RIGID AUTOMORPHISMS OF LINKING SYSTEMS

GEORGE GLAUBERMAN AND JUSTIN LYND

Abstract. A rigid automorphism of a linking system is an automorphism which restricts to the identity on the Sylow subgroup. A rigid inner automorphism is conjugation by an element in the center of the Sylow subgroup. At odd primes, it is known that each rigid automorphism of a centric linking system is inner. We prove that the group of rigid outer automorphisms of a linking system at the prime 2 is elementary abelian, and that it splits over the subgroup of rigid inner automorphisms. In a second result, we show that if an automorphism of a finite group G restricts to the identity on the centric linking system for G, then it is of p'-order modulo the group of inner automorphisms, provided G has no nontrivial normal p'-subgroups. We present two applications of this last result, one to tame fusion systems.

1. Introduction

A saturated fusion system $\mathcal{F}$ is a category in which the objects are the subgroups of a fixed finite p-group S, and the morphisms are injective group homomorphisms between subgroups which are subject to axioms first outlined by Puig [Pui06, AKO11]. When G is a finite group with Sylow p-subgroup S, there is a saturated fusion system $\mathcal{F} = \mathcal{F}_S(G)$ in which the morphisms are the G-conjugation maps between subgroups. One of the important properties of this category is that it keeps precisely the data required to recover the homotopy type of the Bousfield-Kan p-completion $BG^\wedge_p$ of the classifying space of G, as shown in the Martino-Priddy Conjecture, proved by Oliver [Oli04, Oli06]. Recovery of $BG^\wedge_p$, or a p-complete space denoted $B\mathcal{F}$ when no group G is associated with $\mathcal{F}$, is based on the construction of a centric linking system $\mathcal{L}$ for $\mathcal{F}$, an extension category of $\mathcal{F}$ whose existence and uniqueness up to rigid isomorphism was first established in general by Chermak [Che13]. From a group theoretic point of view, centric linking systems, or more generally the transporter systems of Oliver-Ventura [OV07] and the localities of Chermak [Che13], provide finer approximations to p-local structure. They abstract the transporter categories of finite groups, and form structures appearing in new recent approaches to revising the classification of finite simple groups.

We study here in more detail the comparison maps between automorphism groups of finite groups, linking systems, and fusion systems. When $\mathcal{L}$ is a centric linking system associated to the fusion system $\mathcal{F}$, there are groups of automorphisms $\mathrm{Aut}(\mathcal{L})$ and $\mathrm{Aut}(\mathcal{F})$, and a map $\tilde{\mu}\colon \mathrm{Aut}(\mathcal{L}) \to \mathrm{Aut}(\mathcal{F})$ given essentially by restriction to the Sylow group S. When $\mathcal{L} = \mathcal{L}^c_S(G)$ and $\mathcal{F} = \mathcal{F}_S(G)$ for some

Date: January 15, 2021.

2010 *Mathematics Subject Classification.* Primary 20D20, Secondary 20D45.

The first author is partially supported by a Simons Foundation Collaboration Grant. The second author was partially supported by NSA Young Investigator Grant H98230-14-1-0312, Marie Curie Fellowship No. 707778, and NSF Grant DMS-1902152 during the preparation of this manuscript. The authors thank these organizations for their support. The first author expresses his gratitude to the Institute of Mathematics at the University of Aberdeen for its hospitality during a research visit in 2017. The authors would also like to thank the Isaac Newton Institute for Mathematical Sciences, Cambridge, for support and hospitality during the programme Groups, representations and applications, where work on this paper was undertaken and supported by EPSRC grant no EP/R014604/1.

finite group G, there is also a comparison map $\tilde{\kappa}_G\colon N_{\mathrm{Aut}(G)}(S) \to \mathrm{Aut}(\mathcal{L})$, where $N_{\mathrm{Aut}(G)}(S)$ consists of those automorphisms of G which leave S invariant. These induce a pair of maps

$$\mathrm{Out}(G) \xrightarrow{\kappa_G} \mathrm{Out}(\mathcal{L}) \xrightarrow{\mu_{\mathcal{L}}} \mathrm{Out}(\mathcal{F})$$

on outer automorphism groups. We write $\mathrm{Aut}_0(\mathcal{L})$ for the group of *rigid automorphisms* of $\mathcal{L}$, namely $\ker(\tilde{\mu}_{\mathcal{L}})$. Similarly, $\mathrm{Out}_0(\mathcal{L})$ is short for $\ker(\mu_{\mathcal{L}})$.

It follows from the exact sequence of [AKO11, III.5.12] and Chermak's Theorem that $\mu_{\mathcal{L}}$ is an isomorphism if p is odd, and is surjective with kernel an abelian 2-group when $p = 2$. Moreover, the surjectivity of κ_G has been studied intensively in articles by Andersen, Oliver, and Ventura, and by Broto, Moller, and Oliver.

Our first result extends the consequences of unique existence of centric linking systems to show that the kernel of $\mu_{\mathcal{L}}$ is in fact of exponent at most 2, in general, when $p = 2$. To make it easier to apply, we state and prove this in the slightly more general setting of a linking locality (defined just below), and in three equivalent ways. Set $k(p) = 1$ if p is odd, and $k(p) = 2$ if $p = 2$. In particular, a group of exponent $k(p)$ is the trivial group if p is odd and is elementary abelian if $p = 2$.

Theorem 1.1 (Linking locality version)**.** *If $(\mathcal{L}, \Delta, S)$ is a linking locality at the prime p, then the group $\mathrm{Out}_0(\mathcal{L})$ of rigid outer automorphisms of $\mathcal{L}$ is abelian of exponent at most $k(p)$. Moreover, the exact sequence*

$$1 \to \mathrm{Aut}_{Z(S)}(\mathcal{L}) \to \mathrm{Aut}_0(\mathcal{L}) \to \mathrm{Out}_0(\mathcal{L}) \to 1$$

splits.

Theorem 1.2 (Linking system version)**.** *If $\mathcal{L}$ is a linking system at the prime p (in the general sense of* [Hen19]*), then the group $\mathrm{Out}_0(\mathcal{L})$ of rigid outer automorphisms of $\mathcal{L}$ is abelian of exponent at most $k(p)$. Moreover, the exact sequence*

$$1 \to \mathrm{Aut}_{Z(S)}(\mathcal{L}) \to \mathrm{Aut}_0(\mathcal{L}) \to \mathrm{Out}_0(\mathcal{L}) \to 1$$

splits.

Theorem 1.3 (Cohomological version)**.** *Let $\mathcal{F}$ be a saturated fusion system over the finite p-group S, let $\mathcal{O}(\mathcal{F}^c)$ be the orbit category of $\mathcal{F}$-centric subgroups, and let $\mathcal{Z}_{\mathcal{F}}\colon \mathcal{O}(\mathcal{F}^c)^{\mathrm{op}} \to \mathsf{Ab}$ denote the center functor. Then $\lim^1 \mathcal{Z}_{\mathcal{F}}$ is of exponent at most $k(p)$. Moreover, the exact sequence*

$$1 \to \widehat{B}(\mathcal{O}(\mathcal{F}^c), \mathcal{Z}_{\mathcal{F}}) \to \widehat{Z}^1(\mathcal{O}(\mathcal{F}^c), \mathcal{Z}_{\mathcal{F}}) \to \lim{}^1 \mathcal{Z}_{\mathcal{F}} \to 1$$

splits.

Here, a *linking locality* in the sense of [Hen19] (also called a *proper locality* in [Che15]), is a locality $(\mathcal{L}, \Delta, S)$ such that Δ contains all subgroups of S which are centric and radical in $\mathcal{F} = \mathcal{F}_S(\mathcal{L})$, the fusion system of $\mathcal{L}$, and such that $C_{N_{\mathcal{L}}(P)}(O_p(N_{\mathcal{L}}(P))) \leqslant O_p(N_{\mathcal{L}}(P))$ for each $P \in \Delta$. Similarly, a *linking system* is a transporter system $\mathcal{L}$ associated with a saturated fusion system $\mathcal{F}$ such that $\mathrm{Ob}(\mathcal{L})$ contains all $\mathcal{F}$-centric radical subgroups and such that $C_{\mathrm{Aut}_{\mathcal{L}}(P)}(O_p(\mathrm{Aut}_{\mathcal{L}}(P))) \leqslant O_p(\mathrm{Aut}_{\mathcal{L}}(P))$ for each $P \in \mathrm{Ob}(\mathcal{L})$. Other definitions of the term "linking system" without further qualification, such as in [AKO11, Definition III.4.1], are special cases of this one.

An automorphism of a locality $\mathcal{L}$ is *inner* if it is induced by conjugation by an element of $N_{\mathcal{L}}(S)$, and a similar remark applies to transporter systems. In the case of a linking locality or linking system, a rigid inner automorphism is conjugation by an element of the center of S. We have denoted the group of rigid inner automorphisms by $\mathrm{Aut}_{Z(S)}(\mathcal{L})$. This helps to explain some of the terminology and notation in Theorems 1.1-1.2. We explain in more detail in Section 2. Terminology used in Theorem 1.3 is recalled in Section 3.

When p is odd and $\mathcal{L}$ is a centric linking system, Theorems 1.1-1.3 follow from either of the alternative proofs of existence and uniqueness of centric linking systems as given in [Oli13] or [GL16]. The latter is based in part on the former, but removes the dependence of the former on the Classification of Finite Simple Groups (CFSG). The connection between existence and uniqueness and the higher limits of the center functor $\mathcal{Z}_{\mathcal{F}}$ over the orbit category $\mathcal{O}(\mathcal{F}^c)$ of $\mathcal{F}$-centric subgroups is described by [AKO11, Proposition III.5.12]. In particular, this result identifies $\mathrm{Out}_0(\mathcal{L})$ with the first derived limit $\lim^1_{\mathcal{O}(\mathcal{F}^c)}\mathcal{Z}_{\mathcal{F}}$ of the center functor. So when p is odd the theorems follow from [Oli13, Theorem 3.4] or [GL16, Theorem 1.1] and an argument, provided in Section 4, which uses Chermak's iterative procedure for extending a given locality to a new locality on a larger object set.

We shall prove Theorem 1.1 first in the case of a centric linking locality, i.e., when Δ is the collection of $\mathcal{F}$-centric subgroups. The proof is applicable for all primes p, and so we obtain an alternative, somewhat simpler proof of the triviality of $\mathrm{Out}_0(\mathcal{L})$ for p odd, independent of the main result of [GL16]. We then deduce Theorem 1.2 in the same special case, along with Theorem 1.3. Afterward, we shall prove in Section 4 that this implies the seemingly more general statements in Theorems 1.1 and 1.2.

Along the way, we extend to transporter systems a result of Oliver on isomorphisms of (quasicentric) linking systems (Proposition 2.5), and we interpret Chermak's work in the Appendix of [Che13] as an equivalence of groupoids between localities and transporter systems (Theorem 2.11). Besides their use in deducing Theorem 1.2 from 1.1, one motivation for these extensions is to make clear that the results of [Oli13, GL16] give existence and uniqueness of centric linking localities up to *rigid* isomorphism in the same way as the main theorem of [Che13]. That this is not clear at first is caused by an ambiguity in which the notion of "isomorphism" of a transporter system commonly in use does not restrict to the notion of "automorphism" commonly in use, but rather to what should be called "rigid automorphism".

Automorphisms of a finite group that centralize a Sylow subgroup have been studied by Glauberman, Gross, and others. The main result here can be seen as a generalization to linking systems of [Gla68, Theorem 10]. The current work bears the same relationship to [Gla68, Theorem 10] as the proof of existence and uniqueness of centric linking systems outlined above does to the work of Gross [Gro82] and to the recent work of the authors with Guralnick and Navarro [GGLN20]. Our proof of Theorem 1.1 is very different from the proof of [Gla68, Theorem 10], however, in part because not all subgroups of S need be objects.

Recall that for a finite group G with Sylow p-subgroup S and centric linking system $\mathcal{L}_S^{\mathrm{c}}(G)$, there is a comparison homomorphism $\kappa_G\colon \mathrm{Out}(G) \to \mathrm{Out}(\mathcal{L}_S^{\mathrm{c}}(G))$. It is induced essentially by restriction to p-local structure modulo p'-cores, at the level of centric subgroups. In the course of trying to recover from the above theorems the corresponding results about finite groups, we were led to the following result, which seems to be of independent interest.

Theorem 1.4. *Let p be a prime and G a finite group with Sylow p-subgroup S. If $O_{p'}(G) = 1$, then the kernel of the map $\kappa_G\colon \mathrm{Out}(G) \to \mathrm{Out}(\mathcal{L}_S^{\mathrm{c}}(G))$ is a p'-group.*

The proof of Theorem 1.4 relies on the Z_p^*-theorem, namely the statement that an element $x \in S$ whose only G-conjugate in S is x itself must lie in the center of G modulo $O_{p'}(G)$. Thus, our proof of Theorem 1.4 depends on the CFSG if p is odd. (This result and its corollaries in Section 5 for p odd are the only results in the paper that depend on the CFSG.)

When G is simple, the cokernel of κ_G has been studied extensively in [AOV12], [BMO19], and elsewhere. In particular, it has now been shown that the fusion system of each finite simple group G is *tame* in the sense of [AOV12], namely, there is a possibly different finite group G' with Sylow subgroup S such that $\mathcal{F}_S(G) \cong \mathcal{F}_S(G')$ such that the map $\kappa_{G'}$ is split surjective. Theorem 1.4 has been shown in several special cases in the context of those works, cf. [BMO19, Lemma 5.9,Theorem 5.16].

Theorem 1.4 is proved as Theorem 5.1 in Section 5, and we give two applications of it: we show that the splitting condition in the definition of a tame fusion system may be removed, and we give an interesting reinterpretation of the first author's work on the Schreier conjecture [Gla66b].

Terminology and notation. When G is a group and $g \in G$, we write c_g for the left-handed conjugation homomorphism $x \mapsto gxg^{-1}$ and its restrictions. The image of a subgroup P under c_g is sometimes written in left-handed exponential notation ${}^gP$. We write $\mathrm{Hom}_G(P,Q)$ for the set $\{c_g \mid g \in G,\ {}^gP \leqslant Q\}$ of conjugation homomorphisms between P and Q induced in G. Given a finite group G with Sylow p-subgroup S, the fusion system $\mathcal{F}_S(G)$ is the category with objects the subgroups of S and with morphism sets $\mathrm{Hom}_{\mathcal{F}_S(G)}(P,Q) := \mathrm{Hom}_G(P,Q) := \{c_g \mid g \in G,\ {}^gP \leqslant Q\}$. Our terminology for fusion systems follows [AKO11]. For example, $\mathcal{F}^c$ denotes the set of $\mathcal{F}$-centric subgroups, $\mathcal{F}^r$ denotes the set of $\mathcal{F}$-radical subgroups, $\mathcal{F}^f$ denotes the set of fully $\mathcal{F}$-normalized subgroups, and concatenation in the superscript denotes the intersection of the relevant sets.

2. Transporter systems and localities

Throughout this section, $\mathcal{F}$ is a saturated fusion system over a p-group S, and Δ is a nonempty collection of subgroups of S which is closed under $\mathcal{F}$-conjugacy and passing to overgroups. Fix also another triple $\mathcal{F}'$, S', and Δ' of this type.

2.1. Transporter systems.

In the case where $\mathcal{F} = \mathcal{F}_S(G)$ for some finite group G with Sylow p-subgroup S, the transporter category $\mathcal{T}_\Delta(G)$ of G with object set Δ is the category with morphisms $\mathrm{Mor}_{\mathcal{T}_\Delta(G)}(P,Q) = N_G(P,Q) = \{g \in G \mid {}^gP \leqslant Q\}$ where composition is given by multiplication in G. There is an inclusion functor $\delta\colon \mathcal{T}_\Delta(S) \to \mathcal{T}_\Delta(G)$, as well as a functor $\pi\colon \mathcal{T}_\Delta(G) \to \mathcal{F}_S(G)$ which is the inclusion on objects and which sends $g \in N_G(P,Q)$ to $c_g \in \mathrm{Hom}_G(P,Q)$, conjugation by g. This is the standard example of a transporter system associated with $\mathcal{F}_S(G)$.

Definition 2.1 ([OV07, Definition 3.1])**.** A *transporter system* associated with $\mathcal{F}$ is a nonempty category $\mathcal{T}$ with object set $\Delta \subseteq \mathrm{Ob}(\mathcal{F})$, together with structural functors

$$\mathcal{T}_\Delta(S) \xrightarrow{\delta} \mathcal{T} \xrightarrow{\pi} \mathcal{F}$$

which satisfy the following axioms.

(A1) Δ is closed under $\mathcal{F}$-conjugacy and upon passing to overgroups, δ is the identity on objects, and π is the inclusion on objects.

(A2) For each $P, Q \in \Delta$, the kernel

$$E(P) := \ker(\pi_{P,P}\colon\ \mathrm{Aut}_\mathcal{T}(P) \to \mathrm{Aut}_\mathcal{F}(P))$$

acts freely on $\mathrm{Mor}_{\mathcal{T}}(P,Q)$ by right composition, and $\pi_{P,Q}$ is the orbit map for this action. In particular, $\pi_{P,Q}$ is surjective. Also, $E(Q)$ acts freely on $\mathrm{Mor}_{\mathcal{T}}(P,Q)$ by left composition. Here, $\mathrm{Aut}_{\mathcal{T}}(P)$ denotes $\mathrm{Mor}_{\mathcal{T}}(P,P)$.

(B) For each $P, Q \in \Delta$, $\delta_{P,Q}\colon N_S(P,Q) \to \mathrm{Mor}_{\mathcal{T}}(P,Q)$ is injective, and the composite $\pi_{P,Q} \circ \delta_{P,Q}$ sends $g \in N_S(P,Q)$ to $c_g \in \mathrm{Hom}_{\mathcal{F}}(P,Q)$.

(C) For each $\varphi \in \mathrm{Mor}_{\mathcal{T}}(P,Q)$ and each $g \in P$, the diagram

$$\begin{array}{ccc} P & \xrightarrow{\ \varphi\ } & Q \\ {\scriptstyle \delta_{P,P}(g)}\uparrow & & \uparrow{\scriptstyle \delta_{Q,Q}(\pi(\varphi)(g))} \\ P & \xrightarrow[\ \varphi\ ]{} & Q \end{array}$$

commutes in $\mathcal{T}$.

(I) $\delta_{S,S}(S)$ is a Sylow p-subgroup of $\mathrm{Aut}_{\mathcal{T}}(S)$.

(II) Let $\varphi \in \mathrm{Iso}_{\mathcal{T}}(P,Q)$, $P \lhd \bar{P} \leqslant S$, and $Q \lhd \bar{Q} \leqslant S$ be such that $\varphi \circ \delta_{P,P}(\bar{P}) \circ \varphi^{-1} \leqslant \delta_{Q,Q}(\bar{Q})$. Then there is $\bar{\varphi} \in \mathrm{Mor}_{\mathcal{T}}(\bar{P}, \bar{Q})$ such that $\bar{\varphi} \circ \delta_{P,\bar{P}}(1) = \delta_{Q,\bar{Q}}(1) \circ \varphi$.

From now on, we abbreviate $\delta_{P,P}$ to δ_P, $\pi_{P,P}$ to π_P, and use similar notation when considering the application of an arbitrary functor on morphism sets. Also, any future reference to axioms (A1)-(II) should be interpreted as reference to the axioms given in Definition 2.1. The following lemma collects some basic properties of morphisms in a transporter system.

Lemma 2.2. *Fix a transporter system $(\mathcal{T}, \delta, \pi)$ associated with $\mathcal{F}$.*

(a) Each morphism in $\mathcal{T}$ is both a monomorphism and an epimorphism in the categorical sense.

(b) (Restrictions are unique) Given objects $P_0 \leqslant P$, $Q_0 \leqslant Q$, and two morphisms φ_0, φ_0' making the diagram

$$\begin{array}{ccc} P & \xrightarrow{\ \varphi\ } & Q \\ {\scriptstyle \delta_{P_0,P}(1)}\uparrow & & \uparrow{\scriptstyle \delta_{Q_0,Q}(1)} \\ P_0 & \xrightarrow[\ \varphi_0,\varphi_0'\ ]{} & Q_0 \end{array}$$

commute, one has $\varphi_0 = \varphi_0'$.

(c) (Extensions are unique) Given objects $P_0 \leqslant P$, $Q_0 \leqslant Q$, and two morphisms φ, φ' making the diagram

$$\begin{array}{ccc} P & \xrightarrow{\ \varphi,\varphi'\ } & Q \\ {\scriptstyle \delta_{P_0,P}(1)}\uparrow & & \uparrow{\scriptstyle \delta_{Q_0,Q}(1)} \\ P_0 & \xrightarrow[\ \varphi_0\ ]{} & Q_0 \end{array}$$

commute, one has $\varphi = \varphi'$.

Proof. Parts (a) and (b) are contained in [OV07, Lemma 3.2], while part (c) is proved in [Che13, Lemma A.5(c)]. □

By a morphism of fusion systems $\mathcal{F} \to \mathcal{F}'$, it is meant a pair (α, Φ) where $\alpha\colon S \to S'$ is a group homomorphism and $\Phi\colon \mathcal{F} \to \mathcal{F}'$ is a functor which together satisfy $\alpha(P) = \Phi(P)$ on objects and $\Phi(\varphi) \circ \alpha = \alpha \circ \varphi$ for each morphism φ in $\mathcal{F}$. If α is an isomorphism, then Φ is determined uniquely by α. So an isomorphism of fusion systems may be regarded as an isomorphism of the underlying p-groups which "preserves fusion".

Definition 2.3 (Isomorphisms of transporter systems)**.** Let $(\mathcal{T}, \delta, \pi)$ and $(\mathcal{T}', \delta', \pi')$ be transporter systems with object sets Δ and Δ', for the saturated fusion systems $\mathcal{F}$ and $\mathcal{F}'$, respectively.

(1) Let $\alpha\colon \mathcal{T} \to \mathcal{T}'$ be an equivalence of categories. It is said that
   - α is *isotypical* if $\alpha(\delta_P(P)) = \delta'_{\alpha(P)}(\alpha(P))$ for each subgroup $P \in \Delta$, and that
   - α *sends inclusions to inclusions* if $\alpha(\delta_{P,Q}(1)) = \delta'_{\alpha(P),\alpha(Q)}(1)$ for each $P, Q \in \Delta$.

(2) An *isomorphism* is an equivalence $\mathcal{T} \to \mathcal{T}'$ which is isotypical and sends inclusions to inclusions. An automorphism is an isomorphism of a transporter system onto itself.

(3) An isomorphism $\alpha\colon \mathcal{T} \to \mathcal{T}'$ is said to be *rigid* if $S = S'$ and $\alpha_S \circ \delta_S = \delta'_S$ as homomorphisms $S \to \mathrm{Aut}_{\mathcal{T}'}(S)$. Here, as before, α_S means $\alpha_{S,S}$.

(4) An automorphism α of $\mathcal{T}$ is *inner* if there is an element $\varphi \in \mathrm{Aut}_{\mathcal{T}}(S)$ such that α is given on objects by $P \mapsto c_\varphi(P) := \pi(\varphi)(P)$ and on morphisms by mapping $\psi\colon P \to Q$ to

$$c_\varphi(\psi) := \varphi|_{Q,c_\varphi(Q)} \circ \psi \circ (\varphi|_{P,c_\varphi(P)})^{-1},$$

where, for example, $\varphi|_{Q,c_\varphi(Q)}$ is the unique morphism from Q to $c_\varphi(Q)$ in $\mathcal{T}$ such that $\varphi \circ \delta_{Q,S}(1) = \delta_{c_\varphi(Q),S}(1) \circ \varphi$, as given by Lemma 2.2(b). We refer to c_φ as *conjugation by* φ. Write $\mathrm{Aut}_{Z(S)}(\mathcal{T})$ for the group of rigid inner automorphisms of $\mathcal{T}$ which are conjugation by elements of $\delta_S(Z(S)) \leqslant \mathrm{Aut}_{\mathcal{T}}(S)$.

Denote by $\mathrm{Aut}(\mathcal{T}) := \mathrm{Aut}(\mathcal{T}, \delta, \pi)$ the group of automorphisms of $\mathcal{T}$. Denote by T the category of transporter systems and isomorphisms.

Remark 2.4. An isomorphism of transporter systems is in particular an invertible functor, and so one sees that $\mathrm{Aut}(\mathcal{T})$ is indeed a group. This was shown for linking systems in [AOV12, Lemma 1.14(a)], and the same argument applies for an arbitrary transporter system.

We have defined isomorphism here in analogy with the definition of an automorphism of a centric linking system [AKO11, III.4.3], but more generally than is usually done. The usual definition of an isomorphism of transporter systems is a functor $\alpha\colon \mathcal{T} \to \mathcal{T}'$ which commutes with the structural functors: $\alpha \circ \delta = \delta'$ and $\pi' \circ \alpha = \pi$. See for example [BLO03, p.799], [OV07, Proposition 3.11], [AKO11, p.146], or [Che13, Definition A.2]. Rather, Definition 2.3 specializes to the definition of an automorphism of a linking system in [AKO11, Section III.4.3].

The following proposition extends Proposition 4.11 of [AKO11] in two ways, but the proof follows the same basic outline. It helps explain that an isomorphism between transporter systems is equivalent to a triple of functors commuting with the structural functors, and that the usual definition of isomorphism of transporter systems is the same as what we are calling a rigid isomorphism.

Proposition 2.5. *Fix transporter systems $(\mathcal{T}, \delta, \pi)$ and $(\mathcal{T}', \delta', \pi')$ associated to $\mathcal{F}$ and $\mathcal{F}'$ with object sets Δ and Δ' which contain $\mathcal{F}^{cr}$ and $\mathcal{F}'^{cr}$. Given an isomorphism $\alpha\colon \mathcal{T} \to \mathcal{T}'$ in the sense of Definition 2.3, there is a unique associated isomorphism $\beta\colon S \to S'$, a unique functor $\beta_*\colon \mathcal{T}_\Delta(S) \to \mathcal{T}_{\Delta'}(S')$, and a unique isomorphism $c_\beta\colon : \mathcal{F} \to \mathcal{F}'$ of fusion systems such that the* *diagram*

$$\begin{array}{ccccc} \mathcal{T}_\Delta(S) & \xrightarrow{\ \delta\ } & \mathcal{T} & \xrightarrow{\ \pi\ } & \mathcal{F} \\ \Big\downarrow{\scriptstyle \beta_*} & & \Big\downarrow{\scriptstyle \alpha} & & \Big\downarrow{\scriptstyle c_\beta} \\ \mathcal{T}_{\Delta'}(S') & \xrightarrow{\ \delta'\ } & \mathcal{T}' & \xrightarrow{\ \pi'\ } & \mathcal{F}'. \end{array} \tag{2.6}$$

commutes and $\beta = (\beta_*)_S$*. Moreover,* α *is a rigid isomorphism if and only if both* β_* *and* c_β *are the identity functors.*

Proof. Let $\alpha\colon \mathcal{T} \to \mathcal{T}'$ be an isomorphism. As S is the only object of $\mathcal{T}$ with the property that $\mathrm{Mor}_{\mathcal{T}}(P, S) \neq \varnothing$ for each object P of $\mathcal{T}$, and the same is true for S' with respect to $\mathcal{T}'$, it follows that $\alpha(S) = S'$. So $\alpha_S(\delta_S(S)) = \delta'_{S'}(S')$ since α is isotypical. By axiom (B) for a transporter system, the map $\delta'_{S'}\colon S' \to \delta'_{S'}(S')$ is an isomorphism, so there is a unique map β from $S = \mathrm{Aut}_{\mathcal{T}_\Delta(S)}(S)$ to $S' = \mathrm{Aut}_{\mathcal{T}_{\Delta'}(S')}(S')$ such that

$$\alpha_S(\delta_S(s)) = \delta'_{S'}(\beta(s)) \tag{2.7}$$

for each $s \in S$. Then $\beta = (\delta')^{-1}_{S'} \circ \alpha_S \circ \delta_S$ is an isomorphism from S to S'. Now α sends inclusions to inclusions, so commutes with restrictions. Hence, for each $P \in \Delta$, as $\alpha(\delta_P(P)) = \delta'_{\alpha(P)}(\alpha(P))$, we have $\alpha_S(\delta_S(P)) = \delta'_{S'}(\alpha(P))$, and this shows with (2.7) and injectivity of δ' that $\beta(P) = \alpha(P)$ for each P.

Let $\beta_*\colon \mathcal{T}_\Delta(S) \to \mathcal{T}_{\Delta'}(S')$ be the functor induced by β. Namely, β_* sends an object P to $\beta(P)$, and it sends a morphism $P \xrightarrow{s} Q$ to $\beta(P) \xrightarrow{\beta(s)} \beta(Q)$. Then $\delta' \circ \beta_* = \alpha \circ \delta$ by construction.

Next, we wish to define a functor $c_\beta\colon \mathcal{F} \to \mathcal{F}'$ via a mapping on objects sending P to $\beta(P)$, and on morphisms sending $P \xrightarrow{\varphi} Q$ to $\beta(P) \xrightarrow{\beta\circ\varphi\circ\beta^{-1}} \beta(Q)$. This is an isomorphism of fusion systems (the one corresponding to the isomorphism β from S to S') with inverse $c_{\beta^{-1}}$, if well-defined. In order to show the assignment is well-defined, we must prove that each $\beta \circ \varphi \circ \beta^{-1}$ is a morphism in $\mathcal{F}'$. This will be done by showing that $c_\beta(\varphi) = \pi'(\alpha(\tilde\varphi))$ for each $\tilde\varphi \in \mathrm{Mor}_{\mathcal{T}}(P, Q)$ with $\pi(\tilde\varphi) = \varphi$, thus simultaneously showing that the right square in (2.6) commutes.

Fix such a lift $\tilde\varphi$ of φ, and let $s \in P$. Consider the following diagrams:

$$\begin{array}{ccc} P & \xrightarrow{\ \tilde\varphi\ } & Q \\ {\scriptstyle \delta_P(s)}\Big\downarrow & & \Big\downarrow{\scriptstyle \delta_Q(\varphi(s))} \\ P & \xrightarrow[\ \tilde\varphi\ ]{} & Q \end{array}, \qquad \begin{array}{ccc} \alpha(P) & \xrightarrow{\ \alpha(\tilde\varphi)\ } & \alpha(Q) \\ {\scriptstyle \alpha(\delta_P(s))}\Big\downarrow & & \Big\downarrow{\scriptstyle \alpha(\delta_Q(\varphi(s)))} \\ \alpha(P) & \xrightarrow[\ \alpha(\tilde\varphi)\ ]{} & \alpha(Q) \end{array}, \qquad \begin{array}{ccc} \beta(P) & \xrightarrow{\ \alpha(\tilde\varphi)\ } & \beta(Q) \\ {\scriptstyle \delta'_{\beta(P)}(\beta(s))}\Big\downarrow & & \Big\downarrow{\scriptstyle \delta'_{\beta(Q)}(\beta(\varphi(s)))} \\ \beta(P) & \xrightarrow[\ \alpha(\tilde\varphi)\ ]{} & \beta(Q) \end{array}$$

By axiom (C) for $\mathcal{T}$, the first diagram commutes, and the second is α applied to the first. As shown above, $\beta(P) = \alpha(P)$ and $\alpha \circ \delta = \delta' \circ \beta_*$, so the third diagram is the same as the second. By axiom (C) for $\mathcal{T}'$ with $\alpha(\tilde\varphi)$ and $\beta(s)$ in the roles of φ and g, the morphism $\delta'_{\beta(Q)}(\pi'(\alpha(\tilde\varphi))(\beta(s)))$ in place of $\delta'_{\beta(Q)}(\beta(\varphi(s)))$ also makes the third diagram commute, so we have

$$\delta'_{\beta(Q)}(\beta(\varphi(s))) \circ \alpha(\tilde\varphi) = \delta'_{\beta(Q)}(\pi'(\alpha(\tilde\varphi))(\beta(s))) \circ \alpha(\tilde\varphi)$$

as morphisms between $\beta(P)$ and $\beta(Q)$ in $\mathcal{T}$. Since each morphism in a transporter system is an epimorphism (Lemma 2.2(a)) and $\delta'_{\beta(Q)}$ is injective (axiom (B)), it follows that

$$\beta(\varphi(s)) = \pi'(\alpha(\tilde\varphi))(\beta(s)), \qquad \text{for } s \in P.$$

Hence, after replacing s by $\beta^{-1}(s)$, we see that $c_\beta(\varphi) = \pi'(\alpha(\tilde{\varphi}))$ as claimed, and this completes the proof of existence of the functors β_* and c_β.

It remains to prove uniqueness. Observe that uniqueness of β would follow from that of β_*. Suppose $\gamma\colon \mathcal{T}_\Delta(S) \to \mathcal{T}_{\Delta'}(S')$ is a functor such that γ in place of β^* makes the left square in (2.6) commute. Since δ and δ' are the identity on objects by axiom (A1), γ agrees with β_* on objects. Similarly they agree on morphisms, given commutativity of the diagram, since ${\delta'}_{P,Q}$ is injective by axiom (B) for each $P, Q \in \Delta$. Hence, $\gamma = \beta_*$. Next, suppose in addition that $\eta\colon \mathcal{F} \to \mathcal{F}'$ is another functor such that right square in (2.6) commutes with η in place of c_β. By axiom (A1), the functors c_β and η agree with α on the objects Δ. For each morphism φ in $\mathcal{T}$ between subgroups in Δ, we have $\eta(\pi(\varphi)) = c_\beta(\pi(\varphi))$, so by axiom (A2) on the surjectivity of π on morphism sets, we see that η and c_β agree on morphisms in $\mathcal{F}$ between subgroups in Δ. By assumption $\mathcal{F}^{cr} \subseteq \Delta$, so the Alperin-Goldschmidt fusion theorem [BLO03, Proposition A.10] or [AKO11, I.3.5] gives equality.

If α is a rigid isomorphism, then by definition $S = S'$. By commutativity of the left square in (2.6), $\delta'_S \circ \beta = \alpha_S \circ \delta_S = \delta'_S$. So $\beta = \mathrm{id}_S$ as δ'_S is injective. It was shown above that β_* and c_β are uniquely determined by β, so β_* and c_β are the identity. Conversely, if β_* is the identity functor, then $S = S'$, and by commutativity of the left square, we have $\alpha_S \circ \delta_S = \delta'_S \circ \mathrm{id}_S = \delta'_S$, so α is rigid. □

As in the setting of (quasicentric) linking systems [AOV12, p.197], one can define a group homomorphism relating automorphisms of a transporter system with automorphisms of the associated fusion system in this more general setting, using Proposition 2.5. Let $(\mathcal{T}, \delta, \pi)$ be a transporter system with object set Δ associated with the saturated fusion system $\mathcal{F}$ on S. Assume that $\mathcal{F}^{cr} \subseteq \Delta$. Define

$$\tilde{\mu}_{\mathcal{T}}\colon \operatorname{Aut}(\mathcal{T}) \to \operatorname{Aut}(\mathcal{F})$$

to be the map which sends $\alpha \in \operatorname{Aut}(\mathcal{T})$ to the automorphism $\delta_S^{-1} \circ \alpha_S \circ \delta_S$ of $S = \operatorname{Aut}_{\mathcal{T}_\Delta(S)}(S)$. Thus, $\tilde{\mu}_{\mathcal{T}}(\alpha)$ is the automorphism β in Proposition 2.5. This is a group homomorphism (using uniqueness of c_β) which maps $\operatorname{Aut}_{\mathcal{T}}(S)$ onto $\operatorname{Aut}_{\mathcal{F}}(S)$ and has kernel $\operatorname{Aut}_0(\mathcal{T})$. It induces a homomorphism

$$\mu_{\mathcal{T}}\colon \operatorname{Out}(\mathcal{T}) \to \operatorname{Out}(\mathcal{F})$$

with kernel $\operatorname{Out}_0(\mathcal{T})$. When $\mathcal{T} = \mathcal{T}_\Delta(G)$ for some finite group G with Sylow p-subgroup S, we sometimes write $\tilde{\mu}_G$ for $\tilde{\mu}_{\mathcal{T}}$ and μ_G for $\mu_{\mathcal{T}}$, provided $\mathcal{T}$ is understood from the context.

2.2. **Localities.** In his proof of the existence and uniqueness of centric linking systems, Chermak introduced localities and showed in [Che13, Appendix] they are essentially equivalent to transporter systems. The purpose of this section is to explain how Chermak's results give an equivalence of categories between transporter systems and localities, with morphisms isomorphisms, while setting up notation.

Let $\mathcal{L}$ be a finite set (we shall consider only finite localities). Write $\mathbf{W}(\mathcal{L})$ for the monoid of words $(f_n, \dots, f_1)$ in the elements of $\mathcal{L}$, where the multiplication is concatenation $\circ$. A *partial group* is a set $\mathcal{L}$ together with a subset $\mathbf{D} := \mathbf{D}(\mathcal{L}) \subseteq \mathbf{W}(\mathcal{L})$, a multivariable product $\Pi\colon \mathbf{D} \to \mathcal{L}$ defined on words in $\mathbf{D}$, and an inversion map $(-)^{-1}\colon \mathcal{L} \to \mathcal{L}$, subject to certain axioms which may be found in [Che13, Definition 2.1]. The product $f_n \cdots f_1$ is *defined* if $(f_n, \dots, f_1) \in \mathbf{D}$, and in this case we set $f_n \cdots f_1 = \Pi(f_n, \dots, f_1)$. A partial group is a group if and only if $\mathbf{D} = \mathbf{W}(\mathcal{L})$, that is, all products are defined. A *partial subgroup* is a subset $\mathcal{L}_0$ of $\mathcal{L}$ with domain $\mathbf{D}_0 \subseteq \mathbf{W}(\mathcal{L}_0) \cap \mathbf{D}$, such that the restriction of the product Π to $\mathbf{D}_0$ is the product Π_0 for $\mathcal{L}_0$. The subgroups of $\mathcal{L}$ are the partial subgroups $\mathcal{L}_0$ with $\mathbf{W}(\mathcal{L}_0) \subseteq \mathbf{D}(\mathcal{L})$. A *homomorphism* of partial groups is a function

$\gamma\colon \mathcal{L} \to \mathcal{M}$ such that $\gamma^*(\mathbf{D}(\mathcal{L})) \subseteq \mathbf{D}(\mathcal{M})$ and $\Pi(\gamma^*(w)) = \gamma(\Pi(w))$ for any word $w \in \mathbf{D}(\mathcal{L})$. Here, $\gamma^*\colon \mathbf{W}(\mathcal{L}) \to \mathbf{W}(\mathcal{M})$ is the map on words determined by γ. Partial groups and partial group homomorphisms form a category, so there is the usual notion of isomorphism in this category. A homomorphism γ as above is an isomorphism if and only if it is a bijective homomorphism satisfying $\gamma^*(\mathbf{D}(\mathcal{L})) = \mathbf{D}(\mathcal{M})$.

There is a natural notion of conjugation in a partial group when defined. Given $f \in \mathcal{L}$, write $\mathbf{D}(f)$ for the set of $x \in \mathcal{L}$ such that $(f, x, f^{-1}) \in \mathbf{D}$. The product $fxf^{-1} = \Pi(f, x, f^{-1})$ is the conjugate of x by f, sometimes written ${}^f x$. A usual convention, which we adopt, is that any such expression carries the tacit assumption that $x \in \mathbf{D}(f)$. Likewise, for any subset $X \subseteq \mathcal{L}$, the expression ${}^f X$ has a similar meaning, including that $X \subseteq \mathbf{D}(f)$.

Definition 2.8. Let $\mathcal{L}$ be a finite partial group, let S be a p-subgroup of $\mathcal{L}$, and let Δ be a collection of subgroups of S. The triple $(\mathcal{L}, \Delta, S)$ is a *locality* if

- (L1a) $\mathbf{D}(\mathcal{L})$ is equal to the set of those $(f_n, \dots, f_1) \in \mathbf{W}(\mathcal{L})$ such that there is $(X_0, \dots, X_n) \in \mathbf{W}(\Delta)$ with ${}^{f_{i+1}}X_i = X_{i+1}$ for each $0 \leqslant i < n$.
- (L1b) If $P \in \Delta$ and $f \in \mathcal{L}$ with $P \leqslant \mathbf{D}(f)$ and ${}^f P \leqslant S$, then $Q \in \Delta$ for each ${}^f P \leqslant Q \leqslant S$.
- (L2) S is a maximal member of the poset of p-subgroups of $\mathcal{L}$.

We next set up some notation when working with a locality $(\mathcal{L}, \Delta, S)$. A word $w = (f_n, \dots, f_1) \in \mathbf{W}(\mathcal{L})$ is *in* $\mathbf{D}(\mathcal{L})$ *via* X_0 if ${}^{f_i \cdots f_1}X_0 \in \Delta$ for each $1 \leqslant i \leqslant n$, compare (L1a). For $f \in \mathcal{L}$, denote by S_f the set of $s \in S$ such that ${}^f s \in S$. By [Che13, Proposition 2.11], $S_f \in \Delta$. In particular, S_f is a subgroup of $\mathcal{L}$ which plays the role of a Sylow intersection. For an object $P \in \Delta$, the normalizer $N_{\mathcal{L}}(P) = \{f \in \mathcal{L} \mid {}^f P = P\}$, and centralizer $C_{\mathcal{L}}(P) = \{f \in \mathcal{L} \mid {}^f x = x \text{ for all } x \in P\}$ are subgroups of $\mathcal{L}$.

The *fusion system* $\mathcal{F}_S(\mathcal{L})$ of $\mathcal{L}$ is the fusion system on S with morphisms being those group monomorphisms between subgroups of S which can be written as compositions of restrictions of the conjugation homomorphisms $c_f\colon P \to Q$, $x \mapsto {}^f x$ between objects $P, Q \in \Delta$. It is said that $\mathcal{L}$ is a locality *on* $\mathcal{F}_S(\mathcal{L})$.

Example 2.9 ([Che13, Example/Lemma 2.10]). Let G be a finite group, let S be a Sylow p-subgroup of G, and let Δ be a collection of subgroups of S which is closed under $\mathcal{F}_S(G)$-conjugacy and upon passing to overgroups, and which contains all $\mathcal{F}_S(G)$-centric radical subgroups. Let $\mathcal{L}$ be the subset of G consisting of those $g \in G$ such that there exists $P \in \Delta$ with ${}^g P \leqslant S$ (so that ${}^g P \in \Delta$). Let $\mathbf{D} \subseteq \mathbf{W}(\mathcal{L})$ denote the collection of all words $(g_n, \dots, g_1) \in \mathbf{W}(\mathcal{L})$ such that there is $(X_0, \dots, X_n) \in \mathbf{W}(\Delta)$ with ${}^{g_i \cdots g_1}X_0 \in \Delta$ for each $0 \leqslant i \leqslant n$. Whenever $(g_n, \dots, g_1)$ is a word in $\mathbf{D}$, define $\Pi(g_n, \dots, g_1) = g_n \cdots g_1$, the product in G. Then $(\mathcal{L}, \Delta, S)$ is a locality on $\mathcal{F}_S(G)$, written $\mathcal{L}_\Delta(G)$.

Definition 2.10 (Isomorphisms of localities)**.** Let $(\mathcal{L}, \Delta, S)$ and $(\mathcal{L}', \Delta', S')$ be localities.

1. An isomorphism from $(\mathcal{L}, \Delta, S)$ to $(\mathcal{L}', \Delta', S')$ is an isomorphism of partial groups $\beta\colon \mathcal{L} \to \mathcal{L}'$ such that $\beta(\Delta) = \Delta'$ (hence, $\beta(S) = S'$). An automorphism of $(\mathcal{L}, \Delta, S)$ is an isomorphism of $(\mathcal{L}, \Delta, S)$ to itself.
2. An isomorphism β is *rigid* if $S = S'$, and β is the identity on S.
3. An automorphism α of $\mathcal{L}$ is *inner* if it is given by conjugation by an element of $N_{\mathcal{L}}(S)$, namely, there is $f \in N_{\mathcal{L}}(S)$ such that $\alpha(x) = fxf^{-1}$ for all $x \in \mathcal{L}$. (Note that the product fxf^{-1} is always defined when $f \in N_{\mathcal{L}}(S)$.)

Write $\mathrm{Aut}(\mathcal{L}) := \mathrm{Aut}(\mathcal{L}, \Delta, S)$ for the group of automorphisms of $\mathcal{L}$, $\mathrm{Aut}_0(\mathcal{L})$ for the subgroup of rigid automorphisms, and $\mathrm{Aut}_{Z(S)}(\mathcal{L})$ for the subgroup of $\mathrm{Aut}_0(\mathcal{L})$ consisting of automorphisms which are conjugation by elements in $Z(S)$. Denote by L the category of localities with isomorphisms.

2.3. **Equivalence between transporter systems and localities.** In [Che13, Appendix], Chermak goes most of the way toward proving that there is an equivalence between the category of transporter systems with rigid isomorphisms (in the sense of Definition 2.3) and the category of localities with rigid isomorphisms. Here, we suggest a mild extension of Chermak's results to an equivalence of the slightly larger categories T and L with the same objects. First, we briefly review how to pass from a locality to a transporter system and vice versa. More details are given in [Che13, Appendix A].

2.3.1. *From localities to transporter systems.* Given a locality $(\mathcal{L}, \Delta, S)$, one can make a transporter system $(\mathcal{T}_\Delta(\mathcal{L}), \delta, \pi)$ associated with $\mathcal{F}_S(\mathcal{L})$ in the following way. Let $\mathcal{T}_\Delta(\mathcal{L})$ have object set Δ, and for each $P, Q \in \Delta$, take

$$\mathrm{Mor}_{\mathcal{T}_\Delta(\mathcal{L})}(P,Q) = \{(f, P, Q) \mid f \in \mathcal{L},\ {}^{f}P \leqslant Q\}.$$

Composition is given by multiplication in $\mathcal{L}$. The functor δ is the identity on objects, and sends $P \xrightarrow{s} Q$ to (s, P, Q). The functor π is the inclusion on objects and sends (f, P, Q) to the conjugation homomorphism $c_f\colon P \to Q$.

2.3.2. *From transporter systems to localities.* Conversely, to make a locality given a transporter system $(\mathcal{T}, \delta, \pi)$, consider the collection of isomorphisms $\mathrm{Iso}(\mathcal{T})$ in $\mathcal{T}$ and the following relation on the set $\mathrm{Mor}(\mathcal{T})$ of morphisms in $\mathcal{T}$: the morphism $\varphi\colon P \to Q$ is an extension of $\varphi_0\colon P_0 \to Q_0$, written $\varphi_0 \uparrow \varphi$, if the diagram

$$\begin{array}{ccc} P & \xrightarrow{\ \varphi\ } & Q \\ {\scriptstyle \delta_{P_0,P}(1)}\uparrow & & \uparrow{\scriptstyle \delta_{Q_0,Q}(1)} \\ P_0 & \xrightarrow[\ \varphi_0\ ]{} & Q_0 \end{array}$$

commutes in $\mathcal{T}$. This is a partial order, and the equivalence relation on $\mathrm{Iso}(\mathcal{T})$ generated by its restriction to $\mathrm{Iso}(\mathcal{T})$ is denoted $\equiv$. It is shown in [Che13, Lemma A.8(a)] that each $\equiv$-class has a unique maximal member with respect to $\uparrow$. Write $[\varphi]$ for the equivalence class of φ, and set $(\mathcal{L}, \Delta, S) = (\mathrm{Iso}(\mathcal{T})/\equiv, \Delta, S)$, where by abuse of notation, S is identified with the set of equivalence classes $\{[\delta_S(s)] \mid s \in S\}$ of elements in $\delta_S(S) \subseteq \mathrm{Aut}_{\mathcal{T}}(S) \subseteq \mathrm{Iso}(\mathcal{T})$. The domain $\mathbf{D}(\mathcal{L}_\Delta(\mathcal{T}))$ for the product is the set of all words $(f_n, \ldots, f_1) \in \mathbf{W}(\mathcal{L}_\Delta(\mathcal{T}))$ such that there exist objects $P_0, \ldots, P_n \in \Delta$ and isomorphisms $\varphi_i\colon P_{i-1} \to P_i$ in $\mathcal{T}$ such that $\varphi_i \in f_i$ for each i. The product $\Pi\colon \mathbf{D}(\mathcal{L}_\Delta(\mathcal{T})) \to \mathcal{L}_\Delta(\mathcal{T})$ is defined by $\Pi(f_n, \ldots, f_1) = [\varphi_n \circ \cdots \circ \varphi_1]$. The inversion map $-^{-1}\colon \mathcal{L}_\Delta(\mathcal{T}) \to \mathcal{L}_\Delta(\mathcal{T})$ is given by $[\varphi]^{-1} = [\varphi^{-1}]$ for each $\varphi \in \mathrm{Iso}(\mathcal{T})$. It can be shown that these operations on $\mathcal{L}$ are well-defined and that $\mathcal{L}_\Delta(\mathcal{T})$ is a locality [Che13, Lemmas A.7,A.9,A.13].

Recall that T denotes the category of transporter systems with isomorphisms and L denotes the category of localities with isomorphisms. We write T_0 and L_0 for the categories of transporter systems and localities with rigid isomorphisms.

Theorem 2.11 (cf. Chermak [Che13, Appendix])**.** *The categories T and L are equivalent via a functor which restricts to an equivalence between T_0 and L_0.*

Remark 2.12. Strictly speaking, in order for the restriction of the functor $\mathsf{T} \to \mathsf{L}$ (to be constructed in the proof) to induce an equivalence between T_0 and L_0, we must make two canonical identifications of S with other incarnations of S. It is possible that a more precise statement could be made involving a category of S-rigid localities, where an S-rigid locality is a locality $\mathcal{L}$ together with an embedding $S \hookrightarrow \mathcal{L}$ of partial groups which satisfies natural conditions. But we do not pursue that, since our interest here is mainly in Corollary 2.13.

Proof of Theorem 2.11. Define functors $\Theta\colon \mathsf{L} \to \mathsf{T}$ and $\Lambda\colon \mathsf{T} \to \mathsf{L}$ as follows. On objects, the functors are as described in Subsections 2.3.1 and 2.3.2. Let $\gamma\colon \mathcal{L} \to \mathcal{L}'$ be an isomorphism between the two localities $(\mathcal{L}, \Delta, S)$ and $(\mathcal{L}', \Delta', S')$. Define a functor $\Theta(\gamma)\colon \mathcal{T}_\Delta(\mathcal{L}) \to \mathcal{T}_{\Delta'}(\mathcal{L}')$ by the rule

$$
\begin{aligned}
P &\mapsto \gamma(P), \\
(f, P, Q) &\mapsto (\gamma(f), \gamma(P), \gamma(Q)).
\end{aligned}
$$

$\Theta(\gamma)$ is an invertible functor with inverse $\Theta(\gamma^{-1})$, it is clearly isotypical, it sends inclusions to inclusions because $\gamma(1) = 1$, and hence it is an isomorphism of transporter systems. Observe that if $\Delta = \Delta'$ (so $S = S'$) and γ is a rigid isomorphism, then $\Theta(\gamma)(\delta_S(s)) = (s, S, S) = \delta'_S(s)$ for each $s \in S$, so $\Theta(\gamma)$ is a rigid isomorphism of transporter systems. It is then clear that Θ determines a functor $\mathsf{L} \to \mathsf{T}$, which restricts to send $\mathsf{L}_0 \to \mathsf{T}_0$.

Conversely, given an isomorphism $\alpha\colon \mathcal{T} \to \mathcal{T}'$, form the associated localities $(\mathcal{L}_\Delta(\mathcal{T}), \Delta, S)$ and $(\mathcal{L}_{\Delta'}(\mathcal{T}'), \Delta', S')$ and define a function $\Lambda(\alpha)\colon \mathcal{L}_\Delta(\mathcal{T}) \to \mathcal{L}_{\Delta'}(\mathcal{T}')$ via $\Lambda(\alpha)([\varphi]) = [\alpha(\varphi)]'$, where here we write $[-]'$ for equivalence classes in $\mathrm{Iso}(\mathcal{T}')$. As α is invertible, it induces a bijection $\Delta \to \Delta'$ sending $S \mapsto S'$ and a bijection $\mathrm{Iso}(\mathcal{T}) \to \mathrm{Iso}(\mathcal{T}')$. Since α sends inclusions to inclusions, it preserves $\uparrow$ and $\equiv$, and hence $\Lambda(\alpha)$ is a well-defined bijection. Given that α is a functor, it follows from the definition of multiplication in $\mathcal{L}_\Delta(\mathcal{T})$ and [Che13, Lemma A.7(b)] that $\Lambda(\alpha)$ is a partial group homomorphism. Then $\Lambda(\alpha)$ restricts to a homomorphism from S to S' (if we identify these with $\{[\delta_S(s)] \mid s \in S\}$ and $\{[\delta'_{S'}(s')] \mid s' \in S'\}$ via δ and δ', respectively), because α is isotypical. Further, if α is rigid, then this translates directly to the condition that $\Lambda(\alpha)$ is a rigid isomorphism of localities. Again, $\Lambda(\alpha^{-1})$ is the inverse of $\Lambda(\alpha)$, and so $\Lambda(\alpha)$ is an isomorphism of localities. Thus Λ is a functor which restricts to send $\mathsf{T}_0 \to \mathsf{L}_0$.

Define $\eta\colon \mathrm{id}_{\mathsf{T}} \to \Theta \circ \Lambda$ as follows. For any transporter system $\mathcal{T}$, $\eta_\mathcal{T}\colon \mathcal{T} \to \Theta(\Lambda(\mathcal{T}))$ sends each object to itself, and it sends a morphism $\varphi\colon P \to Q$ in $\mathcal{T}$ to the triple $([\varphi_0], P, Q)$, where φ_0 is the unique morphism from P to $Q_0 := \pi(\varphi)(P)$ in $\mathcal{T}$ such that $\delta_{Q_0,Q}(1) \circ \varphi_0 = \varphi$. We will show that η is a natural isomorphism of functors. By [Che13, Lemma A.15], $\eta_\mathcal{T}$ is a rigid isomorphism of transporter systems, provided we make the identification of S with the group of equivalence classes $\{([\delta_S(s)], S, S) \mid s \in S\}$ via the canonical isomorphism. Let now $\alpha\colon \mathcal{T} \to \mathcal{T}'$ be any isomorphism of transporter systems, and consider the naturality diagram:

$$
\begin{array}{ccc}
\mathcal{T} & \xrightarrow{\eta_\mathcal{T}} & \Theta(\Lambda(\mathcal{T})) \\
{\scriptstyle\alpha}\big\downarrow & & \big\downarrow{\scriptstyle\Theta(\Lambda(\alpha))} \\
\mathcal{T}' & \xrightarrow[\eta_{\mathcal{T}'}]{} & \Theta(\Lambda(\mathcal{T}')).
\end{array}
$$

Fix a morphism $\varphi\colon P \to Q$ in $\mathcal{T}$. Then

$$\Theta(\Lambda(\alpha))([\varphi_0], P, Q) = ([\alpha(\varphi_0)], \alpha(P), \alpha(Q))$$

while

$$\eta_{\mathcal{T}'}(\alpha(\varphi)) = ([\alpha(\varphi)_0], \alpha(P), \alpha(Q)).$$

where $\alpha(\varphi)_0$ is the unique morphism from $\alpha(P)$ to $Q_1 := \pi'(\alpha(\varphi))(\alpha(P))$ such that $\alpha(\varphi) = \delta_{Q_1,\alpha(Q)}(1) \circ \alpha(\varphi)_0$. Note also that $\alpha(\varphi) = \delta_{\alpha(Q_0),\alpha(Q)}(1) \circ \alpha(\varphi_0)$ as α sends inclusions to inclusions. Thus, to show that η is natural, it suffices by uniqueness of restrictions, Lemma 2.2(b), to show that $Q_1 = \alpha(Q_0)$. To this end, let β be the isomorphism from S to S' associated with α in Proposition 2.5. By Proposition 2.5, $\alpha(P) = \beta(P)$ for each $P \in \Delta$, and we have

$$\pi'(\alpha(\varphi))(\alpha(P)) = c_\beta(\pi(\varphi))(\beta(P)) = \beta(\pi(\varphi)(P)) = \alpha(\pi(\varphi)(P)),$$

as required. This completes the proof that η is a natural isomorphism.

Next, given a locality $(\mathcal{L}, \Delta, S)$ define $\zeta_{\mathcal{L}} \colon \mathcal{L} \to (\Lambda \circ \Theta)(\mathcal{L})$ by

$$\zeta_{\mathcal{L}}(f) = [(f, S_f, {}^fS_f)].$$

We will show that $\zeta = (\zeta_{\mathcal{L}}) \colon \mathrm{id}_{\mathsf{L}} \to \Lambda \circ \Theta$ is a natural isomorphism. Let $(f_n, \dots, f_1) \in \mathbf{D}(\mathcal{L})$, and set $f = \Pi(f_n, \dots, f_1)$. By Definition 2.8(L1a), there are objects $P_0, \dots, P_n \in \Delta$ such that $P_{i-1} \leqslant S_{f_i}$ and ${}^{f_i}P_{i-1} = P_i$ for $i = 1, \dots, n$. Then $[(f_i, S_{f_i}, {}^{f_i}S_{f_i})] = [(f_i, P_{i-1}, P_i)]$ by definition of the equivalence class $[-]$, and this implies that $\zeta^*_{\mathcal{L}}(f_n, \dots, f_1) := (\zeta_{\mathcal{L}}(f_n), \dots, \zeta_{\mathcal{L}}(f_1)) \in \mathbf{D}(\Lambda(\Theta(\mathcal{L})))$. By definition of the product in $\Lambda(\Theta(\mathcal{L}))$, we have

$$\Pi(\zeta^*_{\mathcal{L}}(f_n, \dots, f_1)) = [(\Pi(f_n, \dots, f_1), P_0, P_n)] = [(f, P_0, P_n)] = [(f, S_f, {}^fS_f)] = \zeta_{\mathcal{L}}(\Pi(f_n, \dots, f_1)),$$

so $\zeta_{\mathcal{L}}$ is a partial group homomorphism.

There is an extension of Lemma 3.6 of [Che13] in which S and S' (and Δ and Δ') need not be equal, and for which Chermak's proof remains valid. This will be used to show that $\zeta_{\mathcal{L}}$ is an isomorphism of localities. The typical element of $\Lambda(\Theta(\mathcal{L}))$ has the form $[(f, P, Q)]$ for $f \in \mathcal{L}$, $P \leqslant S_f$, and $Q \geqslant {}^fP$. It is the image of f under $\zeta_{\mathcal{L}}$, since $\zeta_{\mathcal{L}}(f) = [(f, S_f, {}^fS_f)] = [(f, P, Q)]$ by the commutative diagram

$$\begin{array}{ccc} S_f & \xrightarrow{(f, S_f, {}^fS_f)} & {}^fS_f \\ {\scriptstyle (1,P,S_f)}\uparrow & & \uparrow{\scriptstyle (1,Q,{}^fS_f)} \\ P & \xrightarrow[(f,P,Q)]{} & Q \end{array}$$

in $\Theta(\mathcal{L})$, so $\zeta_{\mathcal{L}}$ is surjective.

Set $S' = \{[(s, S, S)] \mid s \in S\} \leqslant \Lambda(\Theta(\mathcal{L}))$, and fix $s \in S$ and $f \in \mathcal{L}$. Then $(f, s, f^{-1}) \in \mathbf{D}(\mathcal{L})$ via $X \in \Delta$ if and only if

$$([(f, {}^{sf^{-1}}X, {}^{fsf^{-1}}X)], [(s, {}^{f^{-1}}X, {}^{sf^{-1}}X)], [(f^{-1}, X, {}^{f^{-1}}X)]) \in \mathbf{D}(\Lambda(\Theta(\mathcal{L})))$$

by definition of the domain of the locality built out of the transporter system $\Theta(\mathcal{L})$. Moreover, in this case, $fsf^{-1} \in S$ via $X \in \Delta$ if and only if

$$[(fsf^{-1}, X, {}^{fsf^{-1}}X)] = [(f, {}^{sf^{-1}}X, {}^{fsf^{-1}}X) \circ (s, {}^{f^{-1}}X, {}^{sf^{-1}}X) \circ (f^{-1}, X, {}^{f^{-1}}X)] \in S'$$

This shows that $\zeta_{\mathcal{L}}(S_f) = S'_{\zeta_{\mathcal{L}}(f)}$.

Let $h \in \ker(\zeta_{\mathcal{L}})$. Then $[(h, S_h, {}^hS_h)] = 1_{\Lambda(\Theta(\mathcal{L}))} = [(1, S, S)]$. This means $(h, S_h, {}^hS_h)$ is a restriction of $(1, S, S)$, that is $(1, S_h, S) = (h, S_h, S)$, and hence $h = 1$. This completes the check of the hypotheses of the extension of [Che13, Lemma 3.6], and so $\zeta_{\mathcal{L}}$ is an isomorphism by that lemma. Moreover, $\zeta_{\mathcal{L}}$ is a rigid isomorphism of localities, provided we make the identification of S with the group of equivalence classes $\{[(s, S, S)] \mid s \in S\}$ via the canonical isomorphism.

Finally, it remains to verify naturality of ζ. Given another locality $(\mathcal{L}', \Delta', S')$ and isomorphism $\gamma\colon \mathcal{L} \to \mathcal{L}'$ mapping S onto S', we have for each $f \in \mathcal{L}$ that

$$\Lambda(\Theta(\gamma))(\zeta_{\mathcal{L}}(f)) = [(\gamma(f), \gamma(S_f), \gamma({}^f S_f))]'$$

while

$$\zeta_{\mathcal{L}}(\gamma(f)) = [(\gamma(f), S_{\gamma(f)}, {}^{\gamma(f)}S_{\gamma(f)})]'$$

As γ is an isomorphism mapping S onto S', $\gamma^*(\mathbf{D}_{\mathcal{L}}(f)) = \mathbf{D}_{\mathcal{L}'}(\gamma(f))$, and hence $\gamma(S_f) = S_{\gamma(f)}$. Also, $\gamma({}^f P) = {}^{\gamma(f)}\gamma(P)$ for each $P \in \Delta$ and $f \in \mathcal{L}$. This establishes naturality and completes the proof of the theorem. □

Corollary 2.13. *Fix a transporter system $(\mathcal{T}, \pi, \delta)$ and let $\mathcal{L}_\Delta(\mathcal{T})$ be the associated locality. Then the map*

$$\Phi\colon \operatorname{Aut}(\mathcal{T}) \longrightarrow \operatorname{Aut}(\mathcal{L}_\Delta(\mathcal{T}))$$

given by sending an automorphism $\alpha \in \operatorname{Aut}(\mathcal{T})$ to the map $\mathcal{L}_\Delta(\mathcal{T}) \to \mathcal{L}_\Delta(\mathcal{T})$ which sends a class $[\varphi]$ to $[\alpha(\varphi)]$, for each $\varphi \in \operatorname{Iso}(\mathcal{T})$, is an isomorphism of groups. Moreover, Φ maps $\operatorname{Aut}_0(\mathcal{T})$ onto $\operatorname{Aut}_0(\mathcal{L}_\Delta(T))$.

Proof. This follows directly from Theorem 2.11. □

Remark 2.14. The obstruction theory for the existence and uniqueness of centric linking systems "up to isomorphism" as given by Broto, Levi, and Oliver [BLO03, Theorem 3.1], see also [AKO11, III.5.11], holds of course with respect to the notion of isomorphism of centric linking systems used there. By Proposition 2.5 and Corollary 2.13, this definition coincides with the notion of "rigid isomorphism" of the associated localities. Thus, Theorem 3.4 of [Oli13] and Theorem 1.1 of [GL16] imply that any two centric linking localities (i.e., Δ-linking systems with $\Delta = \mathcal{F}^c$ in the terminology of [Che13, p.49]) associated to a given saturated fusion system are rigidly isomorphic in the sense of [Che13].

2.4. **Linking systems and linking localities.** Theorems 1.1 and 1.2 do not hold for arbitrary localities and transporter systems, as can be seen by considering an appropriate finite group G of the form $O_{p'}(G) \times H$, with $O_{p'}(G)$ supporting an automorphism of order p^2, and forming a locality as in the standard Example 2.9.

Definition 2.15. A finite group N is of *characteristic p* if $C_N(O_p(N)) \leqslant O_p(N)$. A *linking locality* is a locality $(\mathcal{L}, \Delta, S)$ such that $\mathcal{F}_S(\mathcal{L})$ is saturated, $\mathcal{F}_S(\mathcal{L})^{cr} \subseteq \Delta$, and $N_{\mathcal{L}}(P)$ is of characteristic p for each $P \in \Delta$. A *linking system* is a transporter system $(\mathcal{T}, \delta, \pi)$ associated with a saturated fusion system $\mathcal{F}$ having object set Δ such that $\mathcal{F}^{cr} \subseteq \Delta$ and $\operatorname{Aut}_{\mathcal{T}}(P)$ is of characteristic p for each $P \in \Delta$.

The assumption that $\mathcal{L}$ is a linking locality (in Theorem 1.1) or a linking system (in Theorem 1.2) is necessary when applying [GL16, Lemma 8.2], which says that a rigid automorphism of a finite group of characteristic p is conjugation by an element of the center of a Sylow p-subgroup.

The definition of linking system appearing in Definition 2.15 was given by Henke [Hen19]. It is more general than the usual definition in [AKO11, Definition III.4.1], which forces each object to be $\mathcal{F}$-*quasicentric*. In Henke's definition, the objects are forced merely to be a subset of the larger collection of $\mathcal{F}$-subcentric subgroups of S, namely the subgroups P of S with the property that $O_p(N_{\mathcal{F}}(Q))$ is $\mathcal{F}$-centric for each fully $\mathcal{F}$-normalized conjugate Q of P. The term "linking locality" also appears first in [Hen19] and refers to the same thing as a "proper locality" in [Che15]. By [Hen19, Proposition 1], the equivalence between localities and transporter systems given in Theorem 2.11 restricts to an equivalence between linking localities and linking systems.

Examples of linking localities include localities of finite groups of Lie type in characteristic p, where, by the Borel-Tits theorem, one may take Δ to be the set of nonidentity subgroups of a Sylow subgroup. On the other hand, every finite group G gives rise to a linking locality on the set Δ of $\mathcal{F}_S(G)$-subcentric subgroups of a Sylow subgroup S, the main theorem of [Hen19].

3. Rigid outer automorphisms of centric linking systems

In this section, we prove Theorems 1.1 and 1.2 in the case $\Delta = \mathcal{F}^c$, and we prove Theorem 1.3. Throughout, we fix a saturated fusion system $\mathcal{F}$ over the finite p-group S and a linking locality $(\mathcal{L}, \Delta, S)$ on $\mathcal{F}$.

A version of the Alperin-Goldschmidt fusion theorem for linking localities was proved by Chermak and is needed in the proof of Theorem 1.1. We state a special case of it in a flexible form.

Proposition 3.1. *Let $\mathcal{C}$ be any conjugation family for $\mathcal{F}$ and let $g \in \mathcal{L}$. Then there are $Q_i \in \mathcal{C} \cap \Delta$ and elements $g_i \in N_{\mathcal{L}}(Q_i)$ such that $g = g_n \cdots g_1$.*

Proof. Recall, by definition of a linking locality (proper locality), that $\mathcal{F}^{cr} \subseteq \Delta$. Further, the collection $\mathbf{A}(\mathcal{F})$ defined in [Che16, Notation 3.3] is a subset of $\mathcal{F}^{cr}$ and coincides with the collection of $\mathcal{F}$-essential subgroups [AKO11, Definition I.3.2]. So the assertion is a special case of [Che16, Theorem 3.5], given that the collection of $\mathcal{F}$-essential subgroups is contained in any conjugation family, cf. [AKO11, Proposition I.3.3(b)]. $\square$

Proposition 3.1 has the immediate consequence that an automorphism which is the identity on $N_{\mathcal{L}}(Q)$ for each $Q \in \mathcal{C} \cap \Delta$ is the identity automorphism of $\mathcal{L}$. We take the opportunity to prove below a more general statement which generalizes Lemma 5.4 of [GL16] to the setting of linking localities. We refer to [Cra11, Definition 7.14] for the definition of a positive characteristic p-functor W, which we call a conjugacy functor for short. There is a mistake in the proof of [GL16, Lemma 5.4], in which $W(Q)$ is claimed to be well-placed, given that Q is. This seems unlikely to be true. It is true that $W(Q)$ is conjugate to a well-placed subgroup, and we give a correct argument in the proof of Lemma 3.2.

Lemma 3.2. *Let τ be an automorphism of $\mathcal{L}$. Fix a conjugacy functor W for $\mathcal{F}$, let $\mathcal{C}$ be the associated conjugation family consisting of those subgroups of S which are well-placed with respect to W, and set*

$$\mathcal{W} = \{Q \in \mathcal{C} \cap \Delta \mid W(Q) = Q\}.$$

Assume that $W(Q) \in \Delta$ and $W(W(Q)) = W(Q)$ whenever $Q \in \Delta$. If τ is the identity on $N_{\mathcal{L}}(Q)$ for each $Q \in \mathcal{W}$, then τ is the identity automorphism of $\mathcal{L}$.

Proof. Assume first that W is the identity functor. Then $\mathcal{W} = \mathcal{C} \cap \Delta$. Let $\tau \in \mathrm{Aut}(\mathcal{L})$, and assume that τ is the identity on $N_{\mathcal{L}}(Q)$ for all $Q \in \mathcal{W} = \mathcal{C} \cap \Delta$. For $g \in \mathcal{L}$, there are $Q_i \in \mathcal{C} \cap \Delta$ and $g_i \in N_{\mathcal{L}}(Q_i)$ such that $g = g_n \cdots g_1$ by Proposition 3.1. Then $\tau(g) = \tau(g_n) \cdots \tau(g_1) = g_n \cdots g_1 = g$ by assumption. Thus, τ is the identity automorphism.

Next, we prove the result for general W satisfying the hypotheses. By the previous case with the identity functor in place of W, it suffices to show that τ is the identity on $N_{\mathcal{L}}(Q)$ for each

$Q \in \mathcal{C} \cap \Delta$. Proceed by induction on the index of Q in S. Assume first that $Q = S$. Since $S \in \mathcal{C}$ (it is contained in every conjugation family), $W(Q) = W(S) \in \mathcal{C} \cap \Delta$ by assumption on W. Hence, as $\tau|_{N_{\mathcal{L}}(W(S))} = \mathrm{id}_{N_{\mathcal{L}}(W(S))}$ and $N_{\mathcal{L}}(S) \leqslant N_{\mathcal{L}}(W(S))$, τ is the identity on $N_{\mathcal{L}}(Q)$. Fix now $Q < S$ and assume that τ is the identity on $N_{\mathcal{L}}(R)$ for all $R \in \Delta$ with $|R| > |Q|$. Let $g \in \mathcal{L}$ with ${}^{g}N_S(W(Q)) \leqslant S$ and ${}^{g}W(Q)$ well-placed by [Cra11, Lemma 7.23]. We claim that τ fixes g. Write $g = g_n \cdots g_1$ for subgroups $R_i \in \mathcal{C} \cap \Delta$ and $g_i \in N_{\mathcal{L}}(R_i)$ with $R_i \geqslant {}^{g_i \cdots g_1}N_S(W(Q))$. So $|R_i| \geqslant |N_S(W(Q))| \geqslant |N_S(Q)| > |Q|$. The claim now follows from the inductive hypothesis. As ${}^{g}W(Q)$ is well-placed and Δ is closed under $\mathcal{L}$-conjugation, we have ${}^{g}W(Q) \in \mathcal{C} \cap \Delta$. Now $N_{\mathcal{L}}({}^{g}Q) \leqslant N_{\mathcal{L}}({}^{g}W(Q))$ by the axioms for a conjugacy functor. Since τ is the identity on $N_{\mathcal{L}}({}^{g}W(Q))$ by hypothesis, we see that τ is the identity on $N_{\mathcal{L}}({}^{g}Q)$. Finally, since $\tau(g) = g$, τ is the identity on $N_{\mathcal{L}}(Q)$, as desired. □

Proof of Theorem 1.1 in the case $\Delta = \mathcal{F}^c$. Recall that $k(p) = 1$ if p is odd, and $k(p) = 2$ if $p = 2$. Fix $\tau \in \mathrm{Aut}_0(\mathcal{L})$. For any finite p-group P, we take the abelian version of the Thompson subgroup $J(P)$, namely, $J(P)$ is the subgroup generated by the abelian subgroups of P of order $d(P)$, where $d(P)$ is the maximum of the orders of the abelian subgroups of P.

We proceed in several steps to complete the proof. The main part of the proof consists in showing that if the automorphism τ is the identity on $N_{\mathcal{L}}(J(S))$, then $\tau^{k(p)} = \mathrm{id}_{\mathcal{L}}$. This is carried out in Steps 2-6.

Step 1. We first arrange that τ restricts to the identity automorphism of $N_{\mathcal{L}}(J(S))$. The restriction τ to $N_{\mathcal{L}}(J(S))$ is an automorphism of $N_{\mathcal{L}}(J(S))$ which is identity on $S \leqslant N_{\mathcal{L}}(J(S))$. Since $\mathcal{L}$ is a linking locality and $J(S) \in \Delta = \mathcal{F}^c$, the normalizer $N_{\mathcal{L}}(J(S))$ is of characteristic p. Thus, by [GL16, Lemma 8.2], we may fix $z \in Z(S)$ such that τ is conjugation by z on $N_{\mathcal{L}}(J(S))$. Then upon replacing τ by $c_z^{-1}\tau$, where $c_z \colon \mathcal{L} \to \mathcal{L}$ denotes the rigid inner automorphism which is (everywhere-defined) conjugation by z, we complete the proof of Step 1.

Consider the following ordering on $\mathcal{F}^c$:

$$Q <_J P \quad \Longleftrightarrow \quad d(Q) < d(P) \quad \text{or} \quad d(Q) = d(P) \text{ and } |J(Q)| < |J(P)|.$$

We claim that $\tau^{k(p)}$ is the identity on $\mathcal{L}$. Assume the contrary, and, using Lemma 3.2 with W the identity functor, choose Q maximal under $<_J$ with the property that $N_{\mathcal{L}}(Q)$ is not fixed by $\tau^{k(p)}$.

Step 2. We show that Q may be taken to be well-placed with respect to J. Let $\mathcal{C}$ be the collection of subgroups of S which are well-placed with respect to the Thompson subgroup functor J. Then $\mathcal{C}$ forms a conjugation family for $\mathcal{F}$ by [Cra11, Corollary 7.26]. Let $g \in N_{\mathcal{L}}(Q)$ not fixed by $\tau^{k(p)}$. By Proposition 3.1, we may write g as a product of elements $g_i \in N_{\mathcal{L}}(R_i)$ with $R_i \in \mathcal{C} \cap \Delta$, and where $Q = Q_0 = Q_n$, $Q_i = {}^{g_i}Q_{i-1}$, and $R_i \geqslant \langle Q_{i-1}, Q_i \rangle$ for each i. Since g is not fixed by $\tau^{k(p)}$, some g_i is not fixed by $\tau^{k(p)}$. Now as Q is isomorphic to a subgroup of R_i, we see that $d(Q) \leqslant d(R_i)$. Therefore, equality holds by maximality of Q under $<_J$. Then $|J(Q)| \leqslant |J(R_i)|$, so again equality holds by maximality of Q. Hence, upon replacing Q by R_i, we may assume that $Q \in \mathcal{C}$.

Step 3. Set $H = N_{\mathcal{L}}(Q)$ and $T = N_S(Q)$. We next show that $J(Q) = J(QJ(T))$. As $Q \in \Delta$, H is of characteristic p. By [GL16, Lemma 8.2], we may fix $z \in Z(T)$ such that τ is conjugation by z on H. Then τ^2 is conjugation by z^2 on H. Since $\tau^{k(p)}$ is not the identity on H, we have that $z^{k(p)}$ is not centralized by H. Applying [Gla68, Theorem A], we conclude that $z^{k(p)}$ is not centralized by $N_H(J(T))$. Now $N_H(J(T)) \leqslant N_H(QJ(T))$ since $H = N_H(Q)$, so that $\tau^{k(p)}$ is not the identity on $N_{\mathcal{L}}(QJ(T))$. As $QJ(T) \in \mathcal{F}^c$ and $d(Q) \leqslant d(QJ(T))$, we have equality by maximality of Q under

$<_J$. Then $J(Q) \leqslant J(QJ(T))$, and so

$$J(Q) = J(QJ(T)), \tag{3.3}$$

again by maximality of Q under $<_J$.

Step 4. Here we show $J(T) = J(Q)$. As $d(Q) \leqslant d(T) = d(J(T)) \leqslant d(QJ(T))$, we have equality by Step 3. Thus, $d(Q) = d(T)$ and $Q \leqslant T$ yield that $J(Q) \leqslant J(T) \leqslant J(QJ(T))$, and again we have equality by choice of Q. This completes the proof of Step 4.

Step 5. We next show that $J(Q)$ is $\mathcal{F}$-centric. Suppose on the contrary that $J(Q)$ is not $\mathcal{F}$-centric. By Step 2, Q is well-placed. By definition of well-placed, $J(T)$ is fully $\mathcal{F}$-normalized. Hence, $J(Q)$ is fully $\mathcal{F}$-normalized by Step 4. Since $J(Q)$ is fully $\mathcal{F}$-normalized and not $\mathcal{F}$-centric, we have $C_S(J(Q)) \nleqslant J(Q)$. Note that $C_S(J(Q)) \nleqslant Q$ since $J(Q)$ does contain its centralizer in Q. Hence, $QC_S(J(Q)) > Q$, so with $R := N_{QC_S(J(Q))}(Q)$, we have

$$R > Q.$$

On the other hand, Step 4 shows that

$$R = QN_{C_S(J(Q))}(Q) = QC_T(J(Q)) = QC_T(J(T)) = QZ(J(T)) = QZ(J(Q)) = Q,$$

a contradiction.

Step 6. Lastly, we obtain a contradiction. Among all well-placed, $\mathcal{F}$-centric subgroups maximal under $<_J$ whose normalizer in $\mathcal{L}$ is not centralized by $\tau^{k(p)}$, choose Q of minimum order. By Step 4 and the definition of well-placed, $J(Q) = J(T)$ is well-placed. By Step 5, $J(Q)$ is centric. Note $\tau^{k(p)}$ is not the identity on $N_H(J(Q)) = H$ by choice of Q. Since $d(Q) = d(J(Q))$ and $J(J(Q)) = J(Q)$, we have that $Q = J(Q)$ by minimality of $|Q|$. Therefore, by Step 4,

$$J(Q) = J(T) = J(N_S(Q)) = J(N_S(J(Q))).$$

It now follows that $Q = J(Q) = J(S)$ by [GL16, Lemma 8.5(b)]. Since $N_{\mathcal{L}}(J(S))$ is centralized by τ by Step 1, this is a contradiction.

Step 7. We prove the splitting condition. Since Steps 1-6 show that $\mathrm{Out}_0(\mathcal{L}) = 1$ if p is odd, splitting is trivial in that case. So take $p = 2$. Let E be the subgroup of $\mathrm{Aut}_0(\mathcal{L})$ consisting of those automorphisms which restrict to the identity on $N_{\mathcal{L}}(J(S))$. Step 1 shows that E maps surjectively onto $\mathrm{Out}_0(\mathcal{L})$ via the quotient map $\mathrm{Aut}_0(\mathcal{L}) \to \mathrm{Out}_0(\mathcal{L})$, while Steps 1-6 show that E is a vector space over $\mathbb{F}_2$. There is therefore a subgroup E_0 which is a complement to $C_{\mathrm{Aut}_{Z(S)}(\mathcal{L})}(N_{\mathcal{L}}(J(S)))$ in E and which maps isomorphically onto $\mathrm{Out}_0(\mathcal{L})$. This proves the assertion. □

Proof of Theorem 1.2 when $\mathcal{L}$ is a centric linking system. This follows directly from Theorem 1.1 in the centric linking locality case, given Theorem 2.11. □

Remark 3.4. The method of proof of Theorems 1.1 and 1.2 in case $\Delta = \mathcal{F}^c$ shows the slightly stronger conclusion: if τ is an automorphism of a centric linking locality (centric linking system) which is the identity on $N_{\mathcal{L}}(J(S))$ ($\mathrm{Aut}_{\mathcal{L}}(J(S))$), then $\tau^{k(p)} = \mathrm{id}_{\mathcal{L}}$.

We next want to prove Theorem 1.3, but first recall certain definitions from [AKO11, Section III.5]. Let $\mathcal{O}(\mathcal{F}^c)$ be the category with objects the $\mathcal{F}$-centric subgroups, and with morphism sets

$$\mathrm{Mor}_{\mathcal{O}(\mathcal{F}^c)}(P, Q) = \mathrm{Inn}(Q)\backslash \mathrm{Hom}_{\mathcal{F}}(P, Q),$$

the set of orbits of $\mathrm{Inn}(Q)$ in its left action by composition. The center functor

$$\mathcal{Z}_{\mathcal{F}} \colon \mathcal{O}(\mathcal{F}^c) \to \mathsf{Ab}$$

is the functor which sends a subgroup P to its center $Z(P)$, and sends a morphism $[\varphi]\colon P \to Q$ to the composite $Z(Q) \hookrightarrow Z(\varphi(P)) \xrightarrow{\varphi^{-1}|_{Z(\varphi(P))}} Z(P)$ induced by the restriction of $\varphi^{-1}\colon \varphi(P) \to P$ to $Z(\varphi(P))$.

We refer to Section III.5.1 of [AKO11] for a description of the bar resolution for functor cohomology and write d for the coboundary map. Recall that a 0-cochain for $\mathcal{Z}_\mathcal{F}$ sends an object P of $\mathcal{O}(\mathcal{F}^c)$ to an element in $Z(P)$. A 1-cochain sends a morphism $P \xrightarrow{[\varphi]} Q$ in the orbit category to an element in $Z(P)$. A 1-cochain for $\mathcal{Z}_\mathcal{F}$ is said to be *inclusion-normalized* if it sends the class of each inclusion ι_P^Q to $1 \in Z(P)$. Write $\widehat{Z}^1(\mathcal{O}(\mathcal{F}^c), \mathcal{Z}_\mathcal{F})$ for the group of inclusion-normalized 1-cocycles, and write $\widehat{B}^1(\mathcal{O}(\mathcal{F}^c), \mathcal{Z}_\mathcal{F}) \subseteq \widehat{Z}^1(\mathcal{O}(\mathcal{F}^c), \mathcal{Z}_\mathcal{F})$ for the group of inclusion-normalized 1-coboundaries.

By the proof of [AKO11, III.5.12], there is a group homomorphism

$$\tilde{\lambda}\colon \widehat{Z}^1(\mathcal{O}(\mathcal{F}^c), \mathcal{Z}_\mathcal{F}) \to \mathrm{Aut}(\mathcal{L})$$

given by sending a 1-cocycle t to the automorphism of $\mathcal{L}$ which is the identity on objects, and which sends a morphism $\varphi\colon P \to Q$ in $\mathcal{L}$ to $\varphi \circ \delta_P(t([\varphi]))$. Next, consider the group homomorphisms

$$\mathrm{cnst}\colon Z(S) \to C^0(\mathcal{O}(\mathcal{F}^c), \mathcal{Z}_\mathcal{F}) \quad \text{and} \quad \mathrm{conj}\colon Z(S) \to \mathrm{Aut}_0(\mathcal{L}),$$

where cnst sends an element $z \in Z(S)$ to the constant 0-cochain u_z with value z on each centric subgroup, and conj sends an element z to the conjugation automorphism $c_{\delta_S(z)} \in \mathrm{Aut}_0(\mathcal{L})$.

Lemma 3.5. *There is an isomorphism of short exact sequences*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \widehat{B}^1(\mathcal{O}(\mathcal{F}^c), \mathcal{Z}_\mathcal{F}) & \longrightarrow & \widehat{Z}^1(\mathcal{O}(\mathcal{F}^c), \mathcal{Z}_\mathcal{F}) & \longrightarrow & \lim^1 \mathcal{Z}_\mathcal{F} & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle du \mapsto u(S)Z(\mathcal{F})} & & \downarrow{\scriptstyle \tilde{\lambda}} & & \downarrow{\scriptstyle \lambda} & & \\ 1 & \longrightarrow & Z(S)/Z(\mathcal{F}) & \xrightarrow{\mathrm{conj}} & \mathrm{Aut}_0(\mathcal{L}) & \longrightarrow & \mathrm{Out}_0(\mathcal{L}) & \longrightarrow & 1. \end{array} \tag{3.6}$$

Proof. This is essentially contained in the proof of [AKO11, Proposition III.5.12]. There the groups $\mathrm{Aut}(\mathcal{L})$ and $\mathrm{Out}(\mathcal{L})$ are denoted $\mathrm{Aut}^I_{\mathrm{typ}}(\mathcal{L})$ and $\mathrm{Out}_{\mathrm{typ}}(\mathcal{L})$. The commutative diagram displayed on [AKO11, p.186] is shown to have exact rows and columns. Thus, $\tilde{\lambda}\colon \widehat{Z}^1(\mathcal{O}(\mathcal{F}^c), \mathcal{Z}_\mathcal{F}) \to \mathrm{Aut}(\mathcal{L})$ is injective with image $\ker(\tilde{\mu}) = \mathrm{Aut}_0(\mathcal{L})$. Also, $\tilde{\lambda}$ induces an injective homomorphism $\lambda\colon \lim^1 \mathcal{Z}_\mathcal{F} \to \mathrm{Out}(\mathcal{L})$ with image $\ker(\mu) = \mathrm{Out}_0(\mathcal{L})$, and so $\tilde{\lambda}$ and λ are isomorphisms after restricting to these codomains. Thus, the commutativity of this diagram also gives that the right square in (3.6) commutes.

Second, from the proof of [AKO11, III.5.12], the composite $d \circ \mathrm{cnst}$ has image $\widehat{B}^1(\mathcal{O}(\mathcal{F}^c), \mathcal{Z}_\mathcal{F})$, where, for each $z \in Z(S)$, the image du_z of u_z under the coboundary map is inclusion-normalized, and $\tilde{\lambda}(du_z)$ is conjugation by $\delta_S(z)$ on $\mathcal{L}$. The composite $\widehat{B}^1(\mathcal{O}(\mathcal{F}^c), \mathcal{Z}_\mathcal{F}) \hookrightarrow \widehat{Z}^1(\mathcal{O}(\mathcal{F}^c), \mathcal{Z}_\mathcal{F}) \xrightarrow{\tilde{\lambda}} \mathrm{Aut}_0(\mathcal{L})$ is injective. Thus, the kernel of the composite $d \circ \mathrm{cnst}$ is the same as the kernel of conj. But $\ker(\mathrm{conj}) = Z(\mathcal{F})$ by [AOV12, Lemma 1.14]. Therefore, the inverse $du \mapsto u(S)Z(\mathcal{F})$ of the isomorphism $Z(S)/Z(\mathcal{F}) \to \widehat{B}^1(\mathcal{O}(\mathcal{F}^c), \mathcal{Z}_\mathcal{F})$ induced by $d \circ \mathrm{cnst}$ makes the left square in (3.6) commute. □

Proof of Theorem 1.3. By Theorem 1.2 in the case $\Delta = \mathcal{F}^c$, the sequence $1 \to \mathrm{Aut}_{Z(S)}(\mathcal{L}) \to \mathrm{Aut}_0(\mathcal{L}) \to \mathrm{Out}_0(\mathcal{L}) \to 1$ is split exact. As $\mathrm{Aut}_{Z(S)}(\mathcal{L})$ is the image of the conjugation map $Z(S)/Z(\mathcal{F}) \to \mathrm{Aut}_0(\mathcal{L})$, it follows from Lemma 3.5 that the sequence $1 \to \widehat{B}^1(\mathcal{O}(\mathcal{F}^c), \mathcal{Z}_\mathcal{F}) \to \widehat{Z}^1(\mathcal{O}(\mathcal{F}^c), \mathcal{Z}_\mathcal{F}) \to \lim^1 \mathcal{Z}_\mathcal{F} \to 1$ is also split exact and that $\lim^1 \mathcal{Z}_\mathcal{F} \cong \mathrm{Out}_0(\mathcal{L})$ is elementary abelian. □

4. Extending to larger object sets

In this section, we observe via Chermak descent [Che13, Theorem 5.15] that the group of rigid automorphisms does not change when a centric linking locality is expanded to a larger object set. Recall from [Hen19] that a subgroup P of S is said to be $\mathcal{F}$-*subcentric* if for each fully $\mathcal{F}$-normalized $\mathcal{F}$-conjugate Q of P, the subgroup $O_p(N_{\mathcal{F}}(Q))$ is $\mathcal{F}$-centric. The set of $\mathcal{F}$-subcentric subgroups is denoted $\mathcal{F}^s$.

Proposition 4.1. *Let $\mathcal{L}^+$ be a linking locality with object set Δ^+ and fusion system $\mathcal{F}$ over a p-group S. Let $\Delta \subseteq \Delta^+$ be a subset which contains $\mathcal{F}^{cr}$ and is closed under $\mathcal{F}$-conjugacy and passing to overgroups. Assume that $\mathcal{L}^+|_\Delta = \mathcal{L}$. Then restriction induces an isomorphism $\mathrm{Aut}_0(\mathcal{L}^+) \to \mathrm{Aut}_0(\mathcal{L})$ which restricts to an isomorphism $\mathrm{Aut}_{Z(S)}(\mathcal{L}^+) \to \mathrm{Aut}_{Z(S)}(\mathcal{L})$.*

Proof. This follows from Corollary 5.16 of [Che13], applied in the same way as in [Hen19, Theorem 7.2]. The proof is by induction on $|\Delta^+ - \Delta|$. If $\Delta^+ = \Delta$, then $\mathcal{L}^+ = \mathcal{L}$ and there is nothing to prove. Let $T \in \Delta^+ - \Delta$ be maximal under inclusion. We claim that Hypothesis 5.3 of [Che13] holds. Since Δ and Δ^+ are $\mathcal{F}$-invariant and closed under passing to overgroups, we can replace T by an $\mathcal{F}$-conjugate if necessary and assume that T is fully $\mathcal{F}$-normalized. By induction, we may also assume that $\Delta^+ = \Delta \cup T^{\mathcal{F}}$.

Let $\widehat{T} = O_p(N_{\mathcal{F}}(T))$. Then $T \leqslant \widehat{T}$, and we claim the inclusion is proper. Assume otherwise. As an object of a linking locality, T is $\mathcal{F}$-subcentric by [Hen19, Proposition 1(b)]. So by [Hen19, Proposition 3.18], it follows that $T \in \mathcal{F}^{cr}$. But then $T \in \Delta$, which contradicts the choice of T. Thus, $T < \widehat{T}$, so $\widehat{T} \in \Delta$ by choice of T.

Let $M = N_{\mathcal{L}}(T)$, and set

$$\Delta_T := \{N_P(T) \mid T \leqslant P \in \Delta\} = \{P \in \Delta \mid T \leqslant P \leqslant N_S(T)\},$$

where the second equality comes from maximality of T in $\Delta^+ - \Delta$. By Lemma 7.1 of [Hen19], M is a finite group which is a model for $N_{\mathcal{F}}(T)$. In particular T is normal in M and $N_S(T)$ is a Sylow p-subgroup of M. So indeed, taking the identity $\mathcal{L} \to \mathcal{L}$ as a rigid automorphism, Hypothesis 5.3 of [Che13] holds. Recall the locality $\mathcal{L}_{\Delta_T}(M)$ from Example 2.9, and note that $\mathcal{L}_{\Delta_T}(M) = M$ in the current situation, since each normal p-subgroup of the fusion system of M is normal in M [Hen19, Theorem 2.1(b)]. By Corollary 5.16 of [Che13], there is a unique rigid isomorphism $\mathcal{L}^+(\mathrm{id}_M) \to \mathcal{L}^+$ which restricts to the identity on $\mathcal{L}$, where the former is constructed in [Che13, Theorem 5.14] and defined after the proof of [Che13, Theorem 5.14]. Identify $\mathcal{L}^+(\mathrm{id}_M)$ and $\mathcal{L}^+$ via this isomorphism. The identity automorphism is then the unique rigid automorphism of $\mathcal{L}^+$ which is the identity on $\mathcal{L}$. This shows that the restriction map $\mathrm{Aut}_0(\mathcal{L}^+) \to \mathrm{Aut}_0(\mathcal{L})$ is injective.

To see surjectivity of restriction, take an arbitrary rigid isomorphism β of $\mathcal{L}$. Again by [Che13, Corollary 5.16], there is a rigid isomorphism $\beta^+ \colon \mathcal{L}^+(\beta|_M) \to \mathcal{L}^+$ which restricts to β on $\mathcal{L}$. Taking now $\mathcal{L}^+(\beta_M)$ in the role of $\mathcal{L}^+$, we see that there is also a rigid isomorphism $\mathrm{id}^+ \colon \mathcal{L}^+ = \mathcal{L}^+(\mathrm{id}_M) \to \mathcal{L}^+(\beta_M)$ which is the identity on $\mathcal{L}$. The composition $\beta^+ \circ \mathrm{id}^+ \in \mathrm{Aut}_0(\mathcal{L}^+)$ restricts to β on $\mathcal{L}$, and this shows the restriction map is surjective. □

Proof of Theorems 1.1 and 1.2. Let $(\mathcal{L}, \Delta, S)$ be an arbitrary linking locality. Now $\Delta \subseteq \mathcal{F}^s$ by Proposition 1(b) of [Hen19], so by Theorem 7.2 of [Hen19], there is a linking locality $(\mathcal{L}^+, \mathcal{F}^s, S)$ which restricts to $\mathcal{L}$ on Δ. As $\mathcal{F}^c \subseteq \mathcal{F}^s$, two applications of Proposition 4.1 give an isomorphism of short exact sequences between $1 \to \mathrm{Aut}_{Z(S)}(\mathcal{L}) \to \mathrm{Aut}_0(\mathcal{L}) \to \mathrm{Out}_0(\mathcal{L}) \to 1$ and

$1 \to \mathrm{Aut}_{Z(S)}(\mathcal{L}^+|_{\mathcal{F}^c}) \to \mathrm{Aut}_0(\mathcal{L}^+|_{\mathcal{F}^c}) \to \mathrm{Out}_0(\mathcal{L}^+|_{\mathcal{F}^c}) \to 1$. Theorem 1.1 now follows from the proof in the case $\Delta = \mathcal{F}^c$. Then Theorem 1.2 follows from Theorem 1.1 and Theorem 2.11. □

Remark 4.2. Given the results of this section, the stronger statement mentioned in Remark 3.4 applies verbatim to arbitrary linking localities (linking systems) with object set Δ containing $J(S)$.

5. Comparing automorphisms of groups and linking systems

One may wonder whether it is possible to recover from Theorem 1.2 the analogous theorems about groups, namely [Gla68, Theorem 10] for $p = 2$ and [GGLN20, Theorem 3.3] for p odd. This is possible, but the only way we know how to do it goes through an argument similar to existing arguments for establishing the group case anyway, so our way seems to have little additional value. However, in the process of trying to construct a proof, we obtained Theorem 5.1 below, which appears to be new and of independent interest. It depends for its proof on the Z_p^*-theorem [Gla66a], [GLS98, 7.8.2,7.8.3] that in a finite group with no normal p'-subgroups, any element which is weakly closed in a Sylow p-subgroup is central.

First we need to set up some notation. Let p be a prime and let G be a finite group with Sylow p-subgroup S. We write $\mathcal{L} = \mathcal{L}_S^c(G)$ and $\mathcal{F} = \mathcal{F}_S(G)$ for the centric linking system and fusion system of G. Thus, $\mathcal{L}$ has objects the $\mathcal{F}$-centric subgroups, or equivalently, the p-centric subgroups of G, i.e the subgroups P of S with $C_G(P) = Z(P) \times O_{p'}(C_G(P))$. Morphisms are given by

$$\mathrm{Mor}_{\mathcal{L}}(P,Q) = N_G(P,Q)/O_{p'}(C_G(P)).$$

where $N_G(P,Q) = \{g \in G \mid {}^gP \leqslant Q\}$ is the transporter set, where composition is induced by multiplication in G, and where $O_{p'}(C_G(P))$ acts on $N_G(P,Q)$ from the right. The structural functor δ is the inclusion map, while π sends a coset $gO_{p'}(C_G(P))$ to conjugation by g.

By Sylow's theorem, each outer automorphism of G is represented by an automorphism $\alpha \in N_{\mathrm{Aut}(G)}(S)$. Such an automorphism induces an isomorphism from $O_{p'}(C_G(P))$ to $O_{p'}(C_G(\alpha(P)))$ and a bijection $N_G(P,Q) \to N_G(\alpha(P), \alpha(Q))$, for each pair of centric subgroups P and Q. It is then straightforward to check that α induces an automorphism of $\mathcal{L}$ by restriction in this way. Let

$$\tilde{\kappa}_G \colon N_{\mathrm{Aut}(G)}(S) \to \mathrm{Aut}(\mathcal{L})$$

denote the resulting group homomorphism. This map sends $\mathrm{Aut}_G(S)$ onto $\{c_\gamma \mid \gamma \in \mathrm{Aut}_{\mathcal{L}}(S)\}$, and so there is an induced homomorphism

$$\kappa_G \colon \mathrm{Out}(G) \to \mathrm{Out}(\mathcal{L}).$$

The composition $\tilde{\mu}_G \circ \tilde{\kappa}_G \colon N_{\mathrm{Aut}(G)}(S) \to \mathrm{Aut}(\mathcal{F}_S(G))$ is just restriction to S. Here $\tilde{\mu}_G$ is defined just after Proposition 2.5.

Theorem 5.1. *Fix a prime p, a finite group G, and a Sylow p-subgroup S of G. Let $\mathcal{L}$ be the centric linking system for G. If $O_{p'}(G) = 1$, then $\ker(\kappa_G)$ is a p'-group.*

The proof uses the Z_p^*-theorem only in the semidirect product of G by a p-power automorphism. So if $p = 2$ or the composition factors of G are known, then this does not depend on the CFSG.

Proof. Assume $O_{p'}(G) = 1$. Fix $a \in N_{\mathrm{Aut}(G)}(S)$ with $[a] \in \ker(\kappa_G)$, and recall that $\tilde{\mu}_G \circ \tilde{\kappa}_G$ sends a to $a|_S$. Since $\tilde{\kappa}_G$ maps $N_{\mathrm{Inn}(G)}(S)$ onto $\mathrm{Inn}(\mathcal{L}) = \{c_\gamma \mid \gamma \in \mathrm{Aut}_{\mathcal{L}}(S)\}$, we may adjust a by an element of $N_{\mathrm{Inn}(G)}(S)$ and take $a \in C_{\mathrm{Aut}(G)}(S)$. Then by choice of a, $\tilde{\kappa}_G(a) \in \mathrm{Inn}(\mathcal{L}) \cap \ker(\tilde{\mu}_G) = \mathrm{Aut}_{Z(S)}(\mathcal{L})$. Choose $z \in Z(S)$ such that $\tilde{\kappa}_G(a) = c_z$. Replacing a by $ac_{z^{-1}}$, we may take $a \in \ker(\tilde{\kappa}_G)$. Finally, replacing a by a p'-power, we may take a of p-power order.

We will show that, if $[a] \neq 1$ in $\mathrm{Out}(G)$, then a normalizes but does not centralize $H/O_{p'}(H)$ for some p-local subgroup $H = N_G(Q)$ with Q p-centric in G, that is, with $Q \in \mathcal{F}_S(G)^c$. Thus, $\tilde{\kappa}_G(a)$ does not centralize $\mathrm{Aut}_{\mathcal{L}}(Q)$, and hence $\tilde{\kappa}_G(a) \neq 1$, contrary to our choice of a.

So assume $[a] \neq 1$. Let $\widehat{G} = G\langle a\rangle$ be the semidirect product, and set $\widehat{S} = S\langle a\rangle$. Then $\widehat{S}$ is Sylow in $\widehat{G}$, and $\langle a\rangle \leqslant Z(\widehat{S})$. Also, $\widehat{S} = S \times \langle a\rangle$ and $Z(\widehat{S}) = Z(S) \times \langle a\rangle$. Note that if a is weakly closed in $\widehat{S}$ with respect to $\widehat{G}$, then by the Z_p^*-theorem, we have $a \in Z(\widehat{G})$ since $O_{p'}(\widehat{G}) = O_{p'}(G) = 1$, so that $a = 1$, contrary to assumption.

So a is not weakly closed in $\widehat{S}$ with respect to $\widehat{G}$. By the Alperin-Goldschmidt fusion theorem in $\widehat{G}$, there is a $\mathcal{F}_{\widehat{S}}(\widehat{G})$-centric radical subgroup $\widehat{Q} \leqslant \widehat{S}$ and $\widehat{h} \in N_{\widehat{G}}(\widehat{Q})$ such that $a \in Z(\widehat{S}) \leqslant Z(\widehat{Q})$, and $a \neq {}^{\widehat{h}}a \in Z(\widehat{Q})$. By [LO02, Proposition A.11(c)],

$$Q := \widehat{Q} \cap G \text{ is } \mathcal{F}_S(G)\text{-centric radical.} \tag{5.2}$$

Write $\widehat{h} = ha^k$ for some integer k and some $h \in G$. Since $a^k \in \widehat{Q}$ and $Q = \widehat{Q} \cap G$, we have $h \in N_G(\widehat{Q}) \leqslant N_G(Q)$. Also, $a \neq {}^{\widehat{h}}a = {}^{h}a$. So $[a, h] \in \widehat{S}$. Note that a normalizes $N_G(Q)$, so a normalizes $O_{p'}(N_G(Q))$. If a centralizes h modulo $O_{p'}(N_G(Q))$, then we would have $[a, h] \in O_{p'}(N_G(Q)) \cap \widehat{S} = 1$, a contradiction. Hence, a does not centralize $N_G(Q)/O_{p'}(N_G(Q))$. Together with (5.2), this completes the proof of the proposition. ☐

A saturated fusion system $\mathcal{F}$ over S is said to be *tame* if $\mathcal{F} = \mathcal{F}_S(G)$ for some finite group G with Sylow p-subgroup S such that the map κ_G is split surjective. Theorem 5.1 can be combined with the following lemma of Broto, Møller, and Oliver to show that the splitting condition in the definition of tame is unnecessary. The version we give of this lemma is a little different from the corresponding statement in [BMO19, Lemma 1.5(b)]: two occurences of $O_{p'}(Z(G))$ appearing there (in the statement and proof) have been replaced by $O_{p'}(G)$. This change helps to make clearer the step in the proof of [BMO19, Lemma 1.5(b)] which reduces to the case in which $Z(G)$ is a p-group. The proof of the lemma is otherwise the same.

Lemma 5.3. *Let G be a finite group, p a prime, and S a Sylow p-subgroup of G. Assume κ_G is surjective and* $\ker(\kappa_G)$ *is a p'-group. Then there is $\widehat{G} \geqslant G/O_{p'}(G)$ such that $\kappa_{\widehat{G}}$ is split surjective and such that $\mathcal{F}_S(\widehat{G}) = \mathcal{F}_S(G)$. In particular, $\mathcal{F}_S(G)$ is tame, and it is tamely realized by $\widehat{G}$.*

Proposition 5.4. *Let $\mathcal{F}$ be a saturated fusion system over the p-group S. If $\mathcal{F} \cong \mathcal{F}_S(G)$ for some finite group G such that the map κ_G is surjective, then $\mathcal{F}$ is tame. Moreover, there is an extension $\widehat{G} \geqslant G/O_{p'}(G)$ of $G/O_{p'}(G)$ which tamely realizes $\mathcal{F}$.*

Proof. Fix such a G, let $\bar{G} = G/O_{p'}(G)$, and identify S also with its image in $\bar{G}$. Write $\mathcal{F} = \mathcal{F}_S(G)$, $\bar{\mathcal{F}} = \mathcal{F}_S(\bar{G})$, $\mathcal{L} = \mathcal{L}_S^c(G)$, and $\bar{\mathcal{L}} = \mathcal{L}_S^c(\bar{G})$. The canonical homomorphism $G \to \bar{G}$ induces isomorphisms $\mathcal{L} \to \bar{\mathcal{L}}$ and $\mathcal{F} \to \bar{\mathcal{F}}$. As in the proof of [AOV12, Lemma 2.19], there is a resulting commutative diagram

$$\begin{array}{ccc} \mathrm{Out}(G) & \longrightarrow & \mathrm{Out}(\bar{G}) \\ {\scriptstyle \kappa_G}\downarrow & & \downarrow{\scriptstyle \kappa_{\bar{G}}} \\ \mathrm{Out}(\mathcal{L}) & \xrightarrow{\cong} & \mathrm{Out}(\bar{\mathcal{L}}) \end{array}$$

As κ_G is surjective, also $\kappa_{\bar{G}}$ is surjective, so we may replace G by $\bar{G}$ and take $O_{p'}(G) = 1$. The result now follows from Theorem 5.1 and Lemma 5.3. ☐

In [Gla66b], the first author showed, for a core-free group G with Sylow 2-subgroup S, that the group $C_{\mathrm{Aut}(G)}(S)$ has abelian 2-subgroups and a normal 2-complement. The following proposition gives further information and a reinterpretation of that situation.

Proposition 5.5. *Let G be a finite group with Sylow p-subgroup S, let $\mathcal{L}$ be the centric linking system for G, and set $A = C_{\mathrm{Aut}(G)}(S)/C_{\mathrm{Inn}(G)}(S)$. If $O_{p'}(G) = 1$, then $A \cong O_{p'}(A) \rtimes B$ where $B = 1$ if p is odd, and where B is an elementary abelian 2-group if $p = 2$. The normal p-complement $O_{p'}(A)$ is the subgroup of $N_{\mathrm{Aut}(G)}(S)/N_{\mathrm{Inn}(G)}(S)$ consisting of those classes which have a representative that restricts to the identity on $\mathcal{L}$. In particular, κ_G is injective upon restriction to any Sylow p-subgroup of $\mathrm{Out}(G)$.*

Proof. The group A is the kernel of the composite $\mu_G \circ \kappa_G$, which is induced by restriction to S. By Theorem 1.2, the kernel of μ_G is either 1 or an elementary 2-group in the cases p odd or $p = 2$, respectively. So $\ker(\kappa_G) = O_{p'}(A)$ by Theorem 5.1. The last statement follows immediately. □

References

[AKO11] Michael Aschbacher, Radha Kessar, and Bob Oliver, *Fusion systems in algebra and topology*, London Mathematical Society Lecture Note Series, vol. 391, Cambridge University Press, Cambridge, 2011. MR 2848834

[AOV12] Kasper K. S. Andersen, Bob Oliver, and Joana Ventura, *Reduced, tame and exotic fusion systems*, Proc. Lond. Math. Soc. (3) **105** (2012), no. 1, 87–152. MR 2948790

[BLO03] Carles Broto, Ran Levi, and Bob Oliver, *The homotopy theory of fusion systems*, J. Amer. Math. Soc. **16** (2003), no. 4, 779–856 (electronic).

[BMO19] Carles Broto, Jesper Møller, and Bob Oliver, *Automorphisms of fusion systems of finite simple groups of Lie type*, Mem. Amer. Math. Soc. **262** (2019), no. 1267, iii+117. MR 4044462

[Che13] Andrew Chermak, *Fusion systems and localities*, Acta Math. **211** (2013), no. 1, 47–139. MR 3118305

[Che15] ———, *Finite localities II*, arXiv preprint arXiv:1505.08110 (2015).

[Che16] ———, *Finite localities III*, arXiv preprint arXiv:1610.06161 (2016).

[Cra11] David A. Craven, *Normal subsystems of fusion systems*, J. Lond. Math. Soc. (2) **84** (2011), no. 1, 137–158. MR 2819694

[GGLN20] George Glauberman, Robert Guralnick, Justin Lynd, and Gabriel Navarro, *Centers of Sylow subgroups and automorphisms*, Israel J. Math. **240** (2020), 253–266.

[GL16] George Glauberman and Justin Lynd, *Control of fixed points and existence and uniqueness of centric linking systems*, Invent. Math. **206** (2016), no. 2, 441–484. MR 3570297

[Gla66a] George Glauberman, *Central elements in core-free groups*, J. Algebra **4** (1966), 403–420. MR 0202822

[Gla66b] ———, *On the automorphism groups of a finite group having no non-identity normal subgroups of odd order*, Math. Z. **93** (1966), 154–160. MR 194503

[Gla68] ———, *Weakly closed elements of Sylow subgroups*, Math. Z. **107** (1968), 1–20. MR 0251141

[GLS98] Daniel Gorenstein, Richard Lyons, and Ronald Solomon, *The classification of the finite simple groups. Number 3. Part I. Chapter A*, Mathematical Surveys and Monographs, vol. 40, American Mathematical Society, Providence, RI, 1998, Almost simple K-groups. MR 1490581 (98j:20011)

[Gro82] Fletcher Gross, *Automorphisms which centralize a Sylow p-subgroup*, J. Algebra **77** (1982), no. 1, 202–233. MR 665174

[Hen19] Ellen Henke, *Subcentric linking systems*, Trans. Amer. Math. Soc. **371** (2019), no. 5, 3325–3373. MR 3896114

[LO02] Ran Levi and Bob Oliver, *Construction of 2-local finite groups of a type studied by Solomon and Benson*, Geom. Topol. **6** (2002), 917–990 (electronic).

[Oli04] Bob Oliver, *Equivalences of classifying spaces completed at odd primes*, Math. Proc. Cambridge Philos. Soc. **137** (2004), no. 2, 321–347.

[Oli06] ———, *Equivalences of classifying spaces completed at the prime two*, Mem. Amer. Math. Soc. **180** (2006), no. 848, vi+102pp.

[Oli13] ———, *Existence and uniqueness of linking systems: Chermak's proof via obstruction theory*, Acta Math. **211** (2013), no. 1, 141–175. MR 3118306

[OV07] Bob Oliver and Joana Ventura, *Extensions of linking systems with p-group kernel*, Math. Ann. **338** (2007), no. 4, 983–1043. MR 2317758 (2008k:55029)

[Pui06] Lluis Puig, *Frobenius categories*, J. Algebra **303** (2006), no. 1, 309–357.

Department of Mathematics, University of Chicago, 5734 S. University Ave, Chicago, IL 60637
Email address: gg@math.uchicago.edu

Department of Mathematics, University of Louisiana at Lafayette, Lafayette, LA 70504
Email address: lynd@louisiana.edu