HEIGHT PAIRING ON HIGHER CYCLES AND MIXED HODGE STRUCTURES

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Abstract. For a smooth, projective complex variety, we introduce several mixed Hodge structures associated to higher algebraic cycles. Most notably, we introduce a mixed Hodge structure for a pair of higher cycles which are in the refined normalized complex and intersect properly. In a special case, this mixed Hodge structure is an oriented biextension, and its height agrees with the higher archimedean height pairing introduced in a previous paper by the first two authors. We also compute a non-trivial example of this height given by Bloch-Wigner dilogarithm function. Finally we study the variation of mixed Hodge structures of Hodge-Tate type, and show that the height extends continuously to degenerate situations.

Contents

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Introduction

Main objectives. Let X be a smooth projective variety of dimension d defined over a number field. The height pairing between cycles is an arithmetic analogue of the intersection product and can be seen as a linking number. It plays a central role in arithmetic geometry.

Arakelov theory and concretely arithmetic intersection theory [GS90] provides a general framework to define and study the height pairing exploiting the analogy with the intersection product. Let Z and W be disjoint, homologically trivial algebraic cycles on X of codimension p and $q = d + 1 - p$ respectively. Assume that there is a regular model $\mathcal{X}$ of X over $\mathcal{O}_F$ (if not we can use de Jong alterations to produce a regular alteration of a model of X). Then one can choose liftings $\widehat{Z}$ and $\widehat{W}$ of Z and W in the arithmetic Chow groups $\widehat{\mathrm{CH}}^p(\mathcal{X})$ and $\widehat{\mathrm{CH}}^q(\mathcal{X})$ respectively satisfying certain technical conditions. In this setting the height pairing is given by

$$\langle Z, W\rangle_{\mathrm{ht}} = \widehat{\deg}(\widehat{Z} \cdot \widehat{W}).$$

This height pairing can be written as a sum of components

$$\langle Z, W\rangle_{\mathrm{ht}} = \langle Z, W\rangle_{\mathrm{fin}} + \langle Z, W\rangle_{\mathrm{Arch}} \in \mathbb{R},$$

where $\langle Z, W\rangle_{\mathrm{fin}}$ is the *finite contribution* that is defined using intersection theory on the model $\mathcal{X}$, while $\langle Z, W\rangle_{\mathrm{Arch}}$ is the archimedean height pairing and can be computed using Green currents in the complex manifold associated to X. Note that, even if $\langle Z, W\rangle_{\mathrm{ht}}$ depends only on the rational equivalence class of Z and W, the finite and archimedean components depend of the actual cycles Z and W.

In the paper [Hai90] R. Hain has given a Hodge theoretical interpretation of the archimedean height pairing. Namely, to the pair of cycles Z and W one can associate a *biextension* $B_{Z,W}$ of mixed Hodge structure. The isomorphism classes of biextension mixed Hodge structures are classified by a single real invariant and the archimedean height pairing agrees with this invariant.

S. Bloch has introduced the higher Chow groups $\mathrm{CH}^p(X, n)$ in [Blo86] as a concrete example of motivic cohomology theory. While in [BGF12] E. Feliu and the first author have introduced the higher arithmetic Chow groups. These groups have been further studied by the first and second author in [BGG19]. Moreover, they have introduced a height pairing between higher cycles whose real regulators are zero. Although there are many differences between the case of algebraic cycles and the case of higher cycles, the height pairing between higher cycles still decomposes as a sum

$$\langle Z, W\rangle_{\mathrm{ht}} = \langle Z, W\rangle_{\mathrm{geom}} + \langle Z, W\rangle_{\mathrm{Arch}}, \tag{0.1}$$

of an *archimedean contribution*, that will be called the archimedean higher height pairing, and a *geometric contribution* that, although is very different in nature to the finite contribution in the case of ordinary cycles, is also related to an intersection product.

The archimedean higher height pairing depends only on the complex manifold associated to X and can be defined for higher cycles on a smooth projective complex variety. The aim of the present paper is to generalize Hain's result and give a Hodge theoretical interpretation of the archimedean higher height pairing between certain higher cycles. More precisely, as we review below (0.2), $\mathrm{CH}^*(X, *)$ can be computed as the homology of a complex $(Z^*(X, *)_{00}, \delta)$. The main result of this paper is the following (Theorem 4.7):

Theorem A. *Let X be a smooth complex projective variety of dimension d and $Z \in Z^p(X, 1)_{00}$ and $W \in Z^q(X, 1)_{00}$ be elements which satisfy the following conditions:*

(i) $p + q = d + 2$,
(ii) $\delta Z = \delta W = 0$,
(iii) *Z and W intersect properly,*
(iv) *the intersection of Z and W satisfies some extra technical conditions.*

Then, in analogy with Hain's construction, there is a canonical mixed Hodge structure $B_{Z,W}$ attached to Z and W from which one can extract a Hodge theoretical height pairing $\langle Z, W\rangle_{\mathrm{Hodge}}$. Moreover, if Z and W both have real regulator zero then

$$\langle Z, W\rangle_{\mathrm{Hodge}} = \langle Z, W\rangle_{\mathrm{Arch}}.$$

Regarding condition *(i)*, much of our analysis carries through the case where $Z \in Z^p(X, n)_{00}$ and $W \in Z^q(X, m)_{00}$ provided that $2(p+q-d-1) =$

$m+n$. However, condition *(iii)* allows for non-trivial intersections of Z and W which contribute to the mixed Hodge structure $B_{Z,W}$. In the case $m=n=1$, this intersection is just a finite set of points and is easy to handle provided we assume some extra technical conditions that are satisfied generically (see Assumption 3.27).

At first glance, the contribution from the intersection of Z and W might appear to be just a technical issue arising during the construction of $B_{Z,W}$. However, on reflection, it is exactly this new contribution which allows $B_{Z,W}$ to have interesting deformations which satisfy Griffiths horizontality.

The asymptotic behavior of the archimedean component of the height pairing has been extensively studied by the third author in [Pea06a] using the Hodge theoretical interpretation. Moreover, in collaboration with P. Brosnan, in [BP19] he has given an explanation of the height jump phenomenon. The asymptotic behavior of the height and the height jump phenomenon has also been studied by the first author in collaboration with R. de Jong and D. Holmes in [BGHdJ18].

A second objective of this paper is to use the Hodge theoretical interpretation of the archimedean higher height pairing to start the study of its asymptotic behavior. In section 5.2 we study an example in dimension 2 in which $n=m=1$ and the cohomology of X is of Hodge–Tate type and we observe that the height can be extended continuously to the degenerate situations. This is in sharp contrast with the usual height pairing that has logarithmic singularities when approaching degenerate situations. We show that this is a general phenomenon of higher heights for Hodge–Tate variations of mixed Hodge structures (Theorem 6.1).

Theorem B. *Let S be a Zariski open subset of a complex manifold $\bar{S}$ such that $D=\bar{S}-S$ is a normal crossing divisor. Let $\mathcal{V}\to S$ be an oriented graded polarized Hodge–Tate variation with length $\ell(\mathcal{V})\geq 4$. Assume $\mathcal{V}$ is admissible with respect to $\bar{S}$ and has unipotent local monodromy about D. Let $p\in D$. Then the limit mixed Hodge structure $\mathcal{V}_p$ of $\mathcal{V}$ at $p\in D$ is an oriented Hodge–Tate structure with the same weight filtration as $\mathcal{V}$. Moreover,*

$$\lim_{s\to p}\operatorname{ht}(\mathcal{V}_s)=\operatorname{ht}(\mathcal{V}_p).$$

In this result oriented means that the top and the bottom graded pieces are constant variations of rank one (see the beginning of Section 6 for a more precise definition), $\operatorname{ht}(\mathcal{V}_s)$ denotes the height of the mixed Hodge structure $\mathcal{V}_s$ (see Definition 2.3), while admissibility and unipotent local monodromy are standard technical conditions to ensure that the variation behaves nicely. The important hypothesis are first, that the length $\ell(\mathcal{V})\geq 4$ that is, the difference between the minimal and maximal weight is at least 4 (hence we are dealing with a higher height) and second that the whole variation is of Hodge–Tate type. In Example 6.9 we show that this last hypothesis is necessary.

Background for usual cycles. Before giving a more precise statement of the main results of the paper we briefly recall the case of ordinary cycles.

For simplicity we will state all the results for varieties over $\mathbb{Q}$. Let X be a smooth and projective variety over $\mathbb{Q}$ of dimension d. Assume that there is a regular model $\mathcal{X}$ of X defined over $\mathbb{Z}$. Let p, q non negative integers satisfying $p+q = d+1$. Beilinson in [Beĭ87] defined a symmetric and bilinear height pairing between elements $\alpha \in \mathrm{CH}^p_{\mathrm{hom}}(X;\mathbb{Q})$ and $\beta \in \mathrm{CH}^q_{\mathrm{hom}}(X;\mathbb{Q})$. As mentioned before we can interpret this pairing using the arithmetic intersection theory [GS90]. Let $Z \in Z^p_{\mathrm{hom}}(X;\mathbb{Q})$ and $W \in Z^q_{\mathrm{hom}}(X;\mathbb{Q})$ be two cycles representing the classes α and β intersecting properly. In this case this means that they have empty intersection. Choose spreading cycles $\mathcal{Z} \in Z^p(\mathcal{X};\mathbb{Q})$ and $\mathcal{W} \in Z^q(\mathcal{X};\mathbb{Q})$ on the model such that their restriction to X are the cycles Z and W and their restrictions to each component of each fiber of the model are numerically equivalent to zero. Choose also Green currents for Z and W whose associated form is zero. For instance a Green current for W with associated form zero is a current g_W of type $q-1, q-1$ satisfying

$$dd^c g_W + \delta_W = 0,$$
$$\overline{g_W} = (-1)^{q-1} g_W,$$

and being smooth outside the support of W. Then

$$\widehat{Z} = (\mathcal{Z}, g_Z) \in \widehat{\mathrm{CH}}^p(\mathcal{X})_{\mathbb{Q}}, \qquad \widehat{W} = (\mathcal{W}, g_W) \in \widehat{\mathrm{CH}}^q(\mathcal{X})_{\mathbb{Q}}$$

are arithmetic cycles and

$$\langle \alpha, \beta \rangle_{\mathrm{ht}} = \widehat{\deg}(\widehat{Z} \cdot \widehat{W}) = \langle Z, W \rangle_{\mathrm{fin}} + \langle Z, W \rangle_{\mathrm{Arch}} \in \mathbb{R},$$

where $\langle Z, W \rangle_{\mathrm{fin}}$ is defined using intersection theory in $\mathcal{X}$ and

$$\langle Z, W \rangle_{\mathrm{Arch}} = \int_Z g_W \in \mathbb{R}.$$

Note that $\langle \alpha, \beta \rangle_{\mathrm{ht}}$ only depends on the classes α and β while $\langle Z, W \rangle_{\mathrm{fin}}$ and $\langle Z, W \rangle_{\mathrm{Arch}}$ only depend on the cycles Z and W but not on the spreading nor on the Green currents satisfying the above conditions.

We now discuss Hain's Hodge theoretic interpretation of the above pairing in [Hai90]. Let H be a torsion free integral pure Hodge structure of weight -1. A *biextension* B associated to H is a mixed Hodge structure of non-zero weights $-2, -1, 0$, with the graded pieces satisfying

$$\mathrm{Gr}^W_0 B = \mathbb{Z}(0),$$
$$\mathrm{Gr}^W_{-1} B = H,$$
$$\mathrm{Gr}^W_{-2} B = \mathbb{Z}(1).$$

Let $\mathcal{B}(H)$ denote the set of isomorphism classes of biextensions as before and $\mathcal{B}(H)_{\mathbb{R}}$ the isomorphism classes of real mixed Hodge structures of the same shape. The following results are proved in [Hai90] (Corollary 3.1.6, Corollary 3.2.2 and Corollary 3.2.9)

(i) $\mathrm{Ext}^1_{\mathbf{MHS}}(\mathbb{Z}(0), H)$ and $\mathrm{Ext}^1_{\mathbf{MHS}}(H, \mathbb{Z}(1))$ are dual tori.

(ii) The projection
$$\mathcal{B}(H) \to \mathrm{Ext}^1_{\mathbf{MHS}}(\mathbb{Z}(0), H) \times \mathrm{Ext}^1_{\mathbf{MHS}}(H, \mathbb{Z}(1))$$
given by $B \mapsto (B/W_{-2}, W_{-1})$ has the structure of a principal $\mathbb{C}^*$ bundle.

(iii) $\mathrm{Ext}^1_{\mathbb{R}-\mathbf{MHS}}(\mathbb{R}(0), H_{\mathbb{R}}) = \mathrm{Ext}^1_{\mathbb{R}-\mathbf{MHS}}(H_R, \mathbb{R}(1)) = 0.$

(iv) There is a canonical bijection $\mathcal{B}_{\mathbb{R}}(H) \xrightarrow{\cong} \mathbb{R}$.

In particular if $Z \in Z^p_{\mathrm{hom}}(X)$ and $W \in Z^q_{\mathrm{hom}}(X)$ are two cycles homologous to zero, intersecting properly with $p+q=d+1$, then the Abel-Jacobi images of Z and W define elements

$$e_Z \in \mathrm{Ext}^1_{\mathbf{MHS}}(\mathbb{Z}(0),H),$$
$$e^\vee_W \in \mathrm{Ext}^1_{\mathbf{MHS}}(H,\mathbb{Z}(1)),$$

where $H=H^{2p-1}(X,\mathbb{Z}(p))/\mathrm{torsion}$. The extension class e_Z is defined by a short exact sequence

$$0 \to H \to E_Z \to \mathbb{Z}(0) \to 0,$$

E_Z being a sub-Hodge structure of $H^{2p-1}(X\setminus |Z|,\mathbb{Z}(p))/\mathrm{torsion}$. While $e^\vee_W$ is given by a short exact sequence

$$0 \to \mathbb{Z}(1) \to E^\vee_W \to H \to 0,$$

with $E^\vee_W$ being a quotient of $H^{2p-1}(X,|W|,\mathbb{Z}(p))/\mathrm{torsion}$. Combining both constructions we get a biextension (Proposition 3.3.2 of [Hai90])

$$B_{Z,W} \mapsto (e_Z, e^\vee_W),$$

which is a subquotient of the mixed Hodge structure

$$H^{2p-1}(X\setminus |Z|,|W|,\mathbb{Z}(p))/\mathrm{torsion}.$$

If $\nu\colon \mathcal{B}(H) \to \mathbb{R}$ is the composition of the change of coefficients $\mathcal{B}(H) \to \mathcal{B}(H)_{\mathbb{R}}$ with the bijection above, we have (Proposition 3.3.12 of [Hai90])

$$\nu(B_{Z,W}) = -\langle Z,W\rangle_{\mathrm{Arch}}.$$

Since proper intersection means $|Z|\cap|W|=\phi$, there is a duality

$$H^{2p-1}(X\setminus |Z|,|W|,\mathbb{Q}(p)) \cong \left(H^{2q-1}(X\setminus |W|,|Z|,\mathbb{Q}(q-1))\right)^\vee,$$

which implies that the above pairing is symmetric.

In [Pea06b, Theorem 5.19] the Hodge theoretical interpretation of the archimedean height pairing is used to obtain results about its asymptotic behavior. Let $Z_s, W_s \subset X_s$ be a flat family of cycles homologous to zero over a smooth curve S. Let z be a local holomorphic coordinate on a small disk $\Delta \subset S$ such that, for $0 \neq z \in \Delta$, the variety X_z is smooth and the cycles Z_z and W_z intersect properly and such that the variation of mixed Hodge structures B_{Z_z,W_z} has unipotent monodromy. Then there is a rational number μ that can be read from the monodromy, and such that

$$\langle Z_z, W_z\rangle_{\mathrm{Arch}} = \mu \log|z| + \eta(z),$$

where $\eta(z)$ is real analytic and remains bounded when z goes to zero.

Higher intersection pairing. We recall the construction of the higher height pairing of [BGG19]. Before that we will also have to recall some terminology.

Let now F be any field and X a smooth projective variety over F. There are two isomorphic descriptions of Bloch's higher Chow groups, the simplicial and the cubical versions. The simplicial version is the one originally introduced by Bloch, but the cubical version is the one more well suited for the product structure. In this paper we will use the cubical description.

In the cubical version, in order to compute the right homology, one is forced to normalize the complex in order to get rid of degenerate elements.

There are two versions of the normalization. In fact, there are two quasi-isomorphic complexes

$$Z^p(X,*)_{00} \subset Z^p(X,*)_0, \tag{0.2}$$

whose homologies compute the cubical version of higher Chow groups. We will use the complex $Z^p(X,*)_{00}$ because its cycles are easier to link with relative cohomology.

Let $\square = \mathbb{P}^1 \setminus \{1\}$ denote a copy of the affine line where the role of ∞ is played by the point 1 and let $\square^n$ denote the n-th cartesian product. Recall that there are coface maps $\delta^i_j \colon \square^{n-1} \to \square^n$, $i = 1,\dots,n$, $j = 0,1$, given by

$$\begin{aligned} \delta^i_0(t_1,\dots,t_{n-1}) &= (t_1,\dots,t_{i-1},0,t_i,\dots,t_{n-1}),\\ \delta^i_1(t_1,\dots,t_{n-1}) &= (t_1,\dots,t_{i-1},\infty,t_i,\dots,t_{n-1}). \end{aligned}$$

For any scheme X, we denote also by δ^i_j the induced maps $X \times \square^{n-1} \to X \times \square^n$. Any intersection of images of the maps δ^i_j is called a face.

Let $Z^p(X,n)$ denote the group of algebraic cycles on $X \times \square^n$ that intersect properly all the faces. Then

$$Z^p(X,n)_{00} = \bigcup_{i=1}^{n} (\delta^i_1)^* \cup \bigcup_{i=2}^{n} (\delta^i_0)^*$$

with differential $\delta \colon Z^p(X,n)_{00} \to Z^p(X,n-1)_{00}$ given by $\delta = -(\delta^1_0)^*$. An element of $Z^p(X,n)_{00}$ will be called a pre-cycle, while an element of $Z \in Z^p(X,n)_{00}$ with $\delta Z = 0$is called a cycle. The higher Chow groups of X are the homology of the complex $(Z^p(X,*)_{00},\delta)$:

$$\mathrm{CH}^p(X,n) = H_n(Z^p(X,*)_{00},\delta), \quad n \geq 0,\ p \geq 0.$$

There is a graded commutative product in $\mathrm{CH}^*(X,*)$ given by the intersection product.

Two pre-cycles $Z \in Z^p(X,n)_{00}$ and $W \in Z^q(X,m)_{00}$ are said to intersect properly if $\pi_1^{-1}Z$ and $\pi_2^{-1}W$ intersect properly among them and with all the faces of $X \times \square^{n+m}$. Here $\pi_1 \colon X \times \square^{n+m} \to \square^n$ and $\pi_2 \colon X \times \square^{n+m} \to \square^m$ are the two projections. If Z and W intersect properly, then the intersection product $Z \cdot W$ is a well defined pre-cycle of $Z^{p+q}(X,n+m)$.

Let $\alpha \in \mathrm{CH}^p(X,n)$ and $\beta \in \mathrm{CH}^q(X,m)$. Then there exist representatives $Z \in Z^p(X,n)_{00}$ and $W \in Z^q(X,m)_{00}$ of α and β respectively that intersect properly. The product $\alpha \cdot \beta$ is represented by $Z \cdot W$.

Let now F be a number field and Σ the set of complex immersions of F. To the smooth projective variety X over F we associate a complex variety

$$X_\Sigma = \coprod_{\sigma \in \Sigma} X \times_\sigma \mathbb{C}.$$

This complex manifold has an antilinear involution F_∞ and we denote $X_\mathbb{R} = (X_\Sigma, F_\infty)$ the corresponding real variety.

There are regulator maps $\rho \colon \mathrm{CH}^p(X,n) \to H^{2p-n}_{\mathfrak{D}}(X_\mathbb{R},\mathbb{R}(p))$, where $H_\mathfrak{D}$ denotes Deligne cohomology. In the papers [BGF12] and [BGG19] the higher arithmetic Chow groups $\widehat{\mathrm{CH}}^*(X,*,\mathfrak{D}_{\mathrm{TW}})$ of X are introduced and studied. These groups depend on the choice of a particular complex $\mathfrak{D}_{\mathrm{TW}}$ that computes Deligne cohomology (see section 1.8). These groups satisfy many properties similar to the ones of classical arithmetic Chow groups. We summarize the properties needed in the definition of the height pairing.

(i) The elements of $\widehat{\mathrm{CH}}^p(X, n, \mathfrak{D}_{\mathrm{TW}})$ are represented by pairs (Z, g_Z) with $Z \in Z^p(X, n)_{00}$ with g_Z a Green form for Z in the appropriate sense (Definition 1.33).

(ii) To each Green form g_Z, there is an associated canonical differential form $\omega(g_Z) \in \mathfrak{D}^{2p-n}_{\mathrm{TW}}(X, p)$ that represents the class of the regulator $\rho(Z) \in H^{2p-n}_{\mathfrak{D}}(X_{\mathbb{R}}, \mathbb{R}(p))$.

(iii) There is a $*$-product of Green forms.

(iv) The groups $\widehat{\mathrm{CH}}^*(X, *, \mathfrak{D}_{\mathrm{TW}})$ form a graded commutative algebra, where the product is induced by the intersection product of cycles meeting properly and the star product of Green forms.

(v) If $f\colon X \to Y$ is a smooth morphism of relative dimension e, there are morphisms

$$f_*\colon \widehat{\mathrm{CH}}^p(X, n, \mathfrak{D}_{\mathrm{TW}}) \to \widehat{\mathrm{CH}}^{p-e}(Y, n, \mathfrak{D}_{\mathrm{TW}}), \qquad n, p \geq 0.$$

(vi) Writing $X_F = \mathrm{Spec}(F)$, there is a short exact sequence

(0.3)
$$0 \to \frac{H^1_{\mathfrak{D}}(X_{F,\mathbb{R}}, \mathbb{R}(p))}{\mathrm{Im}(\rho)} \to \widehat{\mathrm{CH}}^p(X_F, 2p-2, \mathfrak{D}_{\mathrm{TW}}) \to \mathrm{CH}^p(X_F, 2p-2) \to 0.$$

In the above exact sequence $X_{F,\mathbb{R}}$ is the real variety associated to X_F. So $H^1_{\mathfrak{D}}(X_{F,\mathbb{R}}, \mathbb{R}(p))$ is a real vector space of dimension $r_1 + r_2$ if p is odd and r_2 if p is even, where r_1 is the number of real immersions of F and $2r_2$ is the number of non-real complex immersions. Moreover ρ agrees with Borel's regulator up to a normalization factor [BG02]. Hence $\mathrm{Im}(\rho)$ is a lattice in $H^1_{\mathfrak{D}}(X_{F,\mathbb{R}}, \mathbb{R}(p))$. Also $\mathrm{CH}^p(X_F, 2p-2)$ is torsion. Thus $\widehat{\mathrm{CH}}^p(X_F, 2p-2, \mathfrak{D}_{\mathrm{TW}})$ is an extension of a torsion group by a real torus.

Let now $\alpha \in \mathrm{CH}^p(X, n)$ and $\beta \in \mathrm{CH}^q(X, m)$ be two classes satisfying

$$2(p + q - d - 1) = n + m \tag{0.4}$$

and $\rho(\alpha) = \rho(\beta) = 0$. We can find representatives $Z \in Z^p(X, n)_{00}$ and $W \in Z^q(X, m)_{00}$ intersecting properly. Moreover we can choose Green forms g_Z and g_W for Z and W satisfying $\omega(g_Z) = \omega(g_W) = 0$. Using the properties of the arithmetic Chow groups we obtain an element

$$\pi_*((Z, g_Z) \cdot (W, g_W)) \in \widehat{\mathrm{CH}}^{p+q-d}(X_F, n+m, \mathfrak{D}_{\mathrm{TW}}).$$

The condition (0.4) assures us that the target group of the fits in a short exact sequence like (0.3). After tensoring with $\mathbb{Q}$ to get rid of the torsion group on the right of the exact sequence, the height pairing of α and β is defined as

$$\langle \alpha, \beta \rangle_{\mathrm{ht}} := \pi_*((Z, g_Z) \cdot (W, g_W)) \in \frac{H^1_{\mathfrak{D}}(X_{F,\mathbb{R}}, \mathbb{R}(p+q-d))}{\mathrm{Im}(\rho)} \otimes \mathbb{Q}.$$

The pairs

$$\langle Z, W \rangle_{\mathrm{geom}} := (\pi_*(Z \cdot W), 0), \quad \text{and} \quad \langle Z, W \rangle_{\mathrm{Arch}} := (0, \pi_*(g_Z * g_W))$$

give well defined elements of $\widehat{\mathrm{CH}}^{p+q-d}(X_F, n+m, \mathfrak{D}_{\mathrm{TW}})_{\mathbb{Q}}$ obtaining the decomposition (0.1).

We note several differences between the higher height pairing and the usual height pairing.

(i) The higher arithmetic Chow groups are defined for the variety X over F and not for a model $\mathcal{X}$ of X. This is due to the fact that the good properties of the higher arithmetic Chow groups are only established for varieties over a field. At first glance this may seem a big loss of information. Nevertheless, if F is a number field and $\mathcal{O}_F$ is its ring of integers, then for $i > 1$,

$$K_i(F) \otimes \mathbb{Q} = K_i(\mathcal{O}_F) \otimes \mathbb{Q}.$$

Therefore, for the purpose of defining higher heights, there is no great benefit on considering an integral model of the variety. More so if we consider that in the classical case, in order to have a well defined intersection product on the model one has to tensor with $\mathbb{Q}$.

(ii) Since we are working over a field, we can define the product without tensoring with $\mathbb{Q}$. Nevertheless we can tensor with $\mathbb{Q}$ to eliminate the torsion group $\mathrm{CH}^{p+q-d}(X_F, n+m)$.

(iii) Even if there is no model over the ring of integers involved, there is still a geometric contribution of the height pairing coming from the intersection of the cycles. By contrast, the definition of the archimedean higher height pairing is formally identical to the classical case.

(iv) In order for the height pairing to be independent on the choice of the Green forms, we need the condition that the real regulator is zero. By contrast the Hodge theoretical invariant associated to the pair of cycles can be defined even when the regulator of the cycles is non zero.

(v) The higher height pairing is not a real number but an element of the quotient of $H^1_{\mathfrak{D}}(X_{F,\mathbb{R}}, \mathbb{R}(p+q-d))$ by the image of the regulator. The main difference with the classical case is that, for $p+q-d > 1$ the image of the regulator is a full rank lattice. Therefore we can not obtain a well defined real number.

Although the height pairing is an arithmetic invariant of the rational equivalence class of the higher cycles and is well defined up to the image of the regulator, the archimedean higher height pairing can be defined purely in the complex case.

Definition A. Let X be a smooth projective variety over $\mathbb{C}$ and d and $Z \in Z^p(X, n)_{00}$ and $W \in Z^q(X, m)_{00}$ be elements which satisfy the following conditions:

(i) $2(p+q-d-2) = n+m$,
(ii) $\delta Z = \delta W = 0$,
(iii) Z and W intersect properly.
(iv) $\rho(Z) = \rho(W) = 0$.

Let g_Z and g_W be Green forms for Z and W satisfying $\omega(g_Z) = \omega(g_W) = 0$. Then the archimedean height pairing of Z and W is defined as

$$\langle Z, W\rangle_{\mathrm{Arch}} \coloneqq \int_X g_Z * g_W \in H^1_{\mathfrak{D}}\big(\mathrm{Spec}(\mathbb{C}), \mathbb{R}(p+q-d)\big).$$

Mixed Hodge structures associated to higher cycles. From now on we consider a smooth projective variety X over $\mathbb{C}$, and we discuss several mixed Hodge structures associated to a pair of higher cycles.

Given cycles $Z \in Z^p(X,n)_{00}$, $W \in Z^q(X,m)_{00}$, with p,q,n,m satisfying

$$2(p+q-d-1) = n+m \tag{0.5}$$

and intersecting properly. Let $\pi_1 \colon X \times (\mathbb{P}^1)^n \times (\mathbb{P}^1)^m \to X \times (\mathbb{P}^1)^n$ and $\pi_2 \colon X \times (\mathbb{P}^1)^n \times (\mathbb{P}^1)^m \to X \times (\mathbb{P}^1)^m$ be the two projections and

$$S = \pi_1^{-1}(|Z|) \cap \pi_2^{-1}(|W|)$$

be the intersection of the pull backs of supports of Z and W. Notice that unlike the usual algebraic cycle scenario, proper intersection of Z and W no longer means that S is empty.

We first construct a mixed Hodge structure E_Z for Z, fitting in the short exact sequence

$$0 \longrightarrow H^{2p-n-1}(X;p) \longrightarrow E_Z \longrightarrow \mathbb{Q}(0) \longrightarrow 0,$$

and hence defining an element in

$$\operatorname{Ext}^1_{\mathbf{MHS}}\left(\mathbb{Q}(0), H^{2p-n-1}(X;p)\right) = H^{2p-n}_{\mathfrak{D}}(X, \mathbb{Q}(p)).$$

This element agrees with the regulator of Z. Next for W we consider the dual $E_W^\vee$ extension, fitting in the short exact sequence

$$0 \longrightarrow \mathbb{Q}(0) \longrightarrow E_W^\vee \longrightarrow H^{2d-2q+m+1}(X; d-q) \longrightarrow 0.$$

We stress the fact that, in giving a geometric interpretation of $E_W^\vee$ we face the technical problem that the duality in Lemma 1.11 requires the hypothesis of local product situation. We will address this problem latter in the main body when we give more details on the construction of the mixed Hodge structures.

Note that condition (0.4) implies that

$$H^{2p-n-1}(X;p) = H^{2d-2q+m+1}(X; d-q+(m+n)/2+1).$$

Hence, after the appropriate twist, the cohomology groups appearing in both extensions agree and one may hope to glue together E_Z and $E_W^\vee$ is a biextension. Here the presence of the non trivial intersection S makes life more interesting. In fact, under several assumptions aimed to keep the contribution from S under control, we associate to the pair (Z,W) a mixed Hodge structure $B_{Z,W}$, which fits in Figure 2. In the special case of $n=m=1$, and under Assumption 3.27, $B_{Z,W}$ is a generalized biextension (Definition 2.5 and Corollary 3.31), with three non-zero weight graded pieces.

In Section 2.1 we define the height of an oriented mixed Hodge structure using the Deligne splitting (Definition 2.3). This is a real number $\operatorname{ht}(B_{Z,W}) \in \mathbb{R}$. In order to compare it with the archimedean height, that in the case $n=m=1$ lives in $H^1_{\mathfrak{D}}(\operatorname{Spec}(\mathbb{C}), \mathbb{R}(2)) = \mathbb{C}/(2\pi i)^2\mathbb{R}$, we make the following definition.

Definition B. Let $\rho_2 \colon \mathbb{C}/(2\pi i)^2\mathbb{R} \to \mathbb{R}$ be the isomorphism given by

$$\rho_2(v) = \operatorname{Im}(v/(2\pi i)^2),$$

then the Hodge height pairing of Z and W is

$$\langle Z, W\rangle_{\text{Hodge}} = \rho_2^{-1}(\text{ht}(B_{Z,W})).$$

We give a little bit more details on the construction of the above mixed Hodge structures. On $(\mathbb{P}^1)^n$ we have two divisors

$$A = \{(t_1, \cdots, t_n) \mid \exists i, t_i = 1\},$$
$$B = \{(t_1, \cdots, t_n) \mid \exists i, t_i \in \{0, \infty\}\}.$$

Then $A \cup B$ is a simple normal crossing divisor. Moreover, since $\square := \mathbb{P}^1 \setminus \{1 : 1\}$, we have

$$(\mathbb{P}^1)^n \setminus A = \square^n, \quad (\mathbb{P}^1)^n \setminus B = (\mathbb{C}^\times)^n \text{ and } B \cap \square^n = \partial\square^n.$$

Further for $A_X := X \times A$, $B_X := X \times B$, we get isomorphisms of Hodge structures

$$H^r(X \times (\mathbb{P}^1)^n \setminus A_X, B_X) \cong H^{r-n}(X), \tag{0.6}$$
$$H^r(X \times (\mathbb{P}^1)^n \setminus B_X, A_X) \cong H^{r-n}(X; -n). \tag{0.7}$$

Since A_X and B_X are in local product situation (Definition 1.9), the above isomorphisms are compatible with duality

$$H^r(X \times (\mathbb{P}^1)^n \setminus A_X, B_X, \mathbb{Q}(p)) \cong \\ \Big(H^{2d+2n-r}\left(X \times (\mathbb{P}^1)^n \setminus B_X, A_X, \mathbb{Q}(d+n-p)\right)\Big)^\vee.$$

Since $Z \in Z^p(X, n)_{00}$ belongs to the refined normalized complex, then the restriction $Z|_{B_X \setminus A_X}$ is zero. Therefore, the cycle Z defines an unique class (Proposition 3.3)

$$[Z] \in H^{2p}_{|Z| \setminus A_X}\left(X \times (\mathbb{P}^1)^n \setminus A_X, B_X \setminus A_X; p\right)_{\mathbb{Q}}.$$

By Lemma 3.5 its image in $H^{2p}\left(X \times (\mathbb{P}^1)^n \setminus A_X, B_X \setminus A_X; p\right)_{\mathbb{Q}}$ is zero. Pulling back the long exact sequence 3.7 of mixed Hodge structures, by the class $[Z]$ and using the isomorphism (0.6) we obtain the extension E_Z. We remark that the mixed Hodge structure E_Z is a sub Hodge structure of

$$H^{2p-1}(X \times (\mathbb{P}^1)^n \setminus A_X \cup |Z|, B_X; p).$$

We now consider the dual construction for W. As mentioned before, in order to dualize this construction we face the problem that, in general, $A_X \cup |W|$ and B_X are not in local product situation. Therefore, to dualize we need first to blow up $|W| \cap B_X$ until a local product situation is obtained. Let $\mathcal{X}_W$ be such a blow up with $\widehat{A}_X$, $\widehat{B}_X$ and $\widehat{W}$ being the strict transforms of A_X, B_X and $|W|$. Let D be the exceptional divisor. Naively one would expect the mixed Hodge structure to be a quotient of

$$H^{2d+2m-2q+1}(X \times (\mathbb{P}^1)^m \setminus B_X, A_X \cup |W|; d+m-q) = \\ H^{2d+2m-2q+1}(\mathcal{X}_W \setminus \widehat{B}_X \cup D, \widehat{A}_X \cup \widehat{W}; d+m-q),$$

but in fact $E_W^\vee$ is a quotient of

$$H^{2d+2m-2q+1}(\mathcal{X}_W \setminus \widehat{B}_X, \widehat{A}_X \cup \widehat{W} \cup D; d+m-q).$$

Note that the exceptional divisor is in a different position. See Section 3.6 for more details.

Finally, for the construction of $B_{Z,W}$ we refer to Section 3.7 and 3.8. We just remark that in this construction we have to deal, not only with the duality problem mentioned above but also with the contribution of the intersection S of Z and W. Although the methods of this paper can be extended to much more general situations, for the moment we have only made the complete study in the case $n = m = 1$ and Assumption 3.27. One of the main reasons is that we want $B_{Z,W}$ to be a generalized biextension (Definition 2.5), so there is a clean definition of the height of $B_{Z,W}$ that we can compare with the height pairing of Z and W. This forces us to keep S under control to avoid many spurious components in $B_{Z,W}$. For instance, even if S is a point, if it is contained in the singular locus of $|Z|$ and $|W|$, the cohomology with support on S can be very complicated and mask the classes of Z and W.

Nevertheless, using the Deligne splitting one can define a more general height attached to an oriented mixed Hodge structure (Definition 2.3). One would expect that the main result of this paper can be extended to a more general situation using this generalization of the height of a mixed Hodge structure.

Examples. We compute two examples of the higher height pairing. The first one is in dimension 0 with $p = q = n = m = 1$. In this case we find that the higher height pairing is always zero.

The second more interesting example in in dimension 2, with $X = \mathbb{P}^2$, $p = q = 2$ and $n = m = 1$. A method of constructing higher cycles in $\mathbb{P}^2$ is to consider three section s_0, s_1 and s_2 of $\mathcal{O}(1)$. They determine a triangle in $\mathbb{P}^2$ and a higher cycle as explained in Definition 5.1. For two such higher cycles Z and W in general position we compute their higher height pairing. It turns out to be given by a linear combination of values of the Bloch Wigner dilogarithm function. A remarkable feature of this example is that, in the space of parameters of such pair of divisors, the height function can be extended continuously to the degenerate situations. A second observation from the example is that, when both triangles are defined over $\mathbb{R}$ the higher height pairing vanishes. Both phenomenons turn out to hold in more general situations. With respect to the second one, we show in Proposition 5.10 that the higher height pairing between cycles defined over $\mathbb{R}$ should be zero as long as $(n + m)/2$ is odd.

With respect to the continuity of the height function, this is the starting point of the study of the asymptotic behavior. As mentioned previously, we show that the higher height of an admissible variation $\mathcal{V}$ of oriented Hodge-Tate mixed Hodge structures extends continuously to the boundary. It is important to note that this is no longer true if the variation is not of Hodge-Tate type (Example 6.9) or if $\ell(\mathcal{V}) = 2$.

Layout of the paper. Our paper is organized as follows. Sections 1 and 2 are preliminary in nature where we set up notations and collect all the necessary results and definitions needed for the rest of the sections. In Section 3 we study the mixed Hodge structure associated to higher cycles, the key among them is a mixed Hodge structure associated to a pair of higher cycles satisfying a numerical condition. In Section 4 we compute the invariants associated to these mixed Hodge structures in a special case scenario. A key result in this section is the equality of higher archimedean height pairing and the height of the biextension, in case the higher cycles have trivial real regulators. Section 5 is devoted towards computing these invariants in specific examples arising from non-degenerate triangles in $\mathbb{P}^2$. We see that the height of the biextension in this case is given by a sum of Bloch-Wigner Dilogarithm functions. Finally in Section 6 we study the asymptotic behavior of variations of oriented mixed Hodge structures of Hodge–Tate type and as well as arbitrary admissible variations.

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1. Preliminaries

In this section we gather all the conventions, notations and known results that will be used throughout the paper.
All through the section X will denote a smooth complex variety of dimension d. To avoid cumbersome notation, we will not distinguish notationally between a complex algebraic variety and its associated complex space. That is the symbol X will also denote the associated analytic manifold with the classical topology. It will always be clear from the context whether X denotes the algebraic variety or the complex manifold.

1.1. **Mixed Hodge structures.**

A $\mathbb{Q}$-mixed Hodge structure is a triple

$$H = ((H_{\mathbb{Q}}, W), (H_{\mathbb{C}}, W, F), \alpha)$$

where $(H_{\mathbb{Q}}, W)$ is a $\mathbb{Q}$-vector space with an increasing filtration W, while $(H_{\mathbb{C}}, W, F)$ is a complex vector space with an increasing filtration W and a decreasing filtration F, and that $\alpha\colon (H_{\mathbb{Q}}, W) \otimes \mathbb{C} \to (H_{\mathbb{C}}, W)$ is a filtered isomorphism. These data are subjected to several axioms. See for instance [PS08, Definition 3.13]. The vector space $H_{\mathbb{Q}}$ is called the *Betti component* and $H_{\mathbb{C}}$ the *de Rham component*, while α is the comparison isomorphism. The *rank* of a mixed Hodge structure H is the complex dimension of $H_{\mathbb{C}}$ that agrees with the dimension of $H_{\mathbb{Q}}$ over $\mathbb{Q}$.

One can also consider real mixed Hodge structures, where instead of a $\mathbb{Q}$-vector space $H_{\mathbb{Q}}$ one has a real vector space $H_{\mathbb{R}}$. In fact, given a mixed

$\mathbb{Q}$-Hodge structure H we will denote $H_{\mathbb{R}} = H_{\mathbb{Q}} \otimes \mathbb{R}$ obtaining an $\mathbb{R}$-mixed Hodge structure. Usually one identifies $H_{\mathbb{Q}}$ and $H_{\mathbb{R}}$ with its image in $H_{\mathbb{C}}$ through α.

When studying variations of mixed Hodge structures it is convenient to fix the underlying vector space and move the filtrations F and W. Thus if we fix an ($\mathbb{R}$ or $\mathbb{Q}$) vector space V, then a pair of filtrations (F, W) on $V \otimes \mathbb{C}$ and V respectively is called a mixed Hodge structure if the triple

$$((V, W), (V \otimes \mathbb{C}, W, F), \mathrm{Id}_{V \otimes \mathbb{C}})$$

is a mixed Hodge structure.

For $a \in \mathbb{Z}$, the Tate mixed Hodge structure $\mathbb{Q}(a)$ is the mixed Hodge structure given by the following data

$$\begin{gathered} \mathbb{Q}(a)_{\mathbb{Q}} = \mathbb{Q}, \quad W_{-2a-1}\mathbb{Q}(a)_{\mathbb{Q}} = 0, \quad W_{-2a}\mathbb{Q}(a)_{\mathbb{Q}} = \mathbb{Q} \\ \mathbb{Q}(a)_{\mathbb{C}} = \mathbb{C}, \quad F^{-a}\mathbb{Q}(a)_{\mathbb{C}} = \mathbb{C}, \quad F^{-a+1}\mathbb{Q}(a)_{\mathbb{C}} = 0 \\ \alpha(1) = (2\pi i)^a \in \mathbb{C}. \end{gathered}$$

Note that on $\mathbb{Q}(a)_{\mathbb{C}} = \mathbb{C}$ we have two possible complex conjugations. The usual conjugation of $\mathbb{C}$ and the one induced by the isomorphism α. The first one will be called the *de Rham conjugation* and denoted $z \mapsto \overline{z}^{\mathrm{dR}}$ and the second will be called the *Betti conjugation* and denoted $z \mapsto \overline{z}^{\mathrm{B}}$. These two conjugations are related by

$$\overline{z}^{\mathrm{B}} = (-1)^a \overline{z}^{\mathrm{dR}}.$$

In the sequel we will mainly use the Betti conjugation and write $\overline{z} = \overline{z}^{\mathrm{B}}$. Moreover, the mixed Hodge structure $\mathbb{Q}(a)$ comes equipped with the choice of two generators.

$$\begin{gathered} \mathbb{1}(a)_{\mathbb{Q}} = 1 \in \mathbb{Q} = \mathbb{Q}(a)_{\mathbb{Q}} \\ \mathbb{1}(a)_{\mathbb{C}} = 1 \in \mathbb{C} = \mathbb{Q}(a)_{\mathbb{C}}. \end{gathered}$$

These generators are called the Betti and the de Rham generators. They satisfy

$$\overline{\mathbb{1}(a)_{\mathbb{Q}}} = \mathbb{1}(a)_{\mathbb{Q}}, \qquad \overline{\mathbb{1}(a)_{\mathbb{C}}} = (-1)^a \mathbb{1}(a)_{\mathbb{C}}, \qquad \mathbb{1}(a)_{\mathbb{Q}} = (2\pi i)^a \mathbb{1}(a)_{\mathbb{Q}}.$$

Remark 1.1. Note that, although the isomorphisms class of $\mathbb{Q}(a)$ does not depend on the choice of a square root of -1, $i = \sqrt{-1}$, when a is odd, the ratio of the chosen generators $\mathbb{1}(a)_{\mathbb{Q}}/\mathbb{1}(a)_{\mathbb{C}}$ does.

Remark 1.2. Let H be a $\mathbb{Q}$-mixed Hodge structure of rank one. Then H is necessarily pure of even weight, say $2a$. It follows that it is isomorphic to $\mathbb{Q}(-a)$. The choice of an isomorphism $H \to \mathbb{Q}(a)$ is equivalent to the choice of a generator e of $H_{\mathbb{Q}}$.

If $Z \subset X$ is a closed subvariety and $r \in \mathbb{Z}$, then the cohomology groups

$$H^r(X; \mathbb{Q}), \quad H^r(X, Z; \mathbb{Q}) \text{ and } H^r_Z(X; \mathbb{Q}) = H^r(X, X \setminus Z; \mathbb{Q})$$

are all the Betti part of $\mathbb{Q}$-mixed Hodge structures that we denote as

$$H^r(X), \quad H^r(X, Z) \text{ and } H^r_Z(X) = H^r(X, X \setminus Z)$$

respectively. We will use the shorthand

$$H^r(X; p) = H^r(X) \otimes \mathbb{Q}(p).$$

Then $H^r(X;p)_{\mathbb{Q}}$, $H^r(X;p)_{\mathbb{R}}$ and $H^r(X;p)_{\mathbb{C}}$ will denote the rational and real Betti component and the complex de Rham component respectively.

Frequently, in the sequel we will use complexes that compute relative cohomology of a complex projective variety, but they only have information about the real structure and the Hodge filtration, and not about the weight filtration. To work with these at ease we introduce the following notation.

Definition 1.3. A *weak* $\mathbb{R}$*-Hodge complex* is a complex (A^*, d) of $\mathbb{C}$-vector spaces together with an anti-linear involution $\omega \mapsto \overline{\omega}$ commuting with d and a decreasing filtration F (called the Hodge filtration) compatible with d. If A^* is a weak $\mathbb{R}$-Hodge complex, we denote by $A^*_{\mathbb{R}}$ the subcomplex of elements fixed by the involution.

Given a weak $\mathbb{R}$-Hodge complex A^*, the *Tate twisted* weak $\mathbb{R}$-Hodge complex is defined as $A^*(a) = A^* \otimes \mathbb{Q}(a)_{\mathbb{C}}$. Using the identification $\mathbb{Q}(a)_{\mathbb{C}} = \mathbb{C}$, the complex $A^*(a)$ is given by the following data

$$A^*(a) = A^*, \qquad \overline{z}^{\text{new}} = (-1)^a \overline{z}^{\text{old}}, \qquad F^b A^*(a) = F^{a+b} A^*.$$

The superindexes new and old are written here for clarity but will not be used in the sequel. Due to the identification $A^*(a) = A^* \otimes \mathbb{Q}(a)_{\mathbb{C}} = A^* \otimes \mathbb{C} = A^*$ there is a potential ambiguity in the use of the symbol $\overline{\omega}$, as it depends on whether we consider ω as an element of A^* or of $A^*(a)$. In some rare cases, for clarity, an element $\omega \in A^*(a)$ will be written as $\omega \otimes \mathbb{1}(a)_{\mathbb{C}}$.

Remark 1.4. Any *Dolbeault complex* as in [BG97, Definition 2.2] defines a weak $\mathbb{R}$-Hodge complex.

Recall that the shifted complex $A^*[r]$ is defined by $A^n[r] = A^{n+r}$ with differential $(-1)^r d$.

1.2. **Conventions on differential forms and currents.** When dealing with differential forms, currents and cohomology classes, one can use the topologist's convention, where the emphasis is put on having real or integral valued classes in singular cohomology. For instance, in this convention the first Chern class of a line bundle will have integral coefficients. In algebraic geometry, the fact that rational de Rham classes are not rational in singular cohomology, the ubiquitous appearance of the period $2\pi i$, and the fact that the choice of a particular square root of -1 is non canonical, makes it useful to use a different convention. This algebro-geometric convention aims to control the obvious powers of $2\pi i$ and to be independent of the choice of the imaginary unit $i = \sqrt{-1}$.

Of course using one convention or the other is a matter of taste and one can go easily from one to the other by a normalization factor. In this paper we will follow the algebro-geometric convention. Therefore, it is useful to incorporate different powers of $2\pi i$ in the standard operations regarding forms and currents as in [BGKK07, §5.4]. We summarize here the conventions used because they differ from commonly used notations.

We will denote by E^*_X the differential graded algebra of complex valued differential forms on X, by $E^*_{X,\mathbb{R}}$ the subalgebra of real valued forms and by $E^*_{X,c}$ and $E^*_{X,\mathbb{R},c}$ the subalgebras of differential forms with compact support. The complexes of currents are defined as the topological duals of the latter ones. Namely E'^{-n}_X and $E'^{-n}_{X,\mathbb{R}}$ are the topological dual of $E^n_{X,c}$ and $E^n_{X,\mathbb{R},c}$ respectively, with differential given by

$$dT(\eta) = (-1)^{n+1}T(d\eta).$$

Recall that X is smooth of dimension d. We write

$$D^*_X = E'^*_X[-2d](-d).$$

This implies that

$$D^n_{X,\mathbb{R}} = \{T \in D^n_X \mid \forall \eta \in E^{2d-n}_{X,\mathbb{R},c},\ T(\eta) \in (2\pi i)^{-d}\mathbb{R}\}.$$

Hence one can see $D^n_{X,\mathbb{R}}(p)$ as the topological dual of $E^{2d-n}_{X,\mathbb{R}}(d-p)$.

We now consider the current $\int_X$ given by

$$\omega \mapsto \int_X \omega.$$

Then $\int_X \in E'^{-2d}_{X,\mathbb{R}} = D^0_{X,\mathbb{R}}(d)$. This suggest to define

$$\delta_X \coloneqq \textstyle\int_X \otimes \mathbb{1}(-d)_{\mathbb{Q}} = \dfrac{1}{(2\pi i)^d}\textstyle\int_X \otimes \mathbb{1}(-d)_{\mathbb{C}} \in D^0_{X,\mathbb{R}}.$$

Remark 1.5. The current δ_X has two advantages over the current $\int_X$. The first one is that δ_X is independent of the choice of square root of -1 while the current $\int_X$ is not. Indeed, If $z_1, \dots, z_d$ are local complex coordinates with $z_j = x_j + iy_j$, then the standard orientation is given by the volume form

$$\mathrm{Vol} = dx_1 \wedge dy_1 \wedge \cdots \wedge dx_d \wedge dy_d.$$

If we change the choice of the square root of -1 from i to $-i$ then Vol is sent to $(-1)^d\,\mathrm{Vol}$, which is the same change of sign suffered by $(2\pi i)^d$. Of course this explains the presence of i^d but not the presence of $(2\pi)^d$. The second advantage of δ_X is that, if X is defined over $\mathbb{Q}$ and ω is a differential form representing a rational class in $H^{2d}_{\mathrm{Zar}}(X, \Omega^*_{X_{\mathbb{Q}}})$, then $\delta_X(\omega) \in \mathbb{Q}$.

To be consistent with the previous choice we also need to adjust the definition of the current associated to a locally integrable form and to an algebraic cycle. Given a locally integrable differential form ω of degree n, there is a current

$$\int_X \omega \wedge \cdot \in E'^{n-2d}_X = D^n_X(d).$$

we will denote by $[\omega] \in D^n_X$ the current defined by

$$[\omega] = \int_X \omega \wedge \cdot \otimes \mathbb{1}(-d)_{\mathbb{Q}} = \frac{1}{(2\pi i)^d}\int_X \omega \wedge \cdot \otimes \mathbb{1}(-d)_{\mathbb{C}} \in D^n_X. \tag{1.1}$$

In other words $[\omega] = \delta_X \wedge \omega$. With this convention, the morphism of complexes $[\cdot]\colon E^*_X \to D^*_X$ respects the structure of weak Hodge complexes on both sides.

If $f\colon X \to Y$ is a proper map of smooth complex varieties, of dimensions d, d' and relative dimension $e = d - d'$, then the push-forward of currents $f_*\colon E'^*_X \to E'^*_Y$ is defined, for $T \in D^n_X$ and $\eta \in E^{2d-n}_{Y,c}$ by

$$f_*T(\eta) = T(f^*\eta).$$

It induces a map $f_*\colon D^*_X \to D^*_Y[-2e](-e)$.

Finally, let $Z \subset X$ be a codimension p irreducible subvariety of X. Let $\iota\colon \widetilde{Z} \to X$ be a resolution of singularities of Z. Then the current integration along Z is defined as

$$\delta_Z = \iota_*(\delta_{\widetilde{Z}}) \in D^{2p}_{X,\mathbb{R}}(p)$$

Remark 1.6. Since Z is irreducible, $H^{2p}_Z(X,p) = \mathbb{Q}(0)$ and the class of δ_Z is at the same time the Betti and the de Rham generator of $\mathbb{Q}(0)$.

Given any cycle $\zeta \in Z^p(X)$ we define δ_ζ by linearity. Following Remark 1.5, the symbols $[\omega]$ and δ_Y do not depend on a particular choice of $\sqrt{-1}$.

Example 1.7. To see how this conventions, together with the convention in definition 1.3 work in practice, we review the classical example of the logarithm. Consider $X = \mathbb{P}^1$ with absolute coordinate t, so $\operatorname{div}(t) = [0] - [\infty]$, and let $U = X \setminus \{0,\infty\}$. Write

$$\log(t\bar{t}) \in E^0_U(1), \quad \frac{dt}{t},\ \frac{d\bar{t}}{\bar{t}} \in E^0_U(1).$$

Note that, if we want to stress the fact that these elements belong to the twisted complex we will denote them like $\log(t\bar{t}) \otimes \mathbb{1}(1)_{\mathbb{C}}$. These elements satisfy

$$[\log t\bar{t}] \in D^0_{\mathbb{P}^1}(1), \quad \left[\frac{dt}{t}\right],\ \left[\frac{d\bar{t}}{\bar{t}}\right] \in D^1_{\mathbb{P}^1}(1).$$

Moreover,

$$\begin{aligned}
\overline{\log(t\bar{t})} &= -\log(t\bar{t}) \\
\overline{\frac{dt}{t}} &= -\frac{d\bar{t}}{\bar{t}} \\
d\left[\frac{dt}{t}\right] &= \delta_{\operatorname{div} t} = \delta_0 - \delta_\infty, \\
d\left[\frac{d\bar{t}}{\bar{t}}\right] &= -\delta_{\operatorname{div} t} = \delta_\infty - \delta_0. \\
\partial\bar{\partial}\,[\log t\bar{t}] &= -\delta_{\operatorname{div} t} = \delta_\infty - \delta_0
\end{aligned} \tag{1.2}$$

Note how, in the above formulae all the $(2\pi i)$ factors are now implicit.

Recall also the potential ambiguity on the sign of the conjugation mentioned at the end of Definition 1.3. The typical example to keep in mind would be the form

$$\eta = \frac{1}{2}\left(\frac{dt}{t} - \frac{d\bar{t}}{\bar{t}}\right) \in E^1_U(1),$$

that represents a generator of $H^1(U;1)$. Since η is an element of $E^1_U(1)$ then $\overline{\eta} = \eta$. Hence $\eta \in E^1_U(1)_{\mathbb{R}}$. By contrast, if $\eta_0 \in E^1_U$ is the differential form with the same values as η, but this time belonging to E^1_U, then $\overline{\eta_0} = -\eta_0$. Thus η_0 is purely imaginary.

1.3. **Local product situation and duality.** Assume in this subsection that X is projective in order to have Poincaré duality. Let $A \subset X$ be a Zariski closed subset and $a, r \in \mathbb{Z}$. Then Lefschetz duality tells us that there is an isomorphism of Mixed Hodge structures

$$H^r(X \setminus A; a) \cong H^{2d-r}(X, A; d-a)^\vee.$$

If B is a second Zariski closed subset one may ask if there is a refined duality

$$H^r(X \setminus A, B; a) \cong H^{2d-r}(X \setminus B, A; d-a)^\vee? \tag{1.3}$$

In general the answer is no as the following example shows.

Example 1.8. In this example we put $X = \mathbb{P}^2$. Let ℓ_0, ℓ_1 and ℓ_2 be three different lines passing through the same point p and write $A = \ell_0 \cup \ell_1$ and $B = \ell_2$. Then

$$\begin{aligned} H^1(X \setminus A, B) &= \mathbb{Q}(-1), \quad H^3(X \setminus A, B) = 0, \\ H^1(X \setminus B, A) &= \mathbb{Q}(0), \qquad H^3(X \setminus B, A) = 0. \end{aligned}$$

Thus, the answer to question (1.3) is negative.

Nevertheless, if we add some hypothesis to the sets A and B we can have a positive answer.

Definition 1.9. Let A and B be closed subvarieties of X. We say that A and B are in a *local product situation* if, for any point $x \in X$ there is a neighborhood U of x, a decomposition $U = U_A \times U_B$, where U_A and U_B are open disks of smaller dimension, and analytic subvarieties $A' \subset U_A$ and $B' \subset U_B$ such that

$$A \cap U = A' \times U_B, \qquad B \cap U = U_A \times B'.$$

Remark 1.10. The sets A and B of Example 1.8 are not in a local product situation. By contrast, if A and B are divisors without common components such that $A \cap B$ is a normal crossing divisor, then A and B are in local product situation.

The following result is proved in [BKV15, Lemma 6.1.1].

Lemma 1.11. *Let A and B be closed subvarieties of X in local product situation. Then, for every $a, r \in \mathbb{Z}$, there is an isomorphism of mixed Hodge structures*

$$H^r(X \setminus A, B; a) \xrightarrow{\cong} H^{2d-r}(X \setminus B, A; d-a)^\vee.$$

In the next section we will explain how to realize this isomorphism explicitly, after tensoring with $\mathbb{R}$, using differential forms.

We give now two applications of duality.

Lemma 1.12. *Let $Z \subset X$ be a closed subvariety and let $\pi\colon \widetilde{X} \to X$ be a blow-up with center contained in Z such that $\widetilde{X}$ is smooth. Write $\widetilde{Z} = \pi^{-1}(Z)$. Then, for all $a, r \in \mathbb{Z}$, the maps*

$$H^r(X \setminus Z; a) \xrightarrow{\pi^*} H^r(\widetilde{X} \setminus \widetilde{Z}; a) \tag{1.4}$$

$$H^r(X, Z; a) \xrightarrow{\pi^*} H^r(\widetilde{X}, \widetilde{Z}; a) \tag{1.5}$$

are isomorphisms.

Proof. The fact that (1.4) is an isomorphism is obvious because $X \setminus Z = \widetilde{X} \setminus \widetilde{Z}$. By the functoriality of duality, the morphism (1.5) is the composition

$$H^r(X, Z; a) \xrightarrow{\cong} H^{2d-r}(X \setminus Z; d-a)^\vee \xrightarrow{(\pi_*)^\vee} H^{2d-r}(\widetilde{X} \setminus \widetilde{Z}; d-a))^\vee \xrightarrow{\cong} H^r(\widetilde{X}, \widetilde{Z}; a).$$

Since the map π_* in the middle is also an isomorphism by the same reason as before, we conclude that (1.5) is an isomorphism. □

The next result tell us the surprising fact that, under some conditions, we can shift, in the isomorphism of Lemma 1.11, part of the closed subset A to the closed subset B.

Lemma 1.13. *Let A, B be two divisors without common components such that $A \cup B$ is a normal crossing divisor. Let $\pi \colon \widetilde{X} \to X$ be a blow up with center contained in $A \cap B$ such that $\widetilde{X}$ is smooth and $\pi^{-1}(A \cup B)$ is a normal crossing divisor. Let $\widehat{A}$ and $\widehat{B}$ be the strict transforms of A and B respectively and C the exceptional divisor of π. Then, for all $a, r \in \mathbb{Z}$ there are isomorphism*

$$H^r(X \setminus A, B; a) \xrightarrow{\pi^*} H^r(\widetilde{X} \setminus \widehat{A} \cup C, \widehat{B}; a), \tag{1.6}$$

$$H^r(X \setminus A, B; a) \xrightarrow{\cong} H^r(\widetilde{X} \setminus \widehat{A}, \widehat{B} \cup C; a). \tag{1.7}$$

Proof. The fact that π^* is an isomorphism is a consequence of the equalities

$$X \setminus A = \widetilde{X} \setminus (\widehat{A} \cup C), \qquad B \setminus A = \widehat{B} \setminus (\widehat{A} \cup C)$$

The isomorphism (1.7) is the composition of the isomorphisms

$$\begin{aligned} H^r(X \setminus A, B; a) &\xrightarrow{\cong} H^{2d-r}(X \setminus B, A; d-a)^\vee \\ &\xrightarrow{\cong} H^{2d-r}(\widetilde{X} \setminus \widehat{B} \cup C, \widehat{A}; d-a)^\vee \\ &\xrightarrow{\cong} H^r(\widetilde{X} \setminus \widehat{A}, \widehat{B} \cup C; a), \end{aligned}$$

where the the existence of the first and third isomorphism is a consequence of Lemma 1.11 and the second isomorphism agrees with the isomorphism (1.6) applied with A and B interchanged. □

1.4. **Differential forms with zeros and logarithmic poles.** Let $Y \subset X$ be a closed subvariety, $\widetilde{Y}$ a resolution of singularities of Y and $\iota \colon \widetilde{Y} \to X$ the induced map. We denote

$$\Sigma_Y E_X^* = \{\omega \in E_X^* \mid \iota^* \omega = 0\}.$$

Then $\Sigma_Y E_X^*$ is an example of a Dolbeault complex. In particular is a weak $\mathbb{R}$-Hodge complex. Therefore we can apply to it the notation of Definition 1.3. We begin with a basic observation.

Proposition 1.14. *Let $Y \subset X$ be a smooth subvariety. Then The complexes $\Sigma_Y E_X^*$ and $s(E_X^* \xrightarrow{\iota^*} E_{\widetilde{Y}}^*)$ are quasi-isomorphic.*

Proof. For smooth Y the sequence

$$0 \to \Sigma_Y E_X^* \to E_X^* \to E_Y^* \to 0$$

is exact, what implies the result. □

Note that we do not put a weight filtration on $\Sigma_Y E_X^*$. Nevertheless in good conditions the complex $\Sigma_Y E_X^*$ allows us to compute part of the mixed Hodge structure of the relative cohomology of the pair (X, Y).

Proposition 1.15. *Assume that X is projective. Let A be a normal crossing divisor of X and let W be a smooth closed subvariety that intersects transversely all intersections among the components of A. Write $Y = A \cup W$. Assume furthermore that all possible intersections among components of Y are smooth and irreducible. Then, there is a mixed Hodge complex K that computes the relative cohomology groups $H^*(X, Y)$, a quasi-isomorphism*

$$\Sigma_Y E^*_{X,\mathbb{R}} \longrightarrow K_{\mathbb{R}}$$

and a compatible filtered quasi-isomorphism

$$(\Sigma_Y E^*_X, F) \longrightarrow (K_{\mathbb{C}}, F).$$

Proof. Let $Y = Y_1 \cup \cdots \cup Y_r$ be the decomposition of Y into irreducible components. For $I \subset \{1, \dots, r\}$ we write $Y_I = \bigcap_{i \in I} Y_i$. Then there is an exact sequence

$$0 \to \Sigma_Y E^*_X \to E^*_X \to \bigoplus_{|I|=1} E^*_{Y_I} \to \bigoplus_{|I|=2} E^*_{Y_I} \to \cdots \tag{1.8}$$

Moreover this sequence remains exact after taking the F^p subcomplex at each degree. Since the total complex of the sequence

$$\bigoplus_{|I|=1} E^*_{Y_I} \to \bigoplus_{|I|=2} E^*_{Y_I} \to \cdots \to \bigoplus_{|I|=k} E^*_{Y_I} \to \cdots$$

is the de Rham part of a mixed Hodge complex that computes $H^*(Y)$, the result follows. $\square$

Let $A \subset X$ be a normal crossing divisor and $E^*_X(\log A)$ the complex of differential forms on X with logarithmic singularities along A introduced in [BG94a]. This is also a Dolbeault complex, so it has a real structure and a Hodge filtration. Although in this case it also has a weight filtration. We will use the shorthand

$$E^*_X(\log A; a) := E^*_X(\log A)(a)$$

The following result is proved in [BG94a].

Proposition 1.16. *Assume again that X is projective and that $A \subset X$ is a normal crossing divisor. Then, $((E^*_X(\log A)_{\mathbb{R}}, W), (E^*_X(\log A), W, F))$ is a mixed Hodge complex computing the real mixed Hodge structure $H^*(X \setminus A)$.*

Proposition 1.16 can be applied to general subvarieties of X by using resolution of singularities. In order to get a complex that does not depend on the choice of a particular resolution one can take a limit with respect to all possible resolutions. In the sequel we will have a mixed situation where there is already present a normal crossing divisor A that we want to preserve as much as possible and an arbitrary subvariety Z that meets A properly. In this case we use the following notation

$$E^*_X(\log A \cup Z) = \varinjlim_{\widetilde{X}} E^*_X(\log A') \tag{1.9}$$

where the limit runs over all proper modifications $\pi\colon \widetilde{X} \to X$ such that $A' = \pi^{-1}(A \cup Z)$ is a normal crossing divisor and that the restriction $\pi|_{\widetilde{X} \setminus \pi^{-1}(Z)} \colon \widetilde{X} \setminus \pi^{-1}(Z) \to X \setminus Z$ is an isomorphism. In other words we are only allowed to make blow ups supported on Z. The complex $E^*_X(\log A \cup Z)$ inherits a real structure, a Hodge filtration and a weight filtration. Proposition 1.16 easily implies the next result.

Corollary 1.17. *Assume that X is projective, that $A \subset X$ is a normal crossing divisor and that $Z \subset X$ is a closed subvariety. Then,*

$$((E^*_X(\log A \cup Z)_{\mathbb{R}}, W), (E^*_X(\log A \cup Z), W, F))$$

is a mixed Hodge complex computing the real mixed Hodge structure $H^*(X \setminus A \cup Z)$.

We can now combine Proposition 1.15 and Corollary 1.17.

Definition 1.18. Let A be a normal crossing divisor of X. Let $Z, W \subset X$ be closed subvarieties such that non component of W is contained in $A \cup Z$. Let $\iota\colon \widetilde{W} \to X$ be a resolution of singularities of $W \setminus A \cup Z$. Then we write

$$\Sigma_W E^*_X(\log A \cup Z) = \{\omega \in E^*_X(\log A \cup Z) \mid \iota^*\omega = 0\} \subset E^*_X(\log A \cup Z).$$

We again use the shorthand, for $a \in \mathbb{Z}$,

$$\Sigma_W E^*_X(\log A \cup Z; a) := \Sigma_W E^*_X(\log A \cup Z)(a).$$

The complex $\Sigma_W E^*_X(\log A \cup Z)$ has a real structure and a Hodge filtration but not a weight filtration.

Theorem 1.19. *Assume that X is projective and that A, B are divisors without common components such that $A \cup B$ is a normal crossing divisor. Let W be a smooth subvariety intersecting transversely all intersections of components of $A \cup B$ and such that all intersections between components of $A \cup B \cup W$ are smooth and irreducible. Let Z be a closed subvariety. Then, there is a mixed Hodge complex K that computes the relative cohomology groups $H^*(X \setminus (B \cup Z), (A \cup W) \setminus (B \cup Z))$, a quasi-isomorphism*

$$\Sigma_{A\cup W} E^*_X(\log B \cup Z)_{\mathbb{R}} \longrightarrow K_{\mathbb{R}}$$

and a compatible filtered quasi-isomorphism

$$(\Sigma_{A\cup W} E^*_X(\log B \cup Z)_{\mathbb{C}}, F) \longrightarrow (K_{\mathbb{C}}, F).$$

Proof. The proof is essentially the same as the proof of Proposition 1.15 by using Corollary 1.17 on each intersection among components of $A \cup W$. □

Corollary 1.20. *With the hypothesis of Theorem 1.19. For each $a, r \in \mathbb{Z}$ there is a canonical isomorphism*

$$H^r(\Sigma_{A\cup W} E^*_X(\log B \cup Z, a)) \xrightarrow{\cong} H^r(X \setminus (B \cup Z), (A \cup W) \setminus (B \cup Z); a)_{\mathbb{C}}$$

compatible with the Hodge filtration and the real structure. Moreover, the spectral sequence associated to the Hodge filtration F degenerates at the term E_1. Therefore the differential d in the complex $\Sigma_Y E^*_X$ is strict with respect to the filtration F.

Proof. The first statement is a direct consequence of Theorem 1.19. The second statement is also consequence of Theorem 1.19 and standard properties of mixed Hodge complexes. □

Finally we explain how to use differential forms with zeros and poles to make effective the duality of Lemma 1.11 in the normal crossing case.

Proposition 1.21. *Assume that X is projective. Let A and B be two divisors of X without common components such that $A \cup B$ is a normal crossing divisor. For $\eta \in \Sigma_A E^r_X(\log B)$ and $\omega \in \Sigma_B E^{2d-r}_X(\log A)$, the top differential form $\eta \wedge \omega$ is locally integrable. Moreover, the pairing*

$$H^r(X \setminus B, A)_{\mathbb{C}} \otimes H^{2d-r}(X \setminus A, B)_{\mathbb{C}} \longrightarrow \mathbb{R}(-d)_{\mathbb{C}}$$

given, for η and ω closed, by

$$\langle \eta, \omega \rangle = [\eta \wedge \omega](1) = \frac{1}{(2\pi i)^d} \int_X \eta \wedge \omega$$

is a perfect pairing inducing an isomorphism as in Lemma 1.13.

1.5. **Currents on a subvariety.** Let Z be a subvariety of X. We denote by $\Sigma_Z E^*_{X,c} \subset \Sigma_Z E^*_X$ the subspace of differential forms with compact support on X that vanish on Z and we write

$$E'^{-n}_{X,Z} = \{T \in E'^{-n}_X \mid T(\omega) = 0,\ \forall \omega \in \Sigma_Z E^*_{X,c}\}$$

The space $E'^{-n}_{X,Z}$ has been introduced by Bloom and Herrera in [BH69] and, in the case when Z is smooth, it agrees with E'^{-n}_Z. This space is a Dolbeault complex and we write

$$D^*_{X,Z} = E'^*_{X,Z}[-2d](-d), \qquad D^*_{X/Z} = D^*_X / D^*_{X,Z}.$$

Again, the complex $D^*_{X,Z}$ has a real structure and a Hodge filtration but not a weight filtration.

Let $A \subset X$ be a normal crossing divisor and Z a closed subvariety, write $Y = A \cup Z$. If $\omega \in E^*_X(\log A \cup Z; a)$ and $\eta \in \Sigma_Y E^*_X$ then the differential form $\omega \wedge \eta$ is locally integrable in any proper modification $\widetilde{X} \to X$ where ω is defined. This induces a map

$$[\cdot]\colon E^*_X(\log A \cup Z; a) \to D^*_{X/Y}(a)$$

given by

$$[\omega](\eta) = \frac{1}{(2\pi i)^d} \int_X \omega \wedge \eta.$$

Proposition 1.22. *Let A, Z and Y be as before. Assume that Z is smooth and that meets transversely all the strata of A. Then the map*

$$(E^*_X(\log A \cup Z; a), F) \longrightarrow (D^*_{X/Y}(a), F)$$

is a filtered quasi-isomorphism compatible with the real structure.

Proof. The case when $A \cup Z$ is a normal crossing divisor has been proved in [BGKK07, Theorem 5.44] using the techniques from [Fuj80] and [HL71]. Let $\pi\colon \widetilde{X} \to X$ be the blow-up of X along Z. The conditions on Z imply that $\widetilde{Y} \coloneqq \pi^{-1}(Y)$ is a normal crossing divisor. Consider the commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & D^*_{\widetilde{X},\widetilde{Y}} \arrow[r] \arrow[d] & D^*_{\widetilde{X}} \arrow[r] \arrow[d] & D^*_{\widetilde{X}/\widetilde{Y}} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & D^*_{X,Y} \arrow[r] & D^*_X \arrow[r] & D^*_{X/Y} \arrow[r] & 0.
\end{tikzcd}$$

The formula for the cohomology of a blow-up implies that the total complex associated to the diagram of complexes

$$\begin{tikzcd} D^*_{\widetilde{X},\widetilde{Y}} \arrow[r] \arrow[d] & D^*_{\widetilde{X}} \arrow[d] \\ D^*_{X,Y} \arrow[r] & D^*_X \end{tikzcd}$$

is acyclic. Even more, every subcomplex defined by the Hodge filtration is acyclic. This implies that the arrow

$$(D^*_{\widetilde{X}/\widetilde{Y}}, F) \to (D^*_{X/Y}, F)$$

is a filtered quasi-isomorphism. Thus the result follows from the normal crossing case. □

Corollary 1.23. *With the hypothesis of Proposition 1.22, for every $a, r \in \mathbb{Z}$, there is a canonical isomorphism*

$$H^r(X \setminus Y; a)_{\mathbb{C}} = H^r(D^*_{X/Y}(a))_{\mathbb{C}}$$

compatible with the Hodge filtration and the real structure.

1.6. **Wave front sets.** A current T can be viewed as a differential form with *distribution* coefficients or as a *generalized section* of a vector bundle. As such, it has a wave front set that is denoted by $\mathrm{WF}(T)$. The theory of wave front sets of distributions is developed in [Hör90] chap. VIII. For the theory of wave front sets of generalized sections, the reader can consult [GS77] chap. VI. Although we will work with currents and hence with generalized sections of vector bundles, we will follow mainly [Hör90].

Denote he conormal bundle of X minus the zero section as $T^\vee_0 X = T^\vee X \setminus \{0\}$. The wave front set of a current T is a closed conical subset of $T^\vee_0 X$. This set describes the points and directions of the singularities of T and it allows us to define certain products and inverse images of currents. For a concise description of the basic properties of the wave front set we refer to [BGL10, §4].

Let $\mathcal{S} \subset T^\vee_0 X$ be a closed conical subset. We denote by $D^*_{X;\mathcal{S}}$ the space of currents on X with wave front set contained in $\mathcal{S}$. Then [BGL10, Theorem 4.5] implies that

Proposition 1.24. *Assume that X is projective. Then the morphisms*

$$(E^*_X, F) \to (D^*_{X;\mathcal{S}}, F) \to (D^*_X, F)$$

are filtered quasi-isomorphism.

We will need an analogue of Theorem 1.19 for currents with controlled wave front sets. Although the theory of wave front sets depends only of the underlying structure of differentiable manifolds we will state the needed notations and results in the complex case.

Definition 1.25. Let $f\colon Y \to X$ be a morphism of complex manifolds, and let $\mathcal{S} \subset T_0^\vee X$ and $\mathcal{R} \subset T_0^\vee$ closed conical subsets. Then we denote

$$\begin{aligned} N_0^\vee f &= \{(x,\xi) \in T_0^\vee X \mid x = f(y),\ df(y)^t\xi = 0\},\\ f^*\mathcal{S} &= \{(y,\eta) \in T_0^\vee Y \mid \exists (x,\xi) \in \mathcal{S},\ x = f(y),\ \eta = df(y)^t\xi\},\\ f_*\mathcal{R} &= N_0^\vee f\{(x,\xi) \in T_0^\vee X \mid \exists (y,\eta) \in \mathcal{R},\ x = f(y),\ \eta = df(y)^t\xi\}. \end{aligned}$$

Then

$$f_*f^*\mathcal{S} = N_0^\vee f \cup \{(x,\xi) \in T_0^\vee X \mid x = f(y),\ \exists (x,\xi') \in \mathcal{S} df(y)^t\xi = df(y)^t\xi'\}.$$

Clearly, $f_*f^*\mathcal{S} = f_*f^*f_*f^*\mathcal{S}$. We call $f_*f^*\mathcal{S}$ the *saturation* of $\mathcal{S}$ with respect to f. If $\mathcal{S} = f_*f^*\mathcal{S}$ we say that $\mathcal{S}$ is *saturated.* If Y is a smooth submanifold of X and f the corresponding closed immersion, we write $N_0^\vee Y = N_0^\vee f$.

The basic functoriality properties of currents and wave front are the following (see [Hör90, chap. VIII, sect. 2]).

Proposition 1.26. *Let $f\colon Y \to X$ be a morphism of complex manifolds of relative dimension e, and let $\mathcal{S} \subset T_0^\vee X$ and $\mathcal{R} \subset T_0^\vee$ closed conical subsets.*

(i) If $T \in D^r_{X;\mathcal{S}}$ and $N_f \cap \mathcal{S} = \emptyset$, then there is a well defined pullback current $f^*T \in D^r_{Y;f^*\mathcal{S}}$.
(ii) If $T \in D^r_{Y;\mathcal{R}}$, then $f_*T \in D_{X;f_*\mathcal{R}}$.

Let $\iota\colon A \hookrightarrow X$ be a smooth hypersurface and $\mathcal{S} \subset T_0^\vee X$ a closed conical subset. We will denote

$$D^*_{X,A;\mathcal{S}} = D^*_{X,A} \cap D^*_{X;\mathcal{S}} \qquad D^*_{X/A;\mathcal{S}} = D^*_{X;\mathcal{S}} \big/ D^*_{X,A;\mathcal{S}}.$$

Lemma 1.27. *Let $\mathcal{R} \subset T_0^\vee A$ be a closed conical subset. The morphism ι_* induces an isomorphism*

$$\iota_*\colon D^*_{A;\mathcal{R}}[-2](-1) \longrightarrow D^*_{X,A;\iota_*\mathcal{R}}. \tag{1.10}$$

Therefore, if $\mathcal{S} \subset T_0^\vee X$ is saturated, we obtain an isomorphism

$$\iota_*\colon D^*_{A;\iota^*\mathcal{S}}[-2](-1) \longrightarrow D^*_{X,A;\mathcal{S}}.$$

Proof. By Proposition 1.26, the map (1.10) is well defined. Since A is smooth, by [BH69] the map

$$\iota_*\colon D^*_{A}[-2](-1) \longrightarrow D^*_{X,A}$$

is an isomorphism. This implies directly that the map (1.10) is injective. It follows easily from the definition of wave front set, that if $\mathrm{WF}(\iota_*T) \subset f_*\mathcal{R}$ then $\mathrm{WF}(T) \subset \mathcal{R}$ which implies surjectivity. □

When taking the current associated to a differential form with logarithmic singularities, it is easy to control the wave front set. In fact, the map $E^*_X(\log A) \to D^*_{X/A}$ factors as a composition

$$E^*_X(\log A) \longrightarrow D^*_{X/A;N_0^\vee A} \longrightarrow D^*_{X/A}$$

Let $\iota'\colon B \to X$ be another smooth hypersurface such that $\mathcal{S} \cap N_0^\vee B = \emptyset$. By Proposition 1.26 there is a map $(\iota')^*\colon D^*_{X;\mathcal{S}} \to D^*_{B;(\iota')^*\mathcal{S}}$ and we define

$$\Sigma_B D^*_{X;\mathcal{S}} = \ker((\iota')^*).$$

Definition 1.28. We say that $\mathcal{S}$ and B are in *good position* if, for every $p \in B$ there is an open neighborhood $U \subset X$ of p and a smooth retraction $r\colon U \to U \cap B$ such that

$$r^*((\iota')^*\mathcal{S}|_{U\cap B}) \subset \mathcal{S}|_U.$$

Lemma 1.29. *If $\mathcal{S}$ and B are in good position, then the map*

$$(\iota')^*\colon D^*_{X;\mathcal{S}} \to D^*_{B;(\iota')^*\mathcal{S}}$$

is surjective.

Proof. By a partition of unity argument, the statement is local on B. Let $p \in B$ and U and r the neighborhood and smooth retraction that exist because $\mathcal{S}$ and B are in good position. Let $T \in D^*_{B;(\iota')^*\mathcal{S}}$. Then

$$r^*T \in D^*_{U;r^*(\iota')^*\mathcal{S}} \subset D^*_{U;\mathcal{S}}, \text{ and } (\iota')^*r^*T = T$$

proving surjectivity. □

We now put all the ingredients together. Let X be a smooth projective complex variety, $\iota\colon A \hookrightarrow X$ and $\iota'\colon B \hookrightarrow X$ two smooth disjoint hypersurfaces of X and $\mathcal{S} \subset T_0^\vee X$ a closed conical subset that is, at the same time, saturated with respect to ι and in good position with respect to B. We define

$$\Sigma_B D^*_{X/A;\mathcal{S}} = \{T \in D^*_{X/A;\mathcal{S}} \mid T|_B = 0\}.$$

Theorem 1.30. *Let X, A, B and $\mathcal{S}$ be as before. Then the map*

$$(\Sigma_B E^*_X(\log A), F) \longrightarrow (\Sigma_B D^*_{X/A;\mathcal{S}}, F) \tag{1.11}$$

is a filtered quasi-isomorphism.

Proof. By Lemma 1.27, since $\mathcal{S}$ is saturated with respect to ι, we have an isomorphism

$$\iota_*\colon D^*_{A;\iota^*\mathcal{S}}[-2](-1) \longrightarrow D^*_{X,A;\mathcal{S}}.$$

Since $(D^*_{A;\iota^*\mathcal{S}}, F) \to (D^*_A, F)$ is a filtered quasi-isomorphism and the map $D^*_A[-2](-1) \to D^*_{X,A}$ is an isomorphism, we deduce that $(D^*_{X,A;\mathcal{S}}, F) \to (D^*_{X,A}, F)$ is a filtered quasi-isomorphism. We consider the commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & D^*_{X,A;\mathcal{S}} \arrow[r] \arrow[d] & D^*_{X;\mathcal{S}} \arrow[r] \arrow[d] & D^*_{X/A;\mathcal{S}} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & D^*_{X,A} \arrow[r] & D^*_X \arrow[r] & D^*_{X/A} \arrow[r] & 0
\end{tikzcd}$$

As we have discussed, the first vertical arrow is a filtered quasi-isomorphism. By Proposition 1.24 the second vertical arrow is also filtered quasi isomorphism. We deduce that the third arrow also is one. Using now Proposition 1.22 we obtain that the map

$$(E^*_X(\log A), F) \longrightarrow (D^*_{X/A;\mathcal{S}}, F)$$

is a filtered quasi-isomorphism. Consider next the commutative diagram with exact rows.

$$\begin{tikzcd}
0 \arrow[r] & \Sigma_B E^*_X(\log A) \arrow[r] \arrow[d] & E^*_X(\log A) \arrow[r] \arrow[d] & E^*_B \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Sigma_B D^*_{X,A;\mathcal{S}} \arrow[r] & D^*_{X,A;\mathcal{S}} \arrow[r] & D^*_{B;(\iota')^*\mathcal{S}} \arrow[r] & 0
\end{tikzcd}$$

Note that surjectivity of the map $D^*_{X,A;\mathcal{S}} \to D^*_{B;(\iota')^*\mathcal{S}}$ is Lemma 1.29. We already know that the second and third vertical arrows are filtered quasi-isomorphism, hence the first is also one, proving the result. $\square$

1.7. **Higher Chow groups.** We recall here the definition and main properties of the *higher Chow groups* defined by Bloch in [Blo86]. Initially, they were defined using the chain complex associated to a simplicial abelian group, but the description using the cubical complex is more user friendly to define the product structure. We stick to notations and conventions followed in §3 of [BGG19].

Fix a base field k and let $\mathbb{P}^1$ be the projective line over k. Let $\square = \mathbb{P}^1 \setminus \{1\}\,(\cong \mathbb{A}^1)$. The cartesian product $(\mathbb{P}^1)^{\cdot}$ has a cocubical scheme structure. For $i = 1, \dots, n$, we denote by $t_i \in (k \cup \{\infty\}) \setminus \{1\}$ the absolute coordinate of the i-th factor. Then the coface maps are defined as

$$\begin{aligned}
\delta_0^i(t_1, \dots, t_n) &= (t_1, \dots, t_{i-1}, 0, t_i, \dots, t_n),\\
\delta_1^i(t_1, \dots, t_n) &= (t_1, \dots, t_{i-1}, \infty, t_i, \dots, t_n).
\end{aligned}$$

Then, $\square^{\cdot}$ inherits a cocubical scheme structure from that of $(\mathbb{P}^1)^{\cdot}$. An r-dimensional *face* F of $\square^n$ is any subscheme of the form $\delta_{j_1}^{i_1} \cdots \delta_{j_{n-r}}^{i_{n-r}}(\square^r)$. By convention, $\square^n$ is a face of dimension n. The codimension of an r-dimensional face of $\square^n$ is $n - r$.

Let X be an equidimensional quasi-projective scheme of dimension d over the field k. Let $Z^p(X, n)$ be the free abelian group generated by the codimension p closed irreducible subvarieties of $X \times \square^n$, which intersect properly $X \times F$ for every face F of $\square^n$. We call the elements of $Z^p(X, n)$ *admissible* cycles. The pull-back by the coface and codegeneracy maps of $\square^{\cdot}$ endow $Z^p(X, \cdot)$ with a cubical abelian group structure, given by

$$\begin{aligned}
\delta_i^j &= (\delta_j^i)^*,\\
\delta &= \sum_{i=1}^n \sum_{j=0,1} (-1)^{i+j} \delta_i^j.
\end{aligned}$$

Note that the indexes have been raised or lowered to reflect the change from cocubical to cubical structures.

Let $(Z^p(X, *), \delta)$ be the associated chain complex and consider the *normalized* and *refined normalized* chain complexes associated to $Z^p(X, *)$,

$$\begin{aligned}
Z^p(X, n)_0 &\coloneqq \bigcap_{i=1}^n \ker \delta_i^1,\\
Z^p(X, n)_{00} &\coloneqq \bigcap_{i=1}^n \ker \delta_i^1 \cap \bigcap_{i=2}^n \ker \delta_i^0.
\end{aligned}$$

The differential of these normalized complexes are also denoted by δ. One can show that the inclusion

$$Z^p(X,n)_{00} \hookrightarrow Z^p(X,n)_0$$

is a quasi-isomorphism of cubical chain complexes. An element in the above two complexes will be called a *pre-cycle*, and will be called a (higher) *cycle* if it also satisfies $\delta(Z) = 0$.

Definition 1.31. Let X be a quasi-projective equidimensional scheme over a field k. The *higher Chow groups* defined by Bloch are

$$\mathrm{CH}^p(X,n) := H_n(Z^p(X,*)_0) \cong H_n(Z^p(X,*)_{00}).$$

Since we will often come across the notion of proper intersection of higher cycles in this paper, for the sake of easy reference, we recall its definition.

Definition 1.32. Let X be a smooth quasi-projective scheme over k, and let $p,q,n,m \geq 0$ be non-negative integers. If $Z \in Z^p(X,n)$, $W \in Z^q(X,m)$, we say that Z and W *intersect properly* if, for any face F of $\square^{n+m}$,

$$\mathrm{codim}_{X\times F}\left(\pi_1^{-1}|Z| \cap \pi_2^{-1}|W| \cap (X \times F)\right) \geq p+q,$$

where

$$\pi_1 \colon X \times \square^n \times \square^m \to X \times \square^n, \quad \pi_2 \colon X \times \square^n \times \square^m \to X \times \square^m$$

are the projections.

Let $W \in Z^q(X,m)$ be an admissible cycle. We denote by $Z^p_W(X,n) \subset Z^p(X,n)$ the subgroup generated by the codimension p irreducible subvarieties $Z \subset X \times \square^n$, such that Z and W intersect properly. Then it can be shown that the inclusions

$$Z^p_W(X,*)_0 \hookrightarrow Z^p(X,*)_0, \qquad Z^p_W(X,*)_{00} \hookrightarrow Z^p(X,*)_{00}$$

are quasi-isomorphisms.

1.8. **Survey of Deligne–Beilinson cohomology.** As in [BGG19, Example 4.17], given a Dolbeault complex A we can associate to it a diagram of complexes and morphisms

$$\mathfrak{D}_{\mathrm{TW}}(A,p) \underset{E}{\overset{I}{\rightleftarrows}} \mathfrak{D}_t(A,p) \underset{G}{\overset{H}{\rightleftarrows}} \mathfrak{D}(A,p)\ , \tag{1.12}$$

where the three complexes compute the Deligne cohomology of A and all the arrows are homotopy equivalences. The leftmost complex has the advantage that, when A is a Dolbeault algebra, has also a structure of an associative and graded commutative algebra. On the middle complex, we have several product structures, but none is at the same time graded commutative and associative. The rightmost complex is the smallest one and gives a more concise description of Deligne cohomology but again has the disadvantage that the product is only associative up to homotopy.

In particular, if X is a smooth projective variety over $\mathbb{C}$, we can specialize diagram (1.12) to the case $A = E^*_X$ to obtain a diagram

$$\mathfrak{D}_{\mathrm{TW}}(X,p) \rightleftarrows \mathfrak{D}_t(X,p) \rightleftarrows \mathfrak{D}(X,p)\ , \tag{1.13}$$

computing the real Deligne cohomology $H^*_{\mathfrak{D}}(X, \mathbb{R}(p))$ of X. We recall a few pieces of this diagram. Denote by $L_{\mathbb{R}} = (L^*_{\mathbb{R}}, d)$ the algebraic de Rham complex of $\mathbb{A}^1_{\mathbb{R}}$, that is,

$$L^0_{\mathbb{R}} = \mathbb{R}[\varepsilon], \quad L^1_{\mathbb{R}} = \mathbb{R}[\varepsilon]d\varepsilon,$$

where ε is an indeterminate. For a Dolbeault complex A we write

$$\mathfrak{D}_{\mathrm{TW}}(A, p) = \left\{ \omega \in L^*_{\mathbb{R}} \otimes A^*(p)_{\mathbb{C}} \;\middle|\; \begin{array}{c} \omega|_{\varepsilon=0} \in A^*(p)_{\mathbb{R}}, \\ \omega|_{\varepsilon=1} \in F^0 A^*(p)_{\mathbb{C}}. \end{array} \right\} \tag{1.14}$$

and

$$\mathfrak{D}^n(A, p) = \begin{cases} A^{n-1}(p-1)_{\mathbb{R}} \cap \bigoplus\limits_{\substack{p'+q'=n-1 \\ p'<p,\ q'<p}} A^{p',q'}_{\mathbb{C}}, & \text{if } n < 2p, \\ A^n(p)_{\mathbb{R}} \cap \bigoplus\limits_{\substack{p'+q'=n \\ p'\geq p,\ q'\geq p}} A^{p',q'}_{\mathbb{C}}, & \text{if } n \geq 2p. \end{cases}$$

Note that, for $n < 2p$ we can also write

$$\mathfrak{D}^n(A, p) = \frac{A^{n-1}(p)_{\mathbb{C}}}{A^{n-1}(p)_{\mathbb{R}} \cap F^0 A^{n-1}(p)_{\mathbb{C}}}. \tag{1.15}$$

We will denote by

$$\pi_p \colon A^{n-1}(p)_{\mathbb{C}} \longrightarrow \mathfrak{D}^n(A, p) \tag{1.16}$$

the projection map. Then, for $n < 2p$, (see [BGW98] paragraphs (6.1) and (6.2)) the map ${\mathfrak{D}_{\mathrm{TW}}}^n(A, p) \to \mathfrak{D}^n(A, p)$ is given by

$$f(\varepsilon) \otimes \omega_1 + g(\varepsilon)d\varepsilon \otimes \omega_2 \mapsto \int_0^1 g(\varepsilon)d\varepsilon \cdot \pi_p(\omega_2). \tag{1.17}$$

1.9. **Goncharov regulator and higher archimedean height pairing.** Here we give a quick revision of the cubical Goncharov regulator and of the higher archimedean height pairing for sake of ready reference. More details about the regulator can be found in [BGG19, Section 5], more details about Green currents and forms, in [BGG19, Section 6] and about the height pairing in [BGG19, Subsection 7.5]. From now on we denote the differential in the Thom-Whitney complex by $d_{\mathfrak{D}}$, to distinguish it from the differential in the de-Rham complex.

In the paper [BGG19], Goncharov regulator

$$\mathcal{P} \colon \mathrm{CH}^p(X, n) \longrightarrow H^{2p-n}_{\mathfrak{D}}(X, \mathbb{R}(p))$$

is given by a morphism of complexes, also denoted $\mathcal{P}$

$$Z^p(X, *)_0 \to \mathfrak{D}^{2p-*}_{\mathrm{TW},D}(X, p).$$

Recall the complex L form Section 1.7 Let $\lambda \in (L_{\mathbb{C}} \otimes E_{\mathbb{P}^1}(\log B))^1$ be the element given by

$$\lambda = -\frac{1}{2}\left((\varepsilon + 1) \otimes \frac{dt}{t} + (\varepsilon - 1) \otimes \frac{d\bar{t}}{\bar{t}} + d\varepsilon \otimes \log t\bar{t} \right). \tag{1.18}$$

Then $\lambda \in \mathfrak{D}^1_{\mathrm{TW}}(E^*_{\mathbb{P}^1}(\log B), 1)$.

On $(\mathbb{P}^1)^n \setminus B$, for $n \geq 0$, we consider the Wang forms

$$W_0 = 1$$
$$W_n = \pi_1^*\lambda \cdots \pi_n^*\lambda, \ n > 0,$$

where $\pi_i \colon (\mathbb{P}^1)^n \to \mathbb{P}^1$ is the projection onto the i-th factor. Clearly $W_n \in \mathfrak{D}^n_{\mathrm{TW}}(\Sigma_A E^*_{(\mathbb{P}^1)^n}(\log B), n)$. See [BGG19, §5] for the main properties of these forms. By abuse of notation we will also denote by W_n the pull-back of W_n to any variety of the form $X \times (\mathbb{P}^1)^n$. If Z is an irreducible subvariety of $X \times \square^n$ intersecting properly all the faces and $\widetilde{Z}$ is a resolution of singularities of the closure $\overline{Z}$, then the pull-back of W_n is locally integrable. Therefore, for any cycle $Z \in Z^p(X, *)_0$, Writing

$$\delta_{Z,\mathrm{TW}} \coloneqq 1 \otimes \delta_{\overline{Z}} \in \mathfrak{D}^{2p}_{\mathrm{TW}}(D^*_{X \times (\mathbb{P}^1)^n}, p), \tag{1.19}$$

we have a well defined current

$$\delta_{Z,\mathrm{TW}} \cdot W_n \in \mathfrak{D}^{2p+n}_{\mathrm{TW}}(D^*_{X \times (\mathbb{P}^1)^n}, p+n).$$

Then, Goncharov regulator is given by

$$\mathcal{P}(Z) = (\pi_X)_*(\delta_{Z,\mathrm{TW}} \cdot W_n) \in \mathfrak{D}^{2p-n}_{\mathrm{TW}}(D^*_X, p), \tag{1.20}$$

where $\pi_X \colon X \times (\mathbb{P}^1)^n \to X$ is the projection.

Given a cycle $Z \in Z^p(X, n)_0$, we call any current $g_Z \in \mathfrak{D}^{2p-n-1}_{\mathrm{TW},D}(X, p)$ a Green current for Z if it satisfies.

$$\mathcal{P}(Z) + d_{\mathfrak{D}} g_Z = [\omega_Z], \ \text{for } \omega_Z \in \mathfrak{D}^{2p-n}_{\mathrm{TW}}(X, p).$$

A class of Green currents is the class of a Green current in

$$\widetilde{\mathfrak{D}}^{2p-n-1}_{\mathrm{TW},D}(X, p) \coloneqq \mathfrak{D}^{2p-n-1}_{\mathrm{TW},D}(X, p)/ \operatorname{Im} d_{\mathfrak{D}},$$

and is denoted by $\widetilde{g}_Z$. A pair $(Z, \widetilde{g}_Z)$, where $\widetilde{g}_Z$ is a Green current for Z is called an arithmetic cycle, and is the building block to define higher arithmetic Chow groups.

To define an intersection theory at the level of higher arithmetic Chow groups, we need the notion of a Green form of logarithmic type for a cycle Z. It acts as a bridge between the current $1 \otimes \delta_Z \in \mathfrak{D}^{2p}_{\mathrm{TW},D}(X \times (\mathbb{P}^1)^n, p)$ and a smooth form that lives in $\mathfrak{D}^{2p-n}_{\mathrm{TW}}(X, p)$, and computes the real Deligne cohomology class $\mathcal{P}(Z)$. For shorthand, in the next proposition we denote $A \coloneqq (\mathbb{P}^1)^n \setminus \square^n$.

Definition 1.33. Given a cycle $Z \in Z^p(X, n)_0$ and the pullback $|Z|_k$ of $|Z|$ in $X \times \square^k$ (see §6.2 of [BGG19] for exact definition of $|Z|_k$), a *Green form of logarithmic type* for Z is an n-tuple

$$\mathfrak{g}_Z \coloneqq (g_n, g_{n-1}, \cdots, g_0) \in \bigoplus_{k=n}^{0} \mathfrak{D}^{2p-n+k-1}_{\mathrm{TW},\log}(X \times \square^k \setminus |Z|_k, p)_0,$$

Such that, if $n > 0$,

(i) the equation $\delta_Z + d_{\mathfrak{D}}[g_n] = 0$ holds in the complex

$$\mathfrak{D}^{2p}_{\mathrm{TW},D,X \times (\mathbb{P}^1)^n / X \times A}(p)$$

In other words, g_n is a Green form for Z in $X \times \square^n$.

(ii) $(-1)^{n-k+1}\delta g_k + d_{\mathfrak{D}} g_{k-1} = 0, \quad k = 2, \cdots, n.$

(iii) $(-1)^n\delta g_1 + d_{\mathfrak{D}} g_0 =: \omega(\mathfrak{g}_Z) \in \mathfrak{D}^{2p-n}_{\mathrm{TW}}(X,p)$. In other words, the form $(-1)^n\delta g_1 + d_{\mathfrak{D}} g_0$ extends to a smooth form on the whole X. It can be shown that $\omega(\mathfrak{g}_Z)$ is closed and belongs to class $\mathcal{P}(Z)$ in $H^{2p-n}_{\mathfrak{D}}(X,\mathbb{R}(p))$.

While, if $n=0$ the previous conditions collapse to the condition

$$\delta_Z + d_{\mathfrak{D}}[g_n] \in [\mathfrak{D}^{2p}_{\mathrm{TW}}(X,p)].$$

If $Z \in Z^p(X,n)_{00}$ is a cycle in the refined normalized complex, then a *refined Green form* is defined as a Green form satisfying the stronger condition

$$\mathfrak{g}_Z \in \bigoplus_{k=n}^{0} \mathfrak{D}^{2p-n+k-1}_{\mathrm{TW},\log}(X \times \square^k \setminus |Z|'_k, p)_{00}, \tag{1.21}$$

where $|Z|'_k = (\delta_0^1)^{-1} \overset{n-k}{\cdots} (\delta_0^1)^{-1}|Z|$.

It can be shown that every class $\widetilde{g}_Z$ of Green currents contains a Green form of logarithmic type (cf. [BGG19, propositions 6.12, 6.13]).

Let $Z \in Z^p(X,n)_0$ and $W \in Z^q(X,m)_0$ be two cycles intersecting properly in the sense of definition 1.32. Then for choices of classes of green currents $\widetilde{g}_Z$ and $\widetilde{g}_W$ for Z and W respectively, we define the start product

Definition 1.34. Choosing any representative g_Z of $\widetilde{g}_Z$ and a Green form $\mathfrak{g}_W = \{g'_m, \cdots, g'_0\}$ for W contained in $\widetilde{g}_W$, we define the $*$-product of $\widetilde{g}_Z$ and $\widetilde{g}_W$ as

$$\widetilde{g}_Z * \widetilde{g}_W = \left((-1)^n \left(\sum_{j=0}^{m} (\pi_{X,*}\left(\delta_Z \cdot W_n \cdot g'_j \cdot W_j\right) \right) + g_Z \cdot \omega(\mathfrak{g}_W) \right)^{\sim},$$

where $\delta_Z \cdot W_n \cdot g'_j \cdot W_j$ is seen as a current in $X \times (\mathbb{P}^1)^{n+m}$ and π_X is the projection to X.

Of course, the $*$-product $\widetilde{g}_Z * \widetilde{g}_W$ depends on the choice of the Green currents $\widetilde{g}_Z$ and $\widetilde{g}_W$ and not only on the cycles Z and W. Nevertheless, if the real regulators of Z and W are zero, we can obtain an invariant from the $*$-product that only depends on the cycles Z and W. This is the higher analogue of the Archimedean component of the height pairing.

Definition 1.35. Let $Z \in Z^p(X,n)_0$ and $W \in Z^q(X,m)_0$ be cycles intersecting properly, having real regulator classes zero, and $2(p+q-d-1) = n+m$. Then we can find Green currents for Z and W satisfying the conditions

$$d_{\mathfrak{D}} g_Z + \mathcal{P}(Z) = d_{\mathfrak{D}} g_W + \mathcal{P}(W) = 0. \tag{1.22}$$

and the *higher archimedean height pairing* is defined as

$$\langle Z, W\rangle_{\mathrm{Arch}} \coloneqq (p_{X,*}(g_Z * g_W))^{\sim} \in H^1_{\mathfrak{D}}(\mathrm{Spec}(\mathbb{C}), \mathbb{R}(p+q-d)),$$

for any choice of Green current g_Z for Z and a Green current g_W for W satisfying (1.22). Here $p_X \colon X \to \mathrm{Spec}(\mathbb{C})$ is the structural morphism.

It can be shown ([BGG19, Proposition 7.20]) that the definition is independent of the choice of Green currents g_Z and g_W satisfying condition (1.22).

From the fact that $\omega(\mathfrak{g}_W)$ has been chosen to be zero, we get

$$\langle Z, W\rangle_{\mathrm{Arch}} = (-1)^n \sum_{j=0}^{m} p_* \left(\delta_Z \cdot W_n \cdot g'_j \cdot W_j\right)^{\sim},$$

where $p = p_X \circ \pi_X$. This pairing is graded commutative and linear on both components.

2. Oriented mixed Hodge structures and height

2.1. **The height of a mixed Hodge structure.** Let V be a $\mathbb{Q}$-vector space. A mixed Hodge structure (F, W) on V induces a unique functorial bigrading [CKS86, Theorem 2.13]

$$V_{\mathbb{C}} = \bigoplus_{a,b} I^{a,b} \tag{2.1}$$

of the underlying complex vector space $V_{\mathbb{C}}$ such that

(i) $F^a = \oplus_{\alpha\geq a,\beta} I^{\alpha,\beta}$;
(ii) $W_k = \oplus_{\alpha+\beta\leq k} I^{\alpha,\beta}$;
(iii) $\overline{I^{a,b}} \equiv I^{b,a} \mod \oplus_{\beta<b,\alpha<a} I^{\beta,\alpha}$.

The $I^{a,b}$ is given by

$$I^{a,b} = F^a \cap W_{a+b} \cap \left(\overline{F^b} \cap W_{a+b} + \overline{U^{b-1}_{a+b-2}}\right), \tag{2.2}$$

where

$$U^r_s = \sum_{j\geq 0} F^{r-j} \cap W_{s-j}.$$

Definition 2.1. The bigrading (2.1) will be called the *Deligne bigrading* of (F, W). The associated semisimple endomorphism $Y = Y_{(F,W)}$ of $V_{\mathbb{C}}$ which acts as multiplication by $p + q$ on $I^{p,q}$ will be called the *Deligne grading* of (F, W).

We will denote by Π_k the projector over $\operatorname{Gr}^W_k V_{\mathbb{C}} = \bigoplus_{a+b=k} I^{a,b}$ and $\Pi_{a,b}$ the projector over $I^{a,b}$. So, for instance, Π_k is the composition

$$V_{\mathbb{C}} \longrightarrow \operatorname{Gr}^W_k V_{\mathbb{C}} \hookrightarrow V_{\mathbb{C}}.$$

Moreover the semisimple endomorphism Y is given by

$$Y = \sum_{k\in\mathbb{Z}} k\Pi_k. \tag{2.3}$$

Let

$$\mathfrak{gl}(V_{\mathbb{C}})^{a,b} = \{\, \alpha \in \mathfrak{gl}(V_{\mathbb{C}}) \mid \alpha(I^{c,d}) \subseteq I^{a+c,b+d} \,\} \tag{2.4}$$

be the Hodge decomposition of $\mathfrak{gl}(V)$ and define

$$\Lambda^{-1,-1} = \bigoplus_{a<0,b<0} \mathfrak{gl}(V_{\mathbb{C}})^{a,b}. \tag{2.5}$$

Then, $\overline{\Lambda^{-1,-1}} = \Lambda^{-1,-1}$ [CKS86, Eq. 2.19]. For an element $\lambda \in \mathfrak{gl}(V_{\mathbb{C}})$ we will denote $\lambda = \sum \lambda^{a,b}$ its decomposition into Hodge components.

There exists a unique real element $\delta = \delta_{(F,W)} \in \Lambda^{-1,-1}$ such that

$$\overline{Y_{(F,W)}} = e^{-2i\delta} \cdot Y_{(F,W)} \tag{2.6}$$

where $g \cdot \alpha := \mathrm{Ad}(g)\alpha$ denotes the adjoint action of $\mathrm{GL}(V_{\mathbb{C}})$ on $\mathfrak{gl}(V_{\mathbb{C}})$, [CKS86, Prop 2.20]. The element δ defined by (2.6) will be called the *Deligne splitting* of (F, W).

For an element $g \in \mathrm{GL}(V_{\mathbb{C}})$ we will denote by $g \cdot F$ the filtration given by $(g \cdot F)^p V_{\mathbb{C}} = g(F^p V_{\mathbb{C}})$. In general if (F, W) is a mixed Hodge structure on V, the pair of filtrations $(g \cdot F, W)$ do not form a mixed Hodge structure.

Lemma 2.2 ([Pea00, Lemma 4.11])**.** *Let (F, W) be a mixed Hodge structure on V and $\Lambda^{-1,-1}$ be the associated subalgebra* (2.5)*. Then, $\lambda \in \Lambda^{-1,-1}$ implies that $(e^{\lambda} \cdot F, W)$ is a mixed Hodge structure on V and that*

$$I^{p,q}_{(e^{\lambda}\cdot F,W)} = e^{\lambda}(I^{p,q}_{(F,W)}).$$

A choice of graded-polarization of (F, W) determines a hermitian inner product on $V_{\mathbb{C}}$ by declaring the bigrading $\oplus_{a,b} I^{a,b}$ to be orthogonal and defining the inner product on $I^{a,b}$ using the isomorphism $I^{a,b} \cong H^{a,b}\,\mathrm{Gr}^W_{a+b}$ and the standard Hodge inner product on Gr^W_{a+b}. In this way, we can attach a collection of heights to (F, W) via the norms of the Hodge components $\delta^{a,b}$ of δ [Pea06a, §5.1]. To attach a signed height to (F, W), we need a notion of orientation.

Definition 2.3. Given a mixed Hodge structure $H = (F, W)$ on V, define

$$\max(H) = \max\{k \mid \mathrm{Gr}^W_k(V) \neq 0\}, \quad \min(H) = \min\{k \mid \mathrm{Gr}^W_k(V) \neq 0\}.$$

and define the *length* of H as

$$\ell(H) = \max(H) - \min(H).$$

We say that H is *orientable* if $\mathrm{Gr}^W_{\max(H)}(V)$ and $\mathrm{Gr}^W_{\min(H)}(V)$ are both of rank 1. This implies that $\max(H)$ and $\min(H)$ are both even and, writing $a = \max(H)/2$ and $c = \min(H)/2$, that

$$\mathrm{Gr}^W_{\max(H)}(V) \cong \mathbb{Q}(-a), \quad \mathrm{Gr}^W_{\min(H)}(V) \cong \mathbb{Q}(-c). \tag{2.7}$$

If H is orientable, an *orientation* of H consists of a choice of Betti generators $\mathbb{1}_H$ of $\mathrm{Gr}^W_{\max(H)}(V)$ and $\mathbb{1}^{\vee}_H$ of $\mathrm{Gr}^W_{\min(H)}(V)$. Equivalently, an orientation is a choice of the isomorphisms (2.7). Given an orientation of H we define a signed height by the formula:

$$\delta^{r,r}_H(e) = \mathrm{ht}(H)e^{\vee}, \qquad r = -\ell(H)/2, \tag{2.8}$$

where e is the element of $I^{a,a} \subset V_{\mathbb{C}}$ which projects to $\mathbb{1}_H \in \mathrm{Gr}^W_{\max(H)}(V)$ and $e^{\vee}$ is the image of $\mathbb{1}^{\vee}_H$ in $W_{\min(H)}V_{\mathbb{C}}$.

Remark 2.4. The height functions considered above only depend on the underlying $\mathbb{R}$-mixed Hodge structure.

Definition 2.5. Let H be an oriented mixed Hodge structure on V. We say that H is a *generalized biextension* if H has at most three non trivial weights.

Therefore, if H is a generalized biextension, there are three integers $2a > b > 2c$, and a pure Hodge structure H_b of weight b such that

$$\operatorname{Gr}_k^W(V) = \begin{cases} \mathbb{Q}(-a), & \text{if } k = 2a, \\ H_b, & \text{if } k = b, \\ \mathbb{Q}(-c), & \text{if } k = 2c, \\ 0, & \text{otherwise.} \end{cases}$$

Note that H_b may be zero.

Lemma 2.6. *Let $H = (F, W)$ be a generalized biextension and a, b, c as before. Let $e \in I^{a,a}$ be the unique element that maps to the generator $\mathbb{1}_H$ and $e^\vee$ the image of $\mathbb{1}_H^\vee$ in $I^{c,c}$. Then,*

$$\operatorname{ht}(H)e^\vee = \frac{1}{2}\operatorname{Im}\left(\Pi_{2c}(e - \bar{e})\right).$$

Proof. Write $k_1 = 2a$, $k_2 = b$ and $k_3 = 2c$ for the different weights of H and let $Y = Y_{(F,W)}$ and $\delta = \delta_{(F,W)}$. Since $\delta \in \Lambda^{1,1}$, there is a decomposition $\delta = \delta_1 + \delta_2 + \delta_3$, with

$$\delta_1 = \Pi_{k_2} \circ \delta \circ \Pi_{k_1}, \quad \delta_2 = \Pi_{k_3} \circ \delta \circ \Pi_{k_2}, \quad \delta_3 = \Pi_{k_3} \circ \delta \circ \Pi_{k_1}.$$

The decomposition (2.3) and the fact that the projectors Π_k are orthogonal imply

$$[Y, \delta_1] = (k_2 - k_1)\delta_1, \quad [Y, \delta_2] = (k_3 - k_2)\delta_2, \quad [Y, \delta_3] = (k_3 - k_1)\delta_3.$$

In particular,

$$\begin{aligned}[][\delta, [\delta, Y]] &= [\delta_1 + \delta_2 + \delta_3, (k_1 - k_2)\delta_1 + (k_2 - k_3)\delta_2 + (k_1 - k_3)\delta_3] \\ &= (k_1 + k_3 - 2k_2)\delta_2 \circ \delta_1.\end{aligned}$$

Therefore,

$$\begin{gathered}\overline{Y} = e^{-2i\delta} \cdot Y = Y - 2i[\delta, Y] - 2[\delta, [\delta, Y]] = \\ Y - 2i((k_1 - k_2)\delta_1 + (k_2 - k_3)\delta_2 + (k_1 - k_3)\delta_3) - 2(k_1 + k_3 - 2k_2)\delta_2 \circ \delta_1.\end{gathered} \tag{2.9}$$

Since $e \in I^{a,a}$ is a lift of $\mathbb{1}(-a)_{\mathbb{Q}} \in \mathbb{Q}(-a)_{\mathbb{Q}}$ and $\overline{\mathbb{1}(-a)_{\mathbb{Q}}} = \mathbb{1}(-a)_{\mathbb{Q}}$, we can write

$$\bar{e} = e + a_{k_2} + a_{k_3}, \tag{2.10}$$

where $Y(a_j) = ja_j$. We now compute

$$\begin{aligned}\overline{Y}(e) &= \overline{Y(\bar{e})} = \overline{Y(e + a_{k_2} + a_{k_3})} \\ &= k_1\bar{e} + k_2\bar{a}_{k_2} + k_3\bar{a}_{k_3} = k_1 e + k_1 a_{k_2} + k_1 a_{k_3} + k_2\bar{a}_{k_2} + k_3\bar{a}_{k_3}.\end{aligned} \tag{2.11}$$

On the other hand, by equation (2.9),

$$\overline{Y}(e) = k_1 e - 2i(k_1 - k_2)\delta_1(e) - 2i(k_1 - k_3)\delta_3(e) - 2(k_1 + k_3 - 2k_2)\delta_2(\delta_1(e)). \tag{2.12}$$

By (2.10) we deduce

$$\bar{a}_{k_2} = -a_{k_2} - a_{k_3} - \bar{a}_{k_3}. \tag{2.13}$$

By equations (2.11), (2.12) and (2.13) and using the splitting, we deduce the equations

$$(k_1 - k_2)a_{k_2} = -2i(k_1 - k_2)\delta_1(e), \tag{2.14}$$

$$(k_1 - k_2)a_{k_3} + (k_3 - k_2)\bar{a}_{k_3} = 2i(k_3 - k_1)\delta_3(e) - 2(k_1 + k_3 - 2k_2)\delta_2(\delta_1(e)). \tag{2.15}$$

From equation (2.14), taking into account that $k_1 - k_2 \neq 0$, we obtain

$$\delta_1(e) = \frac{i}{2}a_{k_2}. \tag{2.16}$$

Applying δ_2 to equation (2.16) we get

$$\delta_2(a_{k_2}) = -2i\delta_2(\delta_1(e)). \tag{2.17}$$

Computing $\overline{Y}(a_{k_2})$ in two ways as we have done with $\overline{Y}(e)$ yields the equation

$$-2i\delta_2(a_{k_2}) = a_{k_3} + \bar{a}_{k_3}. \tag{2.18}$$

Combining equations (2.15), (2.17) and (2.18) gives

$$\delta_3(e) = \frac{-1}{2}\frac{a_{k_3} - \bar{a}_{k_3}}{2i},$$

which is equivalent to the lemma. □

2.2. **Some residual results.** We next study the effect of a morphism of mixed Hodge structures on the height we have defined. To this end we first recall the compatibility of the Deligne splitting with morphism of mixed Hodge structures.

Lemma 2.7. *Let A and B be mixed Hodge structures with Deligne splittings δ_A and δ_B respectively. Let $f : A \to B$ be a morphism of mixed Hodge structures. Then, $f \circ \delta_A = \delta_B \circ f$.*

Proof. By [CKS86, Prop 2.20] if C is a mixed Hodge structure then δ_C commutes with all (r,r)-morphisms of C. Let $C = A \oplus B$ and observe that $g(a,b) = (a, b + f(a))$ is a morphism of C. Using the block structure of $\mathfrak{gl}(C) = \mathfrak{gl}(A \oplus B)$ it follows immediately from (2.6) that $\delta_C(a,b) = (\delta_A(a), \delta_B(b))$. Writing out the $g \circ \delta_C = \delta_C \circ g$ shows that $f \circ \delta_A = \delta_B \circ f$. □

Proposition 2.8. *Let A and B be oriented mixed Hodge structures such that $\max(A) = \max(B)$ and $\min(A) = \min(B)$. Let $f : A \to B$ be a morphism of mixed Hodge structures which is injective on $\mathrm{Gr}^W_{\max(A)}$ and $\mathrm{Gr}^W_{\min(A)}$. Then,*

$$\mathrm{ht}(A)d_{min}(f) = \mathrm{ht}(B)d_{max}(f),$$

where $f(\mathbb{1}_A) = d_{max}(f)\mathbb{1}_B$ and $f(\mathbb{1}_A^\vee) = d_{min}(f)\mathbb{1}_B^\vee$.

Proof. Let e_A be a lift of $\mathbb{1}_A$ and $e_A^\vee$ the image of $\mathbb{1}_A^\vee$. Then, $f(e_A) = d_{max}(f)e_B$ where e_B is a lift of $\mathbb{1}_B$. Likewise, $f(e_A^\vee) = d_{min}(f)e_B^\vee$ where $e_B^\vee$ is the image of to $\mathbb{1}_B^\vee$. Moreover, since f is of type $(0,0)$, then $f \circ \delta_A = \delta_B \circ f$

implies that $f \circ \delta_A^{r,r} = \delta_B^{r,r} \circ f$ for any r. Setting $r = (\min(A) - \max(A))/2$ it follows that

$$\begin{array}{ccc} f \circ \delta_A^{r,r}(e_A) & = & \delta_B^{r,r} \circ f(e_A) \\ \| & & \| \\ f(\mathrm{ht}(A)e_A^\vee) & = & \delta_B^{r,r}(d_{max}(f)e_B) \\ \| & & \| \\ \mathrm{ht}(A)d_{min}(f)e_B^\vee & = & \mathrm{ht}(B)d_{max}(f)e_B^\vee. \end{array}$$

□

Example 2.9. We put Proposition 2.8 in practice for usual cycles. Let X and Y be smooth projective varieties of dimensions d_X and d_Y respectively. Let $Z \in Z^p_{\mathrm{hom}}(X)$, $W \in Z^q_{\mathrm{hom}}(Y)$ and $\Gamma \in Z^{d_X+r}(X \times Y)$ be a correspondence of *degree* r, such that $p+q+r = d_Y+1$. We assume that the pullbacks of Z and W intersect Γ properly, so that $\Gamma_*(Z)$ and $\Gamma^*(W) := \Gamma^t_*(W)$ are both defined at the level of cycles. Let $B_{Z,\Gamma^*(W)}$ and $B_{\Gamma_*(Z),W}$ be oriented biextensions as defined by Hain in [Hai90], of graded weights $0, -1, -2$. One can show that Γ defines a morphism of Hodge structures between these biextensions

$$\Gamma_{Z,W} \colon B_{Z,\Gamma^*(W)} \to B_{\Gamma_*(Z),W},$$

with $d_{max}(\Gamma_{Z,W}) = d_{min}(\Gamma_{Z,W}) = 1$. Hence we get

$$\mathrm{ht}(B_{Z,\Gamma^*(W)}) = \mathrm{ht}(B_{\Gamma_*(Z),W}).$$

For later use, we record the following:

Lemma 2.10. *Let N be a $(-1,-1)$-morphism of a mixed Hodge structure (F, W). Then, $\delta_{(e^{tN}\cdot F,W)} = \delta_{(F,W)} + \mathrm{Im}(t)N$.*

Proof. By [CKS86, Prop. 2.20], N and $\delta_{(F,W)}$ commute. Therefore, using Lemma (2.2):

$$\begin{aligned} e^{(\bar{t}-t)N-2i\delta_{(F,W)}} \cdot Y_{(e^{tN}\cdot F,W)} &= e^{\bar{t}N} e^{-2i\delta_{(F,W)}} e^{-tN} \cdot Y_{(e^{tN}\cdot F,W)} \\ &= e^{\bar{t}N} e^{-2i\delta_{(F,W)}} e^{-tN} e^{tN} \cdot Y_{(F,W)} \\ &= e^{\bar{t}N} e^{-2i\delta_{(F,W)}} \cdot Y_{(F,W)} \\ &= e^{\bar{t}N} \cdot \overline{Y_{(F,W)}} \\ &= \overline{e^{tN} \cdot Y_{(F,W)}} \\ &= \overline{Y_{(e^{tN}\cdot F,W)}}. \end{aligned}$$

Accordingly, by (2.6) $-2i\delta_{(e^{tN}\cdot F,W)} = (\bar{t} - t)N - 2i\delta_{(F,W)}$ which implies the stated formula after dividing by $-2i$. □

Corollary 2.11. *Let N be a $(-1,-1)$-morphism of a mixed Hodge structure (F, W) and $r = (\min(F, W) - \max(F, W))/2$. If $r < -1$ then*

$$\mathrm{ht}(e^{tN} \cdot F, W) = \mathrm{ht}(F, W)$$

for all $t \in \mathbb{C}$.

Proof. By Lemma (2.10), $\delta^{r,r}_{(e^{tN}\cdot F,W)} = \delta^{r,r}_{(F,W)}$ and hence the two mixed Hodge structures have the same height. □

2.3. **Dual of a mixed Hodge structure.** A real mixed Hodge structure A induces a mixed Hodge structure A^* on the dual vector space $A^*_{\mathbb{R}}$ by the formula

$$I^{a,b}_{A^*} = \{\, \lambda \in A^*_{\mathbb{C}} \mid \lambda(I^{c,d}) = 0, \ \ (c,d) \neq (-a,-b) \,\}. \tag{2.19}$$

If $\alpha \in \mathfrak{gl}(A_{\mathbb{C}})$ then $\alpha^T \in \mathfrak{gl}(A^*_{\mathbb{C}})$ is the linear map $(\alpha^T(\lambda))(v) = \lambda(\alpha(v))$ for all $\lambda \in A^*_{\mathbb{C}}$ and $v \in A_{\mathbb{C}}$. A short calculation shows that if $\alpha \in \mathfrak{gl}(A_{\mathbb{C}})^{a,b}$ then $\alpha^T \in \mathfrak{gl}(A^*_{\mathbb{C}})^{a,b}$. Tracing through the definitions, one sees that the Deligne grading Y_A of A and Y_{A^*} of A^* are related by the formula

$$Y_{A^*} = -Y_A^T. \tag{2.20}$$

It follows from equations (2.20) and (2.6) that

$$\delta_{A^*} = -\delta_A^T. \tag{2.21}$$

Indeed, since $\operatorname{ad}(X_1^T)\cdots\operatorname{ad}(X_{r-1}^T)X_r^T = (-1)^{r-1}\{\operatorname{ad}(X_1)\cdots\operatorname{ad}(X_{r-1})X_r\}^T$ it follows that

$$\begin{aligned}
e^{-2i\ \operatorname{ad}(-\delta_A^T)}Y_{A^*} &= e^{-2i\ \operatorname{ad}(-\delta_A^T)}(-Y_A^T)\\
&= -\sum_{m\geq 0}\frac{1}{m!}(2i\ \operatorname{ad}(\delta_A^T))^m Y_A^T\\
&= -\sum_{m\geq 0}\frac{(-2i)^m}{m!}((\operatorname{ad}(\delta_A)^m)Y_A)^T\\
&= -\left(\exp(-2i\ \operatorname{ad}(\delta_A))Y_A\right)^T = -\overline{Y_A}^T = \overline{-Y_A^T} = \overline{Y_{A^*}}.
\end{aligned}$$

since the operations of transpose and complex conjugation commute. Therefore, $\delta_{A^*} = -\delta_A^T$ by (2.6).

Now if H is a generalized biextension as defined in Definition 2.5, then its dual H^* is also a generalized biextension with

$$\operatorname{Gr}^W_k(V^*) = \begin{cases} \mathbb{Q}(c), & \text{if } k = -2c,\\ H_b^*, & \text{if } k = -b,\\ \mathbb{Q}(a), & \text{if } k = -2a,\\ 0, & \text{otherwise.}\end{cases}$$

We have the following relation between the heights of H and H^*:

Proposition 2.12. *Let H be a generalized biextension. Then*

$$\operatorname{ht}(H^*) = -\operatorname{ht}(H).$$

Proof. By the definition of the dual of an oriented biextension, the generators of H ad H^* satisfy

$$\langle \mathbb{1}_H, \mathbb{1}^\vee_{H^*}\rangle = 1, \qquad \langle \mathbb{1}^\vee_H, \mathbb{1}_{H^*}\rangle = 1.$$

Let e_H be an element of $I^{a,a}_H \subset V_{\mathbb{C}}$ which projects to $\mathbb{1}_H \in \operatorname{Gr}^W_{2a}(V)$ and $e^\vee_H$ is the image of $\mathbb{1}^\vee_H$ in $W_{2c}V_{\mathbb{C}}$. Correspondingly, for H^* we have elements e_{H^*} and $e^\vee_{H^*}$. These elements also satisfy

$$\langle e_{H^*}, e^\vee_H\rangle = 1, \qquad \langle e^\vee_{H^*}, e_H\rangle = 1.$$

Also, since $\delta^{r,r}_{H^*} = -(\delta^{r,r}_H)^T$, we get

$$\operatorname{ht}(H^*)e^\vee_{H^*} = \delta^{r,r}_{H^*}(e_{H^*}) = -(\delta^{r,r}_H)^T(e_{H^*}).$$

Hence
$$\operatorname{ht}(H^*) = \langle -(\delta_H^{r,r})^T(e_{H^*}), e_H\rangle.$$
Finally, using the action of $(\delta_H^{r,r})^T$, we get
$$\operatorname{ht}(H^*) = -\operatorname{ht}(H)\langle e_{H^*}, e_H^\vee\rangle = -\operatorname{ht}(H).$$
□

3. Mixed Hodge structures associated to higher cycles.

In this section we define extension classes for higher cycles $Z \in Z^p(X,n)_{00}$ in the refined normalized complex. For two higher cycles $Z \in Z^p(X,n)_{00}$ and $W \in Z^q(X,m)_{00}$, with $2(p+q-d-1) = n+m$, we construct, under certain assumptions, an oriented mixed Hodge structure diagram (figure 2) which captures both the extension related to cycle Z and the dual to the extension related to W. In an even more special situation for $n = m = 1$, this diagram defines an oriented biextension.

3.1. Two divisors on $(\mathbb{P}^1)^n$.

Definition 3.1. On $(\mathbb{P}^1)^n$, we define the following divisors:
$$A = \{(t_1,\cdots,t_n) \mid \exists i, t_i = 1\},$$
$$B = \{(t_1,\cdots,t_n) \mid \exists i, t_i \in \{0,\infty\}\}.$$
Then $A \cup B$ is a simple normal crossing divisor. Moreover
$$(\mathbb{P}^1)^n \setminus A = \square^n, \quad (\mathbb{P}^1)^n \setminus B = (\mathbb{C}^*)^n \text{ and } B \cap \square^n = \partial\square^n.$$
For any variety X we also denote
$$A_X := X \times A, \qquad B_X := X \times B.$$

The following cohomology groups are easy to compute.

$$H^r((\mathbb{P}^1)^n \setminus A, B) = \begin{cases} 0, & \text{if } r \neq n, \\ \mathbb{Q}(0), & \text{if } r = n. \end{cases} \tag{3.1}$$

$$H^r((\mathbb{P}^1)^n \setminus B, A) = \begin{cases} 0, & \text{if } r \neq n, \\ \mathbb{Q}(-n), & \text{if } r = n. \end{cases} \tag{3.2}$$

In order to fix the isomorphism (3.2) we choose the generator of $H^n((\mathbb{P}^1)^n \setminus B, A; n)_{\mathbb{Q}}$, that is represented by the differential form
$$(-1)^n \frac{dt_1}{t_1} \wedge \cdots \wedge \frac{dt_n}{t_n} \in F^0\Sigma_A E^n_{(\mathbb{P}^1)^n}(\log B; n), \tag{3.3}$$
where t_i is the coordinate of the $\mathbb{P}^1$ in position i. This choice also fixes the isomorphism (3.1). The reason of the sign $(-1)^n$ is to make it compatible with the normalizations chosen in [BGG19]. See for instance Proposition 3.8 below. The Künneth formula and the computations (3.1) and (3.2) produce, for $a, r \in \mathbb{Z}$, isomorphisms of mixed Hodge structures
$$H^r(X \times (\mathbb{P}^1)^n \setminus A_X, B_X; a) \cong H^{r-n}(X, a), \tag{3.4}$$
$$H^r(X \times (\mathbb{P}^1)^n \setminus B_X, A_X; a) \cong H^{r-n}(X, a-n). \tag{3.5}$$

Since A_X and B_X are in product situation (see [BKV15, Lemma 6.1.1 and Remark 6.1.2]), the above isomorphisms are compatible with duality

$$H^r(X \times (\mathbb{P}^1)^n \setminus A_X, B_X, \mathbb{Q}(p)) \cong \Big(H^{2d+2n-r}\left(X \times (\mathbb{P}^1)^n \setminus B_X, A_X, \mathbb{Q}(d+n-p)\right)\Big)^\vee .$$

We fix the isomorphism (3.4) using the generator (3.3) and Proposition 1.21.

Definition 3.2. For any $a, r, n \in \mathbb{Z}$, we denote by

$$\Psi \colon H^r \colon H^r(X \times (\mathbb{P}^1)^n \setminus A_X, B_X; a) \longrightarrow H^{r-n}(X, a)$$

the isomorphism determined by the generator (3.3). This isomorphism sends the class of a closed form

$$\omega \in \Sigma_{B_X} E^r_{X\times(\mathbb{P}^1)^n}(\log A_X)$$

to the class represented by the current

$$(-1)^n(\pi_X)_*\left[\omega \wedge \frac{dt_1}{t_1} \wedge \cdots \wedge \frac{dt_n}{t_n}\right] = \left[\frac{(-1)^n}{(2\pi i)^n}\int_{(\mathbb{P}^1)^n} \omega \wedge \frac{dt_1}{t_1} \wedge \cdots \wedge \frac{dt_n}{t_n}\right],$$

where $\pi_X \colon X \times (\mathbb{P}^1)^n$ is the first projection.

3.2. **The extension associated to a higher cycle.** In this section we show how to associate, to a cycle $Z \in Z^p(X, n)_{00}$, $n \geq 1$, an extension

$$e_Z \in \operatorname{Ext}^1_{\mathbb{Q}-\mathbf{MHS}}(\mathbb{Q}(0), H^{2p-n-1}(X; p)).$$

By definition Z is a codimension p algebraic cycle in $X \times (\mathbb{P}^1)^n \setminus A_X$, which intersects properly all the faces of $B_X \setminus (A_X \cap B_X)$. We write

$$B_X = B_{01} \cup \cdots \cup B_{0n} \cup B_{\infty 1} \cup \cdots \cup B_{\infty n},$$

as the decomposition of B_X into irreducible components.

Since $Z \in Z^p(X, n)_{00}$, we have $Z\cdot(B_{ij}\setminus(A_X \cap B_{ij}))$ well defined. Moreover, Z being a higher cycle, we have

$$Z \cdot (B_{ij} \setminus (A_X \cap B_{ij})) = 0,\ \forall i = 0, \infty,\ \forall j = 1, \cdots, n. \tag{3.6}$$

We denote by $\overline{Z}$ the closure of Z as an algebraic cycle in $X \times (\mathbb{P}^1)^n$. There is a cycle class with support

$$\operatorname{cl}(\overline{Z}) \in H^{2p}_{|\overline{Z}|}(X \times (\mathbb{P}^1)^n; p)_{\mathbb{Q}}$$

and, by restriction, a class

$$\operatorname{cl}(Z) \in H^{2p}_{|Z|}(X \times (\mathbb{P}^1)^n \setminus A_X; p)_{\mathbb{Q}}.$$

Now we have the following

Proposition 3.3. *Under the above setting, there is a unique cycle class*

$$[Z] \in H^{2p}_{|Z|\setminus A_X}\left(X \times (\mathbb{P}^1)^n \setminus A_X, B_X \setminus A_X; p\right)_{\mathbb{Q}},$$

that is sent to $\operatorname{cl}(Z)$ *under the obvious map*

$$H^{2p}_{|Z|\setminus A_X}\left(X \times (\mathbb{P}^1)^n \setminus A_X, B_X \setminus A_X; p\right) \to H^{2p}_{|Z|\setminus A_X}\left(X \times (\mathbb{P}^1)^n \setminus A_X; p\right).$$

Proof. Consider the long exact sequence of relative cohomology with supports

$$\begin{aligned}\cdots \to H^{2p-1}_{(|Z|\cap B_X)\setminus A_X}(B_X \setminus A_X; p) &\to H^{2p}_{|Z|\setminus A_X}(X \times (\mathbb{P}^1)^n \setminus A_X, B_X \setminus A_X; p)\\ \to H^{2p}_{|Z|\setminus A_X}(X \times (\mathbb{P}^1)^n \setminus A_X; p) &\to H^{2p}_{(|Z|\cap B_X)\setminus A_X}(B_X \setminus A_X; p) \to \cdots\end{aligned}$$

The proof will follow if we show

(i) $H^{2p-1}_{(|Z|\cap B_X)\setminus A_X}(B_X \setminus A_X; p) = 0$,

(ii) $\mathrm{cl}(Z) \mapsto 0$, in $H^{2p}_{(|Z|\cap B_X)\setminus A_X}(B_X \setminus A_X; p)$.

Notice that for *(i)*, we cannot use semi-purity directly since $B_X \setminus A_X$ is not smooth. Instead we use the following lemma.

Lemma 3.4. *Let D be a complex space that can be covered by a finite number of smooth closed subvarieties. That is $D = \cup_{i=1}^r D_i$, with D_i Zariski closed and smooth. Put $D_I = \cap_{i\in I} D_i$ for $I \subseteq \{1, \cdots, r\}$, and let Z be a Zariski closed subset such that $Z \cap D_I$ has codimension p for all I. Then*

$$H^k_Z(D; p) = 0, \text{ for all } k < 2p$$

and the map

$$H^{2p}_Z(D; p) \longrightarrow \bigoplus_{i=1}^{r} H^{2p}_{Z\cap D_i}(D_i; p)$$

is a monomorphism.

Proof. The Mayer-Vietoris property for closed coverings gives a first quadrant spectral sequence

$$E_1^{a,b} = \bigoplus_{|I|=a+1} H^b_{Z\cap D_I}(D_I; p) \Longrightarrow H^{a+b}_Z(D; p).$$

Each D_I is smooth and $\mathrm{codim}(Z \cap D_I) = p$. Hence using semi-purity we conclude that $H^b_{Z\cap D_I}(D_I; p) = 0$ for $b < 2p$. Since $a \geq 0$, the first statement follows. The second statement is just the fact that edge morphism of a spectral sequence is a monomorphism. □

The first statement of Lemma 3.4, implies directly condition *(i)*.

The property (3.6) implies that the the class $\mathrm{cl}(Z)$ is sent to 0 in all the groups $H^{2p}_{|Z|\cap B_{ij}}(B_{ij} \setminus A_X; p)$. Therefore condition *(ii)* follows from the second statement of Lemma 3.4. □

Lemma 3.5. *For $Z \in Z^p(X, n)_{00}$, the image of the class $[Z]$ in*

$$H^{2p}(X \times (\mathbb{P}^1)^n \setminus A_X, B_X; p),$$

is zero.

Proof. By the isomorphism (3.4) we know that the mixed Hodge structure

$$H^{2p}(X \times (\mathbb{P}^1)^n \setminus A_X, B_X; p) \cong H^{2p-n}(X; p)$$

is pure of weight $-n$. Since the image of the class $[Z]$ belongs to

$$F^0 H^{2p-n}(X; p)_{\mathbb{C}} \cap H^{2p-n}(X; p)_{\mathbb{R}}.$$

Since in a pure Hodge structure of weight $-n < 0$ this group is zero, we conclude the result. □

There is a long exact sequence of mixed Hodge structures

$$(3.7)\quad 0 \to H^{2p-n-1}(X;p) \to H^{2p-1}(X\times(\mathbb{P}^1)^n\setminus A_X\cup|Z|, B_X;p) \to H^{2p}_{|Z|\setminus A_X}(X\times(\mathbb{P}^1)^n\setminus A_X, B_X;p) \to H^{2p}(X\times(\mathbb{P}^1)^n\setminus A_X, B_X;p)\to\cdots,$$

where the zero on the left hand side follows from

$$H^{2p-1}_{|Z|}(X\times(\mathbb{P}^1)^n\setminus A_X, B_X;p)=0 \qquad \text{(semi-purity)}.$$

By Proposition 3.3 and Lemma 3.5, the cycle class $[Z]$ defines a map

$$\phi_Z\colon \mathbb{Q}(0)\longrightarrow H^{2p}_{|Z|\setminus A_X}(X\times(\mathbb{P}^1)^n\setminus A_X, B_X;p), \tag{3.8}$$

whose image of ϕ_Z in $H^{2p}(X\times(\mathbb{P}^1)^n\setminus A_X, B_X;p)$ is zero. Therefore, pulling back the above long exact sequence through ϕ_Z, we get an extension

$$0\longrightarrow H^{2p-n-1}(X;p)\longrightarrow E_Z\longrightarrow \mathbb{Q}(0)\longrightarrow 0. \tag{3.9}$$

By abuse of notation, we also denote as

$$E_Z \coloneqq \left[0\to H^{2p-n-1}(X;p)\to E_Z\to\mathbb{Q}(0)\to 0\right]$$

the class of this extension in $\operatorname{Ext}^1_{\mathbb{Q}-\mathbf{MHS}}\left(\mathbb{Q}(0), H^{2p-n-1}(X;p)\right)$.

3.3. **Differential forms attached to the extension E_Z.** The extension E_Z induces an extension

$$E_{Z,\mathbb{R}}\in \operatorname{Ext}^1_{\mathbb{R}-\mathbf{MHS}}\left(\mathbb{R}(0), H^{2p-n-1}(X;p)\right).$$

For shorthand we write $H=H^{2p-n-1}(X;p)$, that is a mixed Hodge structure pure of weight $-n$. Recall that there is an isomorphism

$$\operatorname{Ext}^1_{\mathbb{R}-\mathbf{MHS}}(\mathbb{R}(0),H)\xrightarrow{\simeq}\frac{H_{\mathbb{C}}}{F^0_{\mathbb{C}}+H_{\mathbb{R}}}. \tag{3.10}$$

This isomorphism works as follows. Let $E\in\operatorname{Ext}^1_{\mathbb{R}-\mathbf{MHS}}$, so E is the class of a short exact sequence

$$0\to H\to E\to\mathbb{R}(0)\to 0.$$

Let $\mathbb{1}(0)$ be the canonical generator of $\mathbb{R}(0)$. Choose $v\in F^0E$ an element that is sent to $\mathbb{1}(0)$. Then $h=(v-\overline{v})/2$ is sent to zero in $\mathbb{R}(0)$ and therefore belongs to H. The class of h in the quotient at the right hand side of (3.10) does not depend on the choice of v and represents the image of E under the isomorphism (3.10). In this section, given an element $h\in H_{\mathbb{C}}$, we will denote by

$$\widetilde{h}\in\frac{H_{\mathbb{C}}}{F^0_{\mathbb{C}}+H_{\mathbb{R}}} \tag{3.11}$$

its class in the quotient.

We will now construct several differential forms related to the extension $E_{Z,\mathbb{R}}$ and, in particular a representative of its class. To this end we will use the complexes of differential forms with zeros and logarithmic poles

$$\Sigma_{B_X}E^*_{X\times(\mathbb{P}^1)^n}(\log A_X;p),\ \text{ and }\Sigma_{B_X}E^*_{X\times(\mathbb{P}^1)^n}(\log A_X\cup|Z|;p).$$

The relevance of these complexes is clear because, for instance the class $[Z]$ belongs to

$$F^0H^{2p}_{|Z|}(X\times(\mathbb{P}^1)^n\setminus A_X, B_X;p)_{\mathbb{C}}$$

And the underlying cohomology group can be computed using the simple of the morphism of complexes

$$\Sigma_{B_X} E^*_{X\times(\mathbb{P}^1)^n}(\log A_X;p) \xrightarrow{\iota} \Sigma_{B_X} E^*_{X\times(\mathbb{P}^1)^n}(\log A_X \cup |Z|;p). \tag{3.12}$$

Proposition 3.6. *Let X and Z be as in the previous section. Then there are differential forms*

(i) $\eta_Z \in F^0\Sigma_{B_X} E^{2p-1}_{X\times(\mathbb{P}^1)^n}(\log A_X \cup |Z|;p)$ *such that* $d\eta_Z = 0$ *so the pair* $(0,\eta_Z)$ *is a cycle in the simple* $\mathrm{s}(\iota)$ *and the corresponding class satisfies*

$$\{(0,\eta_Z)\} = [Z] \in H^{2p}_{|Z|}(X\times(\mathbb{P}^1)^n \setminus A_X, B_X;p)_{\mathbb{C}}. \tag{3.13}$$

Moreover, on the complex of currents $D^*_{X\times(\mathbb{P}^1)^n/A_X}$ *there is an equality of currents*

$$d[\eta_Z] + \delta_Z = 0. \tag{3.14}$$

(ii) $\theta_Z \in F^{-n}\Sigma_{B_X} E^{2p-1}_{X\times(\mathbb{P}^1)^n}(\log A_X;p)$ *with* $d\theta_Z = 0$ *and* $\overline{\theta}_Z = -\theta_Z$. *Moreover, if we denote by* $\widetilde{\{\theta_Z\}}$ *the image of the class* $\{\theta_Z\}$ *under the composition*

$$H^{2p-1}(X\times(\mathbb{P}^1)^n \setminus A_X, B_X;p)_{\mathbb{C}} \xrightarrow{\simeq} H_{\mathbb{C}} \to \frac{H_{\mathbb{C}}}{F^0H_{\mathbb{C}} + H_{\mathbb{R}}} = \mathrm{Ext}^1\left(\mathbb{R}(0), H\right),$$

where we have used again the shorthand $H = H^{2p-n-1}(X;p)$, *then*

$$\widetilde{\{\theta_Z\}} = E_{Z,\mathbb{R}}. \tag{3.15}$$

(iii) $g_Z \in F^{-1}\cap\overline{F}^{-1}\Sigma_{B_X} E^{2p-2}_{X\times(\mathbb{P}^1)^n}(\log A_X \cup |Z|;p)$ *satisfying* $\overline{g}_Z = -g_Z$ *and*

$$dg_Z = \frac{1}{2}(\eta_Z - \overline{\eta}_Z) - \theta_Z, \tag{3.16}$$

Remark 3.7. Before starting the proof, we recall how the notation in Definition 1.3 works. The conditions

$$g_Z \in F^{-1}\cap\overline{F}^{-1}\Sigma_{B_X} E^{2p-2}_{X\times(\mathbb{P}^1)^n}(\log A_X \cup |Z|;p), \text{ and } \overline{g}_Z = -g_Z$$

are equivalent to

$$g_Z \in \Sigma_{B_X} E^{p-1,p-1}_{X\times(\mathbb{P}^1)^n}(\log A_X \cup |Z|), \text{ and } \overline{g_Z}^{\mathrm{dR}} = (-1)^{p-1}g_Z,$$

where $\overline{g_Z}^{\mathrm{dR}}$ is the original conjugation of differential forms.

Proof of Proposition 3.6. We first note that the equality (3.15) is a consequence of (3.13) and (3.16). Recall the explicit construction the isomorphism (3.10) at the beginning of the section. The mixed Hodge structure E_Z is a substructure of $H^{2p-1}(X\times(\mathbb{P}^1)^n\setminus A_X\cup|Z|, B_X;p)$. Condition (3.13) implies that the class $\{\eta_Z\}$ belongs to $F^0E_{Z,\mathbb{C}}$ and is a choice of the class v. Then equation (3.16) implies that $\{\theta_Z\}$ agrees with the class $(v-\overline{v})/2$, and we deduce (3.15).

The class $[Z]$ belongs to $F^0H^{2p}_{|Z|}(X\times(\mathbb{P}^1)^n\setminus A_X, B_X;p)_{\mathbb{C}}$, and we compute the underlying cohomology group using the simple of morphism ι in (3.12). Therefore, there should be an element $(\alpha_1,\beta_1) \in F^0\,\mathrm{s}(\iota)$ that represents $[Z]$.

By Lemma 3.5 the form α_1 has to be exact. Since by Corollary 1.20 the differential d is strict with respect to the Hodge filtration we deduce that there is

$$\alpha_2 \in F^0 E^{2p-1}_{X\times(\mathbb{P}^1)^n}(\log A_X; p)$$

with $d\alpha_2 = \alpha_1$. Writing $\beta = \beta_1 - \alpha_2$ we deduce that $[Z]$ is represented by $(0,\beta) = (\alpha_1, \beta_1) - d(\alpha_2, 0)$ with

$$\beta \in F^0 \Sigma_{B_X} E^{2p-1}_{X\times(\mathbb{P}^1)^n}(\log A_X \cup |Z|; p).$$

Since the class $\{(0,\overline{\beta})\}$ also agrees with $[Z]$, we get

$$\{(0,\beta - \overline{\beta})\} = 0.$$

Hence

$$\begin{aligned}\{\beta - \overline{\beta}\} \in W_{-1}H^{2p-1}(&X \times (\mathbb{P}^1)^n \setminus A_X \cup |Z|, B_X; p)\\ &= H^{2p-1}(X \times (\mathbb{P}^1)^n \setminus A_X, B_X; p) = H.\end{aligned}$$

Since this last mixed Hodge structure is pure of weight $-n-1$, we can decompose

$$\{\beta - \overline{\beta}\}/2 = c - \overline{c} + t, \tag{3.17}$$

with

$$c \in F^0 H_{\mathbb{C}}, \qquad \overline{c} \in \overline{F}^0 H_{\mathbb{C}}, \qquad t \in F^{-n} H_{\mathbb{C}}$$

and $\overline{t} = -t$. The class c can be represented by a cycle

$$\gamma \in F^0 \Sigma_{B_X} E^{2p-1}_{X\times(\mathbb{P}^1)^n}(\log A_X; p).$$

Hence $\overline{\gamma}$ represents $\overline{c}$. Next choose a representative

$$\theta_1 \in F^{-n} \Sigma_{B_X} E^{2p-1}_{X\times(\mathbb{P}^1)^n}(\log A_X; p)$$

of t. As a form in $\Sigma_{B_X} E^{2p-1}_{X\times(\mathbb{P}^1)^n}(\log A_X)$, it has components of bidegree $(a, 2p-1-a)$ for $a \geq p-n$. We observe that $-\overline{\theta_1}$ also represents t. Hence, there is an $u \in \Sigma_{B_X} E^{2p-2}_{X\times(\mathbb{P}^1)^n}(\log A_X; p)$ such that $du = \theta_1 + \overline{\theta}_1$. Since the bidegrees of θ_1 and $\overline{\theta}_1$ only overlap in the range

$$(p-n, p+n-1), \dots, (p+n-1, p-n) \tag{3.18}$$

we see that some components of du will kill some components of θ_1. Let $F^{n-1}u$ denote the sum of the components of u of bidegree (a,b) with $a \geq p+n-1$. Then $\theta_2 \coloneqq \theta_1 - dF^{n-1}u$ only has components of bidegrees in the range (3.18). This implies that $\overline{\theta}_2$ belongs to F^{-n}

Writing $\theta_Z = (\theta_2 - \overline{\theta}_2)/2$ we obtain a differential form satisfying

$$\theta_Z \in F^{-n} \Sigma_{B_X} E^{2p-1}_{X\times(\mathbb{P}^1)^n}(\log A_X; p), \quad d\theta_Z = 0 \text{ and } \overline{\theta}_Z = -\theta_Z$$

and still representing t.

The decomposition (3.17) implies that there is form

$$g_1 \in \Sigma_{B_X} E^{2p-2}_{X\times(\mathbb{P}^1)^n}(\log A_X \cup |Z|; p)$$

such that

$$dg_1 = \frac{1}{2}\left((\beta - \overline{\beta}) - (\gamma - \overline{\gamma})\right) - \theta_Z$$

and $\overline{g}_1 = -g_1$. We decompose g_1 in bidegrees

$$g_1 = g_1^{p-1,p-1} + F^0 g_1 + \overline{F}^0 g_1.$$

and define

$$g_Z = g_1^{p-1,p-1} \quad \text{and} \quad \eta_Z = \beta - 2\gamma - 2dF^0 g_1.$$

By construction, equation (3.16) is satisfied. Therefore g_Z satisfies all the conditions of the theorem. On the other hand

$$(0, \eta_Z) = (0, \beta) + d(-2\gamma, 2F^0 g_1),$$

so η_Z satisfies condition (3.13). As explained in the beginning, this implies that θ_Z satisfies equation (3.15).

It remains to show equation (3.14). The argument is adapted from [BG94b, Theorem 4.4]. By construction of the class $[Z]$ we see that forgetting the vanishing at B_X, the pair $(0, \eta_Z)$ represents the class $\mathrm{cl}(Z) \in H^{2p}_{|Z|}(X \times (\mathbb{P}^1)^n \setminus A_X; p)_{\mathbb{C}}$. Using resolution of singularities we can construct a smooth complex variety $\widetilde{X}$, a normal crossing divisor D and a codimension p cycle Z' with $|Z'|$ smooth and intersecting transversely all intersections of components of D and a birational map $\pi \colon \widetilde{X} \to X \times (\mathbb{P}^1)^n$, such that $\pi_* Z' = Z$, D being the union of the exceptional divisor of π and the preimage of A_X. The cohomology group $H^{2p}_{|Z'|}(\widetilde{X} \setminus D; p)_{\mathbb{C}}$ can be computed as the simple of the morphism of complexes

$$D^*_{\widetilde{X}/D}(p) \xrightarrow{\iota'} D^*_{\widetilde{X}/(D \cup Z')}(p).$$

moreover there is a morphism of complexes $\mathrm{s}(\iota) \to \mathrm{s}(\iota')$ given by the commutative diagram

$$\begin{tikzcd}
E^*_{X \times (\mathbb{P}^1)^n}(\log A_X; p) \arrow[r, "\iota"] \arrow[d] & \Sigma_{B_X} E^*_{X \times (\mathbb{P}^1)^n}(\log A_X \cup |Z|; p) \arrow[d] \\
D^*_{\widetilde{X}/D}(p) \arrow[r, "\iota'"] & D^*_{\widetilde{X}/(D \cup Z')}(p).
\end{tikzcd}$$

In the complex $\mathrm{s}(\iota)$ the class $\mathrm{cl}(Z')$ is represented by the pair $(\delta_{Z'}, 0)$. Therefore there are currents u, v such that

$$(\delta_{Z'}, 0) - (0, [\pi^* \eta_Z]) = d(u, v) = (du, u - dv).$$

Hence

$$\delta_{Z'} = du, \qquad [\pi^* \eta_Z] = dv - u,$$

which implies the result, thanks to the projection formula. $\square$

3.4. **The class of the extension and Goncharov regulator.** In this section we will use the form θ_Z to relate the class of E_Z with the Goncharov regulator $\mathcal{P}(Z)$ of section 1.9.

Proposition 3.8. *Let $\mathcal{P}$ be the cubical Goncharov regulator normalized as in* [BGG19, Definition 5.1] *and Ψ the isomorphism of Definition 3.2. Under the isomorphism*

$$H^{2p-n}_{\mathfrak{D}}(X, \mathbb{R}(p)) \xrightarrow{\cong} \frac{H^{2p-n-1}(X, \mathbb{C})}{F^p H^{2p-n-1}(X, \mathbb{C}) + H^{2p-n-1}(X, \mathbb{R}(p))}, \tag{3.19}$$

the class $\mathcal{P}(Z)$ is mapped to $\widetilde{\Psi(\theta_Z)}$.

Proof. In this proof, to compute real Deligne cohomology we use the Thom-Whitney Deligne complex $\mathfrak{D}_{\mathrm{TW}}$ of Section 1.7 (see [BGG19, Definition 4.14]). This complex has the advantage to have a well defined graded commutative and associative product.

From the forms constructed in Proposition 3.6 we can define the following Thom-Whitney versions, to complement $\delta_{Z,\mathrm{TW}}$ given by equation (1.19).

$$\begin{aligned} g_{Z,\mathrm{TW}} &:= \epsilon\otimes\eta_Z + (1-\epsilon)\otimes(\eta_Z+\overline{\eta}_Z)/2 + d\epsilon\otimes g_Z \\ &\in \mathfrak{D}^{2p-1}_{\mathrm{TW}}(\Sigma_{B_X} E^*_{X\times(\mathbb{P}^1)^n}(\log A_X\cup|Z|),p), \\ \theta_{Z,\mathrm{TW}} &:= d\epsilon\otimes\theta_Z \in \mathfrak{D}^{2p}_{\mathrm{TW}}(\Sigma_{B_X} E^*_{X\times(\mathbb{P}^1)^n}(\log A_X\cup|Z|),p). \end{aligned} \tag{3.20}$$

Equations (3.14) and (3.16) and the fact that $\overline{\delta_Z}=\delta_Z$ imply that

$$d[g_{\mathrm{TW},Z}] = -\delta_{\mathrm{TW},Z} + [\theta_{\mathrm{TW},Z}]. \tag{3.21}$$

Equations (1.20) and (3.21), together with equation [BGG19, (5.7)] and the fact that $g_{Z,\mathrm{TW}}$ vanishes when restricted to B_X imply the equality of cohomology classes

$$\mathcal{P}(Z) = \{(\pi_X)_*[\theta_{Z,\mathrm{TW}}\cdot W_n]\}.$$

So we are left to compare the classes $\{(\pi_X)_*[\theta_{Z,\mathrm{TW}}\wedge W_n]\}$ with $\{\Psi(\theta_Z)\}$. To this end we will use the explicit description of Wang forms in [BGW98, Definition 6.5]. We note that the form denoted by W_n here is the form $(-1)^n W_n^3$ in [BGW98].

Using (1.17), the image of $\mathcal{P}(Z)$ is represented by the form

$$\sum_{i=1}^{n}\int_0^1 \frac{(-1)^n(\epsilon+1)^i(\epsilon-1)^{n-i}}{2^n i!(n-i)!} d\epsilon\cdot(\pi_X)_*[\theta_Z\wedge P_n^i], \tag{3.22}$$

Where $P_n^i=\sum_{\sigma\in\mathfrak{S}_n}(-1)^\sigma P^i_{n,\sigma}$ and, for a permutation $\sigma\in\mathfrak{S}_n$.

$$P^i_{n,\sigma} = \frac{dt_{\sigma(1)}}{t_{\sigma(1)}}\wedge\dots\wedge\frac{dt_{\sigma(i)}}{t_{\sigma(i)}}\wedge\frac{d\bar t_{\sigma(i+1)}}{\bar t_{\sigma(i+1)}}\wedge\dots\wedge\frac{d\bar t_{\sigma(n)}}{\bar t_{\sigma(n)}}.$$

We now use that

$$[P^i_{n,\sigma}] = -[P^n_{n,\sigma}] + \text{boundaries} + \text{currents in } B_X,$$

$$(-1)^\sigma P^n_{n,\sigma} = \bigwedge_{i=1}^n \frac{dt_i}{t_i},$$

that

$$\sum_{i=0}^{n}\frac{n!(-1)^n(\epsilon+1)^i(\epsilon-1)^{n-i}(-1)^{n-i}}{2^n i!(n-i)!} = \frac{(-1)^n}{2^n}(\epsilon+1-(\epsilon-1))^n = (-1)^n,$$

and that the form θ_Z vanishes on B_X to deduce that the current (3.22) is cohomologous to

$$(-1)^n(\pi_X)_*\left[\theta_Z\wedge\bigwedge_{i=1}^n\frac{dt_i}{t_i}\right] = \Psi(\theta_Z).$$

$\square$

Corollary 3.9. *Let $Z \in Z^p(X,n)_{00}$, be a cycle such that its real regulator class is zero. Then we can choose g_Z, η_Z and θ_Z as in Proposition 3.6 with the additional property $\theta_Z = 0$. Therefore*

$$dg_Z = \frac{1}{2}(\eta_Z - \overline{\eta}_Z).$$

Proof. Let g'_Z, η'_Z and θ'_Z a choice of forms as in Proposition 3.6. If the real regulator class of Z is zero, then Proposition 3.8 implies that the cohomology class of θ'_Z belongs to

$$F^0H^{2p-1}(X \times (\mathbb{P}^1)^n \setminus A_X, B_X; p) + H^{2p-1}(X \times (\mathbb{P}^1)^n \setminus A_X, B_X; p)_{\mathbb{R}}.$$

Hence there exist differential forms

$$\begin{aligned} h_1 &\in F^0\Sigma_{B_X} E^{2p-1}_{X\times(\mathbb{P}^1)^n}(\log A_X; p), \\ h_2 &\in \Sigma_{B_X} E^{2p-1}_{X\times(\mathbb{P}^1)^n,\mathbb{R}}(\log A_X; p), \\ \gamma &\in \Sigma_{B_X} E^{2p-2}_{X\times(\mathbb{P}^1)^n}(\log A_X; p), \end{aligned}$$

with h_1 and h_2 closed, such that

$$\theta'_Z = h_1 + h_2 + d\gamma.$$

We write $\gamma_1 = (\gamma - \overline{\gamma})/2$ and we decompose

$$\gamma_1 = \gamma_1^{p-1,p-1} + F^0\gamma_1 + \overline{F}^0\gamma_1.$$

Then $\overline{F}^0\gamma_1 = -\overline{F^0\gamma_1}$. Moreover, since $\overline{\theta'_Z} = -\theta'_Z$,

$$d\gamma_1^{p-1,p-1} = \theta'_Z - \frac{1}{2}\left((h_1 + 2dF^0\gamma_1) - \overline{(h_1 + 2dF^0\gamma_1)}\right).$$

Thus, if we write

$$g_Z = g'_Z + \gamma_1^{p-1,p-1}, \qquad \eta_Z = \eta'_Z - h_1 - 2F^0\gamma_1 \qquad \theta_Z = 0,$$

then, is easy to verify that the triple $\eta_Z, \theta_Z, \gamma_Z$ satisfies the properties of Proposition 3.6. □

Remark 3.10. When the real regulator class of a higher cycle $Z \in Z^p(Z,n)_{00}$ is zero, and the forms η_Z and g_Z are as in Corollary 3.9, then $\eta_Z = 2\partial g_Z$.

3.5. **Comparison with** [BGG19]**.** This subsection acts as a bridge between the Hodge theoretic forms obtained above, and the higher Green forms and currents used in [BGG19]. We will use it later to connect the higher archimedean height pairing to the height of a mixed Hodge structure associated to a pair of higher cycles. We will follow the notations of [BGG19].

For each n, consider the complex given by

$$\tau\,\mathfrak{D}^{*,-s}_{\mathrm{TW},\mathbb{A}}(X,p) = \tau_{\leq 2p}\,\mathfrak{D}^*_{\mathrm{TW}}(E^*_{X\times(\mathbb{P}^1)^s}(\log B), p).$$

It has a cubical structure and we can form the associated refined normalizes double complex $\tau\,\mathfrak{D}^{*,*}_{\mathrm{TW},\mathbb{A},\log}(X,p)_{00}$ and the corresponding total complex $\tau\,\mathfrak{D}^{*}_{\mathrm{TW},\mathbb{A},\log}(X,p)_{00}$. See [BGG19, 5.2] for more details.

There is a quasi-isomorphism

$$\tau_{\leq 2p}\,\mathfrak{D}^*_{\mathrm{TW}}(X,p) \hookrightarrow \tau\,\mathfrak{D}^*_{\mathrm{TW},\mathbb{A},\log}(X,p)_{00}$$

that is given by the inclusion as the column $n = 0$.

Let Z, $\theta_{Z,\mathrm{TW}}$ and $g_{Z,\mathrm{TW}}$ be as in the previous section and write

$$\underline{\theta_Z} = (\pi_X)_*[\theta_{Z,\mathrm{TW}} \cdot W_n] \in \mathfrak{D}^{2p-n}_{\mathrm{TW}}(X,p) = \mathfrak{D}^{2p-n,0}_{\mathrm{TW},\mathbb{A},\log}(X,p)_{00}.$$

In the complex $\tau\,\mathfrak{D}^*_{\mathrm{TW},\mathbb{A},\log}(X,p)_{00}$, the forms $\theta_{Z,\mathrm{TW}}$ and $\underline{\theta_Z}$ are cohomologous as both represent the class $\{\mathcal{P}(Z)\}$. Therefore we obtain an element

$$(\alpha_n, \cdots, \alpha_0) \in \mathfrak{D}^{2p-n-1}_{\mathrm{TW},\mathbb{A},\log}(X,p)_{00},$$

satisfying

$$(0, \cdots, \underline{\theta_Z}) - (\theta_{Z,\mathrm{TW}}, 0, \cdots, 0) = d(\alpha_n, \cdots, \alpha_0). \tag{3.23}$$

We obtain an n-tuple of forms

$$\mathfrak{g}_Z \coloneqq (g_{Z,\mathrm{TW}} + \alpha_n, \dots, \alpha_0).$$

Lemma 3.11. *The n-tuple of forms*

$$\mathfrak{g}_Z \in \bigoplus_{i=n}^{0} \mathfrak{D}^{2p-n+i-1}_{\mathrm{TW}}(E^*_{X\times(\mathbb{P}^1)^i}(\log A \cup |Z|_i), p)_{00}$$

is a refined Green form for Z, as in [BGG19, Definition 6.5].

Proof. Equations (3.23) and (3.21) when written componentwise, imply the conditions of [BGG19, Definition 6.5]. □

Remark 3.12. Notice that in the Green form $\mathfrak{g}_Z$ only the component over $X \times (\mathbb{P}^1)^n$ has singularities along $|Z|$, while the rest are smooth on $X \times \square^i$, with logarithmic singularities along A_X.

After constructing a higher Green form out of $g_{Z,\mathrm{TW}}$ we also construct a Green current. Let $\underline{g_Z} \coloneqq (\pi_X)_*[g_Z \cdot W_n] \in \mathfrak{D}^{2p-n-1}_{\mathrm{TW},D}(X,p)$. Then, in the complex $\mathfrak{D}^{2p-n-1}_{\mathrm{TW},D}(X,p)$ the equation

$$d\underline{g_Z} = -\mathcal{P}(Z) + \underline{\theta_Z},$$

is satisfied. Hence $\underline{g_Z}$ is a Green current for the cycle Z as in [BGG19, Definition 6.1]).

Let now W be a cycle in $Z^q(X,m)_{00}$, which intersects Z properly and $\underline{g_W}$ a Green current for W in the Thom-Whitney complex. We now can give a second (and simplified) definition of star product:

Definition 3.13. Let $g_{Z,\mathrm{TW}}$, $\underline{g_Z}$ and $\underline{g_W}$ be as before. Then we define the product

$$\underline{g_Z} *_2 \underline{g_W} = (-1)^n (\pi_X)_* \left(\delta_{Z,\mathrm{TW}} \cdot W_m \cdot g_{W,\mathrm{TW}} \cdot W_n\right) + \underline{g_Z} \cdot \underline{\theta_W}$$

We note here that the products are taking place in the ambient space $X \times (\mathbb{P}^1)^m \times (\mathbb{P}^1)^n$, and the notations should be interpreted accordingly. For example $g_{\mathrm{TW},Z}$ really means the pullback of this form in the ambient space. We avoid the pullback notations to simplify the exposition. This note will hold true whenever we take product between elements in a priori different spaces.

We next show that the star product $*_2$ is compatible with the star product $*$ in [BGG19, §6.4],

Proposition 3.14. *Let $\mathfrak{g}'_W$ be a Green form for W in the Thom-Whitney complex, such that $\underline{g_W}^{\sim} = [\mathfrak{g}_W]^{\sim}$. Then for any Green current g_Z of Z, we have*

$$\left(g_Z *_2 \underline{g_W}\right)^{\sim} = (g_Z * \mathfrak{g}'_W)^{\sim}.$$

Proof. Since the product $(g_Z * \mathfrak{g}'_W)^{\sim}$ is independent on the choice of $\mathfrak{g}'_W$ we can make a particular choice. We consider the elements $(\alpha_m, \dots, \alpha_0)$ satisfying (3.23). We write

$$\alpha = \sum_{i=0}^{m} (\pi_X)_*(\alpha_i \cdot W_i).$$

Then α is closed. Indeed by (3.23) and [BGG19, (5.7)]

$$d\alpha = \sum_{i=0}^{m} (\pi_X)_*(d\alpha_i \cdot W_i) + \sum_{i=0}^{m} (-1)^{2p-i-1} (\pi_X)_*(\alpha_i \cdot dW_i) = \\ - (\pi_X)_*(\theta_{W,\mathrm{TW}} \cdot W_n) + \underline{\theta_W} = 0.$$

we define

$$\mathfrak{g}'_W = (g_{W,\mathrm{TW}} + \alpha_m, \alpha_{m-1}, \dots, \alpha_1, \alpha_0 - \alpha).$$

With this choice

$$[\mathfrak{g}'_W] = (\pi_X)_*(g_{W,\mathrm{TW}} \cdot W_n) + \sum_{i=0}^{m} (\pi_X)_*(\alpha_i \cdot W_i) - \alpha = \underline{g_W}.$$

Moreover

$$(-1)^n (g_Z * \mathfrak{g}'_W - g_Z *_2 \underline{g_W}) = \\ \sum_{i=0}^{m} (\pi_X)_*(\delta_{Z,\mathrm{TW}} \cdot W_m \cdot \alpha_i \cdot W_i) - (\pi_X)_*(\delta_{Z,\mathrm{TW}} \cdot W_m \cdot \alpha) = 0.$$

proving the proposition. □

As a consequence we obtain the following formula for the higher archimedean height pairing of Definition 1.35.

Corollary 3.15. *If $Z \in Z^p(X, n)_{00}$ and $W \in Z^q(X, m)_{00}$ be two higher cycles whose real regulator classes are zero with $2(p + q - d - 1) = n + m$, then*

$$\langle Z, W \rangle_{\mathrm{Arch}} = (-1)^n (p)_* \left(\delta_{Z,\mathrm{TW}} \cdot W_m \cdot g_{W,\mathrm{TW}} \cdot W_n\right)^{\sim},$$

where $p\colon X \times (\mathbb{P}^1)^n \times (\mathbb{P}^1)^m \to \mathrm{Spec}(\mathbb{C})$ is the structural morphism.

Proof. The key point is that we can use the second definition of Green current using Proposition 3.14 for the particular choice of Green form for W, since higher archimedean height pairing is independent of the choice of Green form for a higher cycle. Next the real regulator class of W being zero allows us to choose $\theta_W = 0$ by Corollary 3.9. This concludes the proof. □

3.6. **The dual extension.** Let now $q \geq 0$ and $m \geq 1$ be integers and let $W \in Z^q(X,m)_{00}$ be a cycle. We apply the construction of sections 3.2 and 3.3 to this setting, obtaining an extension E_W and the corresponding differential forms. We can dualize the extension E_W to get a dual extension

$$E_W^\vee = \mathrm{Hom}_{\mathbf{MHS}}(E_W, \mathbb{Q}(0)).$$

This extension is given by the short exact sequence

$$0 \longrightarrow \mathbb{Q}(0) \longrightarrow E_W^\vee \longrightarrow H^{2d-2q+m+1}(X; d-p) \longrightarrow 0,$$

dual to (3.9). By construction E_W is a sub-mixed Hodge structure of

$$H^{2q-1}(X \times (\mathbb{P}^1)^m \setminus A_X \cup |W|, B_X; p) \tag{3.24}$$

By duality we would like to see $E_W^\vee$ as a quotient mixed Hodge structure. A naive idea would be to think that $E_W^\vee$ should be a quotient of

$$H^{2d-2q+m+1}(X \times (\mathbb{P}^1)^m \setminus B_X, A_X \cup |W|; d+m-q).$$

But the problem is that the above group does not need to be the dual to (3.24) because B_X and $A_X \cup |W|$ may fail to be in a local product situation. To remedy this situation, we consider a composition of blow ups as in the next lemma.

Lemma 3.16. *There exists a proper transform*

$$\pi \colon \mathcal{X}_W \to X \times (\mathbb{P}^1)^m,$$

that is a composition of blow-ups with smooth centers whose image in $X \times (\mathbb{P}^1)^m$ is contained in $|W| \cap B_X$, such that, if we denote by $\widehat{W}$, $\widehat{A}_X$ and $\widehat{B}_X$ the strict transforms of $|W|$, A_X and B_X respectively, and by D the exceptional divisor, then

(i) The strict transforms $\widehat{W}$ and $\widehat{B}_X$ do not meet.
(ii) The divisor $\widehat{A}_X \cup D \cup \widehat{B}_X$ is a simple normal crossing divisor.

The previous conditions imply that the pair of closed subsets $\widehat{A}_X \cup D$ and $\widehat{B}_X$ are in local product situation and the same is true for the pair $\widehat{A}_X \cup D \cup \widehat{W}$ and $\widehat{B}_X$.

Proof. Let $\mathcal{I}_W$ be the ideal sheaf of $|W|$ and $\mathcal{I}_B$ the ideal sheaf of B_X by blowing up $\mathcal{I}_W + \mathcal{I}_B$ we obtain a proper transform $X_1 \to X \times (\mathbb{P}^1)^m$ such that the strict transform of $|W|$ and B_X do not meet ([Har77, Chapter II, Exercise 7.12]). This proper transform is an isomorphism outside $|W| \cap B_X$ but X_1 is possibly singular. By the use of strong resolution of singularities in the elimination of indeterminacies, there is proper transform

$$\pi \colon \mathcal{X}_W \to X \times (\mathbb{P}^1)^m,$$

that is a composition of blow-ups with smooth centers whose image in $X \times (\mathbb{P}^1)^m$ is contained in $|W| \cap B_X$, with a map $\mathcal{X}_W \to X_1$ making the diagram

$$\begin{tikzcd} \mathcal{X}_W \arrow[r] \arrow[dr] & X_1 \arrow[d] \\ & X \times (\mathbb{P}^1)^m \end{tikzcd}$$

commutative and satisfying the conditions of the lemma. □

Let $\pi\colon \mathcal{X}_W \to X \times (\mathbb{P}^1)^m$ be a map provided by Lemma 3.16,

Notation 3.17. In the sequel we will use the following shorthands.

$$\begin{aligned} \square^m &= (\mathbb{P}^1)^m \setminus A, & \boxbox^m &= ((\mathbb{P}^1)^m \setminus A, B), \\ \square^m_X &= X \times (\mathbb{P}^1)^m \setminus A_X, & \boxbox^m_X &= (X \times (\mathbb{P}^1)^m \setminus A_X, B_X) \\ \widetilde{\square^m_X} &= \mathcal{X}_W \setminus \widehat{A}_X & \widetilde{\boxbox^m_X} &= (\mathcal{X}_W \setminus \widehat{A}_X, \widehat{B}_X). \end{aligned}$$

and the dual ones

$$\begin{aligned} G^m &= (\mathbb{P}^1)^m \setminus B, & \mathbb{G}^m &= ((\mathbb{P}^1)^m \setminus B, A), \\ G^m_X &= X \times (\mathbb{P}^1)^m \setminus B_X, & \mathbb{G}^m_X &= (X \times (\mathbb{P}^1)^m \setminus B_X, A_X) \\ \widetilde{G^m_X} &= \mathcal{X}_W \setminus \widehat{B}_X & \widetilde{\mathbb{G}^m_X} &= (\mathcal{X}_W \setminus \widehat{B}_X, \widehat{A}_X). \end{aligned}$$

Moreover, in the relative schemes like $\boxbox^m_X$, the notation $(\boxbox^m_X \setminus S, T)$ will mean

$$(X \times (\mathbb{P}^1)^m \setminus A_X \cup S, B_X \cup T).$$

We have the following

Lemma 3.18. *The cohomology of $\mathcal{X}_W$ satisfies*

(i) the morphism

$$H^r(X \times (\mathbb{P}^1)^m \setminus A_X \cup |W|, B_X) \xrightarrow{\pi^*} H^r(\mathcal{X}_W \setminus \widehat{A}_X \cup D \cup \widehat{W}, \widehat{B}_X),$$

is an isomorphism for all $r \geq 0$;

(ii) the morphism

$$H^r(X \times (\mathbb{P}^1)^m \setminus A_X, B_X) \longrightarrow H^r(\mathcal{X}_W \setminus \widehat{A}_X \cup D, \widehat{B}_X)$$

is an isomorphism for $r \leq 2q$, and injective for $r = 2q+1$.

Proof. Since the map π gives isomorphisms

$$\begin{aligned} \mathcal{X}_W \setminus \widehat{A}_X \cup D \cup \widehat{W} &\cong X \times (\mathbb{P}^1)^m \setminus A_X \cup |W| \\ \widehat{B}_X \setminus \widehat{A}_X \cup D \cup \widehat{W} &\cong B_X \setminus A_X \cup |W|, \end{aligned}$$

we get *(i)* immediately.

For *(ii)*, let C be the center of the blow ups. by the same reason as before, π^* gives isomorphisms

$$H^r(X \times (\mathbb{P}^1)^m \setminus A_X \cup C, B_X) \xrightarrow{\cong} H^r(\mathcal{X}_W \setminus \widehat{A}_X \cup D, \widehat{B}_X).$$

Moreover, using Notation 3.17 we have a diagram of mixed Hodge structures with exact rows and commutative squares

$$\begin{tikzcd} H^{r-1}(\square^m_X) \arrow[r] \arrow[d, "①"] & H^{r-1}(B_X \setminus A_X) \arrow[r] \arrow[d, "②"] & H^r(\boxbox^m_X) \arrow[r] \arrow[d, "③"] & H^r(\square^m_X) \arrow[r] \arrow[d, "④"] & H^r(B_X \setminus A_X) \arrow[d, "⑤"] \\ H^{r-1}(\widetilde{\square^m_X} \setminus D) \arrow[r] & H^{r-1}(\widehat{B}_X \setminus \widehat{A}_X \cup D) \arrow[r] & H^r(\widetilde{\boxbox^m_X} \setminus D) \arrow[r] & H^r(\widetilde{\square^m_X} \setminus D) \arrow[r] & H^r(\widehat{B}_X \setminus \widehat{A}_X \cup D) \end{tikzcd}$$

Since C has codimension $\geq q+1$ in $X \times (\mathbb{P}^1)^m$ and $C \cap B_W$ has codimension $\geq q+1$ in B, the arrows ①, ②, ④ and ⑤ are isomorphisms. Hence the arrow ③ is also an isomorphism for $r \leq 2q$. For $r = 2q+1$, the arrows ① and ② are isomorphisms, while the arrow ④ is injective. Hence the arrow ③ is also injective. □

Corollary 3.19. *The morphism*

$$H^r(\square^m_X) \longrightarrow H^r(\widetilde{\square^m_X}\setminus D)$$

is an isomorphism for $r \le 2q$ and injective for $r = 2q+1$. Dually, the map

$$H^s(\widetilde{\mathbb{G}^m_X}, D) \longrightarrow H^s(\mathbb{G}^m_X)$$

is an isomorphism for $s \ge 2d+2m-2q$ and surjective for $s = 2d+2m-2q-1$.

We now consider the commutative diagram with exact rows

$$\begin{tikzcd}
H^{2q-1}(\square^m_X) \arrow[r, hook] \arrow[d, "\cong"] & H^{2q-1}(\square^m_X \setminus |W|) \arrow[r] \arrow[d, "\cong"] & H^{2q}_{|W|}(\square^m_X) \arrow[r] \arrow[d, "\cong"] & H^{2q}(\square^m_X) \arrow[d, "\cong"] \\
H^{2q-1}(\widetilde{\square^m_X}\setminus D) \arrow[r, hook] & H^{2q-1}(\widetilde{\square^m_X}\setminus D\cup \widehat{W}) \arrow[r] & H^{2q}_{\widehat{W}}(\widetilde{\square^m_X}\setminus D) \arrow[r] & H^{2q}(\widetilde{\square^m_X}\setminus D)
\end{tikzcd}$$

where the vertical arrows are isomorphisms thanks to lemma 3.18,

In the bottom row of the above diagram, all the relevant relative schemes are in a product situation. Hence the dual to this bottom row, after twisting by $\mathbb{Q}(-d-m)$ to make the twist disappear, writing $d_W = \dim(W) = d+m-q$, and taking into account that $H^{2d_W}(\widehat{W}) = H^{2d_W}(\widehat{W}, \widehat{W}\cap(D\cup\widehat{A}_X))$, reads

$$H^{2d_W}(\widetilde{\mathbb{G}^m_X}, D) \to H^{2d_W}(\widehat{W}) \to H^{2d_W+1}(\widetilde{\mathbb{G}^m_X}, D\cup\widehat{W}) \twoheadrightarrow H^{2d_W+1}(\widetilde{\mathbb{G}^m_X}, D).$$

After unfolding Notation 3.17 we obtain

$$\begin{multline}\tag{3.25} H^{2d_W}(\mathcal{X}_W\setminus\widehat{B}_X, \widehat{A}_X\cup D) \to H^{2d_W}(\widehat{W}) \to \\ H^{2d_W+1}(\mathcal{X}_W\setminus\widehat{B}_X, \widehat{A}_X\cup D\cup\widehat{W}) \to H^{2d_W+1}(\mathcal{X}_W\setminus\widehat{B}_X, \widehat{A}_X\cup D) \to 0.\end{multline}$$

Just as a sanity check, note that in this exact sequence the first arrow is well defined because $\widehat{W}\cap\widehat{B}_X = \emptyset$ and there is a zero at the end because $\dim\widehat{W} = d_W$. We now use that

$$H^{2d_W}(\widehat{W}, \widehat{W}\cap(\widehat{A}_X\cup D)) \cong H^{2d_W}(\widehat{W}),$$

since $\dim(\widehat{W}\cap(\widehat{A}_X\cup D)) < d_W$.

The class of W produces a morphism of mixed Hodge structure

$$\phi^\vee_W\colon H^{2d_W}(\widehat{W}; d_W) \longrightarrow \mathbb{Q}(0), \tag{3.26}$$

that is the dual of the map (3.8). The fact that the image of the class $[W]$ in $H^{2q}(X\times(\mathbb{P}^1)^m\setminus A_X, B_X; p)$ is zero implies that

$$\phi^\vee_W\left(H^{2d_W}(\mathcal{X}_W\setminus\widehat{B}_X, \widehat{A}_X\cup D; d_W)\right) = 0.$$

Hence, taking the push-forward through $\phi^\vee_W$ of the exact sequence (3.25), we obtain a short exact sequence

$$0 \to \mathbb{Q}(0) \to E^\vee_W \to H^{2d_W+1}(\mathcal{X}_W\setminus\widehat{B}_X, \widehat{A}_X\cup D; d_W) \to 0.$$

By Lemma 3.18 *(ii)*, the fact that $\widehat{B}_X$ and $\widehat{A}_X \cup D$ are in local product situation and the isomorphism (3.5) we have

$$\begin{aligned} H^{2d_W+1}(\mathcal{X}_W \setminus \widehat{B}_X, \widehat{A}_X \cup D; d_W) &= H^{2q-1}(\mathcal{X}_W \setminus \widehat{A}_X \cup D, \widehat{B}_X; p)^\vee \\ &= H^{2q-1}(X \times (\mathbb{P}^1)^m \setminus A_X, B_X; p)^\vee \\ &= H^{2d_W+1}(X \times (\mathbb{P}^1)^m \setminus B_X, A_X; d_W) \\ &= H^{2d-(2q-m-1)}(X; d-q). \end{aligned}$$

Therefore the above short exact sequence can be written as

$$0 \to \mathbb{Q}(0) \to E_W^\vee \to H^{2d-(2q-m-1)}(X; d-q) \to 0. \tag{3.27}$$

By construction this exact sequence is the dual sequence to 3.9. We denote by

$$e_W^\vee \in \operatorname{Ext}^1_{\mathbf{MHS}}\left(H^{2d+m-2q+1}(X; d-q), \mathbb{Q}(0)\right),$$

to be the class of this extension.

3.7. **Oriented MHS attached to a pair of higher cycles.** Let $n, m \geq 1$, and $p, q \geq 0$ be integers with

$$2(p+q-d-1) = n+m. \tag{3.28}$$

Let $Z \in Z^p(X,n)_{00}$, and $W \in Z^q(X,m)_{00}$, be two cycles in the refined normalized complex intersecting properly. We want to attach an oriented rational mixed Hodge structure to this pair. This mixed Hodge structure is similar to the one constructed by Hain in [Hai90], with one significant difference: In the case for usual cycles homologous to zero, proper intersection and the numerical relation $p+q = d+1$ mean that the supports of the cycles are disjoint, which is no longer the case here. So one should expect the new mixed Hodge structure to reflect this phenomenon. Moreover, the use of proper modification in order to use duality will add another technical difficulty.

Let

$$\begin{aligned} \pi_1 \colon X \times (\mathbb{P}^1)^n \times (\mathbb{P}^1)^m &\longrightarrow X \times (\mathbb{P}^1)^n \\ \pi_2 \colon X \times (\mathbb{P}^1)^n \times (\mathbb{P}^1)^m &\longrightarrow X \times (\mathbb{P}^1)^m \end{aligned}$$

be the two projections. Then the fact that Z and W meet properly means precisely that $p_1^{-1}(|Z|) \cap p_2^{-1}(|W|) \cap X \times \square^{n+m}$ has codimension $p+q$ and intersects properly all the faces of $\square^{n+m}$. Hence there is a well defined intersection pre-cycle

$$Z \cdot W \in Z^{p+q}(X, n+m)_0$$

Since Z and W are cycles in the refined normalized complex, the same is true for $Z \cdot W$.

Let $\pi \colon \mathcal{X}_W \to X \times (\mathbb{P}^1)^m$ be a proper modification as in Lemma 3.16 applied to W. Let $C \subset |W|$ be the support of the center of π. Then π is an isomorphism outside C. On $\mathcal{X}_W$, $\widehat{W}$, $\widehat{A}_X$ and $\widehat{B}_X$ are the strict transforms of $|W|$, A_X and B_X and D is the exceptional divisor.

We will assume the following technical conditions.

Assumption 3.20. The intersection $\pi_1^{-1}(|Z|) \cap \pi_2^{-1}(C) = \emptyset$.

Remark 3.21. The assumption 3.20 is more and more restrictive the bigger n and m are. In the case $n = m = 1$, this condition is satisfied generically but this is not the case for higher values of n and m.

The sought mixed Hodge structure will appear in a diagram that contains at the same time the exact sequence (3.9) for the cycle Z and the dual exact sequence (3.27) for the cycle W. For the main diagram to fit in one page we need to complement Notation 3.17.

Notation 3.22. We have already introduced the projections π_1 and π_2 and consider also the projection

$$\pi_3\colon \mathcal{X}_W \times (\mathbb{P}^1)^n \longrightarrow \mathcal{X}_W.$$

Moreover we also consider the proper transform

$$\pi'\colon \mathcal{X}_W \times (\mathbb{P}^1)^n \longrightarrow X \times (\mathbb{P}^1)^n \times (\mathbb{P}^1)^m.$$

Note that this map involves a change in the order of the variables. We write,

$$\begin{aligned}
A_1 &= \pi_1^{-1} A_X, & A_2 &= \pi_2^{-1} A_X,\\
B_1 &= \pi_1^{-1} B_X, & B_2 &= \pi_2^{-1} B_X,\\
\overline{A}_2 &= \pi_3^{-1} \widehat{A}_X, & \overline{B}_2 &= \pi_3^{-1} \widehat{B}_X,\\
\overline{A}_1 &= (\pi')^{-1} A_1, & \overline{B}_1 &= (\pi')^{-1} B_1,\\
\overline{D} &= \pi_3^{-1} D, & C_2 &= \pi_2^{-1}(C),\\
\overline{Z} &= |Z| \times (\mathbb{P}^1)^m, & \overline{W} &= \pi_3^{-1} \widehat{W}.
\end{aligned}$$

Note that the spaces marked with an overline are subsets of $\mathcal{X}_W \times (\mathbb{P}^1)^n$ while the others are subsets of $X \times (\mathbb{P}^1)^n \times (\mathbb{P}^1)^m$. We will also consider the relative schemes

$$\begin{aligned}
\square_X^{n,m} &= \square_X^n \times_X \mathbb{G}_X^m = (X \times (\mathbb{P}^1)^n \times (\mathbb{P}^1)^m \setminus A_1 \cup B_2, B_1 \cup A_2),\\
\widetilde{\square_X^{n,m}} &= \square_X^n \times_X \widetilde{\mathbb{G}_X^m} = (\mathcal{X}_W \times (\mathbb{P}^1)^n \setminus \overline{A}_1 \cup \overline{B}_2, \overline{B}_1 \cup \overline{B}_2),\\
\underline{Z} &= (\overline{Z} \setminus \overline{A}_1 \cup \overline{B}_2, \overline{B}_1 \cup \overline{A}_2) \subset \widetilde{\square_X^{n,m}},\\
\underline{W} &= (\overline{W} \setminus \overline{A}_1, \overline{B}_1 \cup \overline{A}_2) \subset \widetilde{\square_X^{n,m}}.
\end{aligned}$$

The relative schemes will always be denoted, either with double line typography or with an underline. Finally we write $\underline{S} = \underline{Z} \cap \underline{W}$. Note that, by Assumption 3.20 the relative schemes $\underline{Z}$ and $\underline{S}$ can be seen as subschemes of either $\square_X^{n,m}$ or $\widetilde{\square_X^{n,m}}$. Note also that in the definition of $\underline{W}$, the divisor $\overline{B}_2$ does not appears because $\widehat{W}$ and $\widehat{B}_X$ are disjoint.

We consider the commutative diagram with exact rows and columns of figure 1. In that diagram, we have omitted $\overline{D}$ in the last column because, by Assumption 3.20, it is disjoint with $\overline{Z}$.

$$\begin{tikzcd}
 & 1 & 2 & 3 & \\
4 \arrow[r] & H^r(\widetilde{\square^{n,m}_X}, \overline{D}) \arrow[r] \arrow[u] & H^r(\widetilde{\square^{n,m}_X} \setminus \overline{Z}, \overline{D}) \arrow[r] \arrow[u] & H^{r+1}_{\underline{Z}}(\widetilde{\square^{n,m}_X}) \arrow[r] \arrow[u] & 5 \\
6 \arrow[r] & H^r(\widetilde{\square^{n,m}_X}, \overline{W} \cup \overline{D}) \arrow[r] \arrow[u] & H^r(\widetilde{\square^{n,m}_X} \setminus \overline{Z}, \overline{W} \cup \overline{D}) \arrow[r] \arrow[u] & H^{r+1}_{\underline{Z}}(\widetilde{\square^{n,m}_X}, \overline{W}) \arrow[r] \arrow[u] & 7 \\
8 \arrow[r] & H^{r-1}(\underline{W}, \overline{W} \cap \overline{D}) \arrow[r] \arrow[u] & H^{r-1}(\underline{W} \setminus S, \overline{W} \cap \overline{D}) \arrow[r] \arrow[u] & H^r_{\underline{S}}(\underline{W}) \arrow[r] \arrow[u] & 9 \\
 & 10 \arrow[u] & 11 \arrow[u] & 12 \arrow[u] &
\end{tikzcd}$$

FIGURE 1. The main diagram

We now analyze the different terms in that diagram for $r = 2p + m - 1$. We start with the top left corner

$$\begin{aligned}
H^{2p+m-1}(\widetilde{\square^{n,m}_X}, \overline{D}) &= H^{2p+m-1}((\widetilde{\mathbb{G}^m_X}, D) \times \square^n) \\
&= H^{2p+m-n-1}(\widetilde{\mathbb{G}^m_X}, D) \\
&= H^{2p+m-n-1}(\mathbb{G}^m_X) \\
&= H^{2p-n-1}(X; -m).
\end{aligned}$$

The first equality is true at the level of relative schemes. The second equality follows from (3.4) and Künneth formula. Since, by (3.28), $2p+m-n-1 = 2d+2m-2q+1$, hence the third equality follows form Corollary 3.19. The last one follows from (3.5). This computation means in particular that the composition

$$H^{2p+m-1}(\widetilde{\square^{n,m}_X}, \overline{D}) \overset{\cong}{\longleftarrow} H^{2p+m-1}(\square^{n,m}_X, C_2) \longrightarrow H^{2p+m-1}(\square^{n,m}_X) \tag{3.29}$$

is an isomorphism. The fact that the composition (3.29) is an isomorphism, together with the fact that $\overline{D}$ and Z are disjoint by Assumption 3.20 imply that the compositions

$$\begin{aligned}
H^{2p+m-1}(\widetilde{\square^{n,m}_X} \setminus \underline{Z}, \overline{D}) \overset{\cong}{\longleftarrow} H^{2p+m-1}(\square^{n,m}_X \setminus \underline{Z}, C_2) \longrightarrow \\
H^{2p+m-1}(\square^{n,m}_X \setminus \underline{Z}) \overset{\cong}{\longrightarrow} H^{2p1}(\square^n_X \setminus Z; -m)
\end{aligned} \tag{3.30}$$

and

$$H^{2p}_Z(\square^n_X; -m) \longrightarrow H^{2p+m}_{\underline{Z}}(\square^{n,m}_X) \longrightarrow H^{2p+m}_{\underline{Z}}(\widetilde{\square^{n,m}_X}) \tag{3.31}$$

are isomorphisms. So we can identify the top row of the diagram with the exact sequence (3.7). In fact this argument also implies that 4 is zero and that the image of the class of Z in 5 is also zero.

Since $2d_W = 2d + 2m - 2q = 2p - 2 + m - n$, using the isomorphisms

$$H^{2d_W+1}((\widetilde{\mathbb{G}^m_X}, \widehat{W} \cup D)), \xrightarrow{\cong} \\ H^{2p+m-1}((\widetilde{\mathbb{G}^m_X}, \widehat{W} \cup D) \times \square^n) = H^{2p+m-1}(\widetilde{\square^{n,m}_X}, \underline{W} \cup \overline{D})$$

and

$$H^{2d_W}(\widehat{W}, \widehat{W} \cap D) = H^{2d_W}(\widehat{W}) \xrightarrow{\cong} \\ H^{2p+m-2-1}((\widehat{W}, \widehat{W} \cap D) \times \square^n) = H^{2p+m-2-1}(\underline{W}, \underline{W} \cap \overline{D})$$

we can identify the first column of the diagram with the exact sequence (3.25). By dimension reasons, these identifications also imply that 1 is zero and that the image of 10 twisted by $\mathbb{Q}(d_W)$ under the map $\phi^\vee_W$ in (3.26) is zero.

Note that the group 9 agrees with 1 and the group 12 agrees with 4 so they both vanish.

Next we face the technical problem that, in general, the groups $H^*_{\underline{S}}(\underline{W})$ are difficult to control. Even if S is one point, if W is singular, it can be very complicated. So in order to proceed we need to add another technical assumption. Afterwards we will give an example of geometrical conditions that assure the fulfillment of the technical assumption.

Assumption 3.23. Assume that the main diagram satisfies the following conditions:

(i) the image of the class of Z in $H^{2p+m}_{\underline{S}}(\underline{W})$ is zero;
(ii) the map $\phi^\vee_W$ sends the image of $H^{2p+m-2}_{\underline{S}}(\underline{W}; d_W)$ to zero;
(iii) the mixed Hodge structure $H^{2p+m-1}_{\underline{S}}(\underline{W})$ has weights contained in the interval $[2p+m-n-1, 2p+2m-1]$.

Proposition 3.24. *Let S_0 be the union of components of S that are not contained in $\overline{A}_1 \cup A_2$. If the conditions*

(i) the subset S_0 is contained in $\widehat{W}_{\mathrm{sm}}$, the open subset of smooth points;
(ii) the pair of subsets S_0 and $\widehat{W} \cap (\overline{B}_1 \cup \overline{D} \cup \overline{A}_2)$ are in local product situation inside $\widehat{W}$;
(iii) we are in the symmetric situation $n = m$;

are satisfied, then the conditions of Assumption 3.23 are also satisfied.

Proof. By resolving singularities of $\widehat{W}$ and using Lemma 1.11, the conditions *(i)* and *(ii)* on the proposition imply that

$$H^r_{\underline{S}}(\underline{W}) = H^{2d+2n+2m-2q-r}(S_0 \setminus \overline{B}_1 \cup \overline{D} \cup \overline{A}_2, \overline{A}_1; d+n+m-q)^\vee.$$

Since $\dim S_0 = (n+m)/2 - 1$, by [GNAPGP88, Chapter IV, Proposition (3.5)] the cohomology of S has weights in the interval $[0, n+m-2]$. Therefore the weights of $H^r_{\underline{S}}(\underline{W})$ are contained in the interval $[2p, 2p+n+m-2]$. If we add the condition $n = m$, then this interval is contained in the interval of Assumption 3.23 *(iii)*.

The class of Z in $H^{2p+m}_{\underline{Z}}(\widetilde{\square^{n,m}_X})$ has weight $2p+2m$ (recall the isomorphism (3.5)) since $H^{2p+m}_{\underline{S}}(\underline{W})$ has weight at most $2p+2m-2$ (here as well we are using $n=m$) condition 3.23 *(i)* follows.

Using again $n=m$, the group $H^{2p+m-2}_{\underline{S}}(\underline{W};d_W)$ has weights in the interval $[2,2m]$. Since the image of the map $\phi_W^\vee$ has weight zero, we deduce condition 3.23 *(ii)*. $\square$

Definition 3.25. Let $n=m\geq 1$ and $p,q\geq 0$ satisfying $p+q=d+n+1$ and let $Z\in Z^p(X,n)_{00}$ and $W\in Z^q(X,n)_{00}$ be cycles satisfying assumptions 3.20 and 3.23. Then the *oriented mixed Hodge structure diagram* associated to Z,W is the diagram obtained from the main diagram in Figure 1 by first twisting by $\mathbb{Q}(p+n)$, then taking the pullback by ϕ_Z and then the push-forward by $\phi_W^\vee$ twisted by $\mathbb{Q}(n+1)$. This diagram is depicted in figure 2.

$$\begin{tikzcd}
 & 0 & 0 & 0 & \\
0 \arrow[r] & H^{2p-n-1}(X;p) \arrow[r] \arrow[u] & E_Z \arrow[r] \arrow[u] & \mathbb{Q}(0) \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & E_W^\vee(n+1) \arrow[r] \arrow[u] & B_{Z,W} \arrow[r] \arrow[u] & C_{Z,W} \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & \mathbb{Q}(n+1) \arrow[r] \arrow[u] & D_{Z,W} \arrow[r] \arrow[u] & H^{2p+n-1}_{\underline{S}}(\underline{W};p+n) \arrow[r] \arrow[u] & 0 \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

FIGURE 2. Oriented mixed Hodge structure diagram

Remark 3.26. In general, if we switch Z and W, we do not obtain the dual of the diagram in figure 2. The first problem is obvious: Assumption 3.20 is not symmetric. But even if assumptions 3.20, 3.23 and the symmetric assumptions are satisfied, the two obtained diagrams may not be dual of each other if Z and W are not in local product situation. Later we will investigate in more detail the duality of this diagram in a particular case.

3.8. **The case $n=m=1$.** Due to the technical difficulties arising from the intersection $\pi_1^{-1}(|Z|)\cap\pi_2^{-1}(|W|)$ we will concentrate on the case $n=m=1$. Then equation (3.28) reads

$$p+q=d+2. \tag{3.32}$$

Proper intersection means that the intersection $\pi_1^{-1}(|Z|)\cap\pi_2^{-1}(|W|)\cap X\times\square^2$ is a finite set of points.

To ease the analysis, we make the following stronger assumption.

Assumption 3.27. We assume that $n = m = 1$ and that the whole intersection $S = \pi_1^{-1}(|Z|) \cap \pi_2^{-1}(|W|) \subset X \times (\mathbb{P}^1)^2$ is a finite set of points. Moreover,

(i) the subsets S and $A_1 \cup A_2 \cup B_1 \cup B_2$ are disjoint;
(ii) the subset S is contained in $\pi^{-1}(|Z|_{\mathrm{sm}}) \cap \pi_2^{-1}(|W|_{\mathrm{sm}})$ and the intersection $\pi^{-1}(|Z|) \cap \pi_2^{-1}(|W|)$ is transverse at every point of S.

In particular $\pi^{-1}(|Z|)$ and $\pi_2^{-1}(|W|)$ are in local product situation.

Assumption 3.27 implies that we can define the diagram in figure 2 and also the same diagram with Z and W swapped.

Proposition 3.28. *Assumption 3.27 implies assumptions 3.20 and 3.23 for the pair Z, W and for the reversed pair W, Z.*

Proof. By condition 3.27 *(i)* $\pi_1^{-1}(|Z|)$ and $\pi_2^{-1}(|W| \cap B_X)$ are disjoint. Therefore Assumption 3.20 is satisfied. Since S is a finite set of points contained in the smooth part of $\underline{W}$, the dimension of $\underline{W}$ is $d + 2 - q = p$, and $2p + m - 1 = 2p$, we deduce that

$$H^{2p+m}_{\underline{S}}(\underline{W}) = H^{2p+m-2}_{\underline{S}} = 0,$$

and that $H^{2p+m-1}_{\underline{S}}(\underline{W})$ is pure of weight $2p$. Hence Assumption 3.23 is also satisfied.

Since Assumption 3.27 is symmetric with respect to the swap of Z and W, we deduce assumptions 3.20 and 3.23 for the pair reversed. □

Next we modify the main diagram in figure 1 to achieve two goals. First we want it to be symmetric under the swap of Z and W, and second, we want the strict transforms of Z and W to be smooth in order to easily use differential forms on them.

Using the same method as in Lemma 3.16. we can find a proper transform $\pi_Z \colon \mathcal{X}'_Z \to X \times \mathbb{P}^1$, with centers contained in $|Z|_{\mathrm{sing}} \cup (|Z| \cap B_X)$, with exceptional divisor D_Z such that

(i) the strict transform $\widehat{Z}$ of $|Z|$ is smooth and does not meet $\widehat{B}_X$;
(ii) the divisor $\widehat{A}_X \cup D_Z \cup \widehat{B}_X$ is a simple normal crossing divisor.

Similarly we construct the proper transform $\pi_W \colon \mathcal{X}'_W \to X \times \mathbb{P}^1$ and define

$$\mathcal{X}_{Z,W} \coloneqq \mathcal{X}'_Z \times_X \mathcal{X}'_W,$$

which is smooth under Assumption 3.27. We denote the union of the centers of blow ups for $\mathcal{X}'_W$ and $\mathcal{X}'_Z$ to be C_W and C_Z respectively. Let

$$\pi' \colon \mathcal{X}_{Z,W} \to X \times \mathbb{P}^1 \times \mathbb{P}^1$$

be the proper morphism induced by the maps π_Z and π_W, and let

$$\pi'_1 \colon \mathcal{X}_{Z,W} \to \mathcal{X}'_Z, \qquad \pi'_2 \colon \mathcal{X}_{Z,W} \to \mathcal{X}'_W,$$

be the projections. We summarize the different maps in the following diagram.

(3.33)
$$
\begin{tikzcd}
& & \mathcal{X}_{Z,W} \arrow[dll, "\pi_1'"'] \arrow[drr, "\pi_2'"] \arrow[dd, "\pi'"] & & \\
\mathcal{X}'_Z \arrow[dd, "\pi_Z"'] & & & & \mathcal{X}'_W \arrow[dd, "\pi_W"] \\
& & X \times \mathbb{P}^1 \times \mathbb{P}^1 \arrow[dll, "\pi_1"'] \arrow[drr, "\pi_2"] & & \\
X \times \mathbb{P}^1 \arrow[drr] & & & & X \times \mathbb{P}^1 \arrow[dll] \\
& & X & &
\end{tikzcd}
$$

We adapt Notation 3.22 to this case, and introduce

Notation 3.29.

$$
\begin{aligned}
A_1 &= \pi_1^{-1} A_X, & A_2 &= \pi_2^{-1} A_X, \\
B_1 &= \pi_1^{-1} B_X, & B_2 &= \pi_2^{-1} B_X, \\
\overline{A}_1 &= (\pi_1')^{-1} \widehat{A}_X, & \overline{B}_1 &= (\pi_1')^{-1} \widehat{B}_X, \\
\overline{A}_2 &= (\pi_2')^{-1} \widehat{A}_X, & \overline{B}_2 &= (\pi_2')^{-1} \widehat{B}_X, \\
\overline{D}_Z &= (\pi_1')^{-1} D_Z, & \overline{D}_W &= (\pi_2')^{-1} D_W, \\
C_1 &= \pi_1^{-1}(C_Z), & C_2 &= \pi_2^{-1}(C_W), \\
\overline{Z} &= (\pi_1')^{-1} \widehat{Z}, & \overline{W} &= (\pi_2')^{-1} \widehat{W}.
\end{aligned}
$$

Here we have denoted by $\widehat{A}_X$ and $\widehat{B}_X$ the strict transforms of A_X and B_X in both blow-ups, $\mathcal{X}_Z$ and $\mathcal{X}_W$. Note that the spaces marked with an overline are subsets of $\mathcal{X}_{Z,W}$ while the others are subsets of $X \times \mathbb{P}^1 \times \mathbb{P}^1$. As before, we will consider the relative schemes

$$
\begin{aligned}
\square_X &= \square_X \times_X \mathbb{G}_X = (X \times \mathbb{P}^1 \times \mathbb{P}^1 \setminus A_1 \cup B_2, B_1 \cup A_2), \\
\underline{\mathcal{X}_{Z,W}} &= (\mathcal{X}_{Z,W} \setminus \overline{A}_1 \cup \overline{B}_2 \cup \overline{D}_Z, \overline{B}_1 \cup \overline{A}_2 \cup \overline{D}_W), \\
\underline{Z} &= (\overline{Z} \setminus \overline{A}_1 \cup \overline{B}_2 \cup \overline{D}_Z, \overline{A}_2) \subset \underline{\mathcal{X}_{Z,W}}, \\
\underline{W} &= (\overline{W} \setminus \overline{A}_1, \overline{B}_1 \cup \overline{A}_2 \cup \overline{D}_W) \subset \underline{\mathcal{X}_{Z,W}}.
\end{aligned}
$$

Finally we write $S = Z \cap W$. Note that, by Assumption 3.27, the subset S can be seen as a the relative scheme $\underline{S} := (S \setminus \emptyset, \emptyset)$ that is a relative subscheme of either $\square_X$ or $\underline{\mathcal{X}_{Z,W}}$. As before, $\overline{B}_2$ does not appear in the definition of $\underline{W}$ because $\widehat{W}$ and $\widehat{B}_X$ are disjoint. Similarly, $\overline{B}_1$ does not appear in the definition of $\underline{Z}$.

In figure 3, there is a more symmetric version of the main diagram in figure 1. The analysis of the main diagram carries through, with small

$$\begin{tikzcd}
 & 0 & 0 & 0 & \\
0 \arrow[r] & H^{2p}(\underline{\mathcal{X}_{Z,W}}) \arrow[r] \arrow[u] & H^{2p}(\underline{\mathcal{X}_{Z,W}} \setminus \overline{Z}) \arrow[r] \arrow[u] & H^{2p+1}_{\underline{Z}}(\underline{\mathcal{X}_{Z,W}}) \arrow[r] \arrow[u] & 5 \\
0 \arrow[r] & H^{2p}(\underline{\mathcal{X}_{Z,W}}, \overline{W}) \arrow[r] \arrow[u] & H^{2p}(\underline{\mathcal{X}_{Z,W}} \setminus \overline{Z}, \overline{W}) \arrow[r] \arrow[u] & H^{2p+1}_{\underline{Z}}(\underline{\mathcal{X}_{Z,W}}, \overline{W}) \arrow[r] \arrow[u] & 7 \\
0 \arrow[r] & H^{2p-1}(\underline{W}) \arrow[r] \arrow[u] & H^{2p-1}(\underline{W} \setminus \overline{S}) \arrow[r] \arrow[u] & H^{2p}_{\underline{S}}(\underline{W}) \arrow[r] \arrow[u] & 0 \\
 & 10 \arrow[u] & 11 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

FIGURE 3. A symmetric version of the main diagram for $n = m = 1$.

modifications to the diagram in figure 3. For instance, using Lemma 1.12, the fact that $\operatorname{codim} C_1 \geq p+1$ and $\dim C_2 \leq p-1$, yield

$$\begin{aligned}
H^{2p}(\underline{\mathcal{X}_{Z,W}}) &= H^{2p}(\square_X \setminus C_1, C_2) \\
&= H^{2p}(\square_X) \\
&= H^{2p-2}(X; -1)
\end{aligned}$$

As in the proof of Proposition 3.28, the group $H^{2p}_{\underline{S}}(\underline{W})$ is pure of weight $2p$. In fact more is true. If $s = \#S$ is the number of points in the intersection, then there is a canonical isomorphism

$$H^{2p}_{\underline{S}}(\underline{W}) \cong \mathbb{Q}(-p)^{\oplus s}.$$

Thus, after pulling back through the class of Z, taking the pushforward with respect to the class of W and twisting by $\mathbb{Q}(p+1)$, we obtain, form figure 3, the particular case of figure 2 depicted in figure 4.

Proposition 3.30. *With Assumption 3.27, the dual of the diagram of figure 4, twisted by $\mathbb{Q}(2)$, agrees with the similar diagram with the role of Z and W reversed. In particular*

$$B_{W,Z} = B^{\vee}_{Z,W}(2), \qquad C_{W,Z} = D^{\vee}_{Z,W}(2), \qquad D_{W,Z} = C^{\vee}_{Z,W}(2).$$

Proof. Since, by condition condition (3.32), we have $(p+1)+(q+1)-2 = d+2 = \dim(\mathcal{X}_{Z,W})$, and by Assumption 3.27 all the subspaces appearing in the diagram in figure 3 are in local product situation, if we take that diagram, twist it by $\mathbb{Q}(p+1)$, then take the dual and finally twist by $\mathbb{Q}(2)$, we obtain the analogous diagram, with Z and W swapped and twisted by $\mathbb{Q}(q+1)$. For instance, the central term of the first diagram twisted by

$$\begin{tikzcd}
 & 0 & 0 & 0 & \\
0 \arrow[r] & H^{2p-2}(X;p) \arrow[r] \arrow[u] & E_Z \arrow[r] \arrow[u] & \mathbb{Q}(0) \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & E_W^\vee(2) \arrow[r] \arrow[u] & B_{Z,W} \arrow[r] \arrow[u] & C_{Z,W} \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & \mathbb{Q}(2) \arrow[r] \arrow[u] & D_{Z,W} \arrow[r] \arrow[u] & \mathbb{Q}(1)^{\oplus s} \arrow[r] \arrow[u] & 0 \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

FIGURE 4. The biextension diagram for $n = m = 1$.

$\mathbb{Q}(p+1)$ is

$$H^{2p}(\underline{\mathcal{X}_{Z,W}} \setminus \overline{Z}, \overline{W}; p+1), \tag{3.34}$$

and $B_{Z,W}$ as a sub quotient this mixed Hodge structure. The dual of this cohomology group, twisted by $\mathbb{Q}(2)$ is

$$H^{2q}(\underline{\mathcal{X}_{Z,W}} \setminus \overline{W}, \overline{Z}; q+1).$$

From this the sought duality follows easily. □

From the diagram of figure 4, and the fact that all the maps there are morphisms of mixed Hodge structures, we deduce the next result.

Corollary 3.31. *If Assumption 3.27 is satisfied, then the mixed Hodge structure $B_{Z,W}$ has weights -4, -2 and 0 and the graded pieces are*

$$\begin{aligned}
\mathrm{Gr}_0^W B_{Z,W} &= \mathbb{Q}(0),\\
\mathrm{Gr}_{-2}^W B_{Z,W} &= H^{2p-2}(X, \mathbb{Q}(p)) \oplus \mathbb{Q}(1)^{\oplus s},\\
\mathrm{Gr}_{-4}^W B_{Z,W} &= \mathbb{Q}(2).
\end{aligned}$$

Therefore it is a generalized biextension. Moreover, if $H^{2p-2}(X, \mathbb{Q}(p))$ is of Hodge–Tate type, the same is true for $B_{Z,W}$.

Remark 3.32. In the case $n = m = 1$, the duality in Proposition 3.30 is not only a duality of mixed Hodge structures, but, as we will see in the next proof of the next proposition, this duality preserves the orientation. This is in contrast with the case $n = m = 0$ as shown in [Hai90, Proposition 3.3.4].

Proposition 3.33. *With Assumption 3.27, we have*

$$\mathrm{ht}(B_{Z,W}) = -\,\mathrm{ht}(B_{W,Z}).$$

Proof. By Proposition 2.12 we only need to show that the duality between $B_{Z,W}$ and $B_{W,Z}$ preserves the orientation. The mixed Hodge structure $B_{Z,W}$

is a subquotient of $H^{2p}(\underline{\mathcal{X}_{Z,W}} \setminus \overline{Z}, \overline{W}; p+1)$, Hence its elements can be represented by differential forms in

$$E_1 = \Sigma_{\overline{B}_1 \cup \overline{A}_2 \cup \overline{D}_W \cup \overline{W}} E^{2p}_{\mathcal{X}_{Z,W}}(\log \overline{A}_1 \cup \overline{B}_2 \cup \overline{D}_Z \cup \overline{Z}; p+1),$$

while the elements in $B_{W,Z}$ can be represented by forms in

$$E_2 = \Sigma_{\overline{A}_1 \cup \overline{B}_2 \cup \overline{D}_Z \cup \overline{Z}} E^{2q}_{\mathcal{X}_{Z,W}}(\log \overline{B}_1 \cup \overline{A}_2 \cup \overline{D}_W \cup \overline{W}; q+1).$$

The duality is given by the map

$$\langle \alpha, \beta \rangle = \int_{\mathcal{X}_{Z,W}} \alpha \wedge \beta.$$

The class of Z is represented by a differential form $\nu_Z \in E_1$ and its dual class can be represented by a differential form $\mu_Z \in E_2$. Similarly we have differential forms ν_W and μ_W. These forms satisfy

$$\int_{\mathcal{X}_{Z,W}} \nu_Z \wedge \mu_Z = \int_{\mathcal{X}_{Z,W}} \nu_W \wedge \mu_W.$$

The orientation of $B_{Z,W}$ is given by the classes (ν_Z, μ_W) and the orientation of $B_{W,Z}$ by the classes (ν_W, μ_Z). Since

$$\langle \nu_Z, \mu_Z \rangle = \int_{\mathcal{X}_{Z,W}} \nu_Z \wedge \mu_Z = 1$$

and

$$\langle \mu_W, \nu_W \rangle = \int_{\mathcal{X}_{Z,W}} \mu_W \wedge \nu_W = (-1)^{4pq} = 1 \tag{3.35}$$

we obtain that the duality preserves orientations and hence the result. Note that in equation (3.35) we are using that $n = m = 1$, that implies that the forms μ_W and ν_W have even degree. In the case $n = m = 0$ the differential forms have odd degree, hence the similar duality would not be compatible with the orientations. □

4. Invariants attached to the mixed Hodge structure $B_{Z,W}$

In this section we suppose that Assumption 3.27 is satisfied and compute the Deligne splitting δ of $B_{Z,W}$ (see (2.6)). This map characterizes $B_{Z,W}$ as a real mixed Hodge structure.

4.1. A decomposition of the Deligne splitting of $B_{Z,W}$.

Since we will be considering different mixed Hodge structures we will use the following variant of the notation in Section 2 to keep track of them.

Notation 4.1. For a MHS H, we will denote the Deligne bigrading as $H_{\mathbb{C}} = \bigoplus_{r,s} I^{r,s}_H$, and will denote the various projections to the individual $I^{r,s}_H$ by $\Pi_{I^{r,s}_H}$. Similarly the projection to the piece $\bigoplus_{p+q=k} I^{p,q}_H$ of pure weight k will be denoted $\Pi_{H,k}$. Also, the Deligne splitting of H will be denoted δ_H.

After Corollary 3.31, the Deligne bigrading of $B \coloneqq B_{Z,W}$ (See (2.1)) has the shape

$$B_{\mathbb{C}} = I^{0,0}_B \oplus \left(\bigoplus_{a+b=-2} I^{a,b}_B \right) \oplus I^{-2,-2}_B$$

Similarly, the bigradings of $C := C_{Z,W}$, $D := D_{Z,W}$, E_Z and $E_W^\vee$ are given by

$$\begin{aligned} C_{\mathbb{C}} &= I_C^{0,0} \oplus I_C^{-1,-1}, & D_{\mathbb{C}} &= I_D^{-1,-1} \oplus I_D^{-2,-2}, \\ E_{Z,\mathbb{C}} &= I_{E_Z}^{0,0} \oplus \bigoplus_{a+b=-2} I_{E_Z}^{a,b}, & E_{W,\mathbb{C}}^\vee(2) &= \bigoplus_{a+b=-2} I_{E_W^\vee(2)}^{a,b} \oplus I_{E_W^\vee(2)}^{-2,-2}. \end{aligned}$$

Since $H^{2p-2}(X, \mathbb{Q}(p))$ and $\mathbb{Q}(1)^s$ are pure Hodge structures of weight -2, their Deligne bigradings are given by

$$H^{2p-2}(X, \mathbb{Q}(p))_{\mathbb{C}} = \bigoplus_{a+b=-2} I_1^{a,b}, \qquad \mathbb{Q}(1)^s_{\mathbb{C}} = I_2^{-1,-1}.$$

The functoriality of the Deligne bigrading and the diagram of figure 4, give us canonical identifications

$$\begin{aligned} I_B^{0,0} &= I_C^{0,0} = I_{E_Z}^{0,0}, \\ I_{E_Z}^{a,b} &= I_{E_W^\vee(2)}^{a,b} = I_1^{a,b}, \quad \text{for } a+b=-2, \\ I_C^{-1,-1} &= I_D^{-1,-1} = I_2^{-1,-1}, \\ I_B^{-1,-1} &= I_C^{-1,-1} \oplus I_{E_Z}^{-1,-1}, \\ I_B^{a,b} &= I_{E_Z}^{a,b}, \quad \text{for } a+b=-2,\ a \neq -1, \\ I_B^{-2,-2} &= I_D^{-2,-2} = I_{E_W^\vee(2)}^{-2,-2}. \end{aligned} \tag{4.1}$$

In terms of the graded pieces of the weight filtration we obtain identifications

$$\begin{aligned} \operatorname{Gr}_0^W B &= \operatorname{Gr}_0^W C = \operatorname{Gr}_0^W E_Z = \mathbb{Q}(0), \\ \operatorname{Gr}_{-2}^W B &= \operatorname{Gr}_{-2}^W C \oplus \operatorname{Gr}_{-2}^W E_Z \\ &= \operatorname{Gr}_{-2}^W D \oplus \operatorname{Gr}_{-2}^W E_W^\vee(2) = H^{2p-2}(X, \mathbb{Q}(p)) \oplus \mathbb{Q}(1)^s_{\mathbb{C}}, \\ \operatorname{Gr}_{-4}^W B &= \operatorname{Gr}_{-4}^W D \oplus \operatorname{Gr}_{-4}^W E_W^\vee(2) = \mathbb{Q}(2). \end{aligned} \tag{4.2}$$

As in the proof of Lemma 2.6 there is a decomposition

$$\delta_B = \delta_1 + \delta_2 + \delta_3,$$

with

$$\delta_1 \colon \operatorname{Gr}_0^W B \to \operatorname{Gr}_{-2}^W B, \quad \delta_2 \colon \operatorname{Gr}_{-2}^W B \to \operatorname{Gr}_{-4}^W B, \quad \delta_3 \colon \operatorname{Gr}_0^W B \to \operatorname{Gr}_{-4}^W B.$$

Using the identifications (4.2), we can write

$$\delta_1 = \delta_{E_Z} + \delta_C, \qquad \delta_2 = \delta_{E_W^\vee(2)} + \delta_D.$$

Moreover, $\delta_3 = \delta_B^{-2,-2}$ as in Definition 2.3. Therefore, if e and $e^\vee$ are the generators of $I_B^{0,0}$ and $I_B^{-2,-2}$ given by the orientation of $B_{Z,W}$, then the height of B is determined by the equation

$$\delta_3(e) = \operatorname{ht}(B) e^\vee.$$

In conclusion, the Deligne splitting δ_B is characterized by the invariants δ_{E_Z}, $\delta_{C_{Z,W}}$, $\delta_{E_W^\vee(2)}$, $\delta_{D_{Z,W}}$ and $\operatorname{ht}(B)$. By duality, the invariant $\delta_{E_W^\vee(2)}$ is determined by δ_{E_W} and $\delta_{D_{Z,W}}$ by $\delta_{C_{W,Z}}$. So we will concentrate in the computation of the invariants δ_{E_Z}, δ_C and $\mathrm{ht}(B)$. By Lemma 2.6 and equation (2.16), we get

$$\delta_{E_Z}(e) = \frac{i}{2}\Pi_{E_Z,-2}\left(\overline{e} - e\right) = \frac{i}{2}\Pi_{E_Z,-2}\left(\overline{e}\right), \tag{4.3}$$

$$\delta_C(e) = \frac{i}{2}\Pi_{I_C^{-1,-1}}\left(\overline{e} - e\right) = \frac{i}{2}\Pi_{I_C^{-1,-1}}\left(\overline{e}\right), \tag{4.4}$$

$$\mathrm{ht}(B)e^{\vee} = -\frac{1}{2}\,\mathrm{Im}\left(\Pi_{I_B^{-2,-2}}\left(\overline{e}\right)\right). \tag{4.5}$$

In this section we will concentrate in the computation of $\delta_{E_Z}(e)$, $\delta_{C_{W,Z}}(e)$ and $\mathrm{ht}(B)$. Moreover, we will show that, when the regulators of Z and W are zero, the height $\mathrm{ht}(B)$ is given by the higher archimedean height pairing.

4.2. **Computation of $\delta_{E_Z}(e)$.** We first compute $\delta_{E_Z}(e)$ using the mixed Hodge structure arising from 3.9. To this end, we will find an element $e \in I_{E_Z}^{0,0}$ that is mapped to the standard generator of $\mathbb{Q}(0)$. Most of the job has been done in Section 3.3. Let η_Z, g_Z and θ_Z be the differential forms provided by Proposition 3.6. In particular,

$$\eta_Z \in F^0\Sigma_{B_X}E^{2p-1}_{X\times\mathbb{P}^1}(\log A_X \cap |Z|; p),$$

with $d\eta_Z = 0$. We claim that the class of η_Z,

$$\{\eta_Z\} \in H^{2p-1}(X \times \mathbb{P}^1 \setminus A_X \cup |Z|, B_X; p)$$

gives us the sought element e.

By Proposition 3.6, the pair $(0, \eta_Z)$ is a cycle in the simple complex associated to the morphism (3.12), representing the cohomology class of Z. By the construction of E_Z, this implies that $\{\eta_Z\}$ belongs to E_Z and that it is mapped to the standard generator of $\mathbb{Q}(0)$. We still need to show that this class belongs to $I_{E_Z}^{0,0}$. By (2.2) and the shape of E_Z,

$$I_{E,Z}^{0,0} = F^0 \cap \left(\overline{F^0} + \overline{F^{-1}} \cap W_{-2}\right).$$

By the construction of η_Z, the class $\{\eta_Z\}$ belongs to F^0. From the equation

$$dg_Z = \frac{1}{2}(\eta_Z - \overline{\eta}_Z) - \theta_Z, \tag{4.6}$$

with $\theta_Z \in F^{-1}\Sigma_{B_X}E^{2p-1}_{X\times\mathbb{P}^1}(\log A_X; p)$, and the fact that $H^{2p-1}(X \times \mathbb{P}^1 \setminus A_X, B_X; p) = H^{2p-2}(X; p)$ is pure of weight -2, we conclude that the cohomology class $\{\eta_Z\}$ belongs to $\overline{F^0} + \overline{F^{-1}} \cap W_{-2}$. Hence $e \coloneqq \{\eta_Z\} \in I_{E_Z}^{0,0}$ is the generator we are looking for.

Using again equation (4.6) and the fact that the class $\{\theta_Z\}$ of θ_Z belongs to W_{-2}, we deduce that

$$\delta_{E_Z}(e) = \frac{i}{2}\Pi_{E_Z,-2}(\overline{e} - e) = -i\widetilde{\theta_Z} = -i\widetilde{\Psi(\theta_Z)}, \tag{4.7}$$

where, in the last equation, we are using the map Ψ from Definition 3.2 to identify $H^{2p-1}(X \times \mathbb{P}^1 \setminus A_X, B_X; p)$ with $H^{2p-2}(X; p)$. Recall that, by Proposition 3.8, the class $\widetilde{\Psi(\theta_Z)}$ represents the Goncharov regulator of Z. So, essentially, the invariant $\delta_{E_Z}(e)$ is the regulator of Z. Note that the factor i comes from the fact that in the chosen normalization, the regulator is purely imaginary, while the map δ is chosen to be real.

Remark 4.2. Although we have written the above computation for $n = 1$ to keep parity with the rest of section 4.1, since Section 3.3 is valid for general $n \geq 1$, the same is true for the above computation.

We now make the computation in the mixed Hodge structure $B_{Z,W}$ as the techniques involved will be used latter in the computations of the other invariants. As before, the key point is to find the generator e of $I_B^{0,0}$. We see $B_{Z,W}$ as a subquotient of

$$H^{2p}(\underline{\mathcal{X}_{Z,W}} \setminus \overline{Z}, \overline{W}; p+1).$$

Hence we will work on the smooth projective variety $\mathcal{X}_{Z,W}$ introduced in Section 3.8.

Notation 4.3. We choose (t_1, t_2) affine coordinates of $\square^2$. We denote

$$\frac{dt_1}{t_1}, \frac{dt_2}{t_2} \in F^0\Sigma_A E^1_{(\mathbb{P}^1)^2}(\log B; 1).$$

Recall, as Example 1.7, that this implies

$$\overline{\left(\frac{dt_1}{t_1}\right)} = -\frac{d\bar{t}_1}{\bar{t}_1} \tag{4.8}$$

Moreover, when working with differential forms on the smooth projective variety $\mathcal{X}_{Z,W}$, that come from other spaces in diagram (3.33), we will not write down explicitly the pullback map. For instance we will denote by η_Z the differential form $(\pi_Z \circ \pi_1')^*\eta_Z$. Similarly dt_1/t_1 and dt_2/t_2 will also denote differential forms on $\mathcal{X}_{Z,W}$.

We have the following characterization of $I_B^{0,0}$.

Lemma 4.4. *An element $\xi \in B_{Z,W}$ belongs to $I_B^{0,0}$ if and only if*

(i) the condition $\xi \in F^0B_{Z,W}$ holds;

(ii) the image of ξ in E_Z belongs to $I_{E_Z}^{0,0}$.

Proof. The implication 'only if' is clear from the fact that $\xi \in I_B^{0,0}$ implies that $\xi \in F^0B_{Z,W}$, and that $\rho\colon B_{Z,W} \to E_Z$ is a morphism of mixed Hodge structures. To show the if part, we note first that $\ker(\rho) = D_{Z,W}$. Since $D_{Z,W}$ is an extension of $\mathbb{Q}(1)^s$ by $\mathbb{Q}(2)$, we get

$$\begin{aligned} D_{Z,W} \subset F^{-2} \cap W_{-4}B_{Z,W} + F^{-1} \cap W_{-2}B_{Z,W} \\ \subset F^{-2} \cap W_{-3}B_{Z,W} + F^{-1} \cap W_{-2}B_{Z,W}. \end{aligned}$$

By assumption, $\xi \in F^0B_{Z,W}$, and we need to check that

$$\overline{\xi} \in F^0B_{Z,W} + F^{-1} \cap W_{-2}B_{Z,W} + F^{-2} \cap W_{-3}B_{Z,W}.$$

Now since also by assumption, $\rho(\xi) \in I_{E_Z}^{0,0}$, we obtain a $\xi' \in I_B^{0,0}$, such that $\rho(\xi') = \rho(\xi)$. Hence $\rho(\xi - \xi') = 0$. Since ρ is a real map, we get $\rho(\overline{\xi} - \overline{\xi'}) = 0$. Thus $\overline{\xi} - \overline{\xi'} \in \ker(\rho) = D_{Z,W}$ and

$$\overline{\xi} \in \overline{\xi'} + D_{Z,W} \subset F^0B_{Z,W} + F^{-1} \cap W_{-2}B_{Z,W} + F^{-2} \cap W_{-3}B_{Z,W},$$

as required. Hence $\xi \in I_B^{0,0}$, and the lemma follows. $\square$

Now we have the following

Proposition 4.5. *Let η_Z be as above, and write, using Notation 4.3,*

$$\nu_Z \coloneqq -\eta_Z \wedge \frac{dt_2}{t_2} \in E^{2p}_{\mathcal{X}_{Z,W}}(\log \overline{A_2} \cup \overline{B_1} \cup \overline{D_Z} \cup \overline{Z}; p+1).$$

Then the cohomology class $\{\nu_Z\}$ is the generator e of $I^{0,0}_B$ that is sent to the canonical generator of $\mathbb{Q}(0)$.

Proof. We first have to show that ν_Z belongs to

$$F^0 \Sigma_{\overline{A_1} \cup \overline{B_2} \cup \overline{D_W} \cup \overline{W}} E^{2p}_{\mathcal{X}_{Z,W}}(\log \overline{A_2} \cup \overline{B_1} \cup \overline{D_Z} \cup \overline{Z}; p+1).$$

For this, the only point that has to be checked is that $\nu_Z|_{\overline{W}}$ vanishes. As differential form ν_Z belongs to F^{p+1}, but

$$\dim(\overline{W}) = d + 2 - q = p.$$

Therefore, $\nu_Z|_{\overline{W}} = 0$. Since η_Z is closed, the same is true for ν_Z. By the explicit description of the isomorphism (3.2), we see that the class $\{\nu_Z\}$ is sent to $\{\eta_Z\}$. In particular to the canonical generator of $\mathbb{Q}(0)$. It remains to be shown that it belongs to $I^{0,0}_B$. The map $B_{Z,W} \to E_Z$ sends that class $\{\nu_Z\}$ to the class $\{\eta_Z\}$ that belongs to $I^{0,0}_{E_Z}$. By Lemma 4.4, $\{\nu_Z\}$ belongs to $I^{0,0}_B$ completing the proof. □

To compute $\delta_{E_Z}(e)$ using ν_Z, it is easier to first project to the cohomology group

$$H^{2p}(\underline{\mathcal{X}_{Z,W}} \setminus \overline{Z}; p+1),$$

that is, we remove the condition of vanishing along W. In the complex

$$\Sigma_{\overline{A_1} \cup \overline{B_2}} E^{*}_{\mathcal{X}_{Z,W}}(\log \overline{A_2} \cup \overline{B_1} \cup \overline{D_Z} \cup \overline{Z}; p+1),$$

equations (4.6) and (4.8), and the fact that η_Z has odd degree, imply that

$$\frac{1}{2}(\nu_Z - \overline{\nu}_Z) - \left(-\theta_Z \wedge \frac{dt_2}{t_2}\right) = d\left(-g_Z \wedge \frac{dt_2}{t_2} + \frac{1}{2}(\log(t_2\overline{t}_2))\overline{\eta}_Z\right). \tag{4.9}$$

From this equation, we conclude again that the invariant $\delta_{E_Z}(e)$ is given by equation (4.7).

4.3. **Computation of $\delta_C(e)$.** Since the form ν_Z represents the generator $e \in I^{0,0}_B$ its image in $C_{Z,W}$ represents the generator $e \in I^{0,0}_C$. To compute this image, we project to the cohomology group $H^{2p+1}_{\underline{Z}}(\underline{\mathcal{X}_{Z,W}}, \overline{W})$. The class of ν_Z is sent to the class of $(0, \nu_Z)$. We know that the class of

$$\lambda_Z \coloneqq \frac{i}{2}(0, \overline{\nu}_Z - \nu_Z) \tag{4.10}$$

is sent to zero in the cohomology group $H^{2p+1}_{\underline{Z}}(\underline{\mathcal{X}_{Z,W}})$. Therefore, according to equation (4.4), in order to compute $\delta_C(e)$, we need to find a preimage of the class of λ_Z in the group $H^{2p}_{\underline{S}}(\underline{W})$. Using Proposition 1.14, the fact that $\overline{W}$ is smooth and the standard description of the connection morphism associated to a short exact sequence, the method to find this preimage is the following. First we find a primitive of λ_Z in the complex that computes the cohomology $H^{*}_{\underline{Z}}(\underline{\mathcal{X}_{Z,W}})$, then we restrict this primitive to the relative scheme

$\underline{W}$ and the class of this restriction will agree with $\delta_C(e)$. By equation (4.9), we have

$$\lambda_Z = d\left(i\theta_Z \wedge \frac{dt_2}{t_2}, -ig_Z \wedge \frac{dt_2}{t_2} + \frac{i}{2}\log(t_2\bar{t}_2)\eta_Z\right)$$

Therefore, by the previous discussion, the class $\delta_C(e)$ is represented by

$$\left(i\theta_Z \wedge \frac{dt_2}{t_2}, -ig_Z \wedge \frac{dt_2}{t_2} + \frac{i}{2}\log(t_2\bar{t}_2)\eta_Z\right)\Big|_{\underline{W}}. \tag{4.11}$$

In order to compute explicitly the cohomology class represented by this form, we use that S is disjoint with $\overline{A}_1 \cup \overline{B}_1 \cup \overline{A}_2 \cup \overline{D}_W$, therefore

$$H^*_{\underline{S}}(\underline{W}) = H^*_S(\overline{W}). \tag{4.12}$$

We write $S = \{(x_j, t_{1,j}, t_{2,j})\}_{j=1,\dots,s}$ and denote by e_j the Betti generator of the term $\mathbb{Q}(1)_{\mathbb{Q}}$ corresponding to the point $(x_j, t_{1,j}, t_{2,j})$, for $j = 1, \dots, s$. We also denote by $\mu_{Z,j}$ the multiplicity of the cycle Z in the component of Z containing $(x_j, t_{1,j})$. Using equation (3.14), we have the residue computation

$$d\left[\left(-ig_Z \wedge \frac{dt_2}{t_2} + \frac{i}{2}\log(t_2\bar{t}_2)\eta_Z\right)\Big|_{\underline{W}}\right] = \\ \left[i\theta_Z \wedge \frac{dt_2}{t_2}\Big|_{\underline{W}}\right] - \frac{i}{2}\sum_{j=1}^{s}\log(t_{2,j}\bar{t}_{2,j})\mu_{Z,j}\delta_{(x_j,t_{1,j},t_{2,j})}.$$

Since $\overline{W}$ is smooth, we can compute the cohomology (4.12) using currents. From the residue computation it follows that

(4.13)

$$\delta_C(e) = \frac{i}{2}\sum_{j=1}^{s}\log(t_{2,j}\bar{t}_{2,j})\mu_{Z,j}\delta_{(x_j,t_{1,j},t_{2,j})} = \frac{1}{4\pi}\sum_{j=1}^{r}\mu_{Z,j}\log(t_{2,j}\bar{t}_{2,j})e_j.$$

In the second equality we are using the implicit de Rham generator carried by $\log(t_{2,j}\bar{t}_{2,j})$:

$$\log(t_{2,j}\bar{t}_{2,j}) = \log(t_{2,j}\bar{t}_{2,j}) \otimes \mathbb{1}(1)_{\mathbb{C}} = \frac{1}{2\pi i}\log(t_{2,j}\bar{t}_{2,j}) \otimes \mathbb{1}(1)_{\mathbb{Q}}.$$

As expected, the invariant $\delta_C(e)$ is real.

4.4. **Computation of ht**(B)**.** Since we will need to consider also the dual construction, we denote by $e_{Z,W}$ the generator of $I^{0,0}_{B_{Z,W}}$ previously denoted by e and by $e^\vee_{Z,W}$ the generator of $I^{-2,-2}_{B_{Z,W}}$. By Proposition 4.5, we know that $e_{Z,W}$ is represented by ν_Z. By equation (4.5) we have that

$$\mathrm{ht}(B)e^\vee_{Z,W} = -\frac{1}{2}\,\mathrm{Im}\left(\Pi_{I_B^{-2,-2}}\left(\bar{e}_{Z,W}\right)\right). \tag{4.14}$$

We consider the dual mixed Hodge structure $B_{W,Z}(-2) = B^\vee_{Z,W}$ with decomposition,

$$B_{W,Z}(-2)_{\mathbb{C}} = J^{2,2}\bigoplus\left(\bigoplus_{l+s=2} J^{l,s}\right)\bigoplus J^{0,0}.$$

Let $e_{W,Z}(-2)$ be the generator of $J^{2,2}$ that is mapped to the generator $\mathbb{1}(-2)_{\mathbb{Q}}$ of $\mathbb{Q}(-2)_{\mathbb{Q}}$. It is constructed as in Section 4.2 with Z and W swapped. It satisfies the conditions

$$\langle e_{W,Z}(-2), e^{\vee}_{Z,W}\rangle = 1, \tag{4.15}$$

$$e_{W,Z}(-2) \in \left(\bigoplus_{a+b=-2} I_B^{a,b} \oplus I_B^{0,0}\right)^{\perp}. \tag{4.16}$$

Equations (4.14), (4.15) and (4.16) imply that

$$\operatorname{ht}(B) = -\frac{1}{2}\operatorname{Im}\langle e_{W,Z}(-2), \overline{e}_{Z,W}\rangle.$$

The class $e_{Z,W}$ is represented by the form

$$\nu_Z \in F^0\Sigma_{\overline{A_1}\cup\overline{B_2}\cup\overline{D_W}\cup\overline{W}} E^{2p}_{\mathcal{X}_{Z,W}}(\log \overline{A_2}\cup\overline{B_1}\cup\overline{D_Z}\cup\overline{Z}; p+1).$$

while the class $e_{W,Z}$ is represented by

$$\nu_W = -\eta_W \wedge \frac{dt_1}{t_1} \in F^0\Sigma_{\overline{A_2}\cup\overline{B_1}\cup\overline{D_Z}\cup\overline{Z}} E^{2q}_{\mathcal{X}_{Z,W}}(\log \overline{A_1}\cup\overline{B_2}\cup\overline{D_W}\cup\overline{W}; q+1).$$

Note that the subset where ν_Z may have logarithmic singularities agrees with the subset where ν_W vanishes and reciprocally. Therefore the differential form $\nu_Z \wedge \nu_W$ is locally integrable in $\mathcal{X}_{Z,W}$, and the duality pairing is given by

$$\langle e_{W,Z}(-2), \overline{e}_{Z,W}\rangle = \frac{1}{(2\pi i)^2}(p_{\mathcal{X}_{Z,W}})_*[\nu_W \wedge \overline{\nu}_Z] = \frac{1}{(2\pi i)^{d+4}}\int_{\mathcal{X}_{Z,W}} \nu_W \wedge \overline{\nu}_Z,$$

where $p_{\mathcal{X}_{Z,W}} \colon \mathcal{X}_{Z,W} \to \operatorname{Spec}(\mathbb{C})$ is the structural map. In consequence, the height of $B_{Z,W}$ is given by

$$\begin{aligned}\operatorname{ht}(B) &= -\frac{1}{2}\operatorname{Im}\frac{1}{(2\pi i)^{p+q+2}}\int_{\mathcal{X}_{W,Z}} \nu_W \wedge \overline{\nu}_Z \\ &= \frac{1}{2}\operatorname{Im}\frac{1}{(2\pi i)^{p+q+2}}\int_{\mathcal{X}_{W,Z}} \eta_W \wedge \frac{dt_1}{t_1} \wedge \overline{\eta}_Z \wedge \frac{d\bar{t}_2}{\bar{t}_2}.\end{aligned} \tag{4.17}$$

Recall for the last equality that

$$\overline{\nu}_Z = \overline{-\eta_Z \wedge \frac{t_2}{t_2}} = \eta_Z \wedge \frac{\bar{t}_2}{\bar{t}_2}$$

Using the fact that $g_Z|_{t_1=0} = g_Z|_{t_1=\infty} = 0$ and that $\eta_W|_{t_2=0} = \eta_W|_{t_2=\infty} = 0$, the residue theorem, and the relations

$$d[g_Z] = \left[\frac{1}{2}(\eta_Z - \overline{\eta}_Z) - \theta_Z\right], \quad d[\eta_W] = -\delta_W,$$

we have

$$\begin{aligned} d\left[\eta_W \wedge \frac{dt_1}{t_1} \wedge g_Z \wedge \frac{d\bar{t}_2}{\bar{t}_2}\right] &= -\delta_W \wedge \frac{dt_1}{t_1} \wedge g_Z \wedge \frac{d\bar{t}_2}{\bar{t}_2} + \frac{1}{2}\left[\eta_W \wedge \frac{dt_1}{t_1} \wedge \eta_Z \wedge \frac{d\bar{t}_2}{\bar{t}_2}\right] \\ &\quad - \frac{1}{2}\left[\eta_W \wedge \frac{dt_1}{t_1} \wedge \overline{\eta}_Z \wedge \frac{d\bar{t}_2}{\bar{t}_2}\right] - \left[\eta_W \wedge \frac{dt_1}{t_1} \wedge \theta_Z \wedge \frac{d\bar{t}_2}{\bar{t}_2}\right]. \end{aligned}$$

For type reasons the second term on the right hand side is zero (as differential form, η_Z is in F^p, η_W is in F^q, so the term is in F^{p+q+1}, but $p+q+1 = d+3 > d+2$). Hence, by Stokes' theorem,

$$\begin{aligned}(4.18)\quad \operatorname{ht}(B) = \operatorname{Im}\frac{-1}{(2\pi i)^{p+2}}\int_W \frac{dt_1}{t_1}\wedge g_Z \wedge \frac{d\bar{t}_2}{\bar{t}_2} \\ + \operatorname{Im}\frac{-1}{(2\pi i)^{p+q+2}}\int_{\mathcal{X}_{W,Z}} \eta_W \wedge \frac{dt_1}{t_1}\wedge\theta_Z\wedge\frac{d\bar{t}_2}{\bar{t}_2}.\end{aligned}$$

The first term on the right hand side of the above equation resembles the higher height pairing, and in fact, it agrees with the higher height pairing, in case the real regulators of the cycles are zero.

Remark 4.6. Although, to define the extension $B_{Z,W}$, we needed to go to the blow up $\mathcal{X}_{Z,W}$ in order to be in local product situation and use duality, In the actual computation of $\operatorname{ht}(B_{Z,W})$ we can remain in $X\times\mathbb{P}^1\times\mathbb{P}^1$.

4.5. **Connection to the higher height pairing when the regulators are zero.** In this subsection we want to compare $\operatorname{ht}(B)$ to the higher archimedean height pairing $\langle Z, W\rangle_{\mathrm{Arch}}$, when the real regulator classes of Z and W are both zero and Assumption 3.27 is satisfied. This can be seen as a generalization of Hain's result [Hai90] relating the archimedean height pairing for the usual cycles homologous to zero with biextensions of mixed Hodge structures.

Before doing the comparison, we need to put both invariants in the same place. Recall that

$$\langle Z, W\rangle_{\mathrm{Arch}} \in H^1_{\mathfrak{D}}(\operatorname{Spec}(\mathbb{C});\mathbb{R}(2)) = \mathbb{Q}(2)_{\mathbb{C}}/\mathbb{Q}(2)_{\mathbb{R}},$$

while

$$\operatorname{ht}(B)\in\mathbb{R}.$$

We denote by $\rho_2\colon \mathbb{Q}(2)_{\mathbb{C}}/\mathbb{Q}(2)_{\mathbb{R}}\to\mathbb{R}$ the isomorphism given by

$$(4.19)\qquad \rho_2(v) = \operatorname{Im}\left(\frac{v}{(2\pi i)^2}\right).$$

Theorem 4.7. *If the real regulators of Z and W are zero, then*

$$\rho_2(\langle Z, W\rangle_{\mathrm{Arch}}) = \operatorname{ht}(B).$$

Proof. Since the real regulators of Z and W are zero, by Corollary 3.9, we can choose g_Z and η_Z with $\theta_Z = 0$ and the same for W. With this choice, after changing the order of the terms, equation (4.17) can be written as

$$\operatorname{ht}(B) = \operatorname{Im}\left(\frac{1}{(2\pi i)^2}(p)_*\left(\delta_W\wedge\frac{d\bar{t}_2}{\bar{t}_2}\wedge g_Z\wedge\frac{dt_1}{t_1}\right)\right).$$

Since $n = m = 1$ and the $*$-product is graded commutative, we have that

$$\langle Z, W\rangle_{\mathrm{Arch}} = -\langle W, Z\rangle_{\mathrm{Arch}}.$$

By Corollary 3.15, for $n = m = 1$ we have

$$\langle Z, W\rangle_{\mathrm{Arch}} = -\langle W, Z\rangle_{\mathrm{Arch}} = (p)_*\left(\delta_{W,\mathrm{TW}}\cdot W_1(t_2)\cdot g_{Z,\mathrm{TW}}\cdot W_1(t_1)\right)^{\sim},$$

as an element in $H^1_{\mathfrak{D}}(\mathrm{Spec}(\mathbb{C}), \mathbb{R}(2))$. Here

$$W_1(t_2) := -\frac{1}{2}\left((\epsilon+1)\otimes\frac{dt_2}{t_2} + (\epsilon-1)\otimes\frac{d\bar{t}_2}{\bar{t}_2} + d\epsilon\otimes\log(t_2\bar{t}_2)\right),$$
$$W_1(t_1) := -\frac{1}{2}\left((\epsilon+1)\otimes\frac{dt_1}{t_1} + (\epsilon-1)\otimes\frac{d\bar{t}_1}{\bar{t}_1} + d\epsilon\otimes\log(t_1\bar{t}_1)\right),$$

while

$$g_{Z,\mathrm{TW}} = \frac{\epsilon+1}{2}\otimes\eta_Z - \frac{\epsilon-1}{2}\otimes\overline{\eta}_Z + d\epsilon\otimes g_Z.$$

In order to prove the proposition, we need to unwrap the product in the TW-complex and use Stokes' theorem. Since the pullback of W in $X\times(\mathbb{P}^1)^2$ has dimension p, we get

$$\langle Z, W\rangle_{\mathrm{Arch}} = (p)_*\left(f(\epsilon)d\epsilon\otimes\delta_W\wedge(\Omega_1+\Omega_2+\Omega_3)\right),$$

where $f(\epsilon) = \frac{1}{4}(\epsilon^2-1)$ and

$$\Omega_1 = -\frac{d\bar{t}_2}{\bar{t}_2}\wedge g_Z\wedge\frac{dt_1}{t_1} - \frac{dt_2}{t_2}\wedge g_Z\wedge\frac{d\bar{t}_1}{\bar{t}_1},$$
$$\Omega_2 = \frac{d\bar{t}_2}{\bar{t}_2}\wedge\frac{\eta_Z}{2}\log(t_1\bar{t}_1) - \frac{dt_2}{t_2}\wedge\frac{\overline{\eta}_Z}{2}\log(t_1\bar{t}_1),$$
$$\Omega_3 = \log(t_2\bar{t}_2)\frac{\eta_Z}{2}\wedge\frac{d\bar{t}_1}{\bar{t}_1} - \log(t_2\bar{t}_2)\frac{\overline{\eta}_Z}{2}\wedge\frac{dt_1}{t_1}.$$

In the computation above one has to take into account that $d\epsilon$ anticommutes with forms of odd degree. Now let

$$\Lambda_1 := \delta_W\wedge d(\log(t_2\bar{t}_2))\wedge g_Z\log(t_1\bar{t}_1),$$
$$\Lambda_2 := \delta_W\wedge\log(t_2\bar{t}_2)g_Z\wedge d(\log(t_1\bar{t}_1)).$$

Then one can easily see that

$$d\Lambda_1 = \delta_W\wedge(\Omega_1-\Omega_2), \qquad d\Lambda_2 = \delta_W\wedge(\Omega_3-\Omega_1).$$

Since our higher height pairing is an element of the Deligne cohomology group, we conclude

$$\begin{aligned}\langle Z, W\rangle_{\mathrm{Arch}} &= (p)_*\left(f(\epsilon)d\epsilon\otimes\delta_W\wedge(\Omega_1+\Omega_1+\Omega_1)\right)\\ &= 3f(\epsilon)d\epsilon\otimes(p)_*\left(\delta_W\wedge\Omega_1\right)\end{aligned}$$

After integrating $f(\epsilon)$ from 0 to 1, we arrive at

$$\langle Z, W\rangle_{\mathrm{Arch}} = -\frac{1}{2}(p)_*\left(\delta_W\wedge\Omega_1\right)$$

Finally, using (remember Notation 1.3)

$$\overline{\delta_W\wedge\frac{d\bar{t}_2}{\bar{t}_2}\wedge g_Z\wedge\frac{dt_1}{t_1}} = -\delta_W\wedge\frac{dt_2}{t_2}\wedge g_Z\wedge\frac{d\bar{t}_1}{\bar{t}_1},$$

we conclude

$$\frac{1}{2}(p)_*\left(\delta_W\wedge\Omega_1\right) = -i\,\mathrm{Im}(p)_*\left(\delta_W\wedge\frac{d\bar{t}_2}{\bar{t}_2}\wedge g_Z\wedge\frac{dt_1}{t_1}\right).$$

Hence we get

$$\begin{aligned}\rho_2(\langle Z,W\rangle_{\mathrm{Arch}}) &= \operatorname{Im}\left(\frac{1}{(2\pi i)^2}(p)_*\left(\delta_W\wedge\frac{d\bar t_2}{\bar t_2}\wedge g_Z\wedge\frac{dt_1}{t_1}\right)\right)\\ &= \mathrm{ht}(B).\end{aligned}$$

$\square$

5. Examples of higher height pairing

5.1. **The case of dimension zero.** As a starter we discuss the case when $X=\mathrm{Spec}(\mathbb{C})$, so $d=0$, and $n=m=p=q=1$. Let $a,b\in\mathbb{C}\setminus\{0,1\}$ then a and b define cycles in $Z^1(X,1)_{00}$ that we denote Z and W. Moreover these cycles always have proper intersection and satisfy Assumption 3.27. A choice of differential forms satisfying the conditions of Proposition 3.6 for the cycle Z are

$$\begin{aligned}\eta_Z &= \frac{dt}{t-1}-\frac{dt}{t-a}\in F^0\Sigma_B E^1_{\mathbb{P}^1}(\log A\cup|Z|;1)\\ g_Z &= \log|t-1|-\log|t-a|+\log|a|\frac{1}{1+t\bar t}\in\Sigma_B E^0_{\mathbb{P}^1}(\log A\cup|Z|;1)\\ \theta_Z &= -d\left(\log|a|\frac{1}{1+t\bar t}\right)=\log|a|\frac{\bar t dt+td\bar t}{(1+t\bar t)^2}\in F^{-1}\Sigma_B E^1_{\mathbb{P}^1}(\log A;1).\end{aligned}$$

Note that the third term in the definition of g_Z is added to satisfy the condition $g_Z(0)=0$ and is the responsible for the presence of θ_Z. Recall also Notation 1.3. With this notation the complex conjugate of η_Z is

$$\overline{\eta}_Z=-\frac{d\bar t}{\bar t-1}-\frac{d\bar t}{\bar t-\bar a}.$$

We denote by η_W, g_W and θ_W the corresponding differential forms for W obtained by replacing b for a.

Since $X=\mathrm{Spec}(\mathbb{C})$, the relative products over X are just absolute products. Therefore there should not be any non trivial interaction between Z and W. As we will see, this is indeed the case.

We can choose $\mathcal{X}_{Z,W}=\mathbb{P}^1\times\mathbb{P}^1$. The intersection $\overline{W}\cap\overline{Z}$ is reduced to the point (a,b). Since $H^0(X;1)=\mathbb{Q}(1)$, the biextension $B_{Z,W}$ has the middle graded piece

$$\mathrm{Gr}^W_{-2}B_{Z,W}=\mathbb{Q}(1)\oplus\mathbb{Q}(1).$$

The first factor comes from the cohomology of X and the second from the intersection point.

The different invariants are easy to compute. We start with $\delta_{E_Z}(e)$. This has to be a real element of $H^0(X;1)$. For clarity, as in Definition 1.3, we will use explicitly the generator $\mathbb{1}(1)_{\mathbb{C}}$ and write $\theta_Z=\theta'_Z\otimes\mathbb{1}(1)_{\mathbb{C}}$ with

$$\theta'_Z\in F^0\Sigma_B E^1_{\mathbb{P}^1}(\log A)$$

given by the same formula as θ_Z. Then, by equation (4.7),

$$\begin{aligned}\delta_{E_Z}(e) &= -i\Psi(\theta_Z)=i\frac{1}{2\pi i}\int d\left(-\log|a|\frac{1}{1+t\bar t}\right)\wedge\frac{dt}{t}\otimes\mathbb{1}(1)_{\mathbb{C}}\\ &= i\log|a|\otimes\mathbb{1}(1)_{\mathbb{C}}=\frac{1}{2\pi}\log|a|\otimes\mathbb{1}(1)_{\mathbb{Q}}.\end{aligned}$$

This element is real as expected.

The invariant $\delta_C(e)$ is given by equation (4.13):

$$\delta_C(e) = \frac{1}{2\pi} \log|b| \otimes \mathbb{1}(1)_{\mathbb{Q}} \in \mathbb{Q}(1)_{\mathbb{C}}.$$

Finally we compute the height $\operatorname{ht}(B)$. According to (4.17), it is given by

$$\operatorname{ht}(B) = \\ \frac{1}{2}\frac{1}{(2\pi i)^4} \operatorname{Im} \int_{(\mathbb{P}^1)^2} \left(\frac{dt_2}{t_2 - 1} - \frac{dt_2}{t_2 - b}\right) \wedge \frac{dt_1}{t_1} \wedge \left(\frac{d\bar{t}_1}{\bar{t}_1 - 1} - \frac{d\bar{t}_1}{\bar{t}_1 - \bar{a}}\right) \wedge \frac{d\bar{t}_2}{\bar{t}_2}.$$

This integral can be computed separately in each variable. Since

$$\frac{1}{2\pi i}\int_{\mathbb{P}^1} \left(\frac{dt}{t-1} - \frac{dt}{t-b}\right) \wedge \frac{d\bar{t}}{\bar{t}} = -\log|b|$$

and

$$\frac{1}{2\pi i}\int_{\mathbb{P}^1} \frac{dt}{t} \wedge \left(\frac{d\bar{t}}{\bar{t}-1} - \frac{d\bar{t}}{\bar{t}-\bar{a}}\right) = -\log|a|$$

we obtain

$$\operatorname{ht}(B) = \frac{1}{2(2\pi i)^2} \operatorname{Im}(\log|a|\log|b|) = 0$$

as we were expecting.

5.2. **An example in dimension 2.** We next compute an example in $\mathbb{P}^2$. In this example $d = 2$, $p = q = 2$ and $n = m = 1$. So condition (3.28) is satisfied.

In this subsection we will present the setting, in the next one we will develop the tools needed to perform the computation using currents and in the last one we will compute the main invariant associated with the biextension.

Let $X = \mathbb{P}^2$ and let $[x_0 : x_1 : x_2]$ be homogeneous coordinates of $\mathbb{P}^2$ and let

$$\begin{aligned} s_0 &= a_0x_0 + a_1x_1 + a_2x_2, \\ s_1 &= b_0x_0 + b_1x_1 + b_2x_2, \\ s_2 &= c_0x_0 + c_1x_1 + c_2x_2. \end{aligned}$$

be three linear global sections of $\mathcal{O}_{\mathbb{P}^2}(1)$ in general position. Let $\ell_i = \operatorname{div}(s_i)$, $i = 0, 1, 2$ be the corresponding reduced divisors that we identify with their support. By general position we mean that the lines ℓ_1, ℓ_2 and ℓ_3 form a non-degenerate triangle.

For $i = 0, 1, 2 (\mathrm{mod}\ 3)$ we write

$$f_i = \frac{s_{i+1}}{s_{i+2}}$$

for the rational function and $p_i = \ell_{i+1} \cap \ell_{i+2}$ for the intersection point. Note the equation $f_0 \cdot f_1 \cdot f_2 = 1$, which will be used later.

Definition 5.1. Given a line ℓ and a rational function f whose divisor does not contain ℓ, we denote by $(\ell, f) \in Z^2(X,1)$ the pre-cycle given as the graph of $f|_\ell$. Let s_0, s_1 and s_2 be sections as before. We denote by

$$Z(s_0,s_1,s_2) = \sum_{i=0}^{2}(\ell_i, f_i) - \sum_{i=0}^{2}\pi_X^*(p_i).$$

Moreover, if $\alpha \in \mathbb{C}^\times$, we write

$$Z(s_0,s_1,s_2;\alpha) = (\ell_0, \alpha f_0) + (\ell_1, f_1) + (\ell_2, f_2) - \sum_{i=0}^{2}\pi_X^*(p_i).$$

In particular $Z(s_0,s_1,s_2) = Z(s_0,s_1,s_2;1)$.

The following lemma is an easy verification.

Lemma 5.2. *For s_0, s_1 and s_2 in general position and $\alpha \in \mathbb{C}^\times$, the pre-cycle $Z(s_0,s_1,s_2;\alpha)$ is a cycle and belongs to $Z^2(X,1)_{00}$.*

Proof. The fact that $Z(s_0,s_1,s_2;\alpha)$ is a cycle follows directly from the condition $\sum_i \operatorname{div}(f_i) = 0$. The degenerate components $\sum_{i=0}^2 \pi_X^*(p_i)$ are subtracted precisely in order to fulfill the condition that $Z(s_1,s_2,s_3;\alpha)$ belongs to the refined normalized complex. □

We define

$$W_\beta \coloneqq Z(x_0,x_1,x_2;\beta)$$

and choose section s_0, s_1 and s_2 that are in general position with respect to $\{x_0,x_1,x_2\}$ so that, for any complex number $\alpha \in \mathbb{C}^\times$, if we write $Z_\alpha = Z(s_0,s_1,s_2;\alpha)$ then Z_α and W_β satisfy Assumption 3.27. We will also write $Z = Z_1$ and $W = W_1$.

The real regulator of Z_α is easy to compute.

Proposition 5.3. *The real regulator class of Z_α in*

$$H^3_{\mathfrak{D}}(\mathbb{P}^2, \mathbb{R}(2)) = H^2(\mathbb{P}^2;2)_{\mathbb{C}}/H^2(\mathbb{P}^2;2)_{\mathbb{R}}$$

is represented by the closed current $-\log|\alpha|\delta_\ell$, for any line ℓ in $\mathbb{P}^2$. In particular, if $|\alpha| = 1$ then the regulator class is zero.

Proof. In the Thom-Whitney complex, the regulator of the cycle Z_α is represented by $(\pi_X)_*(\delta_Z \cdot W_1)$. After taking the direct image and integrate with respect to ϵ we obtain

$$\mathcal{P}(Z_\alpha) = -\frac{1}{2}\left(\left(\log|\alpha|^2 + \log|f_0|^2\right)\delta_{\ell_0} + \log|f_1|^2\delta_{\ell_1} + \log|f_2|^2\delta_{\ell_2}\right).$$

Since each δ_{ℓ_i} is cohomologous to δ_ℓ and, by construction $f_0f_1f_2 = 1$ we deduce the result. □

Let ℓ_i, f_i and p_i, $i = 0,1,2$ be the lines, rational functions and intersection points constructed as before for the sections s_0, s_1, s_2 and let ℓ_i', f_i' and p_i', $i = 0,1,2$ be the ones corresponding to the sections x_0, x_i and x_2. For instance $\ell_0' = \{x_0 = 0\}$, $p_0' = [1:0:0]$ and $f_0' = x_1/x_2$.

For $i = 0, 1, 2$ and $j = 0, 1, 2$ we write $p_{i,j} = \ell_i \cap \ell'_j$,

$$\alpha_i = \begin{cases} \alpha, & \text{if } i = 0, \\ 1, & \text{otherwise,} \end{cases} \qquad \beta_j = \begin{cases} \beta, & \text{if } j = 0, \\ 1, & \text{otherwise,} \end{cases}$$

and

$$q_{i,j} = (p_{i,j}.\alpha_i f_i(p_{i,j}), \beta_j f'_j(p_{i,j})) \in X \times \mathbb{P}^1 \times \mathbb{P}^1.$$

By the generality assumption, the set S consist of the nine points $q_{i,j}$. Moreover, $H^{2p-2}(X;p) = H^2(\mathbb{P}^2;2) = \mathbb{Q}(1)$. Therefore, the biextension $B = B_{Z_\alpha, W_\beta}$ has the shape

$$\begin{aligned} \operatorname{Gr}_0 B &= \mathbb{Q}(0), \\ \operatorname{Gr}_{-2} B &= \mathbb{Q}(1) \oplus \mathbb{Q}(1)^{\oplus 9}, \\ \operatorname{Gr}_{-4} B &= \mathbb{Q}(2). \end{aligned}$$

From the description of the real regulator of Z_α above, the invariant $\delta_{E_{Z_\alpha}}$ is given by

$$\delta_{E_{Z_\alpha}}(e) = i\left((\log|\alpha| + \log|f_0|)\delta_{l_0} + \log|f_1|\delta_{l_1} + \log|f_2|\delta_{l_2}\right)^{\sim}.$$

While from equation (4.13) the invariant $\delta_C(e)$ is given by

$$\delta_C(e) = \frac{1}{2\pi} \sum_{i,j} \log|\beta_j f'_j(p_{ij})| e_{i,j},$$

where $e_{i,j}$ is the generator of the cohomology with support on the point $q_{i,j}$. The remaining invariant $\operatorname{ht}(B)$ will be computed in section 5.4 after we discuss how to use currents to ease the computation.

5.3. **Computation using currents.** In classical Arakelov geometry, it is usually simpler to write down explicitly a Green current for a cycle than to write a Green form with logarithmic singularities for the cycle. Although in general, inverse images and products of currents are not defined, the theory of wave front sets sketched in Subsection 1.6 allow us, in some situations, to work with currents with the same ease as with differential forms. We will use the notations and results of section 1.6.

For simplicity we make the following enhancement of Assumption 3.27.

Assumption 5.4. To Assumption 3.27 we add the condition that $|Z|$ and $|W|$ are both union of smooth subvarieties that intersect A_X and B_X transversely.

Note that Assumption 5.4 is satisfied in the example presented in Subsection 5.2.

Hence we assume 5.4 and we consider first the situation of Z in $X \times \mathbb{P}^1$. We denote by t the absolute coordinate of $\mathbb{P}^1$, and, for shorthand, $A = A_X$ and $B = B_X$. Since $|Z| = \bigcup Z_i$ is a union of smooth components, we write $N_0^\vee|Z| = \bigcup N_0^\vee Z_i$. Let $\iota\colon A \hookrightarrow X \times \mathbb{P}^1$ be the inclusion and $\mathcal{S} = \iota_*\iota^* N_0^\vee|Z|$. Then $\mathcal{S}$ is saturated with respect to ι by construction. The fact that the Z_i intersect B transversely readily implies that $\mathcal{S}$ and B_X are in good position. So the hypothesis of Theorem 1.30 are satisfied.

Let g_Z, η_Z and θ_Z be the differential forms obtained in Proposition 3.6, They define currents

$$\begin{aligned}[\eta_Z] &\in F^0\Sigma_B D^{2p-1}_{X\times\mathbb{P}^1/A;\mathcal{S}}(p)\\ [\theta_Z] &\in F^{-1}\Sigma_B D^{2p-1}_{X\times\mathbb{P}^1/A;\mathcal{S}}(p)\\ [g_Z] &\in F^{-1}\cap\overline{F}^{-1}\Sigma_B D^{2p-2}_{X\times\mathbb{P}^1/A;\mathcal{S}}(p)\end{aligned}$$

satisfying the differential equations

$$\begin{aligned}d[\eta_Z] &= -\delta_Z\\ d[g_Z] &= \frac{1}{2}([\eta_Z]-[\overline{\eta}_Z])-[\theta_Z].\end{aligned}$$

In fact, in our situation, as the following result implies, any choice of currents satisfying the above properties is enough to compute the regulator of Z and the invariant $\mathrm{ht}(B)$.

Lemma 5.5. *Let $\mathcal{S}\subset T^\vee_0 X$ be a closed conical subset that is saturated with respect to ι and is in good position with respect to B. Let*

$$\begin{aligned}\eta'_Z &\in F^0\Sigma_B D^{2p-1}_{X\times\mathbb{P}^1/A;\mathcal{S}}(p)\\ \theta'_Z &\in F^{-1}\Sigma_B D^{2p-1}_{X\times\mathbb{P}^1/A;\mathcal{S}}(p)\\ g'_Z &\in F^{-1}\cap\overline{F}^{-1}\Sigma_B D^{2p-2}_{X\times\mathbb{P}^1/A;\mathcal{S}}(p)\end{aligned}$$

be currents satisfying the differential equations

$$d\eta'_Z = -\delta_Z \tag{5.1}$$

$$dg'_Z = \frac{1}{2}(\eta'_Z-\overline{\eta'}_Z)-\theta'_Z. \tag{5.2}$$

then θ'_Z is closed and there are currents

$$\begin{aligned}\upsilon_1 &\in \Sigma_B D^{2p-2}_{X\times\mathbb{P}^1/A;\mathcal{S}}(p)\\ \upsilon_2 &\in F^0\Sigma_B D^{2p-2}_{X\times\mathbb{P}^1/A;\mathcal{S}}(p)\end{aligned}$$

satisfying

$$d\upsilon_1 = [\theta_Z]-\theta'_Z,\qquad d\upsilon_2 = [\eta_Z]-\eta'_Z.$$

In particular θ'_Z represents the class of the regulator of Z.

Proof. By the properties of the involved forms and currents is easy to see that $[\eta_Z]-\eta_Z$ and θ'_Z are both closed. Moreover the current

$$([\eta_Z]-\eta'_Z)/2-([\theta_Z]-\theta'_Z)-([\overline{\eta}_Z]-\overline{\eta'_Z})/2$$

is exact. By Theorem 1.30, the cohomology group

$$H^{2p-1}(\Sigma_B D^*_{X\times\mathbb{P}^1/A;\mathcal{S}}(p))$$

is the de Rham part of a pure Hodge structure H of weight -2. Moreover,

$$([\eta_Z]-\eta'_Z)/2\in F^0,\quad ([\theta_Z]-\theta'_Z)\in F^{-1}\cap\overline{F}^{-1},\quad\text{and}\quad ([\overline{\eta}_Z]-\overline{\eta'_Z})/2\in\overline{F}^0.$$

Since H is pure of weight -2, there is a direct sum decomposition

$$H = F^0H\oplus F^{-1}\cap\overline{F}^{-1}H\oplus\overline{F}^0H.$$

Therefore the three terms $[\eta_Z] - \eta'_Z$, $[\theta_Z] - \theta'_Z$ and $[\overline{\eta}_Z] - \overline{\eta'_Z}$ are exact. In particular we obtain the current v_1 in the statement. By theorems 1.19 and 1.30, the differential of the above complex is strict with respect to the Hodge filtration. Therefore we can find a primitive v_2 of $[\eta_Z] - \eta_Z$ belonging to F^0, completing the proof of the result. □

Remark 5.6. Since $\mathrm{WF}(\delta_Z) = N_0^\vee |Z|$ and the differential does not increase the wave front set, equation (5.1) implies that, for the currents in the lemma to exist, a necessary condition is that $N_0^\vee|Z| \subset \mathcal{S}$. Clearly $\iota_*\iota^* N_0^\vee |Z| \subset \mathcal{S}$ is a sufficient condition for the currents to exist. In the explicit computation of next section it will be handy to have the freedom to enlarge $\mathcal{S}$.

We now put together Z and W to obtain the next result. Recall that we are implicitly taking the pullbacks to $X \times (\mathbb{P}^1)^2$. Let $\mathcal{S}_Z, \mathcal{S}_W \subset T_*^\vee X \times \mathbb{P}^1$ be closed conical subsets that are saturated with respect to A, in good position with respect to B and such that $\pi_1^*\mathcal{S}_Z \cap \pi_2^* \mathcal{S}_W = \emptyset$.

Corollary 5.7. *Assuming 5.4, let η'_Z, θ'_Z and g'_Z (respectively η'_W, θ'_W and g'_W) be currents satisfying the hypothesis of Lemma 5.5 for the cycle Z and the set $\mathcal{S}_Z$ (respectively W and the set $\mathcal{S}_W$). Then*

$$\begin{aligned}\mathrm{ht}(B) &= \frac{1}{2} \operatorname{Im} \frac{1}{(2\pi i)^2} p_* \left(\eta'_W \wedge \frac{dt_1}{t_1} \wedge \overline{\eta}'_Z \wedge \frac{d\bar{t}_2}{\bar{t}_2}\right) \\ &= -\operatorname{Im} \frac{1}{(2\pi i)^2} p_* \left(\delta_W \wedge \frac{dt_1}{t_1} \wedge g'_Z \wedge \frac{d\bar{t}_2}{\bar{t}_2} + \eta'_W \wedge \frac{dt_1}{t_1} \wedge \theta'_Z \wedge \frac{d\bar{t}_2}{\bar{t}_2} \right),\end{aligned}$$

where $p\colon X \times (\mathbb{P}^1)^2 \to \operatorname{Spec}(\mathbb{C})$ is the structural map.

Proof. That the product current is well defined follows from the fact that the wave front sets of the involved currents are disjoint. By (4.17), we have

$$\mathrm{ht}(B) = \frac{1}{2} \operatorname{Im} \frac{1}{(2\pi i)^2} p_* \left([\eta_W] \wedge \frac{dt_1}{t_1} \wedge \overline{[\eta_Z]} \wedge \frac{d\bar{t}_2}{\bar{t}_2}\right). \tag{5.3}$$

By Lemma 5.5 there are currents

$$v_Z \in F^0 \Sigma_B D^{2p-2}_{X/A;\mathcal{S}}(p), \qquad v_W \in F^0 \Sigma_B D^{2q-2}_{X/A;\mathcal{S}}(q),$$

such that

$$dv_Z = [\eta_Z] - \eta'_Z, \qquad dv_W = [\eta_W] - \eta'_W. \tag{5.4}$$

Since v_Z belongs to F^0 it has at least p holomorphic differentials. As $W \times \mathbb{P}^1 \subset X \times (\mathbb{P}^1)^2$ has dimension p, we obtain

$$\delta_W \wedge \frac{dt_1}{t_1} \wedge \overline{v}_Z \wedge \frac{d\bar{t}_2}{\bar{t}_2} = 0 \tag{5.5}$$

Similarly

$$v_W \wedge \frac{dt_1}{t_1} \wedge \delta_Z \wedge \frac{d\bar{t}_2}{\bar{t}_2} = 0. \tag{5.6}$$

Then the result follows from Stokes' theorem by using equations (5.3), (5.4), (5.5) and (5.6) and the fact that the forms v_Z and η_Z vanish for $t_1 = 0$ and $t_1 = \infty$, while the forms v_W and η_W vanish for $t_2 = 0$ and $t_2 = \infty$. □

5.4. **The invariant** $\operatorname{ht}(B)$ **of the example in dimension two.** Now that we have set up the theory, we are ready to compute the remaining invariant $\operatorname{ht}(B)$ for the pair of higher cycles Z_α and W_β described in 5.2.

The first task is to compute a set of currents satisfying the conditions of Lemma 5.5 for the cycle Z_α. The currents for the cycle W_β will be constructed in a similar way. Since, for the moment we work with a single cycle we denote by t the absolute coordinate of $\mathbb{P}^1$ and we omit any needed pullback to $X \times \mathbb{P}^1$ from the formulas.

We start with a classical Green current for the cycle Z_α in $X \times \square$:

$$g_{Z_\alpha,0} := -\sum_{i=0}^{2} \left(\log \frac{|t-\alpha_i f_i|}{|t-1|} \right) \delta_{\ell_i} \in F^{-1} D^2_{X\times\mathbb{P}^1}(2).$$

Then one can check that

$$2\partial\bar{\partial} g_{Z_\alpha,0} = \sum_{i=0}^{2} \delta_{(\ell_i,\alpha_i f_i)} - \delta_{p_i} - \delta_{\ell_i\times\{1\}}.$$

Hence

$$2\partial\bar{\partial} g_{Z_\alpha,0}|_{X\times\square} = \delta_{Z_\alpha}. \tag{5.7}$$

Moreover $g_{Z_\alpha,0}|_{t=\infty} = 0$. But in general $g_{Z,0}|_{t=0} \neq 0$. In fact,

$$g_{Z_\alpha,0}|_{t=0} = -\sum \log|f_i|\delta_{\ell_i} - \log|\alpha|\delta_{\ell_0}. \tag{5.8}$$

The two terms appearing in this decomposition have a different nature. The first one, the sum, is a boundary, hence we will be able to get rid of it without altering equation (5.7), while the second one is the responsible for the real regulator of Z_α therefore will force us a non zero current θ_{Z_α}.

To see that the first term is a boundary We introduce the current

$$u_Z = [\log|f_0|\partial\log|f_1| - \log|f_1|\partial\log|f_0|] \in F^{-1}D^1_X(2). \tag{5.9}$$

This current does not depend on the choice of α. Using the fact that $2\partial\bar{\partial}[\log|f_i|] = \delta_{\ell_{i+2}} - \delta_{\ell_{i+1}}$, we get

$$\overline{\partial} u_Z - \partial\overline{u}_Z = -\log|f_0|\delta_{\ell_0} - \log|f_1|\delta_{\ell_1} + (\log|f_0| + \log|f_1|)\delta_{\ell_2}.$$

Finally, using the relation $f_0 \cdot f_1 \cdot f_2 = 1$, we get

$$\overline{\partial} u_Z - \partial\overline{u}_Z = -\sum_{i=0}^{2} \log|f_i|\delta_{\ell_i}. \tag{5.10}$$

Let h be the function

$$h(t) = \frac{1}{1+|t|^2}.$$

It is smooth in the whole $\mathbb{P}^1$ and satisfies

$$h(0) = 1, \qquad h(\infty) = 1. \tag{5.11}$$

We define the currents

$$\begin{aligned} g_{Z_\alpha,1} &= -\sum_{i=0}^{2}\left(\log\frac{|t-\alpha_i f_i|}{|t-1|}\right)\delta_{\ell_i} - \left(\overline{\partial}(h(t)u_Z) - \partial(h(t)\overline{u}_Z)\right), \\ g_{Z_\alpha,2} &= h(t)\log|\alpha|\delta_{\ell_0} \\ g_{Z_\alpha} &= g_{Z_\alpha,1} + g_{Z_\alpha,2}. \end{aligned}$$

By equations (5.8), (5.10) and (5.11),

$$g_{Z_\alpha}|_{t=0} = g_{Z_\alpha}|_{t=\infty} = 0.$$

We also write

$$\begin{aligned} \eta_{Z_\alpha} &= 2\partial g_{Z_\alpha,1}, \\ \theta_{Z_\alpha} &= -dg_{Z_\alpha,2}. \end{aligned}$$

Let $\iota\colon A_X \to X \times \mathbb{P}^1$ denote the inclusion and $\mathcal{S} = \iota_*\iota^*\mathrm{WF}(g_{Z_\alpha})$.

Proposition 5.8. *The set $\mathcal{S}$ and the currents g_{Z_α}, θ_{Z_α} and η_{Z_α} satisfy the hypothesis of Lemma 5.5.*

Proof. By construction the set $\mathcal{S}$ is saturated by respect to A_X. By examining the singularities of the different functions, the wave front set of g_{Z_α} is given by

$$\mathrm{WF}(g_{Z_\alpha}) = \bigcup_{i=0}^{2}\left(N_0^\vee(\ell_i \times \mathbb{P}^1) \cup N_0^\vee(\ell_i \times \{1\})\right) \cup N_0^\vee|Z_\alpha|$$

Therefore, if $\iota'\colon B_X \to X \times \mathbb{P}^1$ is the inclusion, then

$$(\iota')^*\mathcal{S} = \bigcup_{i=0}^{2} N_0^\vee(\ell_i \times \{0\}) \cup N_0^\vee(\ell_i \times \{\infty\}).$$

Here the conormal bundle is computed in B_X. Let r_0 be the retraction to $X \times \{0\}$ and r_∞ the retraction to $X \times \{\infty\}$. Since, for $i = 0, 1, 2$ and $j = 0, \infty$,

$$s_j^* N_0^\vee(\ell_i \times \{j\}) N_0^\vee(\ell_i \times \mathbb{P}^1)$$

we deduce that $\mathcal{S}$ and B_X are in good position.

By construction, for $j = \emptyset, 0, 1, 2$, $\overline{g}_{Z_\alpha,j} = -g_{Z_\alpha,j}$. Therefore

$$2\bar{\partial} g_{Z_\alpha,1} = -\overline{\partial g_{Z_\alpha,1}} = -\bar{\eta}_{Z_\alpha}.$$

Therefore

$$dg_{Z_\alpha} = \partial g_{Z_\alpha,1} + \bar{\partial} g_{Z_\alpha,1} + dg_{Z_\alpha,2} = \frac{1}{2}\left(\eta_{Z_\alpha} - \bar{\eta}_{Z_\alpha}\right) - \theta_{Z_\alpha}.$$

The remaining hypothesis follow directly from the construction of the different currents. □

By Corollary 5.7, the height of B_{W_β,Z_α} is given by

$$\mathrm{ht}(B) = -\frac{1}{(2\pi i)^2}\operatorname{Im} p_*\left(\delta_{W_\beta} \wedge \frac{dt_1}{t_1} \wedge g_{Z_\alpha} \wedge \frac{d\bar{t}_2}{\bar{t}_2} + \eta_{W_\beta} \wedge \frac{dt_1}{t_1} \wedge \theta_{Z_\alpha} \wedge \frac{d\bar{t}_2}{\bar{t}_2}\right),$$

The support of the current $g_{Z_\alpha,0}$ is the union of the threefolds $\ell_i \times (\mathbb{P}^1)^2$. Since we are assuming that the intersection of Z_α and W_β is proper, the intersection of $\overline{W}_\beta$ with this support is the union of the lines $p_{ij} \times \mathbb{P}^1 \times$

$\{\beta_j f'_j(p_{ij})\}$ (see Section 5.2 for the notation). Since the pullback of $\frac{d\bar{t}_2}{\bar{t}_2}$ to these lines is zero, we obtain

$$p_*\left(\delta_{W_\beta} \wedge \frac{dt_1}{t_1} \wedge g_{Z_\alpha,0} \wedge \frac{d\bar{t}_2}{\bar{t}_2}\right) = 0.$$

We next compute

$$I_1 = p_*\left(\delta_{W_\beta} \wedge \frac{dt_1}{t_1} \wedge \partial\bar{u}_Z \wedge \frac{d\bar{t}_2}{\bar{t}_2}\right).$$

Using that $W \times \mathbb{P}^1$ has dimension 2, that δ_{W_β} vanishes when restricted to $t_2 = 0$ and $t_2 = \infty$, and that u_Z vanishes when restricted to $t_1 = \infty$, we obtain

$$I_1 = p_*\left(\delta_{W_\beta} \wedge \frac{dt_1}{t_1} \wedge d\bar{u}_Z \wedge \frac{d\bar{t}_2}{\bar{t}_2}\right) = (p_2)_*\left(\delta_{W_\beta} \wedge \bar{u}_Z \wedge \frac{d\bar{t}_2}{\bar{t}_2}\right),$$

where now p_2 is the structural morphism of the product $X \times \mathbb{P}^1$ where W_β lives. Since the support of W_β consist of lines and $\bar{u}_Z$ contains one anti-holomorphic differential we deduce $I_1 = 0$.

Next we consider

$$I_2 = p_*\left(\delta_{W_\beta} \wedge \frac{dt_1}{t_1} \wedge \bar{\partial}u_Z \wedge \frac{d\bar{t}_2}{\bar{t}_2}\right).$$

By the same argument as before

$$I_2 = (p_2)_*\left(\delta_{W_\beta} \wedge u_Z \wedge \frac{d\bar{t}_2}{\bar{t}_2}\right).$$

This time the integral may be non-zero and we will compute it later. The last piece to consider is

$$I_3 = p_*\left(\delta_{W_\beta} \wedge \frac{dt_1}{t_1} \wedge g_{Z_\alpha,2} \wedge \frac{d\bar{t}_2}{\bar{t}_2} + \eta_{W_\beta} \wedge \frac{dt_1}{t_1} \wedge \theta_{Z_\alpha} \wedge \frac{d\bar{t}_2}{\bar{t}_2}\right)$$

Using that $\delta_{W_\beta} = -d\eta_{W_\beta}$, that $dg_{Z_\alpha,2} = -\theta_{Z_\alpha}$, and that

$$\begin{aligned} &\delta_{W_\beta}|_{t_2=0} = \delta_{W_\beta}|_{t_2=\infty} = 0, && g_{Z_\alpha}|_{t_1=0} = \log|\alpha|\delta_{\ell_0},\\ &\eta_{W_\beta}|_{t_2=0} = \eta_{W_\beta}|_{t_2=\infty} = 0 && g_{Z_\alpha}|_{t_1=\infty} = 0,\\ &\theta_{Z_\alpha}|_{t_1=0} = \theta_{Z_\alpha}|_{t_1=\infty} = 0, \end{aligned}$$

we obtain

$$I_3 = (p_2)_*\left(\eta_{W_\beta} \wedge \log|\alpha|\delta_{\ell_0} \wedge \frac{d\bar{t}_2}{\bar{t}_2}\right).$$

Using $\eta_{W_\beta} = 2\partial g_{W_\beta,1}$, δ_{ℓ_0} is closed, Stokes' theorem, and the fact that $\partial g_{W_\beta,1} \wedge \delta_{l_0} \wedge \frac{d\bar{t}_2}{\bar{t}_2}$ is of type $(3,3)$, we deduce

$$\begin{aligned} I_3 &= -\log|\alpha|(p_2)_*\left(g_{W_\beta,1} \wedge \delta_{\ell_0} \wedge d\left[\frac{d\bar{t}_2}{\bar{t}_2}\right]\right)\\ &= \log|\alpha|\left(-\sum_{j=0}^{2}\log|\beta_j f'_j(p_{0j})| - \sum_{j=0}^{2}\log|f'_j(p_{0j})|\right)\\ &= -\log|\alpha|\log|\beta|, \end{aligned}$$

using $f'_0f'_1f'_2 = 1$. So $\mathrm{Im}(I_3) = 0$, and we are reduced to the expression

$$\mathrm{ht}(B) = \frac{1}{(2\pi i)^2}\mathrm{Im}\left((p_2)_*\left(\delta_{W_\beta} \wedge u_Z \wedge \frac{d\bar{t}_2}{\bar{t}_2}\right)\right).$$

Recall that the cycle W has six components. The three degenerate vertical components $V := \sum_{j=0}^{2} p^*(q'_j)$ and the three lines $\sum_{j=0}^{2}(\ell'_j, \beta_j f'_j)$. Since $u_Z|_{q'_j} = 0$, we obtain $\delta_V \wedge u_Z \wedge \frac{d\bar{t}}{\bar{t}} = 0$. Hence, we arrive at

$$\mathrm{ht}(B) = \frac{1}{(2\pi i)^2}\sum_{j=0}^{2}\mathrm{Im}\left(p_{\ell'_j,*}\left[u_Z \wedge \frac{d\bar{t}_2}{\bar{t}_2}\right]\right),$$

where $p_{\ell'_j} : \ell'_j \to \mathrm{Spec}(\mathbb{C})$ is the structural morphism, To compute the contribution of each line we use that $\ell'_j = V(x_j)$ and $f'_j = \frac{x_{j+1}}{x_{j+2}}$ to obtain the parametrizations

$$\begin{aligned}
(\ell'_0, \beta_0 f'_0) &= \{(0:1:t), (1:\beta_0 t)\} \cong \mathbb{P}^1; \qquad && t_2 = \beta_0 t,\\
(\ell'_1, \beta_1 f'_1) &= \{(t:0:1), (1:\beta_1 t)\} \cong \mathbb{P}^1; \qquad && t_2 = \beta_1 t,\\
(\ell'_2, \beta_2 f'_2) &= \{(1:t:0), (1:\beta_2 t)\} \cong \mathbb{P}^1 \qquad && t_2 = \beta_2 t.
\end{aligned}$$

By symmetry we only need to compute the contribution of $(\ell'_0, \beta_0 f'_0)$ as the other two terms will be obtained by a cyclic permutation of $\{0, 1, 2\}$. Restricting to this line we obtain

$$\begin{aligned}
f_0|_{(\ell'_0,\beta_0 f'_0)}(t) &= \frac{b_1 + b_2 t}{c_1 + c_2 t} = \left(\frac{b_2}{c_2}\right)\frac{t - (-\frac{b_1}{b_2})}{t - (-\frac{c_1}{c_2})},\\
f_1|_{(\ell'_0,\beta_0 f'_0)}(t) &= \frac{c_1 + c_2 t}{a_1 + a_2 t} = \left(\frac{c_2}{a_2}\right)\frac{t - (-\frac{c_1}{c_2})}{t - (-\frac{a_1}{a_2})},\\
\frac{d\bar{t}_2}{\bar{t}_2}|_{(\ell'_0,\beta_0 f'_0)} &= \frac{d\bar{t}}{\bar{t}}.
\end{aligned}$$

For shorthand we write

$$\gamma := \frac{b_2}{c_2}, \quad \rho := \frac{c_2}{a_2}, \quad \theta_1 := -\frac{b_1}{b_2}, \quad \theta_2 := -\frac{c_1}{c_2}, \quad \theta_3 := -\frac{a_1}{a_2},$$

and

$$\tilde{f}_0(t) = \frac{t - \theta_1}{t - \theta_2}, \qquad \tilde{f}_1(t) := \frac{t - \theta_2}{t - \theta_3}.$$

The differential form $u_Z|_{(\ell'_0,\beta_0 f'_0)}$ splits up into

$$u_Z|_{(\ell'_0,\beta_0 f'_0)} = u_{1,Z} + u_{2,Z},$$

where

$$\begin{aligned}
u_{1,Z} &= \log|\gamma|\partial\log|\tilde{f}_1(t)| - \log|\rho|\partial\log|\tilde{f}_0(t)|,\\
u_{2,Z} &= \log|\tilde{f}_0(t)|\partial(\log|\tilde{f}_1(t)|) - \log|\tilde{f}_1(t)|\partial(\log|\tilde{f}_0(t)|).
\end{aligned}$$

The current $p_{\ell_0',*}\left[u_{1,Z} \wedge \frac{d\bar{t}}{\bar{t}}\right]$ is simple to compute:

$$\begin{aligned}
&p_{\ell_0',*}\left[u_{1,Z} \wedge \frac{d\bar{t}}{\bar{t}}\right] \\
&\quad = \log|\gamma| p_{\ell_0',*}\left(d\left[\log|\tilde{f}_1|\frac{d\bar{t}}{\bar{t}}\right]\right) - \log|\rho| p_{\ell_0',*}\left(d\left[\log|\tilde{f}_0|\frac{d\bar{t}}{\bar{t}}\right]\right) \\
&\quad = \log|\gamma|\log\frac{|\theta_2|}{|\theta_3|} - \log|\rho|\log\frac{|\theta_1|}{|\theta_2|}.
\end{aligned}$$

Since this expression is purely real it does not contribute to the height of B_{W_β,Z_α}. To compute $p_{\ell_0',*}\left(u_{2,Z}(t) \wedge \frac{d\bar{t}}{\bar{t}}\right)$, we have to make a slight digression to the theory of Bloch-Wigner dilogarithm function. For details the reader is referred to [Zag88]. The dilogarithm function is the holomorphic function defined, over the disk $\mathbb{D} := \{t \in \mathbb{C}\colon |t| < 1\}$ as

$$\mathrm{Li}_2(t) = \sum_{n\geq 1}\frac{t^n}{n^2}.$$

This function can be extended as a holomorphic function to $\mathbb{C}\setminus[1,\infty)$ with jumps $2\pi i\log|t|$. Thus the function $\mathrm{Li}_{2,\mathrm{arg}}(t) := \mathrm{Li}_2(t) + i\arg(1-t)\log|t|$ is continuous. The Bloch-Wigner dilogarithm is defined by taking the imaginary part of $\mathrm{Li}_{2,\mathrm{arg}}$:

$$\begin{aligned}
D_2(t) &= \mathrm{Im}(\mathrm{Li}_2(t)) + \arg(1-t)\log|t| \\
&= \frac{1}{2i}(\mathrm{Li}_2(t) - \mathrm{Li}_2(\bar{t})) + \frac{1}{4i}(\log(1-t) - \log(1-\bar{t}))(\log(t) + \log(\bar{t})).
\end{aligned}$$

We take the branch $-\pi \leq \arg(t) < \pi$. The Bloch-Wigner dilogarithm satisfies the following partial differential equation.

$$\partial i D_2(t) = \log|t|\partial\log|1-t| - \log|1-t|\partial\log|t|.$$

For any two linear rational functions f, g in $\mathbb{C}(\mathbb{P}^1)$, we define

$$S(f,g) := \log|f|\partial(\log|g|) - \log|g|\partial(\log|f|).$$

We make the following observations: Let f, g, h be three linear rational functions. Then

(i) $S(f,g) = -S(g,f)$
(ii) $S(f,gh) = S(f,g) + S(f,h)$
(iii) $S(f,1-f) = S(f,f-1) = \partial i D_2(f)$.

Using the above observations, we can find a boundary formula for $S(\tilde{f}_0,\tilde{f}_1)$. First for rational functions of the form $\frac{t-a}{b-a}$ and $\frac{t-b}{b-a}$ we get

$$S\left(\frac{t-a}{b-a},\frac{t-b}{b-a}\right) = S\left(\frac{t-a}{b-a},\frac{t-a}{b-a}-1\right) = \partial i D_2\left(\frac{t-a}{b-a}\right).$$

Hence

$$\begin{aligned}
S(t-a,t-b) &= \partial i D_2\left(\frac{t-a}{b-a}\right) + S\left(b-a,\frac{t-b}{t-a}\right) \\
&= \partial\left(iD_2\left(\frac{t-a}{b-a}\right) + \log|b-a|\log\frac{|t-b|}{|t-a|}\right).
\end{aligned}$$

Since $u_{2,Z} = S(\tilde{f}_0, \tilde{f}_1)$, we obtain

$$\begin{aligned} u_{2,Z} &= S(t-\theta_1, t-\theta_2) - S(t-\theta_1, t-\theta_3) \\ &\quad - S(t-\theta_2, t-\theta_2) + S(t-\theta_2, t-\theta_3) \\ &= S(t-\theta_1, t-\theta_2) + S(t-\theta_2, t-\theta_3) + S(t-\theta_3, t-\theta_1) \\ &= \partial(G(t)), \end{aligned}$$

where $G(t)$ is given by

$$G(t) = i\left(D_2\left(\frac{t-\theta_1}{\theta_2-\theta_1}\right) + D_2\left(\frac{t-\theta_2}{\theta_3-\theta_2}\right) + D_2\left(\frac{t-\theta_3}{\theta_1-\theta_3}\right)\right)$$
$$+\log|\theta_2-\theta_1|\log\frac{|t-\theta_2|}{|t-\theta_1|} + \log|\theta_3-\theta_2|\log\frac{|t-\theta_3|}{|t-\theta_2|} + \log|\theta_1-\theta_3|\log\frac{|t-\theta_1|}{|t-\theta_3|}.$$

Putting everything in place, we obtain

$$p_{\ell'_{0,*}}\left[u_{2,Z} \wedge \frac{d\bar{t}}{\bar{t}}\right] = p_{\ell'_{0,*}}\left(\left[dG(t)\frac{d\bar{t}}{\bar{t}}\right]\right) = G(0) - G(\infty).$$

Noting that $G(\infty) = 0$ since $D_2(\infty) = \log 1 = 0$, and using the six-fold symmetry of Bloch-Wigner dilogarithm functions, we deduce

$$p_{\ell'_{0,*}}\left[u_{2,Z}(t) \wedge \frac{d\bar{t}}{\bar{t}}\right] = i\left(D_2\left(\frac{\theta_2}{\theta_1}\right) + D_2\left(\frac{\theta_3}{\theta_2}\right) + D_2\left(\frac{\theta_1}{\theta_3}\right)\right)$$
$$+\left(\log|\theta_1|\log\frac{|\theta_1-\theta_3|}{|\theta_1-\theta_2|} + \log|\theta_2|\log\frac{|\theta_2-\theta_1|}{|\theta_2-\theta_3|} + \log|\theta_3|\log\frac{|\theta_3-\theta_2|}{|\theta_3-\theta_1|}\right).$$

After plugging in the values of θ_1, θ_2 and θ_3 and taking the imaginary part,

$$\operatorname{Im} p_{\ell'_{0,*}}\left[u_Z(t) \wedge \frac{d\bar{t}}{\bar{t}}\right] = D_2\left(\frac{b_2c_1}{b_1c_2}\right) + D_2\left(\frac{c_2a_1}{c_1a_2}\right) + D_2\left(\frac{a_2b_1}{a_1b_2}\right).$$

Similarly, for ℓ'_1 and ℓ'_2, we have

$$\begin{aligned} \operatorname{Im} p_{\ell'_{1,*}}\left[u_Z(t) \wedge \frac{d\bar{t}}{\bar{t}}\right] &= D_2\left(\frac{b_0c_2}{b_2c_0}\right) + D_2\left(\frac{c_0a_2}{c_2a_0}\right) + D_2\left(\frac{a_0b_2}{a_2b_0}\right), \\ \operatorname{Im} p_{l'_{2,*}}\left[u_Z(t) \wedge \frac{d\bar{t}}{\bar{t}}\right] &= D_2\left(\frac{b_1c_0}{b_0c_1}\right) + D_2\left(\frac{c_1a_0}{c_0a_1}\right) + D_2\left(\frac{a_1b_0}{a_0b_1}\right). \end{aligned}$$

Summing up, the height of B_{Z_α, W_β} is given by

$$\operatorname{ht}(B) = \frac{1}{(2\pi i)^2}\sum_{\substack{(0,1,2)\\(a,b,c)}} D_2\left(\frac{a_2b_1}{a_1b_2}\right),$$

where the sum is over all cyclic permutations of $(0,1,2)$ and (a,b,c) for a total of nine terms.

The expression above can be reduced to a six dilogarithms one by using the five-term relation for Bloch-Wigner dilogarithm. As a prototype, we show the simplification for the first component of the above sum. Taking

$$u := \frac{b_2c_1}{b_1c_2}, \qquad v := \frac{c_2a_1}{c_1a_2}, \qquad w := \frac{a_2b_1}{a_1b_2},$$

we observe that $uvw = 1$. Hence $D_2(w) = D_2(1 - \frac{1}{w}) = D_2(1 - uv)$. Now using the five-term relation, we conclude

$$D_2(u) + D_2(v) + D_2(w) = D_2\left(\frac{1-uv}{1-u}\right) + D_2\left(\frac{1-uv}{1-v}\right).$$

Plugging back the values of u, v and w, we get

$$D_2\left(\frac{b_2c_1}{b_1c_2}\right) + D_2\left(\frac{c_2a_1}{c_1a_2}\right) + D_2\left(\frac{a_2b_1}{a_1b_2}\right) = \\ D_2\left(\frac{a_2b_1 - a_1b_2}{b_1c_2 - b_2c_1}\left(\frac{c_2}{a_2}\right)\right) + D_2\left(\frac{a_2b_1 - a_1b_2}{a_2c_1 - a_1c_2}\left(\frac{c_1}{b_1}\right)\right).$$

Finally, putting everything together, we get a reduced expression

$$\mathrm{ht}(B) = \frac{1}{(2\pi i)^2} \sum_{(0,1,2)} D_2\left(\frac{a_2b_1 - a_1b_2}{b_1c_2 - b_2c_1}\left(\frac{c_2}{a_2}\right)\right) + D_2\left(\frac{a_2b_1 - a_1b_2}{a_2c_1 - a_1c_2}\left(\frac{c_1}{b_1}\right)\right),$$

where the sum is over the cyclic permutations of $(0, 1, 2)$ only, giving us six terms.

Remark 5.9. From the formula for $\mathrm{ht}(B)$ we can derive two conclusions.

(i) Since $D_2(r) = 0, \forall r \in \mathbb{R}$, we deduce that if the triangles are defined over $\mathbb{R}$ the height pairing is zero. In fact this is a general phenomenon as the Proposition 5.10 shows.

(ii) Since the function D_2 can be extended to a continuous function on $\mathbb{P}^1(\mathbb{C})$, the above height can be extended by continuity to any degenerate situation. We see in the next section that this is a very general phenomenon.

Proposition 5.10. *Let X be a smooth projective variety defined over $\mathbb{R}$ and $X_{\mathbb{C}}$ the corresponding complex variety. Let $Z \in Z^p(X_{\mathbb{C}}, 1)$ and $W \in Z^q(X_{\mathbb{C}}, 1)$ be two higher cycles defined also over $\mathbb{R}$ satisfying Assumption 3.27. Then*

$$\mathrm{ht}(B_{Z,W}) = 0.$$

Proof. The short proof is that, under the hypothesis of the proposition

$$\mathrm{ht}(B_{Z,W}) \in \rho_2(H^1_{\mathfrak{D}}(\mathrm{Spec}(\mathbb{R}); \mathbb{R}(2)))$$

and $H^1_{\mathfrak{D}}(\mathrm{Spec}(\mathbb{R}); \mathbb{R}(2)) = 0$.

In more down to earth terms. Let $\sigma\colon X_{\mathbb{C}} \to X_{\mathbb{C}}$ be the antilinear involution defined by the real structure of X. Assume for the moment that Z and W are not necessarily defined over $\mathbb{R}$. By the functoriality of the construction of mixed Hodge structures, we deduce that $B_{\sigma^*Z,\sigma^*W} = \overline{B_{Z,W}}$, where, for a mixed Hodge structure H, we denote by $\overline{H}$ the mixed Hodge structure obtained by sending i to $-i$.

Let now B be any generalized biextension. Let $r = \ell(B)/2$. Then the operation $B \mapsto \overline{B}$ sends a generator e of $\mathbb{Q}(a)$ to $(-1)^a e$ (see Remark 1.1) and the map Im is sent to $-\,\mathrm{Im}$. Therefore

$$\mathrm{ht}(\overline{B}) = (-1)^{r+1}\,\mathrm{ht}(B).$$

In our case

$$\ell(B_{Z,W})/2 = \frac{n+m}{2} + 1 = 2.$$

Therefore,

$$\mathrm{ht}(B_{\sigma^* Z}, \sigma^* W) = -\,\mathrm{ht}(B_{Z,W}).$$

But if Z and W are defined over $\mathbb{R}$ we also have

$$\mathrm{ht}(B_{\sigma^* Z, \sigma^* W}) = \mathrm{ht}(B_{Z,W}),$$

from which the proposition follows. □

6. Asymptotic Behavior

In this section, we begin the study of the asymptotic behavior of the height of families of higher cycles. In subsection 6.1 we prove the height extends continuously whenever the associated variation of mixed Hodge structure is of Hodge–Tate type. In subsection 6.2 we give a definition of limit height for arbitrary admissible variations of mixed Hodge structures over the punctured disk with unipotent monodromy. In subsection 6.3 we give three examples of heights coming from (i) the dilogarithm variation, (ii) a particular family of triangles in $\mathbb{P}^2$ and (iii) a nilpotent orbit. The first two examples in subsection 6.3 can be read independently of the rest of this section. By definition, an oriented variation of mixed Hodge structure is a variation equipped with a choice of flat, global sections which induce an orientation on each fiber.

6.1. **Hodge–Tate Limits.**

Theorem 6.1. *Let S be a Zariski open subset of a complex manifold $\bar{S}$ such that $D = \bar{S} - S$ is a normal crossing divisor. Let $\mathcal{V} \to S$ be an oriented Hodge–Tate variation (graded-polarized) such that the length $\ell(\mathcal{V}) \geq 4$. Assume $\mathcal{V}$ is admissible with respect to $\bar{S}$ and has unipotent local monodromy about D. Let $p \in D$. Then, the limit mixed Hodge structure $\mathcal{V}_p$ of $\mathcal{V}$ at p is an oriented Hodge–Tate structure with the same weight filtration as $\mathcal{V}$. Moreover,*

$$\lim_{s \to p} \mathrm{ht}(\mathcal{V}_s) = \mathrm{ht}(\mathcal{V}_p). \tag{6.1}$$

To set up the machinery to prove Theorem 6.1, let $p \in \bar{S} - S$. Then we can find a polydisk $\Delta^r \subset \bar{S}$ containing p and local holomorphic coordinates $(s_1, \dots, s_r)$ vanishing at p such that

- *(i)* The image of Δ^r under $(s_1, \dots, s_r)$ is the unit polydisk (coordinate norm < 1) in $\mathbb{C}^r$;
- *(ii)* $D \cap \Delta^r$ is given by the local equation $s_1 \dots s_k = 0$.

Therefore,

$$\Delta^r - D \cap \Delta^r = \Delta^{*k} \times \Delta^{r-k} = \{s \mid s_1, \dots, s_k \neq 0\}.$$

As Theorem 6.1 concerns the asymptotic behavior of the variation, it is sufficient work on $\Delta^{*k} \times \Delta^{r-k}$. We therefore recall the theory of period maps of admissible variations of graded-polarized mixed Hodge structures in this setting following the conventions of [Pea00].

Pick $b \in \Delta^{*k} \times \Delta^{r-k}$ and let $V = \mathcal{V}_b$ be the fiber of $\mathcal{V}$ at b. Let T_j denote the local monodromy of $\mathcal{V}$ about $s_j = 0$. We assume T_j to be unipotent and write $T_j = e^{N_j}$. Note the $[N_a, N_b] = 0$ since the fundamental group of $\Delta^{*k} \times \Delta^{r-k}$ is abelian.

In analogy with the pure case, we can represent $\mathcal{V}$ by a period map

$$\varphi\colon \Delta^r - D \to \Gamma\backslash\mathcal{M},$$

where $\mathcal{M}$ the classifying space of mixed Hodge structure attached to $\mathcal{V}$ with reference fiber V and monodromy group Γ generated by $T_1, \dots, T_k$. As with variations of pure Hodge structures, the classifying space $\mathcal{M}$ is a complex manifold and the period map φ is holomorphic, horizontal and locally liftable.

Let W denote the weight filtration of $\mathcal{V}$ and define

$$\mathrm{GL}(V_{\mathbb{C}})^W = \{g \in \mathrm{GL}(V_{\mathbb{C}}) \mid g(W_k) \subseteq W_k, \quad \forall k\}.$$

Let $\mathcal{S}_j$ denote the graded-polarization of Gr^W_j and define

$$G = \{g \in \mathrm{GL}(V^W_{\mathbb{C}}) \mid \mathrm{Gr}^W(g) \in \mathrm{Aut}_{\mathbb{R}}(\mathcal{S}_\bullet)\}.$$

Then (see §3, [Pea00]) G acts transitively on $\mathcal{M}$ by biholomorphic transformations.

Let $G_{\mathbb{R}} = G \cap \mathrm{GL}(V_{\mathbb{R}})$ and $G_{\mathbb{C}}$ be the complexification of $G_{\mathbb{R}}$. The classifying space $\mathcal{M}$ is a complex analytic open subset of a complex manifold $\check{\mathcal{M}}$ upon which $G_{\mathbb{C}}$ acts transitively by biholomorphisms. Let $\mathfrak{g}_{\mathbb{C}}$ be the Lie algebra of $G_{\mathbb{C}}$ and $\mathfrak{g}^F_{\mathbb{C}}$ denote the isotopy subalgebra of elements which preserve $F \in \check{\mathcal{M}}$. Let $\mathfrak{q}$ be a vector space complement to $\mathfrak{g}^F_{\mathbb{C}}$ in $\mathfrak{g}_{\mathbb{C}}$. Then, by the implicit function theorem, there exists a neighborhood $\mathcal{U}$ of $0 \in \mathfrak{q}$ such that the map

$$u \in \mathcal{U} \longmapsto e^u \cdot F \in \check{\mathcal{M}}$$

is a biholomorphism onto its image.

Let $(z_1, \dots, z_k)$ denote the standard Euclidean coordinates on $\mathbb{C}^k$ and $U^k \subset \mathbb{C}$ denote the product of upper half-planes where $\mathrm{Im}(z_1), \dots, \mathrm{Im}(z_k) > 0$. Let $\Delta^{r-k} \subset \Delta^r$ be the locus where $s_1, \dots, s_k = 0$ and

$$(z, s) = (z_1, \dots, z_k, s_{k+1}, \dots, s_r)$$

be the corresponding coordinate system of $U^k \times \Delta^{r-k}$.

Let $U := U^k \times \Delta^{k-r} \to \Delta^{*k} \times \Delta^{r-k} =: \Delta^*$ be the covering map

$$(z_1, \dots, z_k, s_{k+1}, \dots, s_r) \longrightarrow (e^{2\pi i z_1}, \dots, e^{2\pi i z_k}, s_{k+1}, \dots, s_r),$$

i.e. $s_j = e^{2\pi i z_j}$ for $j = 1, \dots, k$. Let η_j be the covering transformation $\eta_j(z, s) = (z + e_j, s)$ where e_j is the j'th unit coordinate vector in $\mathbb{C}^k$. Set

$$N(z) = z_1 N_1 + \cdots + z_k N_k.$$

By the local liftability of φ there exists a holomorphic map $F\colon U^k \times \Delta^{r-k} \to \mathcal{M}$ such that $F(\eta_j(z, s)) = T_j \cdot F(z, s)$ which makes the following diagram commute

$$\begin{array}{ccc} U & \xrightarrow{F} & \mathcal{M} \\ \downarrow & & \downarrow \\ \Delta^* & \xrightarrow{\varphi} & \Gamma\backslash\mathcal{M}. \end{array}$$

Accordingly, the formula $\tilde{\psi}(z,s) = e^{-N(z)} \cdot F(z,s)$ defines a map $\tilde{\psi}\colon U^k \times \Delta^{r-k} \to \check{\mathcal{M}}$ such that $\tilde{\psi} \circ \eta_j(z,s) = \tilde{\psi}(z,s)$. Therefore, $\tilde{\psi}$ descends to a holomorphic map $\psi\colon \Delta^{*k} \times \Delta^{r-k} \to \check{\mathcal{M}}$. By admissibility [SZ85, Kas86], ψ extends to a holomorphic map $\Delta^r \to \check{\mathcal{M}}$ with limit Hodge filtration

$$F_\infty = \lim_{s\to 0} \psi(s) \in \check{\mathcal{M}}. \tag{6.2}$$

Let N be an element of the monodromy cone

$$\mathcal{C} = \Big\{ \sum_j a_j N_j \,\Big|\, a_1, \dots, a_k > 0 \Big\}.$$

By admissibility, it follows that the relative weight filtration $M = M(N, W)$ of N and W exists, and together with F_∞ define a graded-polarizable mixed Hodge structure (F_∞, M).

The mixed Hodge structure (F_∞, M) induces a mixed Hodge structure on $\mathfrak{g}_{\mathbb{C}}$ with associated Deligne bigrading

$$\mathfrak{g}_{\mathbb{C}} = \bigoplus_{a+b\leq 0} \mathfrak{g}_{\mathbb{C}}^{a,b}.$$

In particular,

$$\mathfrak{g}_{\mathbb{C}}^{F_\infty} = \bigoplus_{\substack{a\geq 0\\ a+b\leq 0}} \mathfrak{g}_{\mathbb{C}}^{a,b}$$

and hence

$$\mathfrak{q}_\infty := \bigoplus_{\substack{a< 0\\ a+b\leq 0}} \mathfrak{g}_{\mathbb{C}}^{a,b} \tag{6.3}$$

is a vector space complement to $\mathfrak{g}_{\mathbb{C}}^{F_\infty}$ in $\mathfrak{g}_{\mathbb{C}}$. Therefore, it follows from equation (6.2) that for $s \sim 0$ we can write

$$\psi(s) = e^{\Gamma(s)} \cdot F_\infty,$$

where $\Gamma(s)$ is a holomorphic function with values in $\mathfrak{q}_\infty$ which vanishes at $s = 0$. Thus,

$$F(z,s) = e^{N(z)} e^{\Gamma(s)} \cdot F_\infty. \tag{6.4}$$

See [Pea00, §6] for a complete account of the constructions outlined in the previous paragraphs.

The final preliminary result we need is the following [HP15, Lemma 5.7]

$$[N_j, \Gamma(s)|_{s_j=0}] = 0, \tag{6.5}$$

which follows from a straightforward consequence of horizontality and the results established above. Accordingly,

$$[N_j, \Gamma(s)] = [N_j, \Gamma(s) - \Gamma(s)|_{s_j=0}]. \tag{6.6}$$

Considering the power series expansion of $\Gamma(s)$ about $s = 0$ we see that $\Gamma(s) - \Gamma(s)|_{s_j=0}$ is divisible by s_j. Thus,

$$s_j | [N_j, \Gamma(s)] \tag{6.7}$$

in $\mathcal{O}(\Delta^r)$.

By induction one has the following result [HP15, 8.11]): Given a multi-index $J = (a_1, \ldots, a_k)$ with non-negative entries define

$$A_J = \prod_j \mathrm{Ad}(N_j)^{a_j}.$$

and

$$s^{|J|} = \prod_{\{j|a_j \neq 0\}} s_j.$$

Then

$$s^{|J|} | A_J \Gamma. \tag{6.8}$$

Let $M(z, \bar{z})$ be a monomial in $z_1, \ldots, z_k$ and $\bar{z}_1, \ldots, \bar{z}_k$. Let $\alpha(s, \bar{s})$ be a real analytic $\mathfrak{g}_{\mathbb{C}}$-valued function on Δ^r in the variables $s_1, \ldots, s_r$ and $\bar{s}_1, \ldots, \bar{s}_r$ which vanishes at $s = 0$. Motivated by (6.8) we say that the product $M(z, \bar{z})\alpha(s, \bar{s})$ is a *tame monomial* if, whenever z_j or $\bar{z}_j$ divide M, then either s_j or $\bar{s}_j$ divides α (note: if f is any $\mathfrak{g}_{\mathbb{C}}$ valued real analytic function, then $z_j s_j f$, $z_j \bar{s}_j f$, $\bar{z}_j s_j f$, $\bar{z}_j \bar{s}_j f$ are all tame monomials). A *tame polynomial* is a finite sum of tame monomials. Let $\mathcal{T}$ denote the set of all tame polynomials.

$\mathcal{T}$ is a complex vector space which is closed under complex conjugation and taking Hodge components with respect to a fixed mixed Hodge structure. If $\beta \in \mathfrak{g}_{\mathbb{C}}$ and $\tau \in \mathcal{T}$ then $[\beta, \tau]$ clearly belongs to $\mathcal{T}$. By equation (6.8), the application of any polynomial in $\mathrm{Ad}(N(z))$ and $\mathrm{Ad}(N(\bar{z}))$ to $\Gamma(s)$ is tame.

To see that $\mathcal{T}$ is closed under Lie bracket, note that if $m_1\alpha_1$ and $m_2\alpha_2$ are tame monomials then

$$[m_1\alpha_1, m_2\alpha_2] = m_1 m_2 [\alpha_1, \alpha_2].$$

If z_j or $\bar{z}_j$ divides $m_1 m_2$ then z_j or $\bar{z}_j$ must divide either m_1 or m_2. If z_j or $\bar{z}_j$ divides m_1 then either s_j or $\bar{s}_j$ divides α_1. As such s_j or $\bar{s}_j$ divides $[\alpha_1, \alpha_2]$. The same argument applies to the case where z_j or $\bar{z}_j$ divides m_2.

Finally, if $\tau \in \mathcal{T}$ then

$$\lim_{\substack{\mathrm{Im}(z) \to \infty \\ s \to 0}} \tau(z, s) = 0, \tag{6.9}$$

where the limit is taken along sequences $(z(m), s(m)) \in U^k \times \Delta^{k-r}$ such that $s(m) \to 0$, $\mathrm{Im}(z_1(m)), \ldots, \mathrm{Im}(z_k(m)) \to \infty$ and $\mathrm{Re}(z_1(m)), \ldots, \mathrm{Re}(z_k(m))$ is constrained to a finite interval.

We now specialize to the case where $\mathcal{V}$ is Hodge–Tate. By the monodromy theorem [Sch73, 6.1] it follows that $N \in C$ acts trivially on each $\mathrm{Gr}^W_{2\ell}$ as $\mathrm{Gr}^W_{2\ell}$ is pure of type (ℓ, ℓ). Therefore, by admissibility and Proposition (2.14) of [SZ85] it follows that the relative weight filtration $M = M(N, W)$ exists and equals W. Accordingly, the limit Hodge filtration F_∞ of $\mathcal{V}$ belongs to $\mathcal{M}$. Therefore, the image of ψ is contained in $\mathcal{M}$.

Remark 6.2. Since every element $N \in C$ acts trivially on Gr^W, the same holds for every

$$N \in \bar{C} = \{ \sum_j a_j N_j \mid a_1, \ldots, a_k \geq 0 \}$$

and hence $N \in \bar{C}$ implies that $M(N, W) = W$. Therefore, $(\psi(s), W)$ is the limit Hodge structure at $s \in D \cap \Delta^r$.

Before continuing, we note that F_∞ depends upon the choice of local coordinates $(s_1, \dots, s_r)$. The permissible changes of coordinates which are compatible with the divisor structure result in the limit Hodge filtration F_∞ only being well defined up to transformation of the form

$$F_\infty \mapsto e^{N(\lambda)} \cdot F_\infty, \qquad N(\lambda) = \sum_j \lambda_j N_j, \tag{6.10}$$

for some complex numbers $\lambda_1, \dots, \lambda_k$. Since $\mathcal{V}$ is Hodge–Tate, $\mathrm{Gr}^W_k = 0$ for odd k. Since $\ell(\mathcal{V}) \geq 4$, by Corollary 2.11 we have

$$\mathrm{ht}(e^{\lambda N} \cdot F_\infty, W) = \mathrm{ht}(F_\infty, W).$$

We conclude this section with the proof of Theorem 6.1.

Proof of Theorem 6.1. By Remark 6.2 and the fact that $F_\infty \in \mathcal{M}$ we deduce that the limit mixed Hodge structure $(F_\infty, M(N, W)) = (F_\infty, W)$ is Hodge–Tate and has the same weight filtration. So it only remains to be shown the continuity condition (6.1).

Returning to the subspace (6.3), we see that since $\mathcal{V}$ is Hodge–Tate and $F_\infty \in \mathcal{M}$, it follows that

$$\mathfrak{q}_\infty = \bigoplus_{a<0} \mathfrak{g}_{\mathbb{C}}^{a,a} = \Lambda^{-1,-1}_{(F_\infty, W)}$$

in this case. Accordingly, by (6.4) and Lemma 2.2 we have

$$Y_{(F(z,s),W)} = Y_{(e^{N(z)}e^{\Gamma(s)} \cdot F_\infty, W)} = e^{N(z)}e^{\Gamma(s)} \cdot Y_{(F_\infty, W)}$$

and hence

$$\overline{Y_{(F(z,s),W)}} = e^{N(\bar{z})}e^{\bar{\Gamma}(s)} \cdot \overline{Y_{(F_\infty, W)}}.$$

Let $\delta = \delta_{(F_\infty, W)}$ and $\delta(z, s) = \delta_{(F(z,s),W)}$ as in (2.6). Then,

$$\overline{Y_{(F(z,s),W)}} = e^{N(\bar{z})}e^{\bar{\Gamma}(s)}e^{-2i\delta} \cdot Y_{(F_\infty, W)}.$$

On the other hand, by definition

$$\overline{Y_{(F(z,s),W)}} = e^{-2i\delta(z,s)} \cdot Y_{(F(z,s),W)} = e^{-2i\delta(z,s)}e^{N(z)}e^{\Gamma(s)} \cdot Y_{(F_\infty, W)}.$$

Comparing these two equations, it follows that

$$e^{N(\bar{z})}e^{\bar{\Gamma}(s)}e^{-2i\delta} \cdot Y_{(F_\infty, W)} = e^{-2i\delta(z,s)}e^{N(z)}e^{\Gamma(s)} \cdot Y_{(F_\infty, W)}. \tag{6.11}$$

By Proposition 2.2 of [CKS86], the group $\exp(W_{-1}\,\mathfrak{gl}(V))$ acts simply transitively on the set of gradings of W. Therefore, equation (6.11) implies that

$$e^{N(\bar{z})}e^{\bar{\Gamma}(s)}e^{-2i\delta} = e^{-2i\delta(z,s)}e^{N(z)}e^{\Gamma(s)}. \tag{6.12}$$

The Hodge components of $\alpha \in \mathfrak{g}_{\mathbb{C}}$ relative to (F_∞, W) will be denoted $\alpha^{-b,-b}$. For the remainder of this proof, we constrain $\mathrm{Re}(z_1), \dots, \mathrm{Re}(z_k)$ to a finite interval.

By the Campbell–Baker–Hausdorff formula (CBH),

$$e^{N(\bar{z})}e^{\bar{\Gamma}(s)} = e^{N(\bar{z}) + \bar{\Gamma}(s) + A(z,s)}, \tag{6.13}$$

where $A(z,s)$ is a Lie polynomial with terms $X = \mathrm{Ad}(X_1) \circ \mathrm{Ad}(X_{m-1})X_m$ where at least one $X_j = \bar{\Gamma}(s)$ and the other X_i are either $N(\bar{z})$ or $\bar{\Gamma}(s)$. Therefore, by the discussion following (6.8), $A(z,s)$ belongs to $\mathcal{T}$. For future use, we observe that $A^{-1,-1}(z,s) = 0$ since $A(z,s)$ is a sum of terms containing at least two elements from $\mathfrak{q}_\infty = \oplus_{k>1} \mathfrak{g}_{\mathbb{C}}^{-k,-k}$.

Before continuing, observe that because each $N_j = N_j^{-1,-1}$ and $\delta = \sum_{k>0} \delta^{-k,-k}$ the equation $[N_j, \delta] = 0$ implies $[N(\bar{z}), \delta^{-k,-k}] = 0$ for all $k > 0$. In particular,

$$\mathrm{Ad}(L_1) \circ \cdots \circ \mathrm{Ad}(L_{m-1})A(z,s) \in \mathcal{T} \tag{6.14}$$

if each L_j is either $-2i\delta$ or $N(\bar{z})$ since $[N(\bar{z}), \delta] = 0$ and $A(z,s)$ is itself constructed from Lie polynomials in $\mathrm{Ad}(N(\bar{z}))$ and $\bar{\Gamma}(s)$.

More generally, any Lie polynomial $U = \mathrm{Ad}(U_1) \circ \cdots \circ \mathrm{Ad}(U_{m-1})U_m$ where each U_j is either $N(\bar{z})$, $\bar{\Gamma}(s)$, $A(z,s)$ and $-2i\delta$ again belongs to $\mathcal{T}$. Indeed, bracketing $\bar{\Gamma}(s)$ or $A(z,s)$ with $-2i\delta$ produces another element of $\mathcal{T}$. By the remarks of the previous paragraph, if $\bar{\Gamma}(s)$ does not appear the result belongs to $\mathcal{T}$. Finally, $\bar{\Gamma}(s)$ belongs to $\mathcal{T}$, and $\mathcal{T}$ is closed under Lie brackets. Application of the Jacobi identity now shows that U belongs to $\mathcal{T}$.

Continuing, by the CBH,

$$e^{N(\bar{z})+\bar{\Gamma}(s)+A(z,s)} e^{-2i\delta} = e^{N(\bar{z})+\bar{\Gamma}(s)+A(z,s)-2i\delta+B(z,s)}, \tag{6.15}$$

where $B(z,s)$ is a Lie polynomial with terms $X = \mathrm{Ad}(X_1) \circ \mathrm{Ad}(X_{m-1})X_m$ where at least one $X_j = -2i\delta$ and the other X_i are either $N(\bar{z}) + \bar{\Gamma}(s) + A(z,s)$ or $-2i\delta$. Expanding out X as a sum of terms $U = \mathrm{Ad}(U_1) \circ \cdots \circ \mathrm{Ad}(U_{r-1})U_r$ where each U_j is either $N(\bar{z})$, $\bar{\Gamma}(s)$, $A(z,s)$ and $-2i\delta$ it follows that $B(z,s)$ belongs to $\mathcal{T}$ by the previous paragraph. As was the case for A, $B^{-1,-1}(z,s) = 0$ since $B(z,s)$ is a sum of terms involving the Lie bracket of at least 2 elements of $\mathfrak{q}_\infty$.

Turning now to the right hand side of (6.12), by (6.13)

$$e^{N(z)} e^{\Gamma(s)} = e^{N(z)+\Gamma(s)+\bar{A}(z,s)}.$$

Therefore,

$$e^{-2i\delta(z,s)} e^{N(z)} e^{\Gamma(s)} = e^{-2i\delta(z,s)+N(z)+\Gamma(s)+\bar{A}(z,s)+C(z,s)}, \tag{6.16}$$

where $C(z,s)$ is a sum of terms $X = \mathrm{Ad}(X_1) \circ \cdots \circ \mathrm{Ad}(X_{m-1})X_m$ with some $X_j = -2i\delta(z,s)$ and the remaining terms X_i either equal to $-2i\delta(z,s)$ or to $N(z) + \Gamma(s) + \bar{A}(z,s)$.

Comparing (6.15) and (6.16) it follows that

$$\begin{aligned} &N(\bar{z}) + \bar{\Gamma}(s) + A(z,s) - 2i\delta + B(z,s) \\ &\qquad = -2i\delta(z,s) + N(z) + \Gamma(s) + \bar{A}(z,s) + C(z,s). \end{aligned} \tag{6.17}$$

Like with A and B, we have $C^{-1,-1}(z,s) = 0$. Accordingly, taking the $(-1,-1)$-component of equation (6.17) yields

$$N(\bar{z}) + (\bar{\Gamma})^{-1,-1}(s) - 2i\delta^{-1,-1} = -2i\delta^{-1,-1}(z,s) + N(z) + \Gamma^{-1,-1}(s).$$

Solving for $\delta^{-1,-1}(z,s)$ gives

$$\delta^{-1,-1}(z,s) = N(\mathrm{Im}(z)) + \mathrm{Im}(\Gamma(s))^{-1,-1} + \delta^{-1,-1}. \tag{6.18}$$

Returning to equation (6.16) and noting that $A^{-1,-1}(z,s) = 0$, upon taking the $(-2,-2)$-component we obtain that

(6.19)
$$\begin{aligned} C^{-2,-2}(z,s) &= \frac{1}{2}[-2i\delta^{-1,-1}(z,s), N(z) + \Gamma^{-1,-1}(s)] \\ &= -i[N(\mathrm{Im}(z)) + \mathrm{Im}(\Gamma(s))^{-1,-1} + \delta^{-1,-1}, N(z) + \Gamma^{-1,-1}(s)] \\ &= -i[N(\mathrm{Im}(z)), \Gamma^{-1,-1}(s)] \\ &\quad - i[\mathrm{Im}(\Gamma(s))^{-1,-1}, N(z) + \Gamma^{-1,-1}(s)] \\ &\quad - i[\delta^{-1,-1}, \Gamma^{-1,-1}(s)]. \end{aligned}$$

In particular, it follows from (6.19) that $C^{-2,-2}(z,s)$ belongs to class $\mathcal{T}$.

Taking $(-2,-2)$ components (6.17) implies

$$\begin{aligned} &(\bar{\Gamma})^{-2,-2}(s) + A^{-2,-2}(z,s) + B^{-2,-2}(z,s) - 2i\delta^{-2,-2} \\ &\qquad = -2i\delta^{-2,-2}(z,s) + \Gamma^{-2,-2}(s) + \bar{A}^{-2,-2}(z,s) + C^{-2,-2}(z,s) \end{aligned}$$

and hence

$$\delta^{-2,-2}(z,s) = \delta^{-2,-2} + D^{-2,-2}(z,s),$$

where $D^{-2,-2}(z,s)$ belongs to the class $\mathcal{T}$. By (6.9) we obtain that

$$\lim_{\substack{\mathrm{Im}(z)\to\infty \\ s\to 0}} \delta^{-2,-2}(z,s) = \delta^{-2,-2}.$$

Therefore, we have completed the proof of Theorem 6.1 in the case where $\ell(\mathcal{V}) = 4$ (e.g. the dilogarithm variation in Example 6.7).

To verify the general statement, we assume by induction that for $a = 2, \cdots, k$ that

(i) $C^{-a,-a}(z,s)$ belongs to class $\mathcal{T}$, and is given by a Lie polynomial with terms

$$\mathrm{Ad}(L_1) \circ \cdots \circ \mathrm{Ad}(L_{r-1}) L_r, \tag{6.20}$$

where each L_j is either $\delta^{-b,-b}$, $N(z)$, $N(\bar{z})$, $\Gamma^{-b,-b}(s)$ or $\bar{\Gamma}^{-b,-b}(s)$.

(ii) $\delta^{-a,-a}(z,s) = \delta^{-a,-a} + D^{-a,-a}(z,s)$ where $D^{-a,-a}(z,s)$ satisfies also condition *(i)*.

The previous paragraphs establish the induction base $a = 2$.

To establish the case $a = k+1$ we recall $C(z,s)$ is a sum of terms $X = \mathrm{Ad}(X_1) \circ \cdots \circ \mathrm{Ad}(X_{m-1}) X_m$ where some $X_j = -2i\delta(z)$ and the remaining terms X_i are either $-2i\delta(z)$ or $N(z) + \Gamma(s) + \bar{A}(z,s)$ (which occurs at least once). In particular, upon expanding $\delta(z,s)$ into Hodge components, it follows that $C^{-a-1,-a-1}(z,s)$ can be expanded into a sum of terms

$$U = \mathrm{Ad}(U_1) \circ \cdots \circ \mathrm{Ad}(U_{m-1}) U_m$$

of the required form *(i)*. It now follows from (6.17) and the previous results about $A(z,s)$, $B(z,s)$ and $C^{-b,-b}(z,s)$ for $b = 1, \ldots, k+1$ that *(ii)* holds. □

6.2. **Heights of Nilpotent Orbits.** Let $\mathcal{U} \to \Delta^*$ be an admissible variation of mixed Hodge structure over the punctured disk Δ^* with weight graded quotients $\mathrm{Gr}_0^W = \mathbb{Z}(0)$, $\mathrm{Gr}_{-1}^W = \mathcal{H}$ and $\mathrm{Gr}_{-2}^W = \mathbb{Z}(1)$. Assume that $\mathcal{U}$ has unipotent monodromy and select an embedding of Δ^* into the coordinate disk

$$\Delta = \{s \in \mathbb{C} \mid |s| < 1\}$$

as the complement of $s = 0$. In §3 of [BP19], the third author and P. Brosnan proved that there exists a rational number μ such that

$$h(s) = \mathrm{ht}(\mathcal{U}_s) + \mu \log |s| \tag{6.21}$$

extends continuously to Δ. Moreover, $h(0)$ can be constructed by pure linear algebra from the data of (N, F_∞, W) of the nilpotent orbit of $\mathcal{U}$.

Consider now an arbitrary oriented admissible variation $\mathcal{V} \to \Delta^*$ with unipotent monodromy. As noted in (6.10), the data (N, F_∞, W) of the associated nilpotent orbit of $\mathcal{V}$ is only well defined up to replacing F_∞ by $e^{\lambda N} \cdot F_\infty$. In this section, we define a height $\mathrm{ht}(N, F_\infty, W)$ of an oriented admissible nilpotent orbit $(e^{zN} \cdot F_\infty, W)$ which generalizes the construction of [BP19] and prove:

Proposition 6.3. *If $\ell(\mathcal{V}) > 2$ then, for any $\lambda \in \mathbb{C}$,*

$$\mathrm{ht}(N, e^{\lambda N} \cdot F_\infty, W) = \mathrm{ht}(N, F_\infty, W).$$

Thus $\mathrm{ht}(N, F_\infty, W)$ only depends on the variation $\mathcal{V}$ and not on a particular choice of limit Hodge filtration F_∞. If moreover N acts trivially on Gr^W then $M(N, W) = W$ and

$$\mathrm{ht}(N, F_\infty, W) = \mathrm{ht}(F_\infty, M).$$

On the right hand side $\mathrm{ht}(F_\infty, M)$ denotes the usual height of the oriented extension (F_∞, M).

Accordingly, we can define the limit height of $\mathcal{V}$ to be $\mathrm{ht}(N, F_\infty, W)$ of the associated nilpotent orbit.

Remark 6.4. Unfortunately, we do not yet have the analog of (6.21) in general. In the next subsection we given an example of an admissible nilpotent variation with weight graded quotients $\mathrm{Gr}_0^W = \mathbb{Z}$, Gr_{-3}^W of rank 2 and $\mathrm{Gr}_{-6}^W \cong \mathbb{Z}(3)$ for which $\mathrm{ht}(\mathcal{V})$ grows like a multiple of $(\log |s|)^3$ as $s \to 0$.

To define the height of a nilpotent orbit, we will freely borrow from section 6 of [BPR17]. The key concept is the notion of a Deligne system, which originates from a letter of P. Deligne to E. Cattani and A. Kaplan:

Definition 6.5. (6.6, [BPR17]) Let K be a field of characteristic zero. A 1-variable Deligne system over K consists of the following data:

(i) an increasing filtration W of a finite dimensional K-vector space V;
(ii) a nilpotent endomorphism N of V which preserves W such that the relative weight filtration $M = M(N, W)$ exists;
(iii) a grading Y of M which preserves W and satisfies $[Y, N] = -2N$.

A morphism of Deligne systems $(W, N, Y) \to (\tilde{W}, \tilde{N}, \tilde{Y})$ is an endomorphism T of the underlying K-vector spaces such that

$$T(W_i) \subseteq \tilde{W}_i, \qquad \tilde{Y} \circ T - T \circ Y = 0 \qquad \text{and} \qquad \tilde{N} \circ T - T \circ N = 0.$$

Given a Deligne system (W, N, Y), each choice of grading Y' of W which commutes with Y determines an $\mathfrak{sl}_2$-triple (N_0, H, N_0^+) where

$$N = \sum_{j \geq 0} N_{-j}, \qquad [Y', N_{-j}] = -jN_{-j}, \tag{6.22}$$

(so N_0 is the 0-eigencomponent of N relative to $\operatorname{Ad} Y'$) and $H = Y - Y'$ (cf. equations 6.8 and 6.9 of [BPR17]). The basic construction of Deligne's letter is the following (see [BPR17] for additional history and references):

Theorem 6.6 ([BPR17, 6.10])**.** *Let (W, N, Y) be a Deligne system. Then, there exists an unique functorial grading $Y' = Y'(N, Y)$ of W which commutes with Y such that*

$$[N - N_0, N_0^+] = 0, \tag{6.23}$$

where (N_0, H, N_0^+) is the associated $\mathfrak{sl}_2$-triple attached to Y' and (W, N, Y).

In particular, given any admissible variation $\mathcal{V}$ of mixed Hodge structure over the punctured disk Δ^* with unipotent monodromy, we obtain a Deligne system (W, N, Y) where W is the weight filtration of $\mathcal{V}$, N is the local monodromy and $Y = Y_{(F_\infty, M)}$ where (F_∞, M) is the limit mixed Hodge structure of $\mathcal{V}$. If $\lambda \in \mathbb{C}$, then $e^{\lambda N}$ is a morphism from (W, N, Y) to $(W, N, Y + 2\lambda N) = (W, N, Y_{(e^{\lambda N} \cdot F_\infty, M)})$. Therefore,

$$Y'(N, Y_{(e^{\lambda N} \cdot F_\infty, M)}) = e^{\lambda N} \cdot Y'(N, Y_{(F_\infty, M)}). \tag{6.24}$$

We next proceed to the definition of the height of a nilpotent orbit. So let $\mathcal{M}$ and $\check{\mathcal{M}}$ be the classifying spaces of mixed Hodge structures of a filtered vector space (V, W) and its compact dual. Let $F \in \check{\mathcal{M}}$ and N a nilpotent endomorphism of V such that $(e^{zN} \cdot F, W)$ is an admissible nilpotent orbit. This means the following conditions

(i) $N(F^r) \subset F^{r-1}$ (horizontality,)
(ii) $e^{zN} \cdot F \in \mathcal{M}$ for $\operatorname{Im}(z) \gg 0$,
(iii) the filtration $M = M(N, W)$ exists.

Let $\max = \max(W)$, $\min = \min(W)$. Assume $(e^{zN} \cdot F, W)$ is oriented and $\ell = \ell(W) > 2$. We have a limit mixed Hodge structure (F, M). Let $Y' = Y'(N, Y_{(F,M)})$ and $\delta = \delta_{(F,M)}$. Write

$$\delta = \sum_{j \geq 0} \delta_{-j}, \qquad [Y', \delta_{-j}] = -j\delta_{-j}. \tag{6.25}$$

Note that this decomposition is with respect to a grading of W and not with respect to a grading of M. We define the height of the admissible nilpotent orbit as

$$\operatorname{ht}(N, F, W)e^\vee = \delta_{-\ell}\, e, \tag{6.26}$$

where e is a lift of the generator of $\operatorname{Gr}^W_{\max}$ and $e^\vee$ projects to the generator of $\operatorname{Gr}^W_{\min}$. We stress here the fact that the generators e and $e^\vee$ as well as the grading Y' correspond to the filtration W, while the operator δ is defined by the mixed Hodge structure (F, M). We proceed in this way because there is no reason for (F, M) to be oriented.

Proof of Proposition 6.3. Let $e^{zN} \cdot F$ be an admissible nilpotent orbit as before and $\lambda \in \mathbb{C}$. Let $\delta = \delta_{(F,M)}$ and $\tilde{\delta} = \delta_{(e^{\lambda N} \cdot F,M)}$. By Lemma 2.10

$$\tilde{\delta} = \delta + \mathrm{Im}(\lambda)N.$$

Moreover, since N is a $(-1,-1)$-morphism of both (F,M) and $(e^{\lambda N} \cdot F, M)$ it follows that both δ and $\tilde{\delta}$ are fixed by the adjoint action of $e^{\lambda N}$.

Let $\delta = \sum_j \delta_j$ and $N = \sum_j N_j$ denote the decomposition of δ and N into eigencomponents with respect to the adjoint action of $Y' = Y'(N, Y_{(F,M)})$ as in (6.25). Then,

$$\tilde{\delta} = e^{\lambda N} \cdot \tilde{\delta} = e^{\lambda N} \cdot \sum_{j \geq 0} \delta_{-j} + \mathrm{Im}(\lambda) N_{-j} = \sum_{j \geq 0} e^{\lambda N} \cdot (\delta_{-j} + \mathrm{Im}(\lambda) N_{-j}). \tag{6.27}$$

Let $\tilde{Y}' = Y'(N, Y_{(e^{\lambda N} \cdot F,M)})$ and

$$\tilde{\delta} = \sum_{j \geq 0} \tilde{\delta}_{-j}, \qquad [\tilde{Y}', \tilde{\delta}_{-j}] = -j \tilde{\delta}_{-j},$$

be the decomposition of $\tilde{\delta}$ into eigencomponents for $\mathrm{Ad}\, \tilde{Y}'$. By equation (6.24), $\tilde{Y}' = e^{\lambda N} \cdot Y'$. Moreover,

$$\begin{aligned}[e^{\lambda N} \cdot Y', e^{\lambda N} \cdot (\delta_{-j} + \mathrm{Im}(\lambda) N_{-j})] &= e^{\lambda N} \cdot [Y', \delta_{-j} + \mathrm{Im}(\lambda) N_{-j}] \\ &= -j e^{\lambda N} \cdot (\delta_{-j} + \mathrm{Im}(\lambda) N_{-j}).\end{aligned}$$

Comparing the previous equation with (6.27) it follows that

$$\tilde{\delta}_{-j} = e^{\lambda N} \cdot (\delta_{-j} + \mathrm{Im}(\lambda) N_{-j}). \tag{6.28}$$

In the notation of (6.26), we are interested in comparing $\tilde{\delta}_{-\ell}$ and $\delta_{-\ell}$. As a first step, we note that N acts trivially on $\mathrm{Gr}^W_{\max}$ and $\mathrm{Gr}^W_{\min}$ as each factor has dimension 1 and N is nilpotent. As N preserves W, it then follows that $e^{\lambda N}$ fixes $\delta_{-\ell}$ and $N_{-\ell}$ under the adjoint action. Thus,

$$\tilde{\delta}_{-\ell} = \delta_{-\ell} + \mathrm{Im}(\lambda) N_{-\ell}.$$

The limit mixed Hodge structure (F,M) induces on Gr^W the limit mixed Hodge structures of the variations of pure Hodge structure on Gr^W. Let $2a = \max$ and $2b = \min$. Then, Gr^W_{2a} is the constant variation of type (a,a) whereas Gr^W_{2b} is the constant variation of type (b,b). Consequently, F^a surjects on Gr^W_{2a} whereas F^{a+1} maps to zero in Gr^W_{2a}. Moreover, $\mathrm{Gr}^W_{2b} = W_{2b}$ and $W_{2b} \subset F^b$ whereas $F^{b+1} \cap W_{2b} = 0$.

By the previous paragraph, it follows that in equation (6.26) we can arrange that $e \in F^a$. By equation (3.20) in [Pea06a], Y' preserves F. Accordingly, since N is horizontal with respect to F, so is each eigencomponent N_{-j}.

Therefore, $N_{-\ell}(e) \in F^{a-1}$. But, $2a - 2b > 2$ implies $a - 1 > b$ and hence $N_{-\ell}(e) \in F^{b+1} \cap W_{2b}$. Thus, $N_{-\ell}(e) = 0$. This proves the first statement of Proposition 6.3.

Finally, if N acts trivially on Gr^W then $N_0 = 0$ and hence $H = Y - Y' = 0$. Therefore, $Y = Y'$ which implies $M = W$ and the decomposition of δ with respect to Y' is just the decomposition of δ with respect to $Y = Y_{(F_\infty,M)}$. □

6.3. **Three Examples.** In this subsection we show that the Bloch–Wigner dilogarithm D_2 is the height of the dilogarithm variation over $\mathbb{P}^1 - \{0, 1, \infty\}$. We then show that up to a multiple of $4\zeta(2)$, we can express D_2 as the height of an elementary family of triangles of the type considered in 5.2. Finally, we show that the height can become unbounded in the case where the underlying variation of mixed Hodge structure is not unipotent in the sense of Hain and Zucker [HZ87].

Example 6.7. Let $\mathcal{V}$ be the dilog variation over $\mathbb{P}^1 - \{0, 1, \infty\}$ (4.13, [HZ87]). Then, $\mathrm{ht}(\mathcal{V}) = -D_2(s)$.

By (4.13, [SZ85]) we may select a basis $\{e_0, e_1, e_2\}$ of $V_{\mathbb{C}} = \mathcal{V}_s$ such that $\mathcal{V}$ has bigrading $I^{a,a} = \mathbb{C}e_{-a}$ and integral structure $V_{\mathbb{Z}}$ generated by

$$\begin{aligned} v_0(s) &= e_0 - \log(1-s)e_1 + L_2(s)e_2, \\ v_1(s) &= (2\pi i)(e_1 + \log(s)e_2), \\ v_2(z) &= (2\pi i)^2 e_2, \end{aligned}$$

where $L_2(s) = \sum_{j=1}^{\infty} \frac{s^j}{j^2}$ is the dilogarithm. By Lemma 2.6 we need to compute $\frac{1}{2}\mathrm{Im}((e_0 - \overline{e_0})_{-4})$.

Abbreviating $v_j(s)$ to v_j, it follows from the previous equations that

$$\begin{aligned} e_2 &= (2\pi i)^{-2} v_2, \\ e_1 &= (2\pi i)^{-1} v_1 - (2\pi i)^{-2} \log(s) v_2, \\ e_0 &= v_0 + (2\pi i)^{-1} \log(1-s) v_1 - (2\pi i)^{-2} [\log(1-s)\log(s) v_2 + L_2(s)] v_2. \end{aligned}$$

Therefore,

$$\begin{aligned} & e_0 - \overline{e_0} \\ &\quad = 2(2\pi i)^{-1} \mathrm{Re}(\log(1-s)) v_1 - 2i(2\pi i)^{-2} \mathrm{Im}(\log(1-s)\log(s) + L_2(s)) v_2 \\ &\quad = 2\,\mathrm{Re}(\log(1-s))(e_1 + \log(s)e_2) - 2i\mathrm{Im}(\log(1-s)\log(s) + L_2(s))e_2. \end{aligned}$$

Accordingly,

$$\begin{aligned} & \frac{1}{2}\mathrm{Im}((e_0 - \overline{e_0})_{-4}) \\ &\qquad = \mathrm{Re}(\log(1-s))\mathrm{Im}(\log(s)) - \mathrm{Im}(\log(1-s)\log(s)) - \mathrm{Im}(L_2(s)). \end{aligned}$$

To simplify the previous equation, let $\log(1-s) = A + iB$ and $\log(s) = C + iD$. Then,

$$\begin{aligned} & \mathrm{Re}(\log(1-s))\mathrm{Im}(\log(s)) - \mathrm{Im}(\log(1-s)\log(s)) \\ &\qquad = AD - (AD + BC) = -BC = -\arg(1-s)\log|s|. \end{aligned}$$

Thus,

$$\mathrm{ht}(\mathcal{V}_s) = -\mathrm{Im}(L_2(s)) - \arg(1-s)\log|s| = -D_2(s).$$

Example 6.8. Returning to the setting of 5.2, let W_β denote the standard triangle and consider the sections

$$\begin{aligned} s_{t,0} &= x_0 + tx_1 + x_2, \\ s_{t,1} &= x_0 + x_1 + tx_2, \\ s_{t,2} &= tx_0 + x_1 + x_2, \end{aligned}$$

of $\mathcal{O}_{\mathbb{P}^2}(1)$, where $t \in S = \mathbb{P}^1 - \{-2,-1,0,1,\infty\}$. Let $\ell_{t,i} = \mathrm{div}(s_{t,i})$ for $i = 0,1,2$ and consider the family of higher cycles $\{Z_\alpha(t)\}_{t\in S}$, with individual $Z_\alpha(t)$ as defined in subsection 5.2. By the choice of t, all the cycles $Z_\alpha(t)$ are non-degenerate and intersect W_β properly and transversely. Moreover, the pair of cycles $Z_\alpha(t)$, W_β satisfies Assumption 3.27. Then,

$$\mathrm{ht}(B_{Z_\alpha(t),W_\beta}) = \frac{3}{(2\pi i)^2}(D_2(t) + D_2(t) + D_2(t^{-2})).$$

To continue, recall that $D_2(z) = D_2(1-1/z)$ and hence $D_2(t^{-2}) = D(1-t^2)$. By the 5 term relation

$$D_2(x) + D_2(y) + D_2\left(\frac{1-x}{1-xy}\right) + D_2(1-xy) + D_2\left(\frac{1-y}{1-xy}\right) = 0.$$

Setting $x = y = t$ it follows that

$$\begin{aligned} D_2(t) + D_2(t) + D_2(1/t^2) &= D_2(t) + D_2(t) + D_2(1-t^2) \\ &= -D_2\left(\frac{1-t}{1-t^2}\right) - D_2\left(\frac{1-t}{1-t^2}\right) \\ &= -2D_2((1+t)^{-1}). \end{aligned}$$

Finally, $D_2(z) = -D_2(1/z)$ and $D_2(z) = -D_2(1-z)$. Therefore,

$$\mathrm{ht}(B_{Z_\alpha(t),W_\beta}) = \frac{6}{(2\pi i)^2}D_2(1+t) = \frac{-D_2(1+t)}{4\zeta(2)} = \frac{D_2(-t)}{4\zeta(2)}.$$

In particular, upon setting $\theta = \pi/2$ in the formula $D_2(e^{i\theta}) = \sum_{n=1}^\infty \frac{\sin(n\theta)}{n^2}$ it follows that $D_2(\sqrt{-1})$ is equal to the Catalan constant C. Thus,

$$\mathrm{ht}(B_{Z_\alpha(-\sqrt{-1}),W_\beta}) = \frac{C}{4\zeta(2)}.$$

Also note that

$$\lim_{t\to p} \mathrm{ht}(B_{Z_\alpha(t),W_\beta}) = 0$$

for $p \in \{-2,-1,0,1,\infty\}$.

To close this subsection, we give an example of an admissible nilpotent orbit $(e^{zN}\cdot F, W)$ with weight graded quotients $\mathrm{Gr}^W_0 \cong \mathbb{Z}(0)$, Gr^W_{-3} of rank 2 and $\mathrm{Gr}^W_{-6} \cong \mathbb{Z}(3)$ such that the height grows like $(\log|s|)^3$ for $s = e^{2\pi i z}$.

Example 6.9. Let $V_{\mathbb{Z}}$ be the lattice generated by e_0, e, f and e_{-6}. Let

$$W_{-6} = \mathbb{Z}e_{-6}, \qquad W_{-3} = W_{-6} \oplus \mathbb{Z}f \oplus \mathbb{Z}e, \qquad W_0 = V_{\mathbb{Z}},$$

with graded-polarizations

$$\mathcal{S}_0([e_0],[e_0]) = \mathcal{S}_{-3}([e],[f]) = \mathcal{S}_{-6}([e_{-6}],[e_{-6}]) = 1.$$

Let N be the nilpotent endomorphism obtained by setting

$$N(e_0) = e, \qquad N(e) = f, \qquad N(f) = e_{-6}, \qquad N(e_{-6}) = 0.$$

Let (F,M) be the mixed Hodge structure defined by setting

$$I^{0,0} = \mathbb{C}e_0, \qquad I^{-1,-1} = \mathbb{C}e, \qquad I^{-2,-2} = \mathbb{C}f, \qquad I^{-3,-3} = \mathbb{C}e_{-6}.$$

Then, $N(I^{a,a}) \subset I^{a-1,a-1}$ and hence N is horizontal with respect to F. We also have $N(M_a) \subseteq M_{a-2}$. To verify that M is the relative weight filtration

N and W it remains to check M induces the monodromy weight filtration of $\mathrm{Gr}(N)$ shifted by $-k$ on Gr^W_k. This is clear for Gr^W_0 and Gr^W_{-6}. Let $\tilde{N}$ be the map induced by N on Gr^W_{-3} then

$$W(\tilde{N})_{-1} = \mathbb{Z}[f], \qquad W(\tilde{N})_1 = \mathrm{Gr}^W_{-3},$$

and hence $W(\tilde{N})[3]_{-4} = W(\tilde{N})_{-1} = \mathbb{Z}[f]$ while $W(\tilde{N})[3]_{-2} = W(\tilde{N})_1 = \mathrm{Gr}^W_{-3}$. Since $I^{-1,-1} = \mathbb{C}e$ and $I^{-2,-2} = \mathbb{C}f$ it follows that M induces the correct filtration on Gr^W_{-3}.

Define

$$\begin{aligned}
\nu_0(z) &= e^{zN}(e_0) = e_0 + ze + \frac{1}{2}z^2 f + \frac{1}{6}z^3 e_{-6},\\
\nu_{-1}(z) &= e^{zN}(e) = e + zf + \frac{1}{2}z^2 e_{-6},\\
\nu_{-2}(z) &= e^{zN}(f) = f + ze_{-6},\\
\nu_{-3}(z) &= e^{zN}(e_{-6}) = e_{-6}.
\end{aligned}$$

Then, $e^{zN}\cdot F^a = \oplus_{b\geq a}\mathbb{C}\nu_b(z)$. Accordingly, $e^{zN}\cdot F$ induces a pure Hodge structure of weight k on Gr^W_k: For Gr^W_0 and Gr^W_{-6} we just take the constant variations of type $(0,0)$ and $(-3,-3)$. The image $e^{zN}\cdot F^{-1}$ in Gr^W_{-3} is $\mathbb{C}[e+zf]$ which gives a variation of pure Hodge structure of weight -3.

Recall (2.12, [CKS86]) that

$$I^{p,q} = F^p \cap W_{p+q} \cap (\bar{F}^q \cap W_{p+q} + \overline{U^{q-1}_{p+q-1}}),$$

where $U^a_b = \sum_{j\geq 0} F^{a-j}\cap W_{b-j}$. In particular,

$$I^{0,0}_{(e^{zN}\cdot F,W)} = \mathbb{C}\nu_0(z), \qquad I^{-1,-2}_{(e^{zN}\cdot F,W)} = \mathbb{C}\nu_{-1}(z),$$

as both $e^{zN}\cdot F^0$ and $(e^{zN}\cdot F^{-1})\cap W_{-3}$ have rank 1.

To determine $I^{-2,-1}$ note that

$$U^{-2}_{-5} = (\mathbb{C}\nu_{-2}(z))\cap W_{-5} \oplus (\mathbb{C}\nu_{-3}(z))\cap W_{-6} = \mathbb{C}e_{-6}.$$

Therefore,

$$\overline{(e^{zN}\cdot F^{-1})}\cap W_{-3} + \overline{U^{-2}_{-5}} = \mathbb{C}\bar{\nu}_{-1}(z)\oplus\mathbb{C}e_{-6} = \mathbb{C}(e+\bar{z}f)\oplus\mathbb{C}e_{-6},$$

and hence

$$\begin{aligned}
(e^{zN}\cdot F^{-2})\cap W_{-3} &\cap (\overline{(e^{zN}\cdot F^{-1})}\cap W_{-3} + \overline{U^{-2}_{-5}})\\
&= (\mathbb{C}\nu_{-1}(z)\oplus\mathbb{C}\nu_{-2}(z))\cap(\mathbb{C}(e+\bar{z}f)\oplus\mathbb{C}e_{-6})\\
&= \mathbb{C}(e+\bar{z}f + z(\bar{z}-\frac{1}{2}z)e_{-6}),
\end{aligned}$$

because $e+\bar{z}f+z(\bar{z}-\frac{1}{2}z)e_{-6} = \nu_{-1}(z)+(\bar{z}-z)\nu_{-2}(z)$. As such, $I^{-1,-2}_{(e^{zN}\cdot F,W)}\oplus I^{-2,-1}_{(e^{zN}\cdot F,W)}$ is spanned by $\nu_{-1}(z)$ and $\nu_{-2}(z)$. Moreover $I^{-3,-3}_{(e^{zN}\cdot F,W)} = I^{-3,-3}$ is generated by e_{-6}.

To finish, observe that

$$\nu_0(z)-\bar{\nu}_0(z) = (z-\bar{z})(e+\frac{1}{2}(z+\bar{z})f + \frac{1}{6}(z^2+z\bar{z}+\bar{z}^2)e_{-6}).$$

Next,

$$e + \frac{1}{2}(z+\bar{z})f + \frac{1}{6}(z^2 + z\bar{z} + \bar{z}^2)e_{-6}$$
$$= \nu_{-1}(z) + \frac{1}{2}\nu_{-2}(z) + \frac{1}{6}(z-\bar{z})^2 e_{-6}.$$

Thus,

$$(\nu_0(z) - \bar{\nu}_0(z))_{-6} = \frac{1}{6}(z-\bar{z})^3 e_{-6}.$$

where $(\cdots)_{-6}$ is projection onto $I^{-3,-3}_{(e^{zN}\cdot F,W)}$ with respect to the Deligne bigrading of $(e^{zN}\cdot F, W)$. Write now $s = e^{2\pi i z}$, then the nilpotent orbit $(e^{zN}\cdot F, W)$ defines a variation of mixed Hodge structures $\mathcal{V}$ over the punctured unit disk with coordinate s. Then, by (2.6),

$$\mathrm{ht}(\mathcal{V}_s) = \frac{1}{12\pi^3}(\log|s|)^3.$$

We note also that, since the mixed Hodge structure (F,M) is split over $\mathbb{R}$, then $\delta_{(F,M)} = 0$. Therefore, in this case

$$\mathrm{ht}(N,F,W) = 0.$$

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