this module structure does not depend on the choice of the projective resolution P, and moreover, for the same reason, it makes no difference whether we precompose an element in $\mathsf{Ext}_A^*(U, V)$ by the endomorphism on U given by z or whether we compose this element with the endomorphism on V given by z.

Theorem 5.3. *Let A be a k-algebra, let U, V be A-modules, and let y, $z \in Z(A)$.*

(i) *Suppose that y and z annihilate U and V. Then $y + z$ annihilates U and V, and we have $D_{y+z} = D_y + D_z$.*
(ii) *Suppose that z annihilates U and V. Then yz annihilates U and V, and we have $D_{yz} = yD_z$.*
(iii) *Suppose that y and z annihilate U and V. Then we have $D_{yz} = 0$.*
(iv) *Let e be an idempotent in $Z(A)$ such that e annihilates U and V. Then $D_e = 0$.*

Proof. For (i), suppose that y and z annihilate U and V. Then clearly $y + z$ annihilates U and V. Let P be a projective resolution of U, with differential δ. Let s, t be a homotopies on P such that $\delta \circ s + s \circ \delta$ is the chain endomorphism of P given by multiplication with y, and such that $\delta \circ t + t \circ \delta$ is the chain endomorphism of P given by multiplication with z. Then $\delta \circ (s+t) + (s+t) \circ \delta$ is the endomorphism of P given by multiplication with $y+z$. Statement (i) follows. For (ii), suppose that z annihilates U and V. Then clearly yz annihilates U and V. As before, let t be a homotopy on P such that $\delta \circ t + t \circ \delta$ is the chain endomorphism of P given by multiplication with z. Denote by $y \cdot t$ the homotopy obtained from composing (or precomposing - this makes no difference) t with the endomorphism given by multiplication with y. Then $\delta \circ (y \cdot t) + (y \cdot t) \circ \delta$ is equal to multiplication on P by yz, which shows that $D_{yz} = yD_z$. Statement (iii) follows from (ii) and the fact that multiplication by y annihilates V, hence annihilates the space $\mathsf{Ext}_A^*(U, V)$. Since $e = e^2$, statement (iv) is a special case of (iii). □

The operators D_z^A are compatible with the restriction to subalgebras B containing z such that A is projective as a B-module.

Proposition 5.4. *Let A be a k-algebra and let B be a subalgebra of A such that A is projective as a left B-module. Let $z \in Z(A) \cap B$, and let U, V be A-modules. Suppose that*

z annihilates U and V. We have a commutative diagram of graded maps

$$\begin{array}{ccc} \mathsf{Ext}^*_A(U,V) & \xrightarrow{D_z^A} & \mathsf{Ext}^{*-1}_A(U,V) \\ \downarrow & & \downarrow \\ \mathsf{Ext}^*_B(\mathrm{Res}^A_B(U),\mathrm{Res}^A_B(V)) & \xrightarrow[D_z^B]{} & \mathsf{Ext}^{*-1}_B(\mathrm{Res}^A_B(U),\mathrm{Res}^A_B(V)) \end{array}$$

where the vertical maps are induced by the restriction to B.

Proof. Let P be a projective resolution of A. By the assumptions on B, the restriction to B of P is a projective resolution of $\mathrm{Res}^A_B(U)$. Thus if s is a homotopy on P which defines D_z^A as in Theorem 5.1, then the restriction to B of s is the corresponding homotopy for D_z^B. The commutativity of the diagram follows immediately from the construction of the maps D_z^A and D_z^B. □

Proposition 5.5. *Let G a finite group, H a subgroup of G and $z \in Z(kG) \cap kH$. Let U, V be kG-modules, and suppose that z annihilates U and V. We have a commutative diagram of graded maps*

$$\begin{array}{ccc} \mathsf{Ext}^*_{kH}(\mathrm{Res}^G_H(U),\mathrm{Res}^G_H(V)) & \xrightarrow{D_z^{kH}} & \mathsf{Ext}^{*-1}_{kH}(\mathrm{Res}^G_H(U),\mathrm{Res}^G_H(V)) \\ {\scriptstyle \mathrm{Tr}^G_H}\downarrow & & \downarrow{\scriptstyle \mathrm{Tr}^G_H} \\ \mathsf{Ext}^*_{kG}(U,V) & \xrightarrow[D_z^{kG}]{} & \mathsf{Ext}^{*-1}_B(U,V) \end{array}$$

Proof. Let P be a projective resolution of the kG-module U, with differential denoted δ. By the assumptions on z and by Theorem 5.1 there is a homotopy s on P such that $\delta \circ s + s \circ \delta$ is equal to the chain endomorphism of P given by multiplication with z. The operator D_z^{kG} is induced by the map sending $f \in \mathsf{Hom}_{kG}(P_n, V)$ to $f \circ s_{n-1}$. Since $\mathrm{Res}^G_H(P)$ is a projective resolution of $\mathrm{Res}^G_H(U)$, it follows that D_z^{kH} is induced by the map sending $f' \in \mathsf{Hom}_{kH}(\mathrm{Res}^G_H(P_n), \mathrm{Res}^G_H(V))$ to $f' \circ s_{n-1}$. Since s_{n-1} is a kG-homomorphism, we have $\mathrm{Tr}^G_H(f') \circ s_{n-1} = \mathrm{Tr}^G_H(f' \circ s_{n-1})$, proving the result. □

The operators D_z^A satisfy a Künneth formula. We suppress the superscripts in what follows, since the central element subscripts determine which algebra we a working in.

Proposition 5.6. *Let A, B be k-algebras such that A, B are finitely generated projective as k-modules, let U, U' be A-modules and V, V' be B-modules, all finitely generated projective as k-modules. Let $z \in Z(A)$ and $w \in Z(B)$ such that z annihilates U, U' and w annihilates V, V'.*

Then $z \otimes 1$ and $1 \otimes w$ annihilate the $A \otimes_k B$-modules $U \otimes_k V$ and $U' \otimes_k V'$, and we have a commutative diagram

$$\begin{array}{ccc} \mathsf{Ext}^i_A(U,U') \otimes_k \mathsf{Ext}^j_B(V,V') & \longrightarrow & \mathsf{Ext}^{i+j}_{A\otimes_k B}(U \otimes_k V, U' \otimes_k V') \\ {\scriptstyle (D_z\otimes 1,\ 1\otimes D_w)}\downarrow & & \downarrow{\scriptstyle D_{z\otimes 1+1\otimes w}} \\ (\mathsf{Ext}^{i-1}_A(U,U') \otimes_k \mathsf{Ext}^j_B(V,V')) \oplus (\mathsf{Ext}^i_A(U,U') \otimes_k \mathsf{Ext}^{j-1}_B(V,V')) & \longrightarrow & \mathsf{Ext}^{i+j-1}_{A\otimes_k B}(U \otimes_k V, U' \otimes_k V') \end{array}$$

where i, j are nonnegative integers. Moreover, we have $D_{z\otimes w} = 0$. In particular, if k is a field, then $D_{z\otimes 1+1\otimes w}$ is determined by D_z, D_w, combined with the Künneth formula.

Proof. Note the following sign convention (as in statement (ii) of Theorem 1.1): the second component $1 \otimes D_w$ of the left vertical map sends $\eta \otimes \theta$ to $(-1)^i \eta \otimes D_w(\theta)$, where $\eta \in \mathsf{Ext}^i_A(U, U')$ and $\theta \in \mathsf{Ext}^j_B(V, V')$.

The assumptions on z and w imply that $z \otimes 1$ and $1 \otimes w$ annihilate the $A \otimes_k B$-modules $U \otimes_k V$ and $U' \otimes_k V'$. Let (P, δ) be a projective resolution of U and (Q, ϵ) a projective resolution of V. Since A and B are projective as k-modules, it follows that $P \otimes_k Q$ is a projective resolution of the $A \otimes_k B$-module $U \otimes_k V$. Note the signs in the differential $\delta \otimes \epsilon$ of $P \otimes_k Q$; more precisely, the differential $\delta \otimes \epsilon$ sends $u \otimes v \in P_i \otimes_k Q_j$ to

$$(\delta_i(u) \otimes v,\ (-1)^i u \otimes \epsilon_j(v))$$

in $(P_{i-1} \otimes_k Q_j) \oplus (P_i \otimes_k Q_{j-1})$.

We have a canonical isomorphism of cochain complexes

$$\mathsf{Hom}_A(P, U') \otimes_k \mathsf{Hom}_B(Q, V') \cong \mathsf{Hom}_{A \otimes_k B}(P \otimes_k Q, U' \otimes_k V')$$

thanks to the assumptions that the involved algebras and modules are finitely generated projective as k-modules. Upon taking cohomology, this induces the horizontal maps

$$\mathsf{Ext}^i_A(U, U') \otimes_k \mathsf{Ext}^j_B(V, V') \to \mathsf{Ext}^{i+j}_{A \otimes_k B}(U \otimes_k V, U' \otimes_k V')$$

in the statement.

Let s be a homotopy on P such that $s \circ \delta + \delta \circ s$ is equal to multiplication by z on P. Similarly, let t be a homotopy on Q such that $t \circ \epsilon + \epsilon \circ t$ is equal to multiplication by w on Q. Define the homotopy σ on $P \otimes_k Q$ by sending $u \otimes v \in P_i \otimes_k Q_j$ to

$$\sigma(u \otimes v) = (s_i(u) \otimes v,\ (-1)^i u \otimes t_j(v))$$

in $(P_{i+1} \otimes_k Q_i) \oplus (P_i \otimes_k Q_{j+1})$. The sign $(-1)^i$ is needed because of the above mentioned sign convention for $1 \otimes D_w$. We need to show that this homotopy has the property that the chain endomorphism of $P \otimes_k Q$ given by

$$\sigma \circ (\delta \otimes \epsilon) + (\delta \otimes \epsilon) \circ \sigma$$

is equal to multiplication by $z \otimes 1 + 1 \otimes w$ on $P \otimes_k Q$.

Let $u \otimes v \in P_i \otimes_k Q_j$. We calculate first the image of $u \otimes v$ under $\sigma \circ (\delta \otimes \epsilon)$. The differential $\delta \otimes \epsilon$ sends $u \otimes v$ to the element

$$(\delta_i(u) \otimes v, (-1)^i u \otimes \epsilon_j(v))$$

in $(P_{i-1} \otimes_k Q_j) \oplus (P_i \otimes_k Q_{j-1})$. The homotopy σ sends this to the element

$$((-1)^{i-1} \delta_i(u) \otimes t_j(v), s_{i-1}(\delta_i(u)) \otimes v + u \otimes t_{j-1}(\epsilon_j(v)), (-1)^i s_i(u) \otimes \epsilon_j(v))$$

in $(P_{i-1} \otimes_k Q_{j+1}) \oplus (P_i \otimes_k Q_j) \oplus (P_{i+1} \otimes_k Q_{j-1})$.

We calculate next the image of $u \otimes v$ under $(\delta \otimes \epsilon) \circ \sigma$. The homotopy σ sends $u \otimes v$ to the element

$$(s_i(u) \otimes v, (-1)^i u \otimes t_j(v))$$

in $(P_{i+1} \otimes_k Q_j) \oplus (P_i \otimes_k Q_{j+1})$. Applying the differential $\delta \otimes \epsilon$ to this element yields

$$((-1)^i \delta_i(u) \otimes t_j(v),\ \delta_{i+1}(s_i(u)) \otimes v + u \otimes \epsilon_{j+1}(t_j(v)), (-1)^{i+1} s_i(u) \otimes \epsilon_j(v))$$

in $(P_{i-1} \otimes_k Q_{j+1}) \oplus (P_i \otimes_k Q_j) \oplus (P_{i+1} \otimes_k Q_{j-1})$. The sum of the images of $u \otimes v$ under the two maps $\sigma \circ (\delta \circ \epsilon)$ and $(\delta \otimes \epsilon) \circ \sigma$ is therefore equal to

$$(0, ((z \otimes 1) + (1 \otimes w))(u \otimes v), 0)$$

as claimed. The first statement follows. Since $z \otimes w = (z \otimes 1)(1 \otimes w)$, the second statement follows from Theorem 5.3 (iii). ☐

Proposition 5.7. *Let A be a k-algebra, let z, $w \in Z(A)$, and let U, V be A-modules. Suppose that z, w annihilate U and V. The following hold.*

(i) $D_z \circ D_z = 0$.
(ii) $D_w \circ D_z = -D_z \circ D_w$.

Proof. Let $P = (P_n)_{n \geqslant 0}$ together with a surjective A-homomorphism $\pi \colon P_0 \to U$ be a projective resolution of U, with differential $\delta = (\delta_n \colon P_n \to P_{n-1})_{n \geqslant 1}$. As above, for notational convenience, we set $P_i = 0$ for $i < 0$ and $\delta_i = 0$ for $i \leqslant 0$.

Let s be a homotopy on P such that $s \circ \delta + \delta \circ s$ is equal to multiplication by z on P. By the construction from Theorem 5.1, the map D_z is induced by the map sending $f \in \mathsf{Hom}_A(P_n, V)$ to $f \circ s$. Thus $D_z \circ D_z$ is induced by the map sending $f \in \mathsf{Hom}_A(P_n, V)$ to $f \circ s \circ s$. In order to show $D_z \circ D_z = 0$, we need to show that if f is a cocycle (that is, $f \circ \delta_{n+1} = 0$), then $f \circ s \circ s = 0$. For this it suffices to show that the graded degree 2 map $s \circ s$ is a chain map from $P[2]$ to P. Indeed, any such chain map is homotopic to zero (as it represents an element in $\mathsf{Ext}_A^{-2}(U, U) = 0$), hence, upon applying the contravariant functor $\mathsf{Hom}_A(-, V)$, it induces a cochain map $\mathsf{Hom}_A(P, V) \to \mathsf{Hom}_A(P[2], V)$ which is still homotopic to zero and which therefore induces the zero map in cohomology.

Composing and precomposing the chain map $s \circ \delta + \delta \circ s$ with s yields the equations (of graded endomorphisms of P of degree 1)

$$s \circ s \circ \delta + s \circ \delta \circ s = s \cdot z \ ,$$

$$s \circ \delta \circ s + \delta \circ s \circ s = z \cdot s \ .$$

Note that the right sides of the two equations are equal, since s is a (graded) A-homomorphism, so commutes with the action of z. Taking the difference of these two equations yields therefore

$$s \circ s \circ \delta - \delta \circ s \circ s = 0 \ .$$

This shows that $s \circ s$ is indeed a chain map $P[2] \to P$, which by the previous paragraph completes the proof of (i). Statement (ii) follows from applying (i) to $z + w$ and using Theorem 5.3 (i). ☐

Let A be a k-algebra, $z \in Z(A)$, and let U, V be A-modules which are annihilated by z. The operator D_z on $\mathsf{Ext}_A^*(U, V)$ can also be described using an injective resolution (I, ϵ) of V instead of a projective resolution (P, δ) of U. Let $p : P \to U$ and $i : V \to I$ be quasi-isomorphisms, where U, V are regarded as complexes concentrated in degree 0. Denote by $K(A)$ the homotopy cateegory of chain complexes of A-modules. By standard facts (see e. g. [3, Section 2.7] or [25, Section 2.7]), the space $\mathsf{Ext}_A^*(U, V)$ can be identified with any of

$$\mathsf{Hom}_{K(A)}(P, V[n]) \xrightarrow{\cong} \mathsf{Hom}_{K(A)}(P, I[n]) \xleftarrow{\cong} \mathsf{Hom}_{K(A)}(U, I[n]) \ \ ,$$

where the isomorphisms are induced by composing with i and precomposing with p. We reindex complexes as chain complexes, if necessary (so in particular, an injective resolution of V is of the form $I_0 \to I_{-1} \to I_{-2} \to \cdots$).

Theorem 5.8. *Let A be a k-algebra, let $z \in Z(A)$, and let U, V be A-modules which are both annihilated by z. Let (I, ϵ) be an injective resolution of V. Let t be a homotopy on I such that $\epsilon \circ t + t \circ \epsilon$ is equal to multiplication by z on the terms of I. The graded k-linear map $t_\vee : \mathsf{Hom}_A(U, I) \to \mathsf{Hom}_A(U[1], I)$ sending $g \in \mathsf{Hom}_A(U, I_{-n})$ to $(-1)^n t_{-n} \circ g$ is a chain map, and the induced map in cohomology is equal to D_z, where we identify the cohomology of $\mathsf{Hom}_A(U, I)$ and $\mathsf{Ext}^*_A(U, V)$ using the isomorphisms preceding the statement.*

Proof. Since z annihilates V, the existence of a homotopy t on I such that $\epsilon \circ t + t \circ \epsilon$ is equal to multiplication by z is obvious. The verification that the assignment $g \mapsto (-1)^n t \circ g$ is a chain map is analogous to the first part of the proof of Theorem 5.1. (The sign $(-1)^n$ comes from the fact that g is regarded as a chain map $U \to I[n]$, and since the differential of $I[n]$ is $(-1)^n \epsilon$, one needs to use the homotopy $(-1)^n t$ in order to obtain multiplication by z as the chain map determined by this homotopy on $I[n]$.)

Let n be a non-negative integer. Let $g \in \mathsf{Hom}_A(U, I_{-n})$ be a cocycle; that is, $\epsilon \circ g = 0$. Note that this is equivalent to stating that $g : U \to I[n]$ is a chain map, and hence $\tilde{g} = g \circ p : P \to I[n]$ is a chain map. Similarly, let $f \in \mathsf{Hom}_A(P_n, V)$ be a cocycle; that is, $f \circ \delta = 0$. As before, this means that $f : P \to V[n]$ is a chain map, and hence $\tilde{f} = i[n] \circ f : P \to I[n]$ is a chain map.

Assume now that f and g represent the same class in $\mathsf{Ext}^n_A(U, V)$. This is equivalent to requiring that the chain maps $\tilde{f}$, $\tilde{g}$ from P to $I[n]$ are homotopic. Thus there is a homotopy u from P to $I[n]$ such that

$$\tilde{f} - \tilde{g} = u \circ \delta + (-1)^n \circ \epsilon \circ u \ ,$$

where the sign $(-1)^n$ comes from the fact that the differential of $I[n]$ is $(-1)^n \epsilon$.

Let s be a homotopy on P such that $\delta \circ s + s \circ \delta$ is equal to multiplication by z on P. By the construction of D_z, the image of the class of f under D_z is represented by $f \circ s$. Note that $f \circ s : P \to V[n]$ is a chain map (this was noted aleady in the proof of D_z being well-defined: since z annihilates V, we have $0 = f \cdot z = f \circ \delta \circ s + f \circ s \circ \delta = f \circ s \circ \delta$, where we use the assumption $f \circ \delta = 0$, and hence $f \circ s \circ \delta = 0$). We need to show that $f \circ s$ and $(-1)^n t \circ g$ represent the same class in $\mathsf{Ext}^{n-1}_A(U, V)$. That is, we need to show that the chain maps $\tilde{f} \circ s$ and $(-1)^n \tilde{g}$ from P to $I[n-1]$ are homotopic, or equivalently, we need to show that their difference $\tilde{f} \circ s - (-1)^n \tilde{g}$ is homotopic to zero.

Note that $f : P \to V[n]$, and hence also $\tilde{f} = i[n] \circ f : P \to I[n]$, is a chain map which is zero in all degrees other than n. Since $I[n]$ is zero in all degrees bigger than n, it follows that $t \circ \tilde{f} = 0$. Similarly, we have $\tilde{g} \circ s = 0$. It follows that

$$\begin{aligned} \tilde{f} \circ s - (-1)^{n-1} t \circ \tilde{g} &= (\tilde{f} - \tilde{g}) \circ s + (-1)^{n-1} t \circ (\tilde{f} - \tilde{g}) \\ &= u \circ \delta \circ s + (-1)^n \epsilon \circ u \circ s + (-1)^{n-1} t \circ u \circ \delta + (-1)^{2n-1} t \circ \epsilon \circ u \\ &= u \circ \delta \circ s + (-1)^n \epsilon \circ u \circ s + (-1)^{n-1} t \circ u \circ \delta - t \circ \epsilon \circ u. \end{aligned}$$

Since $u \cdot z = u \circ (\delta \circ s + s \circ \delta)$ we have $u \circ \delta \circ s = u \cdot z - u \circ s \circ \delta$. Similarly, we have $t \circ \epsilon \circ u = z \cdot u - \epsilon \circ t \circ u$. Inserting these two equations into the displayed equality and cancelling $u \cdot z = z \cdot u$ yields the expression

$$-(u \circ s \circ \delta + (-1)^{n-1} \epsilon \circ u \circ s) + (-1)^{n-1} (t \circ u \circ \delta + (-1)^{n-1} \epsilon \circ t \circ u) \ .$$

The two summands in this equation are contractible chain maps from P to $I[n-1]$, via the homotopies $u \circ s$ and $t \circ u$, respectively. This shows the result. □

Proposition 5.9. *Let A be a k-algebra, let $z \in Z(A)$, and let n be a nonnegative integer.*

(i) *The map $D_z : \mathsf{Ext}_A^n(U,V) \to \mathsf{Ext}_A^{n-1}(U,V)$ is functorial in A-modules U and in V which are annihilated by z.*

(ii) *For any A-module U and any short exact sequence of A-modules*

$$0 \longrightarrow V \longrightarrow W \longrightarrow X \longrightarrow 0$$

such that z annihilates U, V, W, X, the map D_z commutes with the connecting homomorphisms $\gamma^n : \mathsf{Ext}_A^n(U,X) \to \mathsf{Ext}_A^{n+1}(U,V)$; that is, we have

$$\gamma^{n-1} \circ D_z = D_z \circ \gamma^n \colon \mathsf{Ext}_A^n(U,X) \to \mathsf{Ext}_A^n(U,V) \ .$$

(iii) *For any short exact sequence of A-modules*

$$0 \longrightarrow U \longrightarrow V \longrightarrow W \longrightarrow 0$$

and any A-module X such that z annihilates U, V, W, X, the map D_z commutes with the connecting homomorphisms $\gamma^n : \mathsf{Ext}_A^n(U,X) \to \mathsf{Ext}_A^{n+1}(W,X)$. that is, we have

$$\gamma^{n-1} \circ D_z = D_z \circ \gamma^n \colon \mathsf{Ext}_A^n(U,X) \to \mathsf{Ext}_A^n(W,X) \ .$$

Proof. Let P be a projective resolution of U. Since the operator D_z on $\mathsf{Ext}_A^n(U,V)$ is induced by precomposing $f \in \mathsf{Hom}_A(P_n,V)$ with a homotopy s on P, the functoriality in V is obvious. Using an injective resolution I of V and Theorem 5.8 yields the functoriality in U. This proves (i).

Let U be an A-module which is annihilated by z, and let as before P be a projective resolution of U. Let

$$0 \longrightarrow V \stackrel{i}{\longrightarrow} W \stackrel{p}{\longrightarrow} X \longrightarrow 0$$

be a short exact sequence of A-modules V, W, X which are annihilated by z. Since P consists of projective A-modules, applying $\mathsf{Hom}_A(P,-)$ yields a short exact sequence of cochain complexes

$$0 \longrightarrow \mathsf{Hom}_A(P,V) \longrightarrow \mathsf{Hom}_A(P,W) \longrightarrow \mathsf{Hom}_A(P,X) \longrightarrow 0 \ .$$

The connecting homomorphism γ associated to this sequence is constructed as follows. Let $f \in \mathsf{Hom}_A(P_n,X)$ be a cocycle; that is, $f \circ \delta_{n+1} = 0$. This represents a class $\underline{f}$ in $\mathsf{Ext}_A^n(U,X)$. Let $g \in \mathsf{Hom}_A(P_n,W)$ such that $p \circ g = f$. Then $g \circ \delta_{n+1}$ satisfies $p \circ g \circ \delta_{n+1} = f \circ \delta_{n+1} = 0$, so $g \circ \delta_{n+1}$ factors through i. Let $h \in \mathsf{Hom}_A(P_{n+1},V)$ such that

$$i \circ h = (-1)^{n+1} g \circ \delta_{n+1} \ .$$

Then h is a cocycle, and by our sign conventions from Remark 5.2 regarding the differential of $\mathsf{Hom}_A(P,W)$, the class of h in $\mathsf{Ext}_A^{n+1}(U,V)$ is $\gamma^n(\underline{f})$. As before, we suppress subscripts and superscripts to δ, s, γ, and write abusively $h = \gamma(\underline{f})$. Now $D_z(\underline{f})$ is represented by $f \circ s$, and clearly $g \circ s$ lifts $f \circ s$ through p. Therefore, by the same construction as before, applied to $f \circ s$, the class of $\gamma(f \circ s)$ is represented by the map m satisfying $i \circ m = (-1)^n g \circ s \circ \delta$. Since $s \circ \delta + \delta \circ s$ is equal to multiplication by z on P, and since z annihilates W and commutes with g, it follows that $g \circ (s \circ \delta + \delta \circ s) = 0$, and hence

$$i \circ m = (-1)^n g \circ s \circ \delta = -(-1)^n g \circ \delta \circ s = i \circ h \circ s \ .$$

Since i is a monomorphism, this implies $m = h \circ s$. The map m represents the class of $\gamma(D_z(f))$, and $h \circ s$ represents the class of $D_z(\gamma(f))$. Statement (ii) follows. A similar argument, using an injective resolution of X and Theorem 5.8, yields (iii). □

Remark 5.10. With the notation of Theorem 5.1, if A is a finite-dimensional selfinjective algebra over a field k, then the construction principle of degree -1 operators in Theorem 5.1 extends to Tate-Ext, by replacing a projective resolution of U with a complete resolution of U. More precisely, let (P, δ) be a complete resolution of U; that is, P is an acyclic chain complex of projective A-modules together with an isomorphism $\mathrm{Im}(\delta_0) \cong U$. Then $\widehat{\mathsf{Ext}}_A^n(U, V) \cong H^n(\mathsf{Hom}_A(P, V))$ for all integers n; for n positive this coincides with $\mathsf{Ext}_A^n(U, V)$. If $z \in Z(A)$ annihilates U, then multiplication by z on P is a chain endomorphism which is homotopic to zero, and thus there is a homotopy s on P such that $s \circ \delta + \delta \circ s$ is equal to multiplication by z. (In fact, for this part of the construction, it suffices to assume that the endomorphism of U given by multiplication with z factors through a projective module). If z also annihilates V, then just as in the proof of Theorem 5.1 the correspondence sending $f \in \mathsf{Hom}_A(P_n, V)$ to $f \circ s$ induces for any integer n an operator $\hat{D}_z : \widehat{\mathsf{Ext}}_A^n(U, V) \to \widehat{\mathsf{Ext}}_A^{n-1}(U, V)$, which for $n \geqslant 2$ coincides with the operator D_z.

6. The BV operator in terms of homotopies on projective resolutions

Let k be a commutative ring. For G a finite group and an element $g \in Z(G)$ we denote as before for any positive integer n by $\Delta_g = I \circ B \colon H^n(G, k) \to H^{n-1}(G, k)$ the map obtained from the long exact sequences (3.1). The following result shows that Δ_g can be obtained as a special case of the construction described in Theorem 5.1, implying in particular that the component Δ_1 of the BV operator Δ on the summand $H^*(G, k)$ in the centraliser decomposition of $HH^*(kG)$ corresponding to the unit element 1 of G is zero.

Theorem 6.1. *Let G be a finite group and $g \in Z(G)$. With the notation from Theorem 5.1, we have $\Delta_g = D_{g-1}$. In particular, we have $\Delta_1 = D_0 = 0$.*

In order to show this, we will make use of Theorem 4.2. As before, we denote by

$$z_g \colon \mathbb{Z} \times G \to G$$

the group homomorphism sending (n, h) to $g^n h$, where $n \in \mathbb{Z}$ and $h \in G$. Note that z_1 is the canonical projection onto the second component of $\mathbb{Z} \times G$. In order to describe the map induced by z_g on cohomology, we choose a projective resolution $P_{\mathbb{Z}}$ of k as a $k\mathbb{Z}$-module. Here $k\mathbb{Z}$ is the group algebra over k of the infinite cyclic group $(\mathbb{Z}, +)$. We will need to describe quasi-isomorphisms $P_{\mathbb{Z}} \otimes_k P_G \to z_g^*(P_G)$ as complexes of $k(\mathbb{Z} \times G)$-modules.

Identify $k\mathbb{Z} = k[u, u^{-1}]$ for some indeterminate u via the unique algebra isomorphism sending $1_{\mathbb{Z}}$ to u. We choose for $P_{\mathbb{Z}}$ the two-term complex (in degrees 1 and 0) of the form

$$k[u, u^{-1}] \xrightarrow{u-1} k[u, u^{-1}]$$

where the superscript $u - 1$ is the map given multiplication with $u - 1$. This is a projective resolution of k as a $k[u, u^{-1}]$-module, together with the augmentation map $k[u, u^{-1}] \to k$ sending u to 1. Note that pairs consisting of an infinite cyclic group and a generator are unique up to unique isomorphism. Choosing a generator is equivalent to choosing a projective resolution of the form above.

We will make use of the following special case of the Tensor-Hom-adjunction. We adopt the following shorthand: for any kG-module M we write

$$M[u,u^{-1}] = k[u,u^{-1}] \otimes_k M \ .$$

Lemma 6.2. *With the notation above, let M, N be kG-modules. We have a natural isomorphism of k-modules*

$$\mathsf{Hom}_{k(\mathbb{Z}\times G)}(M[u,u^{-1}], z_g^*(N)) \cong \mathsf{Hom}_{kG}(M,N)$$

sending a $k(\mathbb{Z}\times G)$-homomorphism $f\colon M[u,u^{-1}] \to z_g^*(N)$ to the kG-homomorphism $M \to N$ given by $m \mapsto f(1\otimes m)$ for all $m \in M$.

Let $P_G = (P_n)_{n\geqslant 0}$ be a projective resolution of the trivial kG-module, with differential $(\delta_n)_{n\geqslant 1}$. We adopt the convention $P_{-1} = \{0\}$ and $\delta_0 = 0$. Note that the $k(\mathbb{Z}\times G)$-module structure of $z_g^*(P_n)$ is given via $u^i \otimes h$ acting as left multiplication by $g^i h$ on P_n, where $n \geqslant 0$. The degree n term of $P_{\mathbb{Z}} \otimes_k P_G$ is equal to

$$P_n[u,u^{-1}] \oplus P_{n-1}[u,u^{-1}]$$

for any $n \geqslant 0$. Denote by $\eta = (\eta_n)_{n\geqslant 1}$ the differential of $P_{\mathbb{Z}} \otimes_k P_G$. Use the same letter δ_n for the obvious extension $\mathrm{Id}\otimes\delta_n$ of δ_n to $P_n[u,u^{-1}]$. The differential η is given in degree $n \geqslant 1$ by

$$\eta_n : \qquad P_n[u,u^{-1}] \oplus P_{n-1}[u,u^{-1}] \xrightarrow{\begin{pmatrix} \delta_n & u-1 \\ 0 & -\delta_{n-1} \end{pmatrix}} P_{n-1}[u,u^{-1}] \oplus P_{n-2}[u,u^{-1}] \tag{6.3}$$

We describe the identification $H^n(\mathbb{Z}\times G,k) = H^n(G,k) \oplus H^{n-1}(G,k)$ in Theorem 4.2 as follows.

Proposition 6.4. *With the notation above, the canonical split exact sequences*

$$0 \longrightarrow P_n[u,u^{-1}] \longrightarrow P_n[u,u^{-1}] \oplus P_{n-1}[u,u^{-1}] \longrightarrow P_{n-1}[u,u^{-1}] \longrightarrow 0$$

define a degreewise split short exact sequence of chain complexes of $k(\mathbb{Z}\times G)$-modules

$$0 \longrightarrow k[u,u^{-1}] \otimes_k P_G \longrightarrow P_{\mathbb{Z}} \otimes_k P_G \longrightarrow k[u,u^{-1}] \otimes_k P_G[1] \longrightarrow 0 \ .$$

Applying $\mathsf{Hom}_{k(\mathbb{Z}\times G)}(-,k)$, with the appropriate signs for the differentials, yields a short exact sequence of cochain complexes of k-modules

$$0 \longrightarrow \mathsf{Hom}_{kG}(P_G[1],k) \longrightarrow \mathsf{Hom}_{k(\mathbb{Z}\times G)}(P_{\mathbb{Z}} \otimes_k P_G, k) \longrightarrow \mathsf{Hom}_{kG}(P_G,k) \longrightarrow 0$$

which splits canonically, and hence yields a canonical identification

$$H^n(\mathbb{Z}\times G,k) = H^n(G,k) \oplus H^{n-1}(G,k) \ .$$

This is the identification in Theorem 4.2.

Proof. The exact sequence of chain complexes of $k(\mathbb{Z}\times G)$-modules is a special case of a tensor product of the chain complex P_G with a two-term chain complex and easily verified. The exact sequence of cochain complexes of k-modules is obtained by applying the contraviariant functor $\mathsf{Hom}_{k(\mathbb{Z}\times G)}(-,k)$ to the previous sequence and then using the canonical adjunctions

$$\mathsf{Hom}_{k(\mathbb{Z}\times G)}(P_n[u,u^{-1}],k) \cong \mathsf{Hom}_{kG}(P_n,k)$$

from Lemma 6.2. The fact that this sequence splits canonically follows from the observation that multiplication by $u-1$ has image in the kernel of any $k(\mathbb{Z}\times G)$-homomorphism $P_n[u,u^{-1}]\to k$ and so the non-diagonal entry $u-1$ in the differential (6.3) of $P_\mathbb{Z}\otimes_k P_G$ becomes zero upon applying the functor $\mathsf{Hom}_{k(\mathbb{Z}\times G)}(-,k)$. To see that this is the identification in Theorem 4.2, consider the class μ in $H^1(\mathbb{Z},k)$ corresponding to the group homomorphism $\mathbb{Z}\to k$ sending $1_\mathbb{Z}$ to 1_k. This is the class of the 1-cocycle (abusively still denoted by the same letter) $\mu : k[u,u^{-1}]\to k$ sending u to 1. The explicit description of the maps in the statement implies that upon taking cohomology in degree n, the map $\mathsf{Hom}_{kG}(P_G[1],k)\to \mathsf{Hom}_{k(\mathbb{Z}\times G)}(P_\mathbb{Z}\otimes_k P_G,k)$ induces a map which sends $x\in H^{n-1}(G,k)$ to the image of $\mu\otimes x$ in $H^n(\mathbb{Z}\times G,k)$, as required. □

For $n\geqslant 0$, denote by

$$e_n\colon P_n[u,u^{-1}]\to z_g^*(P_n)$$

the $k(\mathbb{Z}\times G)$-homomorphism defined by $e_n(u^i\otimes v)=g^iv$ for all $i\in\mathbb{Z}$ and $v\in P_n$. Equivalently, e_n corresponds to the identity on P_n under the adjunction from Lemma 6.2. The following theorem parametrises homotopies in Theorem 5.1 in terms of certain quasi-isomorphisms $P_\mathbb{Z}\otimes_k P_G\to z_g^*(P_G)$ lifting the identity on k.

Theorem 6.5. *With the notation above, for any $n\geqslant -1$ let $f_n\colon P_n[u,u^{-1}]\to z_g^*(P_{n+1})$ be a $k(\mathbb{Z}\times G)$-homomorphism and let $s_n\colon P_n\to P_{n+1}$ be the corresponding kG-homomorphism sending $a\in P_n$ to $f_n(1\otimes a)$. The following are equivalent.*

(i) *The graded $k(\mathbb{Z}\times G)$-homomorphism*

$$(e_n,f_{n-1})_{n\geqslant 0}\colon P_\mathbb{Z}\otimes_k P_G\to z_g^*(P_G)$$

is a quasi-isomorphism of chain complexes which lifts the identity on k.

(ii) *The graded kG-homomorphism*

$$(s_{n-1})_{n\geqslant 0}\colon P_G[1]\to P_G$$

is a homotopy with the property that the chain map $\delta\circ s+s\circ\delta$ is equal to the endomorphism of P_G given by left multiplication with $g-1$.

Proof. In degree 0, the map e_0 clearly lifts the identity on k, so we need to show that $(e_n,f_{n-1})_{n\geqslant 0}$ is a chain map if and only if $s\circ\delta+\delta\circ s$ is equal to multiplication by $g-1$. By the definition of the differential of $P_\mathbb{Z}\otimes_k P_Q$, we have that $(e_n,f_{n-1})_{n\geqslant 0}$ is a chain map if and only if for any $n\geqslant 1$ we have

$$(e_{n-1},f_{n-2})\circ\begin{pmatrix}\delta_n & u-1\\ 0 & -\delta_{n-1}\end{pmatrix}=\delta_n\circ(e_n,f_{n-1}),$$

where we use as before the same letter δ_n for the extension $\mathrm{Id}\otimes\delta_n$ of δ_n to $P_n[u,u^{-1}]$. This is equivalent to

$$(e_{n-1}\circ\delta_n, e_{n-1}\circ(u-1)-f_{n-2}\circ\delta_{n-1})=(\delta_n\circ e_n,\delta_n\circ f_{n-1})\ ,$$

where we have used the notation $(u-1)$ for the map given by multiplication with $u-1$. In the first component, this holds automatically since $(e_n)_{n\geqslant 0}$ is a chain map. Thus the previous condition is equivalent to

$$e_{n-1}\circ(u-1)=\delta_n\circ f_{n-1}+f_{n-2}\circ\delta_{n-1}\ .$$

Note that the left side is equal to $(g-1)\circ e_{n-1}$, where $(g-1)$ denotes the map given by multiplication with $g-1$. Through the obvious versions of the adjunction from Lemma 6.2, this is equivalent to the statement that $\delta\circ s + s\circ\delta$ is equal to left multiplication by $g-1$ on P_G as stated. □

Proof of Theorem 6.1. The fact that $s^\vee$ is a cochain map is the special case of Theorem 5.1, applied to $A = kG$, $U = V = k$, and $z = g-1$. Thus $H^n(s^\vee) = D_{g-1}$, for $n \geqslant 0$. By Theorem 6.5 (and with the notation of that theorem) the graded map $(s_{n-1})_{n\geqslant 0}\colon P_G[1] \to P_G$ induces a quasi-isomorphism

$$(e_n, f_{n-1})_{n\geqslant 0}\colon P_{\mathbb{Z}} \otimes_k P_G \to z_g^*(P_G)$$

which lifts the identity on k. Applying the functor $\mathsf{Hom}_{k(\mathbb{Z}\times G)}(-,k)$, with the appropriate signs for the differentials, and making use of the adjunction Lemma 6.2, yields a cochain map

$$\mathsf{Hom}_{kG}(P_G, k) \to \mathsf{Hom}_{k(\mathbb{Z}\times G)}(P_{\mathbb{Z}} \otimes_k P_G, k)\ .$$

Taking cohomology, and using the identification in Proposition 6.4, yields for any $n \geqslant 0$ a map

$$H^n(G,k) \to H^n(\mathbb{Z}\times G, k) = H^n(G,k) \oplus H^{n-1}(G,k)\ .$$

By construction, the second component of this map is induced by $s^\vee$, hence equal to D_{g-1}. By Theorem 3.4, the second component is also equal to Δ_g as stated. □

The following result is Theorem 1.1 (ii).

Proposition 6.6. *Let G, H be finite groups, let $g \in Z(G)$ and $h \in Z(H)$. Let i, j be nonnegative integers, let $x \in H^i(G,k)$ and $y \in H^j(H,k)$. Identify $k(G\times H) \cong kG \otimes_k kH$ and identify $x\otimes y$ with its canonical image in $H^{i+j}(G\times H,k)$. Then $\Delta_{(g,h)}(x\otimes y)$ is equal to the canonical image of*

$$\Delta_g(x)\otimes y + (-1)^i x \otimes \Delta_h(y)$$

in $H^{i+j-1}(G\times H,k)$.

Proof. We apply Proposition 5.6 to the case $A = kG$, $U = U' = k$, $B = kH$, $V = V' = k$, $z = g-1$, $w = h-1$. We have

$$\Delta_g(x)\otimes y + (-1)^i x\otimes \Delta_h(y) = D_{g-1}(x)\otimes y + (-1)^i x \otimes D_{h-1}(y)$$

By Proposition 5.6, the image in $H^{i+j}(G\times H,k)$ of this element is equal to

$$D_{(g-1)\otimes 1 + 1\otimes(h-1)}(x\otimes y)\ .$$

It follows from the last statement in Proposition 5.6 that $D_{(g-1)\otimes(h-1)} = 0$. Since

$$(g-1)\otimes(h-1) = (g\otimes h) - (1\otimes 1) - ((g-1)\otimes 1) - (1\otimes(h-1)))$$

this implies that

$$0 = D_{(g-1)\otimes(h-1)} = D_{(g\otimes h)-(1\otimes 1)} - D_{(g-1)\otimes 1} - D_{1\otimes(h-1)}$$

or equivalently,

$$D_{(g\otimes h)-(1\otimes 1)} = D_{(g-1)\otimes 1} + D_{1\otimes(h-1)} = D_{(g-1)\otimes 1 + 1\otimes(h-1)}$$

The left side in the last equation is $\Delta_{(g,h)}$, whence the result. □

The following proposition implies the statements (v) and (iv) in Theorem 1.1.

Proposition 6.7. *Let G be a finite group and g, $h \in Z(G)$.*

(i) *We have $\Delta_{gh} = \Delta_g + \Delta_h$.*
(ii) *We have $\Delta_{g^m} = m\Delta_g$ for any positive integer m.*
(iii) *If the order of g is invertible in k, then $\Delta_g = 0$. In particular, if k is a field of prime characteristic p and g a p'-element in $Z(G)$, then $\Delta_g = 0$.*
(iv) *If k is a field of prime characteristic p, then $\Delta_g = \Delta_{g_p}$, where g_p is the p-part of g.*
(v) *If k is a field of prime characteristic p, denoting by P a Sylow p-subgroup of G, we have $g_p \in Z(P)$, and for any positive integer n the diagram*

$$\begin{array}{ccc} H^n(G,k) & \xrightarrow{\mathrm{Res}_{G,P}} & H^n(P,k) \\ \downarrow{\scriptstyle \Delta_g} & & \downarrow{\scriptstyle \Delta_{g_p}} \\ H^{n-1}(G,k) & \xrightarrow{\mathrm{Res}_{G,P}} & H^{n-1}(P,k) \end{array}$$

is commutative with injective horizontal maps.

Proof. The identity $(g-1)(h-1) = (gh-1) - (g-1) - (h-1)$ implies that $D_{(g-1)(h-1)} = D_{gh-1} - D_{g-1} - D_{h-1}$. By Theorem 5.3 (iii) we have $D_{(g-1)(h-1)} = 0$, and hence $D_{gh-1} = D_{g-1} + D_{h-1}$. By Theorem 6.1 this yields the equality stated in (i), and (ii) is an immediate consequence of (i). By Theorem 6.1 we have $\Delta_1 = 0$. Thus (iii) follows from (ii). Assume that k is a field of prime characterisic p. Note that since g is central, so are the p-part g_p and the p'-part $g_{p'}$ of g. By (i) we have $\Delta_g = \Delta_{g_p} + \Delta_{g_{p'}}$, and by (iii) this is equal to Δ_{g_p}, whence (iv). Since g, and hence also g_p, is central, it follows that g_p is contained in any Sylow p-subgroup of G. Statement (v) follows from (iv), the compatibility of Δ_{g_p} with restriction from G to P by Corollary 4.3 (iii), together with the standard fact that the restriction map $H^*(G,k) \to$ to $H^*(P,k)$ is injective. □

7. The BV operator on the bar resolution

Let k be a commutative ring, G a finite group and $g \in Z(G)$. The purpose of this section is to calculate explicitly a homotopy s on the bar resolution of G as in Theorem 6.1. This will be used in the proof of Theorem 8.1. As before, we denote by $z_g \colon \mathbb{Z} \times G \to G$ the group homomorphism sending (m,h) to $g^m h$.

We choose for P_G the projective resolution of the trivial kG-module which in degree n term is equal to kG^{n+1}, where $n \geqslant 0$ and where G^{n+1} is the direct product of $n+1$ copies of G, with differential δ_n given for $n \geqslant 1$ by

$$\delta_n(a_0, a_1, .., a_n) = \sum_{i=1}^{n-1} (-1)^i (a_0, .., a_i a_{i+1}, .., a_n) + (-1)^n (a_0, a_1, .., a_{n-1})$$

where the a_i are elements in G. The left kG-module structure on the terms kG^{n+1} is given by left multiplication with G on the first copy of G. (This is the resolution obtained from tensoring the Hochschild resolution of kG by $- \otimes_{kG} k$.) In particular, in degree 1, we have

$$\delta_1(a_0, a_1) = a_0 a_1 - a_0 \ .$$

The $k(\mathbb{Z} \times G)$-module structure of $z_g^*(kG^{n+1})$ is given by the action of $u^i \otimes h$ acting as left multiplication by $g^i h$ on the first copy of G in G^{n+1}.

Theorem 7.1. *With the notation above, for any $n \geqslant 1$ denote by $s_{n-1}\colon kG^n \to kG^{n+1}$ the kG-homomorphism defined by*

$$s_{n-1}(a_0, a_1, .., a_{n-1}) = \sum_{i=0}^{n-1}(-1)^i(a_0, .., a_i, g, a_{i+1}, .., a_{n-1}) \ ,$$

where $a_i \in G$ for $0 \leqslant i \leqslant n-1$. Applying the functor $\mathsf{Hom}_{kG}(-, k)$ to the graded map

$$s = (s_{n-1})_{n\geqslant 1}\colon P_G[1] \to P_G$$

yields a map of cochain complexes of k-modules

$$s^\vee\colon \mathsf{Hom}_{kG}(P_G, k) \to \mathsf{Hom}_{kG}(P_G[1], k)$$

such that, for any $n \geqslant 1$, we have

$$\Delta_g = H^n(s^\vee)\colon H^n(G, k) \to H^{n-1}(G, k) \ .$$

Proof. By Theorem 6.5, it suffices to show that s is a homotopy such that $\delta \circ s + s \circ \delta$ is the chain map given by multiplication with $g-1$, or equivalently, that s corresponds through the adjunction in Lemma 6.2 to a quasi-isomorphism $P_\mathbb{Z} \otimes_k P_G \to z_g^*(P_G)$. This is the content of the next theorem, whence the result. □

As in the previous section, for $n \geqslant 0$, denote by

$$e_n\colon kG^{n+1}[u, u^{-1}] \to z_g^*(kG^{n+1})$$

the $k(\mathbb{Z} \times G)$-homomorphism defined by $e_n(u^m \otimes h) = g^m h$ for all $m \in \mathbb{Z}$ and $h \in G$. For $n \geqslant 1$ denote by

$$f_{n-1}\colon kG^n[u, u^{-1}] \to z_g^*(kG^{n+1})$$

the $k(\mathbb{Z} \times G)$-homomorphism given by

$$f_{n-1}(u^m \otimes (a_0, a_1, .., a_{n-1})) = \sum_{j=0}^{n-1}(-1)^j(g^m a_0, .., a_j, g, a_{j+1}, .., a_{n-1})$$

with $m \in \mathbb{Z}$ and the a_j in G. Through an adjunction as in Lemma 6.2, the map f_{n-1} corresponds to the kG-homomorphism $s_{n-1}\colon kG^n \to kG^{n+1}$ defined in the previous theorem.

Theorem 7.2. *With the notation above, the following hold.*

(i) *The graded $k(\mathbb{Z} \times G)$-homomorphism $(e_n, f_{n-1})_{n\geqslant 0}\colon P_\mathbb{Z} \otimes_k P_G \to z_g^*(P_G)$ is a quasi-isomorphism which lifts the identity on k.*

(ii) *The graded kG-homomorphism $(s_{n-1})_{n\geqslant 1}\colon P_G[1] \to P_G$ is a homotopy with the property that $\delta \circ s + s \circ \delta$ is equal to the endomorphism of P_G given by left multiplication with $g-1$.*

Proof. By Theorem 6.5, the two statements are equivalent. We prove the second statement. That is, for $n \geqslant 1$ and $x = (a_0, a_1, .., a_{n-1}) \in P_{n-1}$, we need to prove the equality

$$\delta_n(s_{n-1}(x)) = (g-1)x - s_{n-2}(\delta_{n-1}(x))$$

We start with the left side. We have

$$\delta_n(s_{n-1}(x)) = \sum_{j=0}^{n-1} (-1)^j \delta_n(a_0, .., a_j, g, a_{j+1}, .., a_{n-1}) \ .$$

We need to calculate the summands

$$\delta_n(a_0, .., a_j, g, a_{j+1}, .., a_{n-1}) \ .$$

The definition of δ_n yields an alternating sum over an index i running from 0 to $n-1$, which we will need to break up according to wether $0 \leqslant i < j$, $i = j$, $i = j+1$, $j+1 < i \leqslant n-1$. We have

$$\begin{aligned}\delta_n(a_0, .., a_j, g^{-1}, a_{j+1}, .., a_{n-1}) = &\sum_{i=0}^{j-1}(-1)^i(a_0, .., a_i a_{i+1}, .., a_j, g, a_{j+1}, .., a_{n-1}) \\ &+ (-1)^j(a_0, .., a_j g, a_{j+1}, .., a_{n-1}) \\ &+ (-1)^{j+1}(a_0, .., a_j, g a_{j+1}, .., a_{n-1}) \\ &+ \sum_{i=j+2}^{n-1}(-1)^j(a_0, .., a_j, g, a_{j+1}, .., a_{i-1}a_i, .., a_{n-2}) \\ &+ (-1)^n(a_0, .., a_j, g, a_{j+1}, .., a_{n-2}).\end{aligned}$$

For $j = 0$ or $j = n-1$ or $j = n-2$ this formula takes a slightly different form. If $j = 0$, then the first sum is empty (hence zero by convention). The fourth term (that is, the sum indexed $\sum_{i=j+2}^{n-1}$) is empty if j is one of $n-1$, $n-2$, so zero. In addition, if $j = n-1$, then the third term does not appear (because the component ga_{j+1} is not defined in that case) and the last summand takes the form $(-1)^n(a_0, .., a_{n-1})$, and if $j = n-2$, then the last summand takes the form $(-1)(a_0, .., a_{n-2}, g)$. It follows that

$$\begin{aligned}\delta_n(s_{n-1}(x)) = &\sum_{j=1}^{n-1}\sum_{i=0}^{j-1}(-1)^{i+j}(a_0, .., a_i a_{i+1}, .., a_j, g, a_{j+1}, .., a_{n-1}) \\ &+ \sum_{j=0}^{n-1}(a_0, .., a_j g, a_{j+1}, .., a_{n-1}) \\ &- \sum_{j=0}^{n-2}(a_0, .., a_j, g a_{j+1}, .., a_{n-1}) \\ &+ \sum_{j=0}^{n-3}\sum_{i=j+2}^{n-1}(-1)^{i+j}(a_0, .., a_j, g, a_{j+1}, .., a_{i-1}a_i, .., a_{n-1}) \\ &+ \sum_{j=0}^{n-1}(-1)^{n+j}(a_0, .., a_j, g, a_{j+1}, .., a_{n-2}).\end{aligned}$$

Since g is central, the third sum cancels against the second sum for $j \geqslant 1$. Also, we may reindex the second double sum and replace i by $i-1$. This yields

$$\delta_n(f_n(x)) = \sum_{j=1}^{n-1}\sum_{i=0}^{j-1}(-1)^{i+j}(a_0,..,a_ia_{i+1},..,a_j,g,a_{j+1},..,a_{n-1}) \\ + (a_0g,a_1,...,a_{n-1}) \\ + \sum_{j=0}^{n-3}\sum_{i=j+1}^{n-2}(-1)^{i+j+1}(a_0,..,a_j,g,a_{j+1},..,a_ia_{i+1},..,a_{n-1}) \\ + \sum_{j=0}^{n-1}(-1)^{n+j}(a_0,..,a_j,g,a_{j+1},..,a_{n-2}).$$

We need to show that this is equal to the expression

$$(g-1)x - s_{n-2}(\delta_{n-1}(x)) = (ga_0,a_1,..,a_{n-1}) - (a_0,..,a_{n-1}) \\ - \sum_{i=0}^{n-2}(-1)^i s_{n-2}(a_0,..,a_ia_{i+1},..,a_{n-1}) \\ - (-1)^{n-1}s_{n-2}(a_0,..,a_{n-2}).$$

Expanding s_{n-2} and adjusting indexing yields that this is equal to

$$(ga_0,a_1,..,a_{n-1}) - (a_0,..,a_{n-1}) \\ + \sum_{i=0}^{n-2}\sum_{j=0}^{i-1}(-1)^{i+j+1}(a_0,..,a_j,g,a_{j+1},..,a_ia_{i+1},..,a_{n-1}) \\ + \sum_{i=0}^{n-2}\sum_{j=i+1}^{n-1}(-1)^{i+j}(a_0,..,a_ia_{i+1},..,a_j,g,a_{j+1},..,a_{n-1}) \\ + \sum_{j=0}^{n-2}(-1)^{n+j}(a_0,..,a_j,g,a_{j+1},..,a_{n-2}).$$

We need to match all summands to those for the expression of $\delta_n(s_{n-1}(x))$. The first summand cancels against the second summand in $\delta_n(s_{n-1}(x))$. The second summand cancels agains the summand for $j = n-1$ in the fourth (and last) sum of $\delta_n(s_{n-1}(x))$. The last sum cancels against the remaining summands of the fourth sum in $\delta_n(s_{n-1}(x))$. The first double sum cancels against the second double sum in $\delta_n(s_{n-1}(x))$ because both can be written as sum indexed by pairs (i,j) with $0 \leqslant i < j \leqslant n-2$. Similarly, the remaining double sum cancels against the first double sum in $\delta_n(s_{n-1}(x))$ because both can be written as a sum indexed by pairs (i,j) with $0 \leqslant i < j \leqslant n-1$. □

8. The BV operator in degree 2

For convenience, we restate Theorem 1.1 (xi). For x, y elements in a group G we write $[x,y]$ for $xyx^{-1}y^{-1}$, and we recall that $[x,yz] = [x,y]\,y[x,z]y^{-1}$.

Theorem 8.1. *Let G be a finite group, $g \in Z(G)$, and $x \in H^2(G,k)$. Suppose that x corresponds to a central extension*

$$1 \to k^+ \to K \to G \to 1.$$

For any $h \in G$, choose an inverse image $\hat{h} \in K$. Then identifying $H^1(G,k)$ with $\mathsf{Hom}(G,k)$, we have

$$[\hat{h}, \hat{g}] = \Delta_g(x)(h) \in k^+.$$

We combine Theorem 7.1 and the following Lemma to give a proof of Theorem 8.1.

Lemma 8.2. *Let $1 \to Z \to \hat{G} \to G \to 1$ be a central extension of an abelian group G. For $g \in G$ choose an inverse image $\hat{g}$ of g in $\hat{G}$ such that $\hat{1}_G = 1_{\hat{G}}$. Let $\alpha\colon G \times G \to Z$ be the 2-cocycle defined by*

$$\alpha(g,h) = \hat{g}\hat{h}\widehat{gh}^{-1}$$

for g, $h \in G$. Let $\beta, \gamma\colon G \times G \to Z$ be the maps defined by

$$\beta(g,h) = \alpha(h,g),$$
$$\gamma(g,h) = [\hat{g}, \hat{h}] = \hat{g}\hat{h}\hat{g}^{-1}\hat{h}^{-1}$$

for g, $h \in G$. Then β, γ are 2-cocycles, and we have $\beta\alpha^{-1} = \gamma$.

Proof. All parts are trivial verifications. □

Proof of Theorem 8.1. By Theorem 1.1 (iii), Δ_g commutes with restriction to subgroups of G containing g. Hence, it suffices to consider the case where $G = \langle g, h \rangle$. Therefore, in order to prove the formula for $\Delta_g(x)(h)$ we may assume that G is abelian. Note that with the notation of Theorem 7.1 we have

$$s_1(a_0, a_1) = (a_0, g, a_1) - (a_0, a_1, g)$$

and hence its dual $(s_1)^\vee$ sends a kG-homomorphism $\zeta\colon kG^3 \to k$ to the kG-homomorphism $\zeta \circ s_1\colon kG^2 \to k$ sending (a_0, a_1) to $\zeta(a_0, g, a_1) - \zeta(a_0, a_1, g)$. The identification of $H^2(G;k)$ in terms of classes of cocycles is given via the adjunction map

$$\mathsf{Hom}_{kG}(kG^3, k) \cong \mathsf{Hom}_k(kG^2, k)$$

sending a kG-homomorphism $\zeta\colon kG^3 \to k$ to the k-linear map $kG^2 \to k$ determined by the assignment $(a_1, a_2) \mapsto \zeta(1, a_1, a_2)$. Similarly, the identification $H^1(G,k) = \mathsf{Hom}(G, k^+)$ is given via the adjunction $\mathsf{Hom}_{kG}(kG^2, k) \cong \mathsf{Hom}_k(kG, k)$. Thus if α is a 2-cocyle representing the class x, then the 1-class determined by $\alpha \circ s_1$ is given by the assignment

$$a_1 \mapsto \alpha(g, a_1) - \alpha(a_1, g)$$

By Lemma 8.2 applied with $Z = k^+$ (written additively) this is equal to the map

$$a_1 \mapsto [\hat{a}_1, \hat{g}]$$

and writing h instead of a_1, we get that $\Delta_g(x)(h) = [\hat{h}, \hat{g}]$ for all $h \in G$ as claimed. □

9. Examples

Throughout this section p is a prime and k is a field of characteristic p. As a consequence of Theorem 1.1 (vii), in order to calculate the maps Δ_g, where g is a central element in a finite group G, it suffices to calculate in the case where G is a finite p-group.

Example 9.1. Let G be a finite cyclic p-group and let $g \in G$. Then $H^*(G,k)$ contains a polynomial subalgebra $k[x]$ with x in degree 2, such that $H^*(G,k)$ is generated as a module over $k[x]$ by 1 and a degree 1 element y. Then x is in the image of the map $H^2(G,\mathbb{Z}) \to H^*(G,k)$, and hence $\Delta_g(x) = 0$ by Theorem 1.1 (x). Moreover, by Theorem 1.1 (xi) we have $\Delta_g(y) = y(g)$. Thus, using that Δ_g is a derivation by Theorem 1.1 (i), for any nonnegative integer n, we have $\Delta_g(x^n) = 0$ and $\Delta_g(x^n y) = x^n y(g)$.

Using the canonical identification $HH^*(kG) = kG \otimes_k H^*(G,k)$ from Holm [10], Cibils and Solotar [6], this determines the BV operator Δ on $HH^*(kG)$ as follows: we have

$$\Delta(g \otimes x^n) = 0,$$
$$\Delta(g \otimes x^n y) = y(g).g \otimes x^n.$$

Other papers dealing with BV and Gerstenhaber structures on Hochschild cohomology of cyclic groups include Sánchez-Flores [21], Yang [27], Angel and Duarte [2].

Example 9.2. Let G be a finite abelian p-group, and let $g \in G$. Combining Example 9.1 with the Künneth formula Proposition 6.6 determines Δ_g. Using again the canonical identification of algebras $HH^*(kG) = kG \otimes_k H^*(G,k)$ from [10], [6], the BV operator Δ on $HH^*(kG)$ is determined by $\Delta(g \otimes \zeta) = g \otimes \Delta_g(\zeta)$.

Example 9.3. Let G be the generalised quaternion group Q_{2^n} of order 2^n ($n \geqslant 3$), and let γ be the central element of order two in G. Then $H^*(G,\mathbb{F}_2)$, is generated by two elements x and y in degree one and an element z of degree four. The elements x and y are nilpotent, and generate a finite Poincaré duality algebra with top degree three. Thus we have

$$\sum_{i=0}^{\infty} t^i \dim_{\mathbb{F}_2} H^i(Q_{2^n},\mathbb{F}_2) = \frac{1+2t+2t^2+t^3}{1-t^4}.$$

The exact relations between x and y depend on whether $n = 3$ or $n > 3$. If $n = 3$ then $x^2 + xy + y^2 = 0$ and $x^2y + xy^2 = 0$, whereas if $n > 3$ then the relations are $xy = 0$ and $x^3 + y^3 = 0$. Again using Theorem 1.1 (ix), we have $\Delta_\gamma(x) = \Delta_\gamma(y) = 0$, and it remains to compute $\Delta_\gamma(z)$.

In both cases, $H^3(G,\mathbb{F}_2)$ is one dimensional, and we have $H^3(G,\mathbb{Z}) = 0$. Moreover, there is an element $z' \in H^4(G,\mathbb{Z})$ with $2^n z' = 0$, and such that z is the image of z' under the reduction mod two map $H^4(G,\mathbb{Z}) \to H^4(G,\mathbb{F}_2)$. Therefore $\Delta_\gamma(z') = 0$, and the commutativity of the diagram

$$\begin{array}{ccc} H^4(G,\mathbb{Z}) & \xrightarrow{\Delta_\gamma} & H^3(G,\mathbb{Z}) \\ \downarrow & & \downarrow \\ H^4(G,\mathbb{F}_2) & \xrightarrow{\Delta_\gamma} & H^3(G,\mathbb{F}_2) \end{array}$$

shows that $\Delta_\gamma(z) = 0$. Since Δ_γ is a derivation, it follows that it is zero on all elements of $H^*(G,\mathbb{F}_2)$.

Other papers considering the BV structure on Hochschild cohomology of quaternion groups include Ivanov, Ivanov, Volkov and Zhou [12], Ivanov [11].

Example 9.4. Let G be the dihedral group of order 2^n with $n \geqslant 3$,

$$G = \langle g, h \mid g^2 = h^2 = 1,\ (gh)^{2^{n-1}} = 1\rangle,$$

and let γ be the central involution $(gh)^{2^{n-2}}$ in G. Then $H^*(G, \mathbb{F}_2) = \mathbb{F}_2[x, y, z]/(xy)$, where $|x| = |y| = 1$ and $|z| = 2$. Using Theorem 1.1 (ix) we have $\Delta_\gamma(x) = \Delta_\gamma(y) = 0$, so it remains to evaluate $\Delta_\gamma(z)$. Now z corresponds to the central extension

$$1 \to \mathbb{F}_2^+ \to Q \to G \to 1$$

where Q is the generalised quaternion group of order 2^{n+1}

$$Q = \langle \hat{g}, \hat{h} \mid \hat{g}^2 = \hat{h}^2 = (\hat{g}\hat{h})^{2^{n-1}}, (\hat{g}\hat{h})^{2^n} = 1\rangle.$$

The inverse image of γ in Q is not central, so it follows from Theorem 1.1 (xi) that $\Delta_\gamma(z) \neq 0$. Now G has an automorphism of order two swapping g and h, and swapping x and y. It lifts to an automorphism of Q swapping $\hat{g}$ and $\hat{h}$, and therefore fixes z. So it also fixes $\Delta_\gamma(z)$. The only non-zero fixed element of degree one is $x + y$, and therefore we have

$$\Delta_\gamma(z) = x + y.$$

Since Δ_γ is a derivation, this determines its value on all elements of $H^*(G, k)$. We have

$$\Delta_\gamma(x^i z^j) = jx^{i+1}z^{j-1}, \qquad \Delta_\gamma(y^i z^j) = jy^{i+1}z^{j-1}.$$

Remark 9.5. As well as the proof given in Section 8, Theorem 1.1 (xi) can be proved by a direct computation as follows. Note that $[\hat{h}, \hat{g}] \in k^+$ is central, hence equal to $[\hat{g}, \hat{h}^{-1}]$, and the formula for $\Delta_g(x)$ as stated does indeed define a group homomorphism from G to k^+. Next, since Δ_g commutes with restriction to subgroups of G containing g, it suffices to consider the case where $G = \langle g, h\rangle$. This is an abelian group, and so by Example 9.2, in principle we are done. If $H^1(G, k)$ is one dimensional, both sides are zero. Otherwise, it is two dimensional, and $H^2(G, k)$ modulo the image of $H^2(G, \mathbb{Z}) \to H^2(G, k)$ is one dimensional, spanned by the product of two degree one elements. This reduces the proof to an explicit and slightly tedious computation.

Example 9.6. Let G be the semidihedral group of order 2^n ($n \geqslant 4$). This is the group

$$G = \langle g, h \mid g^2 = 1, h^{2^{n-1}} = 1, ghg = h^{2^{n-2}-1}\rangle.$$

The cohomology ring was computed by Munkholm (unpublished) and Evens and Priddy [9]. We have

$$H^*(G, \mathbb{F}_2) = \mathbb{F}_2[x, y, z, w]/(x^3, xy, xz, z^2 + y^2 w).$$

Evens and Priddy also observed that the cohomology is detected on the dihedral subgroup of order eight, $D = \langle g, h^{2^{n-3}}\rangle$ and the quaternion subgroup of order eight, $Q = \langle gh, h^{2^{n-3}}\rangle$. Let γ be the central element $h^{2^{n-2}}$ of order two. Using Theorem 1.1 (ix) we have $\Delta_\gamma(x) = \Delta_\gamma(y) = 0$.

Applying Δ_γ to the relation $z^2 = y^2 w$ and using the fact that Δ_γ is a derivation, we have $y^2\Delta_\gamma(w) = 0$. Now multiplication by y^2 from $H^2(G, \mathbb{F}_2)$ to $H^4(G, \mathbb{F}_2)$ is injective, so we deduce that $\Delta_\gamma(w) = 0$.

It remains to compute $\Delta_\gamma(z)$. This has degree two, so it is a linear combination of x^2 and y^2. To determine which, we use the information at the bottom of page 71 of [9] on restriction to D and Q. First note that our x and y are their x and $x+y$; this is determined by which non-zero element of $H^1(G,k)$ is nilpotent and what it annihilates. So y restricts to zero on Q, while x and x^2 have non-zero restriction. Since Δ_γ is zero on $H^*(Q,k)$ by Example 9.3, it follows that $\Delta_\gamma(z)$ is a multiple of y^2. The restriction of z to $H^*(D,\mathbb{F}_2)$ is a degree three element which is not in the subring generated by $H^1(D,\mathbb{F}_2)$. So using Example 9.4, it follows that Δ_γ is non-zero on the restriction of z, and hence $\Delta_\gamma(z)$ cannot be zero. Hence we have $\Delta_\gamma(z)=y^2$.

Using the fact that Δ_γ is a derivation, it is determined by the information that $\Delta_\gamma(x)=0$, $\Delta_\gamma(y)=0$, $\Delta_\gamma(z)=y^2$, $\Delta_\gamma(w)=0$.

10. The Gerstenhaber bracket

Throughout this section, let p be a prime and let k be a field of characteristic p. In this section, we combine the formula

$$[x,y]=(-1)^{|x|}\Delta(xy)-(-1)^{|x|}\Delta(x)y-x\Delta(y) \tag{10.1}$$

relating the BV operator Δ to the Gerstenhaber bracket and products in $HH^*(kG)$, with the Siegel–Witherspoon formula

$$xy=\sum_u \mathrm{Tr}^{C_G(guhu^{-1})}_{C_G(g)\cap C_G(uhu^{-1})}(\mathrm{Res}^{C_G(g)}_{C_G(g)\cap C_G(uhu^{-1})}(x)\cdot \mathrm{Res}^{C_G(uhu^{-1})}_{C_G(g)\cap C_G(uhu^{-1})}(u^*(y))) \tag{10.2}$$

for products in $HH^*(kG)$ in terms of the centraliser decomposition from [22]. Here, G is a finite group, g, $h\in G$, $x\in H^*(C_G(g),k)$ and $y\in H^*(C_G(h),k)$, and u runs over a set of double coset representatives of $C_G(g)$ and $C_G(h)$ in G. The notation $u^*(y)$ denotes the image of y in $H^*(C_G(uhu^{-1}),k)$ under conjugation by u. The left side is the product of x and y regarded as elements in $HH^*(kG)$ via the centraliser decomposition. The summand on the right hand side indexed by u is in the summand in the centraliser decomposition corresponding to $guhu^{-1}$, and the multiplication of the restrictions on the right is the usual cup product in $H^*(C_G(g)\cap C_G(guhu^{-1}),k)$.

Combining (10.1) and (10.2), we have

$$\begin{aligned}(10.3)\quad [x,y]=&\\ \sum_u\Big((-1)^{|x|}&\Delta_{guhu^{-1}}\mathrm{Tr}^{C_G(guhu^{-1})}_{C_G(g)\cap C_G(uhu^{-1})}\big(\mathrm{Res}^{C_G(g)}_{C_G(g)\cap C_G(uhu^{-1})}(x)\cdot \mathrm{Res}^{C_G(uhu^{-1})}_{C_G(g)\cap C_G(uhu^{-1})}(u^*(y))\big)\\ &+\mathrm{Tr}^{C_G(guhu^{-1})}_{C_G(g)\cap C_G(uhu^{-1})}\big(-(-1)^{|x|}\mathrm{Res}^{C_G(g)}_{C_G(g)\cap C_G(uhu^{-1})}(\Delta_g(x))\cdot \mathrm{Res}^{C_G(uhu^{-1})}_{C_G(g)\cap C_G(uhu^{-1})}(u^*(y))\\ &\qquad -\mathrm{Res}^{C_G(g)}_{C_G(g)\cap C_G(uhu^{-1})}(x)\cdot \mathrm{Res}^{C_G(uhu^{-1})}_{C_G(g)\cap C_G(uhu^{-1})}(u^*(\Delta_h(y)))\big)\Big)\end{aligned}$$

Note that if $g\in Z(G)$ then this formula simplifies considerably. Namely, $C_G(g)=G$, $C_G(g)\cap C_G(h)=C_G(gh)$, and the transfers do not do anything. There is only one double coset, and we may take $u=1$. The formula then becomes

$$[x,y]=(-1)^{|x|}\Delta_{gh}(\mathrm{Res}^G_{C_G(gh)}(x)\cdot y)-(-1)^{|x|}\mathrm{Res}^G_{C_G(gh)}(\Delta_g(x))\cdot y-\mathrm{Res}^G_{C_G(gh)}(x)\cdot\Delta_h(y),$$

as an element of $H^*(C_G(gh),k)$ in the centraliser decomposition. Using Theorem 1.1 (i) and (v), this simplifies to

$$(-1)^{|x|}\Delta_h \mathrm{Res}^G_{C_G(gh)}(x)\cdot y + \mathrm{Res}^G_{C_G(gh)}(x)\cdot \Delta_g(y). \tag{10.4}$$

If x and y have degree one then using Theorem 1.1 (ix), this formula becomes

$$[x,y] = -x(h)y + y(g)\mathrm{Res}^G_{C_G(gh)}(x). \tag{10.5}$$

Now assume that G is a finite p-group. If $g \in Z(G)\cap\Phi(G)$, where $\Phi(G)$ is the Frattini subgroup of G, then we have $y(g)=0$, and this simplifies further to

$$[x,y] = -x(h)y.$$

Again, these formulas are as elements of the $H^*(C_G(gh),k)$ component in the centraliser decomposition.

If both g and h are in $Z(G)$, then notationally, it helps to keep track of g and h by writing $g\otimes x$ and $h\otimes y$, since the subring of $HH^*(kG)$ corresponding to elements in the centre in the centraliser decomposition is isomorphic to $kZ(G)\otimes H^*(G,k)$. With this notation, equation (10.4) becomes

$$[g\otimes x, h\otimes y] = gh\otimes((-1)^{|x|}\Delta_h(x)\cdot y + x\cdot\Delta_g(y)).$$

In particular, $kZ(G)\otimes H^*(G,k)\subseteq HH^*(kG)$ is a Lie subalgebra. Finally, if x and y have degree one, this becomes

$$[g\otimes x, h\otimes y] = gh\otimes(-x(h)y + y(g)x). \tag{10.6}$$

If g, $h\in Z(G)\cap\Phi(G)$ then the terms $x(h)$ and $y(g)$ vanish, and the Lie bracket is equal to zero.

We record these observations in the following proposition.

Proposition 10.7. *Let G be a finite p-group. Suppose that $g\in Z(G)\cap\Phi(G)$ and $h\in G$. If $x\in H^1(C_G(g),k)$ and $y\in H^1(C_G(h),k)$ in the centraliser decomposition of $HH^1(kG)$ then*

$$[x,y] = -x(h)y$$

as an element of $H^1(C_G(gh),k)$ in the centraliser decomposition.

In particular, the Lie bracket is identically zero on

$$k(Z(G)\cap\Phi(G))\otimes H^1(G,k)\leqslant HH^1(kG).$$

Remark 10.8. If $g\in Z(G)\cap\Phi(G)$ and h is not in $\Phi(G)$, then we can choose $x\in H^1(C_G(g),k)$ such that $x(h)=-1$, and then for all $y\in H^1(C_G(h),k)$ the element $[x,y]$ is y, but as an element of $H^1(C_G(gh),k)$ in the centraliser decomposition.

Taking $g=1$, we see that given $h\notin\Phi(G)$, there exists $x\in H^1(C_G(1),k)$ such that for every $y\in H^1(C_G(h),k)$ we have $[x,y]=y$. It follows, for example, that if G is a non-trivial finite p-group then $HH^1(kG)$ is never nilpotent.

On the other hand, we have the following theorem, whose proof is modelled on the method of Jacobson [13].

Theorem 10.9. *Suppose that G is a finite p-group such that $|Z(G):Z(G)\cap\Phi(G)|\geqslant 3$. Then the Lie subalgebra $kZ(G)\otimes H^1(G,k)\subseteq HH^1(kG)$ is not soluble, and therefore nor is $HH^1(kG)$.*

Proof. We compute inside $kZ(G) \otimes H^*(G,k) \subseteq HH^*(kG)$, as described above. Since by assumption we have $|Z(G) : Z(G) \cap \Phi(G)| \geqslant 3$, either p is odd, or $Z(G)/(Z(G) \cap \Phi(G))$ is non-cyclic. We treat these two cases separately.

(i) Suppose that p is odd. Choose an element $g \in Z(G) \setminus \Phi(G)$, and choose $x \in H^1(G,k)$ with $x(g) = 1$. Set $e = g \otimes x$, $f = -g^{-1} \otimes x$, and $h = -1 \otimes 2x$. We have $g \neq g^{-1}$, so these elements are linearly independent, and using (10.6) we have

$$\begin{aligned}
[e,f] &= -1 \otimes (x(g)x - x(g^{-1})x) = h,\\
[h,e] &= -g \otimes (x(1)2x - 2x(g)x) = 2e,\\
[h,f] &= g^{-1} \otimes (x(1)2x - 2x(g^{-1})x) = -2f.
\end{aligned}$$

Thus e, f and h span a copy of the Lie algebra $\mathfrak{sl}(2)$ inside $HH^1(kG)$. Therefore it is not soluble.

(ii) Suppose that $Z(G)/(Z(G) \cap \Phi(G))$ is non-cyclic. Choose elements g and h in $Z(G)$ so that their images in $Z(G)/(Z(G) \cap \Phi(G))$ generate distinct cyclic subgroups. Choose elements x and y in $H^1(G,k)$ with $x(g) = 1$, $y(g) = 0$, $x(h) = 0$, $y(h) = 1$. Then we compute

$$\begin{aligned}
[g^{-1} \otimes x, g \otimes y] &= 1 \otimes y & [h^{-1} \otimes y, h \otimes x] &= 1 \otimes x\\
[g \otimes x, 1 \otimes x] &= g \otimes x & [h \otimes y, 1 \otimes y] &= h \otimes y\\
[g \otimes y, 1 \otimes x] &= g \otimes y & [h \otimes x, 1 \otimes y] &= h \otimes x\\
[g^{-1} \otimes x, 1 \otimes x] &= -g^{-1} \otimes x & [h^{-1} \otimes y, 1 \otimes y] &= -h^{-1} \otimes y\\
[g^{-1} \otimes y, 1 \otimes x] &= -g^{-1} \otimes y & [h^{-1} \otimes x, 1 \otimes y] &= -h^{-1} \otimes x.
\end{aligned}$$

Thus, letting U be the linear span in $HH^1(kG)$ of the elements appearing in these computations, we see that $[U,U] \supseteq U$. It follows that any Lie subalgebra containing U is not soluble, and hence $kZ(G) \otimes H^1(G,k)$ and $HH^1(kG)$ are not soluble. □

Remark 10.10. If G is the cyclic group of order two and k has characteristic two then $HH^1(kG)$ is a Lie algebra of dimension two, and is therefore soluble. This shows that the condition $|Z(G) : Z(G) \cap \Phi(G)| \geqslant 3$ cannot be weakened to $|Z(G) : Z(G) \cap \Phi(G)| \geqslant 2$.

11. Extraspecial p-groups

As an illustration of the methods developed in this paper, in this section we examine the Lie structure of $HH^1(kG)$ when G is an extraspecial p-group. The methods also cover some other p-groups of class two, so we formulate them in more generality. We begin with a lemma.

Lemma 11.1. *Suppose that G is a finite p-group, and that g, $h \in G$ satisfy*

$$\Phi(C_G(g)) = \Phi(C_G(h)) = \Phi(G),$$

and $C_G(g) \cap C_G(h)$ is a proper subgroup of $C_G(gh)$. If $x \in H^*(C_G(g),k)$, $y \in H^*(C_G(h),k)$ are elements of degree zero or one, then

$$\mathrm{Tr}_{C_G(g)\cap C_G(h)}^{C_G(gh)}(\mathrm{Res}_{C_G(g)\cap C_G(h)}^{C_G(g)}(x) \cdot \mathrm{Res}_{C_G(g)\cap C_G(h)}^{C_G(h)}(y)) = 0.$$

Proof. Since $\Phi(C_G(g)) = \Phi(G)$ we may write $x = \mathrm{Res}^G_{C_G(g)}(\hat{x})$ with $\hat{x} \in H^*(G,k)$. Similarly, $y = \mathrm{Res}^G_{C_G(h)}(\hat{y})$ with $\hat{y} \in H^*(G,k)$. Then

$$\begin{aligned}
&\mathrm{Tr}^{C_G(gh)}_{C_G(g)\cap C_G(h)}(\mathrm{Res}^{C_G(g)}_{C_G(g)\cap C_G(h)}(x)\cdot \mathrm{Res}^{C_G(h)}_{C_G(g)\cap C_G(h)}(y))\\
&\qquad = \mathrm{Tr}^{C_G(gh)}_{C_G(g)\cap C_G(h)}\mathrm{Res}^{G}_{C_G(g)\cap C_G(h)}(\hat{x}\cdot\hat{y})\\
&\qquad = |C_G(gh) : C_G(g)\cap C_G(h)|\mathrm{Res}^G_{C_G(gh)}(\hat{x}\cdot\hat{y})\\
&\qquad = 0
\end{aligned}$$

since $|C_G(gh) : C_G(g)\cap C_G(h)|$ is zero in k. □

Hypothesis 11.2. We suppose that G is a finite p-group satisfying $\Phi(G) = Z(G)$, and g, h are elements of G such that for all $u \in G$, and for all $x \in H^*(C_G(g),k)$, $y \in H^*(C_G(h),k)$ of degree zero or one, we have either

(i) $C_G(g)\cap C_G(uhu^{-1}) = C_G(guhu^{-1})$, or
(ii) $\mathrm{Tr}^{C_G(guhu^{-1})}_{C_G(g)\cap C_G(uhu^{-1})}(\mathrm{Res}^{C_G(g)}_{C_G(g)\cap C_G(uhu^{-1})}(x)\cdot \mathrm{Res}^{C_G(uhu^{-1})}_{C_G(g)\cap C_G(uhu^{-1})}(u^*(y))) = 0$.

We remark that if $\Phi(C_G(g)) = \Phi(C_G(h)) = \Phi(G) = Z(G)$ then Hypothesis 11.2 holds by Lemma 11.1.

Suppose that this hypothesis holds, let $x \in H^1(C_G(g),k)$ and $y \in H^1(C_G(h),k)$, and let u be a double coset representative of $C_G(g)$ and $C_G(h)$ in G. Suppose first that the containment $C_G(g)\cap C_G(uhu^{-1}) \leqslant C_G(guhu^{-1})$ is proper. Then by Hypothesis 11.2, the term corresponding to u in the Siegel–Witherspoon formula (10.2) for xy is zero. The same argument holds for $\Delta_g(x)y$ and $x\Delta_h(y)$, and so the term corresponding to u in (10.3) vanishes.

So the formula (10.3) becomes

$$(11.3)\quad [x,y] = \sum_u \Big(-\Delta_{guhu^{-1}}\big(\mathrm{Res}^{C_G(g)}_{C_G(guhu^{-1})}(x)\cdot \mathrm{Res}^{C_G(uhu^{-1})}_{C_G(guhu^{-1})}(u^*(y))\big)$$
$$+\mathrm{Res}^{C_G(g)}_{C_G(guhu^{-1})}(\Delta_g(x))\cdot \mathrm{Res}^{C_G(uhu^{-1})}_{C_G(guhu^{-1})}(u^*(y)) - \mathrm{Res}^{C_G(g)}_{C_G(guhu^{-1})}(x)\cdot \mathrm{Res}^{C_G(uhu^{-1})}_{C_G(guhu^{-1})}(u^*(\Delta_h(y)))\Big)$$

where u runs over those double coset representatives with $C_G(guhu^{-1}) = C_G(g)\cap C_G(uhu^{-1})$. For such a representative u, g commutes with uhu^{-1}. Since $uhu^{-1}h^{-1} \in [G,G] \leqslant Z(G)$, this implies that g commutes with all conjugates of h. Now applying Theorem 1.1 (iii), we have

$$(11.4)\quad [x,y] = \sum_u \Big(-\Delta_{guhu^{-1}}\big(\mathrm{Res}^{C_G(g)}_{C_G(guhu^{-1})}(x)\cdot \mathrm{Res}^{C_G(uhu^{-1})}_{C_G(guhu^{-1})}(u^*(y))\big)$$
$$+\Delta_g\mathrm{Res}^{C_G(g)}_{C_G(guhu^{-1})}(x)\cdot \mathrm{Res}^{C_G(uhu^{-1})}_{C_G(guhu^{-1})}(u^*(y)) - \mathrm{Res}^{C_G(g)}_{C_G(guhu^{-1})}(x)\cdot \Delta_{uhu^{-1}}\mathrm{Res}^{C_G(uhu^{-1})}_{C_G(guhu^{-1})}(u^*(y))\Big)$$

Set $x' = \mathrm{Res}^{C_G(g)}_{C_G(guhu^{-1})}(x)$, $y' = \mathrm{Res}^{C_G(uhu^{-1})}_{C_G(guhu^{-1})}(u^*(y))$. By Theorem 1.1 (v), we have $\Delta_{guhu^{-1}} = \Delta_g + \Delta_{uhu^{-1}}$ on $H^*(C_G(guhu^{-1}),k)$. Hence, using Theorem 1.1 (i) and (ix), the component of $[x,y]$ coming from u is

$$
\begin{aligned}
&-\Delta_{guhu^{-1}}(x'y') + \Delta_g(x')y' - x'\Delta_{uhu^{-1}}(y') \\
&\quad = -(\Delta_g(x')y' + \Delta_{uhu^{-1}}(x')y' - x'\Delta_g(y') - x'\Delta_{uhu^{-1}}(y')) + \Delta_g(x')y' - x'\Delta_{uhu^{-1}}(y') \\
&\quad = x'\Delta_g(y') - \Delta_{uhu^{-1}}(x')y' \qquad (11.5)\\
&\quad = y(u^{-1}gu)\mathrm{Res}^{C_G(g)}_{C_G(guhu^{-1})}(x) - x(uhu^{-1})\mathrm{Res}^{C_G(uhu^{-1})}_{C_G(guhu^{-1})}(u^*(y))
\end{aligned}
$$

where the multiplications are ordinary cohomology cup products performed inside the summand $H^*(C_G(guhu^{-1}),k)$ corresponding to $guhu^{-1}$ in the centraliser decomposition.

We partition the conjugacy classes of G into subsets C_i, where g is in C_i if $|G:C_G(g)| = p^i$, and for $i \geqslant 0$ we set

$$X_i = \bigoplus_{g\in C_i} H^*(C_G(g),k), \qquad Y_i = \bigoplus_{j\geqslant i} X_j.$$

Here, as usual, the direct sum is over G-conjugacy classes of $g \in C_i$. Then we have

$$HH^*(kG) = \bigoplus_{i=0}^{n} X_i.$$

It follows from (11.5) that if $i \leqslant j$ then $[X_i, X_j] \subseteq Y_j$, and hence $[Y_i, Y_j] \subseteq Y_j$. So the Y_i are Lie ideals, and $HH^1(kG)$ is soluble if and only if each of the Y_i/Y_{i+1} is soluble. Note that by Proposition 10.7, since $Z(G) = \Phi(G)$ we have $[X_0, X_0] = 0$, and $[X_0, X_i] = X_i$ for $i > 0$.

As an example, we apply these methods to extraspecial p-groups.

Theorem 11.6. *Let G be an extraspecial p-group. Then $HH^1(kG)$ is a soluble Lie algebra. The derived length is two, except in the case where G has order p^3 and is isomorphic to $\mathbb{Z}/p^2 \rtimes \mathbb{Z}/p$, in which case it has derived length three.*

Proof. Let G be extraspecial, and let $Z = Z(G) = \Phi(G)$, a group of order p. Every centraliser in G is either equal to G or has index p in G. We divide into two cases. The second case deals with extraspecial groups which are semidirect products $\mathbb{Z}/p^2 \rtimes \mathbb{Z}/p$, and the first case covers all other extraspecial groups. So in the first case, if $|G| = p^3$ then G is either an extraspecial group of exponent p (p odd), or the quaternion group of order eight ($p = 2$).

Case 1. $G \ncong \mathbb{Z}/p^2 \rtimes \mathbb{Z}/p$. In this case, we claim that Hypothesis 11.2 holds for every g and h in G. If $|G| \geqslant p^5$ then for every g and h in G we have $\Phi(C_G(g)) = \Phi(C_G(h)) = \Phi(G) = Z(G)$, so by Lemma 11.1, the hypothesis holds. On the other hand, if $|G| = p^3$ with p odd, and part (i) of the hypothesis does not hold, then neither g nor h is central, and we have $|C_G(g) \cap C_G(uhu^{-1})| = p$ and $|C_G(guhu^{-1})| = p^2$. So $C_G(guhu^{-1})$ is elementary abelian, and then the transfer map from any proper subgroup is zero. Finally, if G is a quaternion group Q_8 then the restriction maps from subgroups of order four to Z is zero. This completes the proof that Hypothesis 11.2 holds for every g and h in G.

So we may apply the theory derived in this section. Using the notation above, we have $HH^1(kG) = X_0 \oplus X_1$, and since $Z(G) = \Phi(G)$ we have $[X_0, X_0] = 0$ and $[X_0, X_1] \leqslant X_1$. It remains to examine $[X_1, X_1]$. Let g and h be non-central elements of G, and let $x \in H^1(C_G(g),k)$ and $y \in H^1(C_G(h),k)$.

If $C_G(g) \neq C_G(h)$ then there are no double coset representatives u satisfying $C_G(g) \cap C_G(uhu^{-1}) = C_G(guhu^{-1})$ in the formula (11.3), because the intersection has index p^2 in G, and so $[x,y] = 0$.

On the other hand, if $C_G(g) = C_G(h)$ then writing C for their common value, the double cosets are just cosets of C. Choosing $v \in G\backslash C$, we may take the double coset representatives to be $1, v, \dots, v^{p-1}$. Now v does not commute with h, so defining $z = vhv^{-1}h^{-1}$, we have $Z = \langle z\rangle$. For some $2 \leqslant m \leqslant p-1$ we have $z^{m-1} = vgv^{-1}g^{-1}$, $z^m = vghv^{-1}(gh)^{-1}$. So for $0 \leqslant i \leqslant p-1$ we have $v^ihv^{-i} = hz^i$, $v^igv^{-i} = gz^{(m-1)i}$, and $v^ighv^{-i} = ghz^{mi}$. Choose n with mn congruent to one modulo p. Then $v^{ni}ghv^{-ni} = ghz^i$.

All contributions to $[x, y]$ are landing in the same summand in the centraliser decomposition, but need conjugating to match the elements being centralised. The contribution coming from v^i is

$$y(v^{-i}gv^i)\,x - x(v^ihv^{-i})\,(v^i)^*(y)$$

in the $C_G(gv^ihv^{-i})$ component, namely the $C_G(ghz^i) = C_G(v^{ni}ghv^{-ni})$ component. So we must conjugate to get

$$y(v^{-i}gv^i)\,(v^{-ni})^*(x) - x(v^ihv^{-i})\,(v^{-(n-1)i})^*(y)$$

in the $C_G(gh)$ component. If $|G| > p^3$, we have $Z \leqslant \Phi(C)$, and so x and y vanish on Z. So conjugating by v^i has no effect. It follows that the above term is independent of i, and when we sum from $i = 0$ to $p-1$ we get zero.

On the other hand, if $|G| = p^3$ then x and y need not vanish on the central element z. In this case, we obtain

$$\begin{aligned} y(v^{-i}gv^i) &= y(gz^{-(m-1)i}) = y(g) - (m-1)iy(z), \\ x(v^ihv^{-i}) &= x(hz^i) = x(h) + ix(z). \end{aligned}$$

So for $\gamma \in C$ we have

$$[x, y](\gamma) = \sum_{i=0}^{p-1} \Big((y(g) - (m-1)iy(z))(x(\gamma) + nirx(z)) - (x(h) + ix(z))(y(\gamma) + (n-1)iry(z)) \Big).$$

Since $p \geqslant 3$, the expressions $\sum_{i=0}^{p-1} 1$, $\sum_{i=0}^{p-1} i$ and are zero in k, and so all but the quadratic term vanish. If $p \geqslant 5$ the quadratic term vanishes too since $\sum_{i=0}^{p-1} i^2 = 0$ in k, but when $p = 3$ it equals -1, and we have

$$\begin{aligned} [x, y](\gamma) &= (-1)(-(m-1)y(z)nrx(z) - x(z)(n-1)ry(z) \\ &= r((m-1)n + (n-1))x(z)y(z) \\ &= r(mn-1)x(z)y(z). \end{aligned}$$

This is equal to zero since m and n are inverses modulo p. This completes the proof in the case $|G| = p^3$, and we are done.

Case 2. $G \cong \mathbb{Z}/p^2 \rtimes \mathbb{Z}/p$. In this case, the failure of Hypothesis 11.2 comes from the fact that transfer from Z to a subgroup $\mathbb{Z}/p^2$ is non-zero in degree one (but zero in degree two). Restriction in degree one from $\mathbb{Z}/p^2$ to Z is zero, however, and transfer from $\mathbb{Z}/p$ to $(\mathbb{Z}/p)^2$ is zero in all degrees, so for the analysis above to fail, we must restrict a degree one element from $(\mathbb{Z}/p)^2$ to Z and then transfer to $\mathbb{Z}/p^2$. So we write $X_1 = X_1' \oplus X_1''$, where X_1' is the sum of the terms in the centraliser decomposition of $HH^1(kG)$ with $C_G(g)$ elementary abelian of order p^2 and X_1'' the sum of the terms with $C_G(g)$ cyclic of order p^2.

As usual we have $[X_0, X_0] = 0$, $[X_0, X_1'] = X_1'$, $[X_0, X_1''] = X_1''$. Furthermore, if g, h and gh have the same centraliser then the same argument as in Case (i) shows that $[x, y] = 0$. So we may assume that $C_G(g) \cap C_G(h) = Z$ and $|C_G(gh)| = p^2$.

If p is odd, then there is a unique subgroup isomorphic to $(\mathbb{Z}/p)^2$ in G, and so if $C_G(g) \cong (\mathbb{Z}/p)^2$ and $C_G(h) \cong \mathbb{Z}/p^2$ then $C_G(gh) \cong \mathbb{Z}/p^2$. Hence $[X_1', X_1'] = 0$, $[X_1', X_1''] \leqslant X_1''$, and $[X_1'', X_1''] = 0$. So $HH^1(kG)$ is soluble of derived length three.

On the other hand, if $p = 2$ then there is a unique subgroup isomorphic to $\mathbb{Z}/4$ in G, and so if $C_G(g) \cong (\mathbb{Z}/2)^2$ and $C_G(h) \cong \mathbb{Z}/4$ then $C_G(gh) \cong (\mathbb{Z}/2)^2$. This time we have $[X_1', X_1'] \leqslant X_1''$ and $[X_1', X_1''] = 0$, and $[X_1'', X_1''] = 0$. So $HH^1(kG)$ is again soluble of derived length three. □

Remark 11.7. The cases of the extraspecial 2-groups D_8 and Q_8 of order eight were also considered in [8, 26].

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