MOTIVIC AND ÉTALE SPANIER-WHITEHEAD DUALITY AND THE BECKER-GOTTLIEB TRANSFER

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Abstract. In this paper, we develop a theory of Becker-Gottlieb transfer based on Spanier-Whitehead duality that holds in both the motivic and étale settings for smooth quasi-projective varieties in as broad a context as possible: for example, for varieties over non-separably closed fields in all characteristics, and also for both the étale and motivic settings. In view of the fact that the most promising applications of the traditional Becker-Gottlieb transfer has been to torsors and Borel-style equivariant cohomology theories, we focus our applications to motivic cohomology theories for torsors as well as Borel-style equivariant motivic cohomology theories, both defined with respect to motivic spectra. We obtain several results in this direction, including a stable splitting in generalized motivic cohomology theories. Various further applications will be discussed in forthcoming papers.

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1. Introduction

Spanier-Whitehead duality in algebraic topology is a classical result formulated and established by E. H. Spanier and J. H. C. Whitehead in the 1950s (see [SpWh55], [SpWh58] and [Sp59]): it was shown there that finite CW complexes have dual complexes if one works in the stable category. This lead to the theory of spectra and much of stable homotopy theory followed. In the 1960s, Atiyah (see [At61]) showed that the Thom-spaces of the normal bundles associated to the imbedding of compact C^∞-manifolds in high dimensional Euclidean spaces provided a Spanier-Whitehead dual for the manifold. A key application of this classical Spanier-Whitehead duality is the construction of a transfer map for fiber bundles, due to Becker and Gottlieb, see [BG75]. The transfer turned out to be a versatile tool in algebraic topology as it often provides stable splittings in classical stable homotopy theory, which are very difficult to obtain otherwise: see [BG75], [Seg73], [Beck74] and [Sn81].

Though the homotopy theory of algebraic varieties in the context of motives and algebraic cycles started only with the work of Voevodsky and Morel (see [MV99]), a closely related theory that only considers algebraic varieties from the point of view of the étale topology has been in existence for over 40 years starting with the work of Artin and Mazur: see [AM69]. The second author's Ph. D thesis (see [?], [J86]) developed the theory of Spanier-Whitehead duality in the context of étale homotopy theory. He also used this to construct a transfer map as a map of stable étale homotopy types for proper smooth maps between algebraic varieties over algebraically closed fields: see [J87]. However, some of the main (potential) applications of the transfer would be to non-proper, but smooth maps, for example fiber-bundles, (as was the case with the traditional Becker-Gottlieb transfer) where the fibers are homogeneous spaces for linear algebraic groups.

In recent years, there has been renewed interest in the homotopy theory of algebraic varieties due to the work of Voevodsky (see [Voev03], [MV99]) on the Milnor conjecture, which introduced several new techniques and the framework of motivic homotopy theory as in [MV99]. It is therefore natural to ask if a suitable theory of Spanier-Whitehead duality in the framework of motivic homotopy theory could be used to construct an analogue of the classical Becker-Gottlieb transfer and if so, if the motivic transfer is compatible with various realizations, for example, the étale and Betti realization. In fact, as we show in sections 8 and 9, the compatibility of the transfer with realization is key to obtaining stable splittings even in the motivic framework. The first author, meanwhile, has been interested in descent questions for algebraic K-theory and formulated a possible approach to understanding these questions using a variant of the Becker-Gottlieb transfer.

A general framework for constructing the Becker-Gottlieb transfer using a variant of Spanier-Whitehead duality was discussed in [DP84] long before stable motivic homotopy theory was invented. A key idea needed here is the notion of objects that are finite in a suitable sense so that they are *dualizable.*

The present paper as well as the sequels, [JP-1] *and* [JP-2], *are devoted to exploring these ideas with the goal of recovering a theory of Becker-Gottlieb type transfers in the motivic and étale contexts along with analogues of several of the classical applications.* In the present paper we obtain a theory of transfer in several distinct contexts:

• for actions of *all* linear algebraic groups on all smooth quasi-projective varieties in characteristic 0 and

• more generally for actions of *all* linear algebraic groups on algebraic varieties that are dualizable in an appropriate stable homotopy theory, i.e. either motivic or étale with certain primes inverted and including large classes of smooth varieties over perfect fields of positive characteristics.

• We show that the transfer then provides stable splittings in the appropriate stable homotopy category, whenever possible. In particular, this applies to certain fibrations between classifying spaces of algebraic groups where the fibers are homogeneous varieties that are dualizable, making use of Spanier-Whitehead duality in the motivic or étale context. It should also be pointed out that, though a version of the transfer has been constructed by Totaro (see [Tot14, Theorem 2.17]), it only holds for Chow groups of classifying spaces of linear algebraic groups in characteristic 0: the question of stable splittings in the motivic stable homotopy category via transfer in any characteristic, as well as the construction of a motivic transfer in positive characteristics were left open there. In related work, Levine [Lev18] has considered transfers for fiber bundles locally trivial in the Nisnevich topology, and with the main results on stable splitting holding only in characteristic 0. However, the construction of the transfer there for Borel style generalized motivic cohomology theories seems not to have taken into account certain key and subtle issues regarding the role of equivariant spectra in the construction of the transfer, as discussed on p. 5 below, and in Remark 2.10. Moreover, a comparison of the techniques used for obtaining splittings in the stable homotopy category is also discussed on p. 4.

The most promising applications of the transfer at present seems to be to G-torsors and to the corresponding Borel style equivariant cohomology theories defined with respect to motivic and étale spectra, which is the focus of the current paper.

The following are some of the main results of the paper. Let G denote a linear algebraic group acting on smooth quasi-projective schemes X and Y of finite type over a perfect field k of arbitrary characteristic. (Here we are *not* assuming that G is *special* in the sense of Grothendieck: see [Ch].) It is also often convenient and necessary to consider a slightly more general situation, where X and Y will denote unpointed simplicial presheaves on the big Nisnevich (or big étale site) of the field k, provided with an action by the linear algebraic group G. Let $\mathbf{Spt}_{\mathrm{mot}}$ ($\mathbf{Spt}_{\mathrm{et}}$, $\mathbf{Spt}_{\mathrm{mot},\mathcal{E}}$, $\mathbf{Spt}_{\mathrm{et},\mathcal{E}}$) denote the category of motivic spectra over k (the corresponding category of spectra on the big étale site of k, the subcategory of $\mathbf{Spt}_{\mathrm{mot}}$ of $\mathcal{E}$-module spectra for a commutative ring spectrum $\mathcal{E} \in \mathbf{Spt}_{\mathrm{mot}}$ and the corresponding subcategory of $\mathbf{Spt}_{\mathrm{et}}$ for a commutative ring spectrum $\mathcal{E} \in \mathbf{Spt}_{\mathrm{et}}$, respectively). (If $\mathrm{S} = \mathrm{Spec}\, \mathrm{k}$, we may often let $\mathbf{Spt}/\mathrm{S}_{\mathrm{mot}}$ ($\mathbf{Spt}/\mathrm{S}_{\mathrm{et}}$) denote $\mathbf{Spt}_{\mathrm{mot}}$ ($\mathbf{Spt}_{\mathrm{et}}$, respectively) to highlight the base field k.) The corresponding stable homotopy category will be denoted $\mathbf{HSpt}_{\mathrm{mot}}$ ($\mathbf{HSpt}_{\mathrm{et}}$, $\mathbf{HSpt}_{\mathrm{mot},\mathcal{E}}$ and $\mathbf{HSpt}_{\mathrm{et},\mathcal{E}}$, respectively). See Definition 3.15 for further details. [1] Throughout, $\mathbf{T}$ will denote $\mathbb{P}^1$ pointed by ∞ and $\mathbf{T}^n$ will denote $\mathbf{T}^{\wedge n}$ for any integer $n \geq 0$. (Similarly $\mathbb{G}_m^v$ will denote $\mathbb{G}_m^{\wedge v}$.)

Moreover, it is important for us that the ring-spectrum $\mathcal{E} \in \mathbf{Spt}_{\mathrm{mot}}$ ($\mathcal{E} \in \mathbf{Spt}_{\mathrm{et}}$) has a lift to an *equivariant ring spectrum* $\mathcal{E}^{\mathrm{G}}$, in the sense of Terminology 3.12. For example, the usual motivic sphere spectrum $\Sigma_{\mathbf{T}}$ lifts to the equivariant sphere spectrum $\mathbb{S}^{\mathrm{G}}$ defined as in Definition 3.4. The only other ring spectra we consider will be $\Sigma_{\mathbf{T}}[p^{-1}]$ for $p = char(k)$, and for a fixed prime $\ell \neq p (= char(k))$, $\Sigma_{\mathbf{T},(\ell)}$ which denotes the localization of $\Sigma_{\mathbf{T}}$ at the prime ideal (ℓ), $\Sigma_{\mathbf{T}}\widehat{}_{\ell}$ which denotes the completion of $\Sigma_{\mathbf{T}}$ at the prime ℓ. $\Sigma_{\mathbf{T}}[p^{-1}]$ ($\Sigma_{\mathbf{T},(\ell)}$, $\Sigma_{\mathbf{T}}\widehat{}_{\ell}$) lifts to the equivariant spectrum $\mathbb{S}^{\mathrm{G}}[p^{-1}]$ ($\mathbb{S}^{\mathrm{G}}_{(\ell)}$, $\widehat{\mathbb{S}^{\mathrm{G}}}_{\ell}$, respectively). In positive characteristic p, we either consider $\mathbf{Spt}_{\mathrm{mot}}[p^{-1}]$, or $\mathbf{Spt}_{\mathrm{mot},\mathcal{E}}$ where $\mathcal{E}$ is any motivic ring spectrum that is ℓ-complete for some prime $\ell \neq p = char(k)$.

We will consider the following three *basic contexts*:

(a) $\mathrm{p} : \mathrm{E} \to \mathrm{B}$ is a G-torsor for the action of a linear algebraic group G with both E and B smooth quasi-projective schemes over k, with B *connected* and

$$\pi_{\mathrm{Y}} : \mathrm{E} \times_{\mathrm{G}} (\mathrm{Y} \times \mathrm{X}) \to \mathrm{E} \times_{\mathrm{G}} \mathrm{Y}$$

the induced map, where G acts diagonally on $\mathrm{Y} \times \mathrm{X}$. One may observe that, on taking $\mathrm{Y} = \mathrm{Spec}\, \mathrm{k}$ with the trivial action of G, the map π_{Y} becomes $\pi_{\mathrm{Y}} : \mathrm{E} \times_{\mathrm{G}} \mathrm{X} \to \mathrm{B}$, which is an important special case.

(b) $\mathrm{BG}^{gm,m}$ will denote the m-th degree approximation to the geometric classifying space of the linear algebraic group G (as in [Tot99], [MV99]), $\mathrm{p} : \mathrm{EG}^{gm,m} \to \mathrm{BG}^{gm,m}$ is the corresponding universal G-torsor and

$$\pi_{\mathrm{Y}} : \mathrm{EG}^{gm,m} \times_{\mathrm{G}} (\mathrm{Y} \times \mathrm{X}) \to \mathrm{EG}^{gm,m} \times_{\mathrm{G}} \times_{\mathrm{G}} \mathrm{Y}$$

is the induced map.

(c) If $\mathrm{p_m}$ ($\pi_{\mathrm{Y,m}}$) denotes the map denoted p (π_{Y}) in (b), here we let $\mathrm{p} = \lim_{\mathrm{m} \to \infty} \mathrm{p_m}$ and let

$$\pi_{\mathrm{Y}} = \lim_{m \to \infty} \pi_{\mathrm{Y,m}} : \mathrm{EG}^{gm} \times_{\mathrm{G}} (\mathrm{Y} \times \mathrm{X}) = \lim_{\mathrm{m} \to \infty} \mathrm{EG}^{gm,m} \times_{\mathrm{G}} (\mathrm{Y} \times \mathrm{X}) \to \lim_{\mathrm{m} \to \infty} \mathrm{EG}^{gm,m} \times_{\mathrm{G}} \mathrm{Y} = \mathrm{EG}^{gm} \times_{\mathrm{G}} \mathrm{Y}.$$

Strictly speaking, the above definitions apply only to the case where G is *special* in the sense of Grothendieck (see [Ch]) and when G is *not special*, the above objects will in fact need to be replaced by the derived push-forward of the above objects viewed as sheaves on the big étale site of k to the corresponding big Nisnevich site of k, as discussed in (6.2.8). However, we will denote these new objects also by the same notation throughout, except when it is necessary to distinguish between them. For G *not special*, we *will assume the base field is also infinite* to prevent certain unpleasant situations.

Theorem 1.1. *Let* $\mathrm{f} : \mathrm{X} \to \mathrm{X}$ *denote a* G*-equivariant map and for each* $m \geq 0$, *let* $\pi_{\mathrm{Y}} : \mathrm{E} \times_{\mathrm{G}} (\mathrm{Y} \times \mathrm{X}) \to \mathrm{E} \times_{\mathrm{G}} \mathrm{Y}$ *denote any one of the maps considered in (a) through (c) above. Let* $\mathrm{f_Y} = \mathrm{id_Y} \times \mathrm{f} : \mathrm{Y} \times \mathrm{X} \to \mathrm{Y} \times \mathrm{X}$ *denote the induced map.*

Then in case (a), we obtain a map (called the transfer)

$$tr(\mathrm{f_Y}) : \Sigma_{\mathbf{T}}(\mathrm{E} \times_{\mathrm{G}} \mathrm{Y})_+ \to \Sigma_{\mathbf{T}}(\mathrm{E} \times_{\mathrm{G}} (\mathrm{Y} \times \mathrm{X}))_+ \quad (tr(\mathrm{f_Y}) : \mathcal{E} \wedge (\mathrm{E} \times_{\mathrm{G}} \mathrm{Y})_+ \to \mathcal{E} \wedge (\mathrm{E} \times_{\mathrm{G}} (\mathrm{Y} \times \mathrm{X}))_+)$$

in $\mathbf{HSpt}_{\mathrm{mot}}$ *(*$\mathbf{HSpt}_{\mathrm{mot},\mathcal{E}}$*, respectively) if* $\Sigma_{\mathbf{T}}\mathrm{X}_+$ *is dualizable in* $\mathbf{Spt}_{\mathrm{mot}}$ *(if* $\mathcal{E} \wedge \mathrm{X}_+$ *is dualizable in* $\mathbf{Spt}_{\mathrm{mot},\mathcal{E}}$*, respectively) having the following properties.*

(i) *If* $tr(\mathrm{f_Y})^{\mathrm{m}} : \Sigma_{\mathbf{T}}(\mathrm{EG}^{gm,m} \times_{\mathrm{G}} \mathrm{Y})_+ \to \Sigma_{\mathbf{T}}(\mathrm{EG}^{gm,m} \times_{\mathrm{G}} (\mathrm{Y} \times \mathrm{X}))_+$ $(tr(\mathrm{f_Y})^{\mathrm{m}} : \mathcal{E} \wedge (\mathrm{EG}^{gm,m} \times_{\mathrm{G}} \mathrm{Y})_+ \to \mathcal{E} \wedge (\mathrm{EG}^{gm,m} \times_{\mathrm{G}} (\mathrm{Y} \times \mathrm{X}))_+)$ *denotes the corresponding transfer maps in case (b), the maps* $\{tr(\mathrm{f_Y})^{\mathrm{m}} | \mathrm{m}\}$ *are compatible as m varies. The corresponding induced map* $\lim_{m \to \infty} tr(\mathrm{f_Y})^{\mathrm{m}}$ *will be denoted* $tr(\mathrm{f_Y})$.

For items (ii) through (iv) we will assume any one of the above contexts (a) through (c).

[1] The homotopy category $\mathbf{HSpt}_{\mathrm{mot}}$ is often denoted $\mathcal{SH}(k)$ in the literature. Our notation of $\mathbf{HSpt}_{\mathrm{mot}}$ hopes to highlight the fact that this is the homotopy category of motivic spectra.

(ii) *If* $h^{*,\bullet}(\quad, \mathcal{E})$ $(h^{*,\bullet}(\quad, M))$ *denotes the generalized motivic cohomology theory defined with respect to the commutative motivic ring spectrum* $\mathcal{E}$ *(a motivic module spectrum* M *over* $\mathcal{E}$*, respectively) then,*

$$tr(f_Y)^*(\pi_Y^*(\alpha).\beta) = \alpha.tr(f_Y)^*(\beta), \quad \alpha \in h^{*,\bullet}(E \times_G Y, M), \beta \in h^{*,\bullet}(E \times_G (Y \times X), \mathcal{E}).$$

Here $tr(f_Y)^*$ (π_Y^*) *denotes the map induced on generalized cohomology by the map* $tr(f_Y)$ $(\pi_Y$*, respectively). Both* $tr(f_Y)^*$ *and* π_Y^* *preserve the degree as well as the weight.*

(iii) *In particular,* $\pi_Y^* : h^{*,\bullet}(E \times_G Y, M) \to h^{*,\bullet}(E \times_G (Y \times X), M)$ *is split injective if* $tr(f_Y)^*(1) = tr(f_Y)^*(\pi_Y^*(1))$ *is a unit, where* $1 \in h^{0,0}(E \times_G Y, \mathcal{E})$ *is the unit of the graded ring* $h^{*,\bullet}(E \times_G Y, \mathcal{E})$.

(iv) *The transfer* $tr(f_Y)$ *is natural with respect to restriction to subgroups of a given group. It is also natural with respect to change of base fields, assuming taking the dual is compatible with such base-change.*

(v) *If* $h^{*,\bullet}(\quad, M)$ *denotes a generalized motivic cohomology theory, then the map* $tr(f_Y)^* : h^{*,\bullet}(EG^{gm} \times_G (Y \times X), M) \to h^{*,\bullet}(EG^{gm} \times_G Y, M)$ *is independent of the choice of a geometric classifying space that satisfies certain basic assumptions (as in 7.1: Proof of Theorem 1.1), and depends only on* X, Y *and the* G*-equivariant map* f.

(vi) *Assume the base field k satisfies the finiteness conditions in* (3.0.3)*. Let* $\mathcal{E}$ *denote a commutative ring spectrum in* $\mathbf{Spt}_{et}$ *which is* ℓ*-complete, in the sense of Definition 1.2 (below), for some prime* $\ell \neq char(k)$*. If* $\mathcal{E} \wedge X_+$ *is dualizable in* $\mathbf{Spt}_{et,\mathcal{E}}$*, then there exists a transfer* $tr(f_Y)$ *in* $\mathbf{Spt}_{et,\mathcal{E}}$ *satisfying similar properties.*

(vii) *Assume the base field k satisfies the finiteness conditions in* (3.0.3)*. Let* $\mathcal{E}$ *denote a commutative ring spectrum in* $\mathbf{Spt}_{mot}$ *which is* ℓ*-complete. Let* $\epsilon^* : \mathbf{Spt}_{mot} \to \mathbf{Spt}_{et}$ *denote the map of topoi induced by the obvious map from the étale site of k to the Nisnevich site of k. Then if* $\mathcal{E} \wedge X_+$ *is is dualizable in* $\mathbf{Spt}_{mot,\mathcal{E}}$ *and* $\epsilon^*(\mathcal{E} \wedge X_+)$ *is dualizable in* $\mathbf{Spt}_{et,\mathcal{E}}$*, the transfer map* $tr(f_Y)$ *is compatible with étale realizations, and for groups* G *that are special,* $\epsilon^*(tr(f_Y)) = tr(\epsilon^*(f_Y))$.

The transfer map in the above theorem is constructed by combining a pre-transfer map defined utilizing a suitable notion of Spanier-Whitehead duality with the Borel construction for the linear algebraic group in question. It also makes use of certain key properties of Thom-spaces of algebraic vector bundles as discussed in the appendix. The construction of the transfer is worked out in detail in section 6 and its key properties worked out in section 7. Then the statement in (i) is verified by an explicit construction: see Step 2 of (6.2). The first statement in (ii) in the Theorem follows readily from a naturality property for the corresponding transfer with respect to change of linear algebraic groups (see Proposition 7.1 and Corollary 7.5). This will imply the property (iii) and the first property in (iv). The second statement in (ii) follows from the fact that the transfer is defined using the pre-transfer (see Examples 2.9) which is a stable map that involves no degree or weight shifts. Property (v) is shown to hold by the way the transfer map, $tr(f_Y)$, is constructed: see 7.1. To make it independent of all the possible choices involved in its construction is the main reason for considering equivariant spectra and the categories $\mathbf{Spt}^G$, $\widetilde{\mathbf{USpt}}^G$, $\mathbf{USpt}^G$: see Remark 6.2 for more on this. The construction of the transfer in the étale framework is similar, though care has to be taken to ensure that affine spaces are contractible in this framework, which accounts partly for the hypotheses in (vi) and in (3.0.3). Property (vii) plays a key-role in the paper, and is discussed in the following paragraphs.

Let $p : E \to B$ denote a torsor as in Theorem 1.1. Observe that one of the main applications of the transfer is to provide sections, whenever possible, to the induced map π_Y^* (in generalized cohomology), and therefore a stable splitting of the map induced by π_Y in the appropriate stable homotopy category. *We provide two distinct strategies to establish such stable splittings, each with its own advantages. (See Theorem 1.5 and Corollaries 1.6 and 1.7 where such splittings are discussed in more details.)* Both start with the observation that the multiplicative property of the transfer as in Theorem 1.1(ii) shows that in order to prove π_Y^* is a split monomorphism, it suffices to show $tr(f_Y)^*(1) = tr(f_Y)^*\pi_Y^*(1)$ is a unit. Both approaches also make use of the base-change property of the transfer as in Proposition 7.1 and then reduce to checking this for simpler situations. They both apply to all linear algebraic groups, irrespective of whether they are special (that is, in Grothendieck's classification as in [Ch]), in particular to all split orthogonal and finite groups, which are known to be non-special.

• *Splittings via the Grothendieck-Witt ring of the base field k.* In this case we make use of the base-change property of the transfer as in Proposition 7.1 to reduce to the case where the torsor $p : E \to B$ is trivial. When the group G is special (is not necessarily special), this means replacing B by a Zariski open cover (étale open cover, respectively) $B' \to B$, over which the torsor p trivializes. Over B', it suffices to prove the splitting at the level of the pre-transfer, so that it is enough to show $\tau_X^*(1)$ is a unit in the Grothendieck-Witt ring of the base field in case $char(k) = 0$ and in the Grothendieck-Witt ring of the base field with p inverted in case $char(k) = p > 0$. (Here $\tau_X = \tau_X(id_X)$ denotes the trace defined in Definition 2.8.)

The *main advantage* of this method is that it provides splittings for all generalized motivic cohomology theories, whenever the above computation of the trace $\tau_X^*(1)$ can be carried out in the Grothendieck-Witt ring of the base field, independent of whether the group G is special.

The main *disadvantages* of this method are as follows. Computing the trace associated to a scheme in the Grothendieck-Witt ring for many schemes is extremely difficult and possibly not do-able with the present technology. The only case where this seems do-able at present is for G/N(T), where G is a split reductive group and N(T) denotes the normalizer of a split maximal torus in G: see [Lev18] for partial results in this direction and see [JP-1, Theorem 1.4] for a computation in the general case. Moreover, the above discussion is only for the case the self-map $f : X \rightarrow X$, (with $X = G/N(T)$) is the identity: it is far from clear how to carry out a similar computation in the Grothendieck-Witt ring for a general self-map f, even for the same X.

• *Splittings for slice-completed generalized motivic cohomology theories.* This method makes strong use of the fact that the transfer map we have constructed is a map in the appropriate stable homotopy category and, therefore induces a map of the corresponding motivic (or étale) Atiyah-Hirzebruch spectral sequences: it suffices to show the transfer induces a splitting at the E_2-terms of this spectral sequence. The multiplicative properties of the slice filtration that a natural pairing of the slices of a motivic spectrum lift to the category $\mathbf{Spt}_{\mathrm{mot}}$ from the corresponding motivic stable homotopy category were verified in [Pel08]. Therefore, it follows that the motivic spectra that define the E_2-terms of the motivic Atiyah-Hirzebruch spectral sequence, that is the slices of the given motivic spectrum, are modules over the motivic Eilenberg-Maclane spectrum $\mathbb{H}(\mathbb{Z})$. Then the multiplicative property of the transfer as in Proposition 7.3 and Corollary 7.5 reduce to checking that we obtain a splitting for motivic cohomology.

Next, we make use of the base-change property of the transfer as in Proposition 7.1 and then reduce to checking this for the action of trivial groups, that is for the pre-transfer. See, for example, Proposition 7.6.

At this point it is often very convenient, as well as necessary, *to know that the transfer is compatible with passage to simpler situations, for example, to a change of the base field to one that is separably or algebraically closed and with suitable realizations, that is either the étale realization or the Betti realization.* A *main advantage* of this approach is that it would be only necessary to compute $tr(f)^*(1)$ and the trace $\tau_X^*(1)$ after such reductions and realizations, which are readily do-able for a large number of schemes X: see Propositions 8.1, 8.3 and Corollary 8.2. *Another advantage* is that it addresses affirmatively the important question if the pre-transfer and transfer are compatible with such reductions and realizations. Moreover, by this method, one can allow any self-map $f : X \rightarrow X$ and compute the corresponding trace $\tau_X(f)$.

The only *disadvantages* for this method seems to be that we need to assume that the base B of the torsor is connected, the object Y is a geometrically connected smooth scheme of finite type over k, and also because this method applies to only slice-completed generalized motivic cohomology theories. However, as several important examples of generalized motivic cohomology theories, such as Algebraic K-theory and Algebraic Cobordism are slice-complete, there do not seem to be any serious disadvantages.

When the base scheme is a field with trivial action by the given group G, the transfer will be referred to as the pre-transfer. The construction of the transfer for Borel-style G-equivariant generalized cohomology theories starts off with a pre-transfer which will have to be a G-equivariant stable map, so that it can be fed into a suitable Borel-construction. The pre-transfer is constructed by making use of Spanier-Whitehead duality as worked out in the classical setting: see [BG75].

• Here is *a particularly tricky aspect* of the construction of the pre-transfer. The Spanier-Whitehead duality one needs to use is in the setting of $\mathbf{Spt}_{\mathrm{mot}}$, $\mathbf{Spt}_{\mathrm{et}}$, $\mathbf{Spt}_{\mathrm{mot},\mathcal{E}}$ or $\mathbf{Spt}_{\mathrm{et},\mathcal{E}}$ and *not* in a corresponding category of equivariant spectra, such as in [CJ14]. There are several reasons for this choice, some of which are:

(i) Currently one does not have Spanier-Whitehead duality for algebraic varieties in the equivariant framework, since one does not have equivariant versions of Gabber's refined alterations, for example.
(ii) For the construction of the transfer in the context of Borel-style generalized equivariant cohomology theories this is all that is needed as shown, for example by [BG75]. i.e. All one needs in this context is Spanier-Whitehead duality in a *non-equivariant setting, but applied to spectra with group actions.*
(iii) On the other hand, we still need the Spanier-Whitehead dual of an object with a G-action to inherit a nice G-action and we need to use sphere-spectra which also have non-trivial G-actions. *In fact, it is crucial that the source of the co-evaluation maps will have to be* G*-equivariant (sphere) spectra: otherwise the spectra showing up as the target of the co-evaluation maps will have no* G*-action*: see Definition 3.4 and 6.1. i.e. Though we only need a non-equivariant form of Spanier-Whitehead duality, one needs to make all the constructions sufficiently equivariant so as to be able to feed them into the Borel construction.
(iv) In [BG75], the way these issues are resolved is by making sure the Thom-Pontrjagin collapse map (which plays the role of the co-evaluation map) can be made equivariant. In our framework, the way we resolve these problems is as follows. First we use G-equivariant spectra to serve as the source of the co-evaluation maps. Then we observe that for objects in $\mathbf{Spt}$, one can find functorial fibrant and cofibrant replacements in $\mathbf{Spt}$ (which generically denotes $\mathbf{Spt}_{\mathrm{mot}}$, $\mathbf{Spt}_{\mathrm{et}}$, $\mathbf{Spt}_{\mathrm{mot},\mathcal{E}}$ or $\mathbf{Spt}_{\mathrm{et},\mathcal{E}}$), and the functoriality implies that

these objects come equipped with compatible G-actions. See the discussion in 3.3.13 as well as the discussion leading up to that, starting with Definition 3.4, for further details. Therefore, the dual we define will be making use of such functorial cofibrant and fibrant replacements and therefore, though they are the duals in **Spt**, they still come equipped with nice G-actions. See Remark 2.10 for a detailed discussion of these issues. It is precisely these issues that make it necessary for us to introduce and work with the categories $\widetilde{\mathbf{USpt}}^{\mathrm{G}}$ and $\mathbf{USpt}^{\mathrm{G}}$ of spectra that come in between, and relate the category of equivariant spectra with the category of ordinary spectra as in section 3.3.

(v) Moreover, though we need to use G-equivariant sphere spectra to serve as the source of the co-evaluation map, we pass to the usual sphere spectra with no-G-action as soon as the construction of the transfer from the pre-transfer is completed: see the details in 6.1 and 6.2.

It should be clear that Theorem 1.1 derives its strength by knowing when certain objects are dualizable in the stable motivic or étale homotopy category and when étale realization as well as change of base fields is compatible with taking duals, (and therefore with the transfer: see section 8), as described in Theorem 1.3 below. The compatibility of the transfer with Betti-realization is similar and will be discussed elsewhere.

Definition 1.2. Let $\mathrm{M} \in \mathbf{Spt}_{\mathrm{mot}}$ ($\mathbf{Spt}_{\mathrm{et}}$). For each prime number ℓ, let $Z_{(\ell)}$ denote the localization of the integers at the prime ideal ℓ and let $Z\hat{}_\ell = \lim_{\infty \leftarrow n} Z/\ell^n$. Then we say M is $Z_{(\ell)}$-local (ℓ-complete, ℓ-primary torsion), if each $[\mathrm{S}^{1^{\wedge s}} \wedge \mathbf{T}^t \wedge \Sigma_{\mathbf{T}}\mathrm{U}_+, \mathrm{M}]$ is a $Z_{(\ell)}$-module ($Z\hat{}_\ell$-module, $Z\hat{}_\ell$-module which is torsion, respectively) as U varies among the objects of the given site, where $[\mathrm{S}^{1^{\wedge s}} \wedge \mathbf{T}^t \wedge \Sigma_{\mathbf{T}}\mathrm{U}_+, \mathrm{M}]$ denotes *Hom* in the stable homotopy category $\mathbf{HSpt}_{\mathrm{mot}}$ ($\mathbf{HSpt}_{\mathrm{et}}$, respectively).

Let $\mathrm{M} \in \mathbf{Spt}_{\mathrm{mot}}$ ($\mathbf{Spt}_{\mathrm{et}}$). Then one may observe that if ℓ is a prime number, and M is ℓ-complete, then M is $Z_{(\ell)}$-local. This follows readily by observing that the natural map $Z \to Z\hat{}_\ell$ factors through $Z_{(\ell)}$ since every prime different from ℓ is inverted in $Z\hat{}_\ell$. One may also observe that if $\mathcal{E}$ is a commutative ring spectrum which is $Z_{(\ell)}$-local (ℓ-complete), then any module spectrum M over $\mathcal{E}$ is also $Z_{(\ell)}$-local (ℓ-complete, respectively). ℓ-completion in the motivic framework is discussed in detail in [CJ14, section 4].

Given the above framework, the following two results are now well-known:

1.1.

(i) If the field k is of characteristic 0, and X is any smooth quasi-projective scheme over k, $\Sigma_{\mathbf{T}}\mathrm{X}_+$ is dualizable in the motivic stable homotopy category $\mathbf{Spt}_{\mathrm{mot}}$.

(ii) If the field k is perfect and of positive characteristic p, and X is any smooth quasi-projective scheme over k, $\Sigma_{\mathbf{T}}\mathrm{X}_+$ is dualizable in $\mathbf{Spt}_{\mathrm{mot}}[p^{-1}]$ which denotes $\mathbf{Spt}_{\mathrm{mot}}$ with the prime p inverted (and where the unit is the sphere spectrum $\Sigma_{\mathbf{T}}[p^{-1}]$).

The first statement is a direct consequence of the fact that the **T**-suspension spectra of smooth quasi-projective varieties over a field of characteristic 0 are dualizable in the motivic homotopy category: see for example, [RO08, Theorem 52]. The second statement is much more involved and for a proof, one needs to invoke Gabber's theory of refined alterations as in [K13] and [Ri13]. (See section 6 for further details.)

We then deduce the following key-result, which when fed into the last theorem, provides various stable splittings in the motivic framework. The stable splittings are discussed in detail in Theorem 1.5 and Corollaries 1.6, 1.7. Assume that the base scheme S = Spec$k$, for a perfect field k satisfying the hypothesis (3.0.3). We will let $\bar{k}$ denote its algebraic closure and let $\bar{\mathrm{S}} = \mathrm{Spec}\,\bar{k}$. Then one obtains the following maps of topoi (where the first two are induced by the obvious morphisms of sites and the last map is induced by change of base fields):

$$\epsilon^* : \mathbf{Spt}/\mathrm{S}_{\mathrm{mot}} \to \mathbf{Spt}/\mathrm{S}_{\mathrm{et}}, \bar{\epsilon}^* : \mathbf{Spt}/\bar{\mathrm{S}}_{\mathrm{mot}} \to \mathbf{Spt}/\bar{\mathrm{S}}_{\mathrm{et}} \text{ and } \eta^* : \mathbf{Spt}/\mathrm{S}_{\mathrm{et}} \to \mathbf{Spt}/\bar{\mathrm{S}}_{\mathrm{et}}. \tag{1.1.1}$$

Since étale cohomology is well-behaved only with torsion coefficients prime to the characteristic, one will need to also consider the functors $\theta : \mathbf{Spt}/\mathrm{S}_{\mathrm{et}} \to \mathbf{Spt}/\mathrm{S}_{\mathrm{et}}$ sending commutative ring spectra $\mathcal{E}$ to $\mathcal{E} \overset{L}{\underset{\Sigma_{\epsilon^*(\mathbf{T})}}{\wedge}} \mathrm{H}(\mathrm{Z}/\ell)$ where $\mathrm{H}(\mathrm{Z}/\ell)$ denotes the mod-ℓ Eilenberg-Maclane spectrum in $\mathbf{Spt}_{\mathrm{et},\epsilon^*(\mathbf{T})}$. If ℓ is a fixed prime different from $char(k)$, and $\mathcal{E}$ is a commutative ring spectrum in $\mathbf{Spt}_{\mathrm{mot}}$, we will also consider the functor sending spectra $\mathrm{M} \in \mathbf{Spt}_{\mathrm{mot},\mathcal{E}}$ to $M \wedge_{\mathcal{E}} \mathcal{E}(\ell^\nu)$, where $\mathcal{E}(\ell^\nu)$ denotes the (homotopy) cofiber of the map $\mathcal{E} \overset{\ell^\nu}{\to} \mathcal{E}$: we will denote this functor by $\phi_{\mathcal{E}}$. We will adopt the convention that the maps of topoi in (5.0.19) in fact denote their corresponding left derived functors.

Theorem 1.3. *(See Propositions 5.1, 5.4, 5.6, 5.9 and Theorem 5.7.) Let k denote a perfect field of arbitrary characteristic satisfying the finiteness hypothesis* (3.0.3) *and let ℓ denote a prime different from* $char(k)$*. Then the following hold (adopting the terminology as in sections 2 and 4):*

(i) *Let $\mathcal{E}$ denote a commutative ring spectrum in $\mathbf{Spt}_{\mathrm{mot}}$ which is $Z_{(\ell)}$-local (ℓ-complete). Then, if* X *is any smooth quasi-projective variety over k, the spectrum $\mathcal{E} \wedge \mathrm{X}_+$ is dualizable in $\mathbf{Spt}_{\mathrm{mot},\mathcal{E}}$. In particular, this holds for any* X *of the form* G/H, *where* G *is a linear algebraic group defined over k and* H *is a closed subgroup and also when the spectrum $\mathcal{E} = \Sigma_{\mathbf{T},(\ell)}$ ($\mathcal{E} = \Sigma_{\mathbf{T}}\widehat{\ }_{\ell}$) which denotes the localization (completion) of the motivic sphere spectrum $\Sigma_{\mathbf{T}}$ at the prime ideal (ℓ) (at the prime ℓ, respectively).*

(ii) *If $\mathcal{E}$ denotes a commutative ring spectrum on the big étale site of k which is ℓ-complete, then $\mathcal{E} \wedge (\Sigma_{\mathbf{T}}\mathrm{G/H}_+)$ is a retract of a finite cellular object in $\mathbf{Spt}_{\mathrm{et},\mathcal{E}}$. Therefore, it is dualizable as an object in $\mathbf{Spt}_{\mathrm{et},\mathcal{E}}$. (For example, $\mathcal{E}$ could be the ℓ-completed S^1-sphere spectrum on the big étale site of Spec k.)*

(iii) *If $\mathcal{E}$ is a commutative motivic ring spectrum so that it is ℓ-primary torsion as in Definition 1.2, then the functors ϵ^*, $\bar{\epsilon}^*$ send the dualizable objects of the form $\mathcal{E} \wedge \mathrm{X}_+$ appearing in (i) to dualizable objects. The same conclusion holds for the functor $\phi_{\mathcal{E}}$ (θ) if $\mathcal{E}$ is a motivic ring spectrum (étale ring spectrum, respectively) that is ℓ-complete. If the ring spectrum $\mathcal{E}$ is ℓ-complete (ℓ-primary torsion), the functor η^* sends the dualizable objects $\mathcal{E} \wedge \mathrm{X}_+$ in (ii) (the dualizable objects of the form $\mathcal{E} \wedge X_+$, respectively) to dualizable objects.*

The first statement is proven in Theorem 5.7. The second statement is proven by first showing that $\mathcal{E}\wedge(\Sigma_{\mathbf{T}}\mathrm{G/H}_+)$ is a compact object in $\mathbf{Spt}_{\mathrm{et},\mathcal{E}}$ and this requires the hypotheses on the spectrum. Then we make use of the observation that the map G $\rightarrow$ G/H is locally trivial in the étale topology to conclude that $\mathcal{E} \wedge (\Sigma_{\mathbf{T}}\mathrm{G/H}_+)$ is a retract of a finite cellular object in $\mathbf{Spt}_{\mathrm{et},\mathcal{E}}$. The proof of the third statement is a bit involved and needs a careful re-examination and analysis of the proof of (i) as in Theorem 5.7 as well as in the proofs of Propositions 5.1 and 5.4 for the special cases considered there. Moreover, the results of sections 2 through 5 set up the necessary foundational results leading up to the main results on the transfer in sections 6 and 7.

Definition 1.4. For a smooth scheme Y (smooth ind-scheme $\mathcal{Y} = \{\mathrm{Y_m}|\mathrm{m}\}$), we define *the slice completed generalized motivic cohomology spectrum* with respect to a motivic spectrum M to be $\hat{\mathrm{h}}(\mathrm{Y},\mathrm{M}) = \underset{\infty\leftarrow \mathrm{n}}{\mathrm{holim}}\mathbb{H}_{\mathrm{Nis}}(\mathrm{Y},\mathrm{s}_{\leq \mathrm{n}}\mathrm{M}) \simeq \mathbb{H}_{\mathrm{Nis}}(\mathrm{Y},\underset{\infty\leftarrow \mathrm{n}}{\mathrm{holim}}\mathrm{s}_{\leq \mathrm{n}}\mathrm{M})$ ($\hat{\mathrm{h}}(\mathcal{Y},\mathrm{M}) = \underset{\infty\leftarrow \mathrm{m}}{\mathrm{holim}}\underset{\infty\leftarrow \mathrm{n}}{\mathrm{holim}}\mathbb{H}_{\mathrm{Nis}}(\mathrm{Y_m},\mathrm{s}_{\leq \mathrm{n}}\mathrm{M}) \simeq \underset{\infty\leftarrow \mathrm{m}}{\mathrm{holim}}\mathbb{H}_{\mathrm{Nis}}(\mathrm{Y_m},\underset{\infty\leftarrow \mathrm{n}}{\mathrm{holim}}\mathrm{s}_{\leq \mathrm{n}}\mathrm{M})$), where $s_{\leq n}\mathrm{M}$ is the homotopy cofiber of the map $f_{n+1}\mathrm{M} \rightarrow \mathrm{M}$ and $\{f_n\mathrm{M}|\mathrm{n}\}$ is the *slice tower* for M with $f_{n+1}\mathrm{M}$ being the $n+1$-th connective cover of M. ($\mathbb{H}_{Nis}(\mathrm{Y},\mathrm{F})$ and $\mathbb{H}_{Nis}(\mathrm{Y_m},\mathrm{F})$ denote the generalized hypercohomology spectrum with respect to a motivic spectrum F computed on the Nisnevich site.) The corresponding homotopy groups for maps from $\Sigma_{\mathbf{T}}(\mathrm{S}^u \wedge \mathbb{G}_{\mathrm{m}}^v)$ to the above spectra will be denoted $\hat{\mathrm{h}}^{u+v,v}(\mathcal{Y},\mathrm{M})$. One may define the completed generalized étale cohomology spectrum of a scheme with respect to an S^1-spectrum by using the Postnikov tower in the place of the slice tower in a similar manner.

The following theorem and corollaries now discuss the main results on splitting in the motivic stable homotopy category obtained using the transfer.

Theorem 1.5. *Let $\pi_{\mathrm{Y}}: \mathrm{E} \times_{\mathrm{G}} (\mathrm{Y} \times \mathrm{X}) \rightarrow \mathrm{E} \times_{\mathrm{G}} \mathrm{Y}$ denote a map as in one of the three cases considered in Theorem 1.1. In case* G *is not special, we will also assume the field k is infinite and we will also assume the field k satisfies the hypothesis* (3.0.3). *Let* M *denote a motivic spectrum.*

(1). *Then the map induced by $tr(id_{\mathrm{Y}})^*$ provides a splitting to the map $\pi_{\mathrm{Y}}^*: \mathrm{h}^{*,\bullet}(\mathrm{E} \times_{\mathrm{G}} \mathrm{Y},\mathrm{M}) \rightarrow \mathrm{h}^{*,\bullet}(\mathrm{E} \times_{\mathrm{G}} (\mathrm{Y} \times \mathrm{X}),\mathrm{M})$ in the following cases:*

(i) *$\Sigma_{\mathbf{T}}\mathrm{X}_+$ is dualizable in the motivic homotopy category $\mathbf{Spt}_{\mathrm{mot}}$, the trace $\tau_{\mathrm{X}}^*(1)$ is a unit in the Grothendieck-Witt ring of the base field k and* M *denotes any motivic ring spectrum. In particular, this holds if* X *and* Y *are smooth schemes of finite type over k and $char(k) = 0$, provided $\tau_{\mathrm{X}}^*(1)$ is a unit in the Grothendieck-Witt ring of k.*

(ii) *$Char(k) = p > 0$. $\mathcal{E}$ denotes any one of the ring spectra $\Sigma_{\mathbf{T}}[p^{-1}]$, for a fixed prime $\ell \neq char(k)$, $\Sigma_{\mathbf{T},(\ell)}$ which denotes the localization of $\Sigma_{\mathbf{T}}$ at the prime ideal (ℓ) or $\Sigma_{\mathbf{T}}\widehat{\ }_{\ell}$ which denotes the completion of $\Sigma_{\mathbf{T}}$ at the prime ℓ).*

$\mathcal{E} \wedge \mathrm{X}_+$ is dualizable in $\mathbf{Spt}_{\mathrm{mot},\mathcal{E}}$ and the corresponding trace $\tau_{\mathrm{X}}: \mathcal{E} \rightarrow \mathcal{E}$ is a unit in the corresponding variant of the Grothendieck-Witt ring, that is, $[\mathcal{E},\mathcal{E}]$, which denotes stable homotopy classes of maps from $\mathcal{E}$ to $\mathcal{E}$ and $\mathrm{M} \in \mathbf{Spt}_{\mathrm{mot},\mathcal{E}}$. In particular, this holds if X *and* Y *are smooth schemes of finite type over k, provided $\tau_{\mathrm{X}}^*(1)$ is a unit in the above variant of Grothendieck-Witt ring of k.*

(2). *Let $\mathrm{f}: \mathrm{X} \rightarrow \mathrm{X}$ denote a* G*-equivariant map and let $f_Y = id_{\mathrm{Y}_+} \wedge \mathrm{f}: \mathrm{Y}_+ \wedge \mathrm{X}_+ \rightarrow \mathrm{Y}_+ \wedge \mathrm{X}_+$. The map induced by $tr(\mathrm{f_Y})^*$ provides a splitting to the map $\pi_{\mathrm{Y}}^*: \hat{\mathrm{h}}^{*,\bullet}(\mathrm{E} \times_{\mathrm{G}} \mathrm{Y},\mathrm{M}) \rightarrow \hat{\mathrm{h}}^{*,\bullet}(\mathrm{E} \times_{\mathrm{G}} (\mathrm{Y} \times \mathrm{X}),\mathrm{M})$ in the following cases:*

(i) *$\Sigma_{\mathbf{T}}\mathrm{X}_+$ is dualizable in the motivic homotopy category $\mathbf{Spt}_{\mathrm{mot}}$ and $tr(\mathrm{f_Y})^*(1)$ is a unit in $\mathrm{H}^{0,0}(\mathrm{E} \times_{\mathrm{G}} \mathrm{Y},\mathbb{Z}) \cong \mathrm{CH}^0(\mathrm{E} \times_{\mathrm{G}} \mathrm{Y})$. In particular, this holds if* X *and* Y *are smooth schemes of finite type over k and $char(k) = 0$, provided $tr(\mathrm{f_Y})^*(1)$ is a unit in $\mathrm{H}^{0,0}(\mathrm{E} \times_{\mathrm{G}} \mathrm{Y},\mathbb{Z})$.*

If the slices of the motivic spectrum M *are all weak modules (see Definition 7.4) over the motivic Eilenberg-Maclane spectrum* $\mathbb{H}(\mathbb{Z}/\ell^{\nu})$ *(for some fixed prime* $\ell \neq char(k)$ *and integer* $\nu > 0$*), then the same conclusion holds if* $tr(f_Y)^*(1)$ *is a unit in* $H^0_{et}(E \times_G Y, Z/\ell^{\nu})$.

(ii) *A corresponding result also holds for the following alternate scenario:*

(a) The field k *is of positive characteristic* p, $\Sigma_{\mathbf{T}}X_+$ *is dualizable in* $\mathbf{Spt}_{mot}[p^{-1}]$, $M \in \mathbf{Spt}_{mot}[p^{-1}]$ *and* $tr(f_Y)^*(1)$ *is a unit in* $H^{0,0}(E \times_G Y, \mathbb{Z}[p^{-1}]) \cong CH^0(E \times_G Y, Z[p^{-1}])$.

(b) The field k *is of positive characteristic* p, $\mathcal{E} = \Sigma_{\mathbf{T},(\ell)}$ *(or* $\mathcal{E} = \Sigma_{\mathbf{T}}\hat{}_{\ell}$*) for some prime* $\ell \neq p$ *and* $\mathcal{E} \wedge X_+$ *is dualizable in* $\mathbf{Spt}_{mot,\mathcal{E}}$, $M \in \mathbf{Spt}_{mot,\mathcal{E}}$ *and* $tr(f_Y)^*(1)$ *is a unit in* $H^{0,0}(B, \mathbb{Z}_{(\ell)}) \cong CH^0(E \times_G Y, Z_{(\ell)})$ *(*$\mathbb{H}^{0,0}(E \times_G Y, \mathbb{Z}\hat{}_{\ell}) \cong CH^0(E \times_G Y, Z\hat{}_{\ell})$*, respectively). (Here* $\Sigma_{\mathbf{T},(\ell)}$ *(*$\Sigma_{\mathbf{T}}\hat{}_{\ell}$*) denotes the localization of the motivic spectrum* $\Sigma_{\mathbf{T}}$ *at the prime ideal* (ℓ) *(the completion at* ℓ*, respectively).) If the slices of the motivic spectrum* M *are all weak modules (see Definition 7.4) over the motivic Eilenberg-Maclane spectrum* $\mathbb{H}(\mathbb{Z}/\ell^{\nu})$ *(for some integer* $\nu > 0$*), then the same conclusion holds if* $tr(f_Y)^*(1)$ *is a unit in* $H^0_{et}(E \times_G Y, Z/\ell^{\nu})$*. In particular, if the above hypotheses on the motivic spectrum* M *and* $tr(f_Y)^*(1)$ *hold, the conclusion holds for any smooth schemes* X *and* Y *of finite type over* k.

(iii) *The field* k *is of characteristic* $p \geq 0$, $\mathcal{E}$ *denotes the* ℓ*-completed* S^1*-sphere spectrum on the big étale site of* k *for some prime* $\ell \neq p$, $\mathcal{E} \wedge X_+$ *is dualizable in* $\mathbf{Spt}_{et,\mathcal{E}}$, $M \in \mathbf{Spt}_{et,\mathcal{E}}$ *with homotopy groups being modules over* Z/ℓ^{ν} *and* $tr(f_Y)^*(1)$ *a unit in* $H^0_{et}(E \times_G Y, Z/\ell^{\nu})$, $\nu \geq 1$*. The slice tower is then replaced by the Postnikov tower.*

For the motivic spectrum representing Algebraic K-theory, the slice completed generalized motivic cohomology identifies with Algebraic K-theory. For a smooth ind-scheme $\mathcal{Y} = \{Y_m|m\}$, we let its algebraic K-theory spectrum be defined as $\mathbf{K}(\mathcal{Y}) = \underset{\infty \leftarrow m}{\text{holim}}\{\mathbf{K}(Y_m)|m\}$. This provides the following corollary.

Corollary 1.6. *Let* $\pi_Y : E \times_G (Y \times X) \to E \times_G Y$ *and* f *denote maps as in Theorem 1.5.*

(i) *Then,* $\pi_Y^* : \mathbf{K}(E \times_G Y) \to \mathbf{K}(E \underset{G}{\times} (Y \times X))$ *is a split injection on homotopy groups, where* $\mathbf{K}$ *denotes the motivic spectrum representing Algebraic K-theory, provided* $\Sigma_{\mathbf{T}}X_+$ *is dualizable in* $\mathbf{Spt}_{mot}$ *and* $tr(f_Y)^*(1)$ *is a unit in* $H^{0,0}(E \times_G Y, \mathbb{Z}) \cong CH^0(E \times_G Y)$*. In particular, this holds for smooth quasi-projective schemes* X, Y *defined over the field* k *with* $char(k) = 0$*, provided the above condition on* $tr(f_Y)^*(1)$ *holds.*

(ii) $\pi_Y^* : \mathbf{K}(E \times_G Y) \wedge M(\ell^{\nu}) \to \mathbf{K}(E \times_G (Y \times X)) \wedge M(\ell^{\nu})$ *is a split injection on homotopy groups, where* $M(\ell^{\nu})$ *denotes the Moore spectrum defined as the homotopy cofiber* $\Sigma_{\mathbf{T}} \overset{\ell^{\nu}}{\to} \Sigma_{\mathbf{T}}$*, provided the following hold: the field* k *satisfies the hypothesis* (3.0.3), $\Sigma_{\mathbf{T}}X_+$ *is dualizable in* $\mathbf{Spt}_{mot}[p^{-1}]$*, and* $tr(f_Y)^*(1)$ *is a unit in* $H^0_{et}(E \times_G Y, Z/\ell^{\nu})$*, with* $\ell \neq char(k)$ *and* $\nu \geq 1$*. In particular, this holds for smooth quasi-projective schemes* X, Y *defined over the field* k*, with* $char(k) = p$*, provided the above condition on* $tr(f_Y)^*(1)$ *holds.*

An important corollary of Theorems 1.1, 1.5 and Corollary 1.6 is the following. (We will use the convention that, for a linear algebraic group G, $BG = BG^{gm,m}$ for some large m or the corresponding colimit as $m \to \infty$.)

Corollary 1.7. [2] *Let* k *denote a perfect field* k *of arbitrary characteristic* ≥ 0 *satisfying the hypothesis* (3.0.3)*. Let* G *denote a split connected reductive group* G*, and split over the field* k *and let* T *denote a split maximal torus with* N(T) *denoting its normalizer. In case* G *is not special, we will assume* k *is infinite.*

(i) Let ℓ *denote a prime different from* $char(k)$*. Then the map* $\Sigma_{\mathbf{T}}BN(T)_+ \to \Sigma_{\mathbf{T}}BG_+$ *induces a split injection on any slice completed generalized motivic cohomology theory defined with respect to a module spectrum over* $\Sigma_{\mathbf{T},(\ell)}$ *as well as on Algebraic K-theory smashed with the Moore spectrum* $M(\ell^{\nu})$, $\nu \geq 1$.

(ii) Assume $char(k) = 0$*. Then, the map* $\Sigma_{\mathbf{T}}BN(T)_+ \to \Sigma_{\mathbf{T}}BG_+$ *induces a split injection on any slice completed generalized motivic cohomology theory defined with respect to a motivic spectrum as well as on Algebraic K-theory, assuming the compatibility of the motivic transfer with the corresponding transfer on the Betti-realization.*

The above theorem, in fact enables, one to restrict the structure group from G to N(T) (and then to T by ad-hoc arguments) in several situations. Taking $G = GL_n$, this becomes a *splitting principle* reducing problems on vector bundles to corresponding problems on line bundles. The motivic Atiyah-Hirzebruch spectral sequence for certain other motivic spectra like MGL also converge strongly, and therefore, the conclusions of the last corollaries would extend to generalized motivic cohomology theories like Algebraic Cobordism.

[2]It has now been established in [JP-1, Theorem 1.6] that for G a connected split reductive group, over a perfect field of arbitrary characteristic p containing a $\sqrt{-1}$, the Euler-characteristic of G/N(T), is a unit in the Grothendieck-Witt group with the prime p inverted. Therefore, Theorem 1.5(1) shows that the conclusions of this corollary hold without having to take slice completions.

Here is a summary of the applications of the transfer that we have already established and which should appear in forthcoming work. In [JP-1] we establish a key remaining property of the transfer and trace, namely the *additivity property.* Making use of this property for the transfer and trace, we obtain a number of further applications such as the motivic analogues of various double coset formulae as well as a solution to the conjecture (of Morel) that the Euler-characteristic of G/N(T) in the Grothendieck-Witt group of the base field k is 1. (Here G is a connected split reductive group over k and N(T) is the normalizer of a split maximal torus in G.) In [JP-1], we also establish a criterion (using actions by $\mathbb{G}_m$) to see if the transfer maps stabilize for infinite families such as $\{\mathrm{BN(T)}_n \rightarrow \mathrm{BGL}_n | n \geq 0\}$, $\{\mathrm{BN(T)}_n \rightarrow \mathrm{BO(2n)} | n \geq 0\}$, (where $\mathrm{N(T)}_n$ denote the corresponding normalizers of split maximal tori), which has important applications to algebraic and Hermitian K-theory. (See [JP-2] for more on this.) In view of these and the fact that the transfer we establish applies to actions by all linear algebraic groups, that is without requiring them to be special, we expect several other applications to Algebraic and Hermitian K-theory as well: see [Ho05], [Sch16], [ST15].

Here is an *outline of the paper.* Sections 2 and 3 of the paper are written in such a way that one can specialize them to various different frameworks readily. Section 2 is devoted to a detailed discussion of the notion of dualizability in a symmetric monoidal model category. We conclude section 2 with a general construction of the transfer. Section 3 sets up the framework for the remainder of the paper. Though the results in this section are of a technical nature, it sets up an important mechanism to relate equivariant and non-equivariant spectra, which is important for constructing a transfer in the setting of generalized Borel-style cohomology theories.

We specialize to the motivic or étale frameworks in the remaining sections. Sections 4 and 5 are devoted to various explicit examples: section 4 briefly discusses cellular objects in the motivic and étale homotopy category and we relate this notion to that of linear schemes and mixed Tate motives. Section 5 discusses various important examples of *dualizable* objects in these homotopy categories and discusses the proof of Theorem 1.3. Section 6, which is a key section, discusses in detail, the construction of transfer maps in the context of G-torsors for linear algebraic groups G, and the corresponding Borel-style equivariant cohomology with respect to general motivic and étale spectra.

We establish various key properties of the transfer in section 7: the base-change property as in Proposition 7.1 enables one to establish multiplicative properties in Proposition 7.3 and Corollary 7.5. These enable one to show that the transfer provides the required splitting if $tr(\mathrm{f_Y})^*(1)$ is a unit. At this point it is often very convenient and also necessary to know that the transfer is compatible with passage to simpler situations, for example, to a change of the base field to one that is separably or algebraically closed or with suitable realizations. This way it would suffice to compute the traces in the simplified situations, which is often easier. Section 8 is devoted to a detailed discussion of this approach. Section 9 then discusses the applications of the transfer to provide stable splittings in Borel-style equivariant motivic and étale cohomology with respect to general motivic and étale spectra and for actions of linear algebraic groups. An Appendix reviews the theory of Thom spaces and Atiyah duality both in the motivic and étale setting.

It should perhaps be obvious that, many applications of the theory developed here are anticipated. In fact, as discussed above, several applications already appear in [JP-1] and [JP-2]. Moreover, as pointed out earlier, Burt Totaro (see [Tot14, 2.6]) has already used the motivic Becker-Gottlieb transfer in characteristic 0 in computing the Chow groups of the classifying spaces of algebraic groups. The restriction to characteristic 0 was forced by the difficulty in proving dualizability for interesting varieties in positive characteristics. Since that difficulty has been removed in the present paper, we anticipate many more interesting applications: in fact the the Chow groups of classifying spaces seem to serve as a testing ground for various conjectures on algebraic cycles that are torsion and in positive characteristics (see [Tot99] and [Tot14].).

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2. Spanier-Whitehead duality and the construction of transfer in a general framework

We begin by recalling the basics of Spanier-Whitehead duality and at the same time clarifying certain key concepts that appear in this framework.

2.1. Weak Dualizability, Reflexivity and Dualizability in a closed symmetric monoidal stable model category. This section is worked out in as broad a generality as possible, so that it becomes easy to define variants of the transfer using different forms of duality. Though this section is clearly based on the discussion in [DP84], it is important to point out various improvements. [DP84], predates the recent developments in the theory of spectra, so that they were forced to work in the stable *homotopy* category. We will always work in a symmetric monoidal stable model category which will often denote a category of spectra. Secondly, we have improved upon the notion of *strong dualizability* as discussed in [DP84] by introducing a weaker notion, namely *weak dualizability*. We show that strong dualizability is equivalent to weak-dualizability and reflexivity.

Accordingly, the basic framework for this paper will be that of a *closed symmetric monoidal stable model category* which will often be denoted **Spt** with the monoidal product denoted $\wedge$ and with an internal Hom functor denoted $\mathcal{H}om$. This means the category is a model category so that its associated homotopy category is a triangulated category and is also provided with a symmetric monoidal structure compatible with the model structure, in the sense that the pushout-product and monoidal axioms are satisfied. (See [Hov99, Chapter 7], [Hov03].) **HSpt** will denote the corresponding homotopy category. Then one may observe the following: (i) **Spt** and **HSpt** are symmetric monoidal categories with the product structure denoted $\wedge$, (ii) the category **Spt** has the structure of a model category which interacts well with the monoidal structure in the sense that the pushout-product axiom and the monoidal axiom (see [SSch98, section 3] for these) are satisfied and (iii) the model category structure is such that the cofibration sequences and the fiber sequences coincide. We will denote the unit of the monoidal structure by Σ. (It is clearly important that the model structure be stable so that cofiber sequences and fiber sequences coincide.) Such cofiber sequences will henceforth be called distinguished triangles.

The right adjoint to $\wedge$ will be denoted $\mathcal{H}om$: this is an internal hom in the category **Spt**, i.e. Given objects $K, X, Y \in$ **Spt**, one has:

(2.1.1) $$Hom(K, \mathcal{H}om(Y, \mathrm{X})) \cong \mathrm{Hom}(\mathrm{K} \wedge \mathrm{Y}, \mathrm{X}).$$

Given any object $\mathrm{E} \in$ **Spt**, one may functorially replace it by a cofibrant object (fibrant object, as well as by an object that is both cofibrant and fibrant) provided with a weak-equivalence with E.

2.1.2. Conventions. (i) *Henceforth we will use the convention that $\wedge$ ($\mathcal{H}om$) in fact denotes the corresponding derived version, which is defined by functorially replacing the appropriate arguments by cofibrant (cofibrant and fibrant) objects.* Moreover, throughout this section, the unit of the monoidal structure will be denotedtr.general Σ. We will let $\mathrm{D(F)} = \mathcal{H}\mathrm{om}(\mathrm{F}, \Sigma)$, for $F \in$ **Spt**.

(ii) Hom now will denote the external derived hom in the corresponding (stable) homotopy category associated to **Spt**. One may let $\mathcal{M}ap$ denote the bi-functor $\mathbf{Spt}^{\mathrm{op}} \times \mathbf{Spt} \rightarrow$ (pointed simplicial sets) provided by the simplicial or quasi-simplicial model structure. Then $Hom(\mathcal{X},\ Y) = \pi_0 Map(\mathcal{X}, \mathcal{Y})$ again after $\mathcal{X}$ ($\mathcal{Y}$) have been replaced by a cofibrant (fibrant, respectively) object.

The following definitions originated in [DP84]. All we do here is to recall basic terminology and definitions, as well as clarify the relationship between the various notions of duality.

Definition 2.1. An object $\mathcal{X}$ in **Spt** is *reflexive* if the obvious map $\mathcal{X} \rightarrow \mathrm{DD}\mathcal{X}$ is a weak-equivalence. Next observe that there is a natural map $e : \mathrm{D}\mathcal{X} \wedge \mathcal{X} = \mathcal{H}\mathrm{om}(\mathcal{X}, \Sigma) \wedge \mathcal{X} \rightarrow \Sigma$ given as adjoint to the identity $\mathrm{D}\mathcal{X} \rightarrow \mathrm{D}\mathcal{X}$. Therefore, one also has a natural map $\mathcal{X} \wedge \mathrm{D}\mathcal{X} \rightarrow \mathcal{H}\mathrm{om}(\mathcal{X}, \mathcal{X})$ given as adjoint to the map $\mathcal{X} \wedge D\mathcal{X} \wedge \mathcal{X} \overset{id_{\mathcal{X}} \wedge e}{\rightarrow} \mathcal{X}$. We say $\mathcal{X}$ is *weakly dualizable* if the above map $\mathcal{X} \wedge \mathrm{D}\mathcal{X} \rightarrow \mathcal{H}\mathrm{om}(\mathcal{X}, \mathcal{X})$ is a weak-equivalence.

Let $\mathcal{X}$ be weakly-dualizable. Observe that, then we obtain the isomorphisms:

$$Hom(\Sigma, \mathcal{X} \wedge \mathrm{D}\mathcal{X}) = \mathrm{Hom}(\Sigma, \mathcal{H}\mathrm{om}(\mathcal{X}, \mathcal{X})) = \mathrm{Hom}(\mathcal{X}, \mathcal{X}).$$

Let $c : \Sigma \rightarrow \mathcal{X} \wedge \mathrm{D}\mathcal{X}$ denote the map that corresponds to the identity $id_{\mathcal{X}}$, under the above isomorphisms. Then one may see that a weak-inverse to the last map $\mathcal{X} \wedge \mathrm{D}\mathcal{X} \rightarrow \mathcal{H}\mathrm{om}(\mathcal{X}, \mathcal{X})$ is given by the map $\mathcal{H}om(\mathcal{X}, \mathcal{X}) \rightarrow \mathcal{H}om(\Sigma, \mathcal{X} \wedge \mathrm{D}\mathcal{X})) = \mathcal{X} \wedge \mathrm{D}\mathcal{X}$ sending $f \mapsto (f \wedge id_{D\mathcal{X}}) \circ c$.

The notion of *weak dualizability* as above, does not appear in [DP84]. Instead, another notion, *strong dualizability*, is discussed. We will consider the notion of strong dualizability (which we will call *dualizability*) mainly to clarify this concept and will relate it to weak-dualizability.

First note that given two objects $\mathcal{X}, \mathcal{Y} \in$ **Spt**, there is a natural map $\mathrm{D}\mathcal{X} \wedge \mathrm{D}\mathcal{Y} \rightarrow \mathrm{D}(\mathcal{Y} \wedge \mathcal{X})$ given by the adjoint of the pairing: $\mathrm{D}\mathcal{X} \wedge \mathrm{D}\mathcal{Y} \wedge \mathcal{Y} \wedge \mathcal{X} \rightarrow \mathrm{D}\mathcal{X} \wedge \Sigma \wedge \mathcal{X} \cong \mathrm{D}\mathcal{X} \wedge \mathcal{X} \rightarrow \Sigma$. Then $\mathcal{X} \wedge \mathrm{D}\mathcal{X}$ is *self-dual* if the composite map $\mathcal{X} \wedge \mathrm{D}\mathcal{X} \rightarrow \mathrm{DD}\mathcal{X} \wedge \mathrm{D}\mathcal{X} \rightarrow \mathrm{D}(\mathcal{X} \wedge \mathrm{D}\mathcal{X})$ is a weak-equivalence. We say $\mathcal{X}$ is *dualizable* if $\mathcal{X}$ is reflexive and if $\mathcal{X} \wedge \mathrm{D}\mathcal{X}$ is self-dual.

Next we proceed to clarify these notions in the following propositions.

Proposition 2.2. *An object $\mathcal{X}$ in* **Spt** *is weakly dualizable and reflexive if and only if $\mathcal{X}$ is dualizable.*

Proof. The key is the commutative square:

(2.1.3)
$$\begin{array}{ccc} \mathcal{X} \wedge \mathrm{D}\mathcal{X} & \longrightarrow & \mathrm{DD}\mathcal{X} \wedge \mathrm{D}\mathcal{X} \\ \downarrow & & \downarrow \\ \mathcal{H}om(\mathcal{X}, \mathcal{X}) & \longrightarrow & \mathcal{H}om(\mathcal{X}, \mathrm{DD}\mathcal{X}) \end{array}$$

Next suppose $\mathcal{X}$ is weakly dualizable and reflexive. Then left vertical map is a weak-equivalence since $\mathcal{X}$ is weakly dualizable and the two horizontal maps are weak-equivalences because $\mathcal{X} \xrightarrow{\simeq} \mathrm{DD}\mathcal{X}$ is a weak-equivalence by reflexivity. Therefore, the right vertical map is also a weak-equivalence. But this identifies with the map $\mathrm{DD}\mathcal{X} \wedge \mathrm{D}\mathcal{X} \to \mathcal{H}\mathrm{om}(\mathcal{X}, \mathrm{DD}\mathcal{X}) = \mathcal{H}\mathrm{om}(\mathcal{X} \wedge \mathrm{D}\mathcal{X}, \Sigma) = \mathrm{D}(\mathcal{X} \wedge \mathrm{D}\mathcal{X})$. Therefore, $\mathcal{X}$ is dualizable.

Next suppose $\mathcal{X}$ is dualizable. Then the fact that $\mathcal{X}$ is reflexive shows that both the horizontal maps in the square (2.1.3) are weak-equivalences, while the fact that $\mathcal{X} \wedge \mathrm{D}\mathcal{X}$ is self-dual, shows that the right vertical map in (2.1.3) is also a weak-equivalence. Therefore, the left vertical map in (2.1.3) is also a weak-equivalence, thereby proving that $\mathcal{X}$ is weakly dualizable. □

Theorem 2.3. *(See* [DP84, 1.3 Theorem]*.) Let $\mathcal{X} \in$* **Spt** *together with a map $e : \mathrm{D}'\mathcal{X} \wedge \mathcal{X} \to \Sigma$ in* **HSpt** *(called evaluation) for some object $\mathrm{D}'\mathcal{X} \in$* **Spt***. Then the following are equivalent:*

(i) The object $\mathcal{X} \in$ **Spt** *is dualizable with $\mathrm{D}'\mathcal{X} \simeq \mathrm{D}\mathcal{X}$.*

(ii) There exists a map $c : \Sigma \to \mathcal{X} \wedge \mathrm{D}'\mathcal{X}$ in **HSpt** *(called co-evaluation) so that the composite maps*

$$\mathcal{X} \simeq \Sigma \wedge \mathcal{X} \overset{c \wedge id_{\mathcal{X}}}{\to} \mathcal{X} \wedge \mathrm{D}'\mathcal{X} \wedge \mathcal{X} \overset{\mathrm{id}_{\mathcal{X}} \wedge \mathrm{e}}{\to} \mathcal{X} \wedge \Sigma \simeq \mathcal{X} \text{ and}$$
$$\mathrm{D}'\mathcal{X} \simeq \mathrm{D}'\mathcal{X} \wedge \Sigma \overset{\mathrm{id}_{\mathrm{D}'\mathcal{X}} \wedge \mathrm{c}}{\to} \mathrm{D}'\mathcal{X} \wedge \mathcal{X} \wedge \mathrm{D}'\mathcal{X} \overset{\mathrm{e} \wedge \mathrm{id}_{\mathrm{D}'\mathcal{X}}}{\to} \Sigma \wedge \mathrm{D}'\mathcal{X} \simeq \mathrm{D}'\mathcal{X}$$

are both homotopic to the identity maps.

(iii) There exists a map $c : \Sigma \to \mathcal{X} \wedge \mathrm{D}'\mathcal{X}$ in **HSpt** *(called co-evaluation) so that for every object $\mathcal{Z}, \mathcal{W} \in$* **Spt***, the maps $Hom(\mathcal{Z}, \mathcal{W} \wedge \mathrm{D}'\mathcal{X}) \to \mathrm{Hom}(\mathcal{Z} \wedge \mathcal{X}, \mathcal{W})$ given by $f \mapsto (id_{\mathcal{W}} \wedge e) \circ (f \wedge id_{\mathcal{X}})$ and $Hom(\mathcal{Z} \wedge \mathcal{X}, \mathcal{W}) \to Hom(\mathcal{Z}, \mathcal{W} \wedge \mathrm{D}'\mathcal{X})$ given by $(g \wedge id_{\mathrm{D}'\mathcal{X}}) \circ (id_{\mathcal{Z}} \wedge c)$ are inverse isomorphisms.*

Moreover, these imply that for every object $\mathcal{Y} \in$ **Spt***, the canonical map $\mathcal{Y} \wedge \mathrm{D}'\mathcal{X} \to \mathcal{H}\mathrm{om}(\mathcal{X}, \mathcal{Y})$ is a weak-equivalence.*

Proof. We merely observe that this is discussed in [DP84, 1.3 Theorem], and the proof given there carries over. □

Corollary 2.4. *let $\mathcal{X}$ be an object in* **Spt***. Then $\mathcal{X}$ is dualizable if and only if the following two conditions are satisfied:*

(i) $\mathcal{X}$ is reflexive (i.e. the natural map $\mathcal{X} \to \mathrm{D}(\mathrm{D}(\mathcal{X}))$ is a weak-equivalence) and

(ii) for every object $\mathcal{Y} \in$ **Spt***, the canonical map $\mathcal{Y} \wedge \mathrm{D}\mathcal{X} \to \mathcal{H}\mathrm{om}(\mathcal{X}, \mathcal{Y})$ is a weak-equivalence.*

Proof. This is clear in view of the above results. □

The following result, however, seems missing in the literature. (See [RO08, proof of Theorem 52] where such a result seems to be implicitly assumed.)

Proposition 2.5. *Let $\mathcal{X} \to \mathcal{Y} \to \mathcal{Z} \to \mathcal{X}[1]$ denote a distinguished triangle in* **Spt***. If two of the three objects $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{Z}$ are dualizable, so is the third.*

Proof. Clearly it suffices to prove that $\mathcal{Z}$ is dualizable if $\mathcal{X}$ and $\mathcal{Y}$ are. Since $\mathcal{X}$ and $\mathcal{Y}$ are assumed to be dualizable, one observes that the natural maps: $\mathcal{Z} \wedge \mathrm{D}\mathcal{Y} \to \mathcal{H}\mathrm{om}(\mathcal{Y}, \mathcal{Z})$ and $\mathcal{Z} \wedge \mathrm{D}\mathcal{X} \to \mathcal{H}\mathrm{om}(\mathcal{X}, \mathcal{Z})$ are weak-equivalences. Now one has the commutative diagram:

$$\begin{array}{ccccccc} \mathcal{Z} \wedge \mathrm{D}\mathcal{Z} & \longrightarrow & \mathcal{Z} \wedge \mathrm{D}\mathcal{Y} & \longrightarrow & \mathcal{Z} \wedge \mathrm{D}\mathcal{X} & \longrightarrow & (\mathcal{Z} \wedge D\mathcal{Z})[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathcal{H}om(\mathcal{Z}, \mathcal{Z}) & \longrightarrow & \mathcal{H}om(\mathcal{Y}, \mathcal{Z}) & \longrightarrow & \mathcal{H}om(\mathcal{X}, \mathcal{Z}) & \longrightarrow & \mathcal{H}om(\mathcal{Z}, \mathcal{Z})[1] \end{array}$$

Since both rows are distinguished triangles and the middle two vertical maps are weak-equivalences, it follows that so is the first vertical map, proving thereby that the map $\mathcal{Z} \wedge \mathrm{D}\mathcal{Z} \to \mathcal{H}\mathrm{om}(\mathcal{Z}, \mathcal{Z})$ is also a weak-equivalence. This proves that $\mathcal{Z}$ is weakly dualizable if $\mathcal{X}$ and $\mathcal{Y}$ are dualizable. Now it suffices to show that $\mathcal{Z}$ is also reflexive.

This follows from a commutative diagram involving the distinguished triangle $\mathcal{X} \to \mathcal{Y} \to \mathcal{Z} \to \mathcal{X}[1]$ and also the corresponding distinguished triangle of the double-duals of $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$. □

Proposition 2.6. *Assume that* $\wedge$ *commutes with all small filtered colimits in either argument and that filtered colimits in* **Spt** *preserve weak-equivalences. Let* $\mathcal{X} \in$ **Spt** *be dualizable. Then for any small filtered direct system of objects* $\{\mathcal{Y}_\alpha|\alpha\}$ *in* **Spt**, *the natural map* $\underset{\alpha}{colim}\mathcal{H}om(\mathcal{X}, \mathcal{Y}_\alpha) \to \mathcal{H}om(\mathcal{X}, \underset{\alpha}{colim}\mathcal{Y}_\alpha)$ *is a weak-equivalence.*

Proof. Let $\{\mathcal{Y}_\alpha|\alpha\}$ denote a small filtered direct system of objects in **Spt**. Since $\mathcal{X}$ is dualizable, we obtain the weak-equivalence for each $\alpha : \mathcal{Y}_\alpha \wedge \mathrm{D}\mathcal{X} \overset{\simeq}{\to} \mathcal{H}\mathrm{om}(\mathcal{X}, \mathcal{Y}_\alpha)$. Taking $\underset{\alpha}{\mathrm{colim}}$, this provides the weak-equivalence:

$$\underset{\alpha}{\mathrm{colim}}(\mathcal{Y}_\alpha \wedge \mathrm{D}\mathcal{X}) \overset{\simeq}{\to} \underset{\alpha}{\mathrm{colim}}\mathcal{H}\mathrm{om}(\mathcal{X}, \mathcal{Y}_\alpha).$$

Since $\wedge$ is assumed to commute with small filtered colimits, the left-hand-side identifies with $\underset{\alpha}{\mathrm{colim}} Y_\alpha \wedge \mathrm{D}\mathcal{X}$. Since $\mathcal{X}$ is dualizable, this identifies with $\mathcal{H}om(\mathcal{X}, \underset{\alpha}{\mathrm{colim}} Y_\alpha)$, so that we obtain the weak-equivalence:

$$\underset{\alpha}{\mathrm{colim}}\mathcal{H}om(\mathcal{X}, \mathcal{Y}_\alpha) \simeq \mathcal{H}om(\mathcal{X}, \underset{\alpha}{\mathrm{colim}} Y_\alpha).$$

□

Next, let $\mathrm{T} : \mathbf{Spt}' \to \mathbf{Spt}$ denote a functor between stable model categories. Then we say T is weakly monoidal if there is a natural map $\mathrm{T(A)} \otimes \mathrm{T(B)} \to \mathrm{T(A \otimes B)}$, for all objects $\mathrm{A, B} \in \mathbf{Spt}'$. We say T is *monoidal* if the above map is a weak-equivalence for all objects A and B in $\mathbf{Spt}'$ and if $\mathrm{T(I')}$ is weakly-equivalent to the unit of **Spt**.

Proposition 2.7. *(See* [DP84, 2.2 Theorem]*.) Assume that the functor* T *is monoidal, induces a functor of the corresponding homotopy categories and that the object* $A \in \mathbf{Spt}'$ *is dualizable, and* I' *is the unit of* $\mathbf{Spt}'$*. Then* $\mathrm{T(A)} \in \mathbf{Spt}'$ *is dualizable and* $\mathrm{T(D(A))} \simeq \mathcal{H}\mathrm{om}(\mathrm{T(A)}, \mathrm{T(I')})$*, where* $\mathcal{H}om$ *again denotes the derived* $\mathcal{H}om$.

2.2. Construction of the transfer in a general framework. Assume that **Spt** denotes a symmetric monoidal stable model category where the monoidal structure is denoted $\wedge$ and where the unit of the monoidal structure is denoted Σ. We will further assume that each object $\mathcal{X}$ in **Spt** comes equipped with a diagonal map $\Delta : \mathcal{X} \to \mathcal{X} \wedge \mathcal{X}$ and a co-unit map $\kappa : \mathcal{X} \to \Sigma$ so that Δ provides $\mathcal{X}$ with the structure of a co-algebra: see [DP84, section 5].

Definition 2.8. (i) Now one may define the *trace* associated to any self-map $\mathrm{f} : \mathcal{X} \to \mathcal{X}$ of an object that is dualizable as follows. Recall that we have denoted the *evaluation* map $\mathrm{D}\mathcal{X} \wedge \mathcal{X} \to \Sigma$ by e. The dual of this map is the *co-evaluation* map $c : \Sigma \to \mathcal{X} \wedge \mathrm{D}\mathcal{X}$. Now the *trace of* f (denoted $\tau_{\mathrm{X}}(\mathrm{f})$ or often just $\tau(\mathrm{f})$) is the composition (in **HSpt**)

(2.2.1) $$\Sigma \overset{c}{\to} \mathcal{X} \wedge \mathrm{D}\mathcal{X} \overset{\tau}{\to} \mathrm{D}\mathcal{X} \wedge \mathcal{X} \overset{\mathrm{id}\wedge \mathrm{f}}{\to} \mathrm{D}\mathcal{X} \wedge \mathcal{X} \overset{\mathrm{e}}{\to} \Sigma.$$

where τ is the map interchanging the two factors.

(ii) Then we define the *transfer* as the composition in **HSpt**:

(2.2.2) $$tr(\mathrm{f}) : \Sigma \overset{c}{\to} \mathcal{X} \wedge \mathrm{D}\mathcal{X} \overset{\tau}{\to} \mathrm{D}\mathcal{X} \wedge \mathcal{X} \overset{\mathrm{id}\wedge\Delta}{\to} \mathrm{D}\mathcal{X} \wedge \mathcal{X} \wedge \mathcal{X} \overset{\mathrm{id}\wedge \mathrm{f}\wedge \mathrm{f}}{\to} \mathrm{D}\mathcal{X} \wedge \mathcal{X} \wedge \mathcal{X} \overset{\mathrm{e}\wedge\mathrm{id}}{\to} \Sigma \wedge \mathcal{X}$$

(iii) If $\mathcal{Y} \,\varepsilon\, \mathbf{Spt}$ is another object, we will also consider the following variant $tr(\mathrm{f}_\mathcal{Y}) = \mathcal{Y} \wedge tr(\mathrm{f}) : \mathcal{Y} \wedge \Sigma \to \mathcal{Y} \wedge \mathcal{X}$.

The composition $\mathrm{D}\mathcal{X} \wedge \mathcal{X} \overset{\mathrm{id}\wedge\Delta}{\to} \mathrm{D}\mathcal{X} \wedge \mathcal{X} \wedge \mathcal{X} \overset{\mathrm{id}\wedge \mathrm{f}\wedge \mathrm{f}}{\to} \mathrm{D}\mathcal{X} \wedge \mathcal{X} \wedge \mathcal{X}$ will often be denoted $id \wedge \Delta(\mathrm{f})$.

The remaining discussion in this section will invoke the categories of spectra considered in the next section. The first observation, as an immediate consequence of the definition is that, if $\mathcal{X}$ in fact is the **T**-suspension spectrum of a smooth quasi-projective variety and $p : \mathcal{X} \to \Sigma_\mathbf{T}$ is the map induced by the structure map of the variety to the base field, then, $p \circ tr(\mathrm{f}) = \tau(\mathrm{f})$. Similarly, if $id \wedge p : \mathcal{Y} \wedge \mathcal{X} \to \mathcal{Y} \wedge \Sigma_\mathbf{T} = \mathcal{Y}$, the composition $(id \wedge p) \circ tr(\mathrm{f}_\mathcal{Y})$ will identify with $id_\mathcal{Y} \wedge \tau(\mathrm{f})$. This will be denoted $\tau(\mathrm{f}_\mathcal{Y})$. (We leave this as an (easy) exercise.)

Examples 2.9. *In these examples, the diagonal* Δ *and the co-unit* κ *are the obvious maps. Since, the evaluation, co-evaluation and the diagonal maps involve no degree or weight shifts, the map induced by the transfer in any generalized cohomology theory preserves both the degree and weight.*

(i) Here we take $\mathbf{Spt} = \mathbf{Spt}_\mathrm{S}$ *for the big Nisnevich site on* $\mathrm{S} = \mathrm{Spec\,k}$*, with* $\wedge$ *denoting the smash product of* **T***-spectra based on the smash product of simplicial presheaves as in* (3.0.4) *so that the transfer is a stable map* $tr(\mathrm{f_Y}) : \Sigma_\mathbf{T} \to \Sigma_\mathbf{T}\mathrm{X}_+$.

(ii) Next we take $\mathbf{Spt} = \mathbf{Spt}/\mathrm{S}$ *for the big Nisnevich site on* S*, with* S *any scheme and* $\wedge$ *denoting a relative smash product of* **T***-spectra over* S *based on the relative smash product of simplicial presheaves as in* (3.0.5)*, so that the transfer should be a stable map* $tr(\mathrm{f_Y}) : \Sigma_\mathbf{T}\mathrm{S}_+ \to \Sigma_\mathbf{T}\mathrm{X}_+$*, inducing a corresponding stable map as in*

(i), $tr(\mathrm{f_s}) : \Sigma_{\mathbf{T}}\mathrm{Spec\,k(s)}_+ \rightarrow \Sigma_{\mathbf{T}}\mathrm{X_s}$*,* $s \in \mathrm{S}$*. Here* $\Sigma_{\mathbf{T}}\mathrm{S}_+$ *(*$\Sigma_{\mathbf{T}}\mathrm{X}_+$*) denotes the* $\mathbf{T}$*-suspension spectrum of* S_+ *(*X_+*, respectively). However, as we point out below, it is far from clear that, then such a transfer exists: in fact it may not exist in general!*

Remark 2.10. In fact it is important to clarify the situation (ii) considered above. Assume that S is a scheme of finite type over a field k. Then it is *important* to point out that the notion of Spanier-Whitehead duality in (ii) is taking place at the level of the fibers over the base scheme S. Therefore, while *the sphere spectrum* Σ, which we need to work with and viewed as an object over S, will reduce to the sphere $\mathbf{T}_s$-spectrum at each point of $s \in \mathrm{S}$, it *will in fact be often not* the *trivial* sphere spectrum defined by $\mathbf{T}_{Spec\,k} \times \mathrm{S}$. Therefore, an object $\mathcal{X} \in \mathbf{Spt}/\mathrm{S}$ that is such that at each $s \in \mathrm{S}$, $\mathcal{X}_s$ is dualizable with respect to the $\mathbf{T}_s$-sphere spectrum, may or may not be dualizable with respect to the given sphere spectrum Σ over S which is the unit of $\mathbf{Spt}/\mathrm{S}$.

For example, let E denote a torsor for a compact Lie group G and let $\mathrm{S} = \mathrm{B} = \mathrm{E}/\mathrm{G}$. One may attempt to construct a transfer by always working *over* $\mathrm{S} = \mathrm{B}$, i.e. for example, attempt to construct a co-evaluation map from the sphere spectrum $\Sigma_{\mathrm{S}^1} \wedge \mathrm{B}_+$ on the base space B. The main problem this approach does not work, is that then this sphere spectrum has no action by G. Therefore, one cannot hope to obtain any non-trivial co-evaluation map with $\Sigma_{\mathrm{S}^1} \wedge \mathrm{B}_+$ as the source, to $\Sigma_{\mathrm{S}^1, \mathrm{E}\underset{\mathrm{G}}{\times}\mathrm{X}}$ which is the S^1-suspension spectrum of the space $\mathrm{E}\underset{\mathrm{G}}{\times}\mathrm{X}$, when X is a space with a non-trivial G-action. (A typical example here is to take $\mathrm{X} = \mathrm{G}/\mathrm{H}$ for some closed subgroup H of G. In this case G acts transitively on X.) (This issue seems to be overlooked in [Lev18]. In fact, the transfer there seems to be constructed starting with the usual motivic sphere spectrum which has only the trivial G-action.)

The solution, at least in principle, *is to always start with a* G*-equivariant sphere spectrum* $\mathbb{S}^{\mathrm{G}}$ *(defined analogously as in Definition 3.4) and then consider* $\mathrm{E} \times_{\mathrm{G}} \mathbb{S}^{\mathrm{G}}$*: this would be the required sphere spectrum over* B. At each fiber over any point of B, there is a G-action, and on forgetting the G-action, the fibers $\mathbb{S}^{\mathrm{G}}_b$ clearly identify with the usual S^1-sphere spectrum (at least in the stable homotopy category).

However, a major complication now is that as $\mathbb{S}^{\mathrm{G}}$ has a non-trivial action by G, $\mathrm{E} \times_{\mathrm{G}} \mathbb{S}^{\mathrm{G}}$ will almost never be the S^1-suspension spectrum of B, $\Sigma_{\mathrm{S}^1} \wedge \mathrm{B}_+$. In order to obtain the latter starting with the former, considerable additional work is needed, as in Steps 2 through 5 in section 6. One may compare with [BG75, section 2] where all these issues show up and are addressed. However, there an equivariant form of the Thom-Pontrjagin collapse simplifies their constructions. The analogous Voevodsky collapse (see Definition 10.8) seems to work well only for smooth projective schemes. Therefore, we proceed somewhat differently and to do this systematically, one needs to first set up a suitable framework for equivariant spectra including a suitable candidate for an equivariant motivic sphere spectrum. The main point of the authors' prior work [CJ14] is indeed to set-up such a framework, and we will adapt that in the following section to provide the required equivariant framework needed for the construction of a transfer map.

3. Basic framework for the rest of the paper

For the remainder of the paper, we will also assume that the category **Spt** is a stable model category associated to a category of pointed simplicial presheaves on an (yet unspecified) category of schemes. The various possible choices are discussed below.

We will fix a *perfect* field k as the base scheme B, and then restrict to the category of smooth schemes of finite type over B. Quite often we will have to restrict to a subcategory of schemes, whose structure map to B factors through another scheme S, which will then become the base scheme for the corresponding subcategory. In this case, we will then always assume that the scheme S is a pointed B-scheme, i.e. provided with a map $\mathrm{B} \rightarrow \mathrm{S}$ which will be a section to the structure map $\mathrm{S} \rightarrow \mathrm{B}$.

In considering the étale site, we will always assume that the base scheme $\mathrm{B} = \mathrm{Spec\,k}$ has the following property:

$$\begin{array}{c}\mathrm{H}^{\mathrm{n}}_{\mathrm{et}}(\mathrm{Spec\,k}, \mathrm{Z}/\ell^{\nu}) \text{ is finitely generated in each degree n and vanishes for } \mathrm{n} >> 0 \\ \text{whenever } \ell \text{ is a prime different from the characteristic of } k \text{ and } \nu \text{ is any positive integer.}\end{array} \tag{3.0.3}$$

One may see from Lemma 5.3 how this hypothesis ensures compactness of objects on the étale site of B, which is in fact the reason for requiring the above property.

The following are some of *the main choices for the category of schemes* we consider. Let S denote a smooth pointed scheme over B. Then Sm/S denotes the category of all smooth schemes of finite type over S. This category will be provided with either the Zariski, Nisnevich or étale topologies and the corresponding site will be denoted $\mathrm{Sm/S_{Zar}}$, $\mathrm{Sm/S_{Nis}}$ or $\mathrm{Sm/S_{et}}$. If B is the field of complex numbers, one also considers the local homeomorphism topology. Here the coverings of an object U are collections $\{\mathrm{U_i} \rightarrow \mathrm{U}(\mathbb{C})|\mathrm{i}\}$, with each $\mathrm{U_i} \rightarrow \mathrm{U}(\mathbb{C})$ a quasi-finite map of topological spaces which are local-homeomorphisms when $U(\mathbb{C})$ is provided with the transcendental topology. $S_{l.h}$ will denote the corresponding site.

Given the above choices for the categories of schemes, we have several different choices for categories of simplicial presheaves on them.

(i) The first choice is the following. A *pointed* simplicial presheaf on any one of the sites considered above will mean a simplicial presheaf P together with a map $S \rightarrow P$. This category will have the symmetric monoidal structure defined by the smash product, i.e. if P and Q are two objects,

$$P \wedge Q = P \times Q/(S \times Q \cup P \times S). \tag{3.0.4}$$

Observe also that, associated to any scheme X over S, one has the pointed scheme $X_{+_S} = X \sqcup S$ over S, often denoted simply by X_+. *For brevity, we will denote any of the above simplicial topoi by* PSh_S.

(ii) A second choice for the category of simplicial presheaves, (which will be important in considering fiber-wise duality), will be the following. We will restrict to the subcategory of pointed simplicial presheaves provided with a map to S, where S is viewed as a simplicial presheaf, by considering the presheaf of sets represented by S on any of the above sites. Alternatively we will also consider the case where S is no longer the base scheme, but denotes a chosen simplicial presheaf and restrict to simplicial presheaves which are provided with a chosen map to S and also pointed over S. (An example of this appears in (6.2.8).) *This subcategory will henceforth be denoted* PSh/S. Since we have already assumed that all the objects are pointed, it follows that for each object $P \in PSh/S$, there are unique maps $s_P : S \rightarrow P$ and $p_P : P \rightarrow S$ so that the composition $p_P \circ s_P$ is the identity. Therefore, s_P sends S isomorphically to a sub-object of P, which we denote by $s_P(S)$. We next define a different monoidal structure on PSh/S as follows. Let $P, Q \in PSh/S$. Then we let:

$$P \wedge^S Q = (P \times_S Q)/(s_P(S) \times_S Q \cup P \times_S s_Q(S)). \tag{3.0.5}$$

It may be important to point out that the term on the right is the quotient over S, i.e. *the pushout* of: $S \leftarrow s_P(S) \times_S Q \cup P \times_S s_Q(S) \rightarrow P \times_S Q$. We skip the verification that PSh/S with above smash product $\wedge^S$ is a closed symmetric monoidal category. If the base scheme S represents a *point* in the site, for example, is the spectrum of a field for the Zariski and Nisnevich sites and is the spectrum of a separably closed field for the étale site, then every simplicial presheaf has an obvious map to S, so that case (ii) reduces to case (i). The main difference between the two cases is therefore, when S is a general scheme or a chosen simplicial presheaf. In this case, the smash product $\wedge^S$ defines what corresponds to *a fiber-wise* smash product. *With a view to keeping our discussion simple, we will discuss mostly this second case.* The discussion of the transfer map in section 6 (see (6.2.19) through (6.2.26)) and Lemma 10.5 show that indeed the fiber-wise smash product is important for us.

(iii) We start with a linear algebraic group G defined over the base scheme B. We will next do a base-extension to S, i.e. replace G by $G_S = G \times_B S$. But we will continue to denote G_S by G. This way, we may assume, without loss of generality that $G \in PSh/S$. Then we will restrict to the category of schemes with G-action, and also to simplicial presheaves in the above categories provided with G-actions: *in all of these, we will view* G *as the corresponding presheaf of groups on the given site.* Here it is important that the base scheme S has trivial action by the group G so that the maps $s : S \rightarrow P$ and $p : P \rightarrow S$ are G-equivariant. Then maps between such G-equivariant simplicial presheaves will be G-equivariant maps of simplicial presheaves, compatible with the structure maps s and p. The subcategory corresponding to (i) ((ii)) will be denoted

$$PSh_S^G (PSh^G/S, \text{respectively}). \tag{3.0.6}$$

Let $U : PSh_S^G \rightarrow PSh_S$ ($PSh^G/S \rightarrow PSh/S$, respectively) denote the forgetful functor forgetting the group action.

Observe that if $P, Q \in PSh_S^G (PSh^G/S)$, then $P \wedge Q$ ($P \wedge^S Q$, respectively) defined above (i.e. with P and Q viewed as objects in PSh_S (PSh/S))) has a natural induced G-action and therefore, defines an object in PSh_S^G (PSh^G/S, respectively). Therefore, we let the monoidal structure on $PSh_S^G (PSh^G/S, \text{respectively})$ be defined by $\wedge$ as in (3.0.4) (be defined by $\wedge^S$ as in (3.0.5), respectively). Similarly, if $P, Q \in PSh_S^G (PSh^G/S)$, then the internal $\mathcal{H}om(P, Q)$ in PSh_S (PSh/S) belongs to PSh_S^G (PSh^G/S, respectively). These basically prove:

$$U(P \wedge Q) = U(P) \wedge U(Q) \text{ and } U(\mathcal{H}om_G(P, Q) = \mathcal{H}om(U(P), U(Q)), P, Q \in PSh_S^G (PSh^G/S), \tag{3.0.7}$$

where $\mathcal{H}om_G$ denotes the internal hom in $\mathcal{C}^G$.

Proposition 3.1. *Let* PSh_S *be provided with one of the model structures defined below. Let* $P \in PSh_S^G$.

i) If $P' \rightarrow U(P)$ *is a functorial cofibrant replacement in* PSh_S*, then* $P' \in PSh_S^G$.

ii) If $U(P) \rightarrow P''$ *is a functorial fibrant replacement in* PSh_S*, then* $P'' \in PSh_S^G$.

The corresponding assertions also hold for objects in $\mathrm{PSh}^{\mathrm{G}}/\mathrm{S}$.

Proof. As the proof for (ii) is entirely similar, we will sketch the main argument for (i) only. Recall G acts on P as a presheaf, i.e. for each scheme S in the given site, G(S) is given an action on P(S). The functoriality in the choice of the cofibrant replacement P′ shows that each $g_s \in \mathrm{G(S)}$ then has an induced action on P′(S), that the square

$$\begin{array}{ccc} \mathrm{P}'(\mathrm{S}) & \xrightarrow{g_s} & \mathrm{P}'(\mathrm{S}) \\ \downarrow & & \downarrow \\ \mathrm{P}(\mathrm{S}) & \xrightarrow{g_s} & \mathrm{P}(\mathrm{S}) \end{array}$$

commutes, and that the corresponding squares for S′ and S, for a map $\mathrm{S}' \to \mathrm{S}$ in the given site are compatible. (See [Hov99, Definition 1.1.1] for details on functorial fibrant and cofibrant replacements.) ☐

3.1. Choice of Model structures. Next one has several possible choices of *model structures* on the categories of simplicial presheaves $\mathrm{PSh}_{\mathrm{S}}$ and PSh/S: see for example, [CJ14, section 2, Propositions 2.2 and 2.4]. For example, one has the *projective* model structure, where the fibrations and weak-equivalences are defined section-wise, with the cofibrations defined by the lifting property. One also has the *injective* model structure where weak-equivalences and cofibrations are defined section-wise, with the fibrations defined by the lifting property. One of the main advantages of the injective model structure is that every object is cofibrant and every injective map of simplicial presheaves is a cofibration. These imply that both the smash products in (3.0.4) and in (3.0.5) are homotopy pushouts. *In view of these, we will always start with the injective model structure on the category of simplicial presheaves.* In addition, we need to modify these model structures, so that the resulting model structure satisfies *the following basic requirements*:

3.1.1.

(i) a map of pointed simplicial presheaves is a weak-equivalence in the model structure if it induces a weak-equivalence at each stalk and
(ii) the pushout-product axiom and the monoidal axiom with respect to the above monoidal structures hold.

In case we are considering the category of pointed simplicial presheaves on the Nisnevich site of the scheme S, where weak-equivalences are defined section-wise to begin with, one needs to apply Bousfield localization with respect to the distinguished squares to obtain the first property: see [Bl01, Lemmas 4.1 and 4.2]. In case we are considering the category of pointed simplicial presheaves on the étale site of the scheme S, where weak-equivalences are defined section-wise to begin with, one needs to apply Bousfield localization with respect to hypercoverings as shown in [DHI04, section 6.3] to obtain the first property. If we begin with the projective (injective) model structures, the corresponding localized model structure will be called the *local projective (the local injective)* model structure. The pushout-product axiom is easy to verify and when every object is cofibrant, the monoidal axiom follows from this. In general, the observations that any trivial cofibration in the local projective model structure is a trivial cofibration in the local injective model structure and the local injective model structure satisfies the monoidal axiom (since every object now is cofibrant) show that the local projective model structure and the local injective model structures satisfy both the above properties.

Since one of the main focus is on motivic applications, we will always refine the weak-equivalences further by Bousfield localization, inverting all maps of the form $\{\mathrm{pr} : \mathrm{X} \times \mathbb{A}^1 \to \mathrm{X}|\mathrm{X}\}$, where X varies over all the schemes in the given site. We will perform this localization even when considering the étale sites, since $\mathbb{A}^1$ is acyclic in the étale topology only with respect to constant sheaves like Z/ℓ^{ν}, where ℓ is different from the residue characteristics.

3.1.2. Key observation. The only (straight-forward) way to put a model structure on the category $\mathrm{PSh}_{\mathrm{S}}^{\mathrm{G}}$ ($\mathrm{PSh}^{\mathrm{G}}/\mathrm{S}$) is to transfer the model structure on $\mathrm{PSh}_{\mathrm{S}}$ (PSh/S) by means of the underlying functor U and a left adjoint to it. This adjoint is given by the functor sending a simplicial presheaf P to $\mathrm{G} \otimes \mathrm{P}$ (which is defined by $(\mathrm{G} \otimes \mathrm{P})(\mathrm{X}) = \underset{\mathrm{G(X)}}{\vee} \mathrm{P(X)}$. However, this will mean the cofibrant objects in $\mathrm{PSh}_{\mathrm{S}}^{\mathrm{G}}$ ($\mathrm{PSh}^{\mathrm{G}}/\mathrm{S}$) will no longer be the same as the cofibrant objects in $\mathrm{PSh}_{\mathrm{S}}$ (PSh/S). *As a result the* $\mathcal{RH}om$ *and the derived* $\wedge$ *in the category* $\mathrm{PSh}_{\mathrm{S}}^{\mathrm{G}}$ *(*$\mathrm{PSh}^{\mathrm{G}}/\mathrm{S}$*) will be distinct from the corresponding objects in* $\mathrm{PSh}_{\mathrm{S}}$ *(PSh/S). Recall that the notion of Spanier-Whitehead duality we will need to use involves stable versions of the corresponding functors in the non-equivariant framework. Therefore, we need to obtain an analogue of Proposition 3.1 for spectra: one of the main goals of the remaining discussion in this section, is to accomplish this.*

3.2. Categories of spectra. Spectra play *two distinct roles* in our context:

(i) One may observe that the definition of the transfer is as a stable map of certain spectra, and its applications are to splitting maps of generalized cohomology theories defined with respect to spectra. Here spectra mean either motivic or étale spectra which *are not necessarily equivariant. Moreover, the notion of Spanier-Whitehead dual that is needed for the transfer is essentially in the non-equivariant setting.*

(ii) In contrast, the construction of the transfer as a stable map starts with a *pre-transfer, which will have to be an equivariant map of equivariant spectra, which is then fed into the Borel-construction to obtain the transfer for generalized (Borel-style) equivariant cohomology theories.* (Equivariant spectra are defined below.)

(iii) *i.e. The spectra that enter into the construction of a pre-transfer (which has to be equivariant) are all equivariant spectra, though the transfer is applied to generalized cohomology theories that are defined with respect to spectra that need not be equivariant.* This dual role of spectra, makes it necessary for us to proceed carefully and explaining how the two roles are related.

3.2.1. *Equivariant spectra.* (See [CJ14, section 3].) Throughout the following discussion, we will adopt the following *terminology*: G denote a fixed smooth affine group scheme defined over the base scheme (which we assume again is a perfect field) and $\mathcal{C}$ will denote the category PSh/S, while $\mathcal{C}^{\mathrm{G}}$ will denote the category $\mathrm{PSh}^{\mathrm{G}}/\mathrm{S}$. Here S could be either the base scheme or a fixed simplicial presheaf, so that all the simplicial presheaves we consider will have a chosen map to it and are pointed over S.

The G-spectra will be indexed not by the non-negative integers, but by the Thom-spaces of finite dimensional representations of the group G. Therefore, we let $\mathrm{Sph}^{\mathrm{G}}$ denote the *subcategory* of $\mathcal{C}^{\mathrm{G}}$ whose *objects* are $\{\mathrm{T_V}|\mathrm{V}\}$, and where V varies over all finite dimensional representations of the group G and $\mathrm{T_V}$ denotes its Thom-space. We let the *morphisms* in this category be given by the maps $\mathrm{T_V} \rightarrow \mathrm{T_{V\oplus W}}$ induced by homothety classes of k-linear injective and G-equivariant maps $\mathrm{V} \rightarrow \mathrm{V} \oplus \mathrm{W}$. One may observe that $\mathrm{T_V}$ identifies with the quotient sheaf $\mathrm{Proj}(\mathrm{V} \oplus 1)/\mathrm{Proj}(\mathrm{V})$, so that there is an injection $\mathrm{V} \rightarrow \mathrm{T_V}$ for every G-representation V.

We also let $\mathrm{USph}^{\mathrm{G}}$ denote the category whose objects are $\{\mathrm{U}(\mathrm{T_V})|\mathrm{T_V} \in \mathrm{Sph}^{\mathrm{G}}\}$, where U is the forgetful functor forgetting the G-action. The morphisms in this category are given by the maps $\mathrm{T_V} \rightarrow \mathrm{T_{V\oplus W}}$ induced by homothety classes of k-linear injective maps $\mathrm{V} \rightarrow \mathrm{V} \oplus \mathrm{W}$.

We will make $\mathrm{Sph}^{\mathrm{G}}$ ($\mathrm{USph}^{\mathrm{G}}$) an *enriched monoidal category, enriched over the category* $\mathcal{C}^{\mathrm{G}}$ *($\mathcal{C}$, respectively)* as follows. First let $\mathrm{S}^0 = (\mathrm{Spec\,k})_+$. Then for V, W that are G-representations, we let the $\mathcal{C}^{\mathrm{G}}$-enriched internal hom in $\mathrm{Sph}^{\mathrm{G}}$ be defined by:

(3.2.2)
$$\begin{aligned}\mathcal{H}om_{\mathcal{C}^{\mathrm{G}}}(\mathrm{T_V}, \mathrm{T_{V\oplus W}}) &= (\bigsqcup_{\alpha:\mathrm{V}\rightarrow\mathrm{V}\oplus\mathrm{W}} \mathrm{T_W}) \sqcup *, \mathrm{W} \neq \{0\}\\ &= (\bigvee_{\alpha:\mathrm{V}\rightarrow\mathrm{V}} \mathrm{S}^0) \sqcup *, \mathrm{W} = \{0\}\end{aligned}$$

Here the sum varies over all *homothety classes of* G*-equivariant and k-linear injective maps* $\mathrm{V} \rightarrow \mathrm{V} \oplus \mathrm{W}$ and $*$ denotes a *base point* added so that the above enriched *$\mathcal{H}$oms* are pointed simplicial sets. The base points in each of the summands $\mathrm{T_W}$ correspond bijectively with the corresponding α and the unique 0-simplex other than the base point in each of the summands S^0 corresponds bijectively with the corresponding α, so that the 0-simplices in $\mathcal{H}om_{\mathcal{C}^{\mathrm{G}}}(\mathrm{T_V}, \mathrm{T_{V\oplus W}})$ correspond bijectively with the morphisms $\mathrm{T_V} \rightarrow \mathrm{T_{V\oplus W}}$ in the *category* underlying the enriched category $\mathrm{Sph}^{\mathrm{G}}$. One defines the $\mathcal{C}$-enriched internal hom in $\mathrm{USph}^{\mathrm{G}}$ by a similar formula as in (3.2.2):

(3.2.3)
$$\begin{aligned}\mathcal{H}om_{\mathcal{C}}(\mathrm{T_V}, \mathrm{T_{V\oplus W}}) &= (\bigsqcup_{\alpha:\mathrm{V}\rightarrow\mathrm{V}\oplus\mathrm{W}} \mathrm{T_W}) \sqcup *, \mathrm{W} \neq \{0\}\\ &= (\bigvee_{\alpha:\mathrm{V}\rightarrow\mathrm{V}} S^0) \sqcup *, \mathrm{W} = \{0\}\end{aligned}$$

where now α varies over homothety classes of k-linear injective maps $\mathrm{V} \rightarrow \mathrm{V} \oplus \mathrm{W}$. But $\mathrm{T_W}$ no longer has any G-action, as W is viewed simply as a k-vector space and not as an G-representation. As a result, the forgetful functor $j : \mathrm{Sph}^{\mathrm{G}} \rightarrow \mathrm{USph}^{\mathrm{G}}$ is a simplicially enriched functor.

Proposition 3.2. *With the above definitions, the category* $\mathrm{Sph}^{\mathrm{G}}$ *is a symmetric monoidal* $\mathcal{C}^{\mathrm{G}}$*-enriched category, where the monoidal structure is given by* $\mathrm{T_V} \wedge \mathrm{T_W} = \mathrm{T_{V\oplus W}}$. *(A corresponding result holds for the category* $\mathrm{USph}^{\mathrm{G}}$*.)*

Proof. We first verify that $\mathrm{Sph}^{\mathrm{G}}$ is a $\mathcal{C}^{\mathrm{G}}$-enriched category. To see this, observe that if $\mathrm{f} : \mathrm{U} \rightarrow \mathrm{U} \oplus \mathrm{V}$ is a G-equivariant injective linear map and $\mathrm{g} : \mathrm{V} \rightarrow \mathrm{V} \oplus \mathrm{W}$ is a G-equivariant injective linear map, the composition $(id \oplus \mathrm{g}) \circ \mathrm{f} : \mathrm{U} \rightarrow \mathrm{U} \oplus \mathrm{V} \oplus \mathrm{W}$ is an injective linear map that is also G-equivariant. The composition $\mathcal{H}om_{\mathcal{C}^{\mathrm{G}}}(\mathrm{T_U}, \mathrm{T_{U\oplus V}}) \times \mathcal{H}\mathrm{om}_{\mathcal{C}^{\mathrm{G}}}(\mathrm{T_V}, \mathrm{T_{V\oplus W}}) \rightarrow \mathcal{H}om_{\mathcal{C}^{\mathrm{G}}}(\mathrm{T_U}, \mathrm{T_{U\oplus V\oplus W}})$ sends the summand $\mathrm{T_V}$ indexed by f and the summand $\mathrm{T_W}$ indexed by g to the summand $\mathrm{T_{V\oplus W}}$ indexed by $(id \oplus \mathrm{g}) \circ \mathrm{f} : \mathrm{U} \rightarrow \mathrm{U} \oplus \mathrm{V} \oplus \mathrm{W}$. One may now see readily that this pairing is associative and unital, so that $\mathrm{Sph}^{\mathrm{G}}$ is a $\mathcal{C}^{\mathrm{G}}$-enriched category. The monoidal structure sends $(\mathrm{T_U}, \mathrm{T_V}) \mapsto$

$T_U \wedge T_V = T_{U \oplus V}$. One may now observe that the associativity isomorphisms $(U \oplus V) \oplus W \cong U \oplus (V \oplus W)$ and the commutativity isomorphism $U \oplus V \cong V \oplus U$ are both G-equivariant maps. Therefore, one observes that the monoidal structure defined by the smash product $(T_U, T_V) \mapsto T_U \wedge T_V = T_{U \oplus V}$ makes the category Sph^G a $\mathcal{C}^G$-symmetric monoidal category. The statements regarding the $\mathcal{C}$-enriched category $USph^G$ may be proven similarly. □

Definition 3.3. (The category $\mathbf{Spt}^G/S$, Smash products and internal Hom in $\mathbf{Spt}^G$). We define $\mathbf{Spt}^G/S$ to denote the $\mathcal{C}^G$-enriched category of $\mathcal{C}^G$-enriched functors $Sph^G \to \mathcal{C}^G$. Observe that the $\mathcal{C}^G$-enriched category Sph^G is *symmetric* monoidal with respect to the smash product of Thom-spaces. As a result (see [Day]), if $\mathcal{X}$, $\mathcal{Y}$ are two G-spectra, viewed as enriched functors $Sph^G \to \mathcal{C}^G$, their smash product $\mathcal{X} \wedge \mathcal{Y}$ defined as the left-Kan extension with respect to the monoidal product $\wedge : Sph^G \times Sph^G \to Sph^G$, will *define* a smash product that is symmetric monoidal on $\mathbf{Spt}^G/S$. i.e. The smash product identifies with the following enriched co-end:

$$\mathcal{X} \wedge \mathcal{Y} = \int^{Ob(Sph^G \otimes Sph^G)} \mathcal{H}om_{Sph^G}(T_V \wedge T_W, \quad) \wedge \mathcal{X}(T_V) \wedge \mathcal{Y}(T_W). \tag{3.2.4}$$

The internal $\mathcal{H}om(\mathcal{X}, \mathcal{Y})$ is defined by the enriched end:

$$\mathcal{H}om(\mathcal{X}, \mathcal{Y})(T_V) = \int_{T_W \in Ob(Sph^G)} \mathcal{H}om_{\mathcal{C}^G}(\mathcal{X}(T_W), \mathcal{Y}(T_{V \oplus W})). \tag{3.2.5}$$

Definition 3.4. (The equivariant sphere spectrum) The equivariant sphere spectrum $\mathbb{S}^G$ will be defined to be the object in $\mathbf{Spt}^G/S$ given by the inclusion functor $Sph^G \to \mathcal{C}^G$, that is, $\mathbb{S}^G(T_V) = T_V$, $T_V \in Sph^G$.

One obtains entirely parallel statements on starting with the category PSh^G_S. The corresponding category of G-equivariant spectra will be denoted $\mathbf{Spt}^G_S$. Moreover, if there is no likelihood for confusion, we will often denote both of these categories by $\mathbf{Spt}^G$ and the corresponding stable homotopy category will be denoted $\mathbf{HSpt}^G$ (associated to the stable model structure discussed below).

Remark 3.5. When the group G is a finite group, the regular representation of G will contain all the irreducible representations (at least in characteristic 0), so that one may define a suspension functor by taking the smash product with the Thom-space of the regular representation. As a result one can then define symmetric G-equivariant spectra readily as one does in the non-equivariant case. Since our interest is mainly when the group G is a linear algebraic group of positive dimension, one cannot adopt this framework of symmetric spectra, which is why we have defined the category $\mathbf{Spt}^G$ as above.

In case $\mathcal{E}^G$ is a commutative ring spectrum in $\mathbf{Spt}^G$, we will let $\mathbf{Spt}^G_{\mathcal{E}^G}$ denote the subcategory of $\mathbf{Spt}^G$, consisting of module spectra over $\mathcal{E}^G$ and their maps. In this case, the smash product $\wedge$ will be replaced by $\wedge_{\mathcal{E}^G}$ which is defined as

$$M \wedge_{\mathcal{E}^G} N = Coeq(M \wedge \mathcal{E}^G \wedge N \rightrightarrows M \wedge N) \tag{3.2.6}$$

where the two maps above make use of the module structures on M and N, respectively. The corresponding internal *Hom* will be denoted $\mathcal{H}om_{\mathcal{E}^G}$.

The main G-equivariant ring spectra of interest to us, other than the sphere spectrum $\mathbb{S}^G$, will be the following:

$$\begin{gathered}(i) \quad \mathbb{S}^G[p^{-1}] \text{ if the base scheme S is a field of characteristic p, (ii)} \quad \mathbb{S}^G_{(\ell)} \text{ and} \\ (iii) \quad \widehat{\mathbb{S}}^G_\ell, \text{ where } \ell \text{ is a prime different from the characteristic of the base field.}\end{gathered} \tag{3.2.7}$$

It is convenient to introduce the following intermediate categories, denoted $\widetilde{\mathbf{USpt}}^G$ and $\mathbf{USpt}^G$, intermediate between $\mathbf{Spt}^G$ and $\mathbf{Spt}$ (which is defined in (3.3.8) below). The objects of the $\mathcal{C}$-enriched category $\widetilde{\mathbf{USpt}}^G$ are $\mathcal{C}$-enriched functors $\tilde{\mathcal{X}}' : Sph^G \to \mathcal{C}$, where $\mathcal{C} = PSh/S$. One may observe that an object in this category is given by $\{\tilde{\mathcal{X}}'(T_V) | T_V \in Sph^G\}$, provided with a compatible family of structure maps $T^\alpha_W \wedge \tilde{\mathcal{X}}'(T_V) \to \tilde{\mathcal{X}}'(T_{W \oplus V})$ in PSh, as α varies over all homothety classes of k-linear injective maps $V \to V \oplus W$. However, these maps are no longer required to be G-equivariant. Observe that there is also a forgetful functor

$$\tilde{U} : \mathbf{Spt}^G \to \widetilde{\mathbf{USpt}}^G \tag{3.2.8}$$

given by sending a $\mathcal{X} \in \mathbf{Spt}^G$ to $For \circ \mathcal{X}$, where $For : PSh^G/S \to PSh/S$ is the forgetful functor. When $\mathcal{E}^G \in \mathbf{Spt}^G$ is a commutative ring spectrum, one defines the category $\widetilde{\mathbf{USpt}}^G_{\mathcal{E}^G}$ similarly by replacing the pairings $T_W \wedge$

$\mathcal{X}'(\mathrm{T_V}) \rightarrow \mathcal{X}'(\mathrm{T_{W\oplus V}})$ with the pairings: $\mathcal{E}^{\mathrm{G}}(\mathrm{T_W}) \wedge \mathcal{X}'(\mathrm{T_V}) \rightarrow \mathcal{X}'(\mathrm{T_{W\oplus V}})$. Then one obtains a forgetful functor $\tilde{\mathrm{U}} : \mathbf{Spt}^{\mathrm{G}}_{\mathcal{E}^{\mathrm{G}}} \rightarrow \widetilde{\mathbf{USpt}}^{\mathrm{G}}_{\mathcal{E}^{\mathrm{G}}}$.

The objects of the $\mathcal{C}$-enriched category $\mathbf{USpt}^{\mathrm{G}}$ are given by $\mathcal{C}$-enriched functors $\mathcal{X}' : \mathrm{USph}^{\mathrm{G}} \rightarrow \mathcal{C}$, where $\mathcal{C}$ denotes the category PSh/S. Again, paraphrasing this, such an object is given by $\{\mathcal{X}'(\mathrm{T_V})|\mathrm{T_V} \in \mathrm{USph}^{\mathrm{G}}\}$, provided with a compatible family of structure maps $\mathrm{T}^{\alpha}_{\mathrm{W}} \wedge \mathcal{X}'(\mathrm{T_V}) \rightarrow \mathcal{X}'(\mathrm{T_{W\oplus V}})$ in PSh associated to each homothety class α of k-linear injective maps of V in $\mathrm{V} \oplus \mathrm{W}$, i.e. these maps are no longer required to be G-equivariant, *but the k-linear automorphisms of* V *act on* $\mathcal{X}'(\mathrm{T_V})$. (In this sense, the category of $\mathbf{USpt}^{\mathrm{G}}$ is similar to the category of what are called *orthogonal spectra*.) Morphisms between two such objects $\{\mathcal{Y}'(\mathrm{T_V})|\mathrm{T_V} \in \mathrm{Sph}^{\mathrm{G}}\}$ and $\{\mathcal{X}'(\mathrm{T_V})|\mathrm{T_V} \in \mathrm{Sph}^{\mathrm{G}}\}$ are given by compatible collections of maps $\{\mathcal{Y}'(\mathrm{T_V}) \rightarrow \mathcal{X}'(\mathrm{T_V})|\mathrm{T_V} \in \mathrm{Sph}^{\mathrm{G}}\}$ which are no longer required to be G-equivariant, but compatible with the pairings: $\mathrm{T_W} \wedge \mathcal{Y}'(\mathrm{T_V}) \rightarrow \mathcal{Y}'(\mathrm{T_{W\oplus V}})$ and $\mathrm{T_W} \wedge \mathcal{X}'(\mathrm{T_V}) \rightarrow \mathcal{X}'(\mathrm{T_{W\oplus V}})$. When $\mathcal{E}^{\mathrm{G}} \in \mathbf{Spt}^{\mathrm{G}}$ is a commutative ring spectrum, one defines the category $\mathbf{USpt}^{\mathrm{G}}_{\mathcal{E}^{\mathrm{G}}}$ similarly by replacing the pairings $\mathrm{T_W} \wedge \mathcal{X}'(\mathrm{T_V}) \rightarrow \mathcal{X}'(\mathrm{T_{W\oplus V}})$ with the pairings: $\mathcal{E}^{\mathrm{G}}(\mathrm{T_W}) \wedge \mathcal{X}'(\mathrm{T_V}) \rightarrow \mathcal{X}'(\mathrm{T_{W\oplus V}})$.

3.2.9. The smash product and the internal hom of spectra in $\mathbf{USpt}^{\mathrm{G}}$ and in $\widetilde{\mathbf{USpt}}^{\mathrm{G}}$ are defined exactly as in the case of $\mathbf{Spt}^{\mathrm{G}}$, so that the functor $\tilde{\mathrm{U}}$ is a strict symmetric monoidal functor. This means in particular that for $\mathcal{X}, \mathcal{Y} \in \mathbf{Spt}^{\mathrm{G}}$,

$$\tilde{\mathrm{U}}(\mathcal{X} \wedge \mathcal{Y}) = \tilde{\mathrm{U}}(X) \wedge \tilde{\mathrm{U}}(Y) \text{ and } \tilde{\mathrm{U}}(\mathcal{H}om_{\mathbf{Spt}^{\mathrm{G}}}(\mathcal{X}, \mathcal{Y})) = \mathcal{H}om_{\widetilde{\mathbf{USpt}}^{\mathrm{G}}}(\tilde{\mathrm{U}}(\mathcal{X}), \tilde{\mathrm{U}}(\mathcal{Y})) \tag{3.2.10}$$

Corresponding results hold for the categories $\mathbf{Spt}^{\mathrm{G}}_{\mathcal{E}^{\mathrm{G}}}$, $\widetilde{\mathbf{USpt}}^{\mathrm{G}}_{\mathcal{E}^{\mathrm{G}}}$.

Of key importance is the observation that the $\tilde{\mathrm{U}}(\mathbb{S}^{\mathrm{G}})$ is the unit of the category $\widetilde{\mathbf{USpt}}^{\mathrm{G}}$ with respect to the above smash product. This follows readily from the fact that $\widetilde{\mathbf{USpt}}^{\mathrm{G}}$ as defined above is a category of $\mathcal{C}$-enriched functors $\mathrm{Sph}^{\mathrm{G}} \rightarrow \mathcal{C}$ (see [Day]). Similarly $\mathrm{U}(\mathbb{S}^{\mathrm{G}})$ is the unit of the category $\mathbf{USpt}^{\mathrm{G}}$ with respect to the above smash product and $\mathrm{U}(\mathcal{E}^{\mathrm{G}})$ is the unit of $\mathbf{USpt}^{\mathrm{G}}_{\mathcal{E}^{\mathrm{G}}}$. In view of this, we will henceforth denote $\tilde{\mathrm{U}}(\mathbb{S}^{\mathrm{G}})$ and $\mathrm{U}(\mathbb{S}^{\mathrm{G}})$ by $\mathbb{S}^{\mathrm{G}}$ and $\mathrm{U}(\mathcal{E}^{\mathrm{G}})$ by $\mathcal{E}^{\mathrm{G}}$.

3.3. Model structures on $\mathbf{USpt}^{\mathrm{G}}$, $\mathbf{USpt}^{\mathrm{G}}_{\mathcal{E}^{\mathrm{G}}}$ and $\widetilde{\mathbf{USpt}}^{\mathrm{G}}$, $\widetilde{\mathbf{USpt}}^{\mathrm{G}}_{\mathcal{E}^{\mathrm{G}}}$. Throughout this discussion, $\mathcal{C} = \mathrm{PSh/S}$ which is the category of pointed simplicial presheaves, pointed over S on either the big étale, the big Nisnevich or the big Zariski site over the fixed base scheme (which will be a perfect field k). It will be provided with a chosen model structure, either projective or injective and where $\mathbb{A}^1$ is inverted.

3.3.1. The level-wise injective model structures. Here we define a map $f : \chi' \rightarrow \chi$ of spectra in $\mathbf{USpt}^{\mathrm{G}}$ to be a *level-wise injective cofibration* (an *injective weak-equivalence*) if the induced map $f(\mathrm{T_V}) : \chi'(\mathrm{T_V}) \rightarrow \chi(\mathrm{T_V})$ is a cofibration (a weak-equivalence, respectively). The *injective fibrations* are defined by the lifting property with respect to trivial cofibrations. One defines the level-wise injective model structure on the category $\widetilde{\mathbf{USpt}}^{\mathrm{G}}$ similarly. The following is proven in [CJ14, Proposition 3.12]

Proposition 3.6. *This defines a combinatorial (in fact, tractable) simplicial monoidal model structure on* $\mathbf{USpt}^{\mathrm{G}}$ *that is left proper. Every injective fibration is a level fibration. The cofibrations are the monomorphisms. The unit of the monoidal structure on* $\mathbf{USpt}^{\mathrm{G}}$ *and in fact every object in* $\mathbf{USpt}^{\mathrm{G}}$ *is cofibrant in this model structure. The corresponding results hold for the category* $\widetilde{\mathbf{USpt}}^{\mathrm{G}}$.

Proof. We will only discuss the proofs for the category $\mathbf{USpt}^{\mathrm{G}}$. We start with the observation that the category $\mathcal{C}$ is a simplicially enriched tractable simplicial model category. The left-properness is obvious, since the cofibrations and weak-equivalences are defined level-wise. The first conclusion follows now from [Lur, Proposition A.3.3.2]: observe that the pushout-product axiom holds since cofibrations (weak-equivalences) are object-wise cofibrations (weak-equivalences, respectively) and the pushout-product axiom holds in the monoidal model category $\mathcal{C}$. The second statement follows from [CJ14, Proposition 3.10(iii)]. Recall the unit of $\mathbf{USpt}^{\mathrm{G}}$ is the inclusion functor $\mathrm{USph}^{\mathrm{G}} \rightarrow \mathcal{C}$. We will denote this by Σ. To prove it is cofibrant, all one has to observe is that $\Sigma(\mathrm{T_V}) = \mathrm{T_V}$ which is cofibrant in $\mathcal{C}$ for every $\mathrm{T_V} \in \mathrm{Sph}^{\mathrm{G}}$. □

Though the projective model structures are of less importance for us, we still need them for the comparison results in Proposition 3.11. Therefore, we provide the following brief discussion of the projective model structure on $\mathbf{USpt}^{\mathrm{G}}$ and $\widetilde{\mathbf{USpt}}^{\mathrm{G}}$. First one recalls that the unstable projective model structure on $\mathcal{C} = \mathrm{PSh}$ has as generating cofibrations (generating trivial cofibrations) all maps of the form $\delta\Delta[n]_+ \wedge \mathrm{U}_+ \rightarrow \Delta[\mathrm{n}]_+ \wedge \mathrm{U}_+$ ($\Lambda[n]_+ \wedge \mathrm{U}_+ \rightarrow \Delta[\mathrm{n}]_+ \wedge \mathrm{U}_+$) as U_+ varies over the objects in the given site. Next we functorially replace every object $\mathrm{T_V}$ in $\mathrm{Sph}^{\mathrm{G}}$ by an object that is cofibrant in $\mathcal{C}$. (The functoriality of the cofibrant replacement shows that then, these functorial cofibrant replacements all come equipped with G-actions. Therefore, we will still denote these cofibrant replacements by $\{\mathrm{T_V}|\mathrm{V}\}$.)

The weak-equivalences (fibrations) in the *level-wise projective model structure* are those maps of spectra $f : \mathcal{X} \to \mathcal{Y}$, for which each $f(\mathrm{T_V}) : \mathcal{X}(\mathrm{T_V}) \to \mathcal{Y}(\mathrm{T_V})$, $\mathrm{T_V} \in \mathrm{Sph^G}$, are weak-equivalences (fibrations, respectively) in $\mathcal{C}$. The cofibrations in this model structure are defined by left-lifting property with respect to the maps that are trivial fibrations in this model structure.

Let $\mathcal{F}_{\mathrm{T_V}}$ denote the left-adjoint to the evaluation functor sending a spectrum $\mathcal{X} \in \mathbf{USpt}^{\mathrm{G}}$ ($\mathcal{X} \in \widetilde{\mathbf{USpt}}^{\mathrm{G}}$) to $\mathcal{X}(\mathrm{T_V})$. One may observe that this is the spectrum defined by

$$(3.3.2) \qquad \begin{aligned} \mathcal{F}_{\mathrm{T_V}}(C)(\mathrm{T_{V\oplus W}}) &= C \wedge \mathrm{T_W} \text{ and} \\ \mathcal{F}_{\mathrm{T_V}}(C)(\mathrm{T_U}) &= *, \mathrm{U} \neq \mathrm{V} \oplus \mathrm{W}, \text{for some W.} \end{aligned}$$

Let I (J) denote the generating cofibrations (generating trivial cofibrations, respectively) of the model category $\mathcal{C}$. We define the generating cofibrations $\mathrm{I}_{\widetilde{\mathbf{USpt}}^{\mathrm{G}}}$ (the generating trivial cofibrations $\mathrm{J}_{\widetilde{\mathbf{USpt}}^{\mathrm{G}}}$) to be

$$(3.3.3) \qquad \bigcup_{\mathrm{T_V}\in \mathrm{Sph^G}} \{\mathcal{F}_{\mathrm{T_V}}(i) \mid i \in \mathrm{I}\} (\bigcup_{\mathrm{T_V}\in \mathrm{Sph^G}} \{\mathcal{F}_{\mathrm{T_V}}(\mathrm{j}) | \mathrm{j} \in \mathrm{J}\}) \;.$$

One defines the generating cofibrations $\mathrm{I}_{\mathbf{USpt}^{\mathrm{G}}}$ (the generating trivial cofibrations $\mathrm{J}_{\mathbf{USpt}^{\mathrm{G}}}$) of the level-wise projective model structure on $\mathbf{USpt}^{\mathrm{G}}$ similarly. Then the following is proven in [CJ14, Proposition 3.10] and in [CJ14, Corollary 3.11].

Proposition 3.7. *The projective cofibrations, the level fibrations and level equivalences define a cofibrantly generated model category structure on $\widetilde{\mathbf{USpt}}^{\mathrm{G}}$ with the generating cofibrations (generating trivial cofibrations) being $\mathrm{I}_{\widetilde{\mathbf{USpt}}^{\mathrm{G}}}$ ($\mathrm{J}_{\widetilde{\mathbf{USpt}}^{\mathrm{G}}}$, respectively). This model structure (called the level-wise projective model structure) has the following properties:*

- (i) *Every projective cofibration (projective trivial cofibration) is a level cofibration (level trivial cofibration, respectively).*
- (ii) *It is left-proper and right proper and is cellular.*
- (iii) *The objects in $\bigcup_{\mathrm{T_V}\in\mathrm{Sph^G}}\{\mathcal{F}_{\mathrm{T_V}}(\mathrm{Sph^G})\}$ are all finitely presented. The category $\widetilde{\mathbf{USpt}}^{\mathrm{G}}$ is symmetric monoidal with the pairing defined in Definition 3.3.*
- (iv) *This category is locally presentable and hence is a tractable (and hence a combinatorial) model category.*
- (v) *With the above structure, $\widetilde{\mathbf{USpt}}^{\mathrm{G}}$ is a symmetric monoidal model category satisfying the monoidal axiom.*
- (vi) *Corresponding results hold for the level-wise projective model structure on $\mathbf{USpt}^{\mathrm{G}}$.*

3.3.4. *Module spectra over a ring spectrum.* Let $\mathcal{E}^{\mathrm{G}} \in \mathbf{USpt}^{\mathrm{G}}$ ($\mathcal{E} \in \widetilde{\mathbf{USpt}}^{\mathrm{G}}$) denote a ring spectrum. One then invokes the free $\mathcal{E}^{\mathrm{G}}$-module functor and the forgetful functor sending an $\mathcal{E}^{\mathrm{G}}$-module spectrum to its underlying spectrum along with [SSch98, Lemma 2.3, Theorem 4.1(2)] to obtain a corresponding cofibrantly generated model category structure on $\mathbf{USpt}^{\mathrm{G}}_{\mathcal{E}^{\mathrm{G}}}$ ($\widetilde{\mathbf{USpt}}^{\mathrm{G}}$). Observe that, in this model structure the fibrations are those maps f in $\mathbf{USpt}^{\mathrm{G}}_{\mathcal{E}^{\mathrm{G}}}$ for which f is a fibration in $\mathbf{USpt}^{\mathrm{G}}$.

3.3.5. The stable model structures on $\mathbf{USpt}^{\mathrm{G}}$, $\mathbf{USpt}^{\mathrm{G}}_{\mathcal{E}^{\mathrm{G}}}$ and $\widetilde{\mathbf{USpt}}^{\mathrm{G}}$, $\widetilde{\mathbf{USpt}}^{\mathrm{G}}_{\mathcal{E}^{\mathrm{G}}}$. (See [CJ14, §3.3.2].) We proceed to define the stable model structure by applying a suitable Bousfield localization to the level-wise injective (projective)model structures considered above. This follows the approach in [Hov01, section 3]. We will explicitly consider only the case of $\widetilde{\mathbf{USpt}}^{\mathrm{G}}$, since essentially the same description applies to the categories $\mathbf{USpt}^{\mathrm{G}}$, $\mathbf{USpt}^{\mathrm{G}}_{\mathcal{E}^{\mathrm{G}}}$ and $\widetilde{\mathbf{USpt}}^{\mathrm{G}}_{\mathcal{E}^{\mathrm{G}}}$. The corresponding model structure will be called the *the injective (projective) stable model structure.* (One may observe that the domains and co-domains of objects of the generating cofibrations are cofibrant, so that there is no need for a cofibrant replacement functor Q as in [Hov01, section 3].)

Let $\mathcal{X} \in \widetilde{\mathbf{USpt}}^{\mathrm{G}}$. Since $\mathcal{X}$ is a $\mathcal{C}$-enriched functor $\mathrm{Sph^G} \to \mathcal{C}$, we obtain a natural map

$$(3.3.6) \qquad (\sqcup_\alpha \mathrm{T}^\alpha_{\mathrm{W}})_+ = \mathcal{H}om_{\mathrm{Sph^G}}(\mathrm{T_V}, \mathrm{T_V} \wedge \mathrm{T_W}) \to \mathcal{H}om_{\mathcal{C}}(\mathcal{X}(\mathrm{T_V}), \mathcal{X}(\mathrm{T_{V\oplus W}}))),$$

where $\mathrm{T}^\alpha_{\mathrm{W}}$ is a copy of $\mathrm{T_W}$ indexed by α, and where α varies over all homothety classes of k-linear injective and G-equivariant maps $\mathrm{V} \to \mathrm{V} \oplus \mathrm{W}$.

Definition 3.8. (Ω-spectra) A spectrum $\chi \in \widetilde{\mathbf{USpt}}^{\mathrm{G}}$ ($\mathcal{X} \in \widetilde{\mathbf{USpt}}^{\mathrm{G}}_{\mathcal{E}^{\mathrm{G}}}$) is an Ω*-spectrum* if it is level-fibrant and each of the natural maps $\chi(\mathrm{T_V}) \to \mathcal{H}\mathrm{om}_{\mathcal{C}}(\mathrm{T}^{\alpha}_{\mathrm{W}}, \chi(\mathrm{T_V} \wedge \mathrm{T_W}))$, for each α as in (3.3.6) is an unstable weak-equivalence in the corresponding model structure on $\mathcal{C}$.

Let $\mathcal{F}_{\mathrm{T_V}}$ denote the left-adjoint to the evaluation functor sending a spectrum $\mathcal{X} \in \widetilde{\mathbf{USpt}}^{\mathrm{G}}$ to $\mathcal{X}(\mathrm{T_V})$: see (3.3.2). Let $\mathrm{C} \in \mathcal{C}$ be an object as above and let $\chi \in \mathbf{USpt}^{\mathrm{G}}$ be fibrant in the level-wise injective model structure. Then

$$Map(\mathrm{C}, \chi(\mathrm{T_V})) = \mathrm{Map}(\mathrm{C}, \mathcal{E}\mathrm{val}_{\mathrm{T_V}}(\chi)) \simeq \mathrm{Map}(\mathcal{F}_{\mathrm{T_V}}(\mathrm{C}), \chi) \text{ and}$$

$$Map(\mathrm{C}, \mathcal{H}\mathrm{om}_{\mathcal{C}}(\mathrm{T}_{\mathrm{W}^{\alpha}}, \chi(\mathrm{T_V} \wedge \mathrm{T_W}))) = \mathrm{Map}(\mathrm{C}, \mathcal{H}\mathrm{om}_{\mathcal{C}}(\mathrm{T}_{\mathrm{W}^{\alpha}}, \mathcal{E}\mathrm{val}_{\mathrm{T_V} \wedge \mathrm{T_W}}(\chi))) \simeq \mathrm{Map}(\mathcal{F}_{\mathrm{T_V} \wedge \mathrm{T_W}}(\mathrm{C} \wedge \mathrm{T}_{\mathrm{W}^{\alpha}}), \chi).$$

Therefore, to convert χ into an Ω -spectrum, it suffices to invert the maps in $\mathbf{S}$, where

$$\mathbf{S} = \{\mathcal{F}_{\mathrm{T_V} \wedge \mathrm{T_W}}(\mathrm{C} \wedge \mathrm{T}_{\mathrm{W}^{\alpha}}) \to \mathcal{F}_{\mathrm{T_V}}(\mathrm{C}) \mid \mathrm{C} \in \text{ Domains or Co-domains of I}, \mathrm{T_V}, \mathrm{T_W} \in \mathrm{Sph}^{\mathrm{G}}, \alpha\} \tag{3.3.7}$$

corresponding to the above maps $C \wedge \mathrm{T}_{\mathrm{W}^{\alpha}} \to \mathrm{C} \wedge \mathcal{H}\mathrm{om}_{\mathrm{Sph}^{\mathrm{G}}}(\mathrm{T_V}, \mathrm{T_V} \wedge \mathrm{T_W})$ by adjunction, as α varies over all homothety classes of k-linear injective G-equivariant maps $\mathrm{V} \to \mathrm{V} \oplus \mathrm{W}$. (Here I denotes the generating cofibrations of $\mathcal{C}$.) Similarly, for a commutative ring spectrum $\mathcal{E}^{\mathrm{G}} \in \mathbf{Spt}^{\mathrm{G}}$, one lets $\mathbf{S}_{\mathcal{E}^{\mathrm{G}}}$ be defined using the corresponding free-functors for $\mathcal{E}^{\mathrm{G}}$-module-spectra. (See [Hov01, Proposition 3.2] that shows it suffices to consider the objects C that form the domains and co-domains of the generating cofibrations in $\mathcal{C}$.)

The *stable injective (projective) model structure* on $\widetilde{\mathbf{USpt}}^{\mathrm{G}}$ ($\widetilde{\mathbf{USpt}^{\mathrm{G}}}_{\mathcal{E}^{\mathrm{G}}}$) is obtained by localizing the level-wise injective (projective) model structure with respect to the maps in $\mathbf{S}$ ($\mathbf{S}_{\mathcal{E}^{\mathrm{G}}}$, respectively). The $\mathbf{S}$-local weak-equivalences ($\mathbf{S}$-local fibrations) will be referred to as the *stable equivalences* (*stable fibrations*, respectively). The cofibrations in the localized model structure are the cofibrations in the level-wise projective or injective model structures on $\widetilde{\mathbf{USpt}}^{\mathrm{G}}$ ($\widetilde{\mathbf{USpt}^{\mathrm{G}}}_{\mathcal{E}^{\mathrm{G}}}$, respectively).

Proposition 3.9. *(See* [CJ14, Proposition 3.16].*) (i) The corresponding stable model structure on* $\widetilde{\mathbf{USpt}}^{\mathrm{G}}$ *(*$\widetilde{\mathbf{USpt}}^{\mathrm{G}}_{\mathcal{E}^{\mathrm{G}}}$*) is cofibrantly generated and left proper. It is also locally presentable, and hence combinatorial (tractable).*

(ii) The fibrant objects in the stable model structure on $\widetilde{\mathbf{USpt}}^{\mathrm{G}}$ *(*$\widetilde{\mathbf{USpt}}^{\mathrm{G}}_{\mathcal{E}^{\mathrm{G}}}$*) are the* Ω*-spectra defined above.*

(iii) The category $\widetilde{\mathbf{USpt}}^{\mathrm{G}}$ *(*$\widetilde{\mathbf{USpt}}^{\mathrm{G}}_{\mathcal{E}^{\mathrm{G}}}$*) is a symmetric monoidal model category (i.e. satisfies the pushout-product axiom: see* [SSch98, Definition 3.1]*) in both the projective and injective stable model structures with the monoidal structure being the same in both the model structures. In the injective model structure, the unit is cofibrant and the monoidal axiom (see* [SSch98, Definition 3.3]*) is also satisfied.*

Let $\mathbf{Spt}$ denote the (usual) category of motivic spectra defined as follows. Its objects are $\mathcal{X} = \{\mathrm{X_n} \in \mathrm{PSh}, \text{ along with structure maps } \mathbf{T}^{\mathrm{m}} \wedge \mathrm{X_n} \to \mathrm{X_{n+m}} | \mathrm{n}, \mathrm{m} \in \mathbb{N}\}$. Morphisms between two such objects $\mathcal{X}$ and $\mathcal{Y}$ are defined as compatible collection of maps $\mathcal{X}_n \to \mathcal{Y}_n$, $n \in \mathbb{N}$ compatible with suspensions by $\mathbf{T}^m$, $m \in \mathbb{N}$. We proceed to relate the category $\mathbf{Spt}$ with $\mathbf{USpt}^{\mathrm{G}}$.

For each natural number n, we choose a trivial representation of G of dimension n. We will denote this representation by n and its Thom space by $\mathrm{T_n} (= \mathbf{T}^{\mathrm{n}})$. We will the identify $\mathbb{N}$ with the $\mathcal{C}$-enriched subcategory of $\mathrm{USph}^{\mathrm{G}}$ consisting of these objects and where

$$\begin{aligned} \mathcal{H}om_{\mathbb{N}}(\mathrm{T_n}, \mathrm{T_{n+m}}) &= \mathrm{T_m}, \text{if } \mathrm{m} \neq 0 \\ &= \mathrm{S}^0, \text{if } \mathrm{m} = 0. \end{aligned} \tag{3.3.8}$$

Thus, we obtain a $\mathcal{C}$-enriched *faithful functor* $i : \mathbb{N} \to \mathrm{USph}^{\mathrm{G}}$, and the functor i^* defines a simplicially enriched functor $\mathbf{USpt}^{\mathrm{G}} \to \mathbf{Spt}$. The functor i^* admits a left adjoint, which we denote by $\mathbb{P} : \mathbf{Spt} \to \mathbf{USpt}^{\mathrm{G}}$. One defines both a projective, as well as an injective model structure on the category $\mathbf{Spt}$, both level-wise and stably: see [CJ14, section 3]. Though for the most part we will only work with the injective model structures, the projective model structures seem helpful for comparing the model categories $\mathbf{Spt}$ and $\mathbf{USpt}^{\mathrm{G}}$.

The free functor $\mathbf{Spt} \to \mathcal{C}$ left adjoint to the evaluation functor $Eval_{\mathrm{T_n}} : \mathbf{Spt} \to \mathcal{C}$, sending $\mathcal{X} \mapsto \mathcal{X}(\mathrm{T_n})$ will be denoted $F_{\mathrm{T_n}}$. The stable model structure on $\mathbf{Spt}$ will be obtained by inverting maps in

$$\mathbf{S}_{\mathbb{N}} = \{F_{\mathrm{T_n} \wedge \mathrm{T_m}}(\mathrm{C} \wedge \mathrm{T_m}) \to \mathrm{F}_{\mathrm{T_n}}(\mathrm{C}) \mid \mathrm{C} \in \text{ Domains or Co-domains of I}, \mathrm{m}, \mathrm{n} \in \mathbb{N}\}. \tag{3.3.9}$$

Let $\mathbf{HSpt}$ denote the corresponding stable homotopy category. We will provide both $\mathbf{USpt}^{\mathrm{G}}$ and $\mathbf{Spt}$ with the projective level-wise and the corresponding projective stable model structures.

Remark 3.10. Observe that as we apply the forgetful $j : \mathrm{Sph}^{\mathrm{G}} \to \mathrm{USph}^{\mathrm{G}}$, we obtain possibly several different objects that are isomorphic in $\mathrm{USph}^{\mathrm{G}}$. Thus one of the differences between the categories $\mathbf{USpt}^{\mathrm{G}}$ and $\mathbf{Spt}$ is that the indexing category for the former has possibly many more objects. A second key difference is that the simplicially enriched hom in the category $\mathrm{USph}^{\mathrm{G}}$ has many symmetries making $\mathrm{USph}^{\mathrm{G}}$ a symmetric monoidal category and much bigger than the indexing category for $\mathbf{Spt}$. Nevertheless we proceed to show that at the homotopy category level, the category $\mathbf{USpt}^{\mathrm{G}}$ and $\mathbf{Spt}$ are equivalent. (This should be viewed as the analogue of the equivalence between the homotopy categories of orthogonal spectra and spectra in the usual sense.)

We proceed to show that the $\mathcal{C}$-enriched stable model categories $\widetilde{\mathbf{USpt}}^{\mathrm{G}}$, $\mathbf{USpt}^{\mathrm{G}}$ and $\mathbf{Spt}$ are Quillen equivalent. The proof will compare both $\widetilde{\mathbf{USpt}}^{\mathrm{G}}$ and $\mathbf{Spt}$ with $\mathbf{USpt}^{\mathrm{G}}$. Since both these comparisons proceed similarly, we deal with them both in the following proposition.

To relate the $\mathcal{C}$-enriched categories, $\mathbf{USpt}^{\mathrm{G}}$ and $\widetilde{\mathbf{USpt}}^{\mathrm{G}}$, one first observes that there is a forgetful functor $j : \mathrm{Sph}^{\mathrm{G}} \to \mathrm{USph}^{\mathrm{G}}$ that sends the Thom-space, $\mathrm{T_V}$, of a G-representation V to $\mathrm{T_V}$ but viewing V as just a k-vector space. Therefore, pull-back by j defines the $\mathcal{C}$-enriched functor $j^* : \mathbf{USpt}^{\mathrm{G}} \to \widetilde{\mathbf{USpt}}^{\mathrm{G}}$. One defines a functor $\tilde{\mathbb{P}}$ as the left-adjoint to j^*. For a $\mathcal{X} \in \widetilde{\mathbf{USpt}}^{\mathrm{G}}$, $\tilde{\mathbb{P}}(\mathcal{X})$ is defined as a $\mathcal{C}$-enriched left Kan-extension along the functor $j : \mathrm{Sph}^{\mathrm{G}} \to \mathrm{USph}^{\mathrm{G}}$. Moreover, the stable projective model structure on $\widetilde{\mathbf{USpt}}^{\mathrm{G}}$ is obtained from the level-wise projective model structure by inverting maps in the collection $\mathbf{S}$ defined in (3.3.7).

Proposition 3.11. [3] *(i) The functors $\mathbb{P}$ and i^* define a Quillen adjunction between the projective stable model structures on $\mathbf{USpt}^{\mathrm{G}}$ and $\mathbf{Spt}$. This is, in fact, a Quillen equivalence.*

(ii) The functors $\tilde{\mathbb{P}}$ and j^* define a Quillen-equivalence between the stable projective model structures on $\mathbf{USpt}^{\mathrm{G}}$ and $\widetilde{\mathbf{USpt}}^{\mathrm{G}}$.

(iii) The functors $\mathbb{P}$ and $\tilde{\mathbb{P}}$ are strict-monoidal functors.

Proof. It should be clear that i^* (j^*) preserves fibrations and weak-equivalences in the level-wise projective model structures. Therefore, its left adjoint $\mathbb{P}$ ($\tilde{\mathbb{P}}$) preserves the cofibrations and trivial cofibrations in the level-wise projective model structures. It is also clear that i^* sends Ω-spectra in $\mathbf{USpt}^{\mathrm{G}}$ to Ω-spectra in $\mathbf{Spt}$ and that j^* sends Ω-spectra in $\mathbf{USpt}^{\mathrm{G}}$ to Ω-spectra in $\widetilde{\mathbf{USpt}}^{\mathrm{G}}$. Therefore, the functors $\mathbb{P}$ and $\tilde{\mathbb{P}}$ preserve stable weak-equivalences between cofibrant objects.

Observe next that the set of generating trivial cofibrations in the projective stable model structure on $\mathbf{USpt}^{\mathrm{G}}$ ($\widetilde{\mathbf{USpt}}^{\mathrm{G}}$) is obtained by taking pushout-products of the maps in $\mathbf{S}$ (in the corresponding family $\mathbf{S}$) with the cofibrations of the form $\Lambda[n]_+ \to \Delta[n]_+$ and $\delta\Delta[n]_+ \to \Delta[n]_+$, $n \geq 0$. Similarly the set of generating trivial cofibrations in the projective stable model structure on $\mathbf{Spt}$ is obtained by taking pushout-products of the maps in $\mathbf{S}_{\mathbb{N}}$ with the cofibrations of the form $\Lambda[n]_+ \to \Delta[n]_+$ and $\delta\Delta[n]_+ \to \Delta[n]_+$, $n \geq 0$. Next, the adjunction between the free functors and the evaluation functors provides the identification:

$$\mathbb{P}(F_{\mathrm{T_n}}) = \mathcal{F}_{i(\mathrm{T_n})} \text{ and } \tilde{\mathbb{P}}(\mathcal{F}_{\mathrm{T_V}}) = \mathcal{F}_{j(\mathrm{T_V})}. \tag{3.3.10}$$

(This follows readily from the identifications $Eval_{\mathrm{T_n}}(i^*(\mathcal{X})) = Eval_{i(\mathrm{T_n})}(\mathcal{X})$ and $Eval_{\mathrm{T_V}}(j^*(\mathcal{X})) = Eval_{j(\mathrm{T_V})}(\mathcal{X})$.) Therefore, it follows that $\mathbb{P}$ ($\tilde{\mathbb{P}}$) sends the generating trivial cofibrations in the projective stable model structure on $\mathbf{Spt}$ ($\widetilde{\mathbf{USpt}}^{\mathrm{G}}$) to the generating trivial cofibrations in the projective stable model structure on $\mathbf{USpt}^{\mathrm{G}}$ ($\mathbf{USpt}^{\mathrm{G}}$, respectively). Since the functor $\mathbb{P}$ ($\tilde{\mathbb{P}}$) also preserves pushouts and filtered colimits, it follows that it preserves trivial cofibrations. Since the cofibrations in the projective stable model structure are the same as in the projective level-wise model structure, it follows that $\mathbb{P}$ ($\tilde{\mathbb{P}}$) also preserves these, thereby proving that the functors $(\mathbb{P}, i^*)$ ($(\tilde{\mathbb{P}}, j^*)$) define a Quillen adjunction of the projective stable model structures on $\mathbf{Spt}$ and $\mathbf{USpt}^{\mathrm{G}}$ ($\widetilde{\mathbf{USpt}}^{\mathrm{G}}$ and $\mathbf{USpt}^{\mathrm{G}}$, respectively).

Next observe that the functor i^* (j^*) being a right Quillen functor preserves trivial fibrations and therefore, (by Ken Brown's lemma: see [Hov01, Lemma 1.1.12]), it preserves all stable weak-equivalences between stably fibrant objects. In fact a stable weak-equivalence between fibrant objects is a level-wise weak-equivalence and i^* (j^*) clearly preserves these. Next we already saw that i^* (j^*) preserves Ω-spectra and therefore all stably fibrant objects. Therefore, suppose $f : \mathcal{X} \to \mathcal{Y}$ is a map in $\mathbf{USpt}^{\mathrm{G}}$ between stably fibrant objects, so that $i^*(f) : i^*(\mathcal{X}) \to i^*(\mathcal{Y})$ is a stable weak-equivalence. Since both $i^*(\mathcal{X})$ and $i^*(\mathcal{Y})$ are stably fibrant, this is a level-wise weak-equivalence of

[3] We skip the proof that the injective and projective stable model structures appearing below are Quillen equivalent, which may be proven in the usual manner.

spectra in **Spt**. i.e. The induced map $i^*(f)(\mathrm{T_n}) : \mathrm{i}^*(\mathcal{X}(\mathrm{T_n})) \to \mathrm{i}^*(\mathcal{Y}(\mathrm{T_n}))$ is a weak-equivalence for every n. Since the objects in $\mathrm{USph}^{\mathrm{G}}$ are also just finite dimensional k-vector spaces (i.e. without any G-action), it follows that f itself is a level-wise weak-equivalence of spectra and therefore also a stable weak-equivalence in $\mathbf{USpt}^{\mathrm{G}}$. Stated another way, this shows that the functor i^* both detects and preserves stable weak-equivalences between fibrant objects. An entirely similar argument proves that j^* both preserves and detects stable weak-equivalences between fibrant objects.

Next we make the following observation:

$$(3.3.11) \qquad i^*(\mathcal{F}_{\mathrm{T_V}}) = F_{\mathrm{T_n}}, \text{ where } n = dim(\mathrm{V}) \text{ and } \mathrm{j}^*(\mathcal{F}_{\mathrm{j(T_V)}}) = \mathcal{F}_{\mathrm{T_V}}.$$

One may see the first by evaluating both sides at $\mathrm{T_m}$, $m \in \mathbb{N}$ and the second by evaluating both sides at $\mathrm{T_W} \in \mathrm{Sph}^{\mathrm{G}}$.

The next step is to show the following holds: let Q denote the fibrant replacement functor in any of the projective stable model category structures on $\widetilde{\mathbf{USpt}}^{\mathrm{G}}$, $\mathbf{USpt}^{\mathrm{G}}$ and **Spt**. Then the functors i^* and j^* strictly commute with Q in the sense

$$(3.3.12) \qquad Q \circ i^* = i^* \circ Q \text{ and } Q \circ j^* = j^* \circ Q.$$

We will only provide a proof for the first equality, since the second equality may be proved in a similar manner. To see this, one needs to recall how a functorial fibrant replacement is constructed making use of the small object argument: see [Hov01, Proposition 2.1.16]. We will consider this for an object $\mathcal{X} \in \mathbf{USpt}^{\mathrm{G}}$. It is defined as the transfinite colimit of a filtered direct system of spectra $\mathcal{X}_\alpha \in \mathbf{USpt}^{\mathrm{G}}$, starting with $\mathcal{X}_0 = \mathcal{X}$. In order to obtain $X_{\alpha+1}$ from X_α, we consider all commutative squares of the form

$$\begin{array}{ccc} A_\alpha & \longrightarrow & X_\alpha \\ \downarrow & & \downarrow \\ B_\alpha & \longrightarrow & * \end{array}$$

with $A_\alpha \to B_\alpha$ one of the generating trivial cofibrations in the projective stable model structure. Then we let $X_{\alpha+1}$ be defined as the corresponding pushout, after having replaced $A_\alpha \to B_\alpha$ by the sum of all such maps as one varies over the generating trivial cofibrations. Since the above pushout and the colimit are taken after evaluating a spectrum at each object $\mathrm{T_V}$, it should be clear that the functor i^* commutes with such colimits and pushouts. Moreover, (3.3.11) shows that the functor i^* sends the generating trivial cofibrations for the stable projective model structure on $\mathbf{USpt}^{\mathrm{G}}$ to the generating trivial cofibrations of the stable projective model structure on **Spt** and that every generating trivial cofibration in this model structure on **Spt** may be obtained by applying the functor i^* to a generating trivial cofibration in the above model structure on $\mathbf{USpt}^{\mathrm{G}}$.

Finally, we now observe from [HSS, Lemma 4.1.7], that it suffices to prove that for any object $\mathcal{X} \in \mathbf{Spt}$ ($\mathcal{Y} \in \widetilde{\mathbf{USpt}}^{\mathrm{G}}$), which is cofibrant in the projective stable model structure there, the composite map $\mathcal{X} \to i^*\mathbb{P}(\mathcal{X}) \to i^*Q(\mathbb{P}(\mathcal{X}))$ ($\mathcal{Y} \to j^*\tilde{\mathbb{P}}(\mathcal{Y}) \to j^*Q(\tilde{\mathbb{P}}(\mathcal{Y}))$ is a stable weak-equivalence. In view of (3.3.12), we obtain the identification $i^*Q(\mathbb{P}(\mathcal{X})) = Q(i^*(\mathbb{P}(\mathcal{X})))$ ($j^*Q(\tilde{\mathbb{P}}(\mathcal{Y})) = Q(j^*(\tilde{\mathbb{P}}(\mathcal{Y})))$, respectively). Clearly the map $i^*\mathbb{P}(\mathcal{X}) \to Q(i^*(\mathbb{P}(\mathcal{X})))$ ($j^*\tilde{\mathbb{P}}(\mathcal{Y}) \to Q(j^*\tilde{\mathbb{P}}(\mathcal{Y})))$ is a stable weak-equivalence, since $Q(i^*(\mathbb{P}(\mathcal{X})))$ ($Q(j^*(\tilde{\mathbb{P}}(\mathcal{Y})))$ is a stably fibrant replacement of $i^*(\mathbb{P}(\mathcal{X}))$ ($j^*(\tilde{\mathbb{P}}(\mathcal{Y}))$, respectively). Therefore, it suffices to show that the natural map $\mathcal{X} \to i^*(\mathbb{P}(\mathcal{X}))$ ($\mathcal{Y} \to j^*(\tilde{\mathbb{P}}(\mathcal{Y}))$, respectively) is a stable weak-equivalence for every cofibrant object $\mathcal{X} \in \mathbf{Spt}$ ($\mathcal{Y} \in \widetilde{\mathbf{USpt}}^{\mathrm{G}}$, respectively). A closely analogous statement is proven in [HSS, Lemma 4.3.11]. Observe that the generating cofibrations are given as in (3.3.3), which are suspension spectra for which the required statement is true by (3.3.10) and (3.3.11). Therefore, one may prove this readily from the construction of a cofibrant replacement using these generating cofibrations and the small object argument as in [Hov01, Theorem 2.1.14]. We skip the remaining details. These complete the proof of the first two statements. Observe that both the functors $\mathbb{P}$ and $\tilde{\mathbb{P}}$ are left-Kan extensions and therefore, commute with the smash-products of spectra, which are also left-Kan extensions. This completes the proof of the proposition. □

Terminology 3.12. *Given a commutative ring spectrum* $\mathcal{E}^{\mathrm{G}} \in \mathbf{Spt}^{\mathrm{G}}$*, we let* $\mathcal{E} = i^*(\tilde{\mathbb{P}}\tilde{\mathrm{U}}(\mathcal{E}^{\mathrm{G}})$*, which is a commutative ring spectrum in* **Spt***. For example, the equivariant sphere spectrum* $\mathbb{S}^{\mathrm{G}}$ *provides* $\mathbb{S} = i^*(\tilde{\mathbb{P}}\tilde{\mathrm{U}}(\mathbb{S}^{\mathrm{G}}))$ *the usual sphere spectrum.*

Remark 3.13. Then one readily proves the existence of a Quillen equivalence between the model categories $\mathbf{USpt}^{\mathrm{G}}_{\mathcal{E}^{\mathrm{G}}}$ and $\mathbf{Spt}_{\mathcal{E}}$, just as in Proposition 3.11.

Proposition 3.14. *Let $\mathcal{X} \in \mathbf{Spt}^{\mathrm{G}}$ and let $\tilde{\mathrm{U}}(\mathcal{X}) \in \widetilde{\mathbf{USpt}}^{\mathrm{G}}$ denote the forgetful functor $\tilde{\mathrm{U}}$ (as in (3.2.8)) applied to $\mathcal{X}$. If $\mathcal{X}'' \to \tilde{\mathrm{U}}(\mathcal{X})$ ($\tilde{\mathrm{U}}(\mathcal{X}) \to \mathcal{X}'$) is a functorial cofibrant (fibrant) replacement in the injective or projective stable model structure on $\widetilde{\mathbf{USpt}}^{\mathrm{G}}$, then both $\mathcal{X}'$ and $\mathcal{X}''$ belong to $\mathbf{Spt}^{\mathrm{G}}$. Moreover, the natural maps $\mathcal{X}'' \to \tilde{\mathrm{U}}(\mathcal{X})$ and $\tilde{\mathrm{U}}(\mathcal{X}) \to \tilde{\mathcal{X}}''$ both belong to $\mathbf{Spt}^{\mathrm{G}}$.*

Proof. Recall that the linear algebraic group G acts on a simplicial presheaf section-wise. Therefore, the functoriality of the cofibrant and fibrant replacements show readily, as in the proof of Proposition 3.1, that the cofibrant and fibrant replacements then inherit G-actions making them belong to $\mathbf{Spt}^{\mathrm{G}}$. □

3.3.13. Derived functors of $\wedge$, the internal $\mathcal{H}om$ and the dual D for equivariant spectra. Recall that the functor $\tilde{\mathrm{U}} : \mathbf{Spt}^{\mathrm{G}} \to \widetilde{\mathbf{USpt}}^{\mathrm{G}}$ is a strict monoidal functor. Let $\mathrm{M}, \mathrm{N} \in \mathbf{Spt}^{\mathrm{G}}$. The fact that one may find functorial cofibrant and fibrant replacements of objects in $\widetilde{\mathbf{USpt}}^{\mathrm{G}}$ shows that one may find a functorial cofibrant replacement $\mathrm{M}'' \to \tilde{\mathrm{U}}(\mathrm{M})$ in $\widetilde{\mathbf{USpt}}^{\mathrm{G}}$ and a functorial fibrant replacement $\tilde{\mathrm{U}}(\mathrm{N}) \to \mathrm{N}'$ in $\widetilde{\mathbf{USpt}}^{\mathrm{G}}$. The functoriality of the cofibrant and fibrant replacements, shows as in Proposition 3.14 that in fact M'', N' and the maps $\mathrm{M}'' \to \mathrm{M}$, $\mathrm{N} \to \mathrm{N}'$ all belong to $\mathbf{Spt}^{\mathrm{G}}$. Therefore, it is possible to define

$$(3.3.14) \qquad \mathrm{M}\overset{\mathrm{L}}{\wedge}\mathrm{N} = \tilde{\mathrm{U}}(\mathrm{M}'') \wedge \tilde{\mathrm{U}}(\mathrm{N}), \quad \mathcal{RH}\mathrm{om}(\mathrm{M}, \mathrm{N}) = \mathcal{H}\mathrm{om}(\tilde{\mathrm{U}}(\mathrm{M}''), \tilde{\mathrm{U}}(\mathrm{N}')), \quad \mathrm{D}(\mathrm{M}) = \mathcal{RH}\mathrm{om}(\tilde{\mathrm{U}}(\mathrm{M}), \tilde{\mathrm{U}}(\mathbb{S}^{\mathrm{G}}))$$

with $\mathrm{M}\overset{\mathrm{L}}{\wedge}\mathrm{N}, \mathcal{RH}\mathrm{om}(\mathrm{M}, \mathrm{N}), \mathrm{D}(\mathrm{M}) \in \mathbf{Spt}^{\mathrm{G}}$. (In fact, since we choose to work with the injective model structures, every object is cofibrant and therefore there is no need for any cofibrant replacements.) Similar conclusions will hold when $\mathcal{E}^{\mathrm{G}} \in \mathbf{Spt}^{\mathrm{G}}$ is a commutative ring spectrum with the corresponding smash product $\wedge_{\mathcal{E}^{\mathrm{G}}}$ and $\mathcal{H}om_{\mathcal{E}^{\mathrm{G}}}$ defined in (3.2.6). (In this case the dual with respect to the ring spectrum $\mathcal{E}$ will denoted $\mathrm{D}_{\mathcal{E}}$.)

We let $\bar{\mathrm{S}} = \mathrm{S} \underset{\mathrm{Spec}\,\mathrm{k}}{\times} \mathrm{Spec}\,\bar{\mathrm{k}}$, where $\bar{k}$ denotes an algebraic closure of k. For a given topology ? = Nis, et, we will let $\widetilde{\mathbf{USpt}}/\mathrm{S}^{\mathrm{G}}_{?}$ ($\widetilde{\mathbf{USpt}}/\bar{S}^{\mathrm{G}}_{?}$) denote the corresponding category of spectra on the site ?. Then we have the following maps of sites: $\epsilon : \mathrm{Sm}/\mathrm{S}_{\mathrm{et}} \to \mathrm{Sm}/\mathrm{S}_{\mathrm{Nis}}$, $\bar{\epsilon} : \mathrm{Sm}/\bar{\mathrm{S}}_{\mathrm{et}} \to \mathrm{Sm}/\bar{\mathrm{S}}_{\mathrm{Nis}}$ and $\eta : \mathrm{Sm}/\bar{\mathrm{S}}_{\mathrm{et}} \to \mathrm{Sm}/\mathrm{S}_{\mathrm{et}}$.

These induce the following maps of topoi:

$$(3.3.15) \quad \epsilon^* : \widetilde{\mathbf{USpt}}^{\mathrm{G}}/\mathrm{S}_{Nis} \to \widetilde{\mathbf{USpt}}^{\mathrm{G}}/\mathrm{S}_{et}, \bar{\epsilon}^* : \widetilde{\mathbf{USpt}}^{\mathrm{G}}/\bar{\mathrm{S}}_{Nis} \to \widetilde{\mathbf{USpt}}^{\mathrm{G}}/\bar{\mathrm{S}}_{et} \text{ and } \eta^* : \widetilde{\mathbf{USpt}}^{\mathrm{G}}/\mathrm{S}_{et} \to \widetilde{\mathbf{USpt}}^{\mathrm{G}}/\bar{\mathrm{S}}_{et}.$$

Definition 3.15. We will let $\widetilde{\mathbf{USpt}}^{\mathrm{G}}_{mot}$ denote $\widetilde{\mathbf{USpt}}^{\mathrm{G}}/\mathrm{S}_{Nis}$. (As observed earlier, when $\mathrm{S} = \mathrm{Spec}\,\mathrm{k}$, and the group G is trivial, the corresponding homotopy category is often denoted $\mathcal{SH}(k)$ in the literature.) If $\mathcal{E}^{\mathrm{G}}$ a commutative ring spectrum in $\widetilde{\mathbf{USpt}}^{\mathrm{G}}_{mot}$, then $\widetilde{\mathbf{USpt}}^{\mathrm{G}}_{mot,\mathcal{E}^{\mathrm{G}}}$ will denote the subcategory of $\widetilde{\mathbf{USpt}}^{\mathrm{G}}_{mot}$ consisting of module spectra over $\mathcal{E}^{\mathrm{G}}$ and where the monoidal structure is given by $\wedge_{\mathcal{E}^{\mathrm{G}}}$. $\widetilde{\mathbf{USpt}}^{\mathrm{G}}_{et}$ and $\widetilde{\mathbf{USpt}}^{\mathrm{G}}_{et,\mathcal{E}^{\mathrm{G}}}$ will denote the corresponding categories of spectra defined on the étale site of the base scheme S.

We summarize the main properties of the category $\widetilde{\mathbf{USpt}}^{\mathrm{G}}/\mathrm{S}$ in the following proposition, which should follow readily from Proposition 3.2. Therefore, we skip its proof.

Proposition 3.16. *(i) $\widetilde{\mathbf{USpt}}^{\mathrm{G}}/\mathrm{S}$ with the above smash product $\wedge^{\mathrm{S}}$ is a closed symmetric monoidal model category. (ii) $\mathbb{S}^{\mathrm{G}}$ is the unit of the monoidal structure $\wedge$ in $\widetilde{\mathbf{USpt}}^{\mathrm{G}}/\mathrm{S}$. (iii) The maps $\epsilon^*, \bar{\epsilon}^*$ and η^* are compatible with the monoidal structures on the above topoi. For $\mathcal{E}^{\mathrm{G}}$ as above, the corresponding results also hold for the category $\widetilde{\mathbf{USpt}}/\mathrm{S}^{\mathrm{G}}_{\mathcal{E}^{\mathrm{G}}}$.*

We conclude this section with the following result, which may be easily proven using the following observations. First one needs to observe that the topological space $\mathbb{P}^1_{\mathbb{C}}$ identifies with the topological space S^2. Over an algebraically closed field $\bar{k}$ of positive characteristic p, one uses $\mathbb{P}^1_{W(\bar{k})}$ (which is the $\mathbb{P}^1$ defined over the ring of Witt vectors of $\bar{k}$) to compare $\mathbb{P}^1_{\bar{k}}$ with $\mathbb{P}^1_{\mathbb{C}}$.

Proposition 3.17. *Let the base scheme $\mathrm{S} = \mathrm{Spec}\,\mathrm{k}$ for a field k with $char(k) = p \geq 0$. Let ℓ denote a prime different from p and let M denote a spectrum in $\mathbf{Spt}_{\mathrm{mot}}$ so that all the homotopy groups $[\mathrm{S}^{1^{\wedge s}} \wedge \mathbf{T}^t \wedge \Sigma_{\mathbf{T}} X_+, \mathrm{M}]$ (as X varies in the site) are ℓ-primary torsion. We will further assume that k has finite ℓ-cohomological dimension. (Here $[\mathrm{S}^{1^{\wedge s}} \wedge \mathbf{T}^t \wedge \Sigma_{\mathbf{T}} X_+, \mathrm{M}]$ denotes Hom in the stable homotopy category $\mathbf{HSpt}_{\mathrm{mot}}$.)*

Then $\eta^*\epsilon^*(\mathrm{M})$ is an S^2-spectrum (i.e. the structure maps defined by smashing with S^2) in $\mathbf{Spt}_{\mathrm{et}}$ whose homotopy groups are also ℓ-primary torsion. In particular, if $\mathrm{M} = \mathbb{H}(\mathbb{Z}/\ell^{\mathrm{n}})$ is the motivic Eilenberg-Maclane spectrum, then

$\eta^*\epsilon^*(\mathrm{M}) = \mathbb{H}(\mathbb{Z}/\ell^{\mathrm{n}})_{\mathrm{et}}$, *which is the corresponding Eilenberg-Maclane spectrum on the étale site with the structure maps defined by smashing with* S^2.

Remark 3.18. The difference between using S^2 for suspensions instead of the usual S^1 is insignificant for objects of $\mathbf{Spt}_{\mathrm{et}}$, since for a given spectrum $\mathrm{M} = \{\mathrm{M_n}|\mathrm{n} \geq 0\}$, $\{\mathrm{M_{2n}}|\mathrm{n} \geq 0\}$ is co-final in $\{\mathrm{M_n}|\mathrm{n} \geq 0\}$.

4. Cellular objects in $\mathbf{Spt}_{\mathrm{mot}}$ and $\mathbf{Spt}_{\mathrm{et}}$, as well as suspension spectra of linear schemes and mixed Tate motives

The notion of cellular objects in the motivic homotopy category has been discussed in [DI05]. On the other hand there is also the notion of linear varieties considered in an early paper of Totaro (see [Tot99]) and also by the second author in [J01]. Finally the category of schemes that are mixed Tate is well-known. *Since the most basic dualizable objects are the finite cellular objects, and as this notion seems to vary depending on the framework (see, for example, the Remark 4.4 below), it seems important to begin with a brief discussion comparing the above notions, which we will do presently.* We will assume throughout this section that the (base) scheme $\mathrm{B} = \mathrm{S}$ is a fixed perfect field k: this is mainly for simplicity. Let $\mathrm{S} = \mathrm{Spec}\,\mathrm{k}$. Let $\epsilon : \mathrm{Sm}/\mathrm{S}_{\mathrm{et}} \to \mathrm{Sm}/\mathrm{S}_{\mathrm{Nis}}$ denote the obvious map of sites and let ϵ^* denote the induced map $\mathbf{Spt}/\mathrm{S}_{\mathrm{Nis}} \to \mathbf{Spt}/\mathrm{S}_{\mathrm{et}}$ of topoi. We will, in fact, let ϵ^* denote the corresponding left derived functor $L\epsilon^*$, which is defined as the composition of a functorial cofibrant replacement functor followed by the functor ϵ^*.

Definition 4.1. Let (Motivic Cells) $= \{\Sigma_{\mathbf{T}}\mathrm{S}^{1^{\wedge s}} \wedge (\mathbb{G}_m)^{\wedge^t}|s,t \geq 0\}$. Then the class of *motivic cellular-objects* in $\mathbf{Spt}_{\mathrm{mot}}$ is the smallest class of objects in $\mathbf{Spt}_{\mathrm{mot}}$ so that (i) it contains (Motivic Cells) (ii) if $\mathcal{X}$ is weakly-equivalent in $\mathbf{Spt}_{\mathrm{mot}}$ to a motivic cellular object, then $\mathcal{X}$ is a motivic cellular object and (iii) if $\{\mathcal{X}_i|i \in I\}$ is a collection of motivic cellular objects indexed by a small category I, then $\underset{I}{\mathrm{hocolim}}\mathcal{X}_i$ is also motivically cellular.

One defines (Étale Cells) similarly by replacing the objects in (Motivic Cells) by the objects in $\mathbf{Spt}_{\mathrm{et}}$ obtained by applying $L\epsilon^*$ to them. One defines the class of *étale cellular objects* similarly using Étale Cells in the place of Motivic Cells. The class of *Mixed Tate motives* is defined similar to motivic cellular objects, but using $\mathbf{T}$ in the place of $\mathbb{G}_m$. We will often use the notation (Cells) to denote either (Motivic Cells) or (Étale Cells).

If $\mathcal{E}$ is a commutative ring spectrum in $\mathbf{Spt}_{\mathrm{mot}}$ ($\mathbf{Spt}_{\mathrm{et}}$) one defines $\mathcal{E}-$Cells to be $\{\{\mathcal{E}\wedge\Sigma_{\mathbf{T}}\mathrm{S}^{1^{\wedge s}}\wedge(\mathbb{G}_m)^{\wedge^t}|s,t \geq 0\}$. Then one defines the class of *$\mathcal{E}$-cellular objects* and *$\mathcal{E}$-Mixed Tate motives* in $\mathbf{Spt}_{\mathrm{mot},\mathcal{E}}$ ($\mathbf{Spt}_{\mathrm{et,E}}$) in a similar manner.

In the above situation, the full subcategory of *finite cellular objects* or equivalently *finite* $\mathbf{T}$*-spectra* is the smallest class of objects in $\mathbf{Spt}_{\mathrm{mot}}$ ($\mathbf{Spt}_{\mathrm{et}}$) containing (Cells) with the following properties: (i) it is closed under finite sums (ii) if $\mathcal{X}$ is weakly-equivalent to a finite cellular object, then $\mathcal{X}$ is a finite cellular object and (iii) it is closed under finite homotopy pushouts.

If $p \geq q \geq 0$, then we let $\mathrm{S^{p,q}} = (\mathrm{S}^1)^{\wedge^{\mathrm{p-q}}} \wedge (\mathbb{G}_{\mathrm{m}})^{\wedge^{\mathrm{q}}}$. Given any pair of integers $a, b \in Z$, one may choose $p = a + 2t$, $q = b + t$, with t a non-negative integer so that $2t \geq -a$, $t \geq -b$ and $a + t \geq b$ (i.e $t \geq b - a$).

Definition 4.2. If $p \geq q \geq 0$, then we let $\Sigma_{\mathbf{T}}^{pq} = \Sigma_{\mathbf{T}}\mathrm{S^{p,q}}$. If $a, b \in Z$ are any pair of integers and t, p, q are chosen as above we let $\Sigma_{\mathbf{T}}^{a,b} = \Sigma_{\mathbf{T}}^{-t}\mathrm{S^{p,q}}$.

Proposition 4.3. *(i) Any spectrum of the form* $\Sigma_{\mathbf{T}}^{a,b}$, $a, b \in Z$ *may be obtained by* $\mathbf{T}$*-suspensions and de-suspensions of objects in* Motivic Cells.

(ii) The full triangulated subcategory of $\mathbf{HSpt}/\mathrm{S}$ *generated by the Mixed Tate objects and closed under infinite co-products identifies with the triangulated full subcategory generated by the Motivic Cellular objects.*

Proof. (i) is clear. (ii) follows since $\mathbf{T}$ identifies with $\mathrm{S}^1 \wedge \mathbb{G}_{\mathrm{m}}$ in $\mathbf{HSpt}$. □

One may readily verify the following properties of cellular objects: (i) If $\mathcal{X}, \mathcal{Y} \in \mathbf{Spt}_{\mathrm{mot}}$ ($\mathbf{Spt}_{\mathrm{et}}$) are cellular, then so is $\mathcal{X} \wedge \mathcal{Y}$ and $\mathcal{X} \times \mathcal{Y}$, (ii) If $\{\mathcal{X}_i|i \in I\}$ is a family with each $\mathcal{X}_i$ cellular, then so is $\bigvee_i \mathcal{X}_i$, which is the co-product of the $\mathcal{X}_i$ and (iii) The suspension-spectra $\Sigma_{\mathbf{T}}^{s+2n,n}(\mathbb{A}_{\mathrm{k}}^m)_+$, $\Sigma_{\mathbf{T}}^{s+2n,n}(\mathbb{A}_{\mathrm{k}}^m - 0)_+$ are cellular. One may consult [DI05, sections 3 and 4] for some of the remaining properties of cellular objects.

Remark 4.4. The following needs to be pointed out. In general, principal G-bundles for linear algebraic groups are locally trivial only in the étale topology. Special cases where such principal bundles are trivial in the Zariski topology (and hence the Nisnevich topology) are when the group is *special*, for example, is a $\mathrm{GL_n}$. Finite groups are known to be *not* special. When a G-principal bundle $\mathrm{p} : \mathrm{E} \to \mathrm{B}$ is locally trivial for the Zariski or Nisnevich topology, one may prove readily that if E is cellular, then so is B. (See [DI05, Proposition 4.3].) The proof is straightforward, but applies largely to the case where G is special. When the principal bundle is only locally trivial for the étale topology, as most often happens, one needs to use the observation that p is smooth and surjective, so that there is an étale cover $B' \to B$ so that p trivializes on the étale cover B'. This observation applies to principal bundles for most groups G, including finite groups.

On the other hand, the *linear schemes* considered in [Tot99] and [J01] are schemes that are built out of affine spaces and tori and are particularly simple objects to study in the setting of algebraic geometry: see [J01]. Though it is not immediately apparent, these two notions are closely related as shown below in Proposition 4.7.

Definition 4.5. (Linear and special linear schemes) A scheme X over k is *linear* if it has a finite filtration $F_0 \subseteq F_1 \subseteq \cdots F_{n-1} \subseteq F_n = X$ by closed sub-schemes so that there exists a sequence $m_0, m_1, \ldots, m_n$, $a_0, a_1, \ldots, a_n$ of non-negative integers with each $F_i - F_{i-1} = \sqcup \mathbb{A}^{a_i}_k \times \mathbb{G}^{m_i}_{m,k}$ which is a finite disjoint union of products of affine spaces and split tori isomorphic to $\mathbb{A}^{a_i}_k \times \mathbb{G}^{m_i}_{m,k}$. Moreover, we require that each of the above summands is a connected component of $F_i - F_{i-1}$. We call $\{F_i | i\}$ a linear filtration.

A linear scheme where the strata $F_i - F_{i-1}$ are all isomorphic to disjoint unions of affine spaces will be called a *a special linear scheme.*

Remark 4.6. Often special linear schemes are called cellular schemes in the literature. We prefer to use the term *special linear schemes* so as not to conflict with the notion of *a cellular object* in $\mathbf{Spt}_{mot}$ and $\mathbf{Spt}_{et}$.

Proposition 4.7. *The* $\mathbf{T}$ *- suspension spectrum of any smooth linear scheme (of finite type over* k*) is a finite cellular object in* $\mathbf{Spt}_{mot}$ *and in* $\mathbf{Spt}_{et}$*.*

Proof. Let $\phi = F_{-1} \subseteq F_0 \subseteq F_1 \subseteq \cdots \subseteq F_{n-1} \subseteq F_n = X$ denote the given filtration so that each $F_i - F_{i-1} = \sqcup \mathbb{A}^{m_i} \times \mathbb{G}^{n_i}_m$, which is a finite sum of products of affine spaces and split tori. The proof is to show by descending induction on i that the suspension spectra $\Sigma_{\mathbf{T}}(X - F_i)$ are all finite cellular. We skip the remaining details. □

5. Further Examples of Dualizable objects and Proof of Theorem 1.3

This section will be devoted to a rather detailed discussion of many examples of dualizable objects in the motivic and étale stable homotopy category. As Theorem 1.3 shows, knowing precisely which objects are dualizable in which context and that the functors of étale realization and base-change to separably closed base fields preserve dualizability is of key importance for the paper. It perhaps needs to be pointed out that [DP84, 2.2 Theorem and 2.4 Corollary] seem to provide a quick proof that the realization functors ought to preserve the notion of Spanier-Whitehead duality, while ignoring various subtleties that show up, such as in Lemma 5.3, Proposition 5.6 or Theorem 5.7. Therefore, we believe it is both necessary and worthwhile to carefully re-examine the proof that an object is dualizable, culminating in the proof of Proposition 5.9 and Theorem 1.3.

We will continue to work in the framework of the stable categories $\mathbf{Spt}_{mot}$ and $\mathbf{Spt}_{et}$. As in the last section, we will continue to assume that the base scheme $B = S = \mathrm{Spec}\, k$ is a perfect field. In the étale framework, we will further assume that k satisfies the finiteness hypotheses in (3.0.3) and that if $\mathcal{E}$ denotes a commutative ring spectrum in $\mathbf{Spt}_{et}$ it is either ℓ-complete or has homotopy groups that are ℓ-primary torsion for some prime $\ell \neq char(k)$. Throughout this section, $\wedge$, $\mathcal{H}om$ and for commutative ring spectra $\mathcal{E}$, $\wedge_{\mathcal{E}}$ and $\mathcal{H}om_{\mathcal{E}}$ will denote their corresponding derived versions. We begin with the following result.

Proposition 5.1. *(i) If* $\mathcal{E}$ *is a commutative ring spectrum in* $\mathbf{Spt}_{mot}$ *and* M *is a finite* $\mathcal{E}$*-cell module spectrum, then* M *is dualizable in* $\mathbf{Spt}_{mot,\mathcal{E}}$*.*

(ii) If M *is an* $\mathcal{E}$*-module spectrum that is dualizable, so is any retract of* M*.*

(iii) Assume the base field k satisfies the finiteness hypotheses in (3.0.3)*. If* $\mathcal{E}$ *is a commutative ring spectrum in* $\mathbf{Spt}_{et}$ *which is* ℓ*-complete for some prime* $\ell \neq char(k)$*, the corresponding results hold for any finite* $\mathcal{E}$*-cell module.*

Proof. We begin with the following observation: in order to prove that an $M \in \mathbf{Spt}_{mot,\mathcal{E}}$ ($M \in \mathbf{Spt}_{et,\mathcal{E}}$) is dualizable, making use of Corollary 2.4, it suffices to prove that the natural maps

$$(a)\, F \wedge_{\mathcal{E}} \mathcal{H}om_{\mathcal{E}}(M, E) \to \mathcal{H}om_{\mathcal{E}}(M, F),\ (b)\, M \to D_{\mathcal{E}}(D_{\mathcal{E}}(M)),\ F \in \mathbf{Spt}_{mot,\mathcal{E}}\ (F \in \mathbf{Spt}_{et,\mathcal{E}},\ \text{respectively})$$

are weak-equivalences for all F as above. We first consider (i). When $M = \Sigma^t_{\mathbf{T}}\mathcal{E}$, the weak-equivalence in (a) is clear. It is also clear when $M = S^s \wedge \Sigma^t_{\mathbf{T}}\mathcal{E}$, since then the left-hand-side of (a) identifies with $F \wedge_{\mathcal{E}} \Omega^s_{S^1}\Sigma^{-t}_{\mathbf{T}}\mathcal{E}$ and the right-hand-side identifies with $\Omega^s_{S^1}\Sigma^{-t}_{\mathbf{T}}F$. The spectral sequence computing the derived smash product on the left-hand-side of (a) will degenerate then showing that the map in (a) is indeed a weak-equivalence. Now the weak-equivalence in (a) follows by induction on the number of E-cells in M. (b) may be proven similarly. It is clear that these arguments work in both $\mathbf{Spt}_{mot}$ and $\mathbf{Spt}_{et}$.

We will skip the proof of (ii) as it follows from routine arguments. As observed above, the proofs of (i) and (ii) extend to the étale case thereby proving (iii). Nevertheless we provide here a few comments clarifying the proofs in the étale case. It is important here that we invert $\mathbb{A}^1$ in $\mathbf{Spt}_{\mathrm{et}}$. Having inverted $\mathbb{A}^1$, one may readily prove as in Theorem 10.15 that if X is a smooth scheme over a perfect field k satisfying the hypothesis in (3.0.3) that has a finite stratification by strata that are affine spaces over k, then $\mathcal{E} \wedge X_+$ is dualizable in $\mathbf{Spt}_{\mathrm{et},\mathcal{E}}$. To complete the proof for all finite cell spectra over $\mathcal{E}$, all one needs to show is that $\mathcal{E} \wedge \mathbb{G}_{m,+}$ is also dualizable in $\mathbf{Spt}_{\mathrm{et},\mathcal{E}}$. This follows now from the fact that $\mathcal{E} \wedge \mathbb{A}^1_+$ is dualizable in $\mathbf{Spt}_{\mathrm{et},\mathcal{E}}$ and the observation that $\mathbb{A}^1 - \mathbb{G}_m = \{0\}$. □

Lemma 5.2. *Let $\mathcal{E}$ denote a commutative ring spectrum in $\mathbf{Spt}_{\mathrm{mot}}$ (in $\mathbf{Spt}_{\mathrm{et}}$) and let $\mathbf{Spt}_{\mathrm{mot},\mathcal{E}}$ ($\mathbf{Spt}_{\mathrm{et},\mathcal{E}}$) denote the subcategory of $\mathcal{E}$-module spectra.*

(i) Let $\mathcal{E}'$ denote a commutative ring spectrum in $\mathbf{Spt}_{\mathrm{mot}}$ ($\mathbf{Spt}_{\mathrm{et}}$) provided with a map of ring spectra $\mathcal{E} \to \mathcal{E}'$. If $\mathrm{M} \in \mathbf{Spt}_{\mathrm{mot},\mathcal{E}}$ ($\mathrm{M} \in \mathbf{Spt}_{\mathrm{et},\mathcal{E}}$) is dualizable, then $\mathrm{M} \wedge_{\mathcal{E}} \mathcal{E}'$ is dualizable in $\mathbf{Spt}_{\mathrm{mot},\mathcal{E}'}$ ($\mathbf{Spt}_{\mathrm{et},\mathcal{E}'}$, respectively).

(ii) For a fixed prime ℓ and $\nu > 0$, let $\mathcal{E}(\ell^\nu)$ denote the homotopy cofiber of the map $\mathcal{E} \overset{\ell^\nu}{\to} \mathcal{E}$. If $\mathrm{M} \in \mathbf{Spt}_{\mathrm{mot},\mathcal{E}}$ ($\mathbf{Spt}_{\mathrm{et},\mathcal{E}}$) is dualizable, then so is $\mathrm{M} \wedge_{\mathcal{E}} \mathcal{E}(\ell^\nu)$.

Proof. We skip the proofs, since both statements may be proved readily. □

The following result shows how the hypothesis (3.0.3) ensures compactness in the étale setting.

Lemma 5.3. *Let k denote a field of finite ℓ-cohomological dimension for some prime $\ell \neq char(k)$. If X is any scheme of finite type over k and $\mathcal{E}$ is a spectrum in $\mathbf{Spt}_{\mathrm{et}}$ which is ℓ-complete, then $\mathcal{E} \wedge \mathrm{X}_+$ is a compact object in $\mathbf{HSpt}_{\mathrm{et},\mathcal{E}}$.*

Proof. Let $RHom_{\mathcal{E}}$ denote the external hom in $\mathbf{HSpt}_{\mathrm{et,E}}$. Then it suffices to prove that if $\{E_\alpha|\alpha\}$ denotes a small filtered direct system of spectra in $\mathbf{Spt}_{\mathrm{et,E}}$, then one obtains an isomorphism:

$$\lim_{\to\,\alpha} RHom_{\mathcal{E}}(\mathcal{E} \wedge \mathrm{X}_+, \mathrm{E}_\alpha) \cong \mathrm{RHom}_{\mathcal{E}}(\mathcal{E} \wedge \mathrm{X}_+, \lim_{\to\,\alpha} \mathrm{E}_\alpha).$$

This readily reduces to showing that one has a weak-equivalence of spectra

$$\lim_{\to\,\alpha} \mathbb{H}(\mathrm{X}, \mathrm{E}_\alpha) \simeq \mathbb{H}(\mathrm{X}, \lim_{\to\,\alpha} \mathrm{E}_\alpha).$$

Here $\mathbb{H}(\mathrm{X},\mathrm{F})$ denotes the hypercohomology spectrum defined as the homotopy inverse limit of the cosimplicial spectrum $\{\Gamma(\mathrm{X}, \mathrm{G^nF})|\mathrm{n}\}$ where $\{\mathrm{G^nF}|\mathrm{n}\}$ denotes the canonical Godement resolution. This follows readily from the assumption that k has finite étale ℓ-cohomological dimension and that $\mathcal{E}$ is a spectrum which is ℓ-complete. Since $\mathcal{E}_\alpha$ are module spectra over $\mathcal{E}$, it follows that the homotopy presheaves of the spectra $\mathcal{E}_\alpha$ are also ℓ-complete, i.e. are modules over $Z\widehat{\ }_\ell$. Therefore, the spectral sequences that compute the generalized cohomology,

$$\mathrm{E}_2^{\mathrm{s,t}} = \mathrm{H}^{\mathrm{s}}_{\mathrm{et}}(\mathrm{X}, \pi_{-\mathrm{t}}(\mathcal{E}_\alpha)) \Rightarrow \pi_{-\mathrm{s}-\mathrm{t}}\mathbb{H}(\mathrm{X}, \mathcal{E}_\alpha)$$

converge strongly for each α, and so does the colimit spectral sequence with respect to the spectrum $\lim_{\to\,\alpha} \mathrm{E}_\alpha$. To see this, observe that since the homotopy sheaves $\pi_{-t}(\mathrm{E}_\alpha)$ are modules over $Z\widehat{\ }_\ell$, the only torsion in the E_2-terms are ℓ-primary torsion. But X has finite ℓ-cohomological dimension, in view of the assumptions, so that there exists an integer $N > 0$ (independent of α) for which $\mathrm{E}_2^{\mathrm{s,t}} = 0$ for all $s > N$.

Finally, since X is a Noetherian scheme, étale cohomology of the scheme X commutes with respect to filtered colimits of abelian sheaves. □

Proposition 5.4. *(i) The $\mathbf{T}$-suspension spectrum of any projective smooth scheme over any affine base scheme S is dualizable in $\mathbf{Spt}_{\mathrm{mot}}$. In fact if X is such scheme over the affine base scheme S, there exists a vector bundle ν on X so that the $\mathbf{T}$-suspension spectrum of the Thom-space of ν is the dual of $\Sigma^m_{\mathbf{T}} X_+$ for some positive integer m. Any vector bundle or affine space bundle over such a projective smooth scheme is also dualizable.*

(ii) If X is a smooth quasi-projective variety over a field of arbitrary characteristic which admits an open immersion into a smooth projective variety so that the complement is a divisor with strict normal crossings, then the $\mathbf{T}$-suspension spectrum of X is dualizable in $\mathbf{Spt}_{\mathrm{mot}}$.

(iii) If the base scheme S is the spectrum of a field of characteristic 0, then the $\mathbf{T}$-suspension spectrum of any quasi-projective variety over S is dualizable in $\mathbf{Spt}_{\mathrm{mot}}$.

(iv) Let $\mathcal{E}$ denote a commutative ring spectrum in $\mathbf{Spt}_{\mathrm{mot}}$. If X denotes any of the schemes appearing in (i) through (iii), then $\mathcal{E} \wedge X_+$ is dualizable in $\mathbf{Spt}_{\mathrm{mot},\mathcal{E}}$. In particular this applies to $\mathcal{E} = \Sigma_{\mathbf{T}}[p^{-1}]$, where $p > 0$ is the characteristic of the base field, $\Sigma_{\mathbf{T},(\ell)}$ and $\widehat{\Sigma}_{\mathbf{T},\ell}$, where $\Sigma_{\mathbf{T},(\ell)}$ ($\widehat{\Sigma}_{\mathbf{T},\ell}$) denotes the localization of $\Sigma_{\mathbf{T}}$ at the prime

ideal (ℓ) *in* Z *(the* ℓ*-completion of* $\Sigma_{\mathbf{T}}$ *as discussed, for example in* [CJ14, section 4]*, respectively). Here* ℓ *is a prime different from the characteristic of the base field.*

(v) Assume the base scheme S *is a perfect field satisfying the hypothesis* (3.0.3)*. Let* $\mathcal{E} \in \mathbf{Spt}_{\mathrm{mot}}$ *denote a commutative ring spectrum whose homotopy groups are* ℓ*-primary torsion, for some prime* $\ell \neq char(k)$ *and let* X *denote any of the schemes appearing in (i) through (iii). If* $\epsilon : \mathbf{Spt}_{\mathrm{mot}} \rightarrow \mathbf{Spt}_{\mathrm{et}}$ *denotes the functor induced by pull-back to the étale site, then* $\epsilon^*(\Sigma_{\mathbf{T}}\mathcal{E} \wedge X_+) \simeq \epsilon^*(\mathcal{E}) \wedge_{\epsilon^*(\Sigma_{\mathbf{T}})} \epsilon^*(\Sigma_{\mathbf{T}}X_+)$ *is dualizable in* $\mathbf{Spt}_{\mathrm{et},\epsilon^*(\mathcal{E})}$.

Proof. The first statement is now well-known in the motivic case and is deduced from Atiyah style duality. This seems to appear originally in [Hu-Kr05], [Ri05], [Ay]. A key idea of the proof is to first prove this for all projective spaces by using ascending induction on their dimension, the case when it is of dimension 0 being that of *Spec k* and then deduce this for all smooth closed projective subvarieties. We have re-worked this to clarify the proof and also to show that it extends to the étale case as well. This is discussed in the appendix. The construction of the motivic Spanier-Whitehead dual in terms of the Thom-spaces as in Theorem 10.14 shows that there exists an algebraic vector bundle ν_{X} over X and a non-negative integer m so that one obtains the co-evaluation and evaluation maps

$$c_X : \Sigma_{\mathbf{T}} \rightarrow \Sigma_{\mathbf{T}}\mathrm{X}_+ \wedge \Sigma_{\mathbf{T}}^{-\mathrm{m}}\mathrm{Th}(\nu_{\mathrm{X}}), \quad \mathrm{e}_{\mathrm{X}} : \Sigma_{\mathbf{T}}^{-\mathrm{m}}\mathrm{Th}(\nu_{\mathrm{X}}) \wedge \Sigma_{\mathbf{T}}\mathrm{X}_+ \rightarrow \Sigma_{\mathbf{T}} \tag{5.0.16}$$

satisfying the hypothesis in Theorem 2.3(ii) with $\mathrm{D}'\mathcal{X}$ replaced by $\Sigma_{\mathbf{T}}^{-m}\mathrm{Th}(\nu_{\mathrm{X}})$ and $\mathcal{X} = \Sigma_{\mathbf{T}}\mathrm{X}_+$. These complete the proof of the first statement. One may also want to observe that a corresponding statement, for the étale case, when the base field is algebraically closed was worked out in detail in the second author's Ph. D thesis and appears in [J86] and [J87], making use of a theory of étale tubular neighborhoods.

The second statement then follows from the first by ascending induction on the number of irreducible components of the normal crossings divisor that is the complement, and the following argument. Assume X is provided with an open immersion into a smooth projective variety $\tilde{\mathrm{X}}$ with complement Y which is also smooth. Then, if the normal bundle associated to the closed immersion $\mathrm{Y} \rightarrow \tilde{\mathrm{X}}$ is N, the homotopy purity theorem (see [MV99, Theorem 3.2.33]) provides the stable cofiber sequence:

$$\mathrm{X} \rightarrow \tilde{\mathrm{X}} \rightarrow \tilde{\mathrm{X}}/\mathrm{X} \simeq \mathrm{P}(\mathrm{N} \oplus 1)/\mathrm{P}(\mathrm{N}).$$

Since $\mathrm{P}(\mathrm{N} \oplus 1)$ and $\mathrm{P}(\mathrm{N})$ are projective and smooth, they are dualizable and hence so is $\tilde{\mathrm{X}}/\mathrm{X}$. Since $\tilde{\mathrm{X}}$ is projective and smooth, it is dualizable and therefore, by Proposition 2.5, it follows that X is also dualizable.

The same argument as in (ii) now applies to prove (iii), since by invoking strong resolution of singularities one may provide X with an open immersion into a projective smooth variety $\tilde{X}$ so that the complement of X is a divisor with strict normal crossings. Lemma 5.2(i) with $\mathcal{E}$ there denoting the motivic sphere spectrum $\Sigma_{\mathbf{T}}$ and $\mathcal{E}'$ denoting the given commutative ring spectrum $\mathcal{E}$ in (iv) then proves the statement in (iv). Observe that, when X is a projective smooth variety, taking the smash product of the co-evaluation and evaluation maps in (5.0.16) with the commutative ring spectrum $\mathcal{E}$ provides the required co-evaluation and evaluation maps for $\mathcal{E} \wedge X_+$.

Clearly it suffices to prove (v) for projective smooth schemes X over k, since the same arguments as above apply to extend this to the other cases. On taking the smash product with the motivic ring spectrum $\mathcal{E}$ and applying ϵ^* to the co-evaluation and evaluation maps in (5.0.16), one obtains the required co-evaluation and evaluation maps for $\epsilon^*(\mathcal{E} \wedge X_+) \simeq \epsilon^*(\mathcal{E}) \wedge_{\epsilon^*(\Sigma_{\mathbf{T}})} \epsilon^*(\Sigma_{\mathbf{T}}X_+)$ which clearly satisfy the hypothesis in Theorem 2.3(ii). As remarked before, étale cohomology is well-behaved only with respect to torsion coefficients away from the characteristics. This is the need to smash with a commutative ring spectrum $\mathcal{E}$ as in (v). □

The following are examples where the Propositions 5.1 and 5.4 apply.

Examples 5.5. *1. Any flag variety,* G/P *for a reductive group* G *and a parabolic subgroup* P *is a projective smooth variety. This includes as special cases all projective spaces. One may also consider projective space bundles and vector bundles over such flag varieties. (For example, one may consider varieties of the form* G/T*, where* G *is a linear algebraic group and* T *is a maximal torus.) These all satisfy the hypotheses in Proposition 5.4(i).*

2. Any split torus T *satisfies the hypothesis in Proposition 5.4(ii). If the rank of the split torus* T *is* n*, one may imbed* T *into a product* $(\mathbb{P}^1)^n$*, where each factor* $\mathbb{G}_m$ *is imbedded in the obvious manner in the corresponding factor* $\mathbb{P}^1$.

3. Let G *denote a reductive group and* H *denote a closed subgroup obtained as the fixed points of an involution, both defined over a field* k *of characteristic different from* 2*. Then the homogeneous variety* G/H *admits a wonderful compactification where the complement of* G/H *is a divisor with strict normal crossings. (See* [DeC-Sp99] *for a proof.) This includes a variety of examples: for example, one may take for the group* G, $\mathrm{G} \times \mathrm{G}$ *and the involution* σ *the automorphism that interchanges the two factors so that* H *is the diagonal copy of* G*. Another example would be* $\mathrm{G} = \mathrm{GL_n}$ *with* $\mathrm{H} = \mathrm{O_n}$ *or* $\mathrm{G} = \mathrm{SL}_{2n}$ *and* $\mathrm{H} = \mathrm{Sp}_{2\mathrm{n}}$.

4. Over a field of characteristic 0*, one may consider any homogeneous variety* G/H *for any linear algebraic group* G *and a closed subgroup* H*. Clearly such a variety is a smooth quasi-projective variety and therefore satisfies the hypotheses in Proposition 5.4(iii). Clearly this includes varieties of the form* G/N(T) *where* G *is a linear algebraic group and* N(T) *denotes the normalizer of a maximal torus in* G*.*

Proposition 5.6. *Assume the base field k satisfies the hypothesis* (3.0.3)*. Let $\mathcal{E}$ denote a commutative ring spectrum in $\mathbf{Spt}_{\mathrm{et}}$ which is ℓ-complete, for some prime $\ell \neq char(k)$. Let* G *denote a split linear algebraic group over k and let* H *denote a closed linear algebraic subgroup which is also split. Then $\mathcal{E} \wedge \mathrm{G/H}_+$ is dualizable in $\mathbf{Spt}_{\mathrm{et},\mathcal{E}}$.*

Proof. The Bruhat decomposition shows that both G and H are linear schemes. Therefore, Proposition 4.7 shows that $\Sigma_{\mathbf{T}}(\mathrm{G})_+$ and $\Sigma_{\mathbf{T}}(\mathrm{H})_+$ are finite cellular objects. Now Remark 4.4 with E, G and B there denoting G, H and G/H, respectively shows that $\mathrm{G} \to \mathrm{G/H}$ is locally trivial in the étale topology and therefore, $\mathcal{E} \wedge \mathrm{G/H}_+$ is cellular in $\mathbf{Spt}_{\mathrm{et},\mathcal{E}}$. On the other hand, G/H is clearly a smooth scheme defined over k (see: [Spr98, 12.2.1]) and since k has finite ℓ cohomological dimension, $\mathrm{G/H}_+ \wedge \mathcal{E}$ is a compact object in $\mathbf{Spt}_{\mathrm{et},\mathcal{E}}$: see Lemma 5.3 above. Therefore, [DI05, Proposition 9.4] applies to prove that $\mathcal{E} \wedge \mathrm{G/H}_+$ is a retract of a finite cellular object in $\mathbf{Spt}_{\mathrm{et},\mathcal{E}}$. (Though [DI05, Proposition 9.4] is stated in the motivic context, its proof shows that the result applies also to the étale framework since the full subcategory of $\mathbf{Spt}_{\mathrm{et},\mathcal{E}}$ consisting of cellular objects is generated by the $\mathcal{E}$-cells. Any compact object in the above full subcategory of $\mathbf{Spt}_{\mathrm{et},\mathcal{E}}$ is a retract of a finite cellular object.) Now Proposition 5.1(ii) shows it is dualizable in $\mathbf{Spt}_{\mathrm{et},\mathcal{E}}$. □

Making strong use of Gabber's refined alterations we proceed to sketch a vast generalization of the second statement in Proposition 5.4 above for quasi-projective schemes over perfect fields with respect to ring spectra $\mathcal{E}$ that are *$Z_{(\ell)}$-local* as in Definition 1.2. Though the arguments we provide below is now rather well-known (see, for example, [K13] or [HKO, 2.5]), it is necessary for us to sketch the relevant arguments in some detail, *so as to show that they indeed carry through under étale realization and change of base fields.*

Assume the base scheme is a perfect field of characteristic $p \geq 0$.

Theorem 5.7. *Let k denote a perfect field of characteristic $p \geq 0$ and let* X *denote a smooth quasi-projective scheme over k. Let ℓ denote a fixed prime different from $char(k)$ and let $\mathcal{E}$ denote a commutative motivic ring spectrum which is $Z_{(\ell)}$-local. Then $\mathcal{E} \wedge \mathrm{X}_+$ is dualizable in the category $\mathbf{Spt}_{\mathrm{mot},\mathcal{E}}$ of module spectra over $\mathcal{E}$ with the same conclusion holding with no conditions on the spectrum $\mathcal{E}$ if* X *is projective and smooth. In particular, this holds for ring spectra $\mathcal{E}$ of the form $\mathrm{K} \overset{\mathrm{L}}{\underset{\Sigma_{\mathbf{T}}}{\wedge}} \mathbb{H}(\mathbb{Z}/\ell^{\nu})$, where* K *is a commutative motivic ring spectrum, ℓ is a prime different from p and $\nu \geq 1$. Here $\mathbb{H}(\mathbb{Z}/\ell^{\nu})$ denotes the usual Z/ℓ^{ν}- motivic Eilenberg-Maclane spectrum, and $\overset{L}{\underset{\Sigma_T}{\wedge}}$ is the derived smash product.*

Proof. We will give two somewhat different proofs of this result, one of which holds only when $\mathcal{E}$ admits weak traces in the sense of [K13] and the other holds more generally making use of [Ri13]. First observe from Corollary 2.4, that the statement we want to prove is that the natural maps

$$\eta_{\mathcal{E}}^{\mathrm{X}} : \mathrm{P} \overset{\mathrm{L}}{\wedge}_{\mathcal{E}} \mathcal{H}om_{\mathcal{E}}(\mathcal{E} \wedge \mathrm{X}_+, \mathcal{E}) \to \mathcal{H}om_{\mathcal{E}}(\mathcal{E} \wedge \mathrm{X}_+, \mathrm{P}), \quad \mathcal{E} \wedge \mathrm{X}_+ \to \mathrm{D}_{\mathcal{E}}(\mathrm{D}_{\mathcal{E}}(\mathcal{E} \wedge \mathrm{X}_+)) \tag{5.0.17}$$

are weak-equivalences for every $\mathcal{E}$-module spectrum P.

In case X is projective and smooth, the results of the appendix (see Theorem 10.14) show that the Thom-space of a *virtual normal* bundle over X de-suspended a finite number of times is a (Spanier-Whitehead) dual of $\Sigma_{\mathbf{T}}\mathrm{X}_+$. Therefore, the above Thom-space de-suspended a finite number of times and smashed with $\mathcal{E}$ will be a (Spanier-Whitehead) dual of $\mathcal{E} \wedge \mathrm{X}_+$ in the category of $\mathcal{E}$-module spectra.

Let $\mathcal{SH}(k, \mathcal{E})$ denote the motivic stable homotopy category of $\mathcal{E}$-module spectra, i.e. the homotopy category associated to $\mathbf{Spt}_{\mathrm{mot},\mathcal{E}}$. Let $\mathcal{SH}_d(k, \mathcal{E})$ denote the localizing subcategory of $\mathcal{SH}(k, \mathcal{E})$ which is generated by the shifted $\mathcal{E}$-suspension spectra of smooth connected schemes of dimension $\leq d$. In general one proceeds by ascending induction on the dimension of X to prove that the maps in (5.0.17) are weak-equivalences, the case of dimension 0 reducing to the case X is projective and smooth. When X is quasi-projective of dimension d, one may assume $j : \mathrm{X} \to \mathrm{Y}$ is an open immersion in a projective scheme Y and let $\mathrm{f} : \mathrm{Y}' \to \mathrm{Y}$ denote the map given by Gabber's refined alteration so that $\mathrm{X}' = \mathrm{f}^{-1}(\mathrm{X})$ is the complement of a divisor with strict normal crossings. Let $\mathrm{U} \subseteq \mathrm{X}$ denote the open subscheme over which f restricts to an $fps\ell'$-cover $\mathrm{g} : \mathrm{V} = \mathrm{f}^{-1}(\mathrm{U}) \to \mathrm{U}$.

Since Y' is smooth and projective, $\mathcal{E} \wedge \mathrm{Y}'_+$ is dualizable in the category of $\mathcal{E}$-module spectra. By homotopy-purity, induction on the number of irreducible components of $\mathrm{Y}' - \mathrm{X}'$ and Proposition 2.5, (see also the proof of

Proposition 5.4(ii)) one observes that $\mathcal{E} \wedge X'_+$ is also dualizable in the same category. Now one considers the stable cofiber sequences:

(5.0.18) $$\mathcal{E} \wedge V_+ \rightarrow \mathcal{E} \wedge X'_+ \rightarrow \mathcal{E} \wedge X'/V, \mathcal{E} \wedge U_+ \rightarrow \mathcal{E} \wedge X_+ \rightarrow \mathcal{E} \wedge X/U.$$

By an argument as in [RO08, Lemma 66] (see also [HKO, 2.5]), both $\mathcal{E} \wedge X'/V$ and $\mathcal{E} \wedge X/U$ belong to $\mathcal{SH}_{d-1}(k, \mathcal{E})$. We will provide some details on this argument, for the convenience of the reader. In case the complement $Z = X' - V$ is also smooth, the homotopy purity Theorem [MV99, Theorem 3.2.33] shows that X'/V is weakly equivalent to the Thom-space of the normal bundle N associated to the closed immersion $Z \rightarrow X'$. Now ascending induction on the number of open sets in a Zariski open covering over which the normal bundle N trivializes will reduce it the case when N is trivial. In this case the conclusion is clear. In general, since the base field is assumed to be perfect, one can stratify Z by a finite number of locally closed subschemes that are smooth. This will give rise to a sequence of motivic spaces filtering X'/V, so that the homotopy cofiber of two successive terms will be of the form considered earlier. An entirely similar argument applies to X/U.

Therefore, by the induction hypotheses, both the maps $\eta_{\mathcal{E}}^{X'/V}$ and $\eta_{\mathcal{E}}^{X/U}$ are weak-equivalences. It follows therefore, by Proposition 2.5, that the map $\eta_V^{\mathcal{E}}$ is also a weak-equivalence. Now we make the key observation proven below that $\mathcal{E} \wedge U_+$ is a retract of $\mathcal{E} \wedge V_+$ at least when U is a sufficiently small Zariski open subscheme and that, therefore, the map $\eta_U^{\mathcal{E}}$ is also a weak-equivalence. Now the second stable cofiber sequence in (5.0.18) together with another application of Proposition 2.5 proves that the map $\eta_X^{\mathcal{E}}$ is also a weak-equivalence. One may prove the second map in (5.0.17) is a weak-equivalence by a similar argument.

It follows straight from the definition that the spectra $K \overset{L}{\underset{\Sigma_{\mathbf{T}}}{\wedge}} \mathbb{H}(\mathbb{Z}/\ell^\nu)$ and $\mathbb{H}(\mathbb{Z}/\ell^\nu)$ are $Z_{(\ell)}$-local. (Observe also that $\mathbb{H}(\mathbb{Z}/\ell^\nu)$ admits weak-traces and that $K \overset{L}{\underset{\Sigma_{\mathbf{T}}}{\wedge}} \mathbb{H}(\mathbb{Z}/\ell^\nu)$ admits weak traces when K admits weak-traces.) □

Lemma 5.8. *(i) Let* V, U *denote two smooth schemes over k and let* $g: V \rightarrow U$ *denote an $fps\ell'$-cover, where ℓ is a fixed prime different from $char(k)$. If $\mathcal{E}$ is a commutative ring spectrum which is $Z_{(\ell)}$-local and which admits weak traces as in* [K13]*, then the map $id_{\mathcal{E}} \wedge g_+ : \mathcal{E} \wedge V_+ \rightarrow \mathcal{E} \wedge U_+$ has a section.*

(ii) More generally the same conclusion holds if U *is a sufficiently small Zariski open subscheme and for any commutative motivic ring spectrum $\mathcal{E}$ that is $Z_{(\ell)}$-local.*

Proof. (i) The first observation is that it suffices to show the induced natural transformation:

$$[\mathcal{E} \wedge U_+, \quad] \overset{g^*}{\rightarrow} [\mathcal{E} \wedge V_+, \quad]$$

has a splitting, where $[K, L]$ denotes homotopy classes of maps in $\mathbf{Spt}_{mot, \mathcal{E}}$, with K cofibrant and L fibrant. Denoting the structure map $U \rightarrow \mathrm{Spec}\, k$ by a, one may identify $[\mathcal{E} \wedge U_+, F]$ ($[\mathcal{E} \wedge V_+, F]$) with $[\mathcal{E}, Ra_*a^*(F)]$ ($[\mathcal{E}, Ra_*Rg_*g^*a^*(F)]$, respectively) for any fibrant $\mathcal{E}$-module spectrum F. Since $\mathcal{E}$ has a structure of traces, so does F. Therefore, the natural map $Ra_*a^*(F) \rightarrow Ra_*Rg_*g^*a^*(F)$ has a splitting provided by the map $d^{-1}Tr(g)$, where d is the degree of the map g and Tr(g) denotes the trace associated to g.

(ii) The proof of (ii) is essentially worked out in [Ri13]. □

We proceed to show that the notion of dualizability is preserved by various standard operations, like change of base fields, or change of sites. Recall that we have already assumed the base scheme S = B is a perfect field k satisfying the hypothesis (3.0.3). We will let $\bar{k}$ denote its algebraic closure. Recall from (3.3.15), the following maps of topoi:

(5.0.19) $$\epsilon^* : \mathbf{Spt}/S_{mot} \rightarrow \mathbf{Spt}/S_{et}, \bar{\epsilon}^* : \mathbf{Spt}/\bar{S}_{mot} \rightarrow \mathbf{Spt}/\bar{S}_{et} \text{ and } \eta^* : \mathbf{Spt}/S_{et} \rightarrow \mathbf{Spt}/\bar{S}_{et}.$$

Since étale cohomology is well-behaved only with torsion coefficients prime to the characteristic, one will need to also consider the functors $\theta : \mathbf{Spt}/S_{et} \rightarrow \mathbf{Spt}/S_{et}$ sending commutative ring spectra $\mathcal{E}$ to $\mathcal{E} \overset{L}{\underset{\Sigma_{\epsilon^*(\mathbf{T})}}{\wedge}} H(Z/\ell)$ where $H(Z/\ell)$ denotes the mod-ℓ Eilenberg-Maclane spectrum in $\mathbf{Spt}_{et, \epsilon^*(\mathbf{T})}$. If ℓ is a fixed prime different from $char(k)$, and $\mathcal{E}$ is a commutative ring spectrum in $\mathbf{Spt}_{et, \mathcal{E}}$, we will also consider the functor sending spectra $M \in \mathbf{Spt}_{et, \mathcal{E}}$ to $M \wedge_{\mathcal{E}} \mathcal{E}(\ell^\nu)$: we will denote this functor by $\phi_{\mathcal{E}}$. We will adopt the convention that the above maps of topoi in fact denote their corresponding left derived functors.

Proposition 5.9. *Let ℓ denote a fixed prime different from $char(k)$, where k is assumed to be a perfect field satisfying the hypothesis* (3.0.3)*. If $\mathcal{E}$ is a commutative motivic ring spectrum so that it is ℓ-primary torsion as in Definition 1.2, then the functors ϵ^*, $\bar{\epsilon}^*$, η^* send the dualizable objects of the form $\mathcal{E} \wedge X_+$ appearing in in*

Propositions 5.1(i) and (ii), 5.4(iv), and Theorem 5.7 to dualizable objects. The same conclusion holds for the functors θ and $\phi_{\mathcal{E}}$ if $\mathcal{E}$ is a motivic ring spectrum that is ℓ-complete. If the ring spectrum $\mathcal{E}$ is ℓ-complete, the functor η^* sends the dualizable objects $\mathcal{E} \wedge X_+$ appearing in Proposition 5.6 to dualizable objects.

Proof. The first observation is that Proposition 5.4(v) already shows the functors ϵ^* ($\bar{\epsilon}^*$) send the dualizable objects in $\mathbf{Spt}/S_{mot}$ ($\mathbf{Spt}/\bar{S}_{mot}$) considered in Proposition 5.4(iv) to dualizable objects in $\mathbf{Spt}/S_{et}$ ($\mathbf{Spt}/\bar{S}_{et}$, respectively). That the functors ϵ^*, $\bar{\epsilon}^*$ and η^* preserve dualizable objects is clear for the finite cellular objects considered in Proposition 5.1. One may make use of the fact that the base field is perfect to see that base-change to the algebraic closure of the base field sends projective smooth schemes to projective smooth schemes and preserves strict normal crossings divisors. Therefore, the functor η^* sends the schemes appearing in Proposition 5.4(v) to dualizable objects.

Observe that the functor ϵ^* sends motivic spectra which are ℓ-primary torsion for a fixed prime different from $char(k)$ to étale spectra which are ℓ-primary torsion and preserves all split maps. It also sends (motivic) spectra with traces to spectra with traces. Therefore, the same argument making use of the stable cofiber sequences in (5.0.18) carries over to prove that ϵ^* and $\bar{\epsilon}^*$ send dualizable objects in Theorem 5.7 to dualizable objects, when the spectrum $\mathcal{E}$ is ℓ-primary torsion. One may prove similarly that the functor η^* sends dualizable objects appearing in Theorem 5.7 to dualizable objects. The conclusion that the functors θ and $\phi_{\mathcal{E}}$ send dualizable objects to dualizable objects follows from Lemma 5.2.

Observe that the base field appearing in Proposition 5.6 is allowed to be arbitrary subject to the hypothesis (3.0.3). It is clear that $\eta^*(G/H) = \bar{G}/\bar{H}$ which also satisfies the hypotheses of Proposition 5.6 over the base field $\bar{k}$. Therefore, the last conclusion follows. □

Proof of Theorem 1.3. The first (second) statement is proven in Theorem 5.7 (Proposition 5.6, respectively). The third statement is proven in Proposition 5.9. □

6. Construction of the Transfer for torsors for linear algebraic groups in Motivic and Étale Cohomology

In this section, we proceed to build on the general transfer map defined in Definition 2.8 to obtain transfer maps for torsors for linear algebraic groups, i.e. when $p : E \to B$ is a smooth map of smooth quasi-projective schemes that is a G-torsor for a linear algebraic group G. By taking $B = BG^{gm,m}$ which is the m-th degree approximation to the classifying space of the group G (its principal G-bundle, respectively) as in [Tot99], [MV99] or [CJ19] and $p_m : E = EG^{gm,m} \to B = BG^{gm,m}$ denoting the corresponding G-torsor, we obtain a transfer for Borel-style generalized equivariant motivic and étale cohomology. In the latter case, one has to deal separately with the issue of obtaining a transfer for the map $\lim_{m\to\infty} p_m$ from the transfers for each of the maps p_m.

For the convenience of the reader, we will quickly review some of the basic framework, discussed earlier. Let k denote a fixed perfect field, $S = Spec\, k$ and let Sm_S denote the category of all smooth quasi-projective schemes of finite type over S. Let G denote a linear algebraic group defined over S, viewed as a presheaf of groups on the same category. We let $\mathbf{Spt}^G_{mot}$ ($\mathbf{Spt}^G_{et}$, respectively) denote the category of G-spectra defined as in section 3: see Definition 3.15. Most of this will be when k is of characteristic 0. $\widetilde{\mathbf{USpt}}^G_{mot}$ ($\widetilde{\mathbf{USpt}}^G_{et}$) will denote the corresponding categories of spectra with $\tilde{U} : \mathbf{Spt}^G_{mot} \to \widetilde{\mathbf{USpt}}^G_{mot}$ and $\tilde{U}_{et} : \mathbf{Spt}^G_{et} \to \widetilde{\mathbf{USpt}}^G_{et}$ denote the corresponding forgetful functors, forgetting the group action.

In case $char(k) = p > 0$, we will let $\mathcal{E}^G \in \mathbf{Spt}^G_{mot}$ be a fixed commutative ring spectrum which is $Z_{(\ell)}$-local for some prime $\ell \neq char(k)$. We will let $\mathbf{Spt}^G_{mot,\mathcal{E}^G}$ denote the subcategory of $\mathcal{E}^G$-module spectra in $\mathbf{Spt}^G_{mot}$. For $\mathbf{Spt}^G_{et}$, the ring spectra $\mathcal{E}^G$ we consider will be ℓ-complete, for some $\ell \neq char(k)$. It may be important to point out that the only spectra in $\mathbf{Spt}^G_{mot,\mathcal{E}^G}$ and $\mathbf{Spt}^G_{et,\mathcal{E}^G}$ that we consider will be the $\mathcal{E}^G$-suspension spectra of schemes with G-action. Moreover, the G-equivariant ring spectra $\mathcal{E}^G$ that we consider will be mostly restricted to the list given in (3.2.7). i.e. Other than the sphere spectrum $\mathbb{S}^G$, the ring spectra $\mathcal{E}^G$ will be one of the following: (i) $\mathbb{S}^G[p^{-1}]$ if the base scheme S is a field of characteristic p, (ii) $\mathbb{S}^G_{(\ell)}$ and (iii) $\widehat{\mathbb{S}}^G_\ell$, where ℓ is a prime different from the characteristic of the base field. It may be important to recall the terminology in 3.12: if $\mathcal{E}^G$ is a commutative G-equivariant ring spectrum $\mathcal{E} = i^*(\tilde{\mathbb{P}}\tilde{U}(\mathcal{E}^G))$ is a ring spectrum in $\mathbf{Spt}$.

Now one may recall, as we observed in (3.3.14), that given $\mathcal{X}, \mathcal{Y} \in \mathbf{Spt}^G_{mot}$, one may find a functorial cofibrant replacement $\tilde{\mathcal{X}} \to \tilde{U}(\mathcal{X})$ in $\widetilde{\mathbf{USpt}}^G_{mot}$ and a functorial fibrant replacement $\tilde{U}(\mathcal{Y}) \to \hat{\mathcal{Y}}$ in $\widetilde{\mathbf{USpt}}^G_{mot}$, with $\tilde{\mathcal{X}} \to \tilde{U}(\mathcal{X}), \tilde{U}(\mathcal{Y}) \to \hat{\mathcal{Y}} \in \widetilde{\mathbf{USpt}}^G_{mot}$. The functoriality of the replacement shows that these replacements come equipped with actions by G making them belong to $\mathbf{Spt}^{\mathrm{G}}$. Therefore, it is possible to define

$$\mathcal{X}\overset{L}{\wedge}\mathcal{Y} = \tilde{\mathrm{U}}(\tilde{\mathcal{X}}) \wedge \mathcal{Y}, \quad \mathcal{RH}om(\mathcal{X}, \mathcal{Y}) = \mathcal{H}om(\tilde{\mathrm{U}}(\tilde{\mathcal{X}}), \tilde{\mathrm{U}}(\hat{\mathcal{Y}})), \quad \mathrm{D}(\mathcal{X}) = \mathcal{RH}om(\mathcal{X}, \mathbb{S}^{\mathrm{G}})$$

with $\mathcal{X}\overset{L}{\wedge}\mathcal{Y}, \mathcal{RH}om(\mathcal{X}, \mathcal{Y}), \mathrm{D}(\mathcal{X}) \in \mathbf{Spt}^{\mathrm{G}}_{\mathrm{mot}}$. Similar conclusions will hold when $\mathcal{E}^{\mathrm{G}} \in \mathbf{Spt}^{\mathrm{G}}_{\mathrm{mot}}$ is a commutative ring spectrum with the corresponding smash product $\wedge_{\mathcal{E}^{\mathrm{G}}}$ and $\mathcal{H}om_{\mathcal{E}^{\mathrm{G}}}$ as well as in the étale case.

Here we make use of the chain of equivalences of stable model category structures on $\widetilde{\mathbf{USpt}}^{\mathrm{G}}_{mot}$, $\mathbf{USpt}^{\mathrm{G}}_{mot}$ and $\mathbf{Spt}_{\mathrm{mot}}$ proven in Proposition 3.11 which are in fact given by weakly monoidal functors. Therefore, [DP84, 2.2 Theorem] (see also Proposition 2.7) shows that the theory of Spanier-Whitehead duality currently known in $\mathbf{Spt}_{\mathrm{mot}}$ carries over to $\widetilde{\mathbf{USpt}}^{\mathrm{G}}_{mot}$: therefore, without the comparison results proven in section 3.2 and 3.3 of this paper, it would not be possible to construct a theory of motivic Becker-Gottlieb transfer that applies to torsors and Borel-style generalized equivariant motivic (and étale) cohomology theories.

6.1. The G-equivariant pre-transfer. Let X denote a smooth quasi-projective scheme, or more generally an unpointed simplicial presheaf defined on $\mathrm{Sm_S}$, subject to the requirement that $\Sigma_{\mathbf{T}}\mathrm{X}_+ \in \mathbf{Spt}_{\mathrm{mot}}$ be dualizable. Corresponding results will hold if $X_+ \wedge \mathcal{E}$ is dualizable in $\mathbf{Spt}_{\mathrm{mot},\mathcal{E}}$ where $\mathcal{E}^{\mathrm{G}} \in \mathbf{Spt}^{\mathrm{G}}$ is a commutative ring spectrum, with $\mathcal{E} = i^*(\tilde{\mathbb{P}}\tilde{\mathrm{U}}(\mathcal{E}^{\mathrm{G}})) \in \mathbf{Spt}$ denoting the corresponding non-equivariant ring spectrum. Then the equivariant sphere spectrum $\mathbb{S}^{\mathrm{G}}$ will be replaced by $\mathcal{E}^{\mathrm{G}}$ everywhere in the construction discussed below.

We will further assume X is provided with an action by the linear algebraic group G. Associated to any G-equivariant self-map $\mathrm{f} : \mathrm{X} \to \mathrm{X}$, over the base field k, we will presently define a pre-transfer map following roughly the definition given in Definition 2.8. The main improvement we need is to make all the maps that enter into the definition of the pre-transfer G-equivariant. We will define the G-equivariant pre-transfer as the composition of a sequence of maps in $\widetilde{\mathbf{USpt}}^{\mathrm{G}}$ which are all G-equivariant. Throughout the following definition we will often abbreviate $\mathbb{S}^{\mathrm{G}} \wedge \mathrm{X}_+$ to just X_+.

Definition 6.1. (i) Accordingly we proceed to first define a G-*equivariant co-evaluation map*, where the source is the G-sphere spectrum $\mathbb{S}^{\mathrm{G}}$. We start with the evaluation map $e : \mathrm{D}(\mathrm{X}_+) \wedge \mathrm{X}_+ \to \mathbb{S}^{\mathrm{G}}$. On taking its dual in $\widetilde{\mathbf{USpt}}^{\mathrm{G}}$, we obtain the map

(6.1.1) $$c : \mathbb{S}^{\mathrm{G}} \simeq \mathrm{D}(\mathbb{S}^{\mathrm{G}}) \to \mathrm{D}(\mathrm{D}(\mathrm{X}_+) \wedge \mathrm{X}_+) \overset{\simeq}{\leftarrow} \mathrm{D}(\mathrm{X}_+) \wedge \mathrm{DD}(\mathrm{X}_+) \overset{\simeq}{\leftarrow} \mathrm{D}(\mathrm{X}_+) \wedge \mathrm{X}_+ \overset{\tau}{\to} \mathrm{X}_+ \wedge \mathrm{D}(\mathrm{X}_+).$$

The above composition will be the *co-evaluation* map c as in Definition 2.2.2. Observe that all the maps above are G-equivariant and the maps going in the wrong-direction are in fact weak-equivalences.

(ii) Now we may compose with the remaining maps in Definition 2.2.2, which are all G-equivariant and go from the term on the left to the term on the right, to obtain *the* G-*equivariant pre-transfer*, denoted $tr(\mathrm{f_Y})'_{\mathrm{G}}$, which will be the following composition:

(6.1.2) $$tr_{\mathrm{G}}(\mathrm{f})' : \mathbb{S}^{\mathrm{G}} \simeq \mathrm{D}(\mathbb{S}^{\mathrm{G}}) \to \mathrm{D}(\mathrm{D}(\mathrm{X}_+) \wedge \mathrm{X}_+) \overset{\simeq}{\leftarrow} \mathrm{D}(\mathrm{X}_+) \wedge \mathrm{DD}(\mathrm{X}_+) \overset{\simeq}{\leftarrow} \mathrm{D}(\mathrm{X}_+) \wedge \mathrm{X}_+ \overset{\tau}{\to} \mathrm{X}_+ \wedge \mathrm{D}(\mathrm{X}_+) \to \mathbb{S}^{\mathrm{G}} \wedge \mathrm{X}_+.$$

Here the last map is the composition

$$\mathrm{X}_+ \wedge \mathrm{D}(\mathrm{X}_+) \overset{\tau}{\to} \mathrm{D}(\mathrm{X}_+) \wedge \mathrm{X}_+ \overset{id\wedge\Delta}{\to} \mathrm{D}(\mathrm{X}_+) \wedge \mathrm{X}_+ \wedge \mathrm{X}_+ \overset{id\wedge f\wedge f}{\to} \mathrm{D}(\mathrm{X}_+) \wedge \mathrm{X}_+ \wedge \mathrm{X}_+ \overset{e\wedge id}{\to} \mathbb{S}^{\mathrm{G}} \wedge \mathrm{X}_+.$$

Observe that all the spectra that make up the above diagram are G-equivariant and therefore, the above diagram could be viewed as a diagram in $\widetilde{\mathbf{USpt}}^{\mathrm{G}}$ where all the spectra and the maps are G-equivariant.[4]

(iii) Given Y, another smooth quasi-projective scheme, or more generally an unpointed simplicial presheaf defined on $\mathrm{Sm_S}$, provided with an action by G, we define $tr(\mathrm{f_Y})'_{\mathrm{G}} : \mathrm{Y}_+ \wedge \mathbb{S}^{\mathrm{G}} \to \mathrm{Y}_+ \wedge \mathbb{S}^{\mathrm{G}} \wedge \mathrm{X}_+$ to be $id_{\mathrm{Y}_+} \wedge tr(\mathrm{f})'_{\mathrm{G}}$.

(iv) We define the *trace*, $\tau_{\mathrm{X}}(\mathrm{f})^{\mathrm{G}}$ to be the composition of the pre-transfer with the map $\mathbb{S}^{\mathrm{G}} \wedge \mathrm{X}_+ \to \mathbb{S}^{\mathrm{G}}$ collapsing all of X_+ to $Spec\, k_+$. Similarly we define $\tau_{\mathrm{X}}(\mathrm{f_Y})^{\mathrm{G}}$ to be the composition of the pre-transfer $tr_{\mathrm{G}}(\mathrm{f_Y})'$ with the map $\mathrm{Y}_+ \wedge \mathbb{S}^{\mathrm{G}} \wedge \mathrm{X}_+ \to \mathrm{Y}_+ \wedge \mathbb{S}^{\mathrm{G}}$.

(v) If $\mathcal{E}^{\mathrm{G}} \in \mathbf{Spt}^{\mathrm{G}}$ is a commutative ring spectrum, and $\mathcal{E} \wedge X_+ \in \mathbf{Spt}$ is dualizable, one defines co-evaluation, transfer and trace maps similarly by replacing $\mathbb{S}^{\mathrm{G}} \wedge \mathrm{X}_+$ ($\mathbb{S}^{\mathrm{G}}$) by $\mathcal{E}^{\mathrm{G}} \wedge \mathrm{X}_+$ ($\mathcal{E}^{\mathrm{G}}$, respectively).

The next goal is to define a transfer map that will define a wrong-way map in generalized cohomology for a G-torsor $\mathrm{p} : \mathrm{E} \to \mathrm{B}$ as well as in Borel-style equivariant generalized motivic (and étale) cohomology associated to actions of linear algebraic groups. Our approach follows closely the construction in [BG75, section 3], in *spirit*.

[4]One can put in a slightly more general form of the diagonal map Δ, which will in fact be important for establishing the localization or Mayer-Vietoris properties of the pre-transfer. This is discussed in [JP-1, Definition 2.4].

6.1.3. Convention. Let G denote a linear algebraic group. We need to carry out the construction of the transfer in two distinct contexts: (i) when the group G is *special* in Grothendieck's terminology: see [Ch]. For example, G could be a $\mathrm{GL_n}$ for some n or a finite product of $\mathrm{GL_n}$s and (ii) when G is not necessarily special. In the first case, every G-torsor is locally trivial on the Zariski (and hence the Nisnevich) topology while in the second case G-torsors are locally trivial only in the étale topology.

In both cases, we will let $\mathrm{BG}^{gm,m}$ ($\mathrm{EG}^{gm,m}$) denote the m-th degree approximation to the classifying space of the group G (its principal G-bundle, respectively) as in [Tot99], [MV99] or [CJ19]. These are, in general, quasi-projective smooth schemes over k. It is important for us to observe that each $\mathrm{EG}^{gm,m}$, with m sufficiently large has k-rational points, where k is the base field. (This will imply that $\mathrm{BG}^{gm,m}$, with m sufficiently large also has k-rational points.)

Next we start with a G-torsor $\mathrm{E} \rightarrow \mathrm{B}$, with both E and B smooth quasi-projective schemes over S. We will further assume that B is *always connected.* Next, we will find affine replacements for these schemes. One may first find an affine replacement $\widetilde{\mathrm{B}}$ for B ($\widetilde{\mathrm{BG}^{gm,m}}$ for $\mathrm{BG}^{gm,m}$) by applying the well-known construction of Jouanolou (see [Joun73]) and then define $\widetilde{\mathrm{E}}$ ($\widetilde{\mathrm{EG}^{gm,m}}$) as the pull-back:

$$\begin{aligned} \widetilde{\mathrm{E}} = \widetilde{\mathrm{B}}\underset{\mathrm{B}}{\times}\mathrm{E}, \tilde{\mathrm{p}} : \widetilde{\mathrm{E}} \rightarrow \widetilde{\mathrm{B}}, \quad (\widetilde{\mathrm{EG}^{gm,m}} = \widetilde{\mathrm{BG}^{gm,m}} \underset{\mathrm{BG}^{gm,m}}{\times} \mathrm{EG}^{gm,m}, \tilde{\mathrm{p}}_m : \widetilde{\mathrm{EG}^{gm,m}} \rightarrow \widetilde{\mathrm{BG}^{gm,m}}) \\ \pi_{\mathrm{Y}} : \widetilde{\mathrm{E}}_{\mathrm{Y}\times\mathrm{X}} = \widetilde{\mathrm{E}} \times_{\mathrm{G}} (\mathrm{Y}\times\mathrm{X}) \rightarrow \widetilde{\mathrm{E}} \times_{\mathrm{G}} \mathrm{Y} = \widetilde{\mathrm{E}}_{\mathrm{Y}}, \quad (\pi_{\mathrm{Y,m}} : \widetilde{\mathrm{EG}^{gm,m}} \times_{\mathrm{G}} (\mathrm{Y}\times\mathrm{X}) \rightarrow \widetilde{\mathrm{EG}^{gm,m}} \times_{\mathrm{G}} \mathrm{Y}) \\ \pi : \widetilde{\mathrm{E}}_{\mathrm{Y}} = \widetilde{\mathrm{E}} \times_{\mathrm{G}} \mathrm{Y} \rightarrow \widetilde{\mathrm{B}}, \quad (\pi_{\mathrm{m}} : \widetilde{\mathcal{E}}_{\mathrm{m,Y}} = \widetilde{\mathrm{EG}^{gm,m}} \times_{\mathrm{G}} \mathrm{Y} \rightarrow \widetilde{\mathrm{BG}^{gm,m}} = \widetilde{\mathcal{B}}_{\mathrm{m}}.) \end{aligned} \tag{6.1.4}$$

6.2. Construction of the transfer. Next we proceed to construct the transfer as a stable map, i.e. a map in $\mathbf{HSpt}_{\mathrm{mot}}$, when $\Sigma_{\mathbf{T}}\mathrm{X}_+$ is dualizable in $\mathbf{HSpt}_{\mathrm{mot}}$ and G is special (and a minor variant of this map when G is non-special):

$$(6.2.1)\quad tr(\mathrm{f_Y}) : \Sigma_{\mathbf{T}}(\widetilde{\mathrm{E}}\times_{\mathrm{G}}\mathrm{Y})_+ \rightarrow \Sigma_{\mathbf{T}}(\widetilde{\mathrm{E}}\times_{\mathrm{G}}(\mathrm{Y}\times\mathrm{X}))_+ \quad (tr(\mathrm{f_Y}) : (\Sigma_{\mathbf{T}}(\widetilde{\mathrm{EG}^{gm,m}}\times_{\mathrm{G}}\mathrm{Y})_+ \rightarrow \Sigma_{\mathbf{T}}(\widetilde{\mathrm{EG}^{gm,m}}\times_{\mathrm{G}}(\mathrm{Y}\times\mathrm{X}))_+.$$

This will be constructed as a composition of several maps in $\mathbf{Spt}_{\mathrm{mot}}$, with some of the maps going the wrong-way, and these wrong-way maps will all be weak-equivalences in $\mathbf{Spt}_{\mathrm{mot}}$. In case $\mathcal{E} \wedge \mathrm{X}_+ \in \mathbf{Spt}_{\mathrm{mot},\mathcal{E}}$ is dualizable for a commutative ring spectrum $\mathcal{E}^{\mathrm{G}} \in \mathbf{Spt}^{\mathrm{G}}$ with $\mathcal{E} = i^*(\tilde{\mathbb{P}}\tilde{\mathrm{U}}(\mathcal{E}^{\mathrm{G}}))$, ($\mathcal{E} \wedge \mathrm{X}_+ \in \mathbf{Spt}_{\mathrm{et},\mathcal{E}}$ is dualizable for a commutative ring spectrum $\mathcal{E}^{\mathrm{G}} \in \mathbf{Spt}^{\mathrm{G}}_{\mathrm{et}}$, so that $\mathcal{E}$ is ℓ-complete for some prime $\ell \neq char(k)$, respectively) the transfer we obtain will be of the following form when G is special (and a variant of this map when G is non-special):

$$(6.2.2)\quad tr(\mathrm{f_Y}) : \mathcal{E}\wedge(\widetilde{\mathrm{E}}\times_{\mathrm{G}}\mathrm{Y})_+ \rightarrow \mathcal{E}\wedge(\widetilde{\mathrm{E}}\times_{\mathrm{G}}(\mathrm{Y}\times\mathrm{X}))_+ \quad (tr(\mathrm{f_Y}) : \mathcal{E}\wedge(\widetilde{\mathrm{EG}^{gm,m}}\times_{\mathrm{G}}\mathrm{Y})_+ \rightarrow \mathcal{E}\wedge(\widetilde{\mathrm{EG}^{gm,m}}\times_{\mathrm{G}}(\mathrm{Y}\times\mathrm{X}))_+.)$$

Remark 6.2. The following remarks may provide some insight and motivation to the construction of the transfer discussed in Steps 0 through 5 below. We have tried to define a transfer that depends only on the G-object X and the G-equivariant self-map f and which does *not* depend on any further choices. This makes it necessary to start with the G-equivariant pre-transfer as in (6.1.2). As a result, we are forced to make use of the framework of the category $\widetilde{\mathbf{USpt}}^{\mathrm{G}}$. However, if one chooses to replace the G-equivariant sphere spectrum $\mathbb{S}^{\mathrm{G}}$ by just the suspension spectrum of the Thom-space $\mathrm{T_V}$, for a fixed (but large enough) representation V of G, then the use of the category $\widetilde{\mathbf{USpt}}^{\mathrm{G}}$ could be circumvented by just using a variant of Proposition 3.1 valid for suspension spectra. The construction of the transfer in [BG75] in fact adopts this latter approach: in their framework, the co-evaluation map corresponds to a Thom-Pontrjagin collapse map associated to the Thom-space of a fixed G-representation. Such an approach does not seem to work in general in the motivic context, though it could be made to work when X denotes a *projective smooth scheme*, provided one makes use of the Voevodsky collapse (see Definition 10.8) in the place of the classical Thom-Pontrjagin collapse.

Step 0: The Borel construction applied to simplicial presheaves with G*-action.* We break this discussion into two cases, depending on whether the group G is *special* in Grothendieck's classification (see [Ch]). In both cases, $\mathrm{PSh^G/S}$ will denote the category of pointed G-equivariant presheaves on *the big Nisnevich site* of S provided with a chosen map to S as in (3.0.6).

Case 1: when G *is special.* Recall this includes all the linear algebraic groups GL_n, SL_n, Sp_{2n}, $n \geq 1$. In this case, we start with the construction (i.e. the functor):

$$\mathrm{PSh^G/S} \rightarrow \mathrm{PSh}/\widetilde{\mathrm{B}}, X \mapsto \widetilde{\mathrm{E}} \times_{\mathrm{G}} X \tag{6.2.3}$$

where the quotient construction is explained below. (If we start with an unpointed simplicial presheaf X, we let $X = \mathrm{X}_+$ and we will always assume that the action by G on X preserves the base point. Therefore, there is a canonical section $\widetilde{\mathrm{B}} \to \widetilde{\mathrm{E}} \times_{\mathrm{G}} X$.) Clearly this extends to a functor:

$$\mathbf{Spt}^{\mathrm{G}}/\mathrm{S} \to \widetilde{\mathbf{USpt}}^{\mathrm{G}}/\widetilde{\mathrm{B}}, \mathcal{X} \mapsto \widetilde{\mathrm{E}} \times_{\mathrm{G}} \mathcal{X}. \tag{6.2.4}$$

In (6.2.3), one cannot view the product $\widetilde{\mathrm{E}} \times X$ as a presheaf on the big Nisnevich site and take the quotient by the action of G, with G again viewed as a Nisnevich presheaf: though such a quotient will be a presheaf on the big Nisnevich site, this will not be the presheaf represented by the scheme (or algebraic space) $\widetilde{\mathrm{E}} \times_{\mathrm{G}} X$, when X is a scheme. In order to get this latter presheaf, when G is special, one needs to start with a Zariski open cover $\{\mathrm{U_i}|\mathrm{i}\}$ of $\widetilde{\mathrm{B}}$ over which $\widetilde{\mathrm{E}}$ is trivial, and then glue together the sheaves $\mathrm{U_i} \times X$ making use of the gluing data provided by the torsor $\widetilde{\mathrm{E}} \to \widetilde{\mathrm{B}}$.

A nice way to view this construction is as follows, at least when X is a Nisnevich sheaf: one needs to in fact take the *quotient sheaf* associated to the presheaf quotient of $\widetilde{\mathrm{E}} \times X$ by the G-action on the big Nisnevich site. Then this produces the right object.

Denoting by $(\widetilde{\mathrm{E}} \times_{\mathrm{G}} \mathcal{X})_{|\mathrm{U_i}}$ the restriction of $\widetilde{\mathrm{E}}$ to $\mathrm{U_i}$, it is clear that $(\widetilde{\mathrm{E}} \times_{\mathrm{G}} \mathcal{X})_{|\mathrm{U_i}}$ identifies with $\mathrm{U_i} \times \mathcal{X}$. Therefore, it is clear that the construction in (6.2.4) sends a G-equivariant map $\alpha : \mathcal{X} \to \mathcal{Y}$ so that $\tilde{\mathrm{U}}(\alpha)$ is a (stable) weak-equivalence in $\widetilde{\mathbf{USpt}}^{\mathrm{G}}/\mathrm{S}$ to a (stable) weak-equivalence in $\widetilde{\mathbf{USpt}}^{\mathrm{G}}/\widetilde{\mathrm{B}}$.

Case 2: Next assume that G is *not necessarily special, in which case we will assume the base field k is infinite* to avoid the issues discussed in [MV99, Example 2.10, 4.2]. Observe that the list of non-special linear algebraic groups includes all the linear algebraic groups such as all finite groups, PGL_n, $\mathrm{O}(n)$, $n \geq 1$ etc. S will denote the base scheme Spec k. Let $\mathbf{H}(\mathrm{Sm}/\mathrm{S_{et}}, \mathbb{A}^1)$ denote the $\mathbb{A}^1$-localized homotopy category of simplicial presheaves on the big étale site $\mathrm{Sm}/\mathrm{S_{et}}$. Let BG denote the simplicial classifying space of G viewed as a simplicial presheaf on the big étale site $\mathrm{Sm}/\mathrm{S_{et}}$ and let $\widetilde{\mathrm{BG}_{et}^{gm,m}}$ denote the scheme $\widetilde{\mathrm{BG}^{gm,m}}$ viewed as a simplicial presheaf on the big étale site $\mathrm{Sm}/\mathrm{S_{et}}$. Then the first observation we make is that one obtains the weak-equivalence

$$\mathrm{BG} \simeq \lim_{m \to \infty} \widetilde{\mathrm{BG}_{et}^{gm,m}} \tag{6.2.5}$$

in $\mathbf{H}(\mathrm{Sm}/\mathrm{S_{et}}, \mathbb{A}^1)$. To prove this one may proceed as follows. Either one may adopt the same arguments as in [MV99, p. 131 and Lemma 2.5, Proposition 2.6 in 4.2] or consider the diagram:

$$\begin{array}{ccccc} & & \mathrm{EG} \times_{\mathrm{G}} \mathrm{EG}^{gm} & & \\ & \overset{p_1}{\swarrow} & & \overset{p_2}{\searrow} & \\ \mathrm{BG} & & & & \mathrm{BG}^{gm}. \end{array} \tag{6.2.6}$$

Then, one may observe that the fibers of both maps p_1 and p_2 over a strictly Hensel ring are acyclic: the fibers of p_1 are acyclic because we have inverted $\mathbb{A}^1$ (and therefore, EG^{gm} is acyclic), and the fibers of p_2 are acyclic because they are the simplicial EG. Thus p_1 and p_2 induce weak-equivalences of the corresponding simplicial sheaves. (See [J20, Theorem 1.5] for a similar argument at the level of equivariant derived categories.) Let $\epsilon : \mathrm{Sm}/\mathrm{S_{et}} \to \mathrm{Sm}/\mathrm{S_{Nis}}$ denote the map of sites from the big étale site of S to the big Nisnevich site of S. It follows therefore that one obtains the identification

$$\mathrm{R}\epsilon_*(\mathrm{BG}) \simeq \lim_{\mathrm{m} \to \infty} \mathrm{R}\epsilon_*(\widetilde{\mathrm{BG}_{\mathrm{et}}^{gm,m}})^5 \tag{6.2.7}$$

in $\mathbf{H}(\mathrm{Sm}/\mathrm{S_{Nis}}, \mathbb{A}^1)$. (Here we will use the injective model structure on simplicial presheaves prior to $\mathbb{A}^1$-localization: see 3.1.) In this case, the construction (6.2.3) now takes on the form:

$$X \mapsto \mathrm{R}\epsilon_*(\widetilde{\mathrm{E}} \times_{\mathrm{G}}^{et} (\mathrm{a} \circ \epsilon^*)(X)), \quad \mathrm{PSh}^{\mathrm{G}}/\mathrm{S} \overset{\epsilon^*}{\to} \mathrm{PSh}^{\mathrm{G}}/\mathrm{S_{et}} \overset{\mathrm{a}}{\to} \mathrm{Sh}^{\mathrm{G}}/\mathrm{S_{et}} \to \mathrm{PSh}/\mathrm{R}\epsilon_*(\widetilde{\mathrm{B}}_{\mathrm{et}}) \tag{6.2.8}$$

Here we have adopted the following conventions: the superscript *et* denotes the fact we are *taking quotient sheaves on the étale site*, *a* denotes the functor sending a presheaf to the associated sheaf, and $\mathrm{PSh}/\mathrm{R}\epsilon_*(\widetilde{\mathrm{B}_{\mathrm{et}}})$ denotes the category of simplicial presheaves on $\mathrm{Sm}/\mathrm{S_{Nis}}$ pointed over the simplicial presheaf $\mathrm{R}\epsilon_*(\widetilde{\mathrm{B}_{\mathrm{et}}})$. Moreover, U is the forgetful functor sending a sheaf to the underlying presheaf.

[5]A main result of [MV99, Proposition 2.6] is that the term on the right is weakly-equivalent to $\lim_{m \to \infty} \epsilon_*(\widetilde{\mathrm{BG}^{\mathrm{gm,m}}})$

Clearly this extends to a functor

(6.2.9) $$\mathcal{X} \mapsto \mathrm{R}\epsilon_*(\widetilde{\mathrm{E}} \times^{et}_{\mathrm{G}} (\mathrm{a} \circ \epsilon^*)(\mathcal{X})), \quad \mathbf{Spt}^{\mathrm{G}}/\mathrm{S}^{\mathrm{U}\circ \mathrm{a} \circ \epsilon^*}{\overset{}{\to}} \mathbf{Spt}^{\mathrm{G}}/\mathrm{S}_{\mathrm{et}} \to \widetilde{\mathbf{USpt}}^{\mathrm{G}}/\mathrm{R}\epsilon_*(\widetilde{\mathrm{B}}_{\mathrm{et}})$$

If $\{\mathrm{U_i}|\mathrm{i} \in \mathrm{I}\}$ is an étale cover of $\widetilde{\mathrm{B}}$ over which $\widetilde{\mathrm{E}}$ is trivial, the same argument as above shows that $(\widetilde{\mathrm{E}} \times^{et}_{\mathrm{G}} (a \circ \epsilon^*)(\mathcal{X}))_{|\mathrm{U}_i} = \mathrm{U_i} \times (\mathrm{a} \circ \epsilon^*)(\mathcal{X})$, so that the functor $\mathcal{X} \mapsto \widetilde{\mathrm{E}} \times^{et}_{\mathrm{G}} (a \circ \epsilon^*)(\mathcal{X})$ sends a G-equivariant map $\alpha : \mathcal{X} \to \mathcal{Y}$ for which $\tilde{\mathrm{U}}(\alpha)$ is a (stable) weak-equivalence in $\widetilde{\mathbf{USpt}}^{\mathrm{G}}/\mathrm{S}$ to a (stable) weak-equivalence in $\widetilde{\mathbf{USpt}}^{\mathrm{G}}_{et}/\widetilde{\mathrm{B}}_{et}$. Therefore, the functor $\mathcal{X} \mapsto \mathrm{R}\epsilon_*(\widetilde{\mathrm{E}} \times^{et}_{\mathrm{G}} (\mathrm{a} \circ \epsilon^*)(\mathcal{X}))$ sends a G-equivariant map $\alpha : \mathcal{X} \to \mathcal{Y}$ for which $\tilde{\mathrm{U}}(\alpha)$ is a (stable) weak-equivalence in $\widetilde{\mathbf{USpt}}^{\mathrm{G}}/\mathrm{S}$ to a (stable) weak-equivalence in $\widetilde{\mathbf{USpt}}^{\mathrm{G}}/\mathrm{R}\epsilon_*(\widetilde{\mathrm{B}}_{\mathrm{et}})$.

In case X is already a sheaf on the big étale site, $(a \circ \epsilon^*)(X) = X$ and therefore, we may replace $(a \circ \epsilon^*)(X)$ in (6.2.8) by just X in the definition of the Borel construction. (This applies to the case where $X = \mathrm{X}$ is a scheme.)

Terminology 6.3. *Throughout the remainder of the paper, we will abbreviate the functor in* (6.2.8) *(* (6.2.9)*) by*

$$X \mapsto \mathrm{R}\epsilon_*(\widetilde{\mathrm{E}} \times^{et}_{\mathrm{G}} X), X \in \mathrm{PSh}^{\mathrm{G}}/\mathrm{S}, \quad (\mathcal{X} \mapsto \mathrm{R}\epsilon_*(\widetilde{\mathrm{E}} \times^{et}_{\mathrm{G}} \mathcal{X}), \mathcal{X} \in \mathbf{Spt}^{\mathrm{G}}/\mathrm{S}, \text{respectively}).$$

Though there is a discussion of the classifying spaces of linear algebraic groups in [MV99, 4.2], it lacks a corresponding discussion on the Borel construction $\mathrm{EG}^{gm,m} \times_{\mathrm{G}} \mathrm{X}$, for X a smooth scheme. We complete our discussion, by providing a comparison of $\mathrm{EG}^{gm,m} \times_{\mathrm{G}} \mathrm{X}$ with $\mathrm{R}\epsilon_*(\widetilde{\mathrm{EG}^{gm,m}} \times^{et}_{\mathrm{G}} \mathrm{X})$ when X is a smooth scheme. We first replace $\lim_{m \to \infty} \mathrm{BG}^{gm,m}_{et}$ and $\lim_{m \to \infty} \mathrm{EG}^{gm,m}_{\mathrm{et}} \times^{\mathrm{et}}_{\mathrm{G}} \mathrm{X}$ by fibrant simplicial presheaves $\widehat{\mathrm{BG}}_{et}$ and $\mathrm{E}\widehat{\mathrm{G}_{\mathrm{et}} \times^{\mathrm{et}}_{\mathrm{G}}} \mathrm{X}$ so that the induced map $\mathrm{E}\widehat{\mathrm{G}_{\mathrm{et}} \times^{\mathrm{et}}_{\mathrm{G}}} \mathrm{X} \to \widehat{\mathrm{BG}}_{et}$ is a fibration with fiber $\hat{\mathrm{X}}$, which is a fibrant replacement for X. Let $\mathrm{U}_\infty = \lim_{\mathrm{m} \to \infty} \mathrm{EG}^{gm,m}$. Now one forms the cartesian square in $\mathrm{PSh}/\mathrm{S}_{\mathrm{et}}$:

(6.2.10) $$\begin{array}{ccc} \mathrm{E}(\mathrm{U}_\infty, \mathrm{G})_{\mathrm{et}} \times^{et}_{\mathrm{G}} \hat{\mathrm{X}} & \longrightarrow & \mathrm{E}\widehat{\mathrm{G}_{et} \times^{et}_{\mathrm{G}}} \mathrm{X} \\ \downarrow & & \downarrow \\ \mathrm{B}(\mathrm{U}_\infty, \mathrm{G})_{\mathrm{et}} & \longrightarrow & \widehat{\mathrm{BG}}_{et}. \end{array}$$

Here $\mathrm{E}(\mathrm{U}_\infty, \mathrm{G})_{\mathrm{et}}$ is the étale simplicial presheaf given in degree n by $\mathrm{U}^{\mathrm{n}+1}_\infty$, and with the structure maps provided by the projections of $\mathrm{U}^{\mathrm{m}}_\infty$ to the various factors U_∞ and by the diagonal maps $\mathrm{U}_\infty \to \mathrm{U}^{\mathrm{m}}_\infty$. $\mathrm{B}(\mathrm{U}_\infty, \mathrm{G})_{\mathrm{et}} = \mathrm{E}(\mathrm{U}_\infty, \mathrm{G})_{\mathrm{et}}/\mathrm{G}$. This square remains a cartesian square on applying the push-forward ϵ_* to the Nisnevich site. [MV99, Lemma 2.5, 4.2] shows that the resulting map in the bottom row is an isomorphism in $\mathbf{H}(\mathrm{Sm}/\mathrm{S}_{\mathrm{Nis}}, \mathbb{A}^1)$, so that so is the resulting map in the top row. Finally an argument exactly as on [MV99, p. 136] shows that one obtains an identification $\epsilon_*(\mathrm{E}(\mathrm{U}_\infty, \mathrm{G})_{\mathrm{et}} \times^{et}_{\mathrm{G}} \hat{\mathrm{X}}) \simeq \epsilon_*(\mathrm{U}_\infty \times^{\mathrm{et}}_{\mathrm{G}} \hat{\mathrm{X}}) = \lim_{\mathrm{m} \to \infty} \epsilon_*(\mathrm{EG}^{gm,m} \times^{\mathrm{et}}_{\mathrm{G}} \hat{\mathrm{X}})$. Therefore, we obtain the identification for a smooth scheme X:

(6.2.11) $$\lim_{m \to \infty} \mathrm{R}\epsilon_*(\widetilde{\mathrm{EG}^{gm,m}} \times^{et}_{\mathrm{G}} \mathrm{X}) = \epsilon_*(\mathrm{E}\widehat{\mathrm{G}_{\mathrm{et}} \times^{\mathrm{et}}_{\mathrm{G}}} \mathrm{X}) \simeq \epsilon_* \lim_{\mathrm{m} \to \infty}(\mathrm{EG}^{gm,m} \times^{\mathrm{et}}_{\mathrm{G}} \hat{\mathrm{X}}).$$

Finally, for convenience in the following steps, *we will denote both the Borel constructions given in* (6.2.4) *and* (6.2.9) *by the notation* $\mathcal{X} \mapsto \widetilde{\mathrm{E}} \times_{\mathrm{G}} \mathcal{X}$. Moreover, we will denote by $\widetilde{\mathrm{B}}$, the object denoted by this symbol in (6.1.4) when G is special, and the object $\mathrm{R}\epsilon_*(\widetilde{\mathrm{B}}_{\mathrm{et}})$ considered in (6.2.8) when G is not special.

Step 1. As the next step in the construction of the transfer map $tr(\mathrm{f_Y})$, we start with the G-equivariant pre-transfer $tr(\mathrm{f_Y})_{\mathrm{G}}$ in (6.1.2) to obtain the stable map over $\widetilde{\mathrm{E}}_{\mathrm{Y}}$, i.e. as a composition of several maps in $\widetilde{\mathbf{USpt}}^{\mathrm{G}}/\widetilde{\mathrm{E}}_{\mathrm{Y}}$, where the wrong-way maps are all weak-equivalences.

(6.2.12) $$\widetilde{\mathrm{E}} \times_{\mathrm{G}} (\mathrm{Y}_+ \wedge \mathbb{S}^{\mathrm{G}}) \overset{\mathrm{id} \times_{\mathrm{G}} tr_{\mathrm{G}}(\mathrm{f_Y})'}{\to} \widetilde{\mathrm{E}} \times_{\mathrm{G}} (\mathrm{Y}_+ \wedge \mathbb{S}^{\mathrm{G}} \wedge \mathrm{X}_+).$$

(Here we are making use of the observation that the above Borel construction preserves weak-equivalences as observed in Step 0, so that we can suppress the fact that the above map is in fact a composition of several maps, some of which go the wrong-way as observed in (6.1.2).) On applying the construction $\widetilde{\mathrm{E}} \times_{\mathrm{G}}$ with a G-equivariant ring spectrum $\mathcal{E}^{\mathrm{G}}$ (as in (3.2.7)) in the place of $\mathbb{S}^{\mathrm{G}}$, the resulting stable map takes on the form:

(6.2.13) $$\widetilde{\mathrm{E}} \times_{\mathrm{G}} (\mathrm{Y}_+ \wedge \mathcal{E}^{\mathrm{G}}) \overset{\mathrm{id} \times_{\mathrm{G}} tr_{\mathrm{G}}(\mathrm{f_Y})'}{\to} \widetilde{\mathrm{E}} \times_{\mathrm{G}} (\mathrm{Y}_+ \wedge \mathcal{E}^{\mathrm{G}} \wedge \mathrm{X}_+).$$

Remark 6.4. The remaining steps in the construction of the transfer may be easily explained by fact that the sphere spectrum $\mathbb{S}^{\mathrm{G}}$ and the ring spectrum $\mathcal{E}^{\mathrm{G}}$ appearing above have non-trivial actions by G, so that neither the source nor the target of the maps in (6.2.12) and (6.2.13) will become suspension spectra of $\widetilde{\mathrm{E}}_{\mathrm{Y}}$ or $\widetilde{\mathrm{E}} \times_{\mathrm{G}} (\mathrm{Y} \times \mathrm{X})_+$ without the considerable efforts in the remaining steps. We will discuss the remaining steps in detail only for the sphere spectrum $\mathbb{S}^{\mathrm{G}}$. This suffices, since the only other ring spectra $\mathcal{E}^{\mathrm{G}}$ we consider will be restricted to those appearing in the list in (3.2.7).

Step 2. Next let V denote a fixed (but arbitrary) finite dimensional representation of the group G.[6] At this point we need to briefly consider two cases, (a) where G is special and (b) where it is not. In case (a), it should be clear that

(6.2.14) $$\widetilde{\mathrm{E}} \times_{\mathrm{G}} \mathrm{V} \text{ is a vector bundle } \xi^{\mathrm{V}} \text{ on the affine scheme } \widetilde{\mathrm{B}},$$

where the quotient construction is done as in (6.2.3), that is on the Zariski site. In case (b), one considers instead:

(6.2.15) $$\widetilde{\mathrm{E}} \times^{et}_{\mathrm{G}} \mathrm{V},$$

where the quotient is taken on the étale topology. Apriori, this is a vector bundle that is locally trivial on the étale topology of $\widetilde{\mathrm{B}}$. But any such vector bundle corresponds to a GL_n-torsor on the étale topology of $\widetilde{\mathrm{B}}$, and hence (by Hilbert's theorem 90: see [Mil, Chapter III, proposition 4.9]), is in fact locally trivial on the Zariski topology of $\widetilde{\mathrm{B}}$. We will denote this vector bundle also by ξ^{V}.

Since $\widetilde{\mathrm{B}}$ is an affine scheme over *Spec k*, we can find a complimentary vector bundle η^{V} on $\widetilde{\mathrm{B}}$ so that

(6.2.16) $$\xi^{\mathrm{V}} \oplus \eta^{\mathrm{V}} \text{ is a trivial bundle over } \widetilde{\mathrm{B}} \text{ and of rank N, for some integer N.}$$

For the remainder of this step, we will consider the case when $\widetilde{\mathrm{E}} = \widetilde{\mathrm{EG}^{gm,m}}$ and $\widetilde{\mathrm{B}} = \widetilde{\mathrm{BG}^{gm,m}}$. We will denote the first by $\mathcal{E}_m$ and the latter by $\mathcal{B}_m$. We will denote the vector bundle $\widetilde{\mathrm{EG}^{gm,m}} \times_{\mathrm{G}} \mathrm{V}$ ($\widetilde{\mathrm{EG}^{gm,m}} \times^{et}_{\mathrm{G}} \mathrm{V}$) on the affine scheme $\mathcal{B}_m = \widetilde{\mathrm{BG}^{gm,m}}$ by ξ^{V}_m. The complimentary vector bundle η^{V} chosen above will now denoted η^{V}_m. We proceed to show that we can choose the integer N independent of m, so that a single choice of N will work for all m. Since the map $\mathrm{EG}^{gm,m} \to \mathrm{BG}^{gm,m}$ is affine, one can readily see that the scheme $\widetilde{\mathrm{EG}^{gm,m}}$ is also an affine scheme. Let $\mathrm{R_m}$ denote the co-ordinate ring of $\widetilde{\mathrm{EG}^{gm,m}}$ and let $\mathrm{R} = \lim_{\infty \leftarrow \mathrm{m}} \mathrm{R_m}$. We proceed to show that, now $(\mathrm{R}\underset{\mathrm{k}}{\otimes}\mathrm{V})^{\mathrm{G}}$ is a *finitely generated projective module over the ring* R^{G}. Let $\mathrm{I_m}$ be the ideal defining $\widetilde{\mathrm{BG}^{gm,m}}$ as a closed subscheme in *Spec*$(\mathrm{R}^{\mathrm{G}})$. Then, $\mathrm{R}^{\mathrm{G}}/\mathrm{I_m}\underset{\mathrm{R}^{\mathrm{G}}}{\otimes}(\mathrm{R}\underset{\mathrm{k}}{\otimes}\mathrm{V})^{\mathrm{G}}$ corresponds to the vector bundle ξ^{V}_m, and therefore, is a finitely generated projective $\mathrm{R}^{\mathrm{G}}/\mathrm{I_m}$-module. In fact, if $\mathfrak{M}$ denotes a maximal ideal in the ring R^{G} and $\bar{\mathrm{I}}_m$ denotes the image of the ideal $\mathrm{I_m}$ in the local ring $\mathrm{R}^{\mathrm{G}}_{(\mathfrak{M})}$, then one can see that the ranks of the inverse system of free modules $\{\mathrm{R}^{\mathrm{G}}_{(\mathfrak{M})}/\bar{\mathrm{I}}_{\mathrm{m}}\underset{\mathrm{R}^{\mathrm{G}}}{\otimes}(\mathrm{R}\underset{\mathrm{k}}{\otimes}\mathrm{V})^{\mathrm{G}}|\mathrm{m}\}$ are the same finite integer. Therefore, their inverse limit, which identifies with $(\mathrm{R}\underset{\mathrm{k}}{\otimes}\mathrm{V})^{\mathrm{G}}_{(\mathfrak{M})}$ is a free $\mathrm{R}^{\mathrm{G}}_{(\mathfrak{M})}$-module. It follows that, $(\mathrm{R}\underset{\mathrm{k}}{\otimes}\mathrm{V})^{\mathrm{G}}$ is a finitely generated projective module over the ring R^{G}.

Therefore, there exists some finitely generated free R^{G}-module F (of rank N) and a *split* surjection

(6.2.17) $$\zeta : \mathrm{F} \twoheadrightarrow (\mathrm{R}\underset{\mathrm{k}}{\otimes}\mathrm{V})^{\mathrm{G}}.$$

Then one sees that the induced maps

(6.2.18) $$\zeta/\mathrm{I_m} : \mathrm{R}^{\mathrm{G}}/\mathrm{I_m}\underset{\mathrm{R}^{\mathrm{G}}}{\otimes}\mathrm{F} \twoheadrightarrow \mathrm{R}^{\mathrm{G}}/\mathrm{I_m}\underset{\mathrm{R}^{\mathrm{G}}}{\otimes}(\mathrm{R}\underset{\mathrm{k}}{\otimes}\mathrm{V})^{\mathrm{G}}$$

are also split surjections for each m, and these splittings are in fact compatible, as they are all induced by the splitting to the map in (6.2.17). Therefore, we obtain a compatible collection of complements to the inverse system of bundles ξ^{V}_m in the trivial bundle of rank N over $\widetilde{\mathrm{BG}^{gm,m}}$, compatible as m varies. We denote the complement to ξ^{V}_m in the trivial bundle of rank N over $\widetilde{\mathrm{BG}^{gm,m}}$ as η^{V}_m.

Next we will consider *the case the group* G *is special*, in which the case the arguments in the following paragraph hold. Denoting by $\mathrm{T_V}$ the Thom-space of the representation V, the bundle $\widetilde{\mathrm{EG}^{gm,m}} \times_{\mathrm{G}} \mathrm{T_V}$ is a sphere-bundle over $\mathcal{B}_m$, which will be denoted $\mathrm{S}(\xi^{\mathrm{V}}_{\mathrm{m}} \oplus 1)$ in the terminology of (10.2.2). Similarly $\mathrm{S}(\eta^{\mathrm{V}}_{\mathrm{m}} \oplus 1)$ denotes the corresponding sphere bundle over $\mathcal{B}_m$. Now Lemma 10.5 (see below) shows that one obtains the identification:

(6.2.19) $$\mathrm{S}(\xi^{\mathrm{V}}_{\mathrm{m}} \oplus 1) \wedge^{\mathcal{B}_{\mathrm{m}}} \mathrm{S}(\eta^{\mathrm{V}}_{\mathrm{m}} \oplus 1) \simeq \mathrm{S}(\xi^{\mathrm{V}}_{\mathrm{m}} \oplus \eta^{\mathrm{V}}_{\mathrm{m}} \oplus 1).$$

Observe that there is a canonical section $s_{\xi^{\mathrm{V}}} : \mathcal{B}_m \to \widetilde{\mathrm{EG}^{gm,m}} \times_{\mathrm{G}} \mathrm{T_V} = \mathrm{S}(\xi^{\mathrm{V}}_{\mathrm{m}} \oplus 1)$, and a canonical section $s_\eta : \mathcal{B}_m \to \mathrm{S}(\eta^{\mathrm{V}}_{\mathrm{m}} \oplus 1)$, which together define a section $s_m : \mathcal{B}_m \to \mathrm{S}(\xi^{\mathrm{V}}_{\mathrm{m}} \oplus 1) \wedge^{\mathcal{B}_{\mathrm{m}}} \mathrm{S}(\eta^{\mathrm{V}}_{\mathrm{m}} \oplus 1)$ of pointed simplicial presheaves over $\mathcal{B}_m = \widetilde{\mathrm{BG}^{gm,m}}$. Then the quotient $(\mathrm{S}(\xi^{\mathrm{V}}_{\mathrm{m}} \oplus 1) \wedge^{\mathcal{B}_{\mathrm{m}}} \mathrm{S}(\eta^{\mathrm{V}}_{\mathrm{m}} \oplus 1))/\mathrm{s}(\mathcal{B}_{\mathrm{m}})$ identifies with the Thom-space

[6] Here we use V to denote both the representation of G and the corresponding symmetric algebra over k.

of the bundle $\xi_m^{\rm V} \oplus \eta_m^{\rm V}$. Since $\eta_m^{\rm V}$ was chosen to be a vector bundle complimentary to $\xi_m^{\rm V}$, $\xi_m^{\rm V} \oplus \eta_m^{\rm V}$ is a trivial bundle (of rank N) so that the above Thom-space identifies with $\mathbf{T}^{\wedge \rm N}(\widetilde{\rm BG}^{gm,m})_+$. Moreover, this holds independent of m.

In case the group G is *not special*, one has to replace $\widetilde{\rm EG}^{gm,m} \times_{\rm G} {\rm T_V}$ by ${\rm R}\epsilon_*(\widetilde{\rm EG}^{gm,m} \underset{\rm G}{\times}^{\rm et} \epsilon^*({\rm T_V}))$ and ${\rm S}(\xi_{\rm m}^{\rm V} \oplus \eta_{\rm m}^{\rm V} \oplus 1)$ with ${\rm R}\epsilon_*({\rm S}(\epsilon^*(\xi_{\rm m}^{\rm V} \oplus \eta_{\rm m}^{\rm V} \oplus 1)))$, making use of Lemma (6.7) to obtain the corresponding statement.

Let $\pi_{\rm Y}$ *denote either of the two projections* $\widetilde{\rm E} \times_{\rm G} ({\rm Y} \times {\rm X}) \to \widetilde{\rm E} \times_{\rm G} ({\rm Y})$ *or* $\mathcal{E}_m = \widetilde{\rm EG}^{gm,m} \times_{\rm G} ({\rm Y} \times {\rm X}) \to \widetilde{\rm EG}^{gm,m} \times_{\rm G} {\rm Y} = \mathcal{B}_{\rm m}$. *Since the second case is subsumed by the first, we will only discuss the first case explicitly in steps 3 through 5.*

Step 3. First we will again assume that the group-scheme G is special. Now observe that the sphere bundle $\widetilde{\rm E} \times_{\rm G} ({\rm Y} \times {\rm T_V})$ identifies with the pullback ${\rm S}(\pi^*(\xi^{\rm V}) \oplus 1) = \pi^*({\rm S}(\xi^{\rm V} \oplus 1))$ and the sphere bundle $\widetilde{\rm E} \times_{\rm G} ({\rm Y} \times {\rm X} \times {\rm T_V})$ identifies with the pullback ${\rm S}(\pi_{\rm Y}^*\pi^*(\xi^{\rm V}) \oplus 1) = \pi_{\rm Y}^*\pi^*({\rm S}(\xi^{\rm V} \oplus 1))$. Next consider

$$ {\rm S}(\pi_{\rm Y}^*\pi^*(\xi^{\rm V}) \oplus 1) \wedge^{\widetilde{\rm E}_{\rm Y}} {\rm S}(\pi_{\rm Y}^*\pi_{\rm Y}^*(\eta^{\rm V}) \oplus 1) = \pi_{\rm Y}^*\pi^*({\rm S}(\xi^{\rm V} \oplus 1)) \wedge^{\widetilde{\rm E}_{\rm Y}} \pi_{\rm Y}^*({\rm S}(\eta^{\rm V} \oplus 1)). $$

This is a sphere bundle over $\widetilde{\rm E}_{\rm Y}$ and it has a canonical section, which we will denote σ, collapsing which provides the Thom-space of the bundle ${\rm S}(\pi_{\rm Y}^*\pi^*(\xi^{\rm V} \oplus \eta^{\rm V} \oplus 1))$. Since $\eta^{\rm V}$ was chosen to be complementary to $\xi^{\rm V}$, it follows that the bundle $\pi_{\rm Y}^*\pi^*(\xi^{\rm V} \oplus \eta^{\rm V})$ is trivial, so that the resulting Thom-space identifies with $\mathbf{T}^{\wedge \rm N}(\widetilde{\rm E} \times_{\rm G} ({\rm Y} \times {\rm X}))_+$.

When the group-scheme G is *not special*, one adopts an argument as in the last paragraph of Step 2 to obtain a corresponding result.

Step 4. Observe that there is section $t' : \widetilde{\rm E}_{\rm Y} \to \widetilde{\rm E} \times_{\rm G} ({\rm Y} \times ({\rm X}_+ \wedge {\rm T_V}))$. Combining that with the canonical section $\widetilde{\rm E}_{\rm Y} \to {\rm S}(\pi^*(\eta^{\rm V} \oplus 1))$ defines a section $t : \widetilde{\rm E}_{\rm Y} \to (\widetilde{\rm E} \times_{\rm G} ({\rm Y} \times ({\rm X}_+ \wedge {\rm T_V})) \wedge^{\widetilde{\rm E}_{\rm Y}} {\rm S}(\pi^*(\eta^{\rm V} \oplus 1))$. Now a key observation is that $(\widetilde{\rm E} \underset{\rm G}{\times} ({\rm Y} \times ({\rm X}_+ \wedge {\rm T_V})) \wedge^{\widetilde{\rm E}_{\rm Y}} {\rm S}(\pi^*(\eta^{\rm V} \oplus 1))$ is an object defined over $\widetilde{\rm E}_{\rm Y}$ and that collapsing the section t also identifies the resulting object with $({\rm S}(\pi_{\rm Y}^*\pi^*(\xi^{\rm V} \oplus 1)) \wedge^{\widetilde{\rm E}_{\rm Y \times X}} {\rm S}(\pi_{\rm Y}^*\pi^*(\eta^{\rm V} \oplus 1)))/\sigma(\widetilde{\rm E}_{\rm Y \times X})$, where $\sigma : \widetilde{\rm E}_{\rm Y \times X} \to {\rm S}(\pi_{\rm Y}^*\pi^*(\xi^{\rm V} \oplus 1)) \wedge^{\widetilde{\rm E}_{\rm Y}} {\rm S}(\pi_{\rm Y}^*\pi^*(\eta^{\rm V} \oplus 1))$ is the canonical section. (See [BG75, (3.7) and (3.8)] for the classical case.)

One may see this as follows, first under the assumption that the group-scheme G is special. Assume that $\{{\rm U_i}|{\rm i}\}$ is a Zariski open cover of $\widetilde{\rm B}$ over which the G-torsor ${\rm p} : \widetilde{\rm E} \to \widetilde{\rm B}$ trivializes. $({\rm S}(\pi_{\rm Y}^*\pi^*(\xi^{\rm V} \oplus 1))_{|{\rm U_i}}$ now is of the form: ${\rm U_i} \times ({\rm Y} \times {\rm X} \times {\rm T_V}) \to {\rm U_i} \times {\rm Y} \times {\rm X}$. We may assume that the vector bundle $\eta^{\rm V}$ also trivializes over the cover $\{{\rm U_i}|{\rm i}\}$. Then $({\rm S}(\pi_{\rm Y}^*\pi^*(\xi^{\rm V} \oplus 1)) \wedge^{\widetilde{\rm E}_{\rm Y \times X}} {\rm S}(\pi_{\rm Y}^*\pi^*(\eta^{\rm V} \oplus 1)))_{|{\rm U_i}} = {\rm U_i} \times (({\rm Y} \times {\rm X}) \times ({\rm T_V} \wedge {\rm T_W}))$, where W corresponds to the fibers of the vector bundle $\eta^{\rm V}$. The section $\sigma_{|{\rm U_i}} : \widetilde{\rm E}_{{\rm Y \times X}|{\rm U_i}} \to ({\rm S}(\pi_{\rm Y}^*\pi^*(\xi^{\rm V} \oplus 1)) \wedge^{\widetilde{\rm E}_{\rm Y \times X}} {\rm S}(\pi_{\rm Y}^*\pi^*(\eta^{\rm V} \oplus 1)))_{|{\rm U_i}}$ now corresponds to the canonical section ${\rm U_i} \times {\rm Y} \times {\rm X} \to {\rm U_i} \times (({\rm Y} \times {\rm X}) \times ({\rm T_V} \wedge {\rm T_W}))$. Intermediate to collapsing the section σ is to take the pushout of

$$ \widetilde{\rm E}_{\rm Y} \leftarrow \widetilde{\rm E} \times_{\rm G} ({\rm Y} \times {\rm X}) \to {\rm S}(\pi_{\rm Y}^*\pi^*(\xi^{\rm V} \oplus 1)) \wedge^{\widetilde{\rm E}_{\rm Y \times X}} {\rm S}(\pi_{\rm Y}^*\pi^*(\eta^{\rm V} \oplus 1))). \tag{6.2.20} $$

Over ${\rm U_i}$, this corresponds to taking the pushout of ${\rm U_i} \times {\rm Y} \leftarrow {\rm U_i} \times ({\rm Y} \times {\rm X}) \to {\rm U_i} \times (({\rm Y} \times {\rm X}) \times ({\rm T_V} \wedge {\rm T_W}))$. The resulting pushout then identifies with ${\rm U_i} \times {\rm Y} \times ({\rm X}_+ \wedge ({\rm T_V} \wedge {\rm T_W}))$, which in fact identities with $(\widetilde{\rm E} \times_{\rm G} ({\rm Y} \times ({\rm X}_+ \wedge {\rm T_V}))) \wedge^{\widetilde{\rm E}_{\rm Y}} {\rm S}(\eta^{\rm V} \oplus 1))_{|{\rm U_i}}$.

Observe that collapsing the section σ can be done in two stages, by first taking the pushout in (6.2.20) and then by collapsing the resulting section from $\widetilde{\rm E}_{\rm Y}$. These complete the verification of the observation in Step 4, at least in the case the group-scheme G is special. When G is not special, one adopts a similar argument using an étale cover $\{{\rm U_i}|{\rm i} \in {\rm I}\}$ of $\widetilde{\rm B}$ over which $\widetilde{\rm E} \to \widetilde{\rm B}$ is trivial.

Step 5. Let $s : \widetilde{\rm E}_{\rm Y} \to \widetilde{\rm E} \times_{\rm G} ({\rm Y}_+ \wedge {\rm T_V}) \wedge^{\widetilde{\rm E}_{\rm Y}} {\rm S}(\pi^*(\eta^{\rm V} \oplus 1))$ denote the canonical section. Then, we proceed to show that the sections s and t are compatible in the sense that the diagram

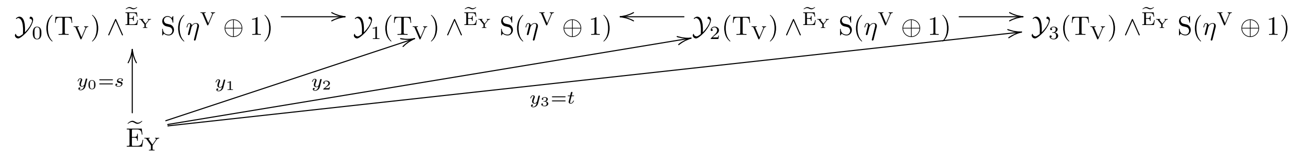
$$ \begin{array}{ccc} \widetilde{\rm E}_{\rm Y} & \xrightarrow{s} & \widetilde{\rm E} \times_{\rm G} ({\rm Y}_+ \wedge {\rm T_V}) \wedge^{\widetilde{\rm E}_{\rm Y}} {\rm S}(\pi^*(\eta^{\rm V} \oplus 1)) \\ & \searrow^{t} & \downarrow {\scriptstyle (id \times_{\rm G} tr_{\rm G}({\rm f_Y})'({\rm T_V})) \wedge^{\widetilde{\rm E}_{\rm Y}} {\rm id}} \\ & & \widetilde{\rm E} \times_{\rm G} (({\rm Y} \times {\rm X})_+ \wedge {\rm T_V}) \wedge^{\widetilde{\rm E}_{\rm Y}} {\rm S}(\pi^*(\eta^{\rm V} \oplus 1))) \end{array} \tag{6.2.21} $$

commutes, that is, in the sense discussed next. Here $(id \underset{G}{\times} tr_G(f_Y)'(T_V)$ is the component of the map of spectra $id \times_G tr_G(f_Y)'$ indexed by T_V. One may break this map into a sequence of maps

(6.2.22)
$$\mathcal{Y}_0(T_V) = \widetilde{E} \times_G (Y_+ \wedge T_V) \to \mathcal{Y}_1(T_V) = \widetilde{E} \times_G (Y_+ \wedge \mathcal{X}_1(T_V)) \leftarrow \mathcal{Y}_2(T_V) = \widetilde{E} \times_G (Y_+ \wedge \mathcal{X}_2(T_V)) \to \mathcal{Y}_3(T_V) = \widetilde{E} \times_G ((Y \times X)_+ \wedge T_V),$$

where the map $\{T_V \to \mathcal{X}_1(T_V) \leftarrow \mathcal{X}_2(T_V) \to X_+ \wedge T_V|V\}$ is the G-equivariant pre-transfer considered in (6.1.2). Observe that each of the objects in (6.2.22) is pointed over $\widetilde{E}_Y$. (When the group is non-special, the quotient sheaves in the diagram (6.2.22), and in the discussion below, are all taken in the étale topology on $\widetilde{E}_Y$ and one will have to replace the diagram in (6.2.21) with $R\epsilon_*$ applied to all the terms there.) One may observe that the corresponding sections from $\widetilde{E}_Y$ are all compatible as the group action leaves the base points of $T_V, \mathcal{X}_1(T_V), \mathcal{X}_2(T_V)$ and $X_+ \wedge T_V$ fixed. This results in the following commutative diagram over $\widetilde{E}_Y$:

(6.2.23)



By the commutativity of the triangle in (6.2.21), we mean the commutativity of all the corresponding triangles that make up the diagram in (6.2.23) and this is now clear in view of the above observations. When $\widetilde{E}_Y = \mathcal{E}_{m,Y} = \widetilde{EG^{gm,m}} \times_G Y$, one may again observe that the corresponding sections from $\mathcal{E}_{m,Y}$ are all compatible as the group action leaves the base points of $T_V, Y_+ \wedge \mathcal{X}_1(T_V), Y_+ \wedge \mathcal{X}_2(T_V)$ and $(Y \times X)_+ \wedge T_V$ fixed. This results in a corresponding diagram over each $\mathcal{E}_{m,Y}$ and the arguments in Step 2 above show that such commutative triangles are compatible as m varies.

Moreover, the commutativity of the diagram (6.2.23) shows that there is an induced map on the quotients by the sections y_i, $i = 0, 1, 2, 3$. Observe that on taking smash product *over* $\widetilde{E}_Y$ with $\widetilde{E} \times_G ((Y \times X)_+ \wedge T_W) \wedge^{\widetilde{E}_Y} S(\pi^*(\eta^W \oplus 1)) = (\widetilde{E}_Y \times \mathbf{T}^{\dim(W) + \mathrm{rank}(\eta^W)})_+$, one obtains a map of the diagram in (6.2.23) to the corresponding diagram with $V \oplus W$ in the place of V. This observation shows that if we define spectra $\mathcal{Z}_i$, $i = 0, 1, 2, 3$ in $\mathbf{Spt}_S$ by

$$\mathcal{Z}_{i,N_V} = (\mathcal{Y}_i(T_V) \wedge^{\widetilde{E}_Y} S(\pi^*(\eta^V \oplus 1)))/y_i(\widetilde{E}_Y), N_V = \dim(V) + \mathrm{rank}(\eta^V) \tag{6.2.24}$$

and if $N_W = \dim(W) + \mathrm{rank}(\eta^V)$, the smash product pairings $\mathbf{T}^{N_W} \wedge \mathcal{Z}_{i,N_V} \to \mathcal{Z}_{i,N_{V \oplus W}}$ are compatible with the maps between the $\mathcal{Z}_i$ considered above. (Note that these spectra are indexed by the integers $\{N_V|V\}$ and not by all the non-negative integers. However, since $\{N_V|V\}$ is cofinal in $\mathbb{N}$, this suffices.) One may also observe that the wrong-way map $\mathcal{Z}_2 \to \mathcal{Z}_1$ is a stable equivalence. Therefore, collapsing out the sections y_i, $i = 0, 3$, then provides the stable map (which in fact is a composition of several maps, with the ones going in the wrong direction being stable weak-equivalences)

(6.2.25)
$$tr(f_Y) : \Sigma_{\mathbf{T}}(\widetilde{E} \times_G Y)_+ \to \Sigma_{\mathbf{T}}(\widetilde{E} \times_G (Y \times X))_+, \quad tr(f_Y)^m : \Sigma_{\mathbf{T}}(\widetilde{EG^{gm,m}} \times_G Y)_+ \to \Sigma_{\mathbf{T}}(\widetilde{EG^{gm,m}} \times_G (Y \times X))_+$$

in case the group G is special, and the following stable map (which in fact is a composition of several maps, with the ones going in the wrong direction being stable weak-equivalences) in case G is not special:

(6.2.26)
$$tr(f_Y) : \Sigma_{\mathbf{T}} R\epsilon_*(\widetilde{E} \times^{et}_G Y)_+ \to \Sigma_{\mathbf{T}} R\epsilon_*(\widetilde{E} \times^{et}_G (Y \times X))_+, \quad tr(f_Y)^m : \Sigma_{\mathbf{T}} R\epsilon_*(\widetilde{EG^{gm,m}} \times^{et}_G Y)_+ \to \Sigma_{\mathbf{T}} R\epsilon_*(\widetilde{EG^{gm,m}} \times^{et}_G (Y \times X))_+.$$

These maps are also compatible as m varies, as observed above and in Step 2.

Definition 6.5. (The transfer.) Therefore, taking the colimit over $m \to \infty$, and making use of the identification in (6.2.11), one obtains the following stable transfer map (in $\mathbf{HSpt}_{mot}$) on the Borel construction:

$$tr(f_Y) : \Sigma_{\mathbf{T}}(\widetilde{E} \times_G Y)_+ \to \Sigma_{\mathbf{T}}(\widetilde{E} \times_G (Y \times X))_+, \tag{6.2.27}$$

$$tr(f_Y) : \Sigma_{\mathbf{T}}(\widetilde{EG^{gm}} \times_G Y)_+ \to \Sigma_{\mathbf{T}}(\widetilde{EG^{gm}} \times_G (Y \times X))_+, \text{ when G is special, and}$$

$$tr(f_Y) : \Sigma_{\mathbf{T}} R\epsilon_*(\widetilde{E_{et}} \times^{et}_G Y)_+ \to \Sigma_{\mathbf{T}} R\epsilon_*(\widetilde{E_{et}} \times^{et}_G (Y \times X))_+, \tag{6.2.28}$$

$$tr(f_Y) : \Sigma_{\mathbf{T}} R\epsilon_*(\widetilde{EG^{gm}_{et}} \times^{et}_G Y)_+ \to \Sigma_{\mathbf{T}} R\epsilon_*(\widetilde{EG^{gm}_{et}} \times^{et}_G (Y \times X))_+, \text{ when G is not special.}$$

Here $\widetilde{\mathrm{EG}^{gm}} \times_{\mathrm{G}} \mathrm{Y} = \lim_{\mathrm{m}\to\infty} \widetilde{\mathrm{EG}^{gm,m}} \times_{\mathrm{G}} \mathrm{Y}$ and $\widetilde{\mathrm{EG}^{gm}_{et}} \times^{et}_{\mathrm{G}} (\mathrm{Y}\times\mathrm{X}) = \lim_{\mathrm{m}\to\infty} \widetilde{\mathrm{EG}^{gm,m}_{\mathrm{et}}} \times_{\mathrm{G}} (\mathrm{Y}\times\mathrm{X})$.

If $\mathcal{E}^{\mathrm{G}}$ denotes a commutative G-equivariant ring spectrum as in (3.2.7), $\mathcal{E} = i^*(\tilde{\mathbb{P}}\tilde{\mathrm{U}}(\mathcal{E}^{\mathrm{G}}))$ is the corresponding ring spectrum in **Spt**, and $\mathcal{E}\wedge X_+$ is dualizable in $\mathbf{USpt}^{\mathrm{G}}_{mot,\mathcal{E}}$ ($\mathbf{USpt}^{\mathrm{G}}_{et,\mathcal{E}}$), the same constructions applied to the G-equivariant pre-transfer (6.1.2) and making use of smashing with the spectrum $\mathcal{E}$ in the place of smashing with $\mathbb{S}$ provides us with the transfer map:

(6.2.29)
$$tr(\mathrm{f_Y})_{\mathcal{E}} : \mathcal{E}\wedge(\tilde{\mathrm{E}}\times_{\mathrm{G}}\mathrm{Y})_+ \to \mathcal{E}\wedge(\tilde{\mathrm{E}}\times_{\mathrm{G}}(\mathrm{Y}\times\mathrm{X}))_+,$$
$$tr(\mathrm{f_Y})_{\mathcal{E}} : \mathcal{E}\wedge(\widetilde{\mathrm{EG}^{\mathrm{gm}}}\times_{\mathrm{G}}\mathrm{Y})_+ \to \mathcal{E}\wedge(\widetilde{\mathrm{EG}^{\mathrm{gm}}}\times_{\mathrm{G}}(\mathrm{Y}\times\mathrm{X}))_+, \text{ when G is special, and}$$
$$tr(\mathrm{f_Y})_{\mathcal{E}} : \mathcal{E}\wedge\mathrm{R}\epsilon_*(\tilde{\mathrm{E}}_{\mathrm{et}}\times^{\mathrm{et}}_{\mathrm{G}}\mathrm{Y})_+ \to \mathcal{E}\wedge\mathrm{R}\epsilon_*(\tilde{\mathrm{E}}_{\mathrm{et}}\times^{\mathrm{et}}_{\mathrm{G}}(\mathrm{Y}\times\mathrm{X}))_+,$$
$$tr(\mathrm{f_Y})_{\mathcal{E}} : \mathcal{E}\wedge\mathrm{R}\epsilon_*(\widetilde{\mathrm{EG}^{\mathrm{gm}}_{\mathrm{et}}}\times^{\mathrm{et}}_{\mathrm{G}}\mathrm{Y})_+ \to \mathcal{E}\wedge\mathrm{R}\epsilon_*(\widetilde{\mathrm{EG}^{\mathrm{gm}}_{\mathrm{et}}}\times^{\mathrm{et}}_{\mathrm{G}}(\mathrm{Y}\times\mathrm{X}))_+, \text{ when G is non-special.}$$

Remark 6.6. Suppose $\mathrm{X} = \mathrm{G/H}$ for a closed linear algebraic subgroup and $\mathrm{Y} = \mathrm{Spec}\, k$. Then the identification
$$\mathrm{R}\epsilon_*(\widetilde{\mathrm{EG}^{gm}_{\mathrm{et}}}\times^{\mathrm{et}}_{\mathrm{G}}\mathrm{G/H})_+ \simeq \mathrm{R}\epsilon_*(\mathrm{BH}^{gm}_{\mathrm{et}}) \simeq \epsilon_*(\lim_{\mathrm{m}\to\infty}\widetilde{\mathrm{BH}^{gm,m}})$$
shows that in this case the target of the transfer map in (6.2.26) is $\Sigma_{\mathbf{T}}(\widetilde{\mathrm{BH}^{gm}})_+$ and the target of the transfer map in (6.2.29) is $\mathcal{E}\wedge(\widetilde{\mathrm{BH}}^{gm})_+$. Observe that this holds irrespective of whether G or H is special.

Lemma 6.7. *Assume the situation as in* (6.2.8)*. Let* P *denote a pointed simplicial presheaf on the big Nisnevich site with* G*-action and let* $\mathrm{S}(\eta\oplus 1)$ *denote the (fiber-wise Thom-space) of a vector bundle* η *on the scheme* $\tilde{\mathrm{B}}$*. Then the natural map*
$$\mathrm{R}\epsilon_*(\tilde{\mathrm{E}}\times_{\mathrm{G}}(\mathrm{a}\circ\epsilon^*)(\mathrm{P}))\underset{\mathrm{R}\epsilon_*(\tilde{\mathrm{B}}_{\mathrm{et}})}{\wedge}\mathrm{R}\epsilon_*(\mathrm{S}((\mathrm{a}\circ\epsilon^*)(\eta\oplus 1)) \to \mathrm{R}\epsilon_*(\tilde{\mathrm{E}}\times_{\mathrm{G}}(\mathrm{a}\circ\epsilon^*)(\mathrm{P})\underset{(\mathrm{a}\circ\epsilon^*)(\tilde{\mathrm{B}})}{\wedge}\mathrm{S}((\mathrm{a}\circ\epsilon^*)(\eta)\oplus 1))$$
is a weak-equivalence.

Proof. First observe that there is a natural map from the left-hand-side to the right-hand-side. Therefore, it suffices to prove this is a weak-equivalence locally on the Zariski site of $\tilde{\mathrm{B}}$. Thus we reduce to the case where η is a trivial bundle. Moreover, as we work in the non-equivariant motivic homotopy theory (see Proposition 3.11 which shows the stable homotopy category associated to $\widetilde{\mathbf{USpt}}^{\mathrm{G}}$ is equivalent to **HSpt**), it suffices to consider the case of a trivial line bundle. In this case $\mathrm{S}(\eta\oplus 1)$ is just $\mathbf{T} = \mathbb{P}^1$.

Let $\mathcal{X}\in\mathbf{Spt}_{\mathrm{mot}}$. Now it suffices to show that one obtains an isomorphism (between Homs in the appropriate stable homotopy category):

(6.2.30)
$$[\mathcal{X}, \mathrm{R}\epsilon_*(\tilde{\mathrm{E}}\times_{\mathrm{G}}(\mathrm{a}\circ\epsilon^*)(\mathrm{P}))\wedge\mathbf{T}] \cong [\mathcal{X}, \mathrm{R}\epsilon_*(\tilde{\mathrm{E}}\times_{\mathrm{G}}(\mathrm{a}\circ\epsilon^*)(\mathrm{P})\wedge(\mathrm{a}\circ\epsilon^*)(\mathbf{T}))]$$

The term on the left identifies with $[\mathcal{X}\wedge\mathbf{T}^{-1}, \mathrm{R}\epsilon_*(\tilde{\mathrm{E}}\times_{\mathrm{G}}(\mathrm{a}\circ\epsilon^*)(\mathrm{P}))]$ which then identifies with $[a(\epsilon^*(\mathcal{X})\wedge\epsilon^*(\mathbf{T}^{-1})), \tilde{\mathrm{E}}\times_{\mathrm{G}}(a\circ\epsilon^*)(\mathrm{P})]$. The term on the right also identifies with the same term, by making use of the adjunction between ϵ^* and $\mathrm{R}\epsilon_*$. This completes the proof. ☐

6.3. Notational terminology for the rest of the paper. Let G denote a linear algebraic group over a perfect base field k acting on a smooth scheme X. Henceforth, we will let BG denote $\lim_{m\to\infty}\widetilde{\mathrm{BG}^{gm,m}}$. Recall this means we start with the finite dimensional scheme $\mathrm{BG}^{gm,m}$ and replace it by an affine scheme making use of Jouanolou's technique. Then EG will denote $\lim_{m\to\infty}\mathrm{EG}^{gm,m}\underset{\mathrm{BG}^{gm,m}}{\times}\widetilde{\mathrm{BG}^{gm,m}}$ and $\mathrm{EG}\times_{\mathrm{G}}\mathrm{X}$ the quotient of $\mathrm{EG}\times\mathrm{X}$ by the diagonal action of G. The ring spectra $\mathcal{E}\in\mathbf{Spt}_{\mathrm{mot}}$ we consider will always be obtained as $\mathcal{E} = i^*(\tilde{\mathbb{P}}\tilde{\mathrm{U}}(\mathcal{E}^{\mathrm{G}}))$ for some commutative ring spectrum $\mathcal{E}^{\mathrm{G}}\in\mathbf{Spt}^{\mathrm{G}}_{\mathrm{mot}}$.

Examples 6.8. *The following are some notable examples of such a transfer.*

(1) Let $i : \mathrm{H}\to\mathrm{G}$ *denote a closed immersion of linear algebraic groups over the base field, neither of which is assumed to special. Let* $\mathrm{X} = \mathrm{G/H}$ *with the obvious* G*-action,* $\mathrm{Y} = \mathrm{Spec}\, k$*, and let* $\mathrm{f} : \mathrm{G/H}\to\mathrm{G/H}$ *denote any* G*-equivariant map. Assume that either* $\Sigma_{\mathbf{T}}\mathrm{G/H}_+$ *is dualizable in* $\mathbf{Spt}_{\mathrm{mot}}$ *or that* $\mathcal{E}$ *is a commutative ring spectrum in* $\mathbf{Spt}_{\mathrm{mot}}$ *and* ℓ *is a prime different from* $char(k)$ *so that* $\mathcal{E}$ *is* $Z_{(\ell)}$*-local with* $\mathcal{E}\wedge\mathrm{G/H}_+$ *dualizable in* $\mathbf{Spt}_{\mathrm{mot},\mathcal{E}}$*. Then one may identify* $\Sigma_{\mathbf{T}}(\mathrm{EG}\times_{\mathrm{G}}\mathrm{G/H})_+\simeq\Sigma_{\mathbf{T}}\mathrm{BH}_+$ *in the motivic stable homotopy category so that the transfer in the first case is* $tr(\mathrm{f_Y}) : \Sigma_{\mathbf{T}}\mathrm{BG}_+\to\Sigma_{\mathbf{T}}\mathrm{BH}_+$ *and in the second case is* $tr(\mathrm{f_Y})_{\mathrm{G}} : \mathcal{E}\wedge\mathrm{BG}_+\to\mathcal{E}\wedge\mathrm{BH}_+$*.*

(2) Let $i : \mathrm{H} \to \mathrm{G}$ *be as above and let* Y *denote a quasi-projective scheme (or an unpointed simplicial presheaf on* $\mathrm{Sm/S}$*) with an action by* H*. Assume further that* $\mathcal{E}$ *is a commutative ring spectrum in* $\mathbf{Spt}_{\mathrm{mot}}$ *and* ℓ *is a prime different from char(k) so that* $\mathcal{E}$ *is* $Z_{(\ell)}$*-local. Then, the* G*-scheme* $\mathrm{G} \times_{\mathrm{H}} \mathrm{Y}$ *identifies as a* G*-scheme with* $\mathrm{G/H} \times \mathrm{Y}$ *(provided with the diagonal action by* G*). Clearly* $\mathrm{G/H}$ *is dualizable in* $\mathbf{Spt}_{\mathrm{mot}}$ *in case* $char(k) = 0$ *and* $\mathcal{E} \wedge \mathrm{G/H}_+$ *is dualizable in* $\mathbf{Spt}_{\mathrm{mot},\mathcal{E}}$ *in case* $char(k) = p > 0$*. The corresponding transfer, when both* G *and* H *are special, is then the stable map* $tr(id_{\mathrm{Y}}) : \Sigma_{\mathbf{T}}(\mathrm{EG} \times_{\mathrm{G}} \mathrm{Y})_+ \to \Sigma_{\mathbf{T}}(\mathrm{EG} \times_{\mathrm{G}} (\mathrm{G} \times_{\mathrm{H}} \mathrm{Y}))_+ \simeq \Sigma_{\mathbf{T}}(\mathrm{EH} \times_{\mathrm{H}} \mathrm{Y})_+$ *in the first case and the map* $tr(id_{\mathrm{Y}}) : \mathcal{E} \wedge (\mathrm{EG} \times_{\mathrm{G}} \mathrm{Y})_+ \to \mathcal{E} \wedge (\mathrm{EG} \times_{\mathrm{G}} (\mathrm{G} \times_{\mathrm{H}} \mathrm{Y}))_+ \simeq \mathcal{E} \wedge (\mathrm{EH} \times_{\mathrm{H}} \mathrm{Y})_+$ *in the second case. In case these are not special, one obtains corresponding stable transfer maps involving* $R\epsilon_*$ *as in* (6.2.26) *and* (6.2.29).

7. Basic properties of the transfer

Throughout this section (and for the remainder of the paper) we will adopt the notational conventions in 6.3.

Proposition 7.1. *(Naturality with respect to base-change and change of groups) Let* G *denote a linear algebraic group over* k *and let* X, Y *denote smooth quasi-projective* G*-schemes over* k *or unpointed simplicial presheaves on* $\mathrm{Sm/S}$ *provided with* G*-actions. Let* $\mathrm{p} : \mathrm{E} \to \mathrm{B}$ *denote a* G*-torsor with* E *and* B *smooth quasi-projective schemes over* k*, with* B *connected and let* $\pi_{\mathrm{Y}} : \mathrm{E} \times_{\mathrm{G}} (\mathrm{Y} \times \mathrm{X}) \to \mathrm{E} \times_{\mathrm{G}} \mathrm{Y}$ *denote any one of the maps considered in Theorem 1.1(a), (b) or (c). Let* $\mathrm{f} : \mathrm{X} \to \mathrm{X}$ *denote a* G*-equivariant map.*

Let G' *denote a closed linear algebraic subgroup of* G, $\mathrm{p}' : \mathrm{E}' \to \mathrm{B}'$ *a* G'*-torsor with* B' *connected, and* Y' *a* G'*-quasi-projective scheme over* k *or an unpointed simplicial sheaf* $\mathrm{Sm/S}$ *provided with an* G'*-actions, so that it comes equipped with a map* $\mathrm{Y}' \to \mathrm{Y}$ *that is compatible with the* G'*-action on* Y' *and the* G*-action on* Y*. Further, we assume that one is provided with a commutative square*

$$\begin{array}{ccc} \mathrm{E}' & \longrightarrow & \mathrm{E} \\ \downarrow & & \downarrow \\ \mathrm{B}' & \longrightarrow & \mathrm{B} \end{array}$$

compatible with the action of G' *(*G*) on* E' *(*E*, respectively). Let* $\pi_{\mathrm{Y}'} : \mathrm{E}' \times_{\mathrm{G}'} (\mathrm{Y}' \times \mathrm{X}) \to \mathrm{E}' \times_{\mathrm{G}'} \mathrm{Y}'$ *denote any one of the maps considered in Theorem 1.1(a), (b) or (c).*

Then if $\Sigma_{\mathbf{T}}\mathrm{X}_+$ *is dualizable in* $\mathbf{Spt}_{\mathrm{mot}}$*, the square*

$$\begin{array}{ccc} \Sigma_{\mathbf{T}}(\mathrm{E}' \times_{\mathrm{G}'} (\mathrm{Y}' \times \mathrm{X}))_+ & \longrightarrow & \Sigma_{\mathbf{T}}(\mathrm{E} \underset{\mathrm{G}}{\times} (\mathrm{Y} \times \mathrm{X}))_+ \\ {\scriptstyle tr(\mathrm{f}_{\mathrm{Y}'})}\uparrow & & \uparrow{\scriptstyle tr(\mathrm{f}_{\mathrm{Y}})} \\ \Sigma_{\mathbf{T}}(\mathrm{E}' \times_{\mathrm{G}'} \mathrm{Y}')_+ & \longrightarrow & \Sigma_{\mathbf{T}}(\mathrm{E} \times_{\mathrm{G}} \mathrm{Y})_+ \end{array} \qquad \begin{array}{ccc} (\Sigma_{\mathbf{T}}\mathrm{R}\epsilon_*(\mathrm{E}' \times^{\mathrm{et}}_{\mathrm{G}'} \epsilon^*(\mathrm{Y}' \times \mathrm{X}))_+ & \longrightarrow & \Sigma_{\mathbf{T}}\mathrm{R}\epsilon_*(\mathrm{E} \times^{\mathrm{et}}_{\mathrm{G}} \epsilon^*(\mathrm{Y} \times \mathrm{X}))_+ \\ {\scriptstyle tr(\mathrm{f}_{\mathrm{Y}'})}\uparrow & & \uparrow{\scriptstyle tr(\mathrm{f}_{\mathrm{Y}})} \\ \Sigma_{\mathbf{T}}\mathrm{R}\epsilon_*(\mathrm{E}' \times^{\mathrm{et}}_{\mathrm{G}'} \epsilon^*(\mathrm{Y}'))_+ & \longrightarrow & \Sigma_{\mathbf{T}}\mathrm{R}\epsilon_*(\mathrm{E} \times^{\mathrm{et}}_{\mathrm{G}} \epsilon^*(\mathrm{Y}))_+) \end{array}$$

commutes in the motivic stable homotopy category when G *is special (*G *is not necessarily special, respectively). Next let* $\mathcal{E}$ *denote a commutative ring spectrum in* $\mathbf{Spt}_{\mathrm{mot}}$ *with* $\mathcal{E} \wedge X_+$ *dualizable in* $\mathbf{Spt}_{\mathrm{mot},\mathcal{E}}$*. Then the square*

$$\begin{array}{ccc} \mathcal{E} \wedge (\mathrm{E}' \times_{\mathrm{G}'} (\mathrm{Y}' \times \mathrm{X}))_+ & \longrightarrow & \mathcal{E} \wedge (\mathrm{E} \underset{\mathrm{G}}{\times} (\mathrm{Y} \times \mathrm{X}))_+ \\ {\scriptstyle tr(\mathrm{f}_{\mathrm{Y}'})}\uparrow & & \uparrow{\scriptstyle tr(\mathrm{f}_{\mathrm{Y}})} \\ \mathcal{E} \wedge (\mathrm{E}' \times_{\mathrm{G}'} \mathrm{Y}')_+ & \longrightarrow & \mathcal{E} \wedge (\mathrm{E} \times_{\mathrm{G}} \mathrm{Y})_+ \end{array} \qquad \begin{array}{ccc} (\mathcal{E} \wedge \mathrm{R}\epsilon_*(\mathrm{E}' \times^{\mathrm{et}}_{\mathrm{G}'} \epsilon^*(\mathrm{Y}' \times \mathrm{X}))_+ & \longrightarrow & \mathcal{E} \wedge \mathrm{R}\epsilon_*(\mathrm{E} \times^{\mathrm{et}}_{\mathrm{G}} \epsilon^*(\mathrm{Y} \times \mathrm{X}))_+ \\ {\scriptstyle tr(\mathrm{f}_{\mathrm{Y}'})}\uparrow & & \uparrow{\scriptstyle tr(\mathrm{f}_{\mathrm{Y}})} \\ \mathcal{E} \wedge \mathrm{R}\epsilon_*(\mathrm{E}' \times^{\mathrm{et}}_{\mathrm{G}'} \epsilon^*(\mathrm{Y}'))_+ & \longrightarrow & \mathcal{E} \wedge \mathrm{R}\epsilon_*(\mathrm{E} \times^{\mathrm{et}}_{\mathrm{G}} \epsilon^*(\mathrm{Y}))_+) \end{array}$$

commutes in the motivic stable homotopy category, $\mathbf{HSpt}_{\mathrm{mot},\mathcal{E}}$ *when* G *is special (*G *is not necessarily special, respectively). (In this case we may require* ℓ *is a prime* $\neq char(\mathrm{k})$ *so that* $\mathcal{E}$ *is* $Z_{(\ell)}$*-local.)*

In case $\mathcal{E}$ *denotes a commutative ring spectrum in* $\mathbf{Spt}_{\mathrm{et}}$ *which is* ℓ*-complete for some prime* ℓ *and* $\mathcal{E} \wedge \mathrm{X}_+$ *is dualizable in* $\mathbf{Spt}_{\mathrm{et},\mathcal{E}}$*, the same conclusions hold in the corresponding étale stable homotopy category* $\mathbf{HSpt}_{\mathrm{et},\mathcal{E}}$*.*

Proof. For each fixed representation V of G, let ξ^{V} (η^{V}) denote the vector bundles on $\widetilde{\mathrm{B}}$ chosen as in (6.2.14) ((6.2.16), respectively). Let ${\xi'}^{\mathrm{V}}$ (${\eta'}^{\mathrm{V}}$) denote the pull-back of these bundles to $\widetilde{\mathrm{B}}'$. Since $\xi^{\mathrm{V}} \oplus \eta^{\mathrm{V}}$ is trivial, so is ${\xi'}^{\mathrm{V}} \oplus {\eta'}^{\mathrm{V}}$. Now the required property follows readily in view of this observation and the definition of the transfer as in Definition 6.5, in view of the cartesian square:

(7.0.1)
$$\begin{array}{ccc} \mathrm{E}' \times_{\mathrm{G}'} (\mathrm{Y}' \times \mathrm{X}) & \longrightarrow & \mathrm{E} \times_{\mathrm{G}} (\mathrm{Y} \times \mathrm{X}) \\ \downarrow{\scriptstyle \pi_{\mathrm{Y}'}} & & \downarrow{\scriptstyle \pi_{\mathrm{Y}}} \\ \mathrm{E}' \times_{\mathrm{G}'} \mathrm{Y}' & \longrightarrow & \mathrm{E} \times_{\mathrm{G}} \mathrm{Y}. \end{array}$$

The main observation here is that the diagrams in (6.2.12) through (6.2.26) for $\widetilde{\mathrm{E}}' \times_{\mathrm{G}} \mathrm{Y}$ map to the corresponding diagrams for $\widetilde{\mathrm{E}} \times_{\mathrm{G}} \mathrm{Y}$ making the resulting diagrams commute, since all the transfers are constructed from the pre-transfer $tr_{\mathrm{G}}(\mathrm{f})'$. □

Remark 7.2. Taking different choices for p' and Y' provides many examples where the last Proposition applies. For example, let $\mathrm{B}' \to \mathrm{B}$ denote a map from another smooth quasi-projective scheme that is also connected, let $\mathrm{E}' = \mathrm{E}\underset{\mathrm{B}}{\times}\mathrm{B}'$ and let $\pi'_{\mathrm{Y}} : \mathrm{E}' \times_{\mathrm{G}} (\mathrm{Y} \times \mathrm{X}) \to \mathrm{E}' \times_{\mathrm{G}} \mathrm{Y}$ denote the induced map. (In particular, B' could be given by $\mathrm{E/H} \cong \mathrm{E} \times_{\mathrm{G}} \mathrm{G/H}$ for a closed subgroup H so that E/H is connected.) Moreover, the above proposition readily provides the following key multiplicative property of the transfer.

Proposition 7.3. *(Multiplicative property) (i) Assume* X *is a smooth quasi-projective variety over* k *or an unpointed simplicial presheaf on* Sm/S *provided with a* G*-action, so that* $\Sigma_{\mathbf{T}}\mathrm{X}_+$ *is a dualizable in* $\mathbf{Spt}_{\mathrm{mot}}$. *Let* H *denote a linear algebraic group and let* $\mathrm{G} = \mathrm{H} \times \mathrm{H}$ *with* $i = \Delta =$ *the diagonal imbedding of* H *in* $\mathrm{G} = \mathrm{H} \times \mathrm{H}$. *Let* $\mathrm{p} : \mathrm{E} \to \mathrm{B}$ *denote an* H*-torsor and let* $\pi_{\mathrm{Y}} : \mathrm{E}\underset{\mathrm{H}}{\times}\mathrm{X} \to \mathrm{B}$ *denote the induced map as in Theorem 1.1(a), (b) or (c). Then the diagram*

$$\begin{array}{ccccc} \Sigma_{\mathbf{T}}\mathrm{E} \times_{\mathrm{H}} (\mathrm{Y} \times \mathrm{X})_+ & \xrightarrow{\mathrm{d}} & \Sigma_{\mathbf{T}}(\mathrm{E} \times \mathrm{E})\underset{\mathrm{H}\times\mathrm{H}}{\times}((\mathrm{Y} \times \mathrm{X}) \times (\mathrm{Y} \times \mathrm{X}))_+ & \xrightarrow{\mathrm{id}\wedge\mathrm{q}\wedge\mathrm{id}} & \Sigma_{\mathbf{T}}(\mathrm{E} \times \mathrm{E})\underset{\mathrm{H}\times\mathrm{H}}{\times}(\mathrm{Y} \times (\mathrm{Y} \times \mathrm{X}))_+ \\ \uparrow{\scriptstyle tr(\mathrm{f_Y})} & & & & \uparrow{\scriptstyle \mathrm{id}\wedge tr(\mathrm{f_Y})} \\ \Sigma_{\mathbf{T}}\mathrm{E} \times_{\mathrm{H}} \mathrm{Y}_+ & & \xrightarrow{\mathrm{d}} & & \Sigma_{\mathbf{T}}(\mathrm{E} \times_{\mathrm{H}} \mathrm{Y})_+ \wedge (\mathrm{E} \times_{\mathrm{H}} \mathrm{Y})_+ \end{array}$$

commutes in the motivic stable homotopy category, when H *is special. Here* d *denotes the diagonal map induced by the diagonal map* $\mathrm{Y} \times \mathrm{X} \to (\mathrm{Y} \times \mathrm{X}) \times (\mathrm{Y} \times \mathrm{X})$ *and* q *denotes the map induced by the projection* $\mathrm{Y} \times \mathrm{X} \to \mathrm{Y}$. *In case* H *is not necessarily special, one obtains a corresponding commutative diagram which is obtained by applying* $\mathrm{R}\epsilon_*$ *to the corresponding terms in the above diagram after the presheaves* Y, $\mathrm{Y} \times \mathrm{X}$ *and* $(\mathrm{Y} \times \mathrm{X}) \times (\mathrm{Y} \times \mathrm{X})$ *have been replaced by their pull-back as sheaves to the étale site and the quotients are taken on the étale site.*

(ii) In case $\mathcal{E}$ *is a commutative ring spectrum in* $\mathbf{Spt}_{\mathrm{mot}}$ *with* $\mathcal{E} \wedge \mathrm{X}_+$ *dualizable in* $\mathbf{Spt}_{\mathrm{mot},\mathcal{E}}$ *and* X *as in (i), then the square*

$$\begin{array}{ccccc} \mathcal{E} \wedge \mathrm{E} \times_{\mathrm{H}} (\mathrm{Y} \times \mathrm{X})_+ & \xrightarrow{\mathrm{d}} & \mathcal{E} \wedge (\mathrm{E} \times \mathrm{E})\underset{\mathrm{H}\times\mathrm{H}}{\times}((\mathrm{Y} \times \mathrm{X}) \times (\mathrm{Y} \times \mathrm{X}))_+ & \xrightarrow{\mathrm{id}\wedge\mathrm{q}\wedge\mathrm{id}} & \mathcal{E} \wedge (\mathrm{E} \times \mathrm{E})\underset{\mathrm{H}\times\mathrm{H}}{\times}(\mathrm{Y} \times (\mathrm{Y} \times \mathrm{X}))_+ \\ \uparrow{\scriptstyle tr(\mathrm{f_Y})} & & & & \uparrow{\scriptstyle \mathrm{id}\wedge tr(\mathrm{f_Y})} \\ \mathcal{E} \wedge \mathrm{E} \times_{\mathrm{H}} \mathrm{Y}_+ & & \xrightarrow{\mathrm{d}} & & \mathcal{E} \wedge (\mathrm{E} \times_{\mathrm{H}} \mathrm{Y})_+ \wedge (\mathrm{E} \times_{\mathrm{H}} \mathrm{Y})_+ \end{array}$$

also commutes in the motivic stable homotopy category, $\mathbf{HSpt}_{\mathrm{mot},\mathcal{E}}$, *, when* H *is special. (Here we may also require that* ℓ *is a prime,* $\neq char(\mathrm{k})$ *so that* $\mathcal{E}$ *is* $Z_{(\ell)}$*-local.) In case* H *is not necessarily special, one obtains a corresponding commutative diagram with all the terms there replaced as in (i). In case* $\mathcal{E}$ *denotes a commutative ring spectrum in* $\mathbf{Spt}_{\mathrm{et}}$ *which is* ℓ*-complete for some prime* $\ell \neq char(k)$, *and* $\mathcal{E} \wedge \mathrm{X}_+$ *is dualizable in* $\mathbf{Spt}_{\mathrm{et},\mathcal{E}}$, *the corresponding square commutes in the étale stable homotopy category* $\mathbf{HSpt}_{\mathrm{et},\mathcal{E}}$.

Proof. We apply Proposition 7.1 with the following choices:

(i) for G, we take $\mathrm{H} \times \mathrm{H}$, for G' we take the diagonal H in $\mathrm{H} \times \mathrm{H}$,
(ii) for p (p') we take $\mathrm{p} \times \mathrm{p} : \mathrm{E} \times \mathrm{E} \to \mathrm{B} \times \mathrm{B}$ (the given $\mathrm{p} : \mathrm{E} \to \mathrm{B}$, respectively),
(iii) for Y we take $\mathrm{Y} \times \mathrm{Y}$ provided with the obvious action of $\mathrm{H} \times \mathrm{H}$ and for Y' we take Y provided with the given action of H.

Then we obtain the cartesian square as in (7.0.1) and the map in the corresponding top row is given by

$$\mathrm{E} \times_{\mathrm{H}} (\mathrm{Y} \times \mathrm{X}) \to (\mathrm{E} \times \mathrm{E})\underset{\mathrm{H}\times\mathrm{H}}{\times}(\mathrm{Y} \times (\mathrm{Y} \times \mathrm{X})).$$

This factors as $E \times_H (Y \times X) \overset{d}{\rightarrow} (E \times E) \underset{H \times H}{\times} ((Y \times X) \times (Y \times X)) \overset{id \times q \times id}{\rightarrow} (E \times E) \underset{H \times H}{\times} (Y \times (Y \times X))$. Therefore, Proposition 7.1 applies. □

Definition 7.4. (Weak module spectra over commutative ring spectra) Let A denote a commutative ring spectrum in $\mathbf{Spt}_{mot}$ ($\mathbf{Spt}_{et}$). Then a spectrum $M \in \mathbf{Spt}_{mot}$ ($\mathbf{Spt}_{et}$) is a weak-module spectrum over A if M is equipped with a pairing $\mu : M \wedge A \rightarrow M$ that is homotopy associative.

Corollary 7.5. *(Multiplicative property of transfer in generalized cohomology theories) Let* $h^{*,\bullet}$ *denote a generalized cohomology theory defined for all smooth schemes of finite type over k with respect to a motivic ring spectrum* A*. Let* H *denote a linear algebraic group. Let* $\pi_Y : E \times_H (Y \times X) \rightarrow E \times_H Y$ *(*$\pi_Y : R\epsilon_*(E \times_H (Y \times X) \rightarrow R\epsilon_*(E \times_H Y)$*) denote the (obvious) map induced by the structure map* $X \rightarrow \operatorname{Spec} k$ *when* H *is special (*H *is not necessarily special) as in Proposition 7.3.*

(i) Assume X *is a smooth quasi-projective variety over k or an unpointed simplicial presheaf on* Sm/S *provided with a* H*-action, so that* $\Sigma_{\mathbf{T}} X_+$ *is a dualizable in* $\mathbf{Spt}_{mot}$*. Then*

$$tr(f_Y)^*(\pi_Y^*(\alpha).\beta) = \alpha.tr(f_Y)^*(\beta), \quad \alpha \in h^{*,\bullet}(E \times_H Y), \beta \in h^{*,\bullet}(E \times_H (Y \times X))$$

$$(\alpha \in h^{*,\bullet}(R\epsilon_*(E \times_H Y)), \beta \in h^{*,\bullet}(R\epsilon_*(E \times_H (Y \times X))))$$

when H *is special (otherwise, respectively). Here* $tr(f_Y)^*$ *(*π_Y^**) denotes the map induced on generalized cohomology by the map* $tr(f_Y)$ *(*π_Y*, respectively). In particular,* π_Y^* *is split injective if* $tr(f_Y)^*(1) = tr(f_Y)^*(\pi_Y^*(1))$ *is a unit, where* $1 \in h^{0,0}(E \times_H Y)$ *(*$1 \in h^{*,\bullet}(R\epsilon_*(E \times_H Y))$*) is the unit of the graded ring* $h^{*,\bullet}(E \times_H Y)$ *(*$h^{*,\bullet}(R\epsilon_*(E \times_H Y))$*, respectively).*

(ii) Assume that X *is a smooth quasi-projective variety over k or an unpointed simplicial presheaf on* Sm/S *provided with a* H*-action and that* A *is a commutative ring spectrum in* $\mathbf{Spt}_{mot}$ *with* $A \wedge X_+$ *dualizable in* $\mathbf{Spt}_{mot,A}$*.*

Suppose M *is a motivic spectrum that is a weak-module spectrum over the commutative motivic ring spectrum* A*. Then*

$$tr(f_Y)^*(\pi_Y^*(\alpha).\beta) = \alpha.tr(f_Y)^*(\beta), \quad \alpha \in h^{*,\bullet}(E \times_H Y, M), \beta \in h^{*,\bullet}(E \times_H (Y \times X)), A),$$

$$(\alpha \in h^{*,\bullet}(R\epsilon_*(E \times_H Y), M), \beta \in h^{*,\bullet}(R\epsilon_*(E \times_H (Y \times X)), A))$$

when H *is special (otherwise, respectively). Here* $tr(f_Y)^*$ *(*π_Y^**) denotes the map induced on generalized cohomology by the map* $tr(f_Y)$ *(*π_Y*, respectively). In particular,*

$$\pi_Y^* : h^{*,\bullet}(E \times_H Y, M) \rightarrow h^{*,\bullet}(E \times_H (Y \times X), M) \quad (\pi_Y^* : h^{*,\bullet}(R\epsilon_*(E \times_H Y), M) \rightarrow h^{*,\bullet}(R\epsilon_*(E \times_H (Y \times X), M)) \text{ is split injective}$$

when H *is special (in general, respectively) if* $tr(f_Y)^*(1) = tr(f_Y)^*(\pi_Y^*(1))$ *is a unit, where* $1 \in h^{0,0}(E \times_H Y, A)$ *(*$1 \in h^{0,0}(R\epsilon_*(E \times_H Y), A)$*) is the unit of the graded ring* $h^{*,\bullet}(E \times_H Y, A)$ *(*$h^{*,\bullet}(R\epsilon_*(E \times_H Y), A)$*, respectively).*

(iii) Assume that A *is a commutative ring spectrum in* $\mathbf{Spt}_{et}$*,* ℓ *is a prime* $\neq char(k)$ *so that* A *is* ℓ*-complete, with* $A \wedge X_+$ *dualizable in* $\mathbf{Spt}_{et,A}$*, and that* M *is a weak module spectrum over* A*. Then the same conclusions as in (ii) hold.*

Proof. The proof of both statements follow by applying the cohomology theory $h^{*,\bullet}$ to all terms in the commutative diagram in Proposition 7.3. We will provide details only for the case H is special, as the general case follows similarly. In the proof of (ii), one needs to start with $h^{*,\bullet}(E \times_H Y, M)$ and $h^{*,\bullet}(E \times_H (Y \times X), M)$ and supplement the arguments for (i) with the module property $h^{*,\bullet}(E \times_H Y, M) \otimes h^{*,\bullet}(E \times_H (Y \times X), A) \overset{\pi_Y^* \otimes id}{\rightarrow} h^{*,\bullet}(E \times_H (Y \times X), M) \otimes h^{*,\bullet}(E \times_H (Y \times X), A) \rightarrow h^{*,\bullet}(E \times_H (Y \times X), M)$ and $h^{*,\bullet}(E \times_H Y, M) \otimes h^{*,\bullet}(E \times_H Y, A) \rightarrow h^*(E \times_H Y, M)$. This provides us with the commutative diagram:
(7.0.2)

$$\begin{array}{ccccc}
h^{*,\bullet}(E \times_H Y, M) \otimes h^{*,\bullet}(E \times_H (Y \times X), A) & & \xrightarrow{id \otimes tr(f_Y)^*} & & h^{*,\bullet}(E \times_H Y, M) \otimes h^*(E \times_H Y, A) \\
\downarrow{\scriptstyle \pi_Y^* \otimes id} & & & & \downarrow{\scriptstyle d^*} \\
h^{*,\bullet}(E \times_H (Y \times X), M) \otimes h^{*,\bullet}(E \times_H (Y \times X)A) & \rightarrow & h^{*,\bullet}(E \times_H (Y \times X), M) & \xrightarrow{tr(f_Y)^*} & h^{*,\bullet}(E \times_H Y, M)
\end{array}$$

Since the multiplicative property is the key to obtaining splittings in the motivic stable homotopy category (see Theorem 1.5), we will provide details on how one deduces commutativity of the above diagram. The commutativity of the above diagram follows from the commutativity of a large diagram which we break up into three squares as follows. *RHom* denotes the derived external Hom in the $\mathbf{Spt}_{mot,A}$.

(7.0.3)
$$\begin{array}{ccc} RHom(\mathrm{E}\times_{\mathrm{H}}\mathrm{Y},\mathrm{M})\wedge \mathrm{RHom}(\mathrm{E}\times_{\mathrm{H}}(\mathrm{Y}\times\mathrm{X}),\mathrm{A}) & \longrightarrow & RHom((\mathrm{E}\times\mathrm{E})_{\mathrm{H}\times\mathrm{H}}(\mathrm{Y}\times(\mathrm{Y}\times\mathrm{X})),\mathrm{M}\wedge\mathrm{A}) \\ \downarrow{\scriptstyle id\wedge tr(\mathrm{f_Y})^*} & & \downarrow{\scriptstyle id\wedge tr(\mathrm{f_Y})^*} \\ RHom(\mathrm{E}\times_{\mathrm{H}}\mathrm{Y},\mathrm{M})\wedge \mathrm{RHom}(\mathrm{E}\times_{\mathrm{H}}\mathrm{Y},\mathrm{A}) & \longrightarrow & RHom((\mathrm{E}\times_{\mathrm{H}}\mathrm{Y})_+\wedge(\mathrm{E}\times_{\mathrm{H}}\mathrm{Y})_+,\mathrm{M}\wedge\mathrm{A}) \end{array}$$

$$\begin{array}{ccc} RHom((\mathrm{E}\times\mathrm{E})_{\mathrm{H}\times\mathrm{H}}(\mathrm{Y}\times(\mathrm{Y}\times\mathrm{X})),\mathrm{M}\wedge\mathrm{A}) & \xrightarrow{d^*\circ(\pi_{\mathrm{Y}}^*\wedge id)} & RHom(\mathrm{E}\times_{\mathrm{H}}(\mathrm{Y}\times\mathrm{X}),\mathrm{M}\wedge\mathrm{A}) \\ \downarrow{\scriptstyle id\wedge tr(\mathrm{f_Y})^*} & & \downarrow{\scriptstyle tr(\mathrm{f_Y})^*} \\ RHom((\mathrm{E}\times_{\mathrm{H}}\mathrm{Y})_+\wedge(\mathrm{E}\times_{\mathrm{H}}\mathrm{Y})_+,\mathrm{M}\wedge\mathrm{A}) & \xrightarrow{d^*} & RHom(\mathrm{E}\times_{\mathrm{H}}\mathrm{Y},\mathrm{M}\wedge\mathrm{A}) \end{array}$$

$$\begin{array}{ccc} RHom(\mathrm{E}\times_{\mathrm{H}}(\mathrm{Y}\times\mathrm{X}),\mathrm{M}\wedge\mathrm{A}) & \xrightarrow{\mu} & RHom(\mathrm{E}\times_{\mathrm{H}}(\mathrm{Y}\times\mathrm{X}),\mathrm{M}) \\ \downarrow{\scriptstyle tr(\mathrm{f_Y})^*} & & \downarrow{\scriptstyle tr(\mathrm{f_Y})^*} \\ RHom(\mathrm{E}\times_{\mathrm{H}}\mathrm{Y},\mathrm{M}\wedge\mathrm{A}) & \xrightarrow{\mu} & RHom(\mathrm{E}\times_{\mathrm{H}}\mathrm{Y},\mathrm{M}) \end{array}$$

The commutativity of the first square is clear from the observation that the transfer $tr(\mathrm{f_Y})$ is a stable map. The commutativity of the second square is essentially the multiplicative property proved in Proposition 7.3. The map μ in the third square is the map induced by the pairing $\mathrm{M}\wedge\mathrm{A}\rightarrow\mathrm{M}$. The commutativity of this square again follows readily from the observation that $tr(\mathrm{f_Y})$ is a stable map. The commutativity of the square in (7.0.2) results by composing the appropriate maps in the first square followed by the second and then the third square. More precisely, one can see that the composition of the maps in the top rows of the three squares followed by the right vertical map in the last square equals $tr(\mathrm{f_Y})^*\circ \mathrm{d}^*\circ(\pi_{\mathrm{Y}}^*\otimes \mathrm{id})$ which is the composition of the left vertical map and the bottom row in the square (7.0.2). Similarly the composition of the left vertical map in the first square and the maps in the bottom rows of the three squares above equals $d^*\circ(id\otimes tr(\mathrm{f_Y})^*)$ which is the composition of the top row and the right vertical map in (7.0.2).

The last statement in (ii) follows by taking $\beta=1=\pi_{\mathrm{Y}}^*(1)\in \mathrm{h}^{0,0}(\mathrm{E}\times_{\mathrm{H}}(\mathrm{Y}\times\mathrm{X}),\mathrm{A})$. The statement in (iii) follows from an entirely similar argument in the étale case. ☐

Next, we proceed to discuss the hypotheses needed to ensure splittings for slice completed generalized motivic cohomology theories. Recall that we have a standing assumption that B is *connected.* In this context, we will *assume* that Y is a *geometrically connected smooth scheme of finite type over* k. We already observed in 6.1.3 that when B denotes a finite degree approximation, $\mathrm{BG^{gm,m}}$, (with m sufficiently large) of the classifying spaces for a linear algebraic group G, it has k-rational points and are therefore *geometrically connected smooth schemes of finite type over* k*: see* [EGA, Tome 24, Chapitre 4, Corollaire 4.5.13]. *Furthermore, we will assume that the generalized cohomology theory* $\mathrm{h}^{*,\bullet}(\quad,\mathrm{A})$ *(defined with respect to the commutative motivic ring spectrum* A*) is such that the restriction map*

(7.0.4)
$$\mathrm{h}^{0,0}(\mathrm{E}\times_{\mathrm{G}}\mathrm{Y},\mathrm{A})\rightarrow \mathrm{h}^{0,0}(\mathrm{Y}_{k'},\mathrm{A}),\ \textit{for } \mathrm{G}\ \textit{special and } \mathrm{h}^{0,0}(\mathrm{R}\epsilon_*(\mathrm{E}\times_{\mathrm{G}}\mathrm{Y}),\mathrm{A})\rightarrow \mathrm{h}^{0,0}(\mathrm{R}\epsilon_*(\mathrm{Y}_{k'}),\mathrm{A}),\ \textit{for } \mathrm{G}\ \textit{not special}$$

is an isomorphism, where $Spec\,k'\rightarrow \mathrm{B}$ *is any point of* B.

Proposition 7.6. *Under the above assumption,* $tr(\mathrm{f_Y})^*(1)=tr(\mathrm{f_Y})^*(\pi_{\mathrm{Y}}^*(1))=(\mathrm{id}_{\mathrm{Y}_{k'}}\wedge\tau_{\mathrm{X}}(\mathrm{f}))^*(1)$ *where* $\tau_{\mathrm{X}}(f)$ *is the trace defined in Definition 6.1 (iii) and* $\pi_{\mathrm{Y}}^*:\mathrm{h}^{*,\bullet}(\mathrm{E}\times_{\mathrm{G}}\mathrm{Y},\mathrm{A})\rightarrow\mathrm{h}^{*,\bullet}(\mathrm{E}\times_{\mathrm{G}}(\mathrm{Y}\times\mathrm{X}),\mathrm{A})$*,* $tr(\mathrm{f_Y})^*:\mathrm{h}^{*,\bullet}(\mathrm{E}\times_{\mathrm{G}}(\mathrm{Y}\times\mathrm{X}),\mathrm{A})\rightarrow\mathrm{h}^{*,\bullet}(\mathrm{E}\times_{\mathrm{G}}\mathrm{Y},\mathrm{A})$ *denote the maps in the case* G *is special and the maps* $\pi_{\mathrm{Y}}^*:\mathrm{h}^{*,\bullet}(\mathrm{R}\epsilon_*(\mathrm{E}\times_{\mathrm{G}}\mathrm{Y}),\mathrm{A})\rightarrow\mathrm{h}^{*,\bullet}(\mathrm{R}\epsilon_*(\mathrm{E}\times_{\mathrm{G}}(\mathrm{Y}\times\mathrm{X})),\mathrm{A})$*,* $tr(\mathrm{f_Y})^*:\mathrm{h}^{*,\bullet}(\mathrm{R}\epsilon_*(\mathrm{E}\times_{\mathrm{G}}(\mathrm{Y}\times\mathrm{X})),\mathrm{A})\rightarrow\mathrm{h}^{*,\bullet}(\mathrm{R}\epsilon_*(\mathrm{E}\times_{\mathrm{G}}\mathrm{Y}),\mathrm{A})$ *in case* G *is not special.*

Proof. We discuss explicitly only the case where $char(k)=0$. In positive characteristics p, one needs to replace the sphere spectrum $\Sigma_{\mathbf{T}}$ everywhere by the corresponding sphere spectrum with the prime p inverted, or completed away from p as discussed in the introduction.

The first equality is clear since π_{Y}^* is a ring homomorphism and therefore, $\pi_{\mathrm{Y}}^*(1)=1$. Next we will consider the case when G is special. The naturality with respect to base-change as in Proposition (7.1), together with the assumption that the restriction $\mathrm{h}^{0,0}(\mathrm{E}\times_{\mathrm{G}}\mathrm{Y})\rightarrow\mathrm{h}^{0,0}(\mathrm{Y}_{k'})$ is an isomorphism shows that $tr(\mathrm{f_Y})^*\pi_{\mathrm{Y}}^*(1)$ is the same for $\mathrm{E}\times_{\mathrm{G}}\mathrm{Y}$ as well as for $\mathrm{Y}_{k'}$. When G is special and the scheme $\mathrm{B}=\mathrm{Spec}\,k$, $tr(\mathrm{f}):\Sigma_{\mathbf{T}}(\mathrm{Spec\,k})_+\rightarrow\Sigma_{\mathbf{T}}\mathrm{X}_+$, (which also identifies with the corresponding pre-transfer $tr(\mathrm{f})'$) so that for $\mathrm{Y}_{k'}$, $tr(\mathrm{f}_{\mathrm{Y}_{k'}})=\mathrm{id}_{\mathrm{Y}_{k'}}\wedge tr(\mathrm{f})'$. Therefore, it is clear that $\pi_{\mathrm{Y}}\circ tr(\mathrm{f}_{\mathrm{Y}_{k'}})=\mathrm{id}_{\mathrm{Y}_{k'}}\wedge\tau_{\mathrm{X}}(\mathrm{f})$ as defined in Definition 6.1 (iii). Therefore, the equality $tr(\mathrm{f_Y})^*(\pi_{\mathrm{Y}}^*(1))=(\mathrm{id}_{\mathrm{Y}_{k'}}\wedge\tau_{\mathrm{X}}(\mathrm{f}))^*(1)$ follows in this case.

Next, we consider the case when G is not necessarily special. In this case, we first recall that the transfer when $\mathrm{B} = \mathrm{Spec}\, k$ is a map, $\Sigma_{\mathbf{T}} \mathrm{R}\epsilon_*(\mathrm{Spec}\, k)_+ \cong \Sigma_{\mathbf{T}} \mathrm{R}\epsilon_*(\mathrm{a} \circ \epsilon^*)(\mathrm{Spec\, k})_+ \rightarrow \Sigma_{\mathbf{T}} \mathrm{R}\epsilon_*(\mathrm{a} \circ \epsilon^*)(\mathrm{X}))_+$. Then we observe the diagram (see the discussion in Step 0: Case 2 in section 6.2):

$$\begin{array}{ccccc}
\mathrm{h}^{0,0}(\mathrm{Y}_{k'}, \mathrm{A}) & \xrightarrow{\pi^*_{\mathrm{Y}_{k'}}} & \mathrm{h}^{0,0}(\mathrm{Y}_{k'} \times \mathrm{X}, \mathrm{A}) & \xrightarrow{tr(\mathrm{f_Y})^*} & \mathrm{h}^{0,0}(\mathrm{Y}_{k'}, \mathrm{A}) \\
\uparrow \cong & & \uparrow & & \uparrow \cong \\
\mathrm{h}^{0,0}(\mathrm{R}\epsilon_*(\mathrm{a} \circ \epsilon^*)(\mathrm{Y}_{k'}), \mathrm{A}) & \xrightarrow{\mathrm{R}\epsilon_*(\mathrm{a} \circ \epsilon^*)\pi^*_{\mathrm{Y}}} & \mathrm{h}^{0,0}(\mathrm{R}\epsilon_*((\mathrm{a} \circ \epsilon^*)(\mathrm{Y}_{k'} \times \mathrm{X})), \mathrm{A}) & \xrightarrow{\mathrm{R}\epsilon_*(\mathrm{a} \circ \epsilon^*) tr(\mathrm{f_Y})^*} & \mathrm{h}^{0,0}(\mathrm{R}\epsilon_*(\mathrm{a} \circ \epsilon^*)(\mathrm{Y}_{k'}), \mathrm{A}) \\
\uparrow \cong & & \uparrow & & \uparrow \cong \\
\mathrm{h}^{0,0}(\mathrm{R}\epsilon_*(\mathrm{E} \times_{\mathrm{G}} \mathrm{Y}), \mathrm{A}) & \xrightarrow{\pi^*_{\mathrm{Y}}} & \mathrm{h}^{0,0}(\mathrm{R}\epsilon_*(\mathrm{E} \times^{\mathrm{et}}_{\mathrm{G}} (\mathrm{a} \circ \epsilon^*)(\mathrm{Y} \times \mathrm{X})), \mathrm{A}) & \xrightarrow{tr(\mathrm{f_Y})^*} & \mathrm{h}^{0,0}(\mathrm{R}\epsilon_*(\mathrm{E} \times_{\mathrm{G}} \mathrm{Y}), \mathrm{A}).
\end{array}$$

The squares that make up the top two rows commute, because the pre-transfer $tr(\mathrm{f_Y})$ is a stable map and by the adjunction between ϵ^* and $\mathrm{R}\epsilon_*$. Therefore, it follows that the composition of the maps in the middle row identifies with the composition of maps in the top row: it is an isomorphism if the composition of the maps in the top row is an isomorphism. Observe that the top row corresponds to the composition of $\pi_{\mathrm{Y}_{k'}}$ and the pre-transfer, which is $id_{\mathrm{Y}_{k'}} \wedge \tau_{\mathrm{X}}(f)$, as defined in Definition 6.1(iv). The vertical maps in the bottom squares correspond to the restriction to the k'-rational point of B. The bottom right square commutes by Proposition (7.1), and the commutativity of the bottom left square is clear. The isomorphisms

$$\mathrm{h}^{0,0}(\mathrm{R}\epsilon_*(\mathrm{E} \times_{\mathrm{G}} \mathrm{Y}), \mathrm{A}) \xrightarrow{\cong} \mathrm{h}^{0,0}(\mathrm{R}\epsilon_*(\mathrm{Y}_{k'}), \mathrm{A}) \cong \mathrm{h}^{0,0}(\mathrm{R}\epsilon_*(\mathrm{a} \circ \epsilon^*)(\mathrm{Y}_{k'}), \mathrm{A}),$$

now show that the composition of the maps in the bottom row, that is, $tr(\mathrm{f_Y})^* \circ \pi^*_{\mathrm{Y}}$ identifies with the composition of the maps in the top row, which is $(id_{\mathrm{Y}_{k'}} \wedge \tau_{\mathrm{X}}(f))^*$. This completes the proof. ☐

Proposition 7.7. *The hypotheses in* (7.0.4) *are satisfied when* $\mathrm{h}^{*,\bullet}$ *denotes motivic cohomology with respect to any commutative ring* R*, and when* B *is any connected smooth scheme, or is a filtered colimit of such schemes.*

Proof. Since Y is assumed to be geometrically connected and B is connected, it follows that $\mathrm{E} \times_{\mathrm{G}} \mathrm{Y}$ is connected. Observe that now, $\mathrm{h}^{0,0} = \mathrm{H}^{0,0}_{\mathrm{M}}$, which denotes motivic cohomology in degree 0 and weight 0. The motivic complex R(0) is the constant sheaf associated to the ring R. Therefore, since $\mathrm{E} \times_{\mathrm{G}} \mathrm{Y}$ is connected, the restriction map $\mathrm{h}^{0,0}(\mathrm{E} \times_{\mathrm{G}} \mathrm{Y}) \rightarrow \mathrm{h}^{0.0}(\mathrm{Y}_{k'})$ is an isomorphism for any point $Spec\, k' \rightarrow \mathrm{B}$. It follows that the first hypothesis in (7.0.4) is always satisfied. Since R(0) is the constant sheaf on the Nisnevich site,

$$\mathrm{h}^{0,0}(\mathrm{R}\epsilon_*(\mathcal{X})) = \mathrm{H}^0(\mathrm{Hom}(\mathrm{R}\Gamma(\mathrm{Spec\, k}, \mathrm{R}\epsilon_*(\mathcal{X}))), \mathrm{R}) = \mathrm{H}^0(\mathrm{Hom}(\mathrm{R}\Gamma(\epsilon^*(\mathrm{Spec\, k}), \mathcal{X})), \mathrm{R})$$

for any smooth scheme $\mathcal{X}$, and the last term on the right provides the connected components of the scheme $\mathcal{X}$ computed on the big étale site. Therefore, the conclusion follows. ☐

7.1. Proof of Theorem 1.1. We will first clarify the terminology used. Recall that $\mathrm{BG}^{gm,m}$ ($\mathrm{EG}^{gm,m}$) denotes the m-th degree approximation to the classifying space of the group G (its principal G-bundle, respectively) as in [Tot99], [MV99] or [CJ19]. If X is a scheme with G-action, one can form the scheme $\mathrm{EG}^{gm,m} \times_{\mathrm{G}} \mathrm{X}$, which is called the Borel construction. In case G is not special, the torsor $\mathrm{EG}^{gm,m} \rightarrow \mathrm{BG}^{gm,m}$ is locally trivial only in the étale topology, so that in this case we replace the Borel construction above by $\mathrm{R}\epsilon_*(\mathrm{EG}^{gm,m} \times^{\mathrm{et}}_{\mathrm{G}} \mathrm{X})$ as discussed in section 6.2, Step 0: Case 2. However, we will continue to denote $\mathrm{R}\epsilon_*(\mathrm{EG}^{gm,m} \times^{\mathrm{et}}_{\mathrm{G}} \mathrm{X})$ by $\mathrm{EG}^{gm,m} \times_{\mathrm{G}} \mathrm{X}$ mainly for the sake of simplicity of notation.

The first statement in the Theorem is the compatibility of the transfer with various degrees of finite dimensional approximations to the classifying space: this has been discussed in Step 2 in the construction of the transfer. The second statement in the Theorem is the multiplicative property proven in Corollary 7.5. This implies the property (iii), and the first statement in (iv) follows from the naturality property for the transfer with respect to base-change as in Proposition 7.1. (See Remark 7.2.) That the transfer is compatible with change of base fields follows from the corresponding property for the pre-transfer: see Proposition 5.9. The second statement in (ii) follows from the fact that the transfer is defined using the pre-transfer (see Examples 2.9) which is a stable map that involves no degree or weight shifts.

Next we will sketch an argument to prove Theorem 1.1(v). Let $\{\mathrm{BG}^{gm,m}(1)|\mathrm{m}\}$, $\{\mathrm{BG}^{gm,m}(2)|\mathrm{m}\}$ denote two sequences of finite degree approximations to the classifying space of the given group G satisfying certain basic assumptions as in [MV99], [Tot99] or [CJ19, Definition 4.1]. Let $\{\mathrm{EG}^{gm,m}(1), \mathrm{EG}^{gm,m}(2)|\mathrm{m}\}$ denote the corresponding universal G-bundles: *the main requirements here are that both these have free actions by* G *and that as* $m \rightarrow \infty$*, these are* $\mathbb{A}^1$*-acyclic.*

Then a key observation is that $\{EG^{gm,m}(1) \times EG^{gm,m}(2)|m\}$ with the diagonal action of the group G also satisfies the same hypotheses so that their quotient by the diagonal action of G will also define approximations to the classifying space of the G. Therefore, after replacing $\{BG^{gm,m}(1)|m\}$ with $\{EG^{gm,m}(1) \times EG^{gm,m}(2)|m\}$, we may assume that one has a direct system of smooth surjective maps $\{EG^{gm,m}(1) \to EG^{gm,m}(2)|m\}$. Now it is straightforward to verify that all the constructions discussed in the above steps for the transfer are compatible with the maps $\{EG^{gm,m}(1) \to EG^{gm,m}(2)|m\}$. Therefore, one obtains a direct system of homotopy commutative diagrams, $m \geq 1$:

$$\begin{array}{ccc} \Sigma_{\mathbf{T}}(EG^{gm,m}(1) \times_G X)_+ & \longrightarrow & \Sigma_{\mathbf{T}}(EG^{gm,m}(2) \times_G X)_+ \\ {\scriptstyle tr(f)^m(1)}\uparrow & & \uparrow{\scriptstyle tr(f)^m(2)} \\ \Sigma_{\mathbf{T}} BG^{gm,m}(1)_+ & \longrightarrow & \Sigma_{\mathbf{T}} BG^{gm,m}(2)_+. \end{array}$$

Finally, one may also verify that the maps $\{EG^{gm,m}(1) \times_G X \to EG^{gm,m}(2) \times_G X|m\}$ and $\{BG^{gm,m}(1) \to BG^{gm,m}(2)|m\}$ induce isomorphisms on generalized motivic cohomology as one takes the $\lim_{m\to\infty}$: see, for example [MV99, §4, Proposition 2.6]. These complete the proof of (v) when $Y = \mathrm{Spec}\, k$: the case when Y is a general smooth G-scheme is similar.

The construction of the transfer in the étale framework is entirely similar, though care has to be taken to ensure that affine spaces are contractible in this framework, which accounts partly for the hypothesis in (vi) and in (3.0.3). Property (vii) is proved in the next section. ☐

8. Computing traces: compatibility of the transfer with realizations

Assume the situation as in Theorem 1.1. Then, very often the main application of the transfer is to prove that π_Y^* is a split injection in generalized cohomology, i.e. one needs to verify that $tr(f_Y)^*(\pi_Y^*(1))$ is a unit. In order to verify that $tr(f_Y)^*(\pi_Y^*(1))$ is a unit, one may adopt the following strategy. First we will show that the transfer constructed above is compatible with passage to a simpler situation, for example passage from over a given base field to its algebraic or separable closure and/or passage to a suitable *realization* functor: we will often use the étale realization. Then, often, $h^{0,0}(B) \simeq h^{0,0}_{real}(B)$ where $h^{*,\bullet}_{real}(B)$ denotes the corresponding generalized cohomology of the realization. Therefore, it will suffice to show that $tr(f_Y)^*_{real}(\pi_Y^*(1))$ is a unit: here $tr(f_Y)_{real}$ denotes the corresponding transfer on the realization. We devote all of this section to a detailed discussion of this technique.

As before we will assume the base scheme is the spectrum of a perfect field k satisfying the assumption (3.0.3). *$\bar{k}$ will denote a fixed separable closure of k and ℓ is a prime different from char(k).* Accordingly $S = \mathrm{Spec}\, k$ and $\bar{S} = \mathrm{Spec}\, \bar{k}$. We first recall the maps of topoi (from (3.3.15)):

(8.0.1) $$\epsilon^* : \mathbf{Spt}/S_{mot} \to \mathbf{Spt}/S_{et}, \bar{\epsilon}^* : \mathbf{Spt}/\bar{S}_{mot} \to \mathbf{Spt}/\bar{S}_{et}, \text{ and } \eta^* : \mathbf{Spt}/S_{et} \to \mathbf{Spt}/\bar{S}_{et}.$$

Let θ and $\phi_{\mathcal{E}}$ denote the functors considered in Proposition 5.9. We let $\mathcal{E} \in \mathbf{Spt}_{mot}$ denote a commutative ring spectrum which is ℓ-complete for a prime $\ell \neq char(k)$. Throughout the following discussion, we will take $Y = \mathrm{Spec}\, k$.

Proposition 8.1. *(Commutativity of the pre-transfer with étale realization) Assume the above situation. Then denoting by $tr(f)'$ the pre-transfer (as in* (2.2.2)*), $\epsilon^*(tr(f)') \simeq tr(\epsilon^*(f)')$ and $\bar{\epsilon}^*(tr(f)') \simeq tr(\bar{\epsilon}^*(f)')$ when applied to the dualizable objects of the form $\mathcal{E} \wedge X_+$ appearing in Propositions 5.1(i) and (ii), 5.4(iv), and Theorem 5.7. The same conclusion holds for ϵ^* and $\bar{\epsilon}^*$ replaced by η^* or any of the two functors θ and $\phi_{\mathcal{E}}$.*

Proof. Implicitly assumed in the proof is the fact that the above maps of topoi all send dualizable objects to dualizable objects. This is already proved in Proposition 5.9. Moreover, as pointed out earlier, [DP84, 2.2 Theorem and 2.4 Corollary] seems to provide a quick proof of the assertion above, so that at least in principle, the results in this proposition should be deducible from op. cit. Nevertheless, it seems best to provide a proof of Proposition 8.1, at least for ϵ^*: the proof for the other functors will be similar. First observe that there is a natural map $\epsilon^*\mathcal{RH}om(K, L) \to \mathcal{RH}om(\epsilon^*(K), \epsilon^*(L))$ for any two objects $K, L \in \mathbf{Spt}_{mot,\mathcal{E}}$. If one takes $L = \mathcal{E}$, $\mathcal{RH}om(K, L)$ will denote $D(K)$. Similarly $\mathcal{RH}om(\epsilon^*(K), \epsilon^*(L))$ will then denote $D(\epsilon^*(K))$.

Now the proof of the assertion for the pre-transfer follows from the commutativity of the following diagrams where the composition of maps in the top row (bottom row) is $\epsilon^*(tr(f_Y)')$ ($tr(\epsilon^*(f)')$, respectively) with the smash products denoting their derived versions and $K = \mathcal{E} \wedge X_+$ as in the Proposition:

$$\begin{array}{ccccc}
\epsilon^*(\mathcal{E}) & \longrightarrow & \epsilon^*(\mathrm{K} \wedge_{\mathcal{E}} \mathrm{DK}) & \overset{\cong}{\longrightarrow} & \epsilon^*(\mathrm{K}) \wedge_{\epsilon^*(\mathcal{E})} \epsilon^*(\mathrm{DK}) \\
\downarrow{\scriptstyle id} & & & \swarrow & \\
\epsilon^*(\mathcal{E}) & \longrightarrow & \epsilon^*(\mathrm{K}) \wedge_{\epsilon^*(\mathcal{E})} \mathrm{D}(\epsilon^*(\mathrm{K})) & &
\end{array}$$

$$\begin{array}{ccccc}
\epsilon^*(\mathrm{K}) \wedge_{\epsilon^*(\mathcal{E})} \epsilon^*(\mathrm{DK}) & \xrightarrow{(id \wedge_{\epsilon^*(\mathcal{E})} \epsilon^*(\mathrm{f}) \wedge_{\epsilon^*(\mathcal{E})} \epsilon^*(\mathrm{f})) \circ (\mathrm{id} \wedge_{\epsilon^*(\mathcal{E})} \Delta) \circ \tau} & \epsilon^*(\mathrm{DK}) \wedge_{\epsilon^*(\mathcal{E})} \epsilon^*(\mathrm{K}) \wedge_{\epsilon^*(\mathcal{E})} \epsilon^*(\mathrm{K}) & \xrightarrow{\epsilon^*(e) \wedge_{\epsilon^*(\mathcal{E})} id} & \epsilon^*(\mathcal{E}) \wedge_{\epsilon^*(\mathcal{E})} \epsilon^*(K) \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle id} \\
\epsilon^*(\mathrm{K}) \wedge_{\epsilon^*(\mathcal{E})} \mathrm{D}(\epsilon^*(\mathrm{K})) & \xrightarrow{(id \wedge_{\epsilon^*(\mathcal{E})} \epsilon^*(\mathrm{f}) \wedge_{\epsilon^*(\mathcal{E})} \epsilon^*(\mathrm{f})) \circ (\mathrm{id} \wedge_{\epsilon^*(\mathcal{E})} \Delta) \circ \tau} & \mathrm{D}(\epsilon^*(\mathrm{K})) \wedge_{\epsilon^*(\mathcal{E})} \epsilon^*(\mathrm{K}) \wedge_{\epsilon^*(\mathcal{E})} \epsilon^*(\mathrm{K}) & \xrightarrow{e \wedge_{\epsilon^*(\mathcal{E})} id} & \epsilon^*(\mathcal{E}) \wedge_{\epsilon^*(\mathcal{E})} \epsilon^*(\mathrm{K})
\end{array}$$

The isomorphism $\epsilon^*(\mathrm{K} \wedge_{\mathcal{E}} \mathrm{DK}) \overset{\cong}{\rightarrow} \epsilon^*(\mathrm{K}) \wedge_{\epsilon^*(\mathcal{E})} \epsilon^*(\mathrm{DK})$ in the top row may be obtained as follows. First one observes that the colimits involved in the definition of the functor ϵ^* is a *sifted* colimit, so that ϵ^* commutes with products. Clearly ϵ^* also commutes with colimits, so that it commutes with the smash products of spectra. □

Corollary 8.2. *Assume that the group* G *is special and that* $\mathrm{f} : \mathrm{X} \rightarrow \mathrm{X}$ *is a* G*-equivariant map and let* $\pi_{\mathrm{Y}} : \mathrm{E} \times_{\mathrm{G}} (\mathrm{Y} \times \mathrm{X}) \rightarrow \mathrm{E} \times_{\mathrm{G}} \mathrm{Y}$ *denote any one of the three cases considered in Theorem 1.1. Then* $\epsilon^*(tr(\mathrm{f_Y})) \simeq tr(\epsilon^*(\mathrm{f_Y}))$*, where* $tr(\mathrm{f})$ *denotes the transfer defined with respect to a motivic ring spectrum* $\mathcal{E}$ *that is* ℓ*-complete for a prime* $\ell \neq char(k)$*.*

Proof. Proposition 8.1 proves the corresponding statement for the pre-transfer when $\mathrm{Y} = \mathrm{Spec}\, \mathrm{k}$. Now the corresponding result holds for a general Y, since the corresponding pre-transfer $tr(\mathrm{f_Y})' = \mathrm{id}_{\mathrm{Y}_+} \wedge tr(\mathrm{f})'$. Now a detailed examination of the various steps in the construction of the transfer show that they all pull-back to define the corresponding construction on the étale site. (In fact, $tr(\mathrm{f_Y})$ as in Definition 6.5 is defined by first taking suitable fiber-wise join, $id \wedge_G tr_{\mathrm{G}}(\mathrm{f_Y})'$, where $tr_{\mathrm{G}}(\mathrm{f_Y})'$ is the G-equivariant pre-transfer.) □

Assume that the generalized cohomology theory $\mathrm{h}^{*,\bullet}$ is defined with respect to the commutative motivic ring spectrum $\mathcal{E}$ which is ℓ-complete for some prime $\ell \neq char(k)$. Then the spectrum $\eta^*(\epsilon^*(\mathcal{E}))$ defines the corresponding generalized étale cohomology theory which will be denoted $\bar{\mathrm{h}}_{et}^{*,\bullet}$. Observe that the natural map $\mathcal{E} \rightarrow R\epsilon_* R\eta_* \eta^* \epsilon^*(\mathcal{E})$ induces a natural map $\mathrm{h}^{*,\bullet} \rightarrow \bar{\mathrm{h}}_{\mathrm{et}}^{*,\bullet}$. We will once again adopt the notations from Theorem 1.1 in the following discussion.

Proposition 8.3. *Assume in addition to the assumption* (7.0.4)*, that the above map* $\mathrm{h}^{0,0}(\mathrm{Y}_{k'}) \rightarrow \bar{\mathrm{h}}_{\mathrm{et}}^{0,0}(\mathrm{Y}_{\bar{k}})$ *is also an isomorphism, where* $\bar{k}$ *denotes an algebraic closure of* k'*. Then, under the above isomorphism* $tr(\mathrm{f}_{\mathrm{Y}_{k'}})^*(1)$ *identifies with* $tr(\eta^*\epsilon^*(\mathrm{f}_{\mathrm{Y}_{\bar{k}}}))^*(1)$*. If follows that,* $\pi_{\mathrm{Y}}^* : \mathrm{h}^{*,\bullet}(\mathrm{E} \times_{\mathrm{G}} \mathrm{Y}) \rightarrow \mathrm{h}^{*,\bullet}(\mathrm{E} \times_{\mathrm{G}} (\mathrm{Y} \times \mathrm{X}))$ *is a split mono-morphism if* $tr(\eta^*\epsilon^*(\mathrm{f}_{\mathrm{Y}_{\bar{k}}}))^*(1)$ *is a unit.*

Proof. We first observe that the transfer $tr(\mathrm{f}_{\mathrm{Y}_{k'}}) = \mathrm{id}_{\mathrm{Y}_{k',+}} \wedge tr(\mathrm{f})'$, where $tr(\mathrm{f})'$ is the pre-transfer associated to X. Therefore, Proposition 8.1 proves the first statement. The remaining statement then follows readily from the first. □

Remark 8.4. Here we are implicitly assuming that the spectra in $\mathbf{Spt}_{\theta^*\epsilon^*(\mathbf{T}),\mathrm{et}}$ identify with the usual spectra where the suspension is with respect to the simplicial S^1. This is discussed in Remark 3.18.

Examples 8.5. *(1) One may take the generalized cohomology* $\mathrm{h}^{*,\bullet}$ *to be mod-*ℓ *motivic cohomology for some prime different from* $char(k)$*. Then the conclusion above says that the induced map in mod*$-\ell$ *motivic cohomology* $\pi_{\mathrm{Y}}^* : \mathrm{h}^{*,\bullet}(\mathrm{E} \times_{\mathrm{G}} \mathrm{Y}, \mathrm{Z}/\ell) \rightarrow \mathrm{h}^{*,\bullet}(\mathrm{E} \times_{\mathrm{G}} (\mathrm{Y} \times \mathrm{X}), \mathrm{Z}/\ell)$ *is a split mono-morphism if the class* $tr(\epsilon^*(\mathrm{f}))^*(1)$ *is a unit in* $\mathrm{H}_{\mathrm{et}}^0(\mathrm{Y}_{k'}, \mu_\ell)$*.*

(2) In this example, we assume that the transfer is compatible with Betti realization. Take $k = \mathbb{C}$*. One may take the generalized cohomology* $\mathrm{h}^{*,\bullet}$ *to be integral (or rational) motivic cohomology. Then the induced map* $\pi_{\mathrm{Y}}^* : \mathrm{h}^{*,\bullet}(\mathrm{E} \times_{\mathrm{G}} \mathrm{Y}) \rightarrow \mathrm{h}^{*,\bullet}(\mathrm{E} \times_{\mathrm{G}} (\mathrm{Y} \times \mathrm{X}))$ *is a split monomorphism if the class* $tr(\zeta^*(\mathrm{f}))^*(1)$ *is a unit in* $Z \cong \mathrm{H}^0(\mathrm{Y}_{k'}, \mathrm{Z})$*, where* ζ^* *now denotes the Betti realization functor. (Observe that it is enough to know that the pre-transfer is compatible with Betti realization. This has been worked out by G. Bainbridge in* [Bain18]*.)*

9. Transfer and stable splittings in the Motivic Stable Homotopy category

We will restrict to the transfers considered in Definition 6.5. Therefore, we may assume G is a linear algebraic group over k, $\mathrm{p}:\mathrm{E}\to\mathrm{B}$ is a G-torsor over a perfect field k, and $\pi_{\mathrm{Y}}:\mathrm{E}\times_{\mathrm{G}}(\mathrm{X}\times\mathrm{Y})\to\mathrm{E}\times_{\mathrm{G}}\mathrm{Y}$ is one of the three maps considered in Theorem 1.1(a), (b) or (c). Recall that X, Y denote unpointed simplicial presheaves (defined on Sm/S) provided with actions by G and so that X is dualizable in the motivic homotopy category. $\mathrm{f}:\mathrm{X}\to\mathrm{X}$ is a G-equivariant map.

In order to obtain splittings in the stable homotopy category, there are essentially *two distinct techniques* we pursue here making use of the transfer as a stable map. Both of these apply to actions of all linear algebraic groups, *irrespective of whether they are special*, but the only draw-back of the first method is that one needs to be able to compute the relevant traces in the Grothendieck-Witt ring of the base field (or the Grothendieck-Witt ring with the characteristic of the base field inverted). As such, currently this works only when the scheme X as in Theorem 1.1 is of the form G/N(T), where G is any split reductive group and N(T) denotes the normalizer of a split maximal torus in G. The second method has the advantage that it is enough to compute the traces after taking étale realizations. The only draw-back of this method is that it only works for slice-completed generalized motivic cohomology theories and certain connectivity assumptions need to be imposed on the schemes considered as in (7.0.4).

9.1. Splittings via the Grothendieck-Witt group: Proof of Theorem 1.5(1). In this approach we show that the class $\tau_{\mathrm{X}}^*(1)$ is a unit in the Grothendieck-Witt group (or the Grothendieck-Witt group with characteristic of the base field inverted) and use that to obtain splittings directly, first at the level of the pre-transfer. This method is rather limited to those schemes X for which it is possible to compute $\tau_{\mathrm{X}}^*(1)$ in the Grothendieck-Witt group. Such a computation is carried out in [JP-1, Theorem 1.6] where $\mathrm{X}=\mathrm{G}/\mathrm{N}(\mathrm{T})$, for a connected split reductive group G over a perfect field and N(T) the normalizer of a split maximal torus in G. Therefore, at present this technique only applies to the above case. In the discussion below, we will only consider the case where $char(k)=0$. In positive characteristics p, the same discussion applies by replacing the sphere spectra everywhere by the corresponding sphere spectra with the prime p inverted, or completed away from p as discussed in the introduction.

Case 1: Here we will assume the group G is *special.* Since the group G is assumed to be special, for each fixed integer $m\geq 1$, the map $\mathrm{p}:\mathrm{E}\to\mathrm{B}$ is a Zariski locally trivial principal G-bundle and let $\tilde{\mathrm{p}}:\widetilde{\mathrm{E}}\to\widetilde{\mathrm{B}}$ denote the induced map where $\widetilde{\mathrm{B}}$ is the affine replacement. Let $\{\mathrm{U_i}|\mathrm{i}\}$ denote a Zariski open cover of B over which the map p trivializes so that $\pi_{\mathrm{Y}|\mathrm{U_i}}=\mathrm{U_i}\times(\mathrm{Y}\times\mathrm{X})\to\mathrm{U_i}\times\mathrm{Y}$.

Let $tr:\Sigma_{\mathbf{T}}(\widetilde{\mathrm{E}}\times_{\mathrm{G}}\mathrm{Y})_+\to\Sigma_{\mathbf{T}}(\widetilde{\mathrm{E}}\times_{\mathrm{G}}(\mathrm{Y}\times\mathrm{X}))_+$ denote the transfer defined in Definition 6.2.26. Then one may observe that $tr_{|\mathrm{U_i}}:\Sigma_{\mathbf{T}}(\mathrm{U_i}\times\mathrm{Y})_+\to\Sigma_{\mathbf{T}}(\mathrm{U_i}\times\mathrm{Y})_+\wedge\Sigma_{\mathbf{T}}\mathrm{X}_+$ is just $id_{\Sigma_{\mathbf{T}}(\mathrm{U_i}\times\mathrm{Y})_+}\wedge tr'_{\mathrm{X}}$, where tr'_{X} denotes the pre-transfer considered in Definition 6.1(ii). Therefore, if $\tau_{\mathrm{X}}^*(1)$ is a unit in the Grothendieck-Witt group (or in the Grothendieck-Witt group with the characteristic of the base field inverted), (where τ_{X} is the trace defined in Definition 6.1(iv)), then the composition, $\Sigma_{\mathbf{T}}\pi_{\mathrm{Y},+}\circ tr_{\mathrm{X}}$, where $\tilde{\pi}_{\mathrm{Y}}$ is the projection $\widetilde{\mathrm{E}}\times_{\mathrm{G}}(\mathrm{Y}\times\mathrm{X})\to\widetilde{\mathrm{E}}\times_{\mathrm{G}}\mathrm{Y}$, will be homotopic to the identity over each $\mathrm{U_i}$.

Now let $\mathrm{h}^{*,\bullet}$ denote a generalized motivic cohomology theory defined with respect to a motivic spectrum. Then the splitting above, over $\mathrm{U_i}$ of the map $\pi_{\mathrm{Y}|\mathrm{U_i}}$ shows, using a Mayer-Vietoris argument and observing that each $\widetilde{\mathrm{B}}$ is quasi-compact, that the composite map $tr^*\circ\pi_{\mathrm{Y}}^*:\mathrm{h}^{*,\bullet}(\widetilde{\mathrm{E}}\underset{\mathrm{G}}{\times}\mathrm{Y})\overset{\pi_{\mathrm{Y}}^*}{\to}\mathrm{h}^{*,\bullet}(\widetilde{\mathrm{E}}\underset{\mathrm{G}}{\times}(\mathrm{Y}\times\mathrm{X}))\overset{\mathrm{tr}^*}{\to}\mathrm{h}^{*,\bullet}(\widetilde{\mathrm{E}}\underset{\mathrm{G}}{\times}\mathrm{Y})$ is an isomorphism. When we vary B over finite dimensional approximations $\{\mathrm{BG}^{\mathrm{gm},\mathrm{m}}|\mathrm{m}\}$, the same therefore holds on taking the colimit of the $\mathrm{BG}^{gm,m}$ over m as $m\to\infty$, as we have shown the transfer maps are all compatible as m varies: see Theorem 1.1(i). (Here the colimit of the $\mathrm{BG}^{gm,m}$ will pullout of the generalized cohomology spectrum as a homotopy inverse limit, and then one uses the usual lim^1-exact sequence to draw the desired conclusion.)

Case 2: Here we will let G denote *any linear algebraic group.* We first recall from Definition 6.1(iii), that the G-equivariant pre-transfer is given by $tr(id_{\mathrm{Y}})'_{\mathrm{G}}:\mathrm{Y}_+\wedge\mathbb{S}^{\mathrm{G}}\to\mathrm{Y}_+\wedge\mathbb{S}^{\mathrm{G}}\wedge\mathrm{X}_+$ to be $id_{\mathrm{Y}_+}\wedge tr(id)'_{\mathrm{G}}$, with $tr(id)_{\mathrm{G}}:\mathbb{S}^{\mathrm{G}}\to\mathbb{S}^{\mathrm{G}}\wedge\mathrm{X}_+$ is the G-equivariant pre-transfer in Definition 6.1(ii). Moreover, the composition of the above pre-transfer and the projection $\mathrm{Y}_+\mathbb{S}^{\mathrm{G}}\wedge\mathrm{X}_+\to\mathrm{Y}_+\wedge\mathbb{S}^{\mathrm{G}}$ is $id_{\mathrm{Y}_+}\wedge\tau_{\mathrm{X}}$. In view of the assumption that $\tau_{\mathrm{X}}^*(1)$ is a unit in the Grothendieck-Witt ring of k, this composite map is a weak-equivalence mapping $\mathrm{Y}_+\wedge\mathbb{S}^{\mathrm{G}}$ to itself. Therefore, it follows that the composite map

$$\widetilde{\mathrm{E}}\times_{\mathrm{G}}^{et}(\mathrm{a}\epsilon^*(\mathrm{Y}_+\wedge\mathbb{S}^{\mathrm{G}}))\overset{\widetilde{\mathrm{E}}\times_{\mathrm{G}}^{\mathrm{et}}(\mathrm{a}\epsilon^* tr_{\mathrm{G}}(id'_{\mathrm{Y}}))}{\longrightarrow}\widetilde{\mathrm{E}}\times_{\mathrm{G}}^{\mathrm{et}}(\mathrm{a}\epsilon^*(\mathrm{Y}_+\wedge\mathbb{S}^{\mathrm{G}}\wedge\mathrm{X}_+))\overset{\widetilde{\mathrm{E}}\times_{\mathrm{G}}^{\mathrm{et}}(\mathrm{a}\epsilon^*(\mathrm{pr}))}{\longrightarrow}\widetilde{\mathrm{E}}\times_{\mathrm{G}}^{\mathrm{et}}(\mathrm{a}\epsilon^*(\mathrm{Y}_+\wedge\mathbb{S}^{\mathrm{G}}))$$

is also a weak-equivalence. Therefore, it remains a weak-equivalence on applying the right derived functor $\mathrm{R}\epsilon_*$. Finally we apply a generalized motivic cohomology theory $\mathrm{h}^{*,\bullet}$ to the above maps to observe that the composite map

$$h^{*,\bullet}(\widetilde{E}\times^{et}_G (a\epsilon^*(Y_+\wedge \mathbb{S}^G)))\overset{\pi_Y^*}{\longrightarrow} h^{*,\bullet}(\widetilde{E}\times^{et}_G (a\epsilon^*(Y_+\wedge \mathbb{S}^G\wedge X_+)))\overset{tr(id_Y^*)}{\longrightarrow} h^{*,\bullet}(\widetilde{E}\times^{et}_G (a\epsilon^*(Y_+\wedge \mathbb{S}^G)))$$

is an isomorphism.

Proof of Theorem 1.5(1): clearly, this follows readily in view of the above discussion.

9.2. Splittings on slice completed generalized cohomology theories: Proof of Theorem 1.5(2). One key observation here is that the map $tr(f_Y)$ being a stable map, it induces a map of the stable slice spectral sequences for $h^{*,\bullet}(E\times_G (X\times Y), M)$ and $h^{*,\bullet}(E\times_G Y, M)$. (One may observe that the slice spectral sequences converge only conditionally, in general: the convergence issues will be discussed below.) Next we will show, under the hypotheses of the theorem, that multiplication by $tr(f_Y)^*(\pi_Y^*(1))$ induces a splitting of the corresponding E_2-terms of the above spectral sequences. For this, recall that the multiplicative properties of the slice filtration, verified in [Pel08], shows that these E_2-terms are modules over the motivic cohomology and that, in fact these E_2-terms are defined by motivic spectra (that is, the slices) that are module spectra over the motivic Eilenberg-Maclane spectrum.

Therefore, under these assumptions, the multiplicative property of the transfer as in Corollary 7.5 with A there denoting the motivic Eilenberg-Maclane spectrum $\mathbb{H}(\mathbb{Z})$ and the module spectrum M there denoting the module spectra defining the above E_2-terms, shows that $tr(f_Y)^*\circ\pi_Y^*$ induces a map of the E_2-terms of the above motivic Atiyah-Hirzebruch spectral sequences. That is, we reduce to proving $tr(f_Y)^*\circ\pi_Y^*$ induces an isomorphism on the motivic cohomology of $E\times_G Y$, modulo the convergence issues of the spectral sequence.

Next we make use of Propositions 7.6, 7.7 which reduces the situation to the case where the group G is trivial, that is, it suffices to prove the splittings for the pre-transfer, under the assumption that $id_{Y_{k',+}}\wedge\tau_X(f)^*(1)$ is a unit in the motivic cohomology $H^{0,0}(Y_{k'})$. A key assumption needed here is the one in (7.0.4). Observe that this reduction holds irrespective of whether the group is special or not. Next we invoke Propositions 8.1 and 8.3 which reduce to proving the splitting for the pre-transfer at the level of the étale realizations with the base field separably closed.

Next we discuss convergence issues of the spectral sequence. Since the map $tr(f_Y)^*\circ\pi_Y^*$ induces an isomorphism at the E_2-terms, and therefore at all the E_r-terms for $r\geq 2$, it follows that it induces an isomorphism of the inverse systems $\{E_r|r\}$ and therefore an isomorphism of the E_∞-terms and the derived E_∞-terms. (See [Board98, (5.1)] for a description of the derived E_∞-terms. It is shown in [CE, Chapter XV, section 2] that both the E_∞-terms and the derived E_∞-terms are determined by the sequence E_r, $r\geq 2$.)

The next observation is that for every fixed integer n and m, on replacing the spectrum M by $s_{\leq n}M$, the corresponding slice spectral sequence for the schemes Y and Y_m converge strongly: this is clear since the $E_1^{u,v}$-terms will vanish for all $u>n$ and also for $u<0$. (See [Board98, Theorem 7.1].) That $E_1^{u,v}=0$ for $u<0$ or $u>n$ follows from the identification of the E_1-terms of the spectral sequence in terms of the slices of the S^1-spectrum forming the 0-th term in the associated Ω-$\mathbb{P}^1$-spectrum: see [Lev08, Proof of Theorem 11.3.3]. Moreover, the abutment of the spectral sequence are the homotopy groups of the slice-completion of the S^1-spectrum forming the corresponding 0-th term. Next, let $\{Y_m|m\}$ denote either one of the following ind-schemes: $Y_m=E\times_G Y$, for all $m\geq 1$ or $Y_m = EG^{gm,m}\times_G Y$, $m\geq 1$. Then, it follows therefore that for each fixed integer m and n, the composite map

$$tr(f_Y)^*\circ\pi_Y^*:\mathbb{H}_{Nis}(Y_m, s_{\leq n}M)\to\mathbb{H}_{Nis}(Y_m, s_{\leq n}M)$$

is a weak-equivalence, where $\mathbb{H}_{Nis}$ denotes the hypercohomology spectrum on the Nisnevich site.

Therefore, from the compatibility of the transfer as m varies as proven in (6.2.25), it follows that the composite map

$$tr(f_Y)^*\circ\pi_Y^*:\underset{\infty\leftarrow m}{\text{holim}}\,\underset{\infty\leftarrow n}{\text{holim}}\,\mathbb{H}_{Nis}(Y_m, s_{\leq n}M)\to\underset{\infty\leftarrow m}{\text{holim}}\,\underset{\infty\leftarrow n}{\text{holim}}\,\mathbb{H}_{Nis}(Y_m, s_{\leq n}M)$$

is a weak-equivalence.

Next we discuss *proof of Theorem 1.5(2).* First observe that the above discussion already proves the first statement in 2(i). Then the second statement there follows by observing that $\Sigma_{\mathbf{T}}X_+$ is dualizable in $\mathbf{Spt}_{mot}$ as observed in Proposition 5.4. We already proved in Propositions 8.1 and 8.3 that the transfer is compatible with étale realizations. In fact, one may take the commutative motivic ring spectrum $\mathcal{E}$ to be the motivic Eilenberg-Maclane spectrum $\mathbb{H}(\mathbb{Z}/\ell^\nu)$. Then Theorem 5.7 shows $\mathcal{E}\wedge X_+$ is dualizable in $\mathbf{Spt}_{\mathcal{E}}$ and Propositions 8.1 and 8.3 show that the pre-transfer is compatible with étale realizations. The last statement in Theorem 1.5(2)(i) follows from these observations and the first statement there already proven.

In order to prove the variant in Theorem 1.5(2)(ii)(a), it suffices to observe that the slices of the module spectrum M are now module spectra over $\Sigma_{\mathbf{T}}[p^{-1}]$ and that the zero-th slice of $\Sigma_{\mathbf{T}}[p^{-1}]=\mathbb{H}(\mathbb{Z}[p^{-1}])$. Then essentially the same arguments as above apply, along with Theorem 5.7 to complete the proof of statement (a). In order to prove the variant (b), it suffices to observe that the slices of the module spectrum M are module spectra over $\Sigma_{\mathbf{T},(\ell)}$ ($\Sigma_{\mathbf{T}}\hat{\ }_{\ell}$) and that the zero-th slice of $\Sigma_{\mathbf{T},(\ell)}$ is $\mathbb{H}(\mathbb{Z}_{(\ell)})$ (the zero-th slice of $\Sigma_{\mathbf{T}}\hat{\ }_{\ell}$ is $\mathbb{H}\mathbb{Z}\hat{\ }_{\ell}$), respectively). In fact, both of these statements follow readily by identifying the slice tower with the coniveau tower as in [Lev08]. The étale variant, (iii) follows first by observing that Postnikov sections of the module spectrum M are module spectra over the ℓ-completed S^1-sphere spectrum $\widehat{\Sigma}_{S^1,\ell}$. □

Proofs of Corollaries 1.6 and 1.7. The slice completed generalized motivic cohomology of any smooth scheme with respect to the motivic spectrum representing Algebraic K-theory, identifies with Algebraic K-theory itself. This proves the first statement in Corollary 1.6. The second statement in Corollary 1.6 now follows from the following observations.

First we observe the weak-equivalence for any motivic spectrum $\mathcal{E}$: $s_p(\mathcal{E}) \wedge \Sigma_{\mathbf{T}}\mathrm{M}(\ell^\nu) \simeq \mathrm{s_p}(\mathcal{E} \wedge_{\Sigma_{\mathbf{T}}} \mathrm{M}(\ell^\nu))$, where $\mathrm{M}(\ell^\nu)$ is defined as the homotopy cofiber of the map $\Sigma_{\mathbf{T}} \stackrel{\ell^\nu}{\rightarrow} \Sigma_{\mathbf{T}}$, and where s_p denotes the p-th slice. This follows from the identification of the slices, with the slices obtained from the coniveau tower as in [Lev08, Theorem 9.0.3]. Let $\mathbf{K}$ denote the motivic spectrum representing algebraic K-theory. Next we recall (see [Lev08, section 11.3]) that the slice $s_0(\mathbf{K}) = \mathbb{H}(\mathbb{Z}) =$ the motivic Eilenberg-Maclane spectrum and that the p-th slice $s_p(\mathbf{K}) = \mathbb{H}(\mathbb{Z}(p)[2p])$, which is the corresponding shifted motivic Eilenberg-Maclane spectrum. Therefore, each $s_p(\mathbf{K})$ has the structure of a module spectrum over the commutative ring spectrum $\mathbb{H}(\mathbb{Z})$. In view of this, one may also observe that the natural map $s_p(\mathbf{K}) \wedge_{\Sigma_{\mathbf{T}}} \mathrm{M}(\ell^\nu) \rightarrow \mathrm{s_p}(\mathbf{K}) \wedge_{\mathbb{H}\mathbb{Z}} \mathbb{H}(\mathbb{Z}/\ell^\nu) = \mathbb{H}(\mathbb{Z}/\ell^\nu(\mathrm{p})[2\mathrm{p}])$ is a weak-equivalence, where $\mathbb{H}(\mathbb{Z}/\ell^\nu)$ denotes the mod$-\ell^\nu$ motivic Eilenberg-Maclane spectrum. Therefore, the slices $s_p(\mathbf{K} \wedge_{\Sigma_{\mathbf{T}}} \mathrm{M}(\ell^\nu)) \simeq \mathrm{s_p}(\mathbf{K}) \wedge_{\Sigma_{\mathbf{T}}} \mathrm{M}(\ell^\nu)$ have the structure of weak-module spectra over the motivic Eilenberg-Maclane spectrum $\mathbb{H}(\mathbb{Z}/\ell^\nu)$. i.e. The hypotheses of the second statement in Theorem 1.5(2)(i) are met, thereby completing the proof of the second statement in Corollary 1.6. (One may also want to observe that the spectrum $\mathbf{K} \wedge_{\Sigma_{\mathbf{T}}} \mathrm{M}(\ell^\nu)$ has cohomological descent on the Nisnevich site of smooth schemes of finite type over k so that the generalized cohomology $\mathrm{h}(X, \mathbf{K} \wedge_{\Sigma_{\mathbf{T}}} \mathrm{M}(\ell^\nu)) \simeq \mathbf{K}(\mathrm{X}) \wedge_{\Sigma_{\mathbf{T}}} \mathrm{M}(\ell^\nu)$ for any smooth scheme X of finite type over k.)

The first statement in Corollary 1.7 follows readily from the statements in Theorem 1.5, Corollary 1.6, Proposition 8.3 and the following observation: the Euler characteristic of G/N(T) (when G and N(T) are defined over a separably closed field) in étale cohomology with Z/ℓ^ν-coefficients (ℓ different from the characteristic) is 1. The second statement there is proven similarly using Betti realization in the place of étale realization: see Examples 8.5(2). □.

10. Appendix: Spanier-Whitehead duality and Thom-spaces in the motivic and étale setting

The main *goal of this section* is to collect together various basic results on Thom spaces of algebraic vector bundles and relate them to Spanier-Whitehead duality in the both the motivic and étale framework. Throughout the following discussion we will let S denote a Noetherian affine scheme and will restrict to smooth schemes of finite type over S.

10.1. Basic results on Thom-spaces.

We begin with the following basic observation on vector bundles over affine schemes.

Proposition 10.1. *(i) Let* X *denote any affine scheme. Then any vector bundle* $\mathcal{E}$ *on* X *has a complement, i.e. there exists another vector bundle* $\mathcal{E}^\perp$ *so that* $\mathcal{E} \oplus \mathcal{E}^\perp$ *is a trivial vector bundle.*

(ii) Assume X *is again an affine scheme. Then, if* $\mathcal{E}$ *and* $\mathcal{F}$ *are two vector bundles on* X*, then they represent the same class in the Grothendieck group* $\mathrm{K}^0(\mathrm{X})$ *if and only if they are stably isomorphic, i.e. isomorphic after the addition of some trivial vector bundles.*

(iii) Let X *denote a quasi-projective scheme, i.e. locally closed in some projective space over an affine base scheme* S*. Then there exists an affine scheme* $\tilde{\mathrm{X}}$ *together with a surjective map* $\tilde{\mathrm{X}} \rightarrow \mathrm{X}$ *so that* $\tilde{\mathrm{X}}$ *is an affine-space bundle over* X*. In particular, the map* $\tilde{\mathrm{X}} \rightarrow \mathrm{X}$ *is an* $\mathbb{A}^1$*-equivalence.*

Proof. (i) is clear from the fact that vector bundles on affine schemes correspond to projective modules over the corresponding coordinate ring. (ii) is discussed in [Voev03, Lemma 2.9]. (iii) is the construction discussed in [Joun73, Lemme 1.5] and often referred to as the Jouanolou trick. □

We will next summarize some well-known facts about Thom-spaces in the stable $\mathbb{A}^1$-homotopy category, $\mathbf{Spt}/\mathrm{S}_{\mathrm{mot}}$. If α is a vector bundle over a smooth scheme X over the scheme S, then one needs to define the Thom-space of α

to be the following canonical homotopy pushout:

(10.1.1)
$$\begin{array}{ccc} \mathrm{E}(\alpha) - \mathrm{X} & \longrightarrow & \mathrm{E}(\alpha) \\ \downarrow & & \downarrow \\ \mathrm{S} & \longrightarrow & \mathrm{Th}(\alpha) \end{array}$$

where $\mathrm{E}(\alpha)$ denotes the total space of the vector bundle α. Since $\mathrm{E}(\alpha)$ and $\mathrm{E}(\alpha) - \mathrm{X}$ map to X and then to S, $\mathrm{Th}(\alpha)$ maps to S. The map $\mathrm{S} \to \mathrm{Th}(\alpha)$ provides a section to the induced map $\mathrm{Th}(\alpha) \to \mathrm{S}$, so that $\mathrm{Th}(\alpha)$ is pointed over S and hence is an object in $\mathbf{Spt}/\mathrm{S}_{\mathrm{mot}}$. (We may often assume that injective maps are cofibrations, in which case the map in the top row is a cofibration, and therefore, it suffices to take the ordinary pushout, in the place of the homotopy pushout.)

When we view $\mathrm{E}(\alpha)$ and $\mathrm{E}(\alpha) - \mathrm{X}$ as sheaves on the big étale site, the corresponding pushout of sheaves on the big étale site will be denoted $\mathrm{Th}(\alpha)_{et}$.

Proposition 10.2. *Let α denote a vector bundle over the scheme X. (i) Viewing $\mathrm{Proj}_{\mathrm{X}}(\alpha \oplus 1) = \mathrm{P}(\alpha \oplus \epsilon^1)$ and $\mathrm{Proj}_{\mathrm{X}}(\alpha) = \mathrm{P}(\alpha)$ as simplicial presheaves over the base scheme S and taking the quotient presheaf, $\mathrm{P}(\alpha \oplus \epsilon^1)/\mathrm{P}(\alpha) \simeq \mathrm{Th}(\alpha)$ where $\mathrm{P}(\beta)$ denotes the projective space bundle associated to a vector bundle β and ϵ^1 denotes a trivial bundle of rank 1. Viewing $\mathrm{P}(\alpha \oplus \epsilon^1)$ and $\mathrm{P}(\alpha)$ as simplicial presheaves over X and taking the quotient presheaf over X, $\mathrm{P}(\alpha \oplus \epsilon^1)/_{\mathrm{X}}\mathrm{P}(\alpha) \simeq \mathrm{S}(\alpha \oplus \epsilon^1)$, a sphere bundle over X. (This may be called the "one-point compactification of the vector bundle α".) The obvious projection $\mathrm{S}(\alpha \oplus \epsilon^1) \to \mathrm{X}$ has a section s that sends a point in X to the point at ∞ in the fiber over that point. Now $\mathrm{S}(\alpha \oplus \epsilon^1)/\mathrm{s}(\mathrm{X}) \cong \mathrm{Th}(\alpha)$.*

(ii) If $\mathrm{X} \to \mathrm{Y}$ is a closed immersion of smooth schemes with $\mathcal{N}$ denoting the corresponding normal bundle, then $\mathrm{Th}(\mathcal{N}) \simeq \mathrm{X}/\mathrm{X} - \mathrm{Y}$.

(iii) Let $g : \mathrm{S}' \to \mathrm{S}$ denote a map of smooth schemes and let $g^*(\alpha)$ denote the induced vector bundle on $\mathrm{X}' = \mathrm{X} \underset{\mathrm{S}}{\times} \mathrm{S}'$. Then g induces a map $\mathrm{Th}(g^*(\alpha)) \to \mathrm{Th}(\alpha)$ compatible with the given map $g : \mathrm{S}' \to \mathrm{S}$. Moreover, the induced map $\mathrm{Th}(g^*(\alpha)) \to \mathrm{Th}(\alpha)$ is natural in g and α.

Proof. The proof of the first statement appears in (10.2.2) and the proof of the second statement appears in [MV99]. Since $\mathrm{Th}(g^*(\alpha))$ is defined by the canonical homotopy pushout

$$\begin{array}{ccc} \mathrm{E}(g^*(\alpha)) - \mathrm{X}' & \longrightarrow & \mathrm{E}(g^*(\alpha)) \\ \downarrow & & \downarrow \\ S' & \longrightarrow & \mathrm{Th}(g^*(\alpha)) \end{array}$$

where $\mathrm{X}' = \mathrm{X} \underset{\mathrm{S}}{\times} \mathrm{S}'$, the existence of the map $\mathrm{Th}(g^*(\alpha)) \to \mathrm{Th}(\alpha)$ is clear. That this is natural in S and α is clear from the from the fact that the canonical homotopy pushout is natural in the arguments defining it. ☐

Notation 10.3. *When we view $\mathrm{P}(\alpha)$ and $\mathrm{P}(\alpha \oplus 1)$ as sheaves on the big étale site the corresponding quotient $\mathrm{P}(\alpha \oplus 1)_{\mathrm{et}}/\mathrm{P}(\alpha)_{\mathrm{et}}$ will be denoted $\mathrm{S}(\alpha \oplus 1)_{\mathrm{et}}$.*

10.2. The fiber-wise join of simplicial presheaves (spectra) fibering over another simplicial presheaf (spectrum). Given maps of simplicial presheaves $\mathrm{Y} \to \mathrm{X}$ and $\mathrm{Z} \to \mathrm{X}$, the *fiber-wise join* $\mathrm{Y} *_{\mathrm{X}} \mathrm{Z}$ is the simplicial presheaf defined as the (canonical) homotopy pushout

(10.2.1)
$$\begin{array}{ccc} \mathrm{Y} \underset{\mathrm{X}}{\times} \mathrm{Z} & \longrightarrow & \mathrm{Z} \\ \downarrow & & \downarrow \\ \mathrm{Y} & \longrightarrow & \mathrm{Y} *_{\mathrm{X}} \mathrm{Z} \end{array}$$

One may readily verify that if X, Y and Z are spectra in any of the above categories of spectra, then one may apply the above construction to the constituent spaces of the above spectra, which will show that the above construction extends to spectra. We elaborate a bit on the above construction and its application to Thom-spaces. First we show that the fiber-wise join indeed does what it is supposed to do.

Lemma 10.4. *Assume X, Y and Z are simplicial presheaves as above. Then (i) there is an induced map $\mathrm{Y} *_{\mathrm{X}} \mathrm{Z} \to \mathrm{X}$. (ii) If $\mathrm{Y}_{\mathrm{x}}, \mathrm{Z}_{\mathrm{x}}$ denote the fibers over $\mathrm{x} \in \mathrm{X}$, $(\mathrm{Y} *_{\mathrm{X}} \mathrm{Z})_{\mathrm{x}} \simeq \mathrm{S}^1 \wedge (\mathrm{Y}_{\mathrm{x}} \wedge \mathrm{Z}_{\mathrm{x}})$. (iii). Therefore, if X denotes the simplicial presheaf represented by a smooth scheme, and Y, Z are pointed simplicial presheaves over X, $\mathrm{Y} *_{\mathrm{X}} \mathrm{Z} \simeq (\mathrm{S}^1 \times \mathrm{X}) \wedge^{\mathrm{X}} (\mathrm{Y} \wedge^{\mathrm{X}} \mathrm{Z}) \simeq \mathrm{Y} \wedge^{\mathrm{X}} ((\mathrm{S}^1 \times \mathrm{X}) \wedge^{\mathrm{X}} \mathrm{Z})$.*

Proof. This is a well-known result. See for example, [CS03, Lemma 2.1]. ☐

Next let X denote a smooth scheme and let α denote a vector bundle over X. We may then consider $\mathrm{Proj}_{\mathrm{X}}(\alpha)$ and $\mathrm{Proj}_{\mathrm{X}}(\alpha \oplus 1)$ as the corresponding projective-space bundles over X. Here 1 denotes the trivial vector bundle of rank 1. We we will view these as simplicial presheaves over the simplicial presheaf represented by X. Next let $\mathrm{S}(\alpha \oplus 1)$ denote the simplicial presheaf quotient of the obvious monomorphism: $\mathrm{Proj}_{\mathrm{X}}(\alpha) \rightarrow \mathrm{Proj}_{\mathrm{X}}(\alpha \oplus 1)$ of simplicial presheaves fibered over X. (Note: the S stands for sphere-bundle.) To be precise, this is defined by the homotopy pushout square (since the map $\mathrm{Proj}_{\mathrm{X}}(\alpha) \rightarrow \mathrm{Proj}_{\mathrm{X}}(\alpha \oplus 1)$ is a monomorphism and we use the injective model structure on simplicial presheaves, one may simply take a pushout square here):

(10.2.2)
$$\begin{array}{ccc} \mathrm{Proj}_{\mathrm{X}}(\alpha) & \longrightarrow & \mathrm{Proj}_{\mathrm{X}}(\alpha \oplus 1) \\ \downarrow & & \downarrow \\ \mathrm{X} & \xrightarrow{s} & S(\alpha \oplus 1) \end{array}$$

where $s : \mathrm{X} \rightarrow \mathrm{S}(\alpha \oplus 1)$ denotes the section sending a point in X to the point at ∞ in the fiber of $\mathrm{S}(\alpha \oplus 1)$. Since the Thom-space $\mathrm{Th}(\alpha)$ identifies with the homotopy cofiber $\mathrm{Proj}_{\mathrm{X}}(\alpha \oplus 1)/\mathrm{Proj}_{\mathrm{X}}(\alpha)$, this is also defined by a similar homotopy pushout, but with X replaced by the base scheme S. i.e. $\mathrm{Th}(\alpha)$ may also be obtained by taking the homotopy pushout $\mathrm{S}(\alpha \oplus 1)/\mathrm{s}(\mathrm{X})$.

If β is another vector bundle over X, we let $\overset{o}{\beta} \rightarrow X$ denote the associated bundle $\beta - 0 \rightarrow X$. Now with $\overset{o}{\beta}_+ = \overset{o}{\beta} \sqcup X$, one obtains $(S^1 \times \mathrm{X}) \wedge^{\mathrm{X}} \overset{o}{\beta}_+ \simeq (\beta \underset{\overset{o}{\beta}}{\sqcup} \mathrm{X}) \cong \mathrm{S}(\beta \oplus 1)$. The last $\cong$ is an isomorphism as simplicial presheaves over X while the $\simeq$ is a weak-equivalence of such simplicial presheaves. The last isomorphism may be seen by working locally on X, so that β is trivial. The $\simeq$ follows from the observation that the fibers of β are acyclic so that $(S^1 \times \mathrm{X}) \wedge^{\mathrm{X}} \overset{o}{\beta}_+ \simeq (\beta \underset{\overset{o}{\beta}}{\sqcup} \mathrm{X})$ as simplicial presheaves over X.

Lemma 10.5. *Let α and β denote two vector bundles over the scheme* X*. Then we obtain the identifications:*

(i) $\mathrm{S}(\alpha \oplus 1) *_{\mathrm{X}} \overset{o}{\beta}_+ \simeq \mathrm{S}(\alpha \oplus 1) \wedge^{\mathrm{X}} \mathrm{S}(\beta \oplus 1) \simeq \mathrm{S}(\alpha \oplus \beta \oplus 1)$ *where* $\overset{o}{\beta}$ *denotes* $\beta - 0 \rightarrow X$*, the associated sphere bundle.*

(ii) The map $\mathrm{S}(\alpha \oplus \beta \oplus 1) \rightarrow \mathrm{X}$ *has a section s sending each point of* X *to the point at* ∞ *in the fiber over that point. Then the quotient* $\mathrm{S}(\alpha \oplus \beta \oplus 1)/\mathrm{s}(\mathrm{X}) = \mathrm{Th}(\alpha \oplus \beta)$ *which is the Thom-space of the vector bundle* $\alpha \oplus \beta$*.*

Proof. Since (ii) is rather straight-forward, we will discuss only (i). In view of the weak-equivalences above between $(S^1 \times \mathrm{X}) \wedge^{\mathrm{X}} \overset{o}{\beta}_+$ and $\mathrm{S}(\beta \oplus 1)$, and the observation that $\wedge^{\mathrm{X}}$ is a homotopy pushout of simplicial presheaves over X (in the injective model structure), it follows that $\mathrm{S}(\alpha \oplus 1) *_{\mathrm{X}} \overset{o}{\beta}_+ \simeq \mathrm{S}(\alpha \oplus 1) \wedge^{\mathrm{X}} ((\mathrm{S}^1 \times \mathrm{X}) \wedge^{\mathrm{X}} \overset{o}{\beta}_+) \simeq \mathrm{S}(\alpha \oplus 1) \wedge^{\mathrm{X}} \mathrm{S}(\beta \oplus 1)$. Since the last fibers over X, one may work locally on X and show readily that it identifies with $\mathrm{S}(\alpha \oplus \beta \oplus 1)$. ☐

Lemma 10.6. *Let* $\mathrm{B} = \mathrm{Spec}\, k$ *denote the base field. Let* G *denote a linear algebraic group defined over* B *and acting on the simplicial presheaves* E *and* X *over the base scheme* B*. Assume that* E *is in fact a smooth scheme of finite type over* B *so that the (geometric) quotient* E/G *exists and is in fact a scheme of finite type over* B*. Let* P *denote a pointed simplicial presheaf in* PSh/B *together with a* G*-action that leaves the base point of* P *fixed. Then* $\mathrm{E} \times_{\mathrm{G}} (\mathrm{P} \wedge \mathrm{X}_+) \cong \mathrm{P}_{\mathrm{E/G}} \wedge^{\mathrm{E/G}} (\mathrm{E} \times_{\mathrm{G}} \mathrm{X}_+)$*, where* $\mathrm{P}_{\mathrm{E/G}} = \mathrm{P} \times_{\mathrm{G}} \mathrm{E}$*.*

Proof. The proof is skipped as one may readily verify the above conclusions. ☐

10.3. Motivic Atiyah duality. The rest of this section will be devoted to summarizing a version of Atiyah-duality (see [At61]) that applies to the motivic and also the étale context, so that for any smooth projective scheme X over a smooth affine base scheme, there exists a vector bundle over the scheme X (which we call the *virtual normal bundle*) whose Thom-space is a Spanier-Whitehead dual of the suspension spectrum $\Sigma_{\mathbf{T}} X$ in the category $\mathbf{Spt}/\mathrm{S}_{\mathrm{mot}}$. The idea of the proof may be summarized as follows: using the evaluation and co-evaluation maps e_{X} and c_{X} defined below, and with the $\mathbf{T}$-suspension spectrum of a smooth projective scheme X denoting $\mathcal{X}$ (the $\mathbf{T}$-suspension spectrum of the Thom-space of a suitable vector bundle over X considered below replacing the dual $\mathrm{D}(\mathcal{X})$), it suffices to show that the conditions in Theorem 2.3(ii) are satisfied.

Under the assumption that the base scheme S satisfies the finiteness hypothesis as in (3.0.3), we may readily observe that the pullback functors $\epsilon^* : \mathbf{Spt}/\mathrm{S}_{\mathrm{Nis}} \rightarrow \mathbf{Spt}/\mathrm{S}_{\mathrm{et}}$, $\bar{\epsilon}^* : \mathbf{Spt}/\bar{\mathrm{S}}_{\mathrm{Nis}} \rightarrow \mathbf{Spt}/\bar{\mathrm{S}}_{\mathrm{et}}$, and $\eta^* : \mathbf{Spt}/\mathrm{S}_{\mathrm{et}} \rightarrow \mathbf{Spt}/\bar{\mathrm{S}}_{\mathrm{et}}$

considered in (5.0.19) as well as the functors θ and $\phi_{\mathcal{E}}$ (discussed in the paragraph below (5.0.19)) send Thom-spaces in the framework of the source, to Thom-spaces in the framework of the target, are compatible with the smash-products and internal Homs in these categories and also send maps that are homotopic to the identity to maps that are homotopic to the identity. Therefore, as shown in the proof of Proposition 5.4(v), the discussion below carries over from the framework of $\mathbf{Spt}/\mathrm{S}_{\mathrm{mot}}$ to all of the other frameworks (at least after inverting $\mathbb{A}^1$ in all these frameworks). i.e. The construction of a Spanier-Whitehead dual from the Thom-space of a vector bundle worked out below in the motivic framework carries over to the étale setting after smashing with an ℓ-complete spectrum, ℓ being prime to the characteristic.

Over algebraically closed fields of arbitrary characteristic, there is already a somewhat different construction valid in the étale setting and making strong of use of étale tubular neighborhoods. This appears in [J86] and [J87].

Definition 10.7. (The diagonal map.) Next we consider the following *diagonal map*. Let α, β denote two vector bundles on the scheme X. Then there is a diagonal map $\mathrm{Th}(\alpha \oplus \beta) \to \mathrm{Th}(\alpha) \wedge \mathrm{Th}(\beta)$. This map is induced by the map $\mathrm{E}(\alpha \oplus \beta) \to \mathrm{E}(\alpha) \times \mathrm{E}(\beta)$ lying over the diagonal map $\mathrm{X} \to \mathrm{X} \times \mathrm{X}$. In this case, one may verify that $E(\alpha \oplus \beta) - \{0\}$ maps to $(E(\alpha) - \{0\}) \times E(\beta) \cup E(\alpha) \times (E(\beta) - \{0\})$. Taking α to be a zero-dimensional bundle, one obtains the diagonal map

$$\Delta : \mathrm{Th}(\beta) \to X_+ \wedge \mathrm{Th}(\beta). \tag{10.3.1}$$

One may interpret the above diagonal map in terms of the associated disk and sphere bundles as follows:

$$\Delta' : \mathrm{Th}(\beta) = \mathrm{P}(\beta \oplus \epsilon^1)/\mathrm{P}(\beta) \to \mathrm{P}(\beta \oplus \epsilon^1)_+ \wedge \mathrm{P}(\beta \oplus \epsilon^1)/\mathrm{P}(\beta) = (\mathrm{P}(\beta \oplus \epsilon^1) \times \mathrm{P}(\beta \oplus \epsilon^1))/\mathrm{P}(\beta \oplus \epsilon^1) \times \mathrm{P}(\beta)$$

Now one composes with the projection $\mathrm{P}(\beta \oplus \epsilon^1) \to \mathrm{X}$ to define the diagonal map in (10.3.1).

10.4. Basic framework: the projective case. Assume next that X and Y are smooth projective schemes with X provided with a closed immersion into Y over the smooth affine scheme S. Y will usually denote a projective space over S, but we denote it by Y for simplicity of notation. Then denoting τ_{X}, τ_Y and $\mathcal{N}$ the tangent bundle to X, the tangent bundle to Y and the normal bundle associated to the imbedding of X in Y, one obtains the short exact sequence

$$0 \to \tau_{\mathrm{X}} \to \tau_{\mathrm{Y}|\mathrm{X}} \to \mathcal{N} \to 0. \tag{10.4.1}$$

Let $\pi_{\mathrm{Y}} : \tilde{\mathrm{Y}} \to \mathrm{Y}$ denote the affine replacement provided by Jouanolou's construction. Then the following are proven in [Voev03, Proposition 2.7 through Theorem 2.11]:

1. (1) There exists a vector bundle V on Y so that $\pi_{\mathrm{Y}}^*(\mathrm{V}) \oplus \pi_{\mathrm{Y}}^*(\tau_{\mathrm{Y}})$ is stably isomorphic to a trivial vector bundle. So we will assume that $\pi_{\mathrm{Y}}^*(\mathrm{V}) \oplus \pi_{\mathrm{Y}}^*(\tau_{\mathrm{Y}}) \oplus \epsilon^{\mathrm{m}} \cong \epsilon^{\mathrm{n}}$ for some m and n. We will replace V by $\mathrm{V} \oplus \epsilon^{\mathrm{m}}$ so that $\pi_{\mathrm{Y}}^*(\mathrm{V}) \oplus \pi_{\mathrm{Y}}^*(\tau_{\mathrm{Y}})$ is the trivial bundle ϵ^n.
2. (2) There exists a collapse map $\mathrm{V} : \mathbf{T}^{\mathrm{n}} \to \mathrm{Th}(\mathrm{V})$. In fact, what Voevodsky shows (see [Voev03, Lemma 2.10 and Theorem 2.11]) is that one considers the Segre-imbedding of $\mathbb{P}^d \times \mathbb{P}^d$ in $\mathbb{P}^{d^2+2d}$ and then shows that $\mathbf{T}^{d^2+2d}$ identifies with a certain quotient sheaf of $\mathbb{P}^{d^2+2d}$ by a certain hyperplane H. Therefore, the n above will be $d^2 + 2d$.)

One may observe that $\pi_{\mathrm{Y}}^*(\mathcal{N} \oplus \mathrm{V}_{|\mathrm{X}}) \oplus \pi_{\mathrm{Y}}^*(\tau_{\mathrm{X}})$ is also stably trivial. If $\pi_{\mathrm{Y}}^*(\mathcal{N} \oplus \mathrm{V}_{|\mathrm{X}}) \oplus \pi_{\mathrm{Y}}^*(\tau_{\mathrm{X}}) \oplus \epsilon^{\mathrm{m}} \cong \epsilon^{\mathrm{N}}$ for some m and N, we will replace V by $\mathrm{V} \oplus \epsilon^{\mathrm{m}}$ and we will make the following definition.

Definition 10.8. (*Virtual normal bundle in the projective case, the Voevodsky collapse and the corresponding co-evaluation map*) We let $\nu_{\mathrm{X}} = \mathcal{N} \oplus \mathrm{V}_{|\mathrm{X}}$ and call it *the virtual normal bundle* to X in Y. Taking Y = X, we see that ν_Y has the property that $\pi_{\mathrm{Y}}^*(\nu_{\mathrm{Y}})$ is a complement to $\pi_{\mathrm{Y}}^*(\tau_{\mathrm{Y}})$ in some trivial bundle over $\tilde{\mathrm{Y}}$.

Clearly $\mathrm{Th}(\nu_{\mathrm{X}}) \simeq \mathrm{V}/\mathrm{V} - \mathrm{X}$ where X is imbedded in V by the composite imbedding $\mathrm{X} \to \mathrm{Y} \overset{0-\text{section}}{\to} \mathrm{E}(\mathrm{V})$. Therefore, one obtains a collapse map $\mathrm{Th}(\mathrm{V}) = \mathrm{V}/\mathrm{V} - \mathrm{Y} \to \mathrm{V}/\mathrm{V} - \mathrm{X} \simeq \mathrm{Th}(\nu_{\mathrm{X}})$. Composing with the collapse $\mathrm{V} : \mathbf{T}^{\mathrm{n}} \to \mathrm{Th}(\mathrm{V})$ one obtains the collapse $\mathrm{V}_{\mathrm{X}} : \mathbf{T}^{\mathrm{n}} \to \mathrm{Th}(\nu_{\mathrm{X}})$. Composing with the diagonal map Δ (considered above), one obtains a map $c : \mathbf{T}^n \to \mathrm{X}_+ \wedge \mathrm{Th}(\nu_{\mathrm{X}})$. The main result that we need is that this map is indeed a co-evaluation map in the sense of Theorem 2.3(ii), so that $\Sigma_{\mathbf{T}}^{-n}\mathrm{Th}(\nu_{\mathrm{X}})$ is indeed a Spanier-Whitehead dual of $\Sigma_{\mathbf{T}}\mathrm{X}_+$. This follows from the following results whose proofs are only briefly sketched as they are rather well-known by now. (See, for example, [Hu-Kr05].)

Lemma 10.9. *Let α denote a vector bundle on* Y. *Then the map π_{Y} induces a weak-equivalence* $\mathrm{Th}(\alpha) \simeq \mathrm{Th}(\pi_{\mathrm{Y}}^*(\alpha))$.

Proposition 10.10. *(i) Assume that* X *and* Y *are smooth projective* S*-schemes with* X *provided with a closed immersion into* Y*. Denoting by* $\pi_{\rm Y}: \tilde{\rm Y} \rightarrow Y$*, an affine replacement provided by Jouanolou's device (as above), the vector bundles* $\pi_{\rm Y}^*(\nu_{\rm Y} \oplus \tau_{\rm Y})$ *and* $\pi_{\rm Y}^*(\nu_{\rm X} \oplus \tau_{\rm X})$ *are trivial, so that there are collapse maps*

$$\hat{d}_Y: {\rm Th}(\nu_{\rm Y}) \wedge \Sigma_{\bf T}{\rm Y}_+ \simeq {\rm Th}(\pi_{\rm Y}^*(\nu_{\rm Y})) \wedge \Sigma_{\bf T}{\rm Y}_+ \rightarrow \Sigma_{\bf T}^{\rm m}\tilde{\rm Y}_+ \simeq \Sigma_{\bf T}^{\rm m}{\rm Y}_+,$$
$$\hat{d}_{\rm X}: {\rm Th}(\nu_{\rm X}) \wedge \Sigma_{\bf T}{\rm X}_+ \simeq {\rm Th}(\pi_{\rm Y}^*(\nu_{\rm X})) \wedge \Sigma_{\bf T}{\rm X}_+ \rightarrow {\rm Th}(\pi_{\rm Y}^*(\nu_{\rm X} \oplus \tau_{\rm X})) = \Sigma_{\bf T}^{\rm m}\tilde{\rm X}_+ \simeq \Sigma_{\bf T}^{\rm m}{\rm X}_+, \text{ and}$$
$$\hat{d}_{\rm X}^Y: {\rm Th}(\nu_{\rm Y}) \wedge \Sigma_{\bf T}{\rm X}_+ \simeq {\rm Th}(\pi_{\rm Y}^*(\nu_{\rm Y})) \wedge \Sigma_{\bf T}{\rm X}_+ \rightarrow \Sigma_{\bf T}^{\rm m}\tilde{\rm X}_+ \simeq \Sigma_{\bf T}^{\rm m}{\rm X}_+.$$

(ii) The relative normal bundle to the composite imbedding ${\rm X} \rightarrow {\rm Y} \rightarrow {\rm E}(\nu_{\rm Y})$ *is* $\mathcal{N} \oplus \nu_{Y|X} = \mathcal{N} \oplus {\rm V}_{|{\rm X}} = \nu_{\rm X}$ *if* $\mathcal{N}$ *denotes the normal bundle for the immersion* ${\rm X} \rightarrow {\rm Y}$*. Therefore, one also obtains a collapse:* $g_{\rm X}^Y: {\rm Th}(\nu_{\rm Y}) \rightarrow {\rm Th}(\nu_{\rm X})$

In this situation, we define the *evaluation map*

(10.4.2) $e_{\rm Y}: \Sigma_{\bf T}^{-m}{\rm Th}(\nu_{\rm Y}) \wedge \Sigma_{\bf T}{\rm Y}_+ \rightarrow \Sigma_{\bf T}$ as $\Sigma_{\bf T}^{-{\rm m}}\hat{\rm d}_{\rm Y}$ composed with the collapse map $\Sigma_{\bf T}{\rm Y}_+ \rightarrow \Sigma_{\bf T}$.

Observe that the same definition applies to define an evaluation map for X:

(10.4.3) $$e_{\rm X}: \Sigma_{\bf T}^{-m}{\rm Th}(\nu_{\rm X}) \wedge \Sigma_{\bf T}{\rm X}_+ \rightarrow \Sigma_{\bf T}$$

since $\nu_{\rm X} = \nu_{Y|X} \oplus \mathcal{N}$. The evaluation map $e_{\rm X}^Y: \Sigma_{\bf T}^{-m}{\rm Th}(\nu_{\rm Y}) \wedge \Sigma_{\bf T}{\rm X}_+ \rightarrow \Sigma_{\bf T}{\rm X}_+$ is defined similarly. Observe that one obtains the following commutative diagram in this case:

(10.4.4) $$\begin{array}{ccccc} \Sigma_{\bf T}^{-m}{\rm Th}(\nu_{\rm Y}) \wedge {\rm Y}_+ & \xleftarrow{id \wedge i} & \Sigma_{\bf T}^{-m}{\rm Th}(\nu_{\rm Y}) \wedge {\rm X}_+ & \xrightarrow{\Sigma_{\bf T}^{-m} g_{\rm X}^Y \wedge id} & \Sigma_{\bf T}^{-m}{\rm Th}(\nu_{\rm X}) \wedge {\rm X}_+ \\ \downarrow{\scriptstyle e_{\rm Y}} & & \downarrow{\scriptstyle e_{\rm X}^Y} & & \downarrow{\scriptstyle e_{\rm X}} \\ \Sigma_{\bf T} & \xrightarrow{id} & \Sigma_{\bf T} & \xrightarrow{id} & \Sigma_{\bf T} \end{array}$$

Observe also that $e_{\rm Y}$ ($e_{\rm X}$, $e_{\rm X}^Y$) corresponds by adjunction to a map

(10.4.5)
$d_{\rm Y}: \Sigma_{\bf T}^{-m}{\rm Th}(\nu_{\rm Y}) \rightarrow {\rm D}(\Sigma_{\bf T}{\rm Y}_+)$ $({\rm d}_{\rm X}: \Sigma_{\bf T}^{-{\rm m}}{\rm Th}(\nu_{\rm X}) \rightarrow {\rm D}(\Sigma_{\bf T}{\rm X}_+), {\rm d}_{\rm X}^{\rm Y}: \Sigma_{\bf T}^{-{\rm m}}{\rm Th}(\nu_{\rm Y}) \rightarrow {\rm D}(\Sigma_{\bf T}{\rm Y}_+)$, respectively.)

Next assume that the maps $d = d_{\rm Y}: \Sigma_{\bf T}^{-m}{\rm Th}(\nu_{\rm Y}) \rightarrow {\rm D}(\Sigma_{\bf T}{\rm Y}_+)$, $\tilde{d}: \Sigma_{\bf T}{\rm Y}_+ \rightarrow \Sigma_{\bf T}^{\rm m}({\rm D}({\rm Th}(\nu_{\rm Y})))$ (where $\tilde{d}$ is obtained by taking the dual of d and then precomposing with obvious map $\Sigma_{\bf T}{\rm Y}_+ \rightarrow {\rm DD}(\Sigma_{\bf T}{\rm Y}_+)$) are weak-equivalences and let $d^{-1}: {\rm D}(\Sigma_{\bf T}{\rm Y}_+) \rightarrow \Sigma_{\bf T}^{-{\rm m}}{\rm Th}(\nu_{\rm Y})$, $\tilde{d}^{-1}: \Sigma_{\bf T}^{m}{\rm D}({\rm Th}(\nu_{\rm Y})) \rightarrow \Sigma_{\bf T}{\rm Y}_+$ denote a homotopy inverse for d ($\tilde{d}$, respectively). Let $\kappa: \Sigma_{\bf T}{\rm Y}_+ = \Sigma_{\bf T}{\rm Y}_+ \rightarrow \Sigma_{\bf T}$ and $\tilde{\kappa}: \Sigma_{\bf T}{\rm Th}(\nu_{\rm Y}) \rightarrow \Sigma_{\bf T}$ denote the obvious maps. Therefore, taking duals, one obtains a map $\kappa': \Sigma_{\bf T} \rightarrow {\rm D}(\Sigma_{\bf T}{\rm Y}_+)$ ($\tilde{\kappa}': \Sigma_{\bf T} \rightarrow {\rm D}({\rm Th}(\nu_{\rm Y}))$, respectively). Composing with d^{-1} ($\tilde{d}^{-1}$), one obtains maps:

(10.4.6) $$c_{\rm Y}': \Sigma_{\bf T} \rightarrow \Sigma_{\bf T}^{-m}{\rm Th}(\nu_{\rm Y}) \text{ and } c_{\rm Y} = \Delta \circ c_{\rm Y}': \Sigma_{\bf T} \rightarrow \Sigma_{\bf T}{\rm Y}_+ \wedge \Sigma_{\bf T}^{-{\rm m}}{\rm Th}(\nu_{\rm Y})$$

We will see below in Theorem 10.14 that the above approach, suffices to define the *co-evaluation* map $c_{\rm Y}$ for any projective smooth scheme Y.

Proposition 10.11. *The composite map*

$$\Sigma_{\bf T}^{-m}{\rm Th}(\nu_{\rm Y}) \wedge \Sigma_{\bf T}{\rm Y}_+ \stackrel{\Delta\wedge{\rm id}}{\rightarrow} \Sigma_{\bf T}{\rm Y}_+ \wedge \Sigma_{\bf T}^{-{\rm m}}{\rm Th}(\nu_{\rm Y}) \wedge \Sigma_{\bf T}{\rm Y}_+ \stackrel{{\rm id}\wedge{\rm e}_{\rm Y}}{\rightarrow} \Sigma_{\bf T}{\rm Y}_+ \wedge \Sigma_{\bf T} \simeq \Sigma_{\bf T}{\rm Y}_+$$

is always homotopic to the map $d_{\rm Y}'$*, i.e. irrespective of whether* ${\rm d}_{\rm Y}$ *is a weak-equivalence. In case* ${\rm d}_{\rm Y}$ *is also a weak-equivalence, the composite map*

$$\Sigma_{\bf T} \wedge \Sigma_{\bf T}{\rm Y}_+ \stackrel{{\rm c}_{\rm Y}\wedge{\rm id}}{\rightarrow} \Sigma_{\bf T}{\rm Y}_+ \wedge \Sigma_{\bf T}^{-{\rm m}}{\rm Th}(\nu_{\rm Y}) \wedge \Sigma_{\bf T}{\rm Y}_+ \stackrel{{\rm id}\wedge{\rm d}_{\rm Y}'}{\rightarrow} \Sigma_{\bf T}{\rm Y}_+ \wedge \Sigma_{\bf T}{\rm Y}_+$$

is homotopic to the diagonal map.

Proof. The diagonal map Δ may be viewed as the map $({\rm E}(\nu_{\rm Y}), {\rm E}(\nu_{\rm Y}) - Y) \rightarrow (Y \times {\rm E}(\nu_{\rm Y}), Y \times {\rm E}(\nu_{\rm Y}) - (Y \times Y))$, sends $(e, e') \mapsto ((p(e), e), (p(e'), e'))$, where $p: {\rm E}(\nu_{\rm Y}) \rightarrow {\rm Y}$ is the obvious projection. The collapse map

$$\hat{d}_{\rm Y}: {\rm Th}(\nu_{\rm Y}) \wedge \Sigma_{\bf T}{\rm Y}_+ \rightarrow {\rm Th}(\nu_{\rm Y} \oplus \tau_{\rm Y}) \simeq \Sigma_{\bf T}^{\rm n}{\rm Y}_+$$

on the other hand identifies with the obvious map

$$({\rm E}(\nu_{\rm Y}) \times Y, {\rm E}(\nu_{\rm Y}) \times Y - (Y \times Y)) \rightarrow ({\rm E}(\nu_{\rm Y}) \times Y, {\rm E}(\nu_{\rm Y}) \times Y - \Delta(Y))$$

where $\Delta(Y)$ is Y imbedded diagonally in $Y \times Y$.

Therefore, it follows that any $(e_y, y) \in \mathrm{E}(\nu_{\mathrm{Y}}) \times Y$ with e_y in the fiber over y is sent by the composition of the map $\Delta \wedge id$ and the collapse d'_{Y} to $(y, y) \in \mathrm{Y}_+ \wedge \mathrm{Y}_+$. Therefore, (e_y, y) (as above) is sent by $(id \wedge e) \circ (\Delta \wedge id)$ to y and $(e_{y'}, y)$, with $y \neq y'$ is collapsed to the base point. This proves the first assertion and the proof of the second is similar: recall that one needs d_{Y} to be a weak-equivalence for c_{Y} to be defined. □

Corollary 10.12. *Assume that the variety* Y *is such that the map* $d : \Sigma_{\mathbf{T}}^{-m}\mathrm{Th}(\nu_{\mathrm{Y}}) \rightarrow D(\Sigma_{\mathbf{T}}\mathrm{Y}_+)$ *is a weak-equivalence. Then the composition*

$$\Sigma_{\mathbf{T}}\mathrm{Y}_+ \simeq \Sigma_{\mathbf{T}} \wedge \Sigma_{\mathbf{T}}\mathrm{Y}_+ \stackrel{\mathrm{c}\wedge\mathrm{id}}{\rightarrow} \Sigma_{\mathbf{T}}\mathrm{Y}_+ \wedge \Sigma_{\mathbf{T}}^{-\mathrm{m}}\mathrm{Th}(\nu_{\mathrm{Y}}) \wedge \Sigma_{\mathbf{T}}\mathrm{Y}_+ \stackrel{\mathrm{id}\wedge\mathrm{e}}{\rightarrow} \wedge \Sigma_{\mathbf{T}}\mathrm{Y}_+$$

is homotopic to the identity map.

Proof. This is clear, since the composite map appearing in the second statement in Proposition 10.11 is homotopic to the diagonal map. □

Remark 10.13. The above corollary, therefore verifies the first condition in Theorem 2.3(ii) under the assumption that the map d is a weak-equivalence. We skip the verification of the second condition there.

We next consider the following rather technical result which when applied suitably will prove that all smooth projective varieties over any smooth affine variety as the base scheme are dualizable in the stable $\mathbb{A}^1$-motivic homotopy category and the corresponding étale category.

Theorem 10.14. *Recall the base scheme is a smooth affine Noetherian scheme* S*. Let* Y *denote any smooth projective scheme over* S *so that* $\Sigma_{\mathbf{T}}\mathrm{Y}_+$ *is dualizable in the stable* $\mathbb{A}^1$*-homotopy category* **Spt**/S *and so that the map* $d_{\mathrm{Y}} : \Sigma_{\mathbf{T}}^{-m}\mathrm{Th}(\nu_{\mathrm{Y}}) \rightarrow \mathrm{D}(\Sigma_{\mathbf{T}}\mathrm{Y}_+)$ *in* (10.4.5) *is a weak-equivalence. Then for any smooth closed subscheme* X *of* Y*,* $\Sigma_{\mathbf{T}}\mathrm{X}_+$ *is also dualizable in the same stable* $\mathbb{A}^1$*-homotopy category* **Spt**/S*. Moreover, the map* $d_{\mathrm{X}} : \Sigma_{\mathbf{T}}^{-m}\mathrm{Th}(\nu_{\mathrm{X}}) \rightarrow \mathrm{D}(\Sigma_{\mathbf{T}}\mathrm{X}_+)$ *is a weak-equivalence.*

Proof. The first step in the proof is to define the *co-evaluation map* $c_{\mathrm{X}} : \Sigma_{\mathbf{T}} \rightarrow \Sigma_{\mathbf{T}}\mathrm{X}_+ \wedge \Sigma_{\mathbf{T}}^{-\mathrm{m}}\mathrm{Th}(\nu_{\mathrm{X}})$. Observe that X imbeds in Y and that therefore, one obtains the collapse map: $g_{\mathrm{X}}^{Y} : \mathrm{Th}(\nu_{\mathrm{Y}}) \rightarrow \mathrm{Th}(\nu_{\mathrm{X}})$. We let $c'_{\mathrm{X}} : \Sigma_{\mathbf{T}} \rightarrow \Sigma_{\mathbf{T}}^{-m}\mathrm{Th}(\nu_{\mathrm{X}})$ be defined as the composition

$$\Sigma_{\mathbf{T}}^{-m} g_{\mathrm{X}}^{Y} \circ c'_{\mathrm{Y}} : \Sigma_{\mathbf{T}} \rightarrow \Sigma_{\mathbf{T}}^{-m}\mathrm{Th}(\nu_{\mathrm{Y}}) \rightarrow \Sigma_{\mathbf{T}}^{-n}\mathrm{Th}(\nu_{\mathrm{X}}). \tag{10.4.7}$$

We let $c_{\mathrm{X}} : \Sigma_{\mathbf{T}} \rightarrow \Sigma_{\mathbf{T}}\mathrm{X}_+ \wedge \Sigma_{\mathbf{T}}^{-\mathrm{m}}\mathrm{Th}(\nu_{\mathrm{X}})$ be defined as $\Delta_{\mathrm{X}} \circ c'_{\mathrm{X}}$. The remainder of the proof is simply to show that with the above definition of the co-evaluation map c_{X} and the evaluation map $e_{\mathrm{X}} : \Sigma_{\mathbf{T}}^{-m}\mathrm{Th}(\nu_{\mathrm{X}}) \wedge \Sigma_{\mathbf{T}}\mathrm{X}_+ \rightarrow \Sigma_{\mathbf{T}}$ defined in (10.4.3) satisfy the hypothesis in Theorem 2.3(ii) when $\mathcal{X}$ ($\mathrm{D}(\mathcal{X})$) is replaced by $\Sigma_{\mathbf{T}}\mathrm{X}_+$ ($\Sigma_{\mathbf{T}}^{-m}\mathrm{Th}(\nu_{\mathrm{X}})$, respectively). We skip the remaining details of a proof of this theorem as this is essentially proved in [Hu-Kr05, Appendix]. One may find related results in [Ay] also. □

Theorem 10.15. *Let* $\mathbb{P}^n$ *denote the* n*-dimensional projective space over the base* S*. Then* $\Sigma_{\mathbf{T}}\mathbb{P}^n_+$ *is dualizable.*

(i) If $\pi_{\mathrm{Y}} : \tilde{\mathbb{P}}^n \rightarrow \mathbb{P}^n$ *is an affine replacement as in 10.1, and* $\tilde{\nu} = \pi_{\mathrm{Y}}^*(\nu)$ *is a complement to* $\pi_{\mathrm{Y}}^*(\tau_{\mathbb{P}^n})$ *in a trivial vector bundle, of rank* m*, then there is a map* $d_{\tilde{P}^n} : \Sigma_{\mathbf{T}}^{-m}\mathrm{Th}(\tilde{\nu}) \rightarrow \mathrm{D}(\Sigma_{\mathbf{T}}\tilde{\mathrm{P}}^{\mathrm{n}}_+)$ *which is a stable* $\mathbb{A}^1$*-equivalence.*

(ii) $\Sigma_{\mathbf{T}}^{-m}\mathrm{Th}(\nu)$ *is also a stable dual to* $\Sigma_{\mathbf{T}}\mathbb{P}^n_+$*, where* ν *denotes the vector bundle denoted* ν_{Y} *(for* $Y = \mathbb{P}^n$*) in Definition 10.8.*

Proof. One may prove both statements in the theorem using ascending induction on n, the case $n = 0$ being obviously true. □

Corollary 10.16. *Any smooth projective scheme over the affine Noetherian base scheme* S *is dualizable in* **Spt**/S.

Proof. Theorem 10.15 shows that any projective space over S satisfies the hypotheses on the scheme Y as in Theorem 10.14. Therefore, Theorem 10.14 shows that every projective smooth scheme over S is dualizable in $\mathbf{Spt}/\mathrm{S}_{\mathrm{mot}}$. □

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