# PUNCTURED GROUPS FOR EXOTIC FUSION SYSTEMS

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Abstract. The transporter systems of Oliver and Ventura and the localities of Chermak are classes of algebraic structures that model the $p$-local structures of finite groups. Other than the transporter categories and localities of finite groups, important examples include centric, quasicentric, and subcentric linking systems for saturated fusion systems. These examples are however not defined in general on the full collection of subgroups of the Sylow group. We study here *punctured groups*, a short name for transporter systems or localities on the collection of nonidentity subgroups of a finite $p$-group. As an application of the existence of a punctured group, we show that the subgroup homology decomposition on the centric collection is sharp for the fusion system. We also prove a Signalizer Functor Theorem for punctured groups and use it to show that the smallest Benson-Solomon exotic fusion system at the prime 2 has a punctured group, while the others do not. As for exotic fusion systems at odd primes $p$, we survey several classes and find that in almost all cases, either the subcentric linking system is a punctured group for the system, or the system has no punctured group because the normalizer of some subgroup of order $p$ is exotic. Finally, we classify punctured groups restricting to the centric linking system for certain fusion systems on extraspecial $p$-groups of order $p^3$.

## 1. Introduction

Let $\mathcal{F}$ be a fusion system over the finite $p$-group $S$. Thus, $\mathcal{F}$ is a category with objects the subgroups of $S$, and with morphisms injective group homomorphisms which contain among them the conjugation homomorphisms induced by elements of $S$ plus one more weak axiom. A fusion system is said to be saturated if it satisfies two stronger "saturation" axioms which were originally formulated by L. Puig [Pui06] and rediscovered by Broto, Levi, and Oliver [BLO03b]. Those axioms hold whenever $G$ is a finite group, $S$ is a Sylow $p$-subgroup of $G$, and $\mathrm{Hom}_{\mathcal{F}}(P,Q) = \mathrm{Hom}_G(P,Q)$ is the set of conjugation maps $c_g$ from $P$ to $Q$ that are induced by elements $g \in G$. The fusion system of a finite group is denoted $\mathcal{F}_S(G)$.

A saturated fusion system $\mathcal{F}$ is said to be *exotic* if it is not of the form $\mathcal{F}_S(G)$ for any finite group $G$ with Sylow $p$-subgroup $S$. The Benson-Solomon fusion systems at $p = 2$ form one family of examples of exotic fusion systems [LO02, AC10]. They are essentially the only known examples at the prime 2, and they are in some sense the oldest known examples, having been studied in the early 1970s by Solomon in the course of the classification of finite simple groups (although not with the more recent categorical framework in mind) [Sol74]. In contrast with the case $p = 2$, a fast-growing literature describes many exotic fusion systems on finite $p$-groups when $p$ is odd.

In replacing a group by its fusion system at a prime, one retains information about conjugation homomorphisms between $p$-subgroups, but otherwise loses information about the group elements themselves. It is therefore natural that a recurring theme throughout the study of saturated fusion

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systems is the question of how to "enhance" or "rigidify" a saturated fusion system to make it again more group-like, and also to study which fusion systems have such rigidifications.

The study of the existence and uniqueness of centric linking systems was a first instantiation of this theme of rigidifying saturated fusion systems. A centric linking system is an important extension category of a fusion system $\mathcal{F}$ which provides just enough algebraic information to recover a $p$-complete classifying space. For example, it recovers the homotopy type of the $p$-completion of $BG$ in the case where $\mathcal{F} = \mathcal{F}_S(G)$. Centric linking systems of finite groups are easily defined, and Oliver proved that the centric linking systems of finite groups are unique [Oli04, Oli06]. Then, Chermak proved that each saturated fusion system, possibly exotic, has a unique associated centric linking system [Che13]. A proof which does not rely on the classification of finite simple groups can be obtained through [Oli13, GL16].

More generally, there are at least two frameworks for considering extensions, or rigidifications, of saturated fusion systems: the *transporter systems* of Oliver and Ventura [OV07] and the *localities* of Chermak [Che13]. In particular, one can consider centric linking systems in either setting. While centric linking systems in either setting have a specific set of objects, the object sets in transporter systems and localities can be any conjugation-invariant collection of subgroups which is closed under passing to overgroups. The categories of transporter systems and isomorphisms and of localities and isomorphisms are equivalent [Che13, Appendix] and [GL21, Theorem 2.11]. However, depending on the intended application, it is sometimes advantageous to work in the setting of transporter systems, and sometimes in localities. The reader is referred to Section 2 for an introduction to localities and transporter systems.

In this paper we study *punctured groups*. These are transporter systems, or localities, with objects the *nonidentity* subgroups of a finite $p$-group $S$. To motivate the terminology, recall that every finite group $G$ with Sylow $p$-subgroup $S$ admits a transporter system $\mathcal{T}_S(G)$ whose objects are *all* subgroups of $S$ and $\mathrm{Mor}_{\mathcal{T}}(P,Q) = N_G(P,Q)$, the transporter set consisting of all $g \in G$ which conjugate $P$ into $Q$. Conversely, [OV07, Proposition 3.11] shows that a transporter system $\mathcal{T}$ whose set of objects consists of all the subgroups of $S$ is necessarily the transporter system $\mathcal{T}_S(G)$ where $G = \mathrm{Aut}\mathcal{T}(1)$, and the fusion system $\mathcal{F}$ with which $\mathcal{T}$ is associated is $\mathcal{F}_S(G)$. Thus, a punctured group $\mathcal{T}$ is a transporter system whose object set is missing the trivial subgroup, an object whose inclusion forces $\mathcal{T}$ to be the transporter system of a finite group.

If we consider localities rather than transporter systems, then the punctured group of $G$ is the *locality* $\mathcal{L}_{\mathscr{S}^*(S)}(G) \subseteq G$ consisting of those elements $g \in G$ which conjugate a nonidentity subgroup of $S$ back into $S$. This is equipped with the multivariable partial product $w := (g_1, \dots, g_n) \mapsto g_1 \cdots g_n$, defined only when each initial subword of the word $w$ conjugates some fixed nonidentity subgroup of $S$ back into $S$. Thus, the product is defined on words which correspond to sequences of composable morphisms in the transporter category $\mathcal{T}_S^*(G)$. See Definition 2.6 for more details.

By contrast with the existence and uniqueness of linking systems, we will see that punctured groups for exotic fusion systems do not necessarily exist. The existence of a punctured group for an exotic fusion system seems to indicate that the fusion system is "close to being realizable" in some sense. Therefore, considering punctured groups might provide some insight into how exotic systems arise.

It is also not reasonable to expect that a punctured group is unique when it does exist. To give one example, the fusion systems $PSL_2(q)$ with $q \equiv 9 \pmod{16}$ all have a single class of involutions and equivalent fusion systems at the prime 2. On the other hand, the centralizer of an involution is dihedral of order $2(q-1)$, so the associated punctured groups are distinct for distinct $q$. Examples like this one occur systematically in groups of Lie type in nondefining characteristic. Later will give examples of realizable fusion systems with punctured groups which do not occur as a full subcategory of the punctured group of a finite group.

We will now describe our results in detail. To start, we present a result which gives some motivation for studying punctured groups.

1.1. **Sharpness of the subgroup homology decomposition.** As an application of the existence of the structure of a punctured group for a saturated fusion system $\mathcal{F}$, we prove that it implies the sharpness of the subgroup homology decomposition for that system. Recall from [BLO03b, Definition 1.8] that given a $p$-local finite group $(S, \mathcal{F}, \mathcal{L})$ its classifying space is the Bousfield-Kan $p$-completion of the geometric realisation of the category $\mathcal{L}$. This space is denoted by $|\mathcal{L}|^\wedge_p$.

The orbit category of $\mathcal{F}$, see [BLO03b, Definition 2.1], is the category $\mathcal{O}(\mathcal{F})$ with the same objects as $\mathcal{F}$ and whose morphism sets $\mathrm{Mor}_{\mathcal{O}(\mathcal{F})}(P,Q)$ is the set of orbits of $\mathcal{F}(P,Q)$ under the action of $\mathrm{Inn}(Q)$. The full subcategory of the $\mathcal{F}$-centric subgroups is denoted $\mathcal{O}(\mathcal{F}^c)$. For every $j \geqslant 0$ there is a functor $\mathcal{H}^j \colon \mathcal{O}(\mathcal{F}^c)^{\mathrm{op}} \to \mathbb{Z}_{(p)}\text{-}\mathfrak{mod}$:

$$\mathcal{H}^j \colon P \mapsto H^j(P; \mathbb{F}_p), \qquad (P \in \mathcal{O}(\mathcal{F}^c)).$$

The stable element theorem for $p$-local finite groups [BLO03b, Theorem B, see also Theorem 5.8] asserts that for every $j \geqslant 0$,

$$H^j(|\mathcal{L}|^\wedge_p; \mathbb{F}_p) \cong \varprojlim_{\mathcal{O}(\mathcal{F}^c)} \mathcal{H}^j = \varprojlim_{P \in \mathcal{O}(\mathcal{F}^c)} H^j(P; \mathbb{F}_p).$$

The proof of this theorem in [BLO03b] is indirect and requires heavy machinery such as Lannes's $T$-functor theory. From the conceptual point of view, the stable element theorem is only a shadow of a more general phenomenon. By [BLO03b, Proposition 2.2] there is a functor

$$\tilde{B} \colon \mathcal{O}(\mathcal{F}^c) \to \mathrm{Top}$$

with the property that $\tilde{B}(P)$ is homotopy equivalent to the classifying space of $P$ (denoted $BP$) and moreover there is a natural homotopy equivalence

$$|\mathcal{L}| \simeq \operatorname*{hocolim}_{\mathcal{O}(\mathcal{F}^c)} \tilde{B}.$$

The Bousfield-Kan spectral sequence for this homotopy colimit [BK72, Ch. XII, Sec. 4.5] takes the form

$$E_2^{i,j} = \varprojlim_{\mathcal{O}(\mathcal{F}^c)^{\mathrm{op}}}{}^{i}\ \mathcal{H}^j \Rightarrow H^{i+j}(|\mathcal{L}|^\wedge_p; \mathbb{F}_p)$$

and is called the *subgroup decomposition* of $(S, \mathcal{F}, \mathcal{L})$. We call the subgroup decomposition *sharp*, see [Dwy98], if the spectral sequence collapses to the vertical axis, namely $E_2^{i,j} = 0$ for all $i > 0$. When this is the case, the stable element theorem is a direct consequence. Indeed, whenever $\mathcal{F}$ is induced from a finite group $G$ with a Sylow $p$-subgroup $S$, the subgroup decomposition is sharp (and the stable element theorem goes back to Cartan-Eilenberg [CE56, Theorem XII.10.1]). This follows immediately from Dwyer's work [Dwy98, Sec. 1.11] and [BLO03a, Lemma 1.3], see for example [DP15, Theorem B].

It is still an open question as to whether the subgroup decomposition is sharp for every saturated fusion system. We will prove the following theorem.

**Theorem 1.1.** *Let $\mathcal{F}$ be a saturated fusion system which affords the structure of a punctured group. Then the subgroup decomposition on the $\mathcal{F}$-centric subgroups is sharp. In other words,*

$$\varprojlim_{\mathcal{O}(\mathcal{F}^c)^{\mathrm{op}}}{}^{i}\ \mathcal{H}^j = 0$$

*for every $i \geqslant 1$ and $j \geqslant 0$.*

We will prove this theorem in Section 3 below. We remark that our methods apply to any functor $\mathcal{H}$ which in the language of [DP15] is the pullback of a Mackey functor on the orbit category of $\mathcal{F}$ denoted $\mathcal{O}(\mathcal{F})$ such that $\mathcal{H}(e) = 0$ where $e \leqslant S$ is the trivial subgroup. In the absence of applications in sight for this level of generality we have confined ourselves to the functors $\mathcal{H} = \mathcal{H}^j$.

1.2. **Signalizer functor theorem for punctured groups.** It is natural to ask for which exotic fusion systems punctured groups exist. We will answer this question for specific families of exotic fusion systems. As a tool for proving the non-existence of punctured groups we define and study signalizer functors for punctured groups thus mirroring a concept from finite group theory.

**Definition 1.2.** Let $(\mathcal{L}, \Delta, S)$ be a punctured group. If $P$ is a subgroup of $S$, write $\mathcal{I}_p(P)$ for the set of elements of $P$ of order $p$. A *signalizer functor of* $(\mathcal{L}, \Delta, S)$ *(on elements of order p)* is a map $\theta$ from $\mathcal{I}_p(S)$ to the set of subgroups of $\mathcal{L}$, which associates to each element $a \in \mathcal{I}_p(S)$ a normal $p'$-subgroup $\theta(a)$ of $C_{\mathcal{L}}(a)$ such that the following two conditions hold:

- (Conjugacy condition) $\theta(a^g) = \theta(a)^g$ for any $g \in \mathcal{L}$ and $a \in \mathcal{I}_p(S)$ such that $a^g$ is defined and an element of $S$.
- (Balance condition) $\theta(a) \cap C_{\mathcal{L}}(b) \leqslant \theta(b)$ for all $a, b \in \mathcal{I}_p(S)$ with $[a, b] = 1$.

Notice in the above definition that, since $(\mathcal{L}, \Delta, S)$ is a punctured group, for any $a \in S$, the normalizer $N_{\mathcal{L}}(\langle a\rangle)$ and thus also the centralizer $C_{\mathcal{L}}(a)$ is a subgroup.

**Theorem 1.3** (Signalizer functor theorem for punctured groups)**.** *Let $(\mathcal{L}, \Delta, S)$ be a punctured group and suppose $\theta$ is a signalizer functor of $(\mathcal{L}, \Delta, S)$ on elements of order p. Then*

$$\widehat{\Theta} := \bigcup_{x \in \mathcal{I}_p(S)} \theta(x)$$

*is a partial normal subgroup of $\mathcal{L}$ with $\widehat{\Theta} \cap S = 1$. In particular, the canonical projection $\rho\colon \mathcal{L} \to \mathcal{L}/\widehat{\Theta}$ restricts to an isomorphism $S \to S^\rho$. Upon identifying $S$ with $S^\rho$, the following properties hold:*

(a) $(\mathcal{L}/\widehat{\Theta}, \Delta, S)$ *is a locality and* $\mathcal{F}_S(\mathcal{L}/\widehat{\Theta}) = \mathcal{F}_S(\mathcal{L})$.
(b) *For each $P \in \Delta$, the projection $\rho$ restricts to an epimorphism $N_{\mathcal{L}}(P) \to N_{\mathcal{L}/\widehat{\Theta}}(P)$ with kernel $\Theta(P)$ and thus induces an isomorphism $N_{\mathcal{L}}(P)/\Theta(P) \cong N_{\mathcal{L}/\widehat{\Theta}}(P)$.*

1.3. **Punctured groups for families of exotic fusion systems.** Let $\mathcal{F}$ be a saturated fusion system on the $p$-group $S$. If $\mathcal{L}$ is a locality or transporter system associated with $\mathcal{F}$, then for each fully $\mathcal{F}$-normalized object $P$ of $\mathcal{L}$, the normalizer fusion system $N_{\mathcal{F}}(P)$ is the fusion system of the group $N_{\mathcal{L}}(P)$ if $\mathcal{L}$ is a locality, and of the group $\mathrm{Aut}_{\mathcal{L}}(P)$ if $\mathcal{L}$ is a transporter system. This gives an easy necessary condition for the existence of a punctured group: for each fully $\mathcal{F}$-normalized nonidentity subgroup $P \leqslant S$, the normalizer $N_{\mathcal{F}}(P)$ is realizable.

Conversely, there is a sufficient condition for the existence of a punctured group: $\mathcal{F}$ is of characteristic $p$-type, i.e. for each fully $\mathcal{F}$-normalized nonidentity subgroup $P \leqslant S$, the normalizer $N_{\mathcal{F}}(P)$ is constrained. This follows from the existence of linking systems (or similarly linking localities) of a very general kind, a result which was shown in [Hen19, Theorem A] building on the existence and uniqueness of centric linking systems.

The Benson-Solomon fusion systems $\mathcal{F}_{\mathrm{Sol}}(q)$ at the prime 2 have the property that the normalizer fusion system of each nonidentity subgroup $P$ is realizable, and moreover, $C_{\mathcal{F}}(Z(S))$ is the fusion system at $p = 2$ of $\mathrm{Spin}_7(q)$, and hence not constrained. So $\mathcal{F}_{\mathrm{Sol}}(q)$ satisfies the obvious necessary condition for the existence of a punctured group, and does not satisfy the sufficient one.

Building on results of Solomon [Sol74], Levi and Oliver showed that $\mathcal{F}_{\mathrm{Sol}}(q)$ is exotic [LO02, Theorem 3.4], i.e., it has no locality with objects *all* subgroups of a Sylow 2-group. In Section 4, we show the following theorem.

**Theorem 1.4.** *For any odd prime power $q$, the Benson-Solomon fusion system $\mathcal{F}_{\mathrm{Sol}}(q)$ has a punctured group if and only if $q \equiv \pm 3 \pmod 8$.*

If $l$ is the nonnegative integer with the property that $2^{l+3}$ is the 2-part of $q^2-1$, then $\mathcal{F}_{\mathrm{Sol}}(q) \cong \mathcal{F}_{\mathrm{Sol}}(3^{2^l})$. So the theorem says that only the smallest Benson-Solomon system, $\mathcal{F}_{\mathrm{Sol}}(3)$, has a punctured group, and the larger ones do not. Further details and a uniqueness statement are given in Theorem 4.1.

When showing the non-existence of a punctured group in the case $q \equiv \pm 1 \pmod 8$, the Signalizer Functor Theorem 1.3 plays an important role in showing that a putative minimal punctured group has no partial normal $p'$-subgroups. To construct a punctured group in the case $q \equiv \pm 3 \pmod 8$, we turn to a procedure we call Chermak descent. It is an important tool in Chermak's proof of the existence and uniqueness of centric linking systems [Che13, Section 5] and allows us (under some assumptions) to "expand" a given locality to produce a new locality with a larger object set. Starting with a linking locality, we use Chermak descent to construct a punctured group $\mathcal{L}$ for $\mathcal{F}_{\mathrm{Sol}}(q)$ in which the centralizer of an involution is $C_{\mathcal{L}}(Z(S)) \cong \mathrm{Spin}_7(3)$.

It is possible that there could be other examples of punctured groups for $\mathcal{F}_{\mathrm{Sol}}(3)$ in which the centralizer of an involution is $\mathrm{Spin}_7(q)$ for certain $q = 3^{1+6a}$; a necessary condition for existence is that each prime divisor of $q^2-1$ is a square modulo 7. However, given this condition, we can neither prove or disprove the existence of an example with the prescribed involution centralizer.

In Section 5, we survey a few families of known exotic fusion systems at odd primes to determine whether or not they have a punctured group. A summary of the findings is contained in Theorem 5.2. For nearly all the exotic systems we consider, either the system is of characteristic $p$-type, or the centralizer of some $p$-element is exotic and a punctured group does not exist.

In particular, when considering the family of Clelland-Parker systems [CP10] in which each essential subgroup is special, we find that $O^{p'}(C_{\mathcal{F}}(Z(S))/Z(S))$ is simple, exotic, and had not appeared elsewhere in the literature as of the time of our writing. We dedicate part of Subsection 5.3 to describing these systems and to proving that they are exotic.

Applying Theorem 1.1 to the results of Sections 4 and 5 establish the sharpness of the subgroup decomposition for new families of exotic fusion systems, notably

- Benson-Solomon's system $\mathcal{F}_{\mathrm{Sol}}(3)$ [LO02],
- all Parker-Stroth systems [PS15],
- all Clelland-Parker systems [CP10] in which each essential subgroup is abelian.

It also recovers the sharpness for certain fusion systems on $p$-groups with an abelian subgroup of index $p$, a result that was originally established in full generality by Dıaz and Park [DP15].

1.4. **Classification of punctured groups over $p_+^{1+2}$.** In general, it seems difficult to classify all the punctured groups associated to a given saturated fusion system. However, for fusion systems over an extraspecial $p$-group of exponent $p$, which by [RV04] are known to contain among them three exotic fusion systems at the prime 7, we are able to work out such an example. There is always a punctured group $\mathcal{L}$ associated to such a fusion system, and when $\mathcal{F}$ has one class of subgroups of order $p$ and the full subcategory of $\mathcal{L}$ with objects the $\mathcal{F}$-centric subgroups is the centric linking system, a classification is obtained in Theorem 6.3. Conversely, the cases we list in that theorem all occur in an example for a punctured group. This demonstrates on the one hand that there can be more than one punctured group associated to the same fusion system and indicates on the other hand that examples for punctured groups are still somewhat limited.

**Outline of the paper and notation.** The paper proceeds as follows. In Section 2 we recall the definitions and basic properties of transporter systems and localities, and we prove the Signalizer Functor Theorem in Subsection 2.8. In Section 3, we prove sharpness of the subgroup decomposition for fusion systems with associated punctured groups. Section 4 examines punctured groups for the Benson-Solomon fusion systems, while Section 5 looks at several families of exotic fusion systems at odd primes. Finally, in Section 6 classifies certain punctured groups over an extraspecial $p$-group of order $p^3$ and exponent $p$. An Appendix A sets notation and provides certain general results on finite groups of Lie type that are needed in Section 4.

Throughout most of the paper we write conjugation like maps on the right side of the argument and compose from left to right. There are two exceptions: when working with transporter systems, such as in Section 3, we compose morphisms from right to left. Also, we apply certain maps in Section 4 on the left of their arguments (e.g. roots, when viewed as characters of a torus). The notation for Section 4 is outlined in more detail in the appendix.

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## 2. Localities and transporter systems

As already mentioned in the introduction, transporter systems as defined by Oliver and Ventura [OV07] and localities in the sense of Chermak [Che13] are algebraic structures which carry essentially the same information. In this section, we will give an introduction to both subjects and outline briefly the connection between localities and transporter systems. At the end we present some signalizer functor theorems for localities.

2.1. **Partial groups.** We refer the reader to Chermak's papers [Che13] or [Che15] for a detailed introduction to partial groups and localities. However, we will briefly summarize the most important definitions and results here. Following Chermak's notation, we write $\mathbf{W}(\mathcal{L})$ for the set of words in a set $\mathcal{L}$, and $\varnothing$ for the empty word. The concatenation of words $u_1,\dots,u_k\in\mathbf{W}(\mathcal{L})$ is denoted by $u_1\circ u_2\circ\cdots\circ u_k$.

**Definition 2.1** (Partial Group)**.** Let $\mathcal{L}$ be a non-empty set, let $\mathbf{D}$ be a subset of $\mathbf{W}(\mathcal{L})$, let $\Pi\colon \mathbf{D}\to\mathcal{L}$ be a map and let $(-)^{-1}\colon\mathcal{L}\to\mathcal{L}$ be an involutory bijection, which we extend to a map

$$(-)^{-1}\colon \mathbf{W}(\mathcal{L})\to\mathbf{W}(\mathcal{L}), w=(g_1,\dots,g_k)\mapsto w^{-1}=(g_k^{-1},\dots,g_1^{-1}).$$

We say that $\mathcal{L}$ is a *partial group* with product $\Pi$ and inversion $(-)^{-1}$ if the following hold:

- $\mathcal{L}\subseteq\mathbf{D}$ (i.e. $\mathbf{D}$ contains all words of length 1), and
$$u\circ v\in\mathbf{D}\Longrightarrow u,v\in\mathbf{D};$$
(so in particular, $\varnothing\in\mathbf{D}$.)
- $\Pi$ restricts to the identity map on $\mathcal{L}$;
- $u\circ v\circ w\in\mathbf{D}\Longrightarrow u\circ(\Pi(v))\circ w\in\mathbf{D}$, and $\Pi(u\circ v\circ w)=\Pi(u\circ(\Pi(v))\circ w)$;
- $w\in\mathbf{D}\Longrightarrow w^{-1}\circ w\in\mathbf{D}$ and $\Pi(w^{-1}\circ w)=\mathbf{1}$ where $\mathbf{1}:=\Pi(\varnothing)$.

Note that any group $G$ can be regarded as a partial group with product defined in $\mathbf{D}=\mathbf{W}(G)$ by extending the "binary" product to a map $\mathbf{W}(G)\to G, (g_1,g_2,\dots,g_n)\mapsto g_1g_2\cdots g_n$.

**For the remainder of this section let $\mathcal{L}$ be a partial group with product $\Pi\colon\mathbf{D}\to\mathcal{L}$ defined on the domain $\mathbf{D}\subseteq\mathbf{W}(\mathcal{L})$.**

Because of the group-like structure of partial groups, the product $\mathcal{X}\mathcal{Y}$ of two subsets $\mathcal{X}$ and $\mathcal{Y}$ of $\mathcal{L}$ is naturally defined by

$$\mathcal{X}\mathcal{Y}:=\{\Pi(x,y)\colon x\in\mathcal{X},\ y\in\mathcal{Y}\text{ such that }(x,y)\in\mathbf{D}\}.$$

Similarly, there is a natural notion of conjugation, which we consider next.

**Definition 2.2.** For every $g\in\mathcal{L}$ we define

$$\mathbf{D}(g)=\{x\in\mathcal{L}\mid(g^{-1},x,g)\in\mathbf{D}\}.$$

The map $c_g\colon\mathbf{D}(g)\to\mathcal{L}$, $x\mapsto x^g=\Pi(g^{-1},x,g)$ is the *conjugation map* by $g$. If $\mathcal{H}$ is a subset of $\mathcal{L}$ and $\mathcal{H}\subseteq\mathbf{D}(g)$, then we set

$$\mathcal{H}^g=\{h^g\mid h\in\mathcal{H}\}.$$

Whenever we write $x^g$ (or $\mathcal{H}^g$), we mean implicitly that $x \in \mathbf{D}(g)$ (or $\mathcal{H} \subseteq \mathbf{D}(g)$, respectively). Moreover, if $\mathcal{M}$ and $\mathcal{H}$ are subsets of $\mathcal{L}$, we write $N_{\mathcal{M}}(\mathcal{H})$ for the set of all $g \in \mathcal{M}$ such that $\mathcal{H} \subseteq \mathbf{D}(g)$ and $\mathcal{H}^g = \mathcal{H}$. Similarly, we write $C_{\mathcal{M}}(\mathcal{H})$ for the set of all $g \in \mathcal{M}$ such that $\mathcal{H} \subseteq \mathbf{D}(g)$ and $h^g = h$ for all $h \in \mathcal{H}$. If $\mathcal{M} \subseteq \mathcal{L}$ and $h \in \mathcal{L}$, set $C_{\mathcal{M}}(h) := C_{\mathcal{M}}(\{h\})$.

**Definition 2.3.** Let $\mathcal{H}$ be a non-empty subset of $\mathcal{L}$. The subset $\mathcal{H}$ is a *partial subgroup* of $\mathcal{L}$ if

- $g \in \mathcal{H} \Longrightarrow g^{-1} \in \mathcal{H}$; and
- $w \in \mathbf{D} \cap \mathbf{W}(\mathcal{H}) \Longrightarrow \Pi(w) \in \mathcal{H}$.

If $\mathcal{H}$ is a partial subgroup of $\mathcal{L}$ with $\mathbf{W}(\mathcal{H}) \subseteq \mathbf{D}$, then $\mathcal{H}$ is called a *subgroup* of $\mathcal{L}$.

A partial subgroup $\mathcal{N}$ of $\mathcal{L}$ is called a *partial normal subgroup* of $\mathcal{L}$ (denoted $\mathcal{N} \trianglelefteq \mathcal{L}$) if for all $g \in \mathcal{L}$ and $n \in \mathcal{N}$,

$$n \in \mathbf{D}(g) \Longrightarrow n^g \in \mathcal{N}.$$

We remark that a subgroup $\mathcal{H}$ of $\mathcal{L}$ is always a group in the usual sense with the group multiplication defined by $hg = \Pi(h, g)$ for all $h, g \in \mathcal{H}$.

2.2. **Localities.** Roughly speaking, localities are partial groups with some some extra structure, in particular with a "Sylow $p$-subgroup" and a set $\Delta$ of "objects" which in a certain sense determines the domain of the product. This is made more precise in Definition 2.5 below. We continue to assume that $\mathcal{L}$ is a partial group with product $\Pi\colon \mathbf{D} \to \mathcal{L}$. We will use the following notation.

**Notation 2.4.** If $S$ is a subset of $\mathcal{L}$ and $g \in \mathcal{L}$, set

$$S_g := \{s \in S \cap \mathbf{D}(g)\colon s^g \in S\}.$$

More generally, if $w = (g_1, \dots, g_n) \in \mathbf{W}(\mathcal{L})$ with $n \geqslant 1$, define $S_w$ to be the set of elements $s \in S$ for which there exists a sequence of elements $s = s_0, s_1, \dots, s_n \in S$ with $s_{i-1} \in \mathbf{D}(g_i)$ and $s_{i-1}^{g_i} = s_i$ for all $i = 1, \dots, n$.

**Definition 2.5.** We say that $(\mathcal{L}, \Delta, S)$ is a *locality* if the partial group $\mathcal{L}$ is finite as a set, $S$ is a $p$-subgroup of $\mathcal{L}$, $\Delta$ is a non-empty set of subgroups of $S$, and the following conditions hold:

(L1) $S$ is maximal with respect to inclusion among the $p$-subgroups of $\mathcal{L}$.

(L2) For any word $w = (f_1, \dots, f_n) \in \mathbf{W}(\mathcal{L})$, we have $w \in \mathbf{D}$ if and only if there exist $P_0, \dots, P_n \in \Delta$ with
  ($*$) $P_{i-1} \subseteq \mathbf{D}(f_i)$ and $P_{i-1}^{f_i} = P_i$ for all $i = 1, \dots, n$.

(L3) The set $\Delta$ is closed under passing to $\mathcal{L}$-conjugates and overgroups in $S$, i.e. $\Delta$ is overgroup-closed in $S$ and, for every $P \in \Delta$ and $g \in \mathcal{L}$ such that $P \subseteq S_g$, we have $P^g \in \Delta$.

If $(\mathcal{L}, \Delta, S)$ is a locality, $w = (f_1, \dots, f_n) \in \mathbf{W}(\mathcal{L})$, and $P_0, \dots, P_n$ are elements of $\Delta$ such that ($*$) holds, then we say that $w \in \mathbf{D}$ via $P_0, \dots, P_n$ (or $w \in \mathbf{D}$ via $P_0$).

It is argued in [Hen19, Remark 5.2] that Definition 2.5 is equivalent to the definition of a locality given by Chermak [Che15, Definition 2.8] (which is essentially the same as the one given in [Che13, Definition 2.9]).

**Example 2.6.** Let $M$ be a finite group and $S \in \mathrm{Syl}_p(M)$. Set $\mathcal{F} = \mathcal{F}_S(M)$ and let $\Delta$ be a non-empty collection of subgroups of $S$, which is closed under $\mathcal{F}$-conjugacy and overgroup-closed in $S$. Set

$$\mathcal{L}_\Delta(M) := \{g \in G\colon S \cap S^g \in \Delta\} = \{g \in G\colon \exists P \in \Delta \text{ with } P^g \leqslant S\}$$

and let $\mathbf{D}$ be the set of tuples $(g_1, \ldots, g_n) \in \mathbf{W}(M)$ such that there exist $P_0, P_1, \ldots, P_n \in \Delta$ with $P_{i-1}^{g_i} = P_i$. Then $\mathcal{L}_\Delta(M)$ forms a partial group whose product is the restriction of the multivariable product on $M$ to $\mathbf{D}$, and whose inversion map is the restriction of the inversion map on the group $M$ to $\mathcal{L}_\Delta(M)$. Moreover, $(\mathcal{L}_\Delta(M), \Delta, S)$ forms a locality.

In the next lemma we summarize the most important properties of localities which we will use throughout, most of the time without reference.

**Lemma 2.7** (Important properties of localities)**.** *Let $(\mathcal{L}, \Delta, S)$ be a locality. Then the following hold:*

(a) *$N_\mathcal{L}(P)$ is a subgroup of $\mathcal{L}$ for each $P \in \Delta$.*
(b) *Let $P \in \Delta$ and $g \in \mathcal{L}$ with $P \subseteq S_g$. Then $Q := P^g \in \Delta$, $N_\mathcal{L}(P) \subseteq \mathbf{D}(g)$ and*

$$c_g \colon N_\mathcal{L}(P) \to N_\mathcal{L}(Q), x \mapsto x^g$$

*is an isomorphism of groups.*
(c) *Let $w = (g_1, \ldots, g_n) \in \mathbf{D}$ via $(X_0, \ldots, X_n)$. Then*

$$c_{g_1} \circ \cdots \circ c_{g_n} = c_{\Pi(w)}$$

*is a group isomorphism $N_\mathcal{L}(X_0) \to N_\mathcal{L}(X_n)$.*
(d) *For every $g \in \mathcal{L}$, we have $S_g \in \Delta$. In particular, $S_g$ is a subgroup of $S$. Moreover, $S_g^g = S_{g^{-1}}$ and $c_g \colon S_g \to S, x \mapsto x^g$ is an injective group homomorphism.*
(e) *For every $g \in \mathcal{L}$, $c_g \colon \mathbf{D}(g) \to \mathbf{D}(g^{-1}), x \mapsto x^g$ is a bijection with inverse map $c_{g^{-1}}$.*
(f) *For any $w \in \mathbf{W}(\mathcal{L})$, $S_w$ is a subgroup of $S$ with $S_w \in \Delta$ if and only if $w \in \mathbf{D}$. Moreover, $w \in \mathbf{D}$ implies $S_w \leqslant S_{\Pi(w)}$.*

*Proof.* Properties (a),(b) and (c) correspond to the statements (a),(b) and (c) in [Che15, Lemma 2.3] except for the fact stated in (b) that $Q \in \Delta$, which is however clearly true if one uses the definition of a locality given above. Property (d) holds by [Che15, Proposition 2.6(a),(b)] and property (e) is stated in [Che13, Lemma 2.5(c)]. Property (f) corresponds to Corollary 2.7 in [Che15]. □

Let $(\mathcal{L}, \Delta, S)$ be a locality. Then it follows from Lemma 2.7(d) that, for every $P \in \Delta$ and every $g \in \mathcal{L}$ with $P \subseteq S_g$, the map $c_g \colon P \to P^g, x \mapsto x^g$ is an injective group homomorphism. The fusion system $\mathcal{F}_S(\mathcal{L})$ is the fusion system over $S$ generated by such conjugation maps. Equivalently, $\mathcal{F}_S(\mathcal{L})$ is generated by the conjugation maps between subgroups of $S$, or by the conjugation maps of the form $c_g \colon S_g \to S, x \mapsto x^g$ with $g \in \mathcal{L}$.

**Definition 2.8.** If $\mathcal{F}$ is a fusion system, then we say that the locality $(\mathcal{L}, \Delta, S)$ is a locality over $\mathcal{F}$ if $\mathcal{F} = \mathcal{F}_S(\mathcal{L})$.

If $(\mathcal{L}, \Delta, S)$ is a locality over $\mathcal{F}$, then notice that the set $\Delta$ is always overgroup-closed in $S$ and closed under $\mathcal{F}$-conjugacy.

**Lemma 2.9.** *Let $(\mathcal{L}, \Delta, S)$ be a locality over a fusion system $\mathcal{F}$ and $P \in \Delta$. Then the following hold:*

(a) *For every $\varphi \in \mathrm{Hom}_\mathcal{F}(P, S)$, there exists $g \in \mathcal{L}$ such that $P \leqslant S_g$ and $\varphi(x) = x^g$ for all $x \in P$.*
(b) *$N_\mathcal{F}(P) = \mathcal{F}_{N_S(P)}(N_\mathcal{L}(P))$.*

*Proof.* For (a) see Lemma 5.6 and for (b) see Lemma 5.4 in [Hen19]. □

Suppose $(\mathcal{L}^+, \Delta^+, S)$ is a locality with partial product $\Pi^+ \colon \mathbf{D}^+ \to \mathcal{L}^+$. If $\Delta$ is a non-empty subset of $\Delta^+$ which is closed under taking $\mathcal{L}^+$-conjugates and overgroups in $S$, we set

$$\mathcal{L}^+|_\Delta := \{f \in \mathcal{L}^+ \colon \exists P \in \Delta \text{ such that } P \subseteq \mathbf{D}^+(f) \text{ and } P^f \leqslant S\}$$

and write $\mathbf{D}$ for the set of words $w = (f_1, \dots, f_n)$ such that $w \in \mathbf{D}^+$ via $P_0, \dots, P_n$ for some $P_0, \dots, P_n \in \Delta$. Note that $\mathbf{D}$ is a set of words in $\mathcal{L}^+|_\Delta$ which is contained in $\mathbf{D}^+$. It is easy to check that $\mathcal{L}^+|_\Delta$ forms a partial group with partial product $\Pi^+|_{\mathbf{D}} \colon \mathbf{D} \to \mathcal{L}^+|_\Delta$, and that $(\mathcal{L}^+|_\Delta, \Delta, S)$ forms a locality; see [Che13, Lemma 2.21] for details. We call $\mathcal{L}^+|_\Delta$ the *restriction* of $\mathcal{L}^+$ to $\Delta$.

2.3. **Morphisms of localities.** Throughout this subsection let $\mathcal{L}$ and $\mathcal{L}'$ be partial groups with products $\Pi \colon \mathbf{D} \to \mathcal{L}$ and $\Pi' \colon \mathbf{D}' \to \mathcal{L}'$ respectively.

**Definition 2.10.** Let $\beta \colon \mathcal{L} \to \mathcal{L}', g \mapsto g^\beta$ be a map. By abuse of notation, we denote by $\beta$ also the induced map on words

$$\mathbf{W}(\mathcal{L}) \to \mathbf{W}(\mathcal{L}'), \quad w = (f_1, \dots, f_n) \mapsto w^\beta = (f_1^\beta, \dots, f_n^\beta)$$

and set $\mathbf{D}^\beta = \{w^\beta \colon w \in \mathbf{D}\}$. We say that $\beta$ is a homomorphism of partial groups if

(1) $\mathbf{D}^\beta \subseteq \mathbf{D}'$; and
(2) $\Pi(w)^\beta = \Pi'(w^\beta)$ for every $w \in \mathbf{D}$.

If moreover $\mathbf{D}^\beta = \mathbf{D}'$ (and thus $\beta$ is in particular surjective), then we say that $\beta$ is a *projection* of partial groups. If $\beta$ is a bijective projection of partial groups, then $\beta$ is called an *isomorphism*.

**Definition 2.11.** Let $(\mathcal{L}, \Delta, S)$ and $(\mathcal{L}', \Delta', S')$ be localities and let $\beta \colon \mathcal{L} \to \mathcal{L}'$ be a projection of partial groups. We say that $\beta$ is a *projection of localities* from $(\mathcal{L}, \Delta, S)$ to $(\mathcal{L}', \Delta', S')$ if, setting $\Delta^\beta = \{P^\beta \mid P \in \Delta\}$, we have $\Delta^\beta = \Delta'$ (and thus $S^\beta = S'$).

If $\beta$ is in addition bijective, then $\beta$ is a called an *isomorphism* of localities. If $S = S'$, then an isomorphism of localities from $(\mathcal{L}, \Delta, S)$ to $(\mathcal{L}', \Delta', S)$ is called a *rigid isomorphism* if it restricts to the identity on $S$.

The notion of a rigid isomorphism will be important later on when talking about the uniqueness of certain localities attached to a given fusion system. We will now describe some naturally occurring projections of localities. Suppose $(\mathcal{L}, \Delta, S)$ is a locality and $\mathcal{N}$ is a partial normal subgroup of $\mathcal{L}$. A *coset* of $\mathcal{N}$ in $\mathcal{L}$ is a subset of the form

$$\mathcal{N}f := \{\Pi(n, f) \colon n \in \mathcal{N} \text{ such that } (n, f) \in \mathbf{D}\}$$

for some $f \in \mathcal{L}$. Unlike in groups, the set of cosets does not form a partition of $\mathcal{L}$ in general. Instead, one needs to focus on the *maximal cosets*, i.e. the elements of the set of cosets of $\mathcal{N}$ in $\mathcal{L}$ which are maximal with respect to inclusion. The set $\mathcal{L}/\mathcal{N}$ of maximal cosets of $\mathcal{N}$ in $\mathcal{L}$ forms a partition of $\mathcal{L}$. Thus, there is a natural map

$$\beta \colon \mathcal{L} \to \mathcal{L}/\mathcal{N}$$

sending each element $g \in \mathcal{L}$ to the unique maximal coset of $\mathcal{N}$ in $\mathcal{L}$ containing $g$. Set $\overline{\mathcal{L}} := \mathcal{L}/\mathcal{N}$ and $\overline{\mathbf{D}} := \mathbf{D}^\beta := \{w^\beta \colon w \in \mathbf{D}\}$. By [Che15, Lemma 3.16], there is a unique map $\overline{\Pi} \colon \overline{\mathbf{D}} \to \overline{\mathcal{L}}$ and a unique involutory bijection $\overline{\mathcal{L}} \to \overline{\mathcal{L}}, \overline{f} \mapsto \overline{f}^{-1}$ such that $\overline{\mathcal{L}}$ with these structures is a partial group, and such that $\beta$ is a projection of partial groups. Moreover, setting $\overline{S} := S^\beta$ and $\overline{\Delta} := \{P^\beta \colon P \in \Delta\}$, the triple $(\overline{\mathcal{L}}, \overline{\Delta}, \overline{S})$ is by [Che15, Corollary 4.5] a locality, and $\beta$ is a projection from $(\mathcal{L}, \Delta, S)$ to $(\overline{\mathcal{L}}, \overline{\Delta}, \overline{S})$. The map $\beta$ is called the *natural projection* from $\mathcal{L} \to \overline{\mathcal{L}}$.

The notation used above suggests already that we will use a "bar notation" similar to the one commonly used in finite groups. Namely, if we set $\overline{\mathcal{L}} := \mathcal{L}/\mathcal{N}$, then for every subset or element $P$ of $\mathcal{L}$, we will denote by $\overline{P}$ the image of $P$ under the natural projection $\beta\colon \mathcal{L} \to \overline{\mathcal{L}}$. We conclude this section with a little lemma needed later on.

**Lemma 2.12.** *Let $(\mathcal{L}, \Delta, S)$ be a locality with partial normal subgroup $\mathcal{N}$. Setting $\overline{\mathcal{L}} := \mathcal{L}/\mathcal{N}$, the preimage of $\overline{S}$ under the natural projection equals $\mathcal{N}S$.*

*Proof.* For every $s \in S$, the coset $\mathcal{N}s$ is by Lemma 3.7(a) and Proposition 3.14(c) in [Che15] maximal. Thus, for every $s \in S$, we have $\overline{s} = \mathcal{N}s$. Hence, the preimage of $\overline{S} = \{\overline{s} \colon s \in S\}$ equals $\bigcup_{s \in S} \mathcal{N}s = \mathcal{N}S$. □

2.4. **Transporter systems.** Throughout this section, fix a finite $p$-group $S$, a fusion system $\mathcal{F}$ over $S$, and a collection $\Delta$ of nonidentity subgroups of $S$ which is overgroup-closed in $S$ and closed under $\mathcal{F}$-conjugacy. As the literature about transporter systems is written in left-hand notation, in this section, we will also *write our maps on the left hand side of the argument.* Accordingly we will conjugate from the left.

The *transporter category* $\mathcal{T}_S(G)$ (at the prime $p$) of a finite group $G$ with Sylow $p$-subgroup $S$ is the category with objects the nonidentity subgroups of $S$ and with morphisms given by the transporter sets $N_G(P, Q) = \{g \in G \mid {}^gP \leqslant Q\}$. More precisely, the morphisms in $\mathcal{T}_S(G)$ between $P$ and $Q$ are the triples $(g, P, Q)$ with $g \in N_G(P, Q)$. We also write $\mathcal{T}_\Delta(G)$ for the full subcategory of $\mathcal{T}_S(G)$ with objects in $\Delta$.

Since we conjugate in this section from the left, for $P, Q \leqslant S$ and $g \in N_G(P, Q)$, we write $c_g$ for the conjugation map from $P$ to $Q$ given by *left conjugation*, i.e.

$$c_g \colon P \to Q,\ x \mapsto {}^gx.$$

**Definition 2.13.** [OV07] A *transporter system* associated to $\mathcal{F}$ is a nonempty finite category $\mathcal{T}$ having object set $\Delta$ together with functors

$$\mathcal{T}_\Delta(S) \xrightarrow{\ \epsilon\ } \mathcal{T} \xrightarrow{\ \rho\ } \mathcal{F}$$

satisfying the following axioms.

(A1) $\epsilon$ is the identity on objects and $\rho$ is the inclusion on objects;

(A2) For each $P, Q \in \Delta$, the kernel

$$E(P) := \ker(\rho_{P,P} \colon \mathrm{Aut}_\mathcal{T}(P) \longrightarrow \mathrm{Aut}_\mathcal{F}(P))$$

acts freely on $\mathrm{Mor}_\mathcal{T}(P, Q)$ by right composition, and $\rho_{P,Q}$ is the orbit map for this action. Also, $E(Q)$ acts freely on $\mathrm{Mor}_\mathcal{T}(P, Q)$ by left composition.

(B) For each $P, Q \in \Delta$, $\epsilon_{P,Q} \colon N_S(P, Q) \to \mathrm{Mor}_\mathcal{T}(P, Q)$ is injective, and the composite $\rho_{P,Q} \circ \epsilon_{P,Q}$ sends $s \in N_S(P, Q)$ to $c_s \in \mathrm{Hom}_\mathcal{F}(P, Q)$.

(C) For all $\varphi \in \mathrm{Mor}_{\mathcal{T}}(P, Q)$ and all $g \in P$, the diagram

$$\begin{tikzcd} P \arrow[r, "\varphi"] \arrow[d, "\epsilon_{P,P}(g)"'] & Q \arrow[d, "\epsilon_{Q,Q}(\rho(\varphi)(g))"] \\ P \arrow[r, "\varphi"'] & Q \end{tikzcd}$$

commutes in $\mathcal{T}$.

(I) $\epsilon_{S,S}(S)$ is a Sylow $p$-subgroup of $\mathrm{Aut}_{\mathcal{T}}(S)$.

(II) Let $\varphi \in \mathrm{Iso}_{\mathcal{T}}(P, Q)$, let $P \trianglelefteq \bar{P} \leqslant S$, and let $Q \trianglelefteq \bar{Q} \leqslant S$ be such that $\varphi \circ \epsilon_{P,P}(\bar{P}) \circ \varphi^{-1} \leqslant \epsilon_{Q,Q}(\bar{Q})$. Then there exists $\bar{\varphi} \in \mathrm{Mor}_{\mathcal{T}}(\bar{P}, \bar{Q})$ such that $\bar{\varphi} \circ \epsilon_{P,\bar{P}}(1) = \epsilon_{Q,\bar{Q}}(1) \circ \varphi$.

If we want to be more precise, we say that $(\mathcal{T}, \epsilon, \rho)$ is a transporter system.

A centric linking system in the sense of [BLO03b] is a transporter system in which $\Delta$ is the set of $\mathcal{F}$-centric subgroups and $E(P)$ is precisely the center $Z(P)$ viewed as a subgroup of $N_S(P)$ via the map $\epsilon_{P,P}$. A more general notion of linking systems will be introduced in Subsection 2.6.

**Definition 2.14.** Let $(\mathcal{T}, \epsilon, \rho)$ and $(\mathcal{T}', \epsilon', \rho')$ be transporter systems. An equivalence of categories $\alpha\colon \mathcal{T} \to \mathcal{T}'$ is called an *isomorphism* if

- $\alpha_{P,P}(\epsilon_{P,P}(P)) = \epsilon'_{\alpha(P),\alpha(P)}(\alpha(P))$ for all objects $P$ of $\mathcal{T}$, and
- $\alpha_{P,Q}(\epsilon_{P,Q}(1)) = \epsilon'_{\alpha(P),\alpha(Q)}(1)$ for all objects $P, Q$ of $\mathcal{T}$.

2.5. **The correspondence between transporter systems and localities.** Throughout this subsection let $\mathcal{F}$ be a fusion system over $S$.

Every locality $(\mathcal{L}, \Delta, S)$ over $\mathcal{F}$ leads to a transporter system associated to $\mathcal{F}$. To see that we need to consider conjugation from the *left*. If $f, x \in \mathcal{L}$ such that $(f, x, f^{-1}) \in \mathbf{D}$ (or equivalently $x \in \mathbf{D}(f^{-1})$), then we set ${}^{f}x := \Pi(f, x, f^{-1}) = x^{f^{-1}}$. Similarly, if $f \in \mathcal{L}$ and $\mathcal{H} \subseteq \mathbf{D}(f^{-1})$, then set

$${}^{f}\mathcal{H} := \mathcal{H}^{f^{-1}} := \{{}^{f}x\colon x \in \mathcal{H}\}.$$

Define $\mathcal{T}_\Delta(\mathcal{L})$ to be the category whose object set is $\Delta$ with the morphism set $\mathrm{Mor}_{\mathcal{T}_\Delta(\mathcal{L})}(P, Q)$ between two objects $P, Q \in \Delta$ given as the set of triples $(f, P, Q)$ with $f \in \mathcal{L}$ such that $P \subseteq \mathbf{D}(f^{-1})$ and ${}^{f}P \leqslant Q$. This leads to a transporter system $(\mathcal{T}_\Delta(\mathcal{L}), \epsilon, \rho)$, where for all $P, Q \in \Delta$, $\epsilon_{P,Q}$ is the inclusion map and $\rho_{P,Q}$ sends $(f, P, Q)$ to the conjugation map $P \to Q$, $x \mapsto {}^{f}x$.

Conversely, Chermak showed in [Che13, Appendix] essentially that every transporter system leads to a locality. More precisely, it is proved in [GL21, Theorem 2.11] that there is an equivalence of categories between the category of transporter systems with morphisms the isomorphisms and the category of localities with morphisms the rigid isomorphisms. The definition of a locality in [GL21] differs slightly from the one given in this paper, but the two definitions can be seen to be equivalent if one uses firstly that conjugation by $f \in \mathcal{L}$ from the left corresponds to conjugation by $f^{-1}$ from the right, and secondly that for every partial group $\mathcal{L}$ with product $\Pi\colon \mathbf{D} \to \mathcal{L}$ the axioms of a partial group yield $\mathbf{D} = \{w \in \mathbf{W}(\mathcal{L})\colon w^{-1} \in \mathbf{D}\}$.

We will consider punctured groups in either setting thus using the term “punctured group” slightly abusively.

**Definition 2.15.** We call a transporter system $\mathcal{T}$ over $\mathcal{F}$ a *punctured group* if the object set of $\mathcal{T}$ equals the set of all non-identity subgroups. Similarly, a locality $(\mathcal{L}, \Delta, S)$ over $\mathcal{F}$ is said to be a *punctured group* if $\Delta$ is the set of all non-identity subgroups of $S$.

Observe that a transporter system over $\mathcal{F}$ which is a punctured group exists if and only if a locality over $\mathcal{F}$ which is a punctured group exists. If it matters it will always be clear from the context whether we mean by a punctured group a transporter system or a locality.

2.6. **Linking localities and linking systems.** As we have seen in the previous subsection, localities correspond to transporter systems. Of fundamental importance in the theory of fusion systems are (centric) linking systems, which are special cases of transporter systems. It is therefore natural to look at localities corresponding to linking systems. Thus, we will introduce special kinds of localities called linking localities. We will moreover introduce a (slightly non-standard) definition of linking systems and summarize some of the most important results about the existence and uniqueness of linking systems and linking localities. Throughout this subsection let $\mathcal{F}$ be a saturated fusion system over $S$.

We refer the reader to [AKO11] for the definitions of $\mathcal{F}$-centric and $\mathcal{F}$-centric radical subgroups denoted by $\mathcal{F}^c$ and $\mathcal{F}^{cr}$ respectively. Moreover, we will use the following definition which was introduced in [Hen19].

**Definition 2.16.** A subgroup $P \leqslant S$ is called $\mathcal{F}$-subcentric if $O_p(N_{\mathcal{F}}(Q))$ is centric for every fully $\mathcal{F}$-normalized $\mathcal{F}$-conjugate $Q$ of $P$. The set of subcentric subgroups is denoted by $\mathcal{F}^s$.

Recall that $\mathcal{F}$ is called constrained if there is an $\mathcal{F}$-centric normal subgroup of $\mathcal{F}$. It is shown in [Hen19, Lemma 3.1] that a subgroup $P \leqslant S$ is $\mathcal{F}$-subcentric if and only if for some (and thus for every) fully $\mathcal{F}$-normalized $\mathcal{F}$-conjugate $Q$ of $P$, the normalizer $N_{\mathcal{F}}(Q)$ is constrained.

**Definition 2.17.**
- A finite group $G$ is said to be of *characteristic* $p$ if $C_G(O_p(G)) \leqslant O_p(G)$.
- Define a locality $(\mathcal{L}, \Delta, S)$ to be of *objective characteristic* $p$ if, for any $P \in \Delta$, the group $N_{\mathcal{L}}(P)$ is of characteristic $p$.
- A locality $(\mathcal{L}, \Delta, S)$ over $\mathcal{F}$ is called a *linking locality*, if $\mathcal{F}^{cr} \subseteq \Delta$ and $(\mathcal{L}, \Delta, S)$ is of objective characteristic $p$.
- A *subcentric linking locality* over $\mathcal{F}$ is a linking locality $(\mathcal{L}, \mathcal{F}^s, S)$ over $\mathcal{F}$. Similarly, a *centric linking locality* over $\mathcal{F}$ is a linking locality $(\mathcal{L}, \mathcal{F}^c, S)$ over $\mathcal{F}$.

If $(\mathcal{L}, \Delta, S)$ is a centric linking locality, then it is shown in [Hen19, Proposition 1] that the corresponding transporter system $\mathcal{T}_\Delta(\mathcal{L})$ is a centric linking system. Also, if $(\mathcal{L}, \Delta, S)$ is a centric linking locality, then it is a centric linking system in the sense of Chermak [Che13], i.e. we have the property that $C_{\mathcal{L}}(P) \leqslant P$ for every $P \in \Delta$.

The term linking system is used in [Hen19] for all transporter systems coming from linking localities, as such transporter systems have properties similar to the ones of linking systems in Oliver's definition [Oli10] and can be seen as a generalization of such linking systems. We adapt this slightly non-standard definition here.

**Definition 2.18.** A *linking system* over $\mathcal{F}$ is a transporter system $\mathcal{T}$ over $\mathcal{F}$ such that $\mathcal{F}^{cr} \subseteq \mathrm{obj}(\mathcal{T})$ and $\mathrm{Aut}_{\mathcal{T}}(P)$ is of characteristic $p$ for every $P \in \mathrm{obj}(\mathcal{T})$. A *subcentric linking system* over $\mathcal{F}$ is a linking system $\mathcal{T}$ whose set of objects is the set $\mathcal{F}^s$ of subcentric subgroups.

Proving the existence and uniqueness of centric linking systems was a long-standing open problem, which was solved by Chermak [Che13]. Building on a basic idea in Chermak's proof, Oliver [Oli13] gave a new one via an earlier developed cohomological obstruction theory. Both proofs depend a priori on the classification of finite simple groups, but work of Glauberman and the third author of this paper [GL16] removes the dependence of Oliver's proof on the classification. The precise theorem proved is the following.

**Theorem 2.19** (Chermak [Che13], Oliver [Oli13], Glauberman–Lynd [GL16])**.** *There exists a centric linking system associated to $\mathcal{F}$ which is unique up to an isomorphism of transporter systems. Similarly, there exists a centric linking locality over $\mathcal{F}$ which is unique up to a rigid isomorphism.*

Using the existence and uniqueness of centric linking systems one can relatively easily prove the following theorem.

**Theorem 2.20** (Henke [Hen19])**.** *The following hold:*

(a) *If $\mathcal{F}^{cr} \subseteq \Delta \subseteq \mathcal{F}^s$ such that $\Delta$ is overgroup-closed in $S$ and closed under $\mathcal{F}$-conjugacy, then there exists a linking locality over $\mathcal{F}$ with object set $\Delta$, and such a linking locality is unique up to a rigid isomorphism. Similarly, there exists a linking system $\mathcal{T}$ associated to $\mathcal{F}$ whose set of objects is $\Delta$, and such a linking system is unique up to an isomorphism of transporter systems. Moreover, the nerve $|\mathcal{T}|$ is homotopy equivalent to the nerve of a centric linking system associated to $\mathcal{F}$.*

(b) *The set $\mathcal{F}^s$ is overgroup-closed in $S$ and closed under $\mathcal{F}$-conjugacy. In particular, there exists a subcentric linking locality over $\mathcal{F}$ which is unique up to a rigid isomorphism, and there exists a subcentric linking system associated to $\mathcal{F}$ which is unique up to an isomorphism of transporter systems.*

The existence of subcentric linking systems stated in part (b) of the above theorem gives often a way of proving the existence of a punctured group. The fusion system $\mathcal{F}$ is said to be of *characteristic $p$-type*, if $N_{\mathcal{F}}(Q)$ is constrained for every non-trivial fully $\mathcal{F}$-normalized subgroup $Q$ of $S$. Equivalently, $\mathcal{F}$ is of characteristic $p$-type, if $\mathcal{F}^s$ contains the set of all non-identity subgroups of $S$. Indeed, if $\mathcal{F}$ is of characteristic $p$-type but not constrained, then the set $\mathcal{F}^s$ equals the set of all non-identity subgroups. So in this case there exists a "canonical" punctured group over $\mathcal{F}$, namely the subcentric linking locality (or the subcentric linking system if one uses the language of transporter systems).

2.7. **Partial normal $p'$-subgroups.** Normal $p'$-subgroups are often considered in finite group theory. We will now introduce a corresponding notion in localities and prove some basic properties. Throughout this subsection let $(\mathcal{L}, \Delta, S)$ be a locality.

**Definition 2.21.** A *partial normal $p'$-subgroup of* $\mathcal{L}$ is a partial normal subgroup $\mathcal{N}$ of $\mathcal{L}$ such that $\mathcal{N} \cap S = 1$. The locality $(\mathcal{L}, \Delta, S)$ is said to be *$p'$-reduced* if there is no non-trivial partial normal $p'$-subgroup of $\mathcal{L}$.

**Remark 2.22.** If $(\mathcal{L}, \Delta, S)$ is a locality over a fusion system $\mathcal{F}$, then for any $p'$-group $N$, the direct product $(\mathcal{L} \times N, \Delta, S)$ is a locality over $\mathcal{F}$ such that $N$ is a partial normal $p'$-subgroup of $\mathcal{L} \times N$ and $(\mathcal{L} \times N)/N \cong \mathcal{L}$; see [Hen17] for details of the construction of direct products of localities. Thus, if we want to prove classification theorems for localities, it is actually reasonable to restrict attention to $p'$-reduced localities.

Recall that, for a finite group $G$, the largest normal $p'$-subgroup is denoted by $O_{p'}(G)$. Indeed, a similar notion can be defined for localities. Namely, it is a special case of [Che15, Theorem 5.1] that the product of two partial normal $p'$-subgroups is again a partial normal $p'$-subgroup. Thus, the following definition makes sense.

**Definition 2.23.** The largest normal $p'$-subgroup of $\mathcal{L}$ is denoted by $O_{p'}(\mathcal{L})$.

We will now prove some properties of partial normal $p'$-subgroups. To start, we show two lemmas which generalize corresponding statements for groups. The first of these lemmas gives a way of passing from an arbitrary locality to a $p'$-reduced locality.

**Lemma 2.24.** *Set $\overline{\mathcal{L}} := \mathcal{L}/O_{p'}(\mathcal{L})$. Then $(\overline{\mathcal{L}}, \overline{\Delta}, \overline{S})$ is $p'$-reduced.*

*Proof.* Let $\mathcal{N}$ be the preimage of $O_{p'}(\overline{\mathcal{L}})$ under the natural projection $\mathcal{L} \to \overline{\mathcal{L}}$. Then by [Che15, Proposition 4.7], $\mathcal{N}$ is a partial normal subgroup of $\mathcal{L}$ containing $O_{p'}(\mathcal{L})$. Moreover, $\overline{\mathcal{N} \cap S} \subseteq \overline{\mathcal{N}} \cap \overline{S} = 1$, which implies $\mathcal{N} \cap S \subseteq O_{p'}(\mathcal{L})$ and thus $\mathcal{N} \cap S \subseteq O_{p'}(\mathcal{L}) \cap S = 1$. Thus, $\mathcal{N}$ is a partial normal $p'$ subgroup of $\mathcal{L}$ and so by definition contained in $O_{p'}(\mathcal{L})$. This implies $O_{p'}(\overline{\mathcal{L}}) = \overline{\mathcal{N}} = 1$. $\square$

**Lemma 2.25.** *Given a partial normal $p'$-subgroup $\mathcal{N}$ of $\mathcal{L}$, the image of $O_{p'}(\mathcal{L})$ in $\mathcal{L}/\mathcal{N}$ under the canonical projection is a partial normal $p'$-subgroup of $\mathcal{L}/\mathcal{N}$. In particular, if $\mathcal{L}/\mathcal{N}$ is $p'$-reduced, then $\mathcal{N} = O_{p'}(\mathcal{L})$.*

*Proof.* Set $\overline{\mathcal{L}} := \mathcal{L}/\mathcal{N}$. Then by [Che15, Proposition 4.7], $\overline{O_{p'}(\mathcal{L})}$ is a partial normal subgroup of $\overline{\mathcal{L}}$. By Lemma 2.12, the preimage of $\overline{S}$ equals $\mathcal{N}S$. As $\mathcal{N} \subseteq O_{p'}(\mathcal{L})$, the preimage of $\overline{O_{p'}(\mathcal{L})} \cap \overline{S}$ is thus contained in $O_{p'}(\mathcal{L}) \cap (\mathcal{N}S)$. By the Dedekind Lemma [Che15, Lemma 1.10], we have $O_{p'}(\mathcal{L}) \cap (\mathcal{N}S) = \mathcal{N}(O_{p'}(\mathcal{L}) \cap S) = \mathcal{N}$. Hence, $\overline{O_{p'}(\mathcal{L})} \cap \overline{S} = \mathbf{1}$ and $\overline{O_{p'}(\mathcal{L})}$ is a normal $p'$-subgroup of $\overline{\mathcal{L}}$. If $\overline{\mathcal{L}} = \mathcal{L}/\mathcal{N}$ is $p'$-reduced, it follows that $\overline{O_{p'}(\mathcal{L})} = \mathbf{1}$ and so $O_{p'}(\mathcal{L}) = \mathcal{N}$. $\square$

We now proceed to prove some technical results which are needed in the next subsection.

**Lemma 2.26.** *If $\mathcal{N}$ is a partial normal $p'$-subgroup of $\mathcal{L}$, then $f \in C_{\mathcal{N}}(S_f)$ for every $f \in \mathcal{N}$.*

*Proof.* Let $f \in \mathcal{N}$, set $P := S_f$ and let $s \in P$. Then $P^f \leqslant S$ and thus $P^{fs} \leqslant S$. Moreover, $P^s = P$. Thus, $w = (s^{-1}, f^{-1}, s, f) \in \mathbf{D}$ via $P^{fs}$. Now $\Pi(w) = (f^{-1})^s f = s^{-1}s^f \in \mathcal{N} \cap S = \mathbf{1}$ and hence $s^f = s$. As $s \in P$ was arbitrary, this proves $f \in C_{\mathcal{N}}(P)$. $\square$

**Lemma 2.27.** *If $\mathcal{N}$ is a non-trivial partial normal $p'$ subgroup of $\mathcal{L}$, then there exists $P \in \Delta$ such that $N_{\mathcal{N}}(P) = C_{\mathcal{N}}(P) \neq 1$. In particular, if $O_{p'}(N_{\mathcal{L}}(P)) = 1$ for all $P \in \Delta$, then $O_{p'}(\mathcal{L}) = 1$.*

*Proof.* Let $\mathcal{N}$ be a non-trivial partial normal $p'$-subgroup and pick $1 \neq f \in \mathcal{N}$. Then $P := S_f \in \Delta$ by Lemma 2.7(e), and it follows from Lemma 2.26 that $1 \neq f \in C_{\mathcal{N}}(P)$. As $N_{\mathcal{N}}(P)$ is a normal $p'$-subgroup of $N_{\mathcal{L}}(P)$ and $P$ is a normal $p$-subgroup of $N_{\mathcal{L}}(P)$, we have $C_{\mathcal{N}}(P) = N_{\mathcal{N}}(P)$. Hence, $C_{\mathcal{N}}(P) = N_{\mathcal{N}}(P) \neq 1$ is a normal $p'$-subgroup of $N_{\mathcal{L}}(P)$ and the assertion follows. $\square$

**Corollary 2.28.** *If $(\mathcal{L}, \Delta, S)$ is a linking locality or, more generally, a locality of objective characteristic $p$, then $O_{p'}(\mathcal{L}) = 1$.*

*Proof.* If, for every $P \in \Delta$, the group $N_{\mathcal{L}}(P)$ is of characteristic $p$, then it is in particular $p'$-reduced. Thus, the assertion follows from Corollary 2.27. $\square$

2.8. **A signalizer functor theorem for punctured groups.** In this section we provide some tools for showing that a locality has a non-trivial partial normal $p'$-subgroup. Corresponding problems for groups are typically treated using signalizer functor theory. A similar language will be used here for localities. We will start by investigating how a non-trivial partial normal $p'$-subgroup can be produced if some information is known on the level of normalizers of objects. We will then use this to show a theorem for punctured groups which looks similar to the signalizer functor theorem for finite groups, but is much more elementary to prove. Throughout this subsection let $(\mathcal{L}, \Delta, S)$ be a locality.

**Definition 2.29.** A *signalizer functor of* $(\mathcal{L}, \Delta, S)$ *on objects* is a map from $\Delta$ to the set of subgroups of $\mathcal{L}$, which associates to $P \in \Delta$ a normal $p'$-subgroup $\Theta(P)$ of $N_{\mathcal{L}}(P)$ such that the following conditions hold:

- (Conjugacy condition) $\Theta(P)^g = \Theta(P^g)$ for all $P \in \Delta$ and all $g \in \mathcal{L}$ with $P \leqslant S_g$.
- (Balance condition) $\Theta(P) \cap C_{\mathcal{L}}(Q) = \Theta(Q)$ for all $P, Q \in \Delta$ with $P \leqslant Q$.

As seen in Lemma 2.27, given a locality $(\mathcal{L}, \Delta, S)$ with $O_{p'}(\mathcal{L}) \neq 1$, there exists $P \in \Delta$ with $O_{p'}(N_{\mathcal{L}}(P)) \neq 1$. The next theorem says basically that, under suitable extra conditions, the converse holds.

**Proposition 2.30.** *If* $\Theta$ *is a signalizer functor of* $(\mathcal{L}, \Delta, S)$ *on objects, then*

$$\widehat{\Theta} := \bigcup_{P \in \Delta} \Theta(P)$$

*is a partial normal $p'$-subgroup of $\mathcal{L}$. In particular, the canonical projection $\rho\colon \mathcal{L} \to \mathcal{L}/\widehat{\Theta}$ restricts to an isomorphism $S \to S^\rho$. Upon identifying $S$ with $S^\rho$, the following properties hold:*

(a) $(\mathcal{L}/\widehat{\Theta}, \Delta, S)$ *is a locality and* $\mathcal{F}_S(\mathcal{L}/\widehat{\Theta}) = \mathcal{F}_S(\mathcal{L})$.

(b) *For each $P \in \Delta$, the restriction $N_{\mathcal{L}}(P) \to N_{\mathcal{L}/\widehat{\Theta}}(P)$ of $\rho$ has kernel $\Theta(P)$ and induces an isomorphism $N_{\mathcal{L}}(P)/\Theta(P) \cong N_{\mathcal{L}/\widehat{\Theta}}(P)$.*

*Proof.* We proceed in three steps, where in the first step, we prove a technical property, which allows us in the second step to show that $\widehat{\Theta}$ is a partial normal $p'$-subgroup, and in the third step to conclude that the remaining properties hold.

*Step 1:* We show $x \in \Theta(S_x)$ for any $x \in \widehat{\Theta}$. Let $x \in \widehat{\Theta}$. Then by definition of $\widehat{\Theta}$, the element $x$ lies in $\Theta(P)$ for some $P \in \Delta$. Choose such $P$ maximal with respect to inclusion. Notice that $[P, x] = 1$. In particular, $P \leqslant S_x$ and $[N_{S_x}(P), x] \leqslant \Theta(P) \cap N_S(P) = \mathbf{1}$. Hence, using the balance condition, we conclude $x \in \Theta(P) \cap C_{\mathcal{L}}(N_{S_x}(P)) = \Theta(N_{S_x}(P))$. So the maximality of $P$ yields $P = N_{S_x}(P)$ and thus $P = S_x$. Hence, $x \in \Theta(S_x)$ as required.

*Step 2:* We show that $\widehat{\Theta}$ is a partial normal $p'$-subgroup of $\mathcal{L}$. Clearly $\widehat{\Theta}$ is closed under inversion, since $\Theta(P)$ is a group for every $P \in \Delta$. Note also that $\Pi(\varnothing) = \mathbf{1} \in \widehat{\Theta}$ as $\mathbf{1} \in \Theta(P)$ for any $P \in \Delta$. Let now $(x_1, \dots, x_n) \in \mathbf{D} \cap \mathbf{W}(\Theta)$ with $n \geqslant 1$. Then $R := S_{(x_1,\dots,x_n)} \in \Delta$ by Lemma 2.7(f). Induction on $i$ together with the balance condition and Step 1 shows $R \leqslant S_{x_i}$ and $x_i \in \Theta(S_{x_i}) \leqslant \Theta(R) \leqslant C_{\mathcal{L}}(R)$ for each $i = 1, \dots, n$. Hence, $\Pi(x_1, x_2, \dots, x_n) \in \Theta(R) \subseteq \widehat{\Theta}$.

Thus, $\widehat{\Theta}$ is a partial subgroup of $\mathcal{L}$. Let $x \in \widehat{\Theta}$ and $f \in \mathcal{L}$ with $(f^{-1}, x, f) \in \mathbf{D}$. Then $X := S_{(f^{-1},x,f)} \in \Delta$ by Lemma 2.7(f). Moreover, $X^{f^{-1}} \leqslant S_x$. By Step 1, we have $x \in \Theta(S_x)$, and then by the balance condition, $x \in \Theta(X^{f^{-1}})$. It follows now from the conjugacy condition that $x^f \in \Theta(X^{f^{-1}})^f = \Theta(X) \subseteq \widehat{\Theta}$. Hence, $\widehat{\Theta}$ is a partial normal subgroup of $\mathcal{L}$. Notice that $\widehat{\Theta} \cap S = \mathbf{1}$, as $\Theta(P) \cap S = \Theta(P) \cap N_S(P) = \mathbf{1}$ for each $P \in \Delta$. Thus, $\widehat{\Theta}$ is a partial normal $p'$-subgroup of $\mathcal{L}$.
*Step 3:* We are now in a position to complete the proof. By [Che15, Corollary 4.5], the quotient map $\rho\colon \mathcal{L} \to \mathcal{L}/\widehat{\Theta}$ is a homomorphism of partial groups with $\ker(\rho) = \widehat{\Theta}$. Moreover, setting $\Delta^\rho := \{P^\rho\colon P \in \Delta\}$, the triple $(\mathcal{L}/\widehat{\Theta}, \Delta^\rho, S^\rho)$ is a locality. Notice that $\rho|_S\colon S \to S^\rho$ is a homomorphism of groups with kernel $S \cap \widehat{\Theta} = 1$ and thus an isomorphism of groups. Upon identifying $S$ with $S^\rho$, it follows now that $(\mathcal{L}/\widehat{\Theta}, \Delta, S)$ is a locality. Moreover, by [Hen19, Theorem 5.7(b)], we have $\mathcal{F}_S(\mathcal{L}) = \mathcal{F}_S(\mathcal{L}/\widehat{\Theta})$. So (a) holds. Let $P \in \Delta$. By [Che15, Theorem 4.3(c)], the restriction of $\rho$ to a map $N_{\mathcal{L}}(P) \to N_{\mathcal{L}/\widehat{\Theta}}(P)$ is an epimorphism with kernel $N_{\mathcal{L}}(P) \cap \widehat{\Theta}$. For any $x \in N_{\mathcal{L}}(P) \cap \widehat{\Theta}$, we have $P \leqslant S_x$ and then $x \in \Theta(S_x) \leqslant \Theta(P)$ by the balance condition and Step 1. This shows $N_{\mathcal{L}}(P) \cap \widehat{\Theta} = \Theta(P)$ and so (b) holds. $\square$

**Remark 2.31.** If $P, Q, R \in \Delta$ such that $P \leqslant Q \leqslant R$ and the balance condition in Definition 2.29 holds for the pairs $P \leqslant Q$ and $Q \leqslant R$, then

$$\Theta(P) \cap C_{\mathcal{L}}(R) = \Theta(P) \cap C_{\mathcal{L}}(Q) \cap C_{\mathcal{L}}(R) = \Theta(Q) \cap C_{\mathcal{L}}(R) = \Theta(R),$$

i.e. the balance condition holds for the pair $P \leqslant R$.

The following proposition is essentially a restatement of [Hen19, Proposition 6.4], but we will give an independent proof building on the previous proposition.

**Proposition 2.32.** *Let $(\mathcal{L}, \Delta, S)$ be a locality such that, setting*

$$\Theta(P) := O_{p'}(N_{\mathcal{L}}(P)) \text{ for every } P \in \Delta,$$

*the group $N_{\mathcal{L}}(P)/\Theta(P)$ is of characteristic $p$ for every $P \in \Delta$. Then the assignment $\Theta$ is a signalizer functor of $(\mathcal{L}, \Delta, S)$ on objects and $O_{p'}(\mathcal{L})$ equals $\widehat{\Theta} := \bigcup\{\Theta(P)\colon P \in \Delta\}$. In particular, the canonical projection $\rho\colon \mathcal{L} \to \mathcal{L}/\widehat{\Theta}$ restricts to an isomorphism $S \to S^\rho$. Upon identifying $S$ with $S^\rho$, the following properties hold:*

(a) *$(\mathcal{L}/\widehat{\Theta}, \Delta, S)$ is a locality of objective characteristic $p$.*
(b) *$\mathcal{F}_S(\mathcal{L}/\widehat{\Theta}) = \mathcal{F}_S(\mathcal{L})$.*
(c) *For every $P \in \Delta$, the restriction $N_{\mathcal{L}}(P) \to N_{\mathcal{L}/\widehat{\Theta}}(P)$ of $\rho$ has kernel $\Theta(P)$ and induces an isomorphism $N_{\mathcal{L}}(P)/\Theta(P) \cong N_{\mathcal{L}/\widehat{\Theta}}(P)$.*

*Proof.* We remark first that, as any normal $p'$-subgroup of $N_{\mathcal{L}}(P)$ centralizes $P$ and $O_{p'}(C_{\mathcal{L}}(P))$ is characteristic in $C_{\mathcal{L}}(P) \trianglelefteq N_{\mathcal{L}}(P)$, we have $\Theta(P) = O_{p'}(C_{\mathcal{L}}(P))$ for every $P \in \Delta$.

We show now that the assignment $\Theta$ is a signalizer functor of $\mathcal{L}$ on objects. It follows from Lemma 2.7(b) that the conjugacy condition holds. Thus, it remains to show the balance condition, i.e. that $\Theta(Q) = \Theta(P) \cap C_{\mathcal{L}}(Q)$ for any $P, Q \in \Delta$ with $P \leqslant Q$. For the proof note that $P$ is subnormal in $Q$. So by induction on the subnormal length and by Remark 2.31, we may assume that $P \trianglelefteq Q$. Set $G := N_{\mathcal{L}}(P)$. Then $Q \leqslant G$ and $C_{\mathcal{L}}(Q) = C_G(Q)$. As $G/\Theta(P) = G/O_{p'}(G)$ has characteristic $p$, it follows from [KS04, 8.2.12] that $O_{p'}(N_G(Q)) = O_{p'}(G) \cap N_G(Q) = O_{p'}(G) \cap C_G(Q)$. Hence, $\Theta(Q) = O_{p'}(C_{\mathcal{L}}(Q)) = O_{p'}(C_G(Q)) = O_{p'}(N_G(Q)) = O_{p'}(G) \cap C_G(Q) =$

$\Theta(P) \cap C_{\mathcal{L}}(Q)$. This proves that the assignment $\Theta$ is a signalizer functor of $(\mathcal{L}, \Delta, S)$ on objects. In particular, by Proposition 2.30, the subset

$$\widehat{\Theta} := \bigcup_{P \in \Delta} \Theta(P)$$

is a partial normal $p'$-subgroup of $\mathcal{L}$. Moreover, upon identifying $S$ with its image in $\mathcal{L}/\widehat{\Theta}$, the triple $(\mathcal{L}/\hat{\Theta}, \Delta, S)$ is a locality and properties (b) and (c) hold. Part (c) and our assumption yield (a). Hence, by Corollary 2.28, we have $O_{p'}(\mathcal{L}/\widehat{\Theta}) = \mathbf{1}$. So by Lemma 2.25, we have $\widehat{\Theta} = O_{p'}(\mathcal{L})$ and the proof is complete. □

The next purely group theoretical lemma will be useful for applying Proposition 2.32. Recall that a finite group $G$ with Sylow $p$-subgroup $T$ is called *$p$-constrained* if $C_T(O_p(G)) \leqslant O_p(G)$.

**Lemma 2.33.** *If $G$ is a $p$-constrained finite group, then $G/O_{p'}(G)$ is of characteristic $p$.*

*Proof.* Writing $T$ for a Sylow $p$-subgroup of $G$ and setting $P := O_p(G)$, the centralizer $C_T(P)$ equals $Z(P)$ and is thus a central Sylow $p$-subgroup of $C_G(P)$. So e.g. by the Schur-Zassenhaus Theorem [KS04, 6.2.1], we have $C_G(P) = Z(P) \times O_{p'}(C_G(P)) = Z(P) \times O_{p'}(G)$. Set $\overline{G} = G/O_{p'}(G)$, write $C$ for the preimage of $C_{\overline{G}}(\overline{P})$ in $G$ and $N$ for the preimage of $N_{\overline{G}}(\overline{P})$ in $G$. As $PO_{p'}(G)$ is a normal subgroup of $N$ with Sylow $p$-subgroup $P$, a Frattini argument yields $N = N_N(P)PO_{p'}(G) = N_N(P)O_{p'}(G)$. As $O_{p'}(G) \leqslant C \leqslant N$, it follows now from a Dedekind argument that $C = N_C(P)O_{p'}(G)$. Observe now that $[P, N_C(P)] \leqslant P \cap O_{p'}(G) = 1$. So $N_C(P) \leqslant C_G(P)$ and $\overline{C} = \overline{N_C(P)} \leqslant \overline{C_G(P)} \leqslant \overline{P}$. Thus, $\overline{G}$ has characteristic $p$. □

We now turn attention to the case that $(\mathcal{L}, \Delta, S)$ is a punctured group and we are given a signalizer functor *on elements of order $p$* in the sense of Definition 1.2 in the introduction. We show first that, if $\theta$ is such a signalizer functor on elements of order $p$ and $a \in \mathcal{I}_p(S)$, the subgroup $\theta(a)$ depends only on $\langle a \rangle$.

**Lemma 2.34.** *Let $(\mathcal{L}, \Delta, S)$ be a punctured group and let $\theta$ be a signalizer functor of $(\mathcal{L}, \Delta, S)$ on elements of order $p$. Then $\theta(a) = \theta(b)$ for all $a, b \in \mathcal{I}_p(S)$ with $\langle a \rangle = \langle b \rangle$.*

*Proof.* If $\langle a \rangle = \langle b \rangle$, then $[a, b] = 1$ and $\theta(a) \subseteq C_{\mathcal{L}}(a) = C_{\mathcal{L}}(b)$. So the balance condition implies $\theta(a) = \theta(a) \cap C_{\mathcal{L}}(b) \subseteq \theta(b)$. A symmetric argument gives the converse inclusion $\theta(b) \subseteq \theta(a)$, so the assertion holds. □

Theorem 1.3 in the introduction follows directly from the following theorem, which explains at the same time how a signalizer functor on objects can be constructed from a signalizer functor on elements of order $p$.

**Theorem 2.35** (Signalizer functor theorem for punctured groups)**.** *Let $(\mathcal{L}, \Delta, S)$ be a punctured group and suppose $\theta$ is a signalizer functor of $(\mathcal{L}, \Delta, S)$ on elements of order $p$. Then a signalizer functor $\Theta$ of $(\mathcal{L}, \Delta, S)$ on objects is defined by*

$$\Theta(P) := \left( \bigcap_{x \in \mathcal{I}_p(P)} \theta(x) \right) \cap C_{\mathcal{L}}(P) \text{ for all } P \in \Delta.$$

*In particular,*

$$\widehat{\Theta} := \bigcup_{P\in\Delta} \Theta(P) = \bigcup_{x\in\mathcal{I}_p(S)} \theta(x)$$

*is a partial normal $p'$ subgroup of $\mathcal{L}$ and the other conclusions in Proposition 2.30 hold.*

*Proof.* Since $\theta(x)$ is a $p'$-subgroup for each $x \in \mathcal{I}_p(S)$, the subgroup $\Theta(P)$ is a $p'$-subgroup for each object $P \in \Delta$. Moreover, it follows from the conjugacy condition for $\theta$ (as stated in Definition 1.2) that $\Theta(P)$ is a normal subgroup of $N_{\mathcal{L}}(P)$, and that the conjugacy condition stated in Definition 2.29 holds for $\Theta$; to obtain the latter conclusion notice that Lemma 2.7(b) implies $C_{\mathcal{L}}(P)^g = C_{\mathcal{L}}(P^g)$ for every $P \in \Theta$ and every $g \in \mathcal{L}$ with $P \leqslant S_g$.

To prove that $\Theta$ is a signalizer functor on objects, it remains to show that the balance condition $\Theta(P) \cap C_{\mathcal{L}}(Q) = \Theta(Q)$ holds for every pair $P \leqslant Q$ with $P \in \Delta$. Notice that $P$ is subnormal in $Q$ whenever $P \leqslant Q$. Hence, if the balance condition for $\Theta$ fails for some pair $P \leqslant Q$ with $P \in \Delta$, then by Remark 2.31, it fails for some pair $P \trianglelefteq Q$ with $P \in \Delta$. Suppose this is the case. Among all pairs $P \trianglelefteq Q$ such that $P \in \Delta$ and the balance condition fails, choose one such that $Q$ is of minimal order.

Notice that $P < Q$, as the balance condition would otherwise trivially hold. So as $1 \neq P_0 := C_P(Q) \in \Delta$ and $P_0 \trianglelefteq P$, the minimality of $|Q|$ yields that the balance condition holds for the pair $P_0 \leqslant P$, i.e. $\Theta(P_0) \cap C_{\mathcal{L}}(P) = \Theta(P)$. If the balance condition holds also for $P_0 \trianglelefteq Q$, then

$$\Theta(Q) = \Theta(P_0) \cap C_{\mathcal{L}}(Q) = \Theta(P_0) \cap C_{\mathcal{L}}(P) \cap C_{\mathcal{L}}(Q) = \Theta(P) \cap C_{\mathcal{L}}(Q)$$

and thus the balance condition holds for $P \leqslant Q$ contradicting our assumption. So the balance condition does not hold for $P_0 \leqslant Q$. Therefore, replacing $P$ by $P_0$, we can and will assume from now on that $P \leqslant Z(Q)$.

It is clear from the definition that $\Theta(Q) \leqslant \Theta(P) \cap C_{\mathcal{L}}(Q)$. Hence it remains to prove the converse inclusion. By definition of $\Theta(Q)$, this means that we need to show $\Theta(P) \cap C_{\mathcal{L}}(Q) \leqslant \theta(b)$ for all $b \in \mathcal{I}_p(Q)$. To show this fix $b \in \mathcal{I}_p(Q)$. As $P \in \Delta$, we have $P \neq 1$ and so we can pick $a \in \mathcal{I}_p(P)$. Since $P \leqslant Z(Q)$, the elements $a$ and $b$ commute. Hence, the balance condition for $\theta$ yields

$$\Theta(P) \cap C_{\mathcal{L}}(Q) \leqslant \theta(a) \cap C_{\mathcal{L}}(b) \leqslant \theta(b).$$

This completes the proof that $\Theta$ is a signalizer functor on objects.

Given $P \in \Delta$, we can pick any $x \in \mathcal{I}_p(P)$ and have $\Theta(P) \subseteq \theta(x)$. Hence, $\hat{\Theta} := \bigcup_{P\in\Delta} \Theta(P)$ is contained in $\bigcup_{x\in\mathcal{I}_p(S)} \theta(x)$. The converse inclusion holds as well, as Lemma 2.34 implies $\theta(x) = \Theta(\langle x\rangle)$ for every $x \in \mathcal{I}_p(S)$. The assertion follows now from Proposition 2.30. □

## 3. Sharpness of the subgroup decomposition

3.1. **Additive extensions of categories.** Let $\mathcal{C}$ be a (small) category. Define a category $\mathcal{C}_{\amalg}$ as follows, see [JM92, Sec. 4]. The objects of $\mathcal{C}_{\amalg}$ are pairs $(I, \mathbf{X})$ where $I$ is a finite set and $\mathbf{X}\colon I \to \mathrm{obj}(\mathcal{C})$ is a function. A morphisms $(I, \mathbf{X}) \to (J, \mathbf{Y})$ is a pair $(\sigma, \mathbf{f})$ where $\sigma\colon I \to J$ is a function and $\mathbf{f}\colon I \to \mathrm{mor}(\mathcal{C})$ is a function such that $\mathbf{f}(i) \in \mathcal{C}(\mathbf{X}(i), \mathbf{Y}(\sigma(i)))$. We leave it to the reader to check that this defines a category.

There is a fully faithful inclusion $\mathcal{C} \subseteq \mathcal{C}_{\amalg}$ by sending $X \in \mathcal{C}$ to the function $\mathbf{X}\colon \{\varnothing\} \to \mathrm{obj}(\mathcal{C})$ with $\mathbf{X}(\varnothing) = X$. We will write $X$ (not boldface) to denote these objects in $\mathcal{C}_{\amalg}$.

The category $\mathcal{C}_{\amalg}$ has a monoidal structure $\coprod$ where $(I, \mathbf{X}) \coprod (J, \mathbf{Y}) \overset{\text{def}}{=} (I \coprod J, \mathbf{X} \coprod \mathbf{Y})$. One checks that this is the categorical coproduct in $\mathcal{C}_{\amalg}$. For this reason we will often write objects of $\mathcal{C}_{\amalg}$ in the form $\coprod_{i \in I} X_i$ where $X_i \in \mathcal{C}$. Also, when the indexing set $I$ is understood we will simply write $\mathbf{X}$ instead of $(I, \mathbf{X})$.

When $(I, \mathbf{X})$ is an object and $J \subseteq I$ we will refer to $(J, \mathbf{X}|_J)$ as a "subobject" of $(I, \mathbf{X})$ and we leave it to the reader to check that the inclusion is a monomorphism, namely for any two morphisms $\mathbf{f}, \mathbf{g} \colon \mathbf{Y} \to \mathbf{X}|_J$, if $\mathrm{incl}_{\mathbf{X}|_J}^{\mathbf{X}} \circ \mathbf{f} = \mathrm{incl}_{\mathbf{X}|_J}^{\mathbf{X}} \circ \mathbf{g}$ then $\mathbf{f} = \mathbf{g}$. One also checks that

$$\mathcal{C}_{\amalg}(\coprod_{i \in I} X_i, Y) = \prod_{i \in I} \mathcal{C}(X_i, Y), \tag{3.1}$$

$$\mathcal{C}_{\amalg}(X, \coprod_{i \in I} Y_i) = \coprod_{i \in I} \mathcal{C}(X, Y_i).$$

**Definition 3.1** (Compare [JM92, p. 123])**.** We say that $\mathcal{C}$ satisfies (PB $\times_{\amalg}$) if the product of each pair of objects in $\mathcal{C}$ exists in $\mathcal{C}_{\amalg}$ and if the pullback of each diagram $c \to e \leftarrow d$ of objects in $\mathcal{C}$ exists in $\mathcal{C}_{\amalg}$.

**Definition 3.2** (Compare [JM92, p. 124 and Lemma 5.13])**.** Assume that $\mathcal{C}$ is a small category satisfying (PB $\times_{\amalg}$). A functor $M \colon \mathcal{C}^{\mathrm{op}} \to \mathbf{Ab}$ is called a *proto-Mackey functor* if there is a functor $M_* \colon \mathcal{C} \to \mathbf{Ab}$ such that the following hold.

(a) $M(C) = M_*(C)$ for any $C \in \mathrm{obj}(\mathcal{C})$.
(b) For any isomorphism $\varphi \in \mathcal{C}$, $M_*(\varphi) = M(\varphi^{-1})$.
(c) By applying $M$ and $M_*$ to a pullback diagram in $\mathcal{C}_{\amalg}$ of the form

$$\begin{tikzcd}
\coprod_{i \in I} B_i \arrow[r, "\sum_i \varphi_i"] \arrow[d, "\sum_i \psi_i"'] & D \arrow[d, "\beta"] \\
C \arrow[r, "\alpha"] & E
\end{tikzcd}$$

where $B_i, C, E \in \mathcal{C}$, there results the following commutative square in $\mathbf{Ab}$

$$\begin{tikzcd}
\bigoplus_{i \in I} M(B_i) \arrow[r, "\sum_i M_*(\varphi_i)"] & M(D) \\
M(C) \arrow[u, "\oplus_i M(\psi_i)"] \arrow[r, "M_*(\alpha)"] & M(E) \arrow[u, "M(\beta)"']
\end{tikzcd}$$

We remark that every pullback diagram in $\mathcal{C}_{\amalg}$ defined by objects in $\mathcal{C}$ is isomorphic in $\mathcal{C}_{\amalg}$ to a commutative square as in (c) in this definition.

Given a small category $\mathbf{D}$ and a functor $M \colon \mathbf{D} \to \mathbf{Ab}$, we write

$$H^*(\mathbf{D}; M) \overset{\text{def}}{=} \varprojlim_{\mathbf{D}}{}^* M.$$

for the derived functors of $M$. We say that $M$ is *acyclic* if $H^i(\mathbf{D}; M) = 0$ for all $i > 0$.

**Proposition 3.3** (See [JM92, Corollary 5.16])**.** *Fix a prime $p$. Let $\mathcal{C}$ be a small category which satisfies (PB $\times_{\amalg}$) and in addition*

(B1) *$\mathcal{C}$ has finitely many isomorphism classes of objects, all morphism sets are finite and all self maps in $\mathcal{C}$ are isomorphisms.*

(B2) *For every object $C \in \mathcal{C}$ there exists an object $D$ such that $|\mathcal{C}(C,D)| \neq 0 \mod p$.*

*Then any proto-Mackey functor $M\colon \mathcal{C}^{\mathrm{op}} \to \mathbb{Z}_{(p)}\text{-}\mathfrak{mod}$ is acyclic, namely $H^i(\mathcal{C}^{\mathrm{op}}, M) = 0$ for all $i > 0$.*

3.2. **Transporter categories.** Let $\mathcal{F}$ be a saturated fusion system over $S$ and let $\mathcal{T}$ be a transporter system associated with $\mathcal{F}$ (Definition 2.13). By [OV07, Lemmas 3.2(b) and 3.8] every morphism in $\mathcal{T}$ is both a monomorphism and an epimorphism. For any $P, Q \in \mathrm{obj}(\mathcal{T})$ such that $P \leqslant Q$ denote $\iota_P^Q = \epsilon_{P,Q}(e) \in \mathrm{Mor}_{\mathcal{T}}(P,Q)$. We think of these as "inclusion" morphisms in $\mathcal{T}$. We obtain a notion of "extension" and "restriction" of morphisms in $\mathcal{T}$ as follows. Suppose $\varphi \in \mathrm{Mor}_{\mathcal{T}}(P,Q)$ and $P' \leqslant P$ and $Q' \leqslant Q$ and $\psi \in \mathrm{Mor}_{\mathcal{T}}(P',Q')$ are such that $\varphi \circ \iota_{P'}^P = \iota_{Q'}^Q \circ \psi$. Then we say that $\psi$ is a restriction of $\varphi$ and that $\varphi$ is an extension of $\psi$. Notice that since $\iota_{Q'}^Q$ is a monomorphism, given $\varphi$ then its restriction $\psi$ if it exists, is unique and we will write $\psi = \varphi|_{P'}^{Q'}$. Similarly, since $\iota_{P'}^P$ is an epimorphism, given $\psi$, if an extension $\varphi$ exists then it is unique. By [OV07, Lemma 3.2(c)], given $\varphi \in \mathrm{Mor}_{\mathcal{T}}(P,Q)$ and subgroups $P' \leqslant P$ and $Q' \leqslant Q$ such that $\rho(\varphi)(P') \leqslant Q'$, then $\varphi$ restricts to a (unique morphism) $\varphi' \in \mathrm{Mor}_{\mathcal{T}}(P',Q')$. We will use this fact repeatedly. In particular, every morphism $\varphi\colon P \to Q$ in $\mathcal{T}$ factors uniquely $P \xrightarrow{\bar{\varphi}} \bar{P} \xrightarrow{\iota_{\bar{P}}^Q} Q$ where $\bar{\varphi}$ is an isomorphism in $\mathcal{T}$ and $\bar{P} = \rho(\varphi)(P)$.

For any $P, Q \in \mathrm{obj}(\mathcal{T})$ set

$$K_{P,Q} = \{(A,\alpha) :\ A \leqslant P,\ A \in \mathrm{obj}(\mathcal{T}),\ \alpha \in \mathrm{Mor}_{\mathcal{T}}(A,Q)\}.$$

This set is partially ordered where $(A,\alpha) \preceq (B,\beta)$ if $A \leqslant B$ and $\alpha = \beta|_A$. Since $K_{P,Q}$ is finite we may consider the set $K_{P,Q}^{\max}$ of the maximal elements.

For any $x \in N_S(P,Q)$ we write $\widehat{x}$ instead of $\epsilon_{P,Q}(x)$. There is an action of $Q \times P$ on $K_{P,Q}$ defined by

$$(y,x) \cdot (A,\alpha) = (xAx^{-1}, \widehat{y} \circ \alpha \circ \widehat{x}^{-1}), \qquad (x \in P,\ y \in Q).$$

This action is order preserving and therefore it leaves $K_{P,Q}^{\max}$ invariant. We will write $\overline{K_{P,Q}^{\max}}$ for the set of orbits. For any $P, Q \in \mathcal{T}$ we will choose *once and for all* a subset

$$\mathcal{K}_{P,Q}^{\max} \subseteq K_{P,Q}^{\max}$$

of representatives for the orbits of $Q \times P$ on $K_{P,Q}^{\max}$.

**Lemma 3.4.** *For any $(A,\alpha) \in K_{P,Q}$ there exists a unique $(B,\beta) \in K_{P,Q}^{\max}$ such that $(A,\alpha) \preceq (B,\beta)$.*

*Proof.* We use induction on $[S:A]$. Fix $(A,\alpha) \in K_{P,Q}$ and $(B_1,\beta_1)$ and $(B_2,\beta_2)$ in $K_{P,Q}^{\max}$ such that $(A,\alpha) \preceq (B_i,\beta_i)$. Thus, $\beta_1|_A = \alpha = \beta_2|_A$. We may assume that $A \lneq B_i$ since if say $A = B_1$ then $(B_1,\beta_1) \preceq (B_2,\beta_2)$ and maximality implies $(B_1,\beta_1) = (B_2,\beta_2)$.

For $i = 1,2$ set $N_i = N_{B_i}(A)$. Then $N_i$ contain $A$ properly and we set $D = \langle N_1, N_2 \rangle$. Then $A \trianglelefteq D$. Set $T = \alpha(A)$ and $\overline{T} = N_Q(T)$. For $i = 1,2$, if $x \in N_i$ then Axiom (C) of [OV07, Def. 3.1] implies

$$\alpha \circ \widehat{x}|_A^A = ((\beta_i|_{N_i}) \circ \widehat{x}|_{N_i}^{N_i})|_A = \widehat{\beta_i(x)}|_Q^Q \circ \beta_i|_A = \widehat{\beta_i(x)}|_Q^Q \circ \alpha.$$

Notice that $\beta_i(x) \in N_Q(T)$, so Axiom (II) of Definition 2.13 implies that $\alpha$ extends to $\delta \in \mathrm{Mor}_{\mathcal{T}}(D,Q)$. Since for $i=1,2$ the morphisms $\iota_A^{N_i}\colon A \to N_i$ are epimorphisms in $\mathcal{T}$, the equality $\beta_i|_{N_i} \circ \iota_A^{N_i} = \alpha = \delta|_A = (\delta|_{N_i}) \circ \iota_A^{N_i}$ shows that $\delta|_{N_i} = \beta_i|_{N_i}$. Now we have $(N_i, \beta_i|_{N_i}) \preceq (D,\delta)$ and $(N_i, \beta_i|_{N_i}) \preceq (B_i,\beta_i)$ in $K_{P,Q}$. Since $|A| < |N_i|$ the induction hypothesis implies that $(B_i,\beta_i)$ is the unique maximal extension of $(N_i, \beta_i|_{N_i})$ for each $i=1,2$, and both must coincide with the unique maximal extension of $(D,\delta)$. It follows that $(B_1,\beta_1) = (B_2,\beta_2)$. $\square$

The *orbit category* of $\mathcal{T}$ is the category $\mathcal{OT}$ with the same set of objects as $\mathcal{T}$. For any $P,Q \in \mathcal{OT}$ the morphism set $\mathrm{Mor}_{\mathcal{OT}}(P,Q)$ is the set of orbits of $\mathrm{Mor}_{\mathcal{T}}(P,Q)$ under the action of $\widehat{Q} = \epsilon_{Q,Q}(Q) \subseteq \mathrm{Mor}_{\mathcal{T}}(Q,Q)$ via postcomposition. See [OV07, Section 4, p. 1010]. Axiom (C) guarantees that composition in $\mathcal{OT}$ is well defined. Given $\varphi \in \mathrm{Mor}_{\mathcal{T}}(P,Q)$ we will denote its image in $\mathrm{Mor}_{\mathcal{OT}}(P,Q)$ by $[\varphi]$.

We notice that every morphism in $\mathcal{OT}$ is an epimorphism, namely for every $[\alpha] \in \mathrm{Mor}_{\mathcal{OT}}(P,Q)$ and $[\beta],[\gamma] \in \mathrm{Mor}_{\mathcal{OT}}(Q,R)$, if $[\beta]\circ[\alpha] = [\gamma]\circ[\alpha]$ then $[\beta]=[\gamma]$. This follows from the fact that every morphism in $\mathcal{T}$ is an epimorphism.

Consider $P,Q \in \mathrm{obj}(\mathcal{OT})$ such that $P \trianglelefteq Q$. Precomposition with $[\iota_P^Q]$ gives a "restriction" map

$$\mathrm{Mor}_{\mathcal{OT}}(Q,S) \xrightarrow{\mathrm{res}} \mathrm{Mor}_{\mathcal{OT}}(P,S).$$

Observe that $Q$ acts on $\mathrm{Mor}_{\mathcal{OT}}(P,S)$ by precomposing morphisms with $[\widehat{x}|_P^P]$ for any $x \in Q$. This action has $P$ in its kernel by Axiom (C) of transporter systems.

**Lemma 3.5.** (a) *For any $P,Q \in \mathrm{obj}(\mathcal{OT})$ such that $P \trianglelefteq Q$ the map $\mathrm{Mor}_{\mathcal{OT}}(Q,S) \to \mathrm{Mor}_{\mathcal{OT}}(P,S)$ induced by the restriction $[\varphi] \mapsto [\varphi|_P]$, gives rise to a bijection*

$$\mathrm{res}\colon \mathrm{Mor}_{\mathcal{OT}}(Q,S) \to \mathrm{Mor}_{\mathcal{OT}}(P,S)^{Q/P} \tag{3.2}$$

(b) *For any $P \in \mathcal{OT}$ we have $|\mathrm{Mor}_{\mathcal{OT}}(P,S)| \neq 0 \mod p$.*

*Proof.* (a) First, observe that if $[\varphi] \in \mathrm{Mor}_{\mathcal{OT}}(Q,S)$ then $[\varphi|_P]$ is fixed by $Q/P$ by Axiom (C), hence the image of res is contained in $\mathrm{Mor}_{\mathcal{OT}}(P,S)^{Q/P}$. Now suppose that $[\varphi] \in \mathrm{Mor}_{\mathcal{OT}}(P,S)^{Q/P}$ and set $\bar{P} = \rho(\varphi)(P)$. Since $[\varphi]$ is fixed by $Q/P$ this exactly means that for every $x \in Q$ there exists $y \in N_S(\bar{P})$ such that $\varphi \circ \widehat{x}|_P^P = \widehat{y}|_{N_S(\bar{P})}^{N_S(\bar{P})} \circ \varphi$ and Axiom (II) implies that $\varphi$ extends to $\psi\colon Q \to S$. This shows that the map res in (3.2) is onto $\mathrm{Mor}_{\mathcal{OT}}(P,S)^{Q/P}$. It is injective because $[\iota_P^Q]$ is an epimorphism in $\mathcal{OT}$.

(b) Use induction on $[S:P]$. If $P=S$ then $\epsilon_{S,S}(S)$ is a Sylow $p$-subgroup of $\mathrm{Aut}_{\mathcal{T}}(S) = \mathrm{Mor}_{\mathcal{T}}(S,S)$ and therefore $|\mathrm{Mor}_{\mathcal{OT}}(S,S)| \neq 0 \mod p$. Suppose $P<S$ and set $Q = N_S(P)$. Then $Q>P$ and since $Q/P$ is a finite $p$-group, $|\mathrm{Mor}_{\mathcal{OT}}(P,S)| = |\mathrm{Mor}_{\mathcal{OT}}(P,S)^{Q/P}| \mod p$. It follows from part (a) and the induction hypothesis on $[S:Q]$ that $|\mathrm{Mor}_{\mathcal{OT}}(P,S)| \neq 0 \mod p$. $\square$

In the remainder of this subsection we will prove that $\mathcal{OT}$ satisfies (PB $\times_{\mathrm{II}}$).

**Definition 3.6.** For $P,Q \in \mathcal{OT}$ let $P \boxtimes Q$ be the following object of $\mathcal{OT}_{\mathrm{II}}$

$$P \boxtimes Q = \coprod_{(L,\lambda)\in \mathcal{K}_{P,Q}^{\max}} L.$$

Let $\pi_1\colon P\boxtimes Q \to P$ and $\pi_2\colon P \boxtimes Q \to Q$ be the morphisms in $\mathcal{OT}_{\mathrm{II}}$ defined by $\pi_1 = \sum_{(L,\lambda)}[\iota_L^P]$ and $\pi_2 = \sum_{(L,\lambda)}[\lambda]$.

**Proposition 3.7.** *$P \boxtimes Q$ is the product in $\mathcal{OT}_{\mathrm{II}}$ of $P, Q \in \mathrm{obj}(\mathcal{OT})$.*

*Proof.* It follows from (3.1) that it suffices to show that

$$\mathcal{OT}_{\mathrm{II}}(R, \mathcal{P} \boxtimes Q) \xrightarrow{(\pi_{1*},\pi_{2*})} \mathrm{Mor}_{\mathcal{OT}}(R,P) \times \mathrm{Mor}_{\mathcal{OT}}(R,Q)$$

is a bijection for any $R \in \mathrm{obj}(\mathcal{OT})$. Write $\pi = (\pi_{1*}, \pi_{2*})$.

*Surjectivity of $\pi$:* Consider $[\varphi] \in \mathrm{Mor}_{\mathcal{OT}}(R,P)$ and $[\psi] \in \mathrm{Mor}_{\mathcal{OT}}(R,Q)$. Set $A = \rho(\varphi)(R)$. Then $A \leqslant P$ and there exists an isomorphism $\bar\varphi \in \mathrm{Mor}_{\mathcal{T}}(R,A)$ such that $\varphi = \iota_A^P \circ \bar\varphi$.

Set $\alpha = \psi \circ (\bar\varphi)^{-1} \in \mathrm{Mor}_{\mathcal{T}}(A,Q)$. Then $(A,\alpha) \in K_{P,Q}$. Choose $(B,\beta) \in K_{P,Q}^{\max}$ such that $(A,\alpha) \preceq (B,\beta)$. There exists a unique $(L,\lambda) \in \mathcal{K}_{P,Q}^{\max}$ and some $x \in P$ and $y \in Q$ such that

$$(L,\lambda) = (y,x) \cdot (B,\beta) = (xBx^{-1}, \widehat{y} \circ \beta \circ (\widehat{x}|_B^L)^{-1}).$$

Set $\mu = (\widehat{x}|_B^L) \circ \iota_A^B \circ \bar\varphi \in \mathrm{Mor}_{\mathcal{T}}(R,L)$. It defines a morphism $[\mu] \colon R \to P \boxtimes Q$ in $\mathcal{OT}_{\mathrm{II}}$ via the inclusion $(L,\lambda) \subseteq P \boxtimes Q$. We claim that $\pi([\mu]) = ([\varphi],[\psi])$, completing the proof of the surjectivity of $\pi$. By definition of $\pi_1 \colon P \boxtimes Q \to P$ and $\pi_2 \colon P \boxtimes Q \to Q$,

$$\begin{aligned}
\pi_{1*}([\mu]) &= [\iota_L^P] \circ [\mu] = [\iota_L^P \circ (\widehat{x}|_B^L) \circ \iota_A^B \circ \bar\varphi] = [(\widehat{x}|_A^P) \circ \bar\varphi] = [(\widehat{x}|_P^P) \circ \varphi] = [\varphi] \\
\pi_{2*}([\mu]) &= [\lambda] \circ [\mu] = [\widehat{y}^{-1} \circ \lambda \circ \mu] = [\widehat{y}^{-1} \circ \lambda \circ (\widehat{x}|_B^L) \circ \iota_A^B \circ \bar\varphi] = \\
&= [\beta \circ \iota_A^B \circ \bar\varphi] = [\alpha \circ \bar\varphi] = [\psi].
\end{aligned}$$

*Injectivity of $\pi$:* Suppose that $h, h' \in \mathcal{OT}_{\mathrm{II}}(R, P \boxtimes Q)$ are such that $\pi(h) = \pi(h')$. From (3.1) there are $(L,\lambda), (L',\lambda') \in \mathcal{K}_{P,Q}^{\max}$ and $\varphi \in \mathrm{Mor}_{\mathcal{T}}(R,L)$ and $\varphi' \in \mathrm{Mor}_{\mathcal{T}}(R,L')$ such that $h = [\varphi]$ and $h' = [\varphi']$ via the inclusions $L, L' \subseteq P \boxtimes Q$. The hypothesis $\pi(h) = \pi(h')$ then becomes $[\iota_L^P \circ \varphi] = [\iota_{L'}^P \circ \varphi']$ and $[\lambda \circ \varphi] = [\lambda' \circ \varphi']$. Thus,

$$\begin{aligned}
&\iota_{L'}^P \circ \varphi' = \widehat{x} \circ \iota_L^P \circ \varphi \qquad \text{for some } x \in P \\
&\lambda' \circ \varphi' = \widehat{y} \circ \lambda \circ \varphi \qquad \text{for some } y \in Q.
\end{aligned} \tag{3.3}$$

Set $A = \rho(\varphi)(R)$ and $A' = \rho(\varphi')(R)$. There are factorizations $\varphi = \iota_A^L \circ \bar\varphi$ and $\varphi' = \iota_{A'}^{L'} \circ \bar{\varphi}'$ for isomorphisms $\bar\varphi \in \mathrm{Mor}_{\mathcal{T}}(R,A)$ and $\bar{\varphi}' \in \mathrm{Mor}_{\mathcal{T}}(R,A')$ in $\mathcal{T}$. We get from (3.3) that $\iota_{A'}^P \circ \bar{\varphi}' = \widehat{x}|_A^P \circ \bar\varphi$. From this we deduce that $A' = xAx^{-1}$ and that $\bar{\varphi}' = \widehat{x}|_A^{A'} \circ \bar\varphi$. The second equation in (3.3) gives

$$\lambda' \circ \iota_{A'}^{L'} = \widehat{y} \circ \lambda \circ \iota_A^L \circ \widehat{x^{-1}}|_{A'}^A.$$

We deduce that $(A', \lambda'|_{A'}) = (y,x) \cdot (A, \lambda|_A)$. Clearly $(A', \lambda'|_{A'}) \preceq (L', \lambda')$ and $(A, \lambda|_A) \preceq (L, \lambda)$ so Lemma 3.4 implies that $(L', \lambda') = (y,x) \cdot (L,\lambda)$. Since $(L,\lambda)$ and $(L',\lambda')$ are elements of $\mathcal{K}_{P,Q}^{\max}$ and are in the same orbit of $Q \times P$ it follows that $(L,\lambda) = (L',\lambda')$. In particular $x \in N_P(L)$, and it follows from (3.3) that $\varphi' = \widehat{x} \circ \varphi$ and that $\lambda = \widehat{y} \circ \lambda \circ \widehat{x}^{-1}$ (since $\varphi$ is an epimorphism in $\mathcal{T}$). By Axiom (II) of Definition 2.13, there exists an extension of $\lambda$ to a morphism $\tilde\lambda \colon \langle L, x\rangle \to Q$ in $\mathcal{T}$. Notice that $\langle L, x \rangle \subseteq P$ so the maximality of $(L,\lambda)$ implies that $x \in L$. Since $\varphi' = \widehat{x} \circ \varphi$ we deduce $[\varphi'] = [\varphi]$ namely $h = h'$ as needed. □

**Definition 3.8.** Let $P \xrightarrow{f} R \xleftarrow{g} Q$ be morphisms in $\mathcal{OT}$. Let $U(f,g)$ be the subobject of $P \boxtimes Q$ obtained by restriction of $P \boxtimes Q \colon \mathcal{K}_{P,Q}^{\max} \xrightarrow{(L,\lambda) \mapsto L} \mathrm{obj}(\mathcal{OT})$ to the set

$$I = \{(L,\lambda) \in \mathcal{K}_{P,Q}^{\max} \ : \ f \circ [\iota_L^P] = g \circ [\lambda]\}.$$

Let $\pi_1\colon U(f,g) \to P$ and $\pi_2\colon U(f,g) \to Q$ denote the restrictions of $\pi_1\colon P \boxtimes Q \to P$ and $\pi_2\colon P \boxtimes Q \to Q$ to $U(f,g)$.

**Proposition 3.9.** *Given any $P \xrightarrow{f} R \xleftarrow{g} Q$ in $\mathcal{OT}$ the construction of $U(f,g)$ in Definition 3.8 is the pullback of $P$ and $Q$ along $f$ and $g$ in $\mathcal{OT}_{\amalg}$. Moreover, the pullback of $P \xrightarrow{\iota_P^R} R \xleftarrow{\iota_Q^R} Q$ is*

$$\coprod_{x \in (Q \backslash R/P)_{\mathcal{T}}} Q^x \cap P$$

*where $x$ runs through representatives of the double cosets such that $Q^x \cap P = x^{-1}Qx \cap P$ is an object of $\mathcal{T}$.*

*Proof.* It follows from (3.1) that in order to check the universal property of $U = U(f,g)$ it suffices to test objects $T \in \mathcal{OT}$. Suppose that we are given morphisms $T \xrightarrow{[\varphi]} P$ and $T \xrightarrow{[\psi]} Q$ which satisfy $f \circ [\varphi] = g \circ [\psi]$. We obtain $T \xrightarrow{([\varphi],[\psi])} P \boxtimes Q$ which factors $T \xrightarrow{\bar{h}} L \subseteq P \boxtimes Q$ for some $(L,\lambda) \in \mathcal{K}_{P,Q}^{\max}$. Then

$$\begin{aligned} f \circ \pi_1|_L \circ \bar{h} &= f \circ \pi_1 \circ ([\varphi],[\psi]) = f \circ [\varphi] \\ g \circ \pi_2|_L \circ \bar{h} &= g \circ \pi_2 \circ ([\varphi],[\psi]) = g \circ [\psi]. \end{aligned}$$

Since $\bar{h}$ is an epimorphism in $\mathcal{OT}$ and since $f \circ [\varphi] = g \circ [\psi]$ by assumption, it follows that $f \circ \pi_1|_L = g \circ \pi_2|_L$ which is the statement $f \circ [\iota_L^P] = g \circ [\lambda]$. This precisely means that $(L,\lambda) \in I$ where $I$ is defined in 3.8, hence $h = ([\varphi],[\psi])$ factors through $U$ and clearly $\pi_1 \circ h = [\varphi]$ and $\pi_2 \circ h = [\psi]$. Since the inclusion $U \subseteq P \boxtimes Q$ is a monomorphism in $\mathcal{OT}_{\amalg}$, there can be only one morphism $h\colon T \to U$ such that $\pi_1 \circ h = [\varphi]$ and $\pi_2 \circ h = [\psi]$. This shows that $U = U(f,g)$ is the pullback.

Now assume we are given $P \xrightarrow{\iota} R \xleftarrow{\iota} Q$. The indexing set of the object $U(f,g)$ consists of $(L,\lambda) \in \mathcal{K}_{P,Q}^{\max}$ such that $[\iota_L^R] = [\iota_Q^R \circ \lambda]$, namely $\iota_Q^R \circ \lambda = \widehat{x}|_L^R$ for some $x \in N_R(L,Q)$, which is furthermore unique. Since $\iota_Q^R$ is a monomorphism, this implies that $\lambda = \widehat{x}|_L^Q$. Since $(L,\lambda)$ is maximal, $L = Q^x \cap P$. We obtain a map $U(\iota_P^R, \iota_Q^R) \to (Q\backslash R/P)_{\mathcal{T}}$ which sends $(L,\lambda)$ to $PxQ$ with $x \in N_R(L,Q)$ described above. This map is injective because if $QxP = Qx'P$ are the images of $(L,\lambda)$ and $(L',\lambda')$ then $x' = qxp$ for some $p \in P$ and $q \in Q$ and it follows that $L' = p^{-1}Lp$ and that $\lambda = \hat{x}|_L^Q$ and $\lambda' = \hat{x'}|_{L'}^Q$ and therefore $\lambda = \hat{q} \circ \lambda' \circ \hat{p}|_L^{L'}$, so $(L,\lambda)$ and $(L',\lambda')$ are in the same orbit of $Q \times P$, hence they must be equal. It is surjective since for any $PxQ \in (Q\backslash R/P)_{\mathcal{T}}$ we obtain a summand in $U(\iota_P^R, \iota_Q^R)$ which is equivalent in $K_{P,Q}$ to $(L,\lambda)$ with $L = Q^x \cap P$ and $\lambda = \widehat{x}|_L^Q$. □

3.3. **The Λ-functors.** Let $\Gamma$ be a finite group and $M$ a (right) $\Gamma$-module. Let $p$ be a fixed prime and let $\mathcal{O}_p(\Gamma)$ be the full subcategory of the category of $\Gamma$-sets whose objects are the transitive $\Gamma$-sets whose isotropy groups are $p$-groups. Let $F_M\colon \mathcal{O}_p(\Gamma)^{\mathrm{op}} \to \mathbf{Ab}$ be the functor which assigns $M$ to the free orbit $\Gamma/1$ and 0 to all orbits with non-trivial isotropy. Define ( [JMO95, Definition 5.3])

$$\Lambda^*(\Gamma, M) \overset{\text{def}}{=} \varprojlim_{\mathcal{O}_p(\Gamma)^{\mathrm{op}}}{}^{*} F_M \qquad \Big( = H^*(\mathcal{O}_p(\Gamma)^{\mathrm{op}}; F_M)\Big).$$

These functors have the following important properties.

**Lemma 3.10.** *Suppose that $M$ is a $\mathbb{Z}_{(p)}[\Gamma]$-module.*

(a) *If $C_\Gamma(M)$ contains an element of order $p$ then $\Lambda^*(\Gamma; M) = 0$.*

(b) *If $\Gamma/C_\Gamma(M)$ has order prime to $p$ then $\Lambda^*(\Gamma; M) = 0$ for all $* \geqslant 1$.*

*Proof.* Point (a) is [JMO95, Proposition 6.1(ii)]. Point (b) follows from [JMO95, Proposition 6.1(ii)] when $p$ divides $|C_\Gamma(M)|$ and from [JMO95, Proposition 6.1(i) and (iii)] when $p$ does not divide $|C_\Gamma(M)|$. □

Observe that $\rho\colon \mathcal{T} \to \mathcal{F}$ reflects isomorphisms. Hence the isomorphism classes of objects of $\mathcal{T}$ and of $\mathcal{OT}$ are $\mathcal{F}$-conjugacy classes.

A functor $\Phi\colon \mathcal{OT}^{\mathrm{op}} \to \mathbf{Ab}$ is called *atomic* if there exists $Q \in \mathrm{obj}(\mathcal{T})$ such that $\Phi$ vanishes outside the $\mathcal{F}$-conjugacy class of $Q$. The fundamental property of $\Lambda$-functors is that they calculate the higher limits of atomic functors:

**Lemma 3.11** ( [OV07, Lemma 4.3])**.** *Let $\mathcal{T}$ be a transporter system associated with a fusion system $\mathcal{F}$ over $S$. Let $\Phi\colon \mathcal{OT}^{\mathrm{op}} \to \mathbb{Z}_{(p)}\text{-}\mathfrak{mod}$ be an atomic functor concentrated on the $\mathcal{F}$-conjugacy class of $Q$. Then there is a natural isomorphism*

$$H^*(\mathcal{OT}^{\mathrm{op}}; \Phi) \cong \Lambda^*(\mathrm{Aut}_{\mathcal{OT}}(Q); \Phi(Q)).$$

We remark that the result holds, in fact, for any functor $\Phi$ into the category of abelian groups (indeed, the proof given by Oliver and Ventura only uses [OV07, Proposition A.2]).

Notice that $\rho\colon \mathcal{T} \to \mathcal{F}$ induces a functor $\bar{\rho}\colon \mathcal{OT} \to \mathcal{O}(\mathcal{F})$. We will write $\mathcal{OT}^c$ for the full subcategory of $\mathcal{T}$ spanned by $P \in \mathcal{T}$ which are $\mathcal{F}$-centric.

**Corollary 3.12.** *Let $\mathcal{T}$ be a transporter category for $\mathcal{F}$. Let $\bar{\Phi}\colon \mathcal{O}(\mathcal{F})^{\mathrm{op}} \to \mathbb{Z}_{(p)}\text{-}\mathfrak{mod}$ be a functor and set $\Phi = \bar{\Phi} \circ \bar{\rho}$. Then $\Phi$ is a functor $\mathcal{OT}^{\mathrm{op}} \to \mathbb{Z}_{(p)}\text{-}\mathfrak{mod}$ and let $\Psi$ be the restriction of $\Phi$ to $\mathcal{OT}^c$. Then the restriction induces an isomorphism*

$$H^*(\mathcal{OT}^{\mathrm{op}}, \Phi) \xrightarrow{\cong} H^*((\mathcal{OT}^c)^{\mathrm{op}}; \Psi).$$

*Proof.* Let $\Phi'\colon \mathcal{OT}^{\mathrm{op}} \to \mathbb{Z}_{(p)}\text{-}\mathfrak{mod}$ be the functor obtained from $\Phi$ by setting $\Phi'(Q) = 0$ for all $Q \in \mathrm{obj}(\mathcal{T} \setminus \mathcal{T}^c)$ and $\Phi'(Q) = \Phi(Q)$ otherwise. This is a well defined functor since the $\mathcal{F}$-centric subgroups are closed to overgroups. It is a standard check that the bar constructions REFERENCE of $\Phi'$ and that of $\Psi$ are equal. It follows that

$$H^*(\mathcal{OT}^{\mathrm{op}}, \Phi') \cong H^*((\mathcal{OT}^c)^{\mathrm{op}}, \Psi).$$

It remains to show that $H^*(\mathcal{OT}^{\mathrm{op}}, \Phi) \cong H^*(\mathcal{OT}^{\mathrm{op}}, \Phi')$.

Suppose that $Q \in \mathrm{obj}(\mathcal{T}\setminus\mathcal{T}^c)$ has minimal order. Set $M = \Phi(Q)$ and let $F_M\colon \mathcal{OT}^{\mathrm{op}} \to \mathbb{Z}_{(p)}\text{-}\mathfrak{mod}$ be the induced atomic functor. The minimality of $Q$ implies that there is an injective natural transformation $F_M \to \Phi$. By possibly replacing it with an $\mathcal{F}$-conjugate, we may assume that $Q$ is fully centralised in $\mathcal{F}$. Since $Q$ is not $\mathcal{F}$-centric, choose some $x \in C_S(Q) \setminus Q$. Its image in $\Gamma = \mathrm{Aut}_{\mathcal{OT}}(Q)$ is a non-trivial element (since $x \notin Q$) of order $p$-power. It acts trivially on $\Phi(Q)$ because its image in $\mathrm{Out}_{\mathcal{F}}(Q)$ is trivial (since the image of $C_S(Q)$ in $\mathrm{Aut}_{\mathcal{F}}(Q)$ is trivial) and because $\Phi = \bar{\Phi} \circ \bar{\rho}$. Lemma 3.10(a) implies that $\Lambda^*(\mathrm{Aut}_{\mathcal{OT}}(Q), \Phi(Q)) = 0$. It follows from Lemma 3.11 and the long exact sequence in derived limits associated with the short exact sequence $0 \to F_M \to \Phi \to \Phi/F_M \to 0$ that $H^*(\mathcal{OT}^{\mathrm{op}}, \Phi) \cong H^*(\mathcal{OT}^{\mathrm{op}}, \Phi/F_M)$. But $\Phi/F_M$ is obtained from

$\Phi$ by annihilating the groups $\Phi(Q')$ for all $Q'$ in the $\mathcal{F}$-conjugacy class of $Q$. We may now apply the same process to $\Phi/F_M$ and continue inductively (on the number of $\mathcal{F}$-conjugacy classes in $\mathcal{T} \setminus \mathcal{T}^c$) to show that $H^*(\mathcal{OT}^{\mathrm{op}}, \Phi) \cong H^*(\mathcal{OT}^{\mathrm{op}}, \Phi')$ as needed. $\square$

*Proof of Theorem 1.1.* Let $\mathcal{F}$ be a saturated fusion system over $S$ which affords a punctured group $\mathcal{T}$. That is, $\mathcal{T}$ is a transporter system associated to $\mathcal{F}$ with object set $\Delta$ containing all the non-trivial subgroups of $S$.

Let $\mathcal{H}^j \colon \mathcal{O}(\mathcal{F})^{\mathrm{op}} \to \mathbb{Z}_{(p)}\text{-}\mathfrak{mod}$ be the functor

$$\mathcal{H}^j \colon P \mapsto H^j(P; \mathbb{F}_p)$$

and let $M^j \colon \mathcal{OT}^{\mathrm{op}} \to \mathbb{Z}_{(p)}\text{-}\mathfrak{mod}$ be the pullback of $\mathcal{H}^j$ along $\mathcal{OT} \xrightarrow{\bar{\rho}} \mathcal{O}(\mathcal{F})$. Our goal is to show that for every $j \geqslant 0$,

$$H^i(\mathcal{O}(\mathcal{F}^c)^{\mathrm{op}}; \mathcal{H}^j) = 0 \qquad \text{for all } i \geqslant 1.$$

Choose a fully normalised $P \in \mathrm{obj}(\mathcal{T})$. Since $N_S(P)$ is a Sylow $p$-subgroup of $\mathrm{Aut}_{\mathcal{T}}(P)$, see [OV07, Proposition 3.4(a)], it follows that $C_S(P)$ is a Sylow $p$-subgroup of the kernel of $\mathrm{Aut}_{\mathcal{T}}(P) \to \mathrm{Aut}_{\mathcal{F}}(P)$ and hence $C_S(P)/Z(P)$ is a Sylow $p$-subgroup of the kernel of $\mathrm{Aut}_{\mathcal{OT}}(P) \to \mathrm{Out}_{\mathcal{F}}(P)$. Thus, if $P$ is $\mathcal{F}$-centric this kernel has order prime to $p$ so [BLO03a, Lemma 1.3] implies the first isomorphism in the display below, while Corollary 3.12 gives the second.

$$H^*(\mathcal{O}(\mathcal{F}^c)^{\mathrm{op}}; \mathcal{H}^j) \cong H^*((\mathcal{OT}^c)^{\mathrm{op}}; M^j) \cong H^*(\mathcal{OT}^{\mathrm{op}}; M^j).$$

It remains to show that $H^*(\mathcal{OT}^{\mathrm{op}}; M^j) = 0$ for all $j \geqslant 0$ and all $* \geqslant 1$.

Assume first that $j \geqslant 1$. We will show that $M^j$ is a proto-Mackey functor for $\mathcal{OT}$.

The transfer homomorphisms give rise to a (covariant) functor $\mathcal{H}^j_* \colon \mathcal{O}(\mathcal{F}) \to \mathbb{Z}_{(p)}\text{-}\mathfrak{mod}$ where $P \mapsto H^j(P; \mathbb{F}_p)$ and to any $\varphi \in \mathcal{F}(P, Q)$ we assign $\mathrm{tr}(\varphi) \colon H^j(P; \mathbb{F}_p) \to \mathcal{H}^j(Q; \mathbb{F}_p)$. Its pullback to $\mathcal{OT}$ via $\bar{\rho}$ gives a covariant functor $M^j_* \colon \mathcal{OT} \to \mathbb{Z}_{(p)}\text{-}\mathfrak{mod}$.

Now, $\mathcal{OT}$ satisfies (PB $\times_{\mathrm{II}}$) by Propositions 3.7 and 3.9. Clearly, $M^j$ and $M^j_*$ have the same values on objects; this is the first condition in Definition 3.2. The transfer homomorphisms have the property that if $\varphi \colon P \to Q$ is an isomorphism then $\mathrm{tr}_P^Q(\varphi) = H^j(\varphi^{-1}; \mathbb{F}_p)$. This is the second condition in Definition 3.2. The factorisation of morphisms in $\mathcal{T}$ as isomorphisms followed by inclusions imply that any pullback diagram $P' \xrightarrow{f} R \xleftarrow{g} Q'$ in $\mathcal{OT}$ is isomorphic to one of the form $P \xrightarrow{[\iota_P^R]} R \xleftarrow{[\iota_Q^R]} Q$. If $U = U([\iota_P^R], [\iota_Q^R])$ is the pullback (Definition 3.8), then, by Proposition 3.9, $U = \coprod_{x \in (Q\backslash R/P)_{\mathcal{T}}} Q^x \cap P$ where $x$ runs through representatives of double cosets such that $Q^x \cap P \neq 1$ (since $\mathrm{obj}(\mathcal{T})$ is the set of all non-trivial subgroups of $S$). Since $j \geqslant 1$ we have that $H^j(\{1\}; \mathbb{F}_p) = 0$ so $\bigoplus_{x \in (Q\backslash R/P)_{\mathcal{T}}} H^j(Q^x \cap P; \mathbb{F}_p) = \bigoplus_{x \in Q\backslash R/P} H^j(Q^x \cap P; \mathbb{F}_p)$. Mackey's formula gives the commutativity of the diagram

$$\begin{tikzcd}
\bigoplus_{x \in (Q\backslash R/P)_{\mathcal{T}}} H^j(Q^x \cap P; \mathbb{F}_p) \arrow[r, "\sum_x \mathrm{tr}^P_{Q^x \cap P}(c_x)"] & Q \\
H^j(P; \mathbb{F}_p) \arrow[u, "\sum_x \mathrm{res}^P_{Q^x \cap P}"] \arrow[r, "\mathrm{res}^R_P"'] & R \arrow[u, "\mathrm{tr}^R_Q"']
\end{tikzcd}$$

This shows that the third condition in Definition 3.2 also holds and that $M^j$ is a proto-Mackey functor. Now, Condition (B1) in Proposition 3.3 clearly holds for $\mathcal{OT}$ and (B2) holds by Lemma 3.5. It follows that $H^i(\mathcal{OT}^{\mathrm{op}}; M^j) = 0$ for all $i \geqslant 1$ as needed.

It remains to deal with the case $j = 0$. In this case $\mathcal{H}^0$ is the constant functor with value $\mathbb{F}_p$. Thus, $\mathrm{Out}_{\mathcal{F}}(P)$ acts trivially on $\mathbb{F}_p$ for any $P \in \mathcal{F}^c$. It follows from Lemma 3.10(b) that if $P = S$ then $\Lambda^i(\mathrm{Out}_{\mathcal{F}}(S), \mathbb{F}_p) = 0$ for all $i > 0$, and if $P < S$ then $\mathrm{Out}_{\mathcal{F}}(P)$ contains an element of order $p$ so $\Lambda^*(\mathrm{Out}_{\mathcal{F}}(P), \mathbb{F}_p) = 0$. Now [BLO03b, Proposition 3.2] together with a filtration of $\mathcal{H}^0$ by atomic functors show that $\mathcal{H}^0$ is acyclic. $\square$

## 4. Punctured groups for $\mathcal{F}_{\mathrm{Sol}}(q)$

The Benson-Solomon systems were predicted to exist by Benson [Ben94], and were later constructed by Levi and Oliver [LO02,LO05]. They form a family of exotic fusion systems at the prime 2 whose isomorphism types are parametrized by the nonnegative integers. Later, Aschbacher and Chermak gave a different construction as the fusion system of an amalgamated free product of finite groups [AC10]. The main result of this section is the following theorem.

**Theorem 4.1.** *A Benson-Solomon system $\mathcal{F}_{\mathrm{Sol}}(q)$ over the 2-group $S$ has a punctured group if and only if $q \equiv \pm 3 \pmod 8$. If $q = \pm 3 \pmod 8$, there is a punctured group $\mathcal{L}$ for $\mathcal{F}_{\mathrm{Sol}}(q)$ which is unique up to rigid isomorphism with the following two properties:*

*(1) $C_{\mathcal{L}}(Z(S)) = \mathrm{Spin}_7(3)$, and*
*(2) $\mathcal{L}|_{\Delta}$ is a linking locality, where $\Delta$ is the set of $\mathcal{F}$-subcentric subgroups of $S$ of 2-rank at least 2.*

4.1. **Notation for $\mathrm{Spin}_7$ and Sol.** It will be usually be most convenient to work with a Lie theoretic description of $\mathrm{Spin}_7$. The notational conventions that we use in this section for algebraic groups and finite groups of Lie type are summarized in Appendix A.

4.1.1. *The maximal torus and root system.* Let $p$ be an odd prime, and set

$$\overline{H} = \mathrm{Spin}_7(\bar{\mathbb{F}}_p).$$

Fix a maximal torus $\bar{T}$ of $\overline{H}$, let $X(\bar{T}) = \mathrm{Hom}(\bar{T}, \bar{\mathbb{F}}_p^\times) \cong \mathbb{Z}^3$ be the character group (of algebraic homomorphisms), and denote by $V = \mathbb{R} \otimes_{\mathbb{Z}} X(\bar{T})$ the ambient Euclidean space which we regard as containing $X(\bar{T})$. Let $\Sigma(\bar{T}) \subseteq X(\bar{T})$ be the set of $\bar{T}$-roots. Denote a $\bar{T}$-root subgroup for the root $\alpha$ by

$$\overline{X}_\alpha = \{x_\alpha(\lambda) \mid \lambda \in \bar{\mathbb{F}}_p\}.$$

As $\overline{H}$ is semisimple, it is generated by its root subgroups [GLS98, Theorem 1.10.1(a)]. We assume that the implicit parametrization $x_\alpha(\lambda)$ of the root subgroups is one like that given by Chevalley, so that the Chevalley relations hold with respect to certain signs $c_{\alpha,\beta} \in \{\pm 1\}$ associated to each pair of roots [GLS98, Theorem 1.12.1].

We often identify $\Sigma(\bar{T})$ with the abstract root system

$$\Sigma = \{\pm e_i \pm e_j, \pm e_i \mid 1 \leqslant i, j \leqslant 3\} \subseteq \mathbb{R}^3$$

of type $B_3$, having base $\Pi = \{\alpha_1, \alpha_2, \alpha_3\}$ with

$$\alpha_1 = e_1 - e_2, \quad \alpha_2 = e_2 - e_3, \quad \alpha_3 = e_3,$$

where the $e_i$ are standard vectors. Write $\Sigma^\vee = \{\alpha^\vee \mid \alpha \in \Sigma\}$ for the dual root system, where $\alpha^\vee = 2\alpha/(\alpha,\alpha)$.

Instead of working with respect to the $\alpha_i$, it is sometimes convenient to work instead with a different set of roots $\{\beta_i\} \subseteq \Sigma$:

$$\beta_1 = \alpha_1, \quad \beta_2 = \alpha_1 + 2\alpha_2 + 2\alpha_3 = e_1 + e_2, \quad \beta_3 = \alpha_3.$$

This is an orthogonal basis of $V$ with respect to the standard inner product $(\,,\,)$ on $\mathbb{R}^3$. An important feature of this basis is that for each $i$ and $j$,

$$\Sigma \cap \{k\beta_i + l\beta_j \mid k,l \in \mathbb{Z}\} = \{\pm\beta_i, \pm\beta_j\}, \tag{4.1}$$

a feature not enjoyed, for example, by an orthogonal basis consisting of short roots. In particular, the $\beta_j$-root string through $\beta_i$ consists of $\beta_i$ only. This implies via Lemma A.1(4), that the corresponding signs involving the $\beta_i$ that appear in the Chevalley relations for $\overline{H}$ are

$$c_{\beta_i,\beta_j} = 1 \text{ if } i \neq j, \text{ and } c_{\beta_i,\beta_i} = -1. \tag{4.2}$$

4.1.2. *The torus and the lattice of coroots.* We next set up notation and state various relations for elements of $\bar{T}$. Let

$$h_\alpha(\lambda) \in \bar{T} \quad \text{and} \quad n_\alpha(\lambda) \in N_{\overline{H}}(\bar{T})$$

be as given in Appendix A as words in the generators $x_\alpha(\lambda)$. By Lemma A.2 and since $\overline{H}$ is of universal type, there is an isomorphism $\mathbb{Z}\Sigma^\vee \otimes \bar{\mathbb{F}}_p^\times \to \bar{T}$ which on simple tensors sends $\alpha^\vee \otimes \lambda$ to $h_\alpha(\lambda)$, and the homomorphisms $h_{\alpha_i} \colon \bar{\mathbb{F}}_p^\times \to \bar{T}$ are injective. In particular, as $\Pi^\vee = \{\alpha_1^\vee, \alpha_2^\vee, \alpha_3^\vee\}$ is a basis for $\mathbb{Z}\Sigma^\vee$, we have $\bar{T} = h_{\alpha_1}(\bar{\mathbb{F}}_p^\times) \times h_{\alpha_2}(\bar{\mathbb{F}}_p^\times) \times h_{\alpha_3}(\bar{\mathbb{F}}_p^\times)$. Define elements $z$ and $z_1 \in \bar{T}$ by

$$z_1 = h_{\alpha_1}(-1) \quad \text{and} \quad z = h_{\alpha_3}(-1)$$

Thus, $z$ and $z_1$ are involutions. Similar properties hold with respect to the $\beta_i$'s. Recall that $\beta_i = \alpha_i$ for $i = 1, 3$. Since $\beta_2^\vee = \alpha_1^\vee + 2\alpha_2^\vee + \alpha_3^\vee$, Lemma A.2(3) yields

$$h_{\beta_2}(-1) = h_{\alpha_1}(-1)h_{\alpha_2}((-1)^2)h_{\alpha_3}(-1) = z_1 z.$$

In particular,

$$h_{\beta_1}(-1)h_{\beta_2}(-1)h_{\beta_3}(-1) = z_1 z_1 z z = 1. \tag{4.3}$$

However, as $\mathbb{Z}$-span of the $\beta_i^\vee$'s is of index 2 in $\mathbb{Z}\Sigma^\vee$ and every element of $\bar{\mathbb{F}}_p^\times$ is a square, we still have

$$\bar{T} = h_{\beta_1}(\bar{\mathbb{F}}_p^\times)h_{\beta_2}(\bar{\mathbb{F}}_p^\times)h_{\beta_3}(\bar{\mathbb{F}}_p^\times). \tag{4.4}$$

So the $h_{\beta_i}(\bar{\mathbb{F}}_p)^\times$ generate $\bar{T}$, but the product is no longer direct.

4.1.3. *The normalizer of the torus and Weyl group.* The subgroup

$$\widehat{W} := \langle n_{\alpha_1}(1), n_{\alpha_2}(1), n_{\alpha_3}(1)\rangle \leqslant N_{\overline{H}}(\bar{T})$$

projects onto the Weyl group

$$W = \langle w_{\alpha_1}, w_{\alpha_2}, w_{\alpha_3}\rangle \cong C_2 \wr S_3 \cong C_2 \times S_4$$

of type $B_3$ in which the $w_{\alpha_i}$ are fundamental reflections. Also, $\widehat{W} \cap \bar{T}$ is the 2-torsion subgroup $\{t \in \bar{T} \mid t^2 = 1\}$ of $\bar{T}$. A subgroup similar to $\widehat{W}$ was denoted "$W$" in [AC10, Lemma 4.3]. It is sometimes called the Tits subgroup [GLS98, Remark 1.12.11].

Let

$$\gamma = c_{\alpha_1, \alpha_2+\alpha_3} \in \{\pm 1\},$$

and fix a fourth root $i \in \mathbb{F}_p^\times$ of 1. (This notation will hopefully not cause confusion with the use of $i$ as an index.) Define elements $w_0, \tau \in N_{\overline{H}}(\bar{T})$ by

$$w_0 = n_{\beta_1}(-\gamma) n_{\beta_2}(1) n_{\beta_3}(1) \quad \text{and} \quad \tau = n_{\alpha_2+\alpha_3}(1) h_{\beta_1}(-i) h_{\beta_2}(i) h_{\beta_3}(i).$$

It will be shown in Lemma 4.2 that $w_0$ and $\tau$ are commuting involutions and that $w_0$ inverts $\bar{T}$.

4.1.4. *Three commuting $SL_2$'s.* Let

$$\bar{L}_i = \langle \overline{X}_{\beta_i}, \overline{X}_{-\beta_i}\rangle,$$

for $i = 1, 2, 3$. Thus, $\bar{L}_i \cong SL_2(\bar{\mathbb{F}}_p)$ for each $i$ by the Chevalley relations, again using that $\overline{H}$ is of universal type when $i = 3$. A further consequence of (4.1) is that the Chevalley commutator formula [GLS98, 1.12.1(b)] yields

$$[\bar{L}_i, \bar{L}_j] = 1 \quad \text{for all} \quad i \neq j.$$

For each $i$, $\bar{L}_i$ has unique involution $h_{\beta_i}(-1)$ which generates the center of $\bar{L}_i$. By (4.3), the center of the commuting product $\bar{L}_1\bar{L}_2\bar{L}_3$ is $\langle z, z_1\rangle$, of order 4. By (4.4), $\bar{T} \leqslant \bar{L}_1\bar{L}_2\bar{L}_3$.

4.1.5. *The Steinberg endomorphism and* $\mathrm{Spin}_7(q)$. We next set up notation for the Steinberg endomorphism we use to descend from $\overline{H}$ to the finite versions. Let $q = p^a$ be a power of $p$. Let $\epsilon \in \{\pm 1\}$ be such that $q \equiv \epsilon \pmod 4$, and let $k$ be the 2-adic valuation of $q - \epsilon$.

The standard Frobenius endomorphism $\zeta$ of $\overline{H}$ is determined by its action $x_\alpha(\lambda)^\zeta = x_\alpha(\lambda^p)$ on the root groups, and so from the definition of the $n_\alpha$ and $h_\alpha$ in (A.1), also $n_\alpha(\lambda)^\zeta = n_\alpha(\lambda^p)$ and $h_\alpha(\lambda)^\zeta = h_\alpha(\lambda^p)$. Write $c_{w_0}$ conjugation map induced by $w_0$, as usual, and define

$$\sigma = \begin{cases} \zeta^a & \text{if } \epsilon = 1 \\ \zeta^a c_{w_0} & \text{if } \epsilon = -1. \end{cases}$$

Then $\sigma$ is a Steinberg endomorphism of $\overline{H}$ in the sense of [GLS98, Definition 1.15.1], and we set

$$H := C_{\overline{H}}(\sigma) = \mathrm{Spin}_7(q).$$

Given that $w_0$ inverts $\bar{T}$, the action of $\sigma$ on $\bar{T}$ is given for each $t \in \bar{T}$ by

$$t^\sigma = t^{\epsilon q},$$

and hence

$$C_{\bar{T}}(\sigma) = \{t \in \bar{T} \mid t^{\epsilon q} = t\} \cong (C_{q-\epsilon})^3$$

Likewise,

$$C_{\bar{T}}(\sigma c_{w_0}) \cong (C_{q+\epsilon})^3.$$

Finally, let $\mu = \mu_q \in \bar{\mathbb{F}}_p^\times$ be a fixed element of 2-power order satisfying $\mu^{\epsilon q} = -\mu$ and powering to the fourth root $i$, and we set

$$c = h_{\beta_1}(\mu)h_{\beta_2}(\mu)h_{\beta_3}(\mu) \in \bar{T}.$$

4.1.6. *A Sylow 2-subgroup.* We next set up notation for Sylow 2-subgroups of $\overline{H}$ and $H$ along with various important subgroups of them. Let

$$\bar{S} = \bar{T}_{2^\infty}\widehat{W}_{\bar{S}},$$

where $\bar{T}_{2^\infty}$ denotes the 2-power torsion in $\bar{T}$ and where $\widehat{W}_{\bar{S}} = \langle n_{\alpha_1}(1), n_{\alpha_2+\alpha_3}(1), n_{\alpha_3}(1)\rangle$. Set

$$S = C_{\bar{S}}(\sigma).$$

Define subgroups

$$Z < U < E < A \leqslant S$$

by

$$Z = \langle z\rangle, \quad U = \langle z, z_1\rangle, \quad E = \langle z, z_1, e\rangle, \quad \text{and } A = E\langle w_0\rangle.$$

Then $Z = Z(S)$, $U$ is the unique four subgroup normal in $S$, and $E = \{t \in \bar{T} \mid t^2 = 1\} = \{t \in T \mid t^2 = 1\}$. It will be shown in Lemma 4.2 that $w_0 \in S$, and hence $A \leqslant S$.

We also write

$$T_S = T \cap S;$$

thus, $T_S = O_2(T) \cong (C_{2^k})^3$ is the $2^k$-torsion in $\bar{T}$, a Sylow 2-subgroup of $T$.

4.2. **Conjugacy classes of elementary abelian subgroups of $\overline{H}$ and $H$.** We state and prove here several lemmas on conjugacy classes of elementary abelian subgroups of $\overline{H}$ and $H$, and on the structure of various 2-local subgroups. Much of the material here is written down elsewhere, for example in [LO02] and [AC10]. Our setup is a little different because of the emphasis on the Lie theoretic approach, so we aim to give more detail in order to make the treatment here as self-contained as possible.

The first lemma is elementary and records several initial facts about the elements we have defined in the previous section. Its proof is mainly an exercise in applying the various Chevalley relations defining $\overline{H}$.

**Lemma 4.2.** *Adopting the notation from §§4.1, we have*

*(1)* $Z(\overline{H}) = Z = \langle z\rangle$*;*
*(2) the elements* $w_0$ *and* $\tau$ *are involutions in* $N_S(\bar{T}) - \bar{T}$*, and* $c \in T_S$ *has order* $2^l$*, powering into* $E - U$*;*
*(3)* $w_0$ *inverts* $\bar{T}$*; and*
*(4)* $[w_0, \tau] = [c, \tau] = 1$*.*

*Proof.* **(1)**: It is well known that $Z(\overline{H})$ has order 2. We show here for the convenience of the reader that the involution generating $Z(\overline{H})$ is $z = h_{\alpha_3}(-1)$. We already observed in §§4.1.3 that $z$ is an involution. For each root $\alpha \in \Sigma$, the inner product of $\alpha$ with $\alpha_3$ is an integer, and so $\langle \alpha, \alpha_3 \rangle = 2(\alpha, \alpha_3) \in 2\mathbb{Z}$. By Lemma A.2(1), $h_{\alpha_3}(-1)$ lies in the kernel of $\alpha$. Thus, the centralizer in $\overline{H}$ of $h_{\alpha_3}(-1)$ contains all root groups by Proposition A.3, and hence $C_{\overline{H}}(h_{\beta_3}(-1)) = \overline{H}$.

**(2)**: We show that $w_0$ is an involution. Using equations (A.6) and (4.2), we see that

$$[n_{\beta_i}(\pm 1), n_{\beta_j}(\pm 1)] = 1 \text{ for each } i, j \in \{1, 2, 3\}. \tag{4.5}$$

So $w_0^2 = 1$ by (A.7) and (4.3).

We next prove that $\tau$ is an involution. Recall

$$\tau = n_{\alpha_2+\alpha_3}(1) h_{\beta_1}(-i) h_{\beta_2}(i) h_{\beta_3}(i).$$

First, note that $n_{\alpha_2+\alpha_3}(1)^2 = z$. To see this, use (A.7) to get $n_{\alpha_2+\alpha_3}(1)^2 = h_{\alpha_2+\alpha_3}(-1)$. Then use $(\alpha_2 + \alpha_3)^\vee = 2\alpha_2 + 2\alpha_3 = 2\alpha_2^\vee + \alpha_3^\vee$ and Lemma A.2(3) to get

$$n_{\alpha_2+\alpha_3}(1)^2 = h_{\alpha_2+\alpha_3}(-1)^2 = h_{\alpha_2}(-1)^2 h_{\alpha_3}(-1) = h_{\alpha_3}(-1) = z$$

as desired. Next, the fundamental reflection $w_{\alpha_2+\alpha_3}$ interchanges $\beta_1$ and $\beta_2$ and fixes $\beta_3$, so $n_{\alpha_2+\alpha_3}(1)$ inverts $h_{\beta_1}(-i)h_{\beta_2}(i)$ by conjugation and centralizes $h_{\beta_3}(i)$ by (A.5). Hence,

$$\begin{aligned}\tau^2 &= n_{\alpha_2+\alpha_3}(1)^2 (h_{\beta_1}(-i)h_{\beta_2}(i)h_{\beta_3}(i))^{n_{\alpha_2+\alpha_3}(1)} (h_{\beta_1}(-i)h_{\beta_2}(i)h_{\beta_3}(i)) \\ &= n_{\alpha_2+\alpha_3}(1)^2 h_{\beta_3}(i)^2 = zz = 1.\end{aligned}$$

We show $c$ is of order $2^l$ and powers into $E - U$. Recall that $k$ is the 2-adic valuation of $q - \epsilon$, and that $C_{\bar{T}}(\sigma) = (C_{q-\epsilon})^3$. The latter has Sylow 2-subgroup of exponent $2^k$. But $c \in C_{\bar{T}}(\sigma)$ since

$$c^\sigma = h_{\beta_1}(\mu^{\epsilon q}) h_{\beta_2}(\mu^{\epsilon q}) h_{\beta_3}(\mu^{\epsilon q}) = h_{\beta_1}(-\mu) h_{\beta_2}(-\mu) h_{\beta_3}(-\mu) \overset{(4.3)}{=} h_{\beta_1}(\mu) h_{\beta_2}(\mu) h_{\beta_3}(\mu) = c.$$

So $c$ has order at most $2^l$. On the other hand,

$$c^{2^{l-1}} = h_{\beta_1}(i) h_{\beta_2}(i) h_{\beta_3}(i).$$

As in §§4.1.4, we have $h_{\beta_2}(i) = h_{\alpha_1}(i) h_{\alpha_2}(i^2) h_{\alpha_3}(i)$, and so

$$c^{2^{l-1}} = h_{\alpha_1}(-1) h_{\alpha_2}(-1) h_{\alpha_3}(-1).$$

Since $\overline{H}$ is of universal type and $U = \langle h_{\alpha_1}(-1), h_{\alpha_3}(-1) \rangle$, it follows from Lemma A.2(2) that $c^{2^{l-1}} \in E - U$, and hence $c$ has order $2^l$ as claimed. In particular, this shows $c \in T_S$.

It remains to show that $w_0, \tau \in S$ in order to complete the proof of (2). For each $\alpha \in \Sigma$, we have $[n_\alpha(\pm 1), \zeta] = 1$ by (A.1), while $[n_{\beta_i}(\pm 1), w_0] = 1$ for $i = 1, 2, 3$ by (A.6) and (4.2). Also, $h_{\beta_1}(\pm i) h_{\beta_2}(\pm i) h_{\beta_3}(\pm i) \in E \leqslant H$ by (4.3). These points combine to give $w_0 \in H$, $\tau^\zeta = \tau$, and $\tau \in \bar{S}$. As $[w_0, \tau] = 1$ by (4) below, we see $\tau \in H$, so indeed $\tau \in H \cap \bar{S} = S$. Finally,

$$n_{\beta_1}(1) n_{\beta_2}(\gamma) n_{\beta_3}(1) = n_{\beta_1}(1) n_{\beta_1}(1)^{n_{\alpha_2+\alpha_3}(1)} n_{\beta_3}(1) \in \bar{S}$$

and this element represents the same coset modulo $E$ as $w_0$ does by (A.7) and (4.5). Since $E \leqslant S$, it follows that $w_0 \in S$.

**(3)**: Since $\{\beta_1, \beta_2, \beta_3\}$ is an orthogonal basis of $V$, the image $w_{\beta_1}w_{\beta_2}w_{\beta_3}$ in $W$ of $w_0$ acts as minus the identity on $V$. In particular, it acts as minus the identity on the lattice of coroots $\mathbb{Z}\Sigma^\vee \subseteq V$. This implies via Lemma A.2(4) that $w_0$ inverts $\overline{T}$, and so (3) holds.

**(4)**: Showing $[w_0, \tau] = 1$ requires some information about the signs appearing in our fixed Chevalley presentation. First,

$$\langle \beta_1, \alpha_2 + \alpha_3 \rangle = \frac{2(\alpha_1, \alpha_2 + \alpha_3)}{(\alpha_2 + \alpha_3, \alpha_2 + \alpha_3)} = -2.$$

So by Lemma A.1(3),

$$c_{\beta_1,\alpha_2+\alpha_3}c_{\beta_2,\alpha_2+\alpha_3} = (-1)^{\langle \beta_1, \alpha_2+\alpha_3\rangle} = (-1)^{-2} = 1,$$

and hence $c_{\beta_2,\alpha_2+\alpha_3} = \gamma \overset{\text{def}}{=} c_{\beta_1,\alpha_2+\alpha_3}$. The root string of $\alpha_2 + \alpha_3 = e_2$ through $\beta_3 = e_3$ is $e_3 - e_2, e_3, e_3 + e_2$ so

$$c_{\beta_3,\alpha_2+\alpha_3} = (-1)^1 = -1$$

by Lemma A.1(4). So $n_{\alpha_2+\alpha_3}(1)$ inverts each of $n_{\beta_1}(-\gamma)n_{\beta_2}(1)$ and $n_{\beta_3}(1)$, by (A.6). Using $[w_0, n_{\alpha_2+\alpha_3}(1)] \in E$, $h_{\beta_1}(\pm i)h_{\beta_2}(\pm i)h_{\beta_3}(\pm i) \in E$, and (4.5), we thus have

$$\begin{aligned}
[w_0, \tau] &= [w_0, h_{\beta_1}(-i)h_{\beta_2}(i)h_{\beta_3}(i)][w_0, n_{\alpha_2+\alpha_3}(1)]^{h_{\beta_1}(-i)h_{\beta_2}(i)h_{\beta_3}(i)} \\
&= [w_0, h_{\beta_1}(-i)h_{\beta_2}(i)h_{\beta_3}(i)][w_0, n_{\alpha_2+\alpha_3}(1)] \\
&= [w_0, n_{\alpha_2+\alpha_3}(1)] \\
&= [n_{\beta_1}(-\gamma)n_{\beta_2}(1), n_{\alpha_2+\alpha_3}(1)]^{n_{\beta_3}(1)}[n_{\beta_3}(1), n_{\alpha_2+\alpha_3}(1)] \\
&= (n_{\beta_1}(-\gamma)^2 n_{\beta_2}(1)^2)^{n_{\beta_3}(1)} n_{\beta_3}(1)^2 \\
&= n_{\beta_1}(\gamma)^2 n_{\beta_2}(-1)^2 n_{\beta_3}(1)^2 \\
&= z_1 z_1 z z \\
&= 1.
\end{aligned}$$

Finally, since $[c, n_{\alpha_2+\alpha_3}(1)] = 1$ by (A.5), we have $[c, \tau] = 1$. $\square$

For any group $X$, write $\mathscr{E}_k(X)$ for the elementary abelian subgroups of $X$ of order $2^k$ and $\mathscr{E}_k(X, Y)$ for the subset of $\mathscr{E}_k(X)$ consisting of those members containing the subgroup $Y$. Denote by $X^\circ$ the connected component of $X$.

We next record information about the conjugacy classes and normalizers of four subgroups containing $Z$.

**Lemma 4.3.** *Let $\overline{B} = N_{\overline{H}}(U)$ and $B = N_H(U)$. Write $\overline{B}^\circ$ for the connected component of $\overline{B}$.*

*(1) $\mathscr{E}_2(\overline{H}, Z) = U^{\overline{H}}$, and*

$$\overline{B} = (\bar{L}_1\bar{L}_2\bar{L}_3)\langle \tau \rangle \quad \text{and} \quad \overline{B}^\circ = C_{\overline{H}}(U) = \bar{L}_1\bar{L}_2\bar{L}_3,$$

*where $\tau$ interchanges $\bar{L}_1$ and $\bar{L}_2$ by conjugation and centralizes $\bar{L}_3$. Moreover $Z(\overline{B}^\circ) = U$.*

*(2) $\mathscr{E}_2(H, Z) = U^H$, and*

$$B = (L_1L_2L_3)\langle c, \tau \rangle \text{ and } C_H(U) = (L_1L_2L_3)\langle c \rangle,$$

*where $L_i = C_{\bar{L}_i}(\sigma)$, and where $c \in N_{\overline{T}}(L_1L_2L_3)$ acts as a diagonal automorphism on each $L_i$.*

*Proof.* Viewing $\overline{H}$ classically, an involution in $\overline{H}/Z$ has involutory preimage in $\overline{H}$ if and only if it has $-1$-eigenspace of dimension 4 on the natural orthogonal module (see, for example, [AC10, Lemma 4.2] or [LO02, Lemma A.4(b)]). It follows that all noncentral involutions are $\overline{H}$-conjugate into $U$, and hence that all four subgroups containing $Z$ are conjugate. Viewing $\overline{H}$ Lie theoretically gives another way to see this: let $V$ be a four subgroup of $\overline{H}$ containing $Z$, and let $v \in V - Z$. By e.g. [Spr09, 6.4.5(ii)], $v$ lies in a maximal torus, and all maximal tori are conjugate. So we may conjugate in $\overline{H}$ and take $v \in E$. Using Lemma 4.4(1) below for example, $N_{\overline{H}}(\bar{T})/C_{N_{\overline{H}}(\bar{T})}(E) \cong S_4$ acts faithfully on $E$ and centralizes $Z$, so as a subgroup of $GL(E)$ it is the full stabilizer of the chain $1 < Z < E$. This implies $N_{\overline{H}}(\bar{T})$ acts transitively on the nonidentity elements of the quotient $E/Z$, so $v$ is $N_{\overline{H}}(T)$-conjugate into $U$.

We next use Proposition A.3 to compute $\overline{B}$. Recall that

$$U = \langle z, z_1 \rangle = \langle h_{\alpha_3}(-1), h_{\alpha_1}(-1) \rangle$$

and that $z = h_{\alpha_3}(-1)$ is central in $\overline{H}$ by Lemma 4.2(1). So $C_{\overline{H}}(U) = C_{\overline{H}}(h_{\alpha_1}(-1))$. By Proposition A.3 and inspection of $\Sigma$,

$$\begin{aligned} C_{\overline{H}}(U)^\circ &= \langle \bar{T}, \overline{X}_\alpha \mid \ \langle \alpha, \alpha_1 \rangle \text{ is even} \rangle \\ &= \langle \bar{T}, \overline{X}_{\pm\alpha} \mid \alpha \in \{\beta_1, \beta_2, \beta_3\} \rangle. \end{aligned}$$

Further, $\bar{T} \leqslant \bar{L}_1 \bar{L}_2 \bar{L}_3$ by (4.4), so

$$C_{\overline{H}}(U)^\circ = \langle \overline{X}_{\beta_i}, \overline{X}_{-\beta_i} \mid i \in \{1,2,3\} \rangle = \bar{L}_1 \bar{L}_2 \bar{L}_3 \tag{4.6}$$

as claimed.

We next prove that $C_{\overline{H}}(U)$ is connected. Since $C_{\overline{H}}(U) = C_{\overline{H}}(z_1)$, this follows directly from a theorem of Steinberg to the effect that the centralizer of a semisimple element in a simply connected reductive group is connected, but it is possible to give a more direct argument in this special case. By Proposition A.3,

$$C_{\overline{H}}(U) = C_{\overline{H}}(U)^\circ C_{N_{\overline{H}}(\bar{T})}(U),$$

and we claim that $C_{N_{\overline{H}}(\bar{T})}(U) \leqslant C_{\overline{H}}(U)^\circ$. By (4.6), $N_{C_{\overline{H}}(U)^\circ}(\bar{T})/\bar{T}$ is elementary abelian of order 8. On the other hand, $C_{N_{\overline{H}}(\bar{T})}(U)/\bar{T}$ stabilizes the flag $1 < Z < U < E$, and so induces a group of transvections on $E$ of order 4 with center $Z$ and axis $U$. The element $w_0$ of $N_{\overline{H}}(\bar{T})$ inverts $\bar{T}$ and is trivial on $E$ by Lemma 4.2(3). It follows that $|N_{C_{\overline{H}}(U)}(\bar{T})/\bar{T}| = |N_{C_{\overline{H}}(U)^\circ}(\bar{T})/\bar{T}|$, and so $N_{C_{\overline{H}}(U)}(\bar{T}) = N_{C_{\overline{H}}(U)^\circ}(\bar{T})$. Thus,

$$C_{N_{\overline{H}}(\bar{T})}(U) = N_{C_{\overline{H}}(U)}(\bar{T}) = N_{C_{\overline{H}}(U)^\circ}(\bar{T}) \leqslant C_{\overline{H}}(U)^\circ,$$

completing the proof of the claim. By (4.6)

$$C_{\overline{H}}(U) = \bar{L}_1 \bar{L}_2 \bar{L}_3. \tag{4.7}$$

For each $\lambda \in \bar{\mathbb{F}}_p$, we have

$$\begin{aligned} x_{\beta_3}(\lambda)^\tau &= x_{\beta_3}(\lambda)^{n_{\alpha_2+\alpha_3}(1)h_{\beta_1}(-i)h_{\beta_2}(i)h_{\beta_3}(i)} \\ &= x_{\beta_3}(-\lambda)^{h_{\beta_3}(i)} \\ &= x_{\beta_3}(i^{\langle \beta_3,\beta_3\rangle}(-\lambda)) \\ &= x_{\beta_3}(i^2(-\lambda)) \\ &= x_{\beta_3}(\lambda) \end{aligned}$$

Similarly, $x_{-\beta_3}(\lambda)^\tau = x_{-\beta_3}(i^{-2}(-\lambda)) = x_{-\beta_3}(\lambda)$. So as $\bar{L}_3 = \langle x_{\pm\beta_3}(\lambda)\rangle$, we have $[\bar{L}_3, \tau] = 1$. Finally, since $w_{\alpha_2+\alpha_3}$ interchanges $\beta_1$ and $\beta_2$, and since $\bar{T}$ normalizes all root groups, $\tau$ interchanges $\bar{L}_1$ and $\bar{L}_2$. In particular, $\tau$ interchanges the central involutions $h_{\beta_1}(-1) = z_1$ and $h_{\beta_2}(-1) = zz_1$ of $\bar{L}_1$ and $\bar{L}_2$. This shows $\tau$ acts nontrivially on $U$, and hence

$$\overline{B} = (\bar{L}_1\bar{L}_2\bar{L}_3)\langle\tau\rangle,$$

completing the proof of (1).

By (1), $C_{\overline{H}}(U)$ is connected, so [GLS98, Theorem 2.1.5] applies to give $\mathscr{E}_2(H,Z) = U^H$. Let $L_i = C_{\bar{L}_i}(\sigma)$ for $i = 1,2,3$, and set $B^\circ = L_1L_2L_3 \leqslant H$. Since $w \in N_H(U) - C_H(U)$, we have $C_H(U) = C_{\overline{B}^\circ}(\sigma)$. Let $\tilde{B}$ denote the direct product of the $\bar{L}_i$, and let $\tilde{\sigma}$ be the Steinberg endomorphism lifting $\sigma|_{\overline{B}^\circ}$ along the isogeny $\tilde{B} \to \overline{B}^\circ$ given by quotienting by $\langle(-1,-1,-1)\rangle$ (see, e.g. [GLS98, Lemma 2.1.2(d,e)]). Then $C_{\tilde{B}}(\tilde{\sigma}) = L_1 \times L_2 \times L_3$. So by [GLS98, Theorem 2.1.8] applied with the pair $\tilde{B}$, $\langle(-1,-1,-1)\rangle$ in the role of $\overline{K}$, $\overline{Z}$, we see that $B^\circ$ is of index 2 in $C_H(U)$ with $C_H(U) = B^\circ(C_H(U) \cap \bar{T}) = B^\circ T$. The element $c = h_{\beta_1}(\mu)h_{\beta_2}(\mu)h_{\beta_3}(\mu) \in T$ lifts to an element $\tilde{c} \in \tilde{B}$ with $[\tilde{c},\tilde{\sigma}] = (-1,-1,-1)$ by definition of $\mu$, and so $c \in C_H(U) - B^\circ$ by [GLS98, Theorem 2.1.8]. Finally as each $L_i$ is generated by root groups on which $c$ acts nontrivially, $c$ acts as a diagonal automorphism on each $L_i$. $\square$

Next we consider the $H$-conjugacy classes of elementary abelian subgroups of order 8 which contain $Z$.

**Lemma 4.4.** *The following hold.*

*(1)* $N_{\overline{H}}(E) = N_{\overline{H}}(\bar{T})$ *and* $C_{\overline{H}}(E) = \bar{T}\langle w_0\rangle$.
*(2)* $N_H(E) = N_H(T)$ *and* $C_H(E) = T\langle w_0\rangle$.
*(3)* $N_{\overline{H}}(\bar{T})/\bar{T} \cong C_2 \times S_4 \cong N_H(T)/T$.

*Proof.* Given that $w_0$ inverts $\bar{T}$ (Lemma 4.2(3)) part (1) is proved in Proposition A.4.

By (1),

$$N_H(E) = N_{\overline{H}}(E) \cap H = N_{\overline{H}}(\bar{T}) \cap H = N_H(\bar{T}),$$

while $N_H(\bar{T}) \leqslant N_H(H \cap \bar{T}) = N_H(T)$. These combine to show the inclusion $N_H(E) \leqslant N_H(T)$. But $N_H(T) \leqslant N_H(E)$ since $E = \Omega_1(O_2(T))$ is characteristic in $T$. Next, by (1),

$$C_H(E) = C_{\overline{H}}(E) \cap H = \bar{T}\langle w_0\rangle \cap H = (\bar{T} \cap H)\langle w_0\rangle$$

with the last equality as $w_0 \in H$ by Lemma 4.2(2). This shows $C_H(E) = T\langle w_0\rangle$.

For part (3) in the case of $\overline{H}$, see Section 4.1.3. We show part (3) for $H$. First, by (1) and (2),

$$N_H(\bar{T}) = C_{N_{\overline{H}}(\bar{T})}(\sigma) = C_{N_{\overline{H}}(E)}(\sigma) = N_H(E) = N_H(T). \tag{4.8}$$

In the special case $\epsilon = 1$, $\sigma$ centralizes $\widehat{W}$, which covers $W = N_{\overline{H}}(\bar{T})/\bar{T}$. Using (4.8), this shows $N_H(T) = N_H(\bar{T})$ projects onto $W$ with kernel $\bar{T} \cap C_{N_{\overline{H}}(\bar{T})}(\sigma) = T$. So $N_H(T)/T \cong W$ in this case.

In any case, $\bar{T}w_0$ generates the center of $N_{\overline{H}}(\bar{T})/\bar{T}$, so $g^\sigma g^{-1} \in \bar{T}$ for each $g \in N_{\overline{H}}(\bar{T})$. Since $\bar{T}$ is connected, for each such $g$ there is $t \in \bar{T}$ with $t^{-\sigma}t = g^\sigma g^{-1}$ by the Lang-Steinberg Theorem, and hence $tg \in C_{N_{\overline{H}}(\bar{T})}(\sigma)$. This shows each coset $\bar{T}g$ contains an element centralized by $\sigma$, and so arguing as in the previous paragraph, we have $N_H(T)/T \cong W$. $\square$

**Lemma 4.5.** *Let $d = cw_0$ and $E' = U\langle d\rangle \leqslant S$. Then $\mathscr{E}_3(H, Z)$ is the disjoint union of $E^H$ and $E'^H$. Moreover, there is a $\sigma$-invariant maximal torus $T'$ of $H$ with $E' = \{t \in T' \mid t^2 = 1\}$ such that the following hold.*

*(1) $O_{2'}(C_H(E)) = O_{2'}(T) \cong (C_{(q-\epsilon)/2^k})^3$, and $N_H(T)/T \cong C_2 \times S_4$ acts faithfully on the $r$-torsion subgroup of $T$ for each odd prime $r$ dividing $q-\epsilon$;*

*(2) $O_{2'}(C_H(E')) = O_{2'}(T') \cong (C_{(q+\epsilon)/2})^3$, and $N_H(T')/T' \cong C_2 \times S_4$ acts faithfully on the $r$-torsion subgroup of $T'$ for each odd prime $r$ dividing $q+\epsilon$; and*

*(3) $C_H(E') = T'\langle w_0'\rangle$ for some involution $w_0'$ inverting $T'$.*

*Proof.* By Lemma 4.2, $w_0$ is an involution inverting $\bar{T}$ and hence inverting $c$. So $d$ is an involution, and indeed, $E'$ is elementary abelian of order 8.

Part of this Lemma is proved by Aschbacher and Chermak [AC10, Lemma 7.8]. We give an essentially complete proof for the convenience of the reader. Let $\overline{X} \in \{\overline{B}^\circ, \overline{H}\}$, and write $X = C_{\overline{X}}(\sigma)$. The centralizer $C_{\overline{X}}(E) = \bar{T}\langle w_0\rangle$ is not connected, but has the two connected components $\bar{T}$ and $\bar{T}w_0$. Thus, there are two $C_{\overline{X}}(\sigma)$-conjugacy classes of subgroups of $X$ conjugate to $E$ in $\overline{H}$ [GLS98, 2.1.5]. A representative of the other $X$-class can be obtained as follows. Since $\overline{X}$ is connected, we may fix by the Lang-Steinberg Theorem $g \in \overline{X}$ such that $w_0 = g^\sigma g^{-1}$. Then $g^\sigma = w_0 g$. In the semidirect product $\overline{X}\langle\sigma\rangle$, we have $\sigma^g = \sigma w_0$. Now as $\bar{T}\langle w_0\rangle$ is invariant under $\sigma w_0$, it follows that $(\bar{T}\langle w_0\rangle)^g$ is $\sigma$-invariant. Indeed by choice of $g$, we have $t^{g\sigma} = t^{\sigma w_0 g}$ for each $t \in \bar{T}$, i.e., the conjugation isomorphism $\bar{T}\langle w_0\rangle \xrightarrow{c_g} \bar{T}^g\langle w_0^g\rangle$ intertwines the actions of $\sigma w_0$ on $\bar{T}\langle w_0\rangle$ and $\sigma$ on $\bar{T}^g\langle w_0^g\rangle$. Then $E$ and $E^g$ are representatives for the $X$-classes of subgroups of $X$ conjugate in $\overline{X}$ to $E$, and

$$X \cap \bar{T}^g = C_{\bar{T}^g}(\sigma) \cong C_{\bar{T}}(\sigma w_0) = \{t \in T \mid t^{-\epsilon q} = t\} \cong (C_{q+\epsilon})^3. \tag{4.9}$$

The above argument shows we may take $g \in \overline{B}^\circ$ even when $\overline{X} = \overline{H}$. By Lemma 4.3, $\overline{B}^\circ$ is a commuting product $\bar{L}_1\bar{L}_2\bar{L}_3$ with $\bar{L}_i \cong SL_2(\bar{\mathbb{F}}_p)$ and $Z(\overline{B}^\circ) = U$. Also, $\overline{B}^\circ \cong \bar{J}/\langle j\rangle$ where $\bar{J}$ is a direct product of the $\bar{L}_i$'s and $j$ the product of the unique involutions of the direct factors (Section 4.1.4). Thus, each involution in $\overline{B}^\circ - U$ is of the form $f_1f_2f_3$ for elements $f_i \in \bar{L}_i$ of order 4. But $\bar{L}_i$ is transitive on its elements of order 4. Hence, all elementary abelian subgroups of $\overline{B}^\circ$ of order 8 containing $U$ are $\overline{B}^\circ$-conjugate. Now $E$ is contained in the normal subgroup $L_1L_2L_3$ of $C_H(U)$, while $E'$ is not since $d$ lies in the coset $L_1L_2L_3c$. It follows that $E^g$ is $C_H(U)$-conjugate to $E'$. Hence, $E$ and $E'$ are representatives for the $X$-conjugacy classes of elementary abelian subgroups of $X$ of order 8 containing $Z$.

Fix $b \in C_H(U)$ with $E^{gb} = E'$. Set $\bar{T}' = \bar{T}^{gb}$ and $T' = C_{\bar{T}^{gb}}(\sigma)$, and $w_0' = w_0^{gb}$. By (4.9), $O_{2'}(T')$ is as described in (a)(ii), and $w_0'$ inverts $T'$. Now $N_H(T)/T \cong C_2 \times S_4$ by Lemma 4.4(3). Since $\bar{T}w_0$ generates the center of $N_{\bar{H}}(\bar{T})/\bar{T}$, it follows by choice of $g$ and [Car85, 3.3.6] that $N_H(T^g)/T^g \cong N_H(T)/T$, and hence $N_H(T')/T' \cong N_H(T)/T$ because $b \in H$.

Fix an odd prime $r$ dividing $q - \epsilon$ (resp. $q + \epsilon$), and let $T_r$ (resp. $T_r'$) be the $r$-torsion subgroup of $\bar{T}$ (resp. $\bar{T}^{gb}$). Then $T_r \leqslant T$ (resp. $T_r' \leqslant T'$). Since $N_{\bar{H}}(\bar{T})/\bar{T}$ (resp. $N_{\bar{H}}(\bar{T}^{gb})/\bar{T}^{gb}$) acts faithfully on $T_r$ (resp. $T_r'$) by Proposition A.4, it follows that the same is true for $N_H(T)/T$ (resp. $N_H(T')/T'$). This completes the proof of (1) and (2), and part (3) then follows. $\square$

4.3. **Conjugacy classes of elementary abelian subgroups in a Benson-Solomon system.** In this subsection we look at the conjugacy classes and automizers of elementary abelian subgroups of the Benson-Solomon systems. We adopt the notation from the first part of this section, so $S$ is a Sylow 2-subgroup of $H = \mathrm{Spin}_7(q)$, $Z = Z(S)$ is of order 2, and $E$ is the 2-torsion in the fixed maximal torus $T$ of $H$.

**Lemma 4.6.** *Let $\mathcal{F} = \mathcal{F}_{\mathrm{Sol}}(q)$ be a Benson-Solomon fusion system over $S$. Then*

*(1) $\mathscr{E}_1(S) = Z^{\mathcal{F}}$, and $N_{\mathcal{F}}(Z) = C_{\mathcal{F}}(Z) \cong \mathcal{F}_S(H)$.*

*(2) For $T_S = T \cap S$, $\mathrm{Out}_{\mathcal{F}}(T_S) = \mathrm{Aut}_{\mathcal{F}}(T_S) \cong C_2 \times GL_3(2)$, and $\mathrm{Out}_{\mathcal{F}}(T_S\langle w_0\rangle) \cong GL_3(2)$ acts naturally on $T_S/\Phi(T_S)$ and on $E$.*

*Proof.* Part (1) follows from the construction of $\mathcal{F}_{\mathrm{Sol}}(q)$. The structure of $\mathrm{Out}_{\mathcal{F}}(T_S)$ also follows by the construction, especially the one of Aschbacher and Chermak [AC10, Section 5]. For a proof of the structure of $\mathrm{Out}_{\mathcal{F}}(T_S\langle w_0\rangle)$, we refer to Lemma 2.38(c) of [HL18]. $\square$

We saw in Lemma 4.5 that $H$ has two conjugacy classes of elementary abelian subgroups of order 8 containing $Z$. As far as we can tell, Aschbacher and Chermak do not discuss the possible $\mathcal{F}$-conjugacy of $E$ and $E'$ explicitly, but this can be gathered from their description of the conjugacy classes of elementary abelian subgroups of order 16. Since we need to show that $E$ and $E'$ are in fact not $\mathcal{F}$-conjugate, we provide an account of that description.

On p.935 of [AC10], $T_S$ is denoted $R_0$. As on pg.935-936, write $R_1 = N_{\bar{T}}(T_S\langle w_0\rangle) \cong (C_{2^{k+1}})^3$. Thus, $T_S$ has index 8 in $R_1$, and $R_1/T_S$ is elementary abelian of order 8. Fix a set

$$\{x_e \mid e \in E\}$$

of coset representatives for $T_S$ in $R_1$, with notation chosen so that $x_1 = 1$ and $x_e^{2^k} = e \in E$ for each $e \in E - \{1\}$, and write

$$A_e = A^{x_e}.$$

Since $w_0$ inverts $\bar{T}$, $A_e = E\langle t_e w_0\rangle$ where $t_e := x_e^{-2} = [x_e, w_0] \in T_S$ also powers to $e \in E$. Since $E \leqslant T_S$ and $[T_S, w_0] = \Phi(T_S)$ there are bijections

$$\{A_e^{T_S} \mid e \in E\} \longrightarrow T_S/\Phi(T_S) \longrightarrow E$$

given by $A_e^{T_S} \mapsto t_e\Phi(T_S) \mapsto e = t_e^{2^{k-1}}$. These maps are $\mathrm{Aut}_{\mathcal{F}}(T_S\langle w_0\rangle)$-equivariant, $\mathrm{Inn}(T_S\langle w_0\rangle)$ acts trivially, and hence by Lemma 4.6(2), $\mathrm{Aut}_{\mathcal{F}}(T_S\langle w_0\rangle)$ has two orbits on $\mathscr{E}_4(T_S\langle w_0\rangle) = \bigcup_{e\in E} A_e^{T_S}$ with representatives $A = A_1$ and $A_e$ with $e \neq 1$.

**Lemma 4.7.** *$\mathscr{E}_4(S)$ is the disjoint union of $A_1^{\mathcal{F}}$ and $A_e^{\mathcal{F}}$, where $e$ is any nonidentity element of $E$. All $A_e$ with $e \neq 1$ are $\mathrm{Aut}_{\mathcal{F}}(T_S\langle w_0\rangle)$-conjugate, and $\mathrm{Aut}_{\mathcal{F}}(A_e) = C_{\mathrm{Aut}(A_e)}(e)$ for each $e \in E$.*

*Proof.* This is Lemma 7.12(c) of [AC10], except for the statement on $\mathrm{Aut}_{\mathcal{F}}(T_S\langle w_0\rangle)$-conjugacy, which is contained in the setup on p.936 of that paper and observed above. $\square$

**Lemma 4.8.** *$\mathscr{E}_3(S)$ is the disjoint union of $E^{\mathcal{F}}$ and $E'^{\mathcal{F}}$, and we have $\mathrm{Aut}_{\mathcal{F}}(E) = \mathrm{Aut}(E)$ and $\mathrm{Aut}_{\mathcal{F}}(E') = \mathrm{Aut}(E')$.*

*Proof.* Assume to get a contradiction that $E$ and $E'$ are $\mathcal{F}_{\mathrm{Sol}}(q)$-conjugate. Since $E$ is normal in $S$, it is fully $\mathcal{F}_{\mathrm{Sol}}(q)$-normalized, so there is $\varphi \in \mathrm{Hom}_{\mathcal{F}_{\mathrm{Sol}}(q)}(C_S(E'), C_S(E))$ with $E'^{\varphi} = E$ by [AKO11, I.2.6]. By Lemma 4.7, post-composing with an element $\mathrm{Aut}_{\mathcal{F}}(T_S\langle w_0\rangle)$, which normalizes $E$, we may take $\varphi$ with $A_e^{\varphi} = A_e$. But then $U^{\varphi} \leqslant E$ and $e^{\varphi} = e$ by Lemma 4.7, so $E^{\varphi} = E$ a contradiction. Now we appeal to [AC10, Lemma 7.8] for the structure of the $\mathcal{F}$-automorphism groups. $\square$

4.4. **Proof of Theorem 4.1.** We now turn to the proof of Theorem 4.1. As an initial observation, note that if $\mathcal{L}$ is a punctured group for $\mathcal{F}_{\mathrm{Sol}}(q')$ for some $q' \equiv 3 \pmod 8$, then $C_{\mathcal{L}}(Z)$ is a group whose 2-fusion system is isomorphic to that of $\mathrm{Spin}_7(q')$. With the help of a result of Levi and Oliver, it follows that $O^{2'}(C_{\mathcal{L}}(Z)/O_{2'}(C_{\mathcal{L}}(Z))) \cong \mathrm{Spin}_7(q)$ for some odd prime power $q \equiv \pm 3 \pmod 8$ (Lemma 4.9). Lemma 4.10 below then gives strong restrictions on $q$. Ultimately it implies that $q = 3^{1+6a}$ for some $a \geqslant 0$ with the property that $q^2 - 1$ is divisible only by primes which are squares modulo 7. Although there are at least several such nonnegative integers $a$ with this property (the first few are $0, 1, 2, 3, 5, 7, 8, 13, 15, \dots$), we are unable to to determine whether a punctured group for $\mathcal{F}_{\mathrm{Sol}}(q)$ exists when $a > 0$.

**Lemma 4.9.** *Let $G$ be a finite group whose 2-fusion system is isomorphic to that of $\mathrm{Spin}_7(q')$ for some odd $q'$. Then $G/O_{2'}(G) \cong \mathrm{Spin}_7(q)\langle\varphi\rangle$ for some odd $q$ with $v_2(q^2 - 1) = v_2(q'^2 - 1)$, and where $\varphi$ induces a field automorphism of odd order.*

*Proof.* It was shown by Levi and Oliver in the course of proving $\mathcal{F}_{\mathrm{Sol}}(q)$ is exotic that $O^{2'}(G/O_{2'}(G))$ is isomorphic to $\mathrm{Spin}_7(q)$ for some odd $q$ [LO02, Proposition 3.4]. If $S'$ and $S$ are the corresponding Sylow 2-subgroups, then $S'$ and $S$ are isomorphic by definition of an isomorphism of a fusion system. If $k$ and $k'$ are one less than the valuations of $q^2-1$ and $q'^2-1$, then the orders of $S$ and $S'$ are $2^{4+3k}$ and $2^{4+3k'}$, so $k = k'$. The description of $G/O_{2'}(G)$ follows, since $\mathrm{Out}(\mathrm{Spin}_7(q)) \cong C_n \times C_2$, where $q = p^n$ and $C_n$ is generated by a field automorphism. $\square$

The extension of $\mathrm{Spin}_7(q)$ by a group of field automorphisms of odd order has the same 2-fusion system as $\mathrm{Spin}_7(q)$, but we will not need this.

**Lemma 4.10.** *Let $q$ be an odd prime power with the property that $GL_3(2)$ has a faithful 3-dimensional representation over $\mathbb{F}_r$ for each prime divisor $r$ of $q^2 - 1$. Then each such $r$ is a square modulo 7, and $q = 3^{1+6a}$ for some $a \geqslant 0$. In particular, $q = 3 \pmod 8$.*

*Proof.* Set $G = GL_3(2)$ for short. We first show that $GL_3(2)$ has a faithful 3-dimensional representation over $\mathbb{F}_r$ if and only if $r$ is a square modulo 7. If $r = 2$, 3, or 7, then as $|SL_3(3)|$ is not divisible by 7 and $G \cong PSL_2(7) \cong \Omega_3(7)$, the statement holds. So we may assume that $p$ does not divide $|G|$, so that $\mathbb{F}_r GL_3(2)$ is semisimple. Let $V$ be a faithful 3-dimensional module with character $\varphi$, necessarily irreducible. From the character table for $GL_3(2)$, we see that $\varphi$ takes values in $\mathbb{F}_r((1+\sqrt{-7})/2)$. By [Fei82, I.19.3], a modular representation is writable over its field of character values, so this extension is a splitting field for $V$. Thus, $V$ is writeable over $\mathbb{F}_r$ if and only if $-7$ is a square modulo $r$, which by quadratic reciprocity is the case if and only if $r$ is a square modulo 7.

Now fix an odd prime power $q$ with the property that $q^2-1$ is divisible only by primes which are squares modulo 7. Since $q(q-1)(q+1)$ is divisible by 3 and 3 is not a square, we have $q=3^l$ for some $l$. Now $q-1$ and $q+1$ are squares, so $q=1$ or $3 \pmod 7$. Assume the former. Then 6 divides $l$, so $q=3^l=\pm 1 \pmod 5$. But then $q^2-1$ is divisible by the nonsquare 5, a contradiction. So the latter holds, $l=1+6a$ for some $a \geqslant 0$, and hence $q=3 \pmod 8$. $\square$

**Lemma 4.11.** *Let $H=\mathrm{Spin}_7(3)$ and $Z=Z(H)$. If $P \geqslant Z$ is a 2-subgroup of $H$ of 2-rank at least 2, then $N_H(P)$ is of characteristic 2.*

*Proof.* Let $P \leqslant S$ with $Z \leqslant V \leqslant P$ and $V$ a four group. By Lemma 4.3(2), we may conjugate in $H$ and take $V=U$, and $C_H(U)=L_1L_2L_3\langle c\rangle$, where $c$ induces a diagonal automorphism on each $L_i \cong SL_2(3)$. Thus, $O_2(C_H(U))$ is a commuting product of three quaternion subgroups of order 8 which contains its centralizer in $C_H(U)$, and hence $C_H(U)$ is of characteristic 2.

Recall that $N_H(P)$ is of characteristic 2 if and only if $C_H(P)$ is of characteristic 2 and that the normalizer of any 2-subgroup in a group of characteristic 2 is of characteristic 2 (see, e.g. [Hen19, Lemma 2.2]). It follows that $N_{C_H(U)}(P)$ is of characteristic 2, so $C_H(P)=C_{C_H(U)}(P)$ is of characteristic 2, so $N_H(P)$ is of characteristic 2. $\square$

We may now prove the main theorem of this section.

*Proof of Theorem 4.1.* ( $\Longrightarrow$ ): Let $\mathcal{F} := \mathcal{F}_{\mathrm{Sol}}(q')$ for some odd prime power $q'$. Suppose first that $q'=\pm 1 \pmod 8$, where we argue by contradiction. Fix a locality $(\mathcal{L},\Delta,S)$ on $\mathcal{F}$ with $\Delta$ the set of nonidentity subgroups of $S$. Among all such localities, take one where $\mathcal{L}$ is of minimum cardinality. Write $Z=Z(S)$ as before, and set $G := N_{\mathcal{L}}(Z)=C_{\mathcal{L}}(Z)$ for short.

Consider the case $O_{2'}(G)=1$. Set $H=O^{2'}(G)$. As $\mathcal{L}$ is a locality on $\mathcal{F}_{\mathrm{Sol}}(q')$, $\mathcal{F}_S(G)=\mathcal{F}_S(N_{\mathcal{L}}(Z))$ is isomorphic to the 2-fusion system of $N_{\mathcal{F}}(Z)$, namely to the fusion system of $\mathrm{Spin}_7(q')$. By Lemma 4.9, $H \cong \mathrm{Spin}_7(q)$ for some odd prime power $q \equiv \pm 1 \pmod 8$. We identify $H$ with the group defined in Section 4.1.5 and we adopt the notation of Sections 4.1-4.3. In particular $H$ has Sylow 2-subgroup $S$, $\epsilon \in \{\pm 1\}$ is such that $q \equiv \epsilon \pmod 4$, and $E_1 := E$ and $E_{-1} := E'$ are the representatives for $\mathcal{F}$-conjugacy classes of elementary abelian subgroups of order 8 in $S$ (Lemmas 4.5 and 4.8). For $\delta=\pm 1$, let $T_\delta$ be the maximal torus containing $E_\delta$ of Lemma 4.5. For each positive integer $r$ dividing $q-\delta\epsilon$, write $T_{\delta,r}$ for the $r$-torsion in $T_\delta$. Moreover, set $T_{\delta,S}=T_\delta \cap S$. Thus, $T_{1,S}=T_S=T_{1,2^k}$ is homocyclic of order $2^{3k}$, and $T_{-1,S}=E_{-1}$.

Now fix $\delta$ and let $N=N_{\mathcal{L}}(T_{\delta,S})$. By Lemmas 4.4(2) and 4.5(3),

$$C_H(E_\delta)=T_\delta\langle w\rangle,$$

where $w$ is an involution inverting $T_\delta$. In particular, since

$$O_{2'}(T_\delta)=[O_{2'}(T_\delta),\langle w\rangle] \leqslant [C_H(E_\delta),C_H(E_\delta)]$$

and $O^{2'}(G)=H$, we have

$$C_H(E_\delta)=O^{2'}(C_H(E_\delta))=O^{2'}(C_G(E_\delta))$$

Also, $C_{\mathcal{L}}(E_\delta) = C_G(E_\delta)$ as $E_\delta$ contains $Z$. It follows that $C_H(E_\delta) = O^{2'}(C_{\mathcal{L}}(E_\delta))$ is normal in $N_{\mathcal{L}}(E_\delta)$, so

$$(4.10) \qquad O_{2'}(C_H(E_\delta)) \text{ is normal in } N_{\mathcal{L}}(E_\delta).$$

Next we show

$$(4.11) \qquad N = N_{\mathcal{L}}(E_\delta).$$

We may assume $T_{\delta,S} > E_\delta$, and so $\delta = 1$, $T_{\delta,S} = T_S$, and $E_\delta = E$. Certainly $N_{\mathcal{L}}(T_S) \leqslant N_{\mathcal{L}}(E)$. For the other inclusion, note $N_{\mathcal{L}}(E)$ acts on $C_H(E)$ by (4.10) so it acts on $T_S$ since $T_S$ is the unique abelian 2-subgroup of maximum order in $C_H(E)$. Thus, $N_{\mathcal{L}}(E) \leqslant N_{\mathcal{L}}(T_S)$, completing the proof (4.11).

Let $r$ be a prime divisor of $q - \delta\epsilon$. By (4.11) and (4.10), $T_{\delta,r}$ is normal in $N$, so $C_N(T_{\delta,r})$ is normal in $N$. Set $\bar{N} = N/C_N(T_{\delta,S})$. Then as $\mathcal{L}$ is a locality on $\mathcal{F}_{\mathrm{Sol}}(q)$, we have $\bar{N} \cong \mathrm{Aut}_{\mathcal{F}}(T_{\delta,S}) \cong C_2 \times GL_3(2)$ when $\delta = 1$ by Lemma 4.6(2), while $\bar{N} \cong \mathrm{Aut}_{\mathcal{F}}(T_{\delta,S}) \cong GL_3(2)$ when $\delta = -1$ by Lemma 4.8 (because $T_{-1,S} = E_{-1}$). By Lemma 4.5, for a parabolic subgroup $\bar{X} \cong C_2 \times S_4$ or $S_4$ of $\bar{N}$ at the prime 2, we have $\bar{X} \cap \overline{C_N(T_{\delta,r})} = 1$. So $\overline{C_N(T_{\delta,S})} = 1$ as this subgroup is normal in $\bar{N}$, and thus $C_N(T_{\delta,r}) = C_N(T_{\delta,S})$. It follows that $GL_3(2) \leqslant \bar{N}$ acts faithfully on $T_{\delta,r} \cong C_r^3$. Since this holds for each $\delta = \pm 1$ and prime $r$, Lemma 4.10 implies that $q \equiv 3 \pmod 8$, and this contradicts our original assumption.

We are reduced to showing $O_{2'}(G) = 1$. Set $\theta(a) = O_{2'}(C_{\mathcal{L}}(a))$ for each involution $a \in S$. By Lemma 2.7(b), $\theta$ is conjugacy invariant. Let $a, b \in S$ be two distinct commuting involutions. By conjugacy invariance, to verify balance, we can assume $b = z$ and $a = u \in U - Z$. Set $X = O_{2'}(C_{\mathcal{L}}(u)) \cap G$, an odd order normal subgroup of $C_{\mathcal{L}}(U)$, and use bars for images modulo $O_{2'}(G)$. By a Frattini argument $\overline{C_{\mathcal{L}}(U)} = \overline{C_G(U)} = C_{\bar{G}}(\bar{U})$, so $\bar{X}$ is normal in the latter group. However, $\bar{G}$ is an extension of $H$ by a cyclic group generated by a field automorphism $\varphi$ of odd order by Lemma 4.9. We may take $\varphi$ to be standard, that is, acting on the root groups via $x_\alpha(\lambda) \mapsto x_\alpha(\lambda^r)$ with $r$ odd. Each component $L_i \cong SL_2(q)$ of $C_{\bar{G}}(\bar{U})$ is generated by a root group and its opposite (Section 4.1.4), it follows that $\varphi$ acts nontrivially as a field automorphism on each such $L_i$, and hence $\bar{X} \leqslant O_{2'}(C_{\bar{G}}(\bar{U})) \leqslant O_{2'}(L_1L_2L_3) = 1$. Equivalently, $X = O_{2'}(C_{\mathcal{L}}(u)) \cap G \leqslant O_{2'}(G)$. This shows that the balance condition holds. For each $P \in \Delta$, set

$$\Theta(P) = \left( \bigcap_{x \in \mathcal{I}_2(P)} \theta(x) \right) \cap C_{\mathcal{L}}(P).$$

Then by Theorem 2.35, $\Theta$ defines a signalizer functor on subgroups. By Theorem 2.32, $\bar{\Theta} = \bigcup_{P \in \Delta} \Theta(P)$ is a partial normal subgroup of $\mathcal{L}$, and $\mathcal{L}/\bar{\Theta}$ is again a punctured group for $\mathcal{F}_{\mathrm{Sol}}(q)$. By minimality of $|\mathcal{L}|$, we have $\bar{\Theta} = 1$, and in particular $O_{2'}(G) = 1$. This completes the proof of the forward direction of the theorem.

$(\Longleftarrow)$ : Let now $\mathcal{F} = \mathcal{F}_{\mathrm{Sol}}(3)$ and $\mathcal{H} = C_{\mathcal{F}}(Z) = \mathcal{F}_S(H)$ with $H = \mathrm{Spin}_7(3)$. Set

$$\Delta = \{P \in \mathcal{F}^s \mid P \text{ is of 2-rank at least 2}\},$$

and $\Delta_Z = \{P \in \Delta \mid P \geqslant Z\}$. Then $\Delta$ is closed under $\mathcal{F}$-conjugacy and passing to overgroups by [Hen19]. So it is also closed under $\mathcal{H}$-conjugacy.

Neither $\mathcal{H}^{cr}$ nor $\mathcal{F}^{cr}$ contains a subgroup $Q$ of $S$ of 2-rank 1. Indeed, assume $Q$ is such. Suppose first that $Q$ is cyclic, or generalized quaternion of order at least 16. Then $\mathrm{Aut}(Q)$ is a 2-group. So since $Q$ is centric, $N_S(Q)/Q = \mathrm{Aut}_S(Q) = \mathrm{Aut}_{\mathcal{F}}(Q)$, for example, has a normal 2-subgroup, so $Q$ is not radical. Assume $Q$ is quaternion of order 8. As $U$ is a normal subgroup of $S$, we have $[Q, U] \leqslant Z = Z(S) \leqslant Q$, so $U \leqslant N_S(Q)$. But $N_S(Q)$ is a 2-group containing $Q$ self-centralizing with index 2, and so $N_S(Q)$ is quaternion or semidihedral of order 16. But neither of these groups has a normal four subgroup, a contradiction.

Each element of $\mathcal{F}^{cr} \cup \mathcal{H}^{cr}$ contains $Z$. Also $\mathcal{F}^s \subseteq \mathcal{H}^s$ by [Hen19, Lemma 3.16]. Thus, we have shown

$$\mathcal{F}^{cr} \cup \mathcal{H}^{cr} \subseteq \Delta_Z \subseteq \Delta \subseteq \mathcal{F}^s \subseteq \mathcal{H}^s. \tag{4.12}$$

The hypotheses of [Hen19, Theorem A] are thus satisfied, so we may fix a linking locality $\mathcal{L}$ on $\mathcal{F}$ with object set $\Delta$, and this $\mathcal{L}$ is unique up to rigid isomorphism.

We shall verify the conditions (1)-(5) of [Che13, Hypothesis 5.3] with $Z$ in the role of "$T$" and $H$ in the role of "$M$". Conditions (1), (2) hold by construction. Condition (4) holds since $Z$ is normal in $H$ and $\mathcal{F}_S(N_{\mathcal{L}}(Z)) \cong \mathcal{H}$ by [LO02]. To see condition (3), first note that $Z$ is fully normalized in $\mathcal{F}$ because it is central in $S$. Let $Z', Z''$ be distinct $\mathcal{F}$-conjugates of $Z$. Then $\langle Z', Z''\rangle$ contains a four group $V$. By Lemma 4.3(2), $V$ is $\mathcal{F}$-conjugate to $U$, and $O_2(N_{\mathcal{F}}(U)) \in \mathcal{F}^c$ is a commuting product of three quaternion groups of order 8. Thus, $V \in \Delta$, and hence $\langle Z', Z''\rangle \in \Delta$. So Condition (3) holds. It remains to verify Condition (5), namely that $N_{\mathcal{L}}(Z)$ and $\mathcal{L}_{\Delta_Z}(H)$ are rigidly isomorphic. By (4.12) and Lemma 4.11, $\mathcal{L}_{\Delta_Z}(H)$ is a linking locality over $\mathcal{H}$ with $\Delta_Z$ as its set of objects.

On the other hand, by [Che13, Lemma 2.19], $N_{\mathcal{L}}(Z)$ is a locality on $\mathcal{H}$ with object set $\Delta_Z$, in which $N_{N_{\mathcal{L}}(Z)}(P) = C_{N_{\mathcal{L}}(P)}(Z)$ for each $P \in \Delta_Z$. As $\mathcal{L}$ a linking locality, $N_{\mathcal{L}}(P)$ is of characteristic 2, and hence the 2-local subgroup $N_{N_{\mathcal{L}}(Z)}(P)$ of $N_{\mathcal{L}}(P)$ is also of characteristic 2. So again this together with (4.12) gives that $N_{\mathcal{L}}(Z)$ is a linking locality over $\mathcal{H}$ with object set $\Delta_Z$. Thus, by [Hen19, Theorem A], we may fix a rigid isomorphism $\lambda\colon \mathcal{L}_{\Delta_Z}(H) \to N_{\mathcal{L}}(Z)$ and complete the proof of (5).

So by [Che13, Theorem 5.14], there is a locality $\mathcal{L}^+$ over $\mathcal{F}$ with object set

$$\Delta^+ := \{P \leqslant S \mid Z^{\varphi} \leqslant P \text{ for some } \varphi \in \mathrm{Hom}_{\mathcal{F}}(Z, S)\},$$

such that $\mathcal{L}^+|_{\Delta} = \mathcal{L}$, and $\mathcal{L}^+$ is unique up to rigid isomorphism with this property. Since each subgroup of $S$ contains an involution, and all involutions in $S$ are $\mathcal{F}$-conjugate (by Lemma 4.6(1)), $\Delta^+$ is the collection of all nontrivial subgroups of $S$. Thus, $\mathcal{L}^+$ is a punctured group for $\mathcal{F}$. □

## 5. Punctured groups for exotic fusion systems at odd primes

In this section, we survey some of the known examples of exotic fusion systems at odd primes in the literature, and determine which ones have associated punctured groups.

Let $\mathcal{F}$ be a saturated fusion system over the $p$-group $S$. A subgroup $Q$ of $S$ is said to be $\mathcal{F}$-*subcentric* if $Q$ is $\mathcal{F}$-conjugate to a subgroup $P$ for which $O_p(N_{\mathcal{F}}(P))$ is $\mathcal{F}$-centric. Equivalently, by [Hen19, Lemma 3.1], $Q$ is $\mathcal{F}$-subcentric if, for any fully $\mathcal{F}$-normalized $\mathcal{F}$-conjugate $P$ of $Q$, the normalizer $N_{\mathcal{F}}(P)$ is constrained. Write $\mathcal{F}^s$ for the set of subcentric subgroups of $\mathcal{F}$. Thus, $\mathcal{F}^s$

contains the set of nonidentity subgroups of $S$ if and only if $\mathcal{F}$ is of characteristic $p$-type (and $\mathcal{F}^s$ is the set of *all* subgroups of $S$ if and only if $\mathcal{F}$ is constrained).

A finite group $G$ is said to be of *characteristic* $p$ if $C_G(O_p(G)) \leqslant O_p(G)$. A *subcentric linking system* is a transporter system $\mathcal{L}^s$ associated to $\mathcal{F}$ such that $\mathrm{Obj}(\mathcal{L}^s) = \mathcal{F}^s$ and $\mathrm{Aut}_{\mathcal{L}^s}(P)$ is of characteristic $p$ for every $P \in \mathrm{Obj}(\mathcal{L}^s)$. By a theorem of Broto, Castellana, Grodal, Levi and Oliver [BCG+05], the constrained fusion systems are precisely the fusion systems of finite groups of characteristic $p$. The finite groups of characteristic $p$, which realize the normalizers of fully normalized subcentric subgroups, can be "glued together" to build a subcentric linking systems associated with $\mathcal{F}$. More precisely, building on the unique existence of centric linking systems, the first author has used Chermak descent to show that each saturated fusion system has a unique associated subcentric linking system.

For each of the exotic systems $\mathcal{F}$ considered in this section, it will turn out that either $\mathcal{F}$ is of characteristic $p$-type, or $S$ has a fully $\mathcal{F}$-normalized subgroup $X$ of order $p$ such that $N_{\mathcal{F}}(X)$ is exotic. In the latter case, there is the following elementary observation.

**Lemma 5.1.** *Let $\mathcal{F}$ be a saturated fusion system over $S$. Assume there is some nontrivial fully $\mathcal{F}$-normalized subgroup $X$ such that $N_{\mathcal{F}}(X)$ is exotic. Then a punctured group for $\mathcal{F}$ does not exist.*

*Proof.* If there were a transporter system $\mathcal{L}$ associated with $\mathcal{F}$ having object set containing $X$, then $\mathrm{Aut}_{\mathcal{L}}(X)$ would be a finite group whose fusion system is $N_{\mathcal{F}}(X)$. □

We restrict attention here to the following families of exotic systems at odd primes, considered in order: the Ruiz-Viruel systems [RV04], the Oliver systems [Oli14], the Clelland-Parker systems [CP10], and the Parker-Stroth systems [PS15]. The results are summarized in the following theorem.

**Theorem 5.2.** *Let $\mathcal{F}$ be a saturated fusion system over a finite $p$-group $S$.*

*(a) If $\mathcal{F}$ is a Ruiz-Viruel system at the prime 7, then $\mathcal{F}$ is of characteristic 7-type, so has a punctured group.*
*(b) If $\mathcal{F}$ is an exotic Oliver system, then $\mathcal{F}$ has a punctured group if and only if $\mathcal{F}$ occurs in cases (a)(i), (a)(iv), or (b) of [Oli14, Theorem 2.8].*
*(c) If $\mathcal{F}$ is an exotic Clelland-Parker system, then $\mathcal{F}$ has a punctured group if and only if each essential subgroup is abelian. Moreover, if so then $\mathcal{F}$ is of characteristic $p$-type.*
*(d) If $\mathcal{F}$ is a Parker-Stroth system, then $\mathcal{F}$ is of characteristic $p$-type, so has a punctured group.*

*Proof.* This follows upon combining Theorem 2.20 or Lemma 5.1 with Lemma 5.4, Proposition 5.7, Propositions 5.9 and 5.11, and Proposition 5.12, respectively. □

When showing that a fusion system is of characteristic $p$-type, we will often use the following elementary lemma.

**Lemma 5.3.** *Let $X$ be a fully $\mathcal{F}$-normalized subgroup of $S$ such that $C_S(X)$ is abelian. Then $N_{\mathcal{F}}(X)$ is constrained.*

*Proof.* Using Alperin's Fusion Theorem [AKO11, Theorem I.3.6], one sees that $C_S(X)$ is normal in $C_{\mathcal{F}}(X)$. In particular, $C_{\mathcal{F}}(X)$ is constrained. Therefore, by [Hen19, Lemma 2.13], $N_{\mathcal{F}}(X)$ is constrained. □

5.1. **The Ruiz-Viruel systems.** Three exotic fusion systems at the prime 7 were discovered by Ruiz and Viruel, two of which are simple. The other contains one of the simple ones with index 2.

**Lemma 5.4.** *Let $\mathcal{F}$ be a saturated fusion system over an extraspecial $p$-group $S$ of order $p^3$ and exponent $p$. Then $N_{\mathcal{F}}(Z(S)) = N_{\mathcal{F}}(S)$. In particular, $\mathcal{F}$ is of characteristic $p$-type.*

*Proof.* Clearly $N_{\mathcal{F}}(S) \subseteq N_{\mathcal{F}}(Z(S))$. Note that $N_{\mathcal{F}}(Z(S))$ is a saturated fusion system over $S$ as well. So by [RV04, Lemma 3.2], if a subgroup of $S$ is centric and radical in $N_{\mathcal{F}}(Z(S))$, then it is either elementary abelian of order $p^2$ or equal to $S$. Moreover, by [RV04, Lemma 4.1], an elementary abelian subgroup $V$ of order $p^2$ is radical in $N_{\mathcal{F}}(Z(S))$ if and only if $\mathrm{Aut}_{\mathcal{F}}(V)$ contains $\mathrm{SL}_2(p)$. However, if $\mathrm{Aut}_{\mathcal{F}}(V)$ contains $\mathrm{SL}_2(p)$, then it does not normalize $Z(S)$. This implies that $S$ is the only subgroup of $S$ which is centric and radical in $N_{\mathcal{F}}(Z(S))$. Hence, by Alperin's Fusion Theorem [AKO11, Theorem I.3.6], we have $N_{\mathcal{F}}(Z(S)) \subseteq N_{\mathcal{F}}(S)$ and thus $N_{\mathcal{F}}(Z(S)) = N_{\mathcal{F}}(S)$. In particular, $N_{\mathcal{F}}(Z(S))$ is constrained. If $X$ is a non-trivial subgroup of $\mathcal{F}$ with $X \neq Z(S)$, then $C_S(X)$ is abelian. So it follows from Lemma 5.3 that $\mathcal{F}$ is of characteristic $p$-type. □

In Section 6, it is shown that for the three exotic Ruiz-Viruel systems, the subcentric linking system is the unique associated punctured group whose full subcategory on the centric subgroups is the centric linking system.

5.2. **Oliver's systems.** A classification of the simple fusion systems $\mathcal{F}$ on $p$-groups with a unique abelian subgroup $A$ of index $p$ is given in [Oli14, COS17, OR20a]. Here we consider only those exotic fusion systems in which $A$ is not essential in $\mathcal{F}$, namely those fusion systems appearing in the statement of [Oli14, Theorem 2.8].

Whenever $\mathcal{F}$ is a saturated fusion system $\mathcal{F}$ on a $p$-group $S$ with a unique abelian subgroup $A$ of index $p$, we adopt Notation 2.2 of [Oli14]. For example,

$$Z = Z(S), \quad Z_2 = Z_2(S), \quad S' = [S,S], \quad Z_0 = Z \cap S', \quad \text{and} \quad A_0 = ZS',$$

and also

$$\mathcal{H} = \{Z\langle x\rangle \mid x \in S - A\} \quad \text{and} \quad \mathcal{B} = \{Z_2\langle x\rangle \mid x \in S - A\}.$$

**Lemma 5.5.** *Let $\mathcal{F}$ be a saturated fusion system on a finite $p$-group $S$ having a unique abelian subgroup $A$ of index $p$.*

- *(a) If $P \leqslant S$ is $\mathcal{F}$-essential, then $P \in \{A\} \cup \mathcal{H} \cup \mathcal{B}$, $|N_S(P)/P| = p$, and each $\alpha \in N_{\mathrm{Aut}_{\mathcal{F}}(P)}(\mathrm{Aut}_S(P))$ extends to an automorphism of $S$.*

*Assume now in addition that $A$ is not essential in $\mathcal{F}$.*

- *(b) If $O_p(\mathcal{F}) = 1$, then $\mathcal{F}^e \cap \mathcal{H} \neq \varnothing$, $Z_0 = Z$ is of order $p$, $S' = A_0$ is of index $p^2$ in $S$, and $S$ has maximal class.*
- *(c) If $P \in \mathcal{H} \cup \mathcal{B}$ is $\mathcal{F}$-essential, then $P \cong C_p^2$ or $p_+^{1+2}$ according to whether $P \in \mathcal{H}$ or $P \in \mathcal{B}$, and $O^{p'}(\mathrm{Out}_{\mathcal{F}}(P)) \cong SL_2(p)$ acts naturally on $P/[P,P]$.*
- *(d) If $P \in \mathcal{F}^e \cap \mathcal{H}$, then each $\alpha \in N_{\mathrm{Aut}_{\mathcal{F}}(P)}(Z)$ extends to an automorphism of $S$.*

*(e) A subgroup $P \leqslant S$ is essential in $N_{\mathcal{F}}(Z)$ if and only if $P \in \mathcal{F}^e \cap \mathcal{B}$.*
*(f) There is $x \in S - A$ such that $A_0\langle x\rangle$ is $\mathrm{Aut}_{\mathcal{F}}(S)$-invariant.*

*Proof.* Parts (a), (b), and (f) are shown in [Oli14, Lemma 2.3,2.4], and (c) follows from [Oli14, Lemma 2.7]. Suppose as in (d) that $P \in \mathcal{F}^e \cap \mathcal{H}$. By (c), $\mathrm{Aut}_{\mathcal{F}}(P)$ is a subgroup of $GL_2(p)$ containing $SL_2(p)$, and the stabilizer of $Z$ in this action normalizes $O^{p'}(C_{\mathrm{Aut}_{\mathcal{F}}(P)}(Z)) = \mathrm{Aut}_S(P)$. So (d) follows from (a).

It remains to prove (e). If $P \in \mathcal{F}^e \cap \mathcal{B}$, then as $Z = [P,P]$ is $\mathrm{Aut}_{\mathcal{F}}(P)$-invariant in this case, $\mathrm{Out}_{N_{\mathcal{F}}(Z)}(P) = \mathrm{Out}_{\mathcal{F}}(P)$ has a strongly $p$-embedded subgroup, and so $P$ is essential in $N_{\mathcal{F}}(Z)$. Conversely, suppose $P$ is $N_{\mathcal{F}}(Z)$-essential. By (a) applied to $N_{\mathcal{F}}(Z)$, $P \in \{A\}\cup\mathcal{H}\cup\mathcal{B}$ and $\mathrm{Out}_S(P)$ is of order $p$, so by assumption $N_{\mathrm{Out}_{N_{\mathcal{F}}(Z)}(P)}(\mathrm{Out}_S(P))$ is strongly $p$-embedded in $\mathrm{Out}_{N_{\mathcal{F}}(Z)}(P)$ by [AKO11, Proposition A.7]. Now each member of $N_{\mathrm{Aut}_{\mathcal{F}}(P)}(\mathrm{Aut}_S(P))$ extends to $S$ by (a), so $Z$ is $N_{\mathrm{Aut}_{\mathcal{F}}(P)}(\mathrm{Aut}_S(P))$-invariant. Thus, $N_{\mathrm{Out}_{\mathcal{F}}(P)}(\mathrm{Out}_S(P)) = N_{\mathrm{Out}_{N_{\mathcal{F}}(Z)}(P)}(\mathrm{Out}_S(P))$ is a proper subgroup of $\mathrm{Out}_{\mathcal{F}}(P)$, and hence strongly $p$-embedded by [AKO11, Proposition A.7] again. So $P$ is essential in $\mathcal{F}$. By assumption $P \neq A$, and $P \notin \mathcal{H}$ by (d). So $P \in \mathcal{B}$. □

**For the remainder of this subsection, we let $\mathcal{F}$ be a saturated fusion system on a $p$-group $S$ with a unique abelian subgroup $A$ of index $p$. Further, we assume that $O_p(\mathcal{F}) = 1$ and $A$ is not essential in $\mathcal{F}$.**

We next set up some additional notation. Fix an element $x \in S - A$ such that $A_0\langle x\rangle$ is $\mathrm{Aut}_{\mathcal{F}}(S)$-invariant, as in Lemma 5.5(f). Since $O_p(\mathcal{F}) = 1$, $S$ is of maximal class by Lemma 5.5(b). In particular $Z = Z_0$ is of order $p$, $A/A_0$ is of order $p$, and $S' = A_0$, so we can adopt [Oli14, Notation 2.5]. As in [Oli14, Notation 2.5], define $\mathcal{H}_i$ and $\mathcal{B}_i$ to be the $S$-conjugacy classes of the subgroups $Z\langle xa^i\rangle$ and $Z_2\langle xa^i\rangle$ for $i = 0, 1, \dots, p-1$, and set

$$\mathcal{H}_* = \mathcal{H}_1 \cup \dots \cup \mathcal{H}_p \quad \text{and} \quad \mathcal{B}_* = \mathcal{B}_1 \cup \dots \cup \mathcal{B}_p,$$

so that $\mathcal{H} = \mathcal{H}_0 \cup \mathcal{H}_*$ and $\mathcal{B} = \mathcal{B}_0 \cup \mathcal{B}_*$.

Set

$$\Delta = (\mathbb{Z}/p\mathbb{Z})^\times \times (\mathbb{Z}/p\mathbb{Z})^\times \quad \text{and} \quad \Delta_i = \{(r, r^i) \mid r \in (\mathbb{Z}/p\mathbb{Z})^\times\}.$$

Define $\mu\colon \mathrm{Aut}_{\mathcal{F}}(S) \to \Delta$ and $\widehat{\mu}\colon \mathrm{Out}_{\mathcal{F}}(S) \to \Delta$ by $\widehat{\mu}([\alpha]) = \mu(\alpha) = (r,s)$, where

$$(xA_0)^\alpha = x^r A_0 \quad \text{and} \quad z^\alpha = z^s.$$

The following lemma looks at the image of homomorphisms analogous to $\mu$ and $\widehat{\mu}$ which are defined instead with respect to $N_{\mathcal{F}}(Z)/Z$ and $C_{\mathcal{F}}(Z)/Z$.

**Lemma 5.6.** *Assume $|S/Z| = p^m$ with $m \geqslant 4$. Let $\mathcal{E} \in \{N_{\mathcal{F}}(Z), C_{\mathcal{F}}(Z)\}$, and let $\mu_{\mathcal{E}}$ be the restriction of $\mu$ to $\mathrm{Aut}_{\mathcal{E}}(S)$. Let $\mu_{\mathcal{E}/Z}\colon \mathrm{Aut}_{\mathcal{E}/Z}(S/Z) \to \Delta$ be the map analogous to $\mu$ but defined instead with respect to $S/Z$. Then*

$$\mathrm{Im}(\mu_{\mathcal{E}/Z}) = \{(r, sr^{-1}) \mid (r,s) \in \mathrm{Im}(\mu_{\mathcal{E}})\}$$

*In particular, if $\mathrm{Im}(\mu_{\mathcal{E}}) = \Delta$, then $\mathrm{Im}(\mu_{\mathcal{E}/Z}) = \Delta$. And if $\mathrm{Im}(\mu_{\mathcal{E}}) = \Delta_i$ for some $i$, then $\mathrm{Im}(\mu_{\mathcal{E}/Z}) = \Delta_{i-1}$, where the indices are taken modulo $p-1$.*

*Proof.* This essentially follows from [COS17, Lemma 1.11(b)]. By assumption, $\mathcal{E}/Z$ is a fusion system over a $p$-group $S/Z$ of order at least $p^4$. So $A/Z$ is the unique abelian subgroup of $S/Z$ of index $p$ by [Oli14, Lemma 1.9]. Since $S$ is of maximal class, so is the quotient $S/Z$. In particular, $Z(S/Z)$ is of order $p$, so we can define $\mu_{\mathcal{E}/Z}$ as suggested with $xZ$ in the role of $x$ and $gZ$ in the role of $z$, where $g \in Z_2 - Z$ is a fixed element.

Let $\alpha \in \mathrm{Aut}_{\mathcal{E}}(S)$ with $\mu(\alpha) = (r, s)$, let $\bar{\alpha}$ be the induced automorphism of $S/Z$, and let $t \in (\mathbb{Z}/p\mathbb{Z})^{\times}$ be such that $a^{\alpha}A_0 = a^t A_0$ (which exists since $A$ and $A_0$ are $\mathrm{Aut}_{\mathcal{F}}(S)$-invariant and $|A/A_0| = |Z_0| = p$). By [COS17, Lemma 1.11(b)], $\alpha$ acts on the $i$-th upper central quotient $Z_i(S)/Z_{i-1}(S)$ by raising a generator to the power $tr^{m-i}$ for $i = 1, \ldots, m-1$. Thus, $s = tr^{m-1}$ and $(gZ)^{\bar{\alpha}} = g^{\alpha}Z = g^{tr^{m-2}}Z$. Hence, $\mu_{\mathcal{E}/Z}(\bar{\alpha}) = (r, sr^{-1})$. Conversely if $\mu_{\mathcal{E}/Z}(\bar{\alpha}) = (r, \bar{s})$, then $\mu_{\mathcal{E}}(\alpha) = (r, \bar{s}r)$. □

In the following proposition, we refer to Oliver's systems according to the itemized list (a)(i-iv), (b) given in [Oli14, Theorem 2.8].

**Proposition 5.7.** *Assume $\mathcal{F}$ is one of the exotic systems appearing in Theorem 2.8 of [Oli14]. Write $|S/Z| = p^m$ with $m \geqslant 3$.*

- *(a) $\mathcal{F}$ is of characteristic $p$-type whenever $\mathcal{F}^e \subseteq \mathcal{H}$. In particular, this holds if $\mathcal{F}$ occurs in case (a)(i), (a)(iv), or (b).*
- *(b) If $\mathcal{F}$ is in case (a)(ii) and $m \geqslant 4$, then $N_{\mathcal{F}}(Z)$ is exotic. Moreover, $\mathcal{F}$ is of component type, and $C_{\mathcal{F}}(Z)/Z$ is simple, exotic, and occurs in (a)(iv) in this case. If $\mathcal{F}$ is in case (a)(ii) with $m = 3$ (and hence $p = 5$), then $N_{\mathcal{F}}(Z)/Z$ is the fusion system of $5^2GL_2(5)$, and $\mathcal{F}$ is of characteristic 5-type.*
- *(c) If $\mathcal{F}$ is in case (a)(iii), then $N_{\mathcal{F}}(Z)$ is exotic. Moreover, $\mathcal{F}$ is of component type with $C_{\mathcal{F}}(Z)/Z$ is simple, exotic, and of type (a)(i).*

*Proof.* Each of Oliver's systems is simple on $S$ with a unique abelian subgroup $A$ of index $p$ which is not essential, so it satisfies our standing assumptions and the hypotheses of Lemmas 5.5 and 5.6, and we can continue the notation from above. In particular, $Z_0 = Z$ is of order $p$, $S' = A_0$, and $S$ is of maximal class.

For each fully $\mathcal{F}$-normalized subgroup $X \leqslant S$ of order $p$ and not equal to $Z$, $C_S(X)$ is abelian: if $X \leqslant A$ this follows since $C_S(X) = A$ ($X$ is not central), while if $X \nleqslant A$, this follows since $C_A(X) = Z$ by Lemma 5.5(b). Thus $N_{\mathcal{F}}(X)$ is constrained in this case by Lemma 5.3. Hence $\mathcal{F}$ is of characteristic $p$-type if and only if $N_{\mathcal{F}}(Z)$ is constrained. By Lemma 5.5(e), if $\mathcal{F}^e \subseteq \mathcal{H}$, then $N_{\mathcal{F}}(Z)$ has no essential subgroups. By the Alperin-Goldschmidt fusion theorem [AKO11, I.3.5], each morphism in $N_{\mathcal{F}}(Z)$ extends to $S$, and hence $S$ is normal in $N_{\mathcal{F}}(Z)$. So if $\mathcal{F}^e \subseteq \mathcal{H}$, then $N_{\mathcal{F}}(Z)$ is constrained.

**Case: $\mathcal{F}$ occurs in (a)(i), (a)(iv), or (b) of [Oli14, Theorem 2.8].** We have $\mathcal{F}^e \subseteq \mathcal{H}$ precisely in these cases. So $\mathcal{F}$ is of characteristic $p$-type.

**Case: $\mathcal{F}$ occurs in (a)(ii).** Here, $m \equiv -1 \pmod{p-1}$, $\widehat{\mu}(\mathrm{Out}_{\mathcal{F}}(S)) = \Delta$, and $\mathcal{F}^e = \mathcal{B}_0 \cup \mathcal{H}_*$. By assumption $\mathcal{F}$ is exotic, so as $\mathcal{F}$ is the fusion system of ${}^3D_4(q)$ when $p = 3$, we have $p \geqslant 5$.

Let $\mathcal{E} \in \{N_{\mathcal{F}}(Z), C_{\mathcal{F}}(Z)\}$ and set $\bar{\mathcal{F}} = N_{\mathcal{F}}(Z)/Z$, $\bar{\mathcal{F}}_1 = C_{\mathcal{F}}(Z)/Z$, and $\bar{S} = S/Z$, so that $\bar{\mathcal{F}}_1 \trianglelefteq \bar{\mathcal{F}}$ is a normal pair of fusion systems on $\bar{S}$. By Lemma 5.5(e), the set of $N_{\mathcal{F}}(Z)$-essential subgroups is $\mathcal{F}^e \cap \mathcal{B}$. A straightforward argument shows that the set of $C_{\mathcal{F}}(Z)$-essential subgroups is then also $\mathcal{F}^e \cap \mathcal{B}$, and that factoring by $Z$ induces a bijection between the set of essential subgroups of $\mathcal{E}$ and of $\bar{\mathcal{E}}$. Thus, $\bar{\mathcal{E}}^e = \bar{\mathcal{B}}_0$, where $\bar{\mathcal{B}}_0 = \{\bar{P} \mid P \in \mathcal{B}_0\}$.

**Subcase:** $m \geqslant 4$. Since $\bar{S}$ has order $p^4$, we know that $\bar{A}$ is the unique abelian subgroup of $\bar{S}$ of index $p$ by [Oli14, Lemma 1.9].

We will prove that $\bar{\mathcal{E}}$ is reduced. Since $|\bar{S}| \geqslant p^4$, the subgroup $\bar{Z}_2\langle\bar{x}\rangle$ is not normal in $\bar{S}$. Since $O_p(\bar{\mathcal{E}})$ is contained in every $\bar{\mathcal{E}}$-essential subgroup, we have $O_p(\bar{\mathcal{E}}) \leqslant \bigcap \bar{\mathcal{B}}_0 = Z(\bar{S})$. By Lemma 5.5(c), $Z_2$ is not $\mathrm{Aut}_{\mathcal{E}}(P)$-invariant for any $P \in \mathcal{B}_0$, and hence $\bar{Z}_2 = Z(\bar{S})$ is not $\mathrm{Aut}_{\bar{\mathcal{E}}}(\bar{P})$-invariant for any $\bar{P} \in \bar{\mathcal{B}}_0$. So $O_p(\bar{\mathcal{E}}) = 1$.

Next, since $\mathrm{Im}(\mu) = \Delta$ in the present case, we have $\mathrm{Im}(\mu_{\mathcal{E}}) = \Delta$ when $\mathcal{E} = N_{\mathcal{F}}(Z)$ since the $\mathrm{Aut}_{N_{\mathcal{F}}(Z)}(S) = \mathrm{Aut}_{\mathcal{F}}(S)$. When $\mathcal{E} = C_{\mathcal{F}}(Z)$, the $\mathcal{E}$-automorphism group of $S$ is the centralizer of $Z$ in $\mathrm{Aut}_{\mathcal{F}}(S)$. By definition of the map $\mu$, this means

$$\mathrm{Im}(\mu_{\mathcal{E}}) = \{(r, 1) \mid r \in (\mathbb{Z}/p\mathbb{Z})^\times\} = \Delta_0$$

So in any case, $\mathrm{Im}(\mu_{\bar{\mathcal{E}}}) \geqslant \Delta_{-1}$ by Lemma 5.6.

We next show that $O^p(\bar{\mathcal{E}}) = \bar{\mathcal{E}}$. By [Oli14, Proposition 1.3(c,d)], the focal subgroup of $\bar{\mathcal{E}}$ is generated by $[\bar{P}, \mathrm{Aut}_{\bar{\mathcal{E}}}(\bar{P})]$ for $\bar{P} \in \bar{\mathcal{B}}_0 \cup \{\bar{S}\}$, and $O^p(\bar{\mathcal{E}}) = \bar{\mathcal{E}}$ if and only if $\mathfrak{foc}(\bar{\mathcal{E}}) = \bar{S}$. Since $\bar{P}$ is a natural module for $O^{p'}(\mathrm{Aut}_{\bar{\mathcal{E}}}(\bar{P})) \cong SL_2(p)$ for each $\bar{P} \in \bar{\mathcal{B}}_0$ (Lemma 5.5(c)), the focal subgroup of $\bar{\mathcal{E}}$ contains $\langle \bar{\mathcal{B}}_0 \rangle = \bar{A}_0\langle\bar{x}\rangle$. Thus, $\mathfrak{foc}(\bar{\mathcal{E}}) = \bar{S}$ if and only if $\bar{a} \in [\bar{S}, \mathrm{Aut}_{\bar{\mathcal{E}}}(\bar{S})]$. But we just saw that $\mathrm{Im}(\mu_{\bar{\mathcal{E}}}) \geqslant \Delta_{-1}$. Further, if $\bar{\alpha}$ is an $\bar{\mathcal{E}}$-automorphism of $\bar{S}$ with $\mu_{\bar{\mathcal{E}}}(\bar{\alpha}) = (r, r^{-1})$, then for the class $t \in (\mathbb{Z}/p\mathbb{Z})^\times$ with $(\bar{a}\bar{A}_0)^{\bar{\alpha}} = \bar{a}^t\bar{A}_0$, we have $r^{-1} = tr^{m-2}$ by [COS17, Lemma 1.11(b)], and hence $t = r^{-(m-1)}$. As $m+1 \equiv 0 \pmod{p-1}$ and $p \geqslant 5$, we have $-(m-1) \not\equiv 0 \pmod{p-1}$. So $\mathrm{Aut}_{\bar{\mathcal{E}}}(\bar{S})$ acts nontrivially on $\bar{A}/\bar{A}_0$, and hence $\mathfrak{foc}(\bar{\mathcal{E}}) = \bar{S}$.

We next show that $O^{p'}(\bar{\mathcal{E}}) = \bar{\mathcal{F}}_1$ using Lemma 1.4 of [Oli14]. Since $\bar{\mathcal{F}}_1$ is a normal subsystem of $\bar{\mathcal{F}}$ on $\bar{S}$, it follows from [AKO11, Theorem I.7.7] that $O^{p'}(\bar{\mathcal{E}})$ is a subsystem of $\bar{\mathcal{F}}_1$. So it will be sufficient to show that $O^{p'}(\bar{\mathcal{F}}_1) = \bar{\mathcal{F}}_1$. Set $\bar{P} = Z(\bar{S})\langle\bar{x}\rangle = \bar{Z}_2\langle\bar{x}\rangle \in \bar{\mathcal{B}}_0$, and let $\bar{\alpha}$ be an $\bar{\mathcal{F}}_1$-automorphism of $\bar{S}$. Since $\bar{A}_0\langle\bar{x}\rangle$ is $\bar{\alpha}$-invariant, it follows that $\bar{\alpha}$ preserves the $\bar{S}$-class $\bar{\mathcal{B}}_0$ under conjugation, and so upon adjusting $\bar{\alpha}$ by an inner automorphism of $\bar{S}$ (which doesn't change the image of $\bar{\alpha}$ under $\mu_{\bar{\mathcal{F}}_1}$), we can assume that $\bar{\alpha}$ normalizes $\bar{P}$. The restriction of $\bar{\alpha}$ to $\bar{P}$ acts via an element of $SL_2(p)$ on $\bar{P}$ since $\mathrm{Im}(\mu_{\bar{\mathcal{F}}_1}) = \Delta_{-1}$, and so this restriction is contained in $O^{p'}(\mathrm{Aut}_{\bar{\mathcal{F}}_1}(\bar{P}))$. Thus, $O^{p'}(\bar{\mathcal{F}}_1) = \bar{\mathcal{F}}_1$ by Lemma 1.4 of [Oli14].

Thus, $\bar{\mathcal{F}}_1$ is reduced. Step 1 of the proof of [Oli14, Theorem 2.8] then shows that $\bar{\mathcal{F}}_1$ is the unique reduced fusion system with the given data, and then Step 2 shows that $\bar{\mathcal{F}}_1$ is simple. So $\bar{\mathcal{F}}_1$ is exotic and occurs in case (a)(iv) of Oliver's classification, since $m-1 = -2 \not\equiv 0, -1 \pmod{p-1}$.

It remains to show that $N_{\mathcal{F}}(Z)$ is exotic. For this it suffices to show that $N_{\mathcal{F}}(Z)/Z = \bar{\mathcal{F}}$ is exotic. For suppose $N_{\mathcal{F}}(Z)$ is realizable by a group $G$ with Sylow $p$-subgroup $S$. Then $Z$ is normal in the fusion system of $G$, and so $N_G(Z)$ also realizes $N_{\mathcal{F}}(Z)$. Hence, $N_G(Z)/Z$ realizes $N_{\mathcal{F}}(Z)/Z$.

Assume to get a contradiction that $\bar{\mathcal{F}}$ is realizable. We will verify the hypotheses of [DRV07, Proposition 2.19] for $\bar{\mathcal{F}}$. Let $\bar{T} \leqslant \bar{S}$ be a nontrivial, strongly $\bar{\mathcal{F}}$-closed subgroup. Since $\bar{\mathcal{F}}_1$ is a subsystem of $\bar{\mathcal{F}}$ on $\bar{S}$, $\bar{T}$ is strongly $\bar{\mathcal{F}}_1$-closed. By [Oli14, Theorem 2.8], either $\bar{T} = \bar{A}_0\langle\bar{x}\rangle$ or $\bar{T} = \bar{S}$. In particular, $\bar{T}$ is nonabelian. Also, since $Z(\bar{T}) = Z(\bar{S})$ by Lemma 5.5(b), it follows that $\bar{T}$ is centric and does not split as a direct product of two of its proper subgroups. Thus, by [DRV07, Proposition 2.19], $\bar{\mathcal{F}} = \mathcal{F}_{\bar{S}}(\bar{G})$ for some finite group $\bar{G}$ with Sylow subgroup $\bar{S}$ and with $F^*(\bar{G})$ simple. Let $\bar{G}_0 = O^{p'}(\bar{G})$. Then $\bar{\mathcal{F}}_0 := \mathcal{F}_{\bar{S}}(\bar{G}_0)$ is normal subsystem of $\bar{\mathcal{F}}$ on $\bar{S}$, so $\bar{\mathcal{F}}_1 \subseteq \bar{\mathcal{F}}_0$ by [AKO11, Theorem I.7.7]. The hyperfocal subgroup of $\bar{\mathcal{F}}_0$ contains the hyperfocal subgroup of $\bar{\mathcal{F}}_1$, which is $\bar{S}$. Thus, $O^p(\bar{G}_0) = \bar{G}_0$. Since $O^{p'}(\bar{G}_0) = \bar{G}_0 = O^p(\bar{G}_0)$ and the outer automorphism group of $F^*(\bar{G}_0)$ is solvable by the Schreier conjecture, it follows that $\bar{G}_0 = F^*(\bar{G}_0)$ is simple.

Now set $\bar{T} = \bar{A}_0\langle \bar{x} \rangle = \langle \bar{\mathcal{B}}_0 \rangle$. Then $\bar{T}$ is indeed strongly $\bar{\mathcal{F}}_0$-closed and proper in $\bar{S}$. By a result of Flores and Foote [FF09], stated as in [AKO11, II.12.12], we have $p = 3 = |\bar{T}| = p^{m-2}$, and this contradicts $m \geqslant 4$. Therefore, $\bar{\mathcal{F}}$ is exotic.

**Subcase: $m = 3$.** Since $m \equiv -1 \pmod{p-1}$, we have $p = 5$. So $\bar{S}$ is extraspecial of order $5^3$ and exponent 5. We saw above that $\bar{\mathcal{F}}^e = \bar{\mathcal{B}}_0$, which is of size 1 in this case. That is, there is a unique essential subgroup, $Z(\bar{S})\langle \bar{x} \rangle$, which is therefore $\mathrm{Aut}_{\bar{\mathcal{F}}}(\bar{S})$-invariant. By the Alperin-Goldschmidt fusion theorem, this subgroup is normal in $\bar{\mathcal{F}}$. So $\bar{\mathcal{F}}$ is constrained, it is isomorphic to the fusion system of $5^2 GL_2(5)$ by Lemma 5.6 and [Oli14, Lemma 2.6(c)], and $\mathcal{F}$ is of characteristic 5-type. This completes the proof of (b).

**Case: $\mathcal{F}$ occurs in (a)(iii).** Then $m \equiv 0 \pmod{p-1}$, $\mathcal{F}^e = \mathcal{H}_0 \cup \mathcal{B}_*$, and $\mathrm{Im}(\mu) = \Delta$. Set $\bar{\mathcal{F}} = N_{\mathcal{F}}(Z)/Z$ and $\bar{\mathcal{F}}_1 = C_{\mathcal{F}}(Z)/Z$. Similarly to the previous case, we can show $\bar{\mathcal{F}}^e = \bar{\mathcal{F}}_1^e = \{\bar{P} \mid P \in \mathcal{B}_*\}$. Denote this set by $\bar{\mathcal{B}}_*$, and let $\bar{\mathcal{E}} \in \{\bar{\mathcal{F}}, \bar{\mathcal{F}}_1\}$. Since $m \geqslant 3$, we have $p \geqslant 5$, and hence in fact $m \geqslant 4$. In particular, $\bar{A}$ is the unique abelian subgroup of $\bar{S}$, $\bar{A}$ is not essential in $\bar{\mathcal{E}}$, and $O_p(\bar{\mathcal{E}}) = 1$ by the same argument as in the previous case.

We can see that $O^{p'}(\bar{\mathcal{F}}) = O^{p'}(\bar{\mathcal{F}}) = \bar{\mathcal{F}}_1$ using $\mathrm{Im}(\mu_{\bar{\mathcal{F}}_1}) = \Delta_{-1}$ and Lemma 5.6 as in the previous case. Also as the previous case, the focal subgroup of $\bar{\mathcal{E}}$ contains $\langle \bar{\mathcal{B}}_* \rangle$, which this time is equal to $\bar{S}$. So $O^p(\bar{\mathcal{E}}) = \bar{\mathcal{E}}$. We've shown $\bar{\mathcal{F}}_1$ is reduced, and so $\bar{\mathcal{F}}_1$ is simple by Steps 1 and 2 of the proof of [Oli14, Theorem 2.8]. As $|\bar{S}/Z(\bar{S})| = p^{m-1}$ and $m - 1 \equiv -1 \pmod{p-1}$, it follows that $\bar{\mathcal{F}}_1$ occurs in case (a)(i) of Oliver's classification. In particular, $\bar{\mathcal{F}}_1$ is exotic.

As in the previous case, to show $N_{\mathcal{F}}(Z)$ is exotic it will be sufficient to show that $\bar{\mathcal{F}}$ is exotic. Suppose instead that $\bar{\mathcal{F}}$ is realizable. As before, we use [DRV07, Proposition 2.19] to see that $\bar{\mathcal{F}}$ is realizable by an almost simple group $\bar{G}$, and then get that $\bar{G}_0 = O^{p'}(\bar{G})$ is simple. Let $\bar{\mathcal{F}}_0 = \mathcal{F}_{\bar{S}}(\bar{G}_0)$. Then $\bar{\mathcal{F}}_0$ is a normal subsystem of $\bar{\mathcal{F}}_1$ on $\bar{S}$, so $\bar{\mathcal{F}}_1 = O^{p'}(\bar{\mathcal{F}}_0) \subseteq \bar{\mathcal{F}}_0$. Further, $\bar{\mathcal{F}}_0 \neq \bar{\mathcal{F}}_1$ since we saw earlier that $\bar{\mathcal{F}}_1$ is exotic. We are thus in the situation of Theorem A of [OR20b]. By that theorem, there is $n \geqslant 2$ and a prime power $q$ with $q \neq 0, \pm 1 \pmod{p}$ such that $G \cong PSL_n^{\pm}(q)$, $PSp_{2n}(q)$, $\Omega_{2n+1}(q)$, or $P\Omega_{2n+2}^{\pm}(q)$. Since $m \geqslant 4$, no member of $\bar{\mathcal{B}} = \{\bar{P} \mid P \in \mathcal{B}\}$ is normal in $\bar{S}$. So $\bar{A}$ is the unique $\bar{\mathcal{F}}_0$-centric abelian subgroup which is normal in $\bar{S}$. By [OR20b, Proposition 4.5], we have $\mathrm{Aut}_{\bar{\mathcal{F}}_0}(\bar{A})/O_{p'}(\mathrm{Aut}_{\bar{\mathcal{F}}_0}(\bar{A})) \cong S_\kappa$, where $\kappa$ is such that $S_\kappa$ has Sylow $p$-subgroup of order $p$. Hence, $\mathrm{Aut}_{\bar{\mathcal{F}}_0}(\bar{A})$ has a strongly $p$-embedded subgroup. This contradicts the fact that $\bar{A}$ is not essential in $\bar{\mathcal{F}}_0$ and completes the proof. □

5.3. **The Clelland-Parker systems.** We now describe the fusion systems constructed by Clelland and Parker in [CP10]. Throughout we fix a power $q$ of the odd prime $p$ and set $k := \mathbb{F}_q$. Let $A := A(n, k)$ be the irreducible module of dimension $2 \leqslant n+1 \leqslant p$ over $k$ for $SL_2(q)$; for example, $A$ can be taken to be the space of homogeneous polynomials of degree $n$ with coefficients in $k$. There is an action of $D := k^\times \times GL_2(k)$ on $A$ that extends that of $SL_2(k)$; we write $G$ for the semidirect product $DA$. Let $U$ be a Sylow $p$-subgroup of $D$ and let $S := S(n, k) := UA$ be the semidirect product of $A$ by $U$.

The center $Z := Z(S)$ is a one-dimensional $k$-subspace of $A$ and by [CP10, Lemma 4.2(iii)], we have

(5.1) $$C_A(X) = Z(S) \text{ for each subgroup } X \text{ not contained in } A.$$

The second center $Z_2(S)$ is a 2-dimension $k$-subspace of $A$. Let $R = ZU$ and $Q = Z_2(S)U$. Then $R \cong q^2$ and $Q$ is special of shape $q^{1+2}$. Let $H_R$ be the stabilizer in $GL_3(k)$ of a one dimensional subspace, and identify its unipotent radical with $R$. Let $H_Q$ be the stabilizer in $GSp_4(k)$ of a one dimensional subspace and identify the corresponding unipotent radical with $Q$. It is shown in [CP10] that $N_G(R)$ is isomorphic to a Borel subgroup $GL_3(k)$, and that $N_G(Q)$ is isomorphic to a Borel subgroup of $GSp_4(k)$. This allows to form the free amalgamated products

$$F(1, n, k, R) := G *_{N_G(R)} H_R$$

and

$$F(1, n, k, Q) := G *_{N_G(Q)} H_Q.$$

Set

$$\mathcal{F}(1, n, k, R) := \mathcal{F}_S(F(1, n, k, R))$$

and

$$\mathcal{F}(1, n, k, Q) := \mathcal{F}_S(F(1, n, k, Q)).$$

More generally, for each $X \in \{R, Q\}$ and each divisor $r$ of $q - 1$, subgroup $F(r, n, k, X)$ of $F(1, n, k, X)$ of index $r$, which contains $O^{p'}(G)$ and $O^{p'}(H_X)$. They set then

$$\mathcal{F}(r, n, k, X) = \mathcal{F}_S(F(r, n, k, X)).$$

As they show, distinct fusion systems are only obtained for distinct divisors $r$ of $(n + 2, q - 1)$ when $X = R$, and for distinct divisors $r$ of $(n, q - 1)$ when $X = Q$. By [CP10, Theorem 4.9], for all $n \geqslant 1$ and each divisor $r$ of $(n + 2, q - 1)$, $\mathcal{F}(r, n, k, R)$ is saturated. Similarly, $\mathcal{F}(r, n, k, Q)$ is saturated for each $n \geqslant 2$ and each divisor $r$ of $(n, q - 1)$. It is determined in Theorem 5.1, Theorem 5.2 and Lemma 5.3 of [CP10] which of these fusion systems are exotic. It turns out that $\mathcal{F}(r, n, k, R)$ is exotic if and only if either $n > 2$ or $n = 2$ and $q \notin \{3, 5\}$. Furthermore, $\mathcal{F}(r, n, k, Q)$ is exotic if and only if $n \geqslant 3$, in which case $p \neq 3$ as $n \leqslant p - 1$.

**For the remainder of this subsection, except in Lemma 5.10, we use the notation introduced above.**

For the problems we will consider here, we will sometimes be able to reduce to the case $r = 1$ using the following lemma.

**Lemma 5.8.** *For any divisor $r$ of $q - 1$, the fusion system $\mathcal{F}(r, n, k, R)$ is a normal subsystem of $\mathcal{F}(1, n, k, R)$ of index prime to $p$, and the fusion system $\mathcal{F}(r, n, k, Q)$ is a normal subsystem of $\mathcal{F}(1, n, k, Q)$ of index prime to $p$.*

*Proof.* For $X \in \{R, Q\}$, the fusion systems $\mathcal{F}(r, n, k, X)$ and $\mathcal{F}(1, n, k, X)$ are both saturated by the results cited above, As $F(r, n, k, X)$ is a normal subgroup of $F(1, n, k, X)$, it is easy to check that $\mathcal{F}(r, n, k, X)$ is $\mathcal{F}(1, n, k, X)$-invariant. As both $\mathcal{F}(1, n, k, X)$ and $\mathcal{F}(r, n, k, X)$ are fusion systems over $S$, the claim follows. □

**Proposition 5.9.** *$\mathcal{F}(r, n, k, R)$ is of characteristic $p$-type for all $1 \leqslant n \leqslant p-1$ and for all divisors $r$ of $(n + 2, q - 1)$.*

*Proof.* Fix $1 \leqslant n \leqslant p-1$ and a divisor $r$ of $(n+2, q-1)$. Set $\mathcal{F} = \mathcal{F}(1, n, k, R)$. By Lemma 5.8, $\mathcal{F}(r, n, k, R)$ is a normal subsystem of $\mathcal{F}$ of index prime to $p$. So by [Hen19, Proposition 2(c)], it suffices to show that $\mathcal{F}$ is of characteristic $p$-type. By [CP10, Lemma 5.3(i,ii)], $\mathcal{F}$ is of realizable and of characteristic $p$-type when $n = 1$, so we may and do assume $n \geqslant 2$.

Using the notation above, set $\mathcal{F}_1 = \mathcal{F}_S(G)$, $S_2 = N_S(R)$, and $\mathcal{F}_2 = \mathcal{F}_{S_2}(H_R)$. The fusion system $\mathcal{F}$ is generated by $\mathcal{F}_1$ and $\mathcal{F}_2$ by [CP10, Theorem 3.1], and so as $\mathcal{F}_1$ and $\mathcal{F}_2$ are both constrained with $O_p(\mathcal{F}_1) = A$ and $O_p(\mathcal{F}_2) = R$, it follows that $\mathcal{F}$ is in turn generated by $\mathrm{Aut}_{\mathcal{F}_1}(A)$, $\mathrm{Aut}_{\mathcal{F}_1}(S)$, $\mathrm{Aut}_{\mathcal{F}_2}(R)$, and $\mathrm{Aut}_{\mathcal{F}_2}(S_2)$. However, the last automorphism group is redundant, since $N_{H_R}(S_2) = N_G(R)$ induces fusion in $\mathcal{F}_1$. Hence

$$\mathcal{F} = \langle \mathrm{Aut}_{\mathcal{F}_1}(S), \mathrm{Aut}_{\mathcal{F}_1}(A), \mathrm{Aut}_{\mathcal{F}_2}(R) \rangle. \tag{5.2}$$

Observe also that the following property is a direct consequence of (5.1):

(5.3) If $X \leqslant S$ with $X \nleqslant Z$, then either $X \leqslant A$ and $C_S(X) = A$, or $|C_S(X)| \leqslant q^2$.

We can now show that $\mathcal{F}$ is of characteristic $p$-type. Let first $X \in \mathcal{F}^f$ such that $X \nleqslant Z$. We show that $N_{\mathcal{F}}(X)$ is constrained. If $X$ is not $\mathcal{F}$-conjugate into $A$ or into $R$, then every morphism in $N_{\mathcal{F}}(X)$ extends by (5.2) to an automorphism of $S$. So $N_{\mathcal{F}}(X) = N_{N_{\mathcal{F}}(S)}(X)$. As $N_{\mathcal{F}}(S)$ is constrained, it follows thus from [Hen19, Lemma 2.11] that $N_{\mathcal{F}}(X)$ is constrained. So we may assume that there exists an $\mathcal{F}$-conjugate $Y$ of $X$ with $Y \leqslant A$ or $Y \leqslant R$. We will show that $C_S(X)$ is abelian so that $N_{\mathcal{F}}(X)$ is constrained by Lemma 5.3. Note that $|C_S(X)| \geqslant |C_S(Y)|$ as $X$ is fully normalized and thus fully centralized in $\mathcal{F}$. Since $X$ is not contained in $Z = Z(S)$, we have in particular $Y \nleqslant Z$. If $Y \leqslant A$, then $A \leqslant C_S(Y)$ and, since $X$ is fully centralized and $n \geqslant 2$, $|C_S(X)| \geqslant |C_S(Y)| \geqslant |A| > q^2$. So by (5.3), $C_S(X) = A$ is abelian. Similarly, by (5.3), if $X \leqslant A$ then $C_S(X) = A$ is abelian. Thus we may assume $Y \leqslant R$ and $X \nleqslant A$. Then $R \leqslant C_S(Y)$ and (5.3) implies $q^2 \geqslant |C_S(X)| \geqslant |C_S(Y)| \geqslant |R| = q^2$. So the inequalities are equalities, $C_S(Y) = R$ and $|C_S(X)| = q^2$. By the extension axiom, there exists $\varphi \in \mathrm{Hom}_{\mathcal{F}}(C_S(Y), C_S(X))$. So it follows that $C_S(X) \in R^{\mathcal{F}}$ is abelian. This completes the proof that $N_{\mathcal{F}}(X)$ is constrained for every $X \in \mathcal{F}^f$ with $X \nleqslant Z$.

Let now $1 \neq X \leqslant Z$. It remains to show that $N_{\mathcal{F}}(X)$ is constrained. If $N_{\mathcal{F}}(X) \subseteq N_{\mathcal{F}}(S)$, then again by [Hen19, Lemma 2.11], $N_{\mathcal{F}}(X) = N_{N_{\mathcal{F}}(S)}(X)$ is constrained since $N_{\mathcal{F}}(S)$ is constrained. We will finish the proof by showing that indeed $N_{\mathcal{F}}(X) \subseteq N_{\mathcal{F}}(S)$. Assume by contradiction that $N_{\mathcal{F}}(X) \nsubseteq N_{\mathcal{F}}(S)$. Then there exists an essential subgroup $E$ of $N_{\mathcal{F}}(X)$. Observe that $Z < E$, since $E$ is $N_{\mathcal{F}}(X)$-centric. As $\mathrm{Aut}_S(E)$ is not normal in $\mathrm{Aut}_{N_{\mathcal{F}}(Z)}(E)$, there exists an element of $\mathrm{Aut}_{N_{\mathcal{F}}(Z)}(E)$ which does not extend to an $\mathcal{F}$-automorphism of $S$. So by (5.2), $E$ is $\mathcal{F}$-conjugate into $A$ or into $R$. Assume first that there exists an $\mathcal{F}$-conjugate $\hat{E}$ of $E$ such that $\hat{E} \leqslant A$. Property (5.1) yields that $R \cap A = Z$. So $E$ is conjugate to $\hat{E} \leqslant A$ via an element of $\mathrm{Aut}_{\mathcal{F}_1}(S)$ by (5.2). Thus, as $C_S(E) \leqslant E$, we have $A \leqslant C_S(\hat{E}) \leqslant \hat{E}$. Hence $A = \hat{E}$ by (5.3). As $A$ is $\mathrm{Aut}_{\mathcal{F}_1}(S)$-invariant, it follows $E = A$. Looking at the structure of $G$, we observe now that $N_G(X) = N_G(S)$ and so $\mathrm{Aut}_S(A)$ is normal in $\mathrm{Aut}_{N_{\mathcal{F}}(X)}(A) = N_{\mathrm{Aut}_{\mathcal{F}}(A)}(X) = N_{\mathrm{Aut}_{\mathcal{F}_1}(A)}(X)$. Hence, $A$ cannot be essential in $N_{\mathcal{F}}(X)$ and we have derived a contradiction. Thus, $E$ is not $\mathcal{F}$-conjugate into $A$. Therefore, again by (5.2), $E$ is conjugate into $R$ under an element of $\mathrm{Aut}_{\mathcal{F}_1}(S)$. Let $\alpha \in \mathrm{Aut}_{\mathcal{F}_1}(S)$

such that $E^\alpha \leqslant R$. As $C_S(E) \leqslant E$, we have then $C_S(E^\alpha) \leqslant E^\alpha$. Since $R$ is abelian, it follows $E^\alpha = R$. Thus, we have

$$\mathrm{Aut}_{N_{\mathcal{F}}(X)}(E)^\alpha = N_{\mathrm{Aut}_{\mathcal{F}}(E)}(X)^\alpha = N_{\mathrm{Aut}_{\mathcal{F}}(R)}(X^\alpha).$$

As $1 \neq X^\alpha \leqslant Z^\alpha = Z \leqslant R$ and $\mathrm{Aut}_{\mathcal{F}}(R) = \mathrm{Aut}_{\mathcal{F}_2}(R)$ acts $k$-linearly on $R$, $N_{\mathrm{Aut}_{\mathcal{F}}(R)}(X^\alpha)$ has a normal Sylow $p$-subgroup. Thus, $\mathrm{Aut}_{N_{\mathcal{F}}(X)}(E) \cong N_{\mathrm{Aut}_{\mathcal{F}}(R)}(X^\alpha)$ has a normal Sylow $p$-subgroup, contradicting the fact that $E$ is essential in $N_{\mathcal{F}}(X)$. This final contradiction shows that $N_{\mathcal{F}}(X) \subseteq N_{\mathcal{F}}(S)$ is constrained. This completes the proof of the assertion. □

Our next goal will be to show that $\mathcal{F} := \mathcal{F}(r, n, k, Q)$ does not have a punctured group for $n \geqslant 3$ (i.e. in the case that $\mathcal{F}$ is exotic). For that we prove that, using the notation introduced at the beginning of this subsection, $N_{\mathcal{F}}(Z)/Z$ is exotic. The structure of $N_{\mathcal{F}}(Z)/Z$ resembles the structure of $\mathcal{F}(r, n-1, k, R)$ except that the elementary abelian normal subgroup of index $q$ is not essential. Indeed, it will turn out that the problem of showing that $N_{\mathcal{F}}(Z)/Z$ is exotic reduces to the situation treated in the following lemma, whose proof of part (c) depends on the classification of finite simple groups.

**Lemma 5.10.** *Fix a power $q$ of $p$ as before. Let $S$ be an arbitrary $p$-group such that $S = U \ltimes A$ splits as a semidirect product of an elementary abelian subgroup $A$ with an elementary abelian subgroup $U$. Assume $|U| = q$, and $|A| = q^n$ for some $3 \leqslant n \leqslant p-1$. Set $P := Z(S)U$, $T := [S,S]U$, and let $\mathcal{F}$ be a saturated fusion system over $S$. Assume the following conditions hold:*

- (i) *$Z(S)$ has order $q$, $[S,S] \not\leqslant Z(S)$, and $Z(S) = C_A(u)$ for every $1 \neq u \in U$.*
- (ii) *$O^{p'}(\mathrm{Aut}_{\mathcal{F}}(P)) \cong \mathrm{SL}_2(q)$ and $P$ is a natural $\mathrm{SL}_2(q)$-module for $O^{p'}(\mathrm{Aut}_{\mathcal{F}}(P))$,*
- (iii) *$\mathcal{F}$ is generated by $\mathrm{Aut}_{\mathcal{F}}(P)$ and $\mathrm{Aut}_{\mathcal{F}}(S)$,*
- (iv) *$\mathrm{Aut}_{\mathcal{F}}(S)$ acts irreducibly on $A/[S,S]$, and*
- (v) *there is a complement to $\mathrm{Inn}(S)$ in $\mathrm{Aut}_{\mathcal{F}}(S)$ which normalizes $U$.*

*Then the following hold:*

- (a) *The non-trivial strongly closed subgroups of $\mathcal{F}$ are precisely $S$ and $T$.*
- (b) *Neither $S$ nor $T$ can be written as the direct product of two non-trivial subgroups.*
- (c) *$\mathcal{F}$ is exotic.*

*Proof.* Observe first that (iii) implies that $P$ is fully normalized. In particular, $\mathrm{Aut}_S(P) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(P))$. As $Z := Z(S)$ has order $q$, it follows from (ii) that $Z(S) = C_P(N_S(P)) = [P, N_S(P)] \leqslant [S,S]$. In particular, $P \leqslant T$. We note also that $C_S(P) = P$ as $C_A(U) = Z(S)$ by (i).

To prove (a), we argue first that $T$ is strongly closed. Observe that $T$ is normal in $S$, since $T$ contains $[S,S]$. As $[S,S]$ is characteristic in $S$, it follows thus from (v) that $T$ is $\mathrm{Aut}_{\mathcal{F}}(S)$-invariant. Thus, as $P \leqslant T$, (iii) implies that $T$ is strongly closed in $\mathcal{F}$. Let now $S_0$ be a non-trivial proper subgroup of $S$ strongly closed in $\mathcal{F}$. Since $S_0$ is normal in $S$, it follows $1 \neq S_0 \cap Z(S) \leqslant P$. By (ii), $\mathrm{Aut}_{\mathcal{F}}(P)$ acts irreducibly on $P$. So $P \leqslant S_0$. Hence, $[S,S] = [A,U] \leqslant [S,P] \leqslant [S,S_0] \leqslant S_0$ and thus $T = [S,S]U = [S,S]P \leqslant S_0$. Suppose $T < S_0$. As $U \leqslant S_0 \leqslant S = AU$, we have $S_0 = (S_0 \cap A)U$ and thus $[S,S] < S_0 \cap A < A$. So $\mathrm{Aut}_{\mathcal{F}}(S)$ does not act irreducibly on $A/[S,S]$, contradicting (iv). This shows (a).

For the proof of (b) let $S^* \in \{S, T\}$ and assume by contradiction that $S^* = S_1 \times S_2$ where $S_1$ and $S_2$ are non-trivial subgroups of $S^*$. Notice that in either case $Z = Z(S^*)$ by (i). Moreover, again using (i), we note that $[S_1, S_1] \times [S_2, S_2] = [S, S] \not\leqslant Z = Z(S)$. So there exists in either case $i \in \{1, 2\}$ with $[S_i, S_i] \not\leqslant Z$ and thus $S_i \cap A \not\leqslant Z$. We assume without loss of generality that $S_1 \cap A \not\leqslant Z$. Setting $\overline{S^*} = S^*/Z$, we note that $\overline{S_1 \cap A}$ is a non-trivial normal subgroup of $\overline{S^*}$, and intersects thus non-trivially with $Z = Z(\overline{S^*})$. Hence, $((S_1 \cap A)Z) \cap Z_2(S^*) \not\leqslant Z$ and so $S_1 \cap A \cap Z_2(S^*) \not\leqslant Z$. Choosing $s \in (S_1 \cap A \cap Z_2(S^*)) \backslash Z$, we have $s \in N_S(P) \backslash P$ as $A \cap P = Z$ and $[s, P] \leqslant [Z_2(S^*), P] \leqslant Z \leqslant P$. Using (ii) and $C_S(P) \leqslant P$, it follows $Z = [P, s]$. So $Z = [P, s] \leqslant [P, S_1] \leqslant S_1$ as $P \leqslant S^*$ and $S_1$ is normal in $S^*$. Since $S_2$ is a non-trivial normal subgroup of $S^*$, we have $S_2 \cap Z = S_2 \cap Z(S^*) \neq 1$. This contradicts $S_1 \cap S_2 = 1$. Thus, we have shown that $S^*$ cannot be written as a direct product of two non-trivial subgroups, i.e., property (b) holds.

Part (c) follows now from a combination of results in the literature, some of which use the classification of finite simple groups. Most notably, we use the Schreier conjecture, Oliver's work on fusion systems over $p$-groups with an abelian subgroup of index $p$ [Oli14], and the work of Flores–Foote [FF09] determining the simple groups having a Sylow $p$-subgroup with a proper non-trivial strongly closed subgroup. To argue in detail, assume that $\mathcal{F}$ is realizable. By (b), neither $S$ nor $T$ can be written as a direct product of two non-trivial subgroups. By (a), $S$ and $T$ are the only non-trivial strongly closed subgroups. The subgroup $T$ is $\mathcal{F}$-centric since $T$ is strongly closed and $P \leqslant T$ is self-centralizing in $S$. Clearly, $S$ is $\mathcal{F}$-centric. So as $\mathcal{F}$ is realizable, it follows from [DRV07, Proposition 2.19] that $\mathcal{F} = \mathcal{F}_S(G)$ for some almost simple group $G$ with $S \in \mathrm{Syl}_p(G)$. Set $G_0 := O^{p'}(G)$ and $\mathcal{F}_0 := \mathcal{F}_S(G_0)$. Notice that $\mathrm{SL}_2(q) \cong O^{p'}(\mathrm{Aut}_{\mathcal{F}}(P))$ is contained in $\mathcal{F}_0$, so in particular, $S$ is not normal in $\mathcal{F}_0$. As $T$ is strongly closed in $\mathcal{F}$, $T$ is also strongly closed in the subsystem $\mathcal{F}_0$ of $\mathcal{F}$. Hence, if $G_0$ were simple, then by the work of [FF09], we would have $p = 3 = |T|$. (We refer the reader to [AKO11, Theorem 12.12], which summarizes for us the relevant part of the work of Flores–Foote.) Clearly $|T| > p$, so $G_0$ is not simple. Observe that $F^*(G_0) = F^*(G)$ and $G_0$ is almost simple, since $G$ is almost simple. Since the outer automorphism group of any simple group is solvable by the Schreier conjecture (see e.g. [GLS98, Theorem 7.11(a)]), it follows that $G_0$ has a normal subgroup of prime index. So since $G_0 = O^{p'}(G_0)$, $G_0$ has a normal subgroup $N$ of index $p$. Then $S \cap N$ is strongly closed. Hence $S \cap N = T$ and $p = |G_0/N| = |S/T| = q$. Hence we are in the situation that $S$ has an abelian subgroup of index $p$. Since $T$ is the only proper non-trivial strongly closed subgroup, we have $T = F^*(G) \cap S$.

Set $G_1 := F^*(G)$ and $\mathcal{F}_1 := \mathcal{F}_T(G_1)$. Notice that $Z_2(S) \leqslant N_S(P)$ and $|N_S(P)/P| = q = p$ by (ii). As $[S, S]$ is normal in $S$, we have $\hat{Z} := Z_2(S) \cap [S, S] \not\leqslant Z$. As $Z = P \cap A$ and $\hat{Z} \leqslant [S, S] \leqslant A$, it follows that $\hat{Z} \not\leqslant P$ and $N_S(P) = \hat{Z}P \leqslant [S, S]P = T$. Since $T$ is strongly closed in $\mathcal{F}$, this implies that $N_S(P^*) \leqslant T$ for every $\mathcal{F}$-conjugate $P^*$ of $P$. Let $P^*$ be an $\mathcal{F}$-conjugate of $P$. As $G_1 = O^{p'}(G_1)$, the morphisms in $O^{p'}(\mathrm{Aut}_{\mathcal{F}}(P^*)) = \langle \mathrm{Aut}_S(P^*)^{\mathrm{Aut}_{\mathcal{F}}(P^*)} \rangle$ lie in $\mathcal{F}_1$. So $\mathrm{Aut}_{\mathcal{F}_1}(P^*)$ is isomorphic to a subgroup of $\mathrm{GL}_2(p)$ containing $\mathrm{SL}_2(p)$ and has thus a strongly $p$-embedded subgroup. Since $C_S(P) \leqslant P$ and $P^{\mathcal{F}} = P^{\mathrm{Aut}_{\mathcal{F}}(S)}$, the subgroup $P^*$ is $\mathcal{F}_1$-centric. Hence, $P^*$ is essential in $\mathcal{F}_1$. Since $N_S(P) \leqslant T$ and $|S : T| = p$, this shows $P^S$ splits into $p$ $T$-conjugacy classes, all of which are essential in $\mathcal{F}_1$. On the other hand, $[S, S]$ is not essential in $\mathcal{F}$ by (iii), so $[S, S]$ is not essential in $\mathcal{F}_1$ either.

Next we reach a contradiction when $n \geqslant 4$. Since we can choose $P^* \in P^S \backslash P^T$, and since $[S,S] = T \cap A$ is an abelian subgroup of $T$ of index $p$ and order $p^{n-1}$ with $n-1 \geqslant 3$, it follows now from [Oli14, Lemma 2.7(a)] applied with $\mathcal{F}_1$ in place of $\mathcal{F}$ that $n-1 \equiv -1 \pmod{p-1}$. Since $n \leqslant p-1$, this implies $n-1 = p-2$. However, since $\mathcal{F}_1$ is the fusion system of the simple group $G_1 = F^*(G)$, [Oli14, Lemma 1.6] gives $p = 3$. So $n = 2$ which contradicts our assumption.

Finally, suppose $n = 3$. Then $T$ is extraspecial of order $p^3$ and exponent $p$, and $\mathcal{F}_1$ has exactly $p$ essential subgroups of index $p$, namely the members of $P^S$ (the only other subgroup of index $p$ in $T$ being $[S,S]$, which is not essential). But there is no saturated fusion system over $T$ with exactly $p$ essential subgroups by the classification of Ruiz and Viruel [RV04, Tables 1.1, 1.2]. This contradiction completes the proof of (c) and the lemma. □

Recall that $\mathcal{F}(r,n,k,Q)$ is realizable in the case $n = 2$ and thus has a punctured group. So the case $n \geqslant 3$, which we consider in the following proposition, is actually the only interesting remaining case.

**Proposition 5.11.** *Let $3 \leqslant n \leqslant p-1$ (and thus $p \geqslant 5$), let $r$ be a divisor of $(n, q-1)$, and set $\mathcal{F} = \mathcal{F}(r,n,k,Q)$. Then $N_{\mathcal{F}}(Z)$ and $N_{\mathcal{F}}(Z)/Z$ are exotic. In particular, $\mathcal{F}$ does not have a punctured group.*

*Proof.* By Lemma 5.1, $\mathcal{F}$ does not have a punctured group if $N_{\mathcal{F}}(Z)$ is exotic. Moreover, if $N_{\mathcal{F}}(Z)$ is realized by a finite group $G$, then $N_{\mathcal{F}}(Z)$ is also realized by $N_G(Z)$, and $N_{\mathcal{F}}(Z)/Z$ is realized by $N_G(Z)/Z$. So it is sufficient to show that $N_{\mathcal{F}}(Z)/Z$ is exotic.

Recall from above that $S = S(n,k)$, $A = A(n,k)$ and $Z := Z(S)$. Set $\mathcal{F}_1 = \mathcal{F}_S(G)$ and $\mathcal{F}_2 = \mathcal{F}_{S_2}(H_Q)$ with $S_2 = N_S(Q)$. Suppose first $r = 1$. Then one argues similarly as in the proof of Proposition 5.9 that $\mathcal{F} = \langle \mathrm{Aut}_{\mathcal{F}_1}(S), \mathrm{Aut}_{\mathcal{F}_1}(A), \mathrm{Aut}_{\mathcal{F}_2}(Q)\rangle$. Namely, $\mathcal{F}$ is generated by $\mathcal{F}_1$ and $\mathcal{F}_2$ by [CP10, Theorem 3.1], and so as $\mathcal{F}_1$ and $\mathcal{F}_2$ are both constrained with $O_p(\mathcal{F}_1) = A$ and $O_p(\mathcal{F}_2) = Q$, it follows that $\mathcal{F}$ is in turn generated by $\mathrm{Aut}_{\mathcal{F}_1}(A)$, $\mathrm{Aut}_{\mathcal{F}_1}(S)$, $\mathrm{Aut}_{\mathcal{F}_2}(Q)$, and $\mathrm{Aut}_{\mathcal{F}_2}(S_2)$. However, the last automorphism group is redundant, since $N_{H_R}(S_2) = N_G(Q)$ induces fusion in $\mathcal{F}_1$. So indeed $\mathcal{F} = \langle \mathrm{Aut}_{\mathcal{F}_1}(S), \mathrm{Aut}_{\mathcal{F}_1}(A), \mathrm{Aut}_{\mathcal{F}_2}(Q)\rangle$ if $r = 1$. This implies $\mathrm{Aut}_{\mathcal{F}_1}(S) = \mathrm{Aut}_{\mathcal{F}}(S)$, $\mathrm{Aut}_{\mathcal{F}}(A) = \mathrm{Aut}_{\mathcal{F}_1}(A)$ and (as $N_G(Q) = N_{H_Q}(S_2)$) $\mathrm{Aut}_{\mathcal{F}}(Q) = \mathrm{Aut}_{\mathcal{F}_2}(Q)$. Moreover, the set of $\mathcal{F}$-essential subgroups comprises $A$ and all $\mathrm{Aut}_{\mathcal{F}}(S)$-conjugates of $Q$. On easily checks that a normal subsystem of $\mathcal{F}$ of index prime to $p$ has the same essential subgroups as $\mathcal{F}$ itself, for any saturated fusion system $\mathcal{F}$. So as, for arbitrary $r$, by Lemma 5.8, $\mathcal{F}$ is a normal subsystem of $\mathcal{F}(1,n,k,Q)$ of index prime to $p$, it follows that, in any case, the $\mathcal{F}$-essential subgroups are $A$ and the $\mathrm{Aut}_{\mathcal{F}_1}(S)$-conjugates of $Q$. Since there is a complement to $S$ in $N_G(S)$ which normalizes $U$ and thus $Q$, the $\mathrm{Aut}_{\mathcal{F}_1}(S)$-conjugates of $Q$ are precisely the $S$-conjugates of $Q$. So, for arbitrary $r$, we have

$$\mathcal{F} = \langle \mathrm{Aut}_{\mathcal{F}}(S), \mathrm{Aut}_{\mathcal{F}}(A), \mathrm{Aut}_{\mathcal{F}}(Q)\rangle. \tag{5.4}$$

Moreover, $\mathrm{Aut}_{\mathcal{F}}(S) \leqslant \mathrm{Aut}_{\mathcal{F}_1}(S)$, $\mathrm{Aut}_{\mathcal{F}}(A) \leqslant \mathrm{Aut}_{\mathcal{F}_1}(A)$, and $\mathrm{Aut}_{\mathcal{F}}(Q) \leqslant \mathrm{Aut}_{\mathcal{F}_2}(Q)$. Recall also that $O^{p'}(H_Q) \leqslant F(r,n,k,Q)$ and thus $\mathrm{SL}_2(q) \cong O^{p'}(\mathrm{Aut}_{\mathcal{F}_2}(Q)) \leqslant \mathrm{Aut}_{\mathcal{F}}(Q)$.

Note that $\mathrm{Aut}_{\mathcal{F}}(Q)$ normalizes $Z$ and lies thus in $N_{\mathcal{F}}(Z)$. We will show next that $N_{\mathcal{F}}(Z)$ is generated by $\mathrm{Aut}_{\mathcal{F}}(S)$ and $\mathrm{Aut}_{\mathcal{F}}(Q)$. By the Alperin–Goldschmidt fusion theorem, it suffices to show that every essential subgroup of $N_{\mathcal{F}}(Z)$ is an $\mathrm{Aut}_{\mathcal{F}}(S)$-conjugate of $Q$. So fix an essential subgroup $E$ of $N_{\mathcal{F}}(Z)$ and assume that $E \notin Q^{\mathrm{Aut}_{\mathcal{F}}(S)}$. As $C_S(E) \leqslant E$, we have $Z < E$. If $E \leqslant A$ then $E = A$. However, $\mathrm{Aut}_{\mathcal{F}}(A) \leqslant \mathrm{Aut}_{\mathcal{F}_1}(A) = \mathrm{Aut}_G(A)$ and one observes that $S$ is normal in $N_G(Z)$. So $\mathrm{Aut}_{N_{\mathcal{F}}(Z)}(A) = N_{\mathrm{Aut}_{\mathcal{F}}(A)}(Z)$ has a normal Sylow $p$-subgroup, which contradicts $E$ being essential. Assume now that $E \leqslant Q$. Suppose first $Z < Z(E)$. The images of the maximal abelian subgroups of $Q$ are precisely the 1-dimensional $k$-subspaces of $Q/Z$. As $\mathrm{Aut}_{\mathcal{F}}(Q)$ fixes $Z$ and acts transitively on the one-dimensional $k$-subspaces of $Q/Z$, we see that $Z(E)$ is conjugate into $Z_2(S) = A \cap Q$ under an element of $\mathrm{Aut}_{\mathcal{F}}(Q)$. So replacing $E$ by a suitable $\mathrm{Aut}_{\mathcal{F}}(Q)$-conjugate, we may assume $Z(E) \leqslant A \cap Q$. As $Z < Z(E)$, it follows then from (5.1) that $E \leqslant A$. As $C_S(E) \leqslant E$ and $A \not\leqslant Q$, this is a contradiction. So we have $Z = Z(E)$. As $[E, Q] \leqslant [Q, Q] \leqslant Z$, it follows $\mathrm{Aut}_Q(E) \leqslant C := C_{\mathrm{Aut}_{N_{\mathcal{F}}(Z)}(E)}(E/Z(E)) \cap C_{\mathrm{Aut}_{N_{\mathcal{F}}(Z)}(E)}(Z(E))$. However, $C$ is a normal $p$-subgroup of $\mathrm{Aut}_{N_{\mathcal{F}}(Z)}(E)$. Thus, as $E$ is radical in $N_{\mathcal{F}}(Z)$, we have $\mathrm{Aut}_Q(E) \leqslant C \leqslant \mathrm{Inn}(E)$. As $C_S(E) \leqslant E$, it follows $E = Q$ contradicting the choice of $E$. So we have shown that $E$ lies neither in $A$ nor in $Q$. Since the choice of $E$ was arbitrary, this means that $E$ is not $\mathrm{Aut}_{\mathcal{F}}(S)$-conjugate into $A$ or $Q$. So by (5.4), every $\mathcal{F}$-automorphism of $E$ extends to an $\mathcal{F}$-automorphism of $S$. This implies that $\mathrm{Aut}_S(E)$ is normal in $\mathrm{Aut}_{\mathcal{F}}(E)$ and thus in $\mathrm{Aut}_{N_{\mathcal{F}}(Z)}(E)$. Again, this contradicts $E$ being essential. So we have shown that $N_{\mathcal{F}}(Z)$ is generated by $\mathrm{Aut}_{\mathcal{F}}(S)$ and $\mathrm{Aut}_{\mathcal{F}}(Q)$.

Set $\overline{S} = S/Z$ and $\overline{\mathcal{F}} = N_{\mathcal{F}}(Z)/Z$. We will check that the hypothesis of Lemma 5.10 is fulfilled with $\overline{\mathcal{F}}$, $\overline{S}$, $\overline{A}$, $\overline{U}$ and $\overline{Q}$ in place of $\mathcal{F}$, $S$, $A$, $U$ and $P$. Part (c) of this Lemma will then imply that $N_{\mathcal{F}}(Z)/Z$ is exotic as required. Notice that $|\overline{U}| = |U| = q$, $|A| = q^{n+1}$ and $|\overline{A}| = q^n$. As $Q = Z_2(S)U$, we have $\overline{Q} = Z(\overline{S})\overline{U}$. By [CP10, Lemma 4.2(i)&(iii)], hypothesis (i) of Lemma 5.10 holds. Recall that $O^{p'}(\mathrm{Aut}_{H_Q}(Q)) \cong \mathrm{SL}_2(q)$ lies in $N_{\mathcal{F}}(Z)$. In particular, hypothesis (ii) in Lemma 5.10 holds with $\overline{\mathcal{F}}$ and $\overline{Q}$ in place of $\mathcal{F}$ and $P$. Since we have shown above that $N_{\mathcal{F}}(Z)$ is generated by $\mathrm{Aut}_{\mathcal{F}}(S)$ and $\mathrm{Aut}_{\mathcal{F}}(Q)$, it follows that $\overline{\mathcal{F}}$ fulfills hypothesis (iii) of Lemma 5.10. Observe that there exists a complement $K$ of $S$ in $N_G(S)$ which normalizes $U$. Then $\mathrm{Aut}_{\mathcal{F}}(S) \leqslant \mathrm{Aut}_G(S) = \mathrm{Inn}(S)\mathrm{Aut}_K(S)$. Thus $\mathrm{Aut}_{\mathcal{F}}(S) = \mathrm{Inn}(S)(\mathrm{Aut}_K(S) \cap \mathrm{Aut}_{\mathcal{F}}(S))$ and $\mathrm{Aut}_K(S) \cap \mathrm{Aut}_{\mathcal{F}}(S)$ is a complement to $\mathrm{Inn}(S)$ in $\mathrm{Aut}_{\mathcal{F}}(S)$ which normalizes $U$. This implies that hypothesis (v) of Lemma 5.10 holds for $\overline{\mathcal{F}}$.

It remains to show hypothesis (iv) of Lemma 5.10 for $\overline{\mathcal{F}}$. Equivalently, we need to show that $\mathrm{Aut}_{\mathcal{F}}(S) = \mathrm{Aut}_{N_{\mathcal{F}}(Z)}(S)$ acts irreducibly on $A/[S, S]$. For the proof, we use the representations Clelland and Parker give for $G$ and $H_Q$, and the way they construct the free amalgamated product; see pp. 293 and pp. 296 in [CP10]. Let $\xi$ be a generator of $k^\times$. We have

$$g := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \xi^{-1} & 0 & 0 \\ 0 & 0 & \xi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in O^{p'}(H_Q) \leqslant N_{F(r,n,k,Q)}(Z).$$

In the free amalgamated product $F(1, n, k, Q)$, the element $g \in H_Q$ is identified with

$$\left(1, \begin{pmatrix} 1 & 0 \\ 0 & \xi \end{pmatrix}, 0_{A(n,k)}\right) \in N_G(Q),$$

and this element can be seen to act by scalar multiplication with $\xi^n$ on $y^n \in A = A(n, k)$ and thus on $A/[S, S]$. As $n \leqslant p - 1$ and $\xi$ has order $q - 1$, the action of $g$ on $A(n, k)/[S, S]$ is thus irreducible. Hence, the action of $\mathrm{Aut}_{\mathcal{F}}(S)$ on $A/[S, S]$ is irreducible. This shows that the hypothesis of Lemma 5.10 is fulfilled with $\overline{\mathcal{F}}$ in place of $\mathcal{F}$, and thus $\overline{\mathcal{F}} = N_{\mathcal{F}}(Z)/Z$ is exotic as required. □

5.4. **The Parker-Stroth systems.** Let $p \geqslant 5$ be a prime and $m = p-4$. Let $A = A(m, \mathbf{F}_p)$ and $D$ be as in §§5.3. The Parker-Stroth systems are fusion systems over the Sylow subgroup of a semidirect product of $Q$ by $D$, where $Q$ is extraspecial of order $p^{1+(p-3)}$ and of exponent $p$, and where $Q/Z(Q) \cong A$ as a $\mathbf{F}_pD$-module. Set $Z := Z(S)$ and observe that $Z = Z(Q)$ is of order $p$, while $Z_2(S) \leqslant Q$ is elementary abelian of order $p^2$.

The fusion systems are generated by $\mathrm{Aut}_{\mathcal{F}}(Q)$, $\mathrm{Aut}_{\mathcal{F}}(S) \cong \mathrm{Inn}(S)C_{p-1}$, and $\mathrm{Aut}_{\mathcal{F}}(W) \cong SL_2(p)$, where $W$ is a certain elementary abelian subgroup of $S$ of order $p^2$. We refer to [PS15, p.317] for more details on the embedding of $W$ in $S$, where our $W$ is denoted $W_0$. For our purposes, we just need to know that

$$W \cap Q = W \cap Z_2(S) = Z(S), \tag{5.5}$$

and thus

$$W \text{ is not } \mathrm{Aut}_{\mathcal{F}}(S)\text{-conjugate into } Q. \tag{5.6}$$

which is deduced from the description of the embedding together with [PS15, Lemma 2.3(iii)].

**Proposition 5.12.** *Each Parker-Stroth system is of characteristic $p$-type, and so has a punctured group in the form of its subcentric linking system.*

*Proof.* Let $Y$ be a subgroup of order $p$ in $S$. Fix an essential subgroup $E$ of $C_{\mathcal{F}}(Y)$ and a $C_{\mathcal{F}}(Y)$-automorphism $\alpha$ of $E$. Then $E \geqslant YZ(S)$, and hence if $Y$ is not $Z$, then $\Omega_1(Z(E))$ is of rank at least 2.

Assume first that $Y \nleqslant Q$. Then $C_{Q/Z}(Y)$ is of order $p$, so $C_Q(Y)$ is elementary abelian (of order $p^2$). Hence $C_S(Y)$ is abelian in this case, and so $C_{\mathcal{F}}(Y)$ is constrained, when saturated.

Assume next that $Y \leqslant Q$ but $Y \nleqslant Z$. Then $C_S(Y)$ is abelian when $p = 5$ as then $m = 1$ and $Y/Z$ is its own orthogonal complement with respect to the symplectic form on $Q/Z$. We may therefore assume $p \geqslant 7$. When $p \geqslant 7$, this centralizer is nonabelian. Indeed, when $p \geqslant 7$, we have two cases. Either $Y \nleqslant Z_2(S)$ and $C_S(Y) = Y \times Q_0$ where $Q_0$ is extraspecial of order $p^{1+(p-5)}$, or $Y \leqslant Z_2(S)$ and $C_S(Y) = Y \times Q_0U$, where $U = S \cap D$. To see this, we refer to the definition of the symplective form defining $Q$ on [PS15, p.312] and note that $Z_2(S) = \langle (0,1), (X^m, 0)\rangle \leqslant A \times \mathbf{F}_p$ in the notation there.

Since $Y$ is in $Q$ but not in $Q \cap W = Z(S)$ by (5.5) and assumption, we know that $Y$ is not in $W$. Now $E$ contains $YZ(S) \cong C_p \times C_p$ and $E$ is contained in $C_S(Y) \leqslant Q$. It follows that $E$ is not $\langle \mathrm{Aut}_{\mathcal{F}}(S), \mathrm{Aut}_{\mathcal{F}}(Q)\rangle$-conjugate into $W$. So each morphism in a decomposition of $\alpha$ lies in $\mathrm{Aut}_{\mathcal{F}}(Q)$ or $\mathrm{Aut}_{\mathcal{F}}(S)$. Hence, since $p \geqslant 7$, $Q$ is the Thompson subgroup of $S$ and so is invariant under $\mathrm{Aut}_{\mathcal{F}}(S)$. We conclude that $\alpha$ extends to $Q$. So $Q \leqslant E$ since $\alpha$ was chosen arbitrarily and $E$ is essential. It follows that $E = Q$ since $Q$ is of index $p$ in $S$. Thus, $Q$ is normal (and centric) in $C_{\mathcal{F}}(Y)$. Therefore, $C_{\mathcal{F}}(Y)$ is constrained.

Assume finally that $Y = Z$. Suppose first that $E$ is $\langle \mathrm{Aut}_{\mathcal{F}}(S), \mathrm{Aut}_{\mathcal{F}}(Q)\rangle$-conjugate into $W$. As $W$ is not contained in $Q$, neither is any $\langle \mathrm{Aut}_{\mathcal{F}}(S), \mathrm{Aut}_{\mathcal{F}}(Q)\rangle$-conjugate of $E$. Hence, in this case, $E$ is in fact $\mathrm{Aut}_{\mathcal{F}}(S)$-conjugate into $W$, and so conjugate to $W$, since as any $\mathrm{Aut}_{\mathcal{F}}(S)$-conjugate of $E$ contains its centralizer in $S$. We may therefore assume that $E = W$. As each member of

$\mathrm{Aut}_{\mathcal{F}}(S)$ fixes $Y$, there exists a $\langle \mathrm{Aut}_{\mathcal{F}}(S), \mathrm{Aut}_{\mathcal{F}}(W)\rangle$-decomposition of $\alpha$ each member of which fixes $Y$. However, each member of $N_{\mathrm{Aut}_{\mathcal{F}}(W)}(Z)$ extends to $S$, whence $\alpha$ extends to $S$. Since $\alpha$ was arbitrary, this contradicts the choice of $E$ essential.

Thus in all cases, $C_{\mathcal{F}}(Y)$ is constrained. We conclude that the Parker-Stroth systems are of characteristic $p$-type and therefore have a punctured group. □

## 6. Punctured groups over $p_+^{1+2}$

The main purpose of this section is to illustrate that there can be several punctured groups associated to the same fusion system, and that the nerves of such punctured groups (regarded as transporter systems) might not be homotopy equivalent to the nerve of the centric linking system. Indeed, working in the language of localities, we will see that there can be several punctured groups extending the centric linking locality. This is the case even though we consider examples of fusion systems of characteristic $p$-type, and so in each case, the subcentric linking locality exists as the "canonical" punctured group extending the centric linking locality. On the other hand, we will see that in many cases, the subcentric linking locality is indeed the only $p'$-reduced punctured group over a given fusion system. Thus, "interesting" punctured groups seem still somewhat rare.

More concretely, we will look at fusion systems over a $p$-group $S$ which is isomorphic to $p_+^{1+2}$. Here $p_+^{1+2}$ denotes the extraspecial group of order $p^3$ and exponent $p$ if $p$ is an odd prime, and (using a somewhat non-standard notation) we write $p_+^{1+2}$ for the dihedral group of order 8 if $p = 2$. Note that every subgroup of order at least $p^2$ is self-centralizing in $S$ and thus centric in every fusion system over $S$. Thus, if $\mathcal{F}$ is a saturated fusion system over $S$ with centric linking locality $(\mathcal{L}, \Delta, S)$, we just need to add the cyclic groups of order $p$ as objects to obtain a punctured group. We will again use Chermak's iterative procedure, which gives a way of expanding a locality by adding one $\mathcal{F}$-conjugacy class of new objects at the time. If all subgroups of order $p$ are $\mathcal{F}$-conjugate, we thus only need to complete one step to obtain a punctured group. Conversely, we will see in this situation that a punctured group extending the centric linking locality is uniquely determined up to a rigid isomorphism by the normalizer of an element of order $p$. Therefore, we will restrict attention to this particular case. More precisely, we will assume the following hypothesis.

**Hypothesis 6.1.** Assume that $p$ is a prime and $S$ is a $p$-group such that $S \cong p_+^{1+2}$ (meaning here $S \cong D_8$ if $p = 2$). Set $Z := Z(S)$. Let $\mathcal{F}$ be a saturated fusion system over $S$ such that all subgroups of $S$ of order $p$ are $\mathcal{F}$-conjugate.

It turns out that there is a fusion system $\mathcal{F}$ fulfilling Hypothesis 6.1 if and only if $p \in \{2, 3, 5, 7\}$; for odd $p$ this can by seen from the classification theorem by Ruiz and Viruel [RV04] and for $p = 2$ the 2-fusion system of $A_6$ is known to be the only fusion system with one conjugacy class of involutions. For $p \in \{5, 7\}$ our two theorems below depend on the classification of finite simple groups.

Assume Hypothesis 6.1. One easily observes that the 2-fusion system of $A_6$ is of characteristic 2-type. Therefore, it follows from Lemma 5.4 that the fusion system $\mathcal{F}$ is always of characteristic $p$-type and thus the associated subcentric linking locality is a punctured group. As discussed in Remark 2.22, this leads to a host of examples for punctured groups $\mathcal{L}^+$ over $\mathcal{F}$ which are modulo a partial normal $p'$-subgroup isomorphic to a subcentric linking locality over $\mathcal{F}$. One can ask whether there are more examples. Indeed, the next theorem tells us that this is the case if and only if $p = 3$.

**Theorem 6.2.** *Under Hypothesis 6.1 there exists a punctured group $(\mathcal{L}^+, \Delta^+, S)$ over $\mathcal{F}$ such that $\mathcal{L}^+/O_{p'}(\mathcal{L}^+)$ is not a subcentric linking locality if and only if $p = 3$.*

It seems that for $p = 3$, the number of $3'$-reduced punctured groups over $\mathcal{F}$ is probably also severely limited. However, since we don't want to get into complicated and lengthy combinatorial arguments, we will not attempt to classify them all. Instead, we will prove the following theorem, which leads already to the construction of interesting examples.

**Theorem 6.3.** *Assume Hypothesis 6.1. Suppose that $\mathcal{L}^+$ is a punctured group over $\mathcal{F}$ such that $\mathcal{L}^+|_{\mathcal{F}^c}$ is a centric linking system over $\mathcal{F}$. Then $\mathcal{L}^+$ is $p'$-reduced. Moreover, up to a rigid isomorphism, $\mathcal{L}^+$ is uniquely determined by the isomorphism type of $N_{\mathcal{L}^+}(Z)$, and one of the following holds.*

(a) *$\mathcal{L}^+$ is the subcentric linking system for $\mathcal{F}$; or*
(b) *$p = 3$, $\mathcal{F}$ is the 3-fusion system of the Tits group ${}^2F_4(2)'$ and $N_{\mathcal{L}^+}(Z) \cong 3S_6$; or*
(c) *$p = 3$, $\mathcal{F}$ is the 3-fusion system of $Ru$ and of $J_4$, and $N_{\mathcal{L}^+}(Z) \cong 3\#\mathrm{Aut}(A_6)$ or an extension of $3L_3(4)$ by a field or graph automorphism.*

*Conversely, each of the cases listed in (a)-(c) occurs in an example for $\mathcal{L}^+$.*

Before beginning the proof, we make some remarks. The 3-fusion systems of $Ru$ and $J_4$ are isomorphic. For $G = Ru$ and $S$ a Sylow 3-subgroup of $G$, one has $N_G(Z(S)) \cong 3\#\mathrm{Aut}(A_6)$ [GLS98, Table 5.3r], so the punctured group $\mathcal{L}^+$ in Theorem 6.3(c) is the punctured group of $Ru$ at the prime 3 (for example, since our theorem tells us that $\mathcal{L}^+$ is uniquely determined by the isomorphism type of $N_{\mathcal{L}^+}(Z)$). Using the classification of finite simple groups, this can be shown to be the only punctured group in (b) or (c) that is isomorphic to the punctured group of a finite group. For example, when $G = J_4$, one has $N_G(Z(S)) \cong (6M_{22}) \cdot 2$. The 3-fusion system of $6M_{22}$ is constrained and isomorphic to that of $3M_{21} = 3L_3(4)$ and also that of $3M_{10} = 3(A_6.2)$, where the extension $A_6.2$ is non-split (see [GLS98, Table 5.3c]). If we are in the situation of Theorem 6.3(c) and $N_{\mathcal{L}^+}(Z(S))$ is an extension of $3L_3(4)$ by a field automorphism, then $N_{\mathcal{L}^+}(Z(S))$ is a section of $N_G(Z(S))$. Also, for $G = {}^2F_4(2)'$, the normalizer in $G$ of a subgroup of order 3 is solvable [Mal91, Proposition 1.2].

As remarked already above, for $p \in \{2, 5, 7\}$, there are also saturated fusion systems over $S$, in which all subgroups of order $p$ are conjugate. For $p = 5$, the only such fusion system is the fusion system of the Thompson sporadic group. However, the Thompson group is of local characteristic 5, and thus its punctured group is just the subcentric linking locality. For $p = 7$, Ruiz and Viruel discovered three exotic fusion systems of characteristic 7-type, in which all subgroups of order 7 are conjugate. As our theorem shows, for each of these fusion systems, the subcentric linking locality is the only associated punctured group extending the centric linking system. We will now start to prove Theorem 6.2 and Theorem 6.3 in a series of lemmas.

If Hypothesis 6.1 holds and $\mathcal{L}^+$ is a punctured group over $\mathcal{F}$, then $M_0 := N_{\mathcal{L}^+}(Z)$ is a finite group containing $S$ as a Sylow $p$-subgroup. Moreover, $Z$ is normal in $M_0$. These properties are preserved if we replace $M_0$ by $M := M_0/O_{p'}(M_0)$ and identify $S$ with its image in $M$. Moreover, we have $O_{p'}(M) = 1$. We analyze the structure of such a finite group $M$ in the following lemma.

Most of our arguments are elementary. However, for $p \geqslant 5$, we need the classification of finite simple groups in the form of knowledge about the Schur multipliers of finite simple groups to show in case (b) that $p = 3$.

**Lemma 6.4.** *Let $M$ be a finite group with a Sylow $p$-subgroup $S \cong p_+^{1+2}$. Assume that $Z := Z(S)$ is normal in $M$ and $O_{p'}(M) = 1$. Then one of the following holds.*

(a) *$S \trianglelefteq M$ and $C_M(S) \leqslant S$, or*

(b) *$p = 3$, $S \leqslant F^*(M)$, and $F^*(M)$ is quasisimple with $Z(F^*(M)) = Z$.*

*Proof.* Assume first that $S \trianglelefteq M$. In this case we have $[S, E(M)] = 1$ and thus $S \cap E(M) \leqslant Z(E(M))$. So by [Asc93, 33.12], $E(M)$ is a $p'$-group. Since we assume $O_{p'}(M) = 1$, this implies $E(M) = 1$ and $F^*(M) = O_p(M) = S$. Therefore (a) holds.

Thus, for the remainder of the proof, we will assume that $S$ is not normal in $M$, and we will show (b). First we prove

$$E(M) \neq 1. \tag{6.1}$$

Suppose $E(M) = 1$ and set $P = O_p(M)$. Note that $Z \leqslant P$. As $O_{p'}(M) = 1$, we have $P = F^*(M)$, so $C_M(P) \leqslant P$ and $P \neq Z$. As we assume that $S$ is not normal in $M$, we have moreover $P \neq S$. If $P$ is elementary abelian of order $p^2$, then $M/P$ acts on $P$ and normalizes $Z$, thus it embeds into a Borel subgroup of $GL_2(p)$. If $p = 2$ and $P$ is cyclic of order 4 then $\mathrm{Aut}(P)$ is a 2-group. So $S$ is in any case normal in $M$ and this contradicts our assumption. Thus (6.1) holds.

We can now show that

$$p \text{ divides } |Z(K)| \text{ for some component } K \text{ of } M. \tag{6.2}$$

First note that $p$ divides $|K|$ for each component $K$ of $M$. For otherwise, if $p$ doesn't divide $|K|$ for some $K$, then $1 < K \leqslant O_{p'}(E(M)) \leqslant O_{p'}(M) = 1$, a contradiction.

Supposing (6.2) is false, $Z(E(M))$ is a $p'$-group and thus by assumption trivial. Hence, $E(M)$ is a direct product of simple groups. Since $Z$ is normal in $M$, $[Z, E(M)] = 1$ and thus $Z \cap E(M) = 1$. As the $p$-rank of $M$ is two and $p$ divides $|K|$ for each component $K$, there can be at most one component, call it $J$, which is then simple and normal in $M$. As $p$ divides $|J|$ and $J$ is normal in $M$, it follows that $S \cap J \neq 1$. But then $[S \cap J, S] \leqslant J \cap Z = 1$ and so $S \cap J = Z$ is normal in $J$, a contradiction. Thus, (6.2) holds.

Next we will show that

$$K = F^*(M) \text{ is quasisimple with } S \leqslant K \text{ and } Z(K) = Z. \tag{6.3}$$

To prove this fix a component $K$ of $M$ such that $p$ divides $|Z(K)|$. Then $p$ divides $|K|/|Z(K)|$ by [Asc93, 33.12]. If $S$ is not a subgroup of $K$, then $K/Z(K)$ is a perfect group with cyclic Sylow $p$-subgroups, so $Z(K)$ is a $p'$-group by [Asc93, 33.14], a contradiction. Therefore $S \leqslant K$. If there were a component $L$ of $M$ different from $K$, then we would have $[S \cap L, L] \leqslant [K, L] = 1$, i.e. $L$ would have a central Sylow $p$-subgroup. However, we have seen above that $p$ divides the order of each component, so we would get a contradiction to [Asc93, 33.12]. Hence, $K = F^*(G)$ is the unique component of $M$. Note that $O_{p'}(Z(K)) \leqslant O_{p'}(M) = 1$ and thus $Z(K)$ is a $p$-group. Since $[Z, K] = 1$, this implies $Z = Z(K)$. Thus (6.3) holds.

To prove (b), it remains to show that $p = 3$. Assume first that $p = 2$ so that $S \cong D_8$. Then $\mathrm{Aut}(S)$ is a 2-group and thus $N_K(S) = SC_K(S)$. Hence, with $\bar{K} = K/Z$, we have $N_{\bar{K}}(\bar{S}) =$

$C_{\bar{K}}(\bar{S})$. Therefore, $\bar{K}$ has a normal $p$-complement by Burnside's Theorem (see e.g. [KS04, 7.2.1]), a contradiction which establishes $p \neq 2$.

For $p \geqslant 5$, we appeal to the account of the Schur multipliers of the finite simple groups given in Chapter 6 of [GLS98] to conclude that, by the classification of the finite simple groups, $K/Z(K)$ must be isomorphic to $L_m(q)$ with $p$ dividing $(m, q-1)$, or to $U_m(q)$ with $p$ dividing $(m, q+1)$. But each group of this form has Sylow $p$-subgroups of order at least $p^4$, a contradiction. □

**Lemma 6.5.** *Assume Hypothesis 6.1 and let $(\mathcal{L}^+, \Delta^+, S)$ be a punctured group over $\mathcal{F}$. Then the following hold:*

(a) *If $P \in \Delta^+$ with $|P| \geqslant p^2$, then $N_{\mathcal{L}^+}(P)$ is $p$-constrained.*
(b) *If $p \neq 3$ then, upon identifying $S$ with its image in $\mathcal{L}^+/O_{p'}(\mathcal{L}^+)$, the triple $(\mathcal{L}^+/O_{p'}(\mathcal{L}^+), \Delta^+, S)$ is a subcentric linking locality over $\mathcal{F}$.*

*Proof.* If $P \in \Delta^+$ with $|P| \geqslant p^2$, then $S = N_S(P)$ is a Sylow $p$-subgroup of $N_{\mathcal{L}^+}(P)$. As $P$ is normal in $N_{\mathcal{L}^+}(P)$ and $C_S(P) \leqslant P$, it follows that $N_{\mathcal{L}^+}(P)$ is $p$-constrained. Thus (a) holds.

Assume now $p \neq 3$. As all subgroups of order $p$ are by assumption $\mathcal{F}$-conjugate, we have by Lemma 2.7(b) and Lemma 2.9(a) that $N_{\mathcal{L}^+}(P) \cong M := N_{\mathcal{L}^+}(Z)$ for every $P \in \Delta^+$ with $|P| = p$. Moreover, by Lemma 6.4, $M/O_{p'}(M)$ has a normal Sylow $p$-subgroup and is thus in particular $p$-constrained. Hence, using (a) and Lemma 2.33, we can conclude that $N_{\mathcal{L}^+}(P)/O_{p'}(N_{\mathcal{L}^+}(P))$ is of characteristic $p$ for every $P \in \Delta^+$. Hence, by Proposition 2.32, the triple $(\mathcal{L}^+/O_{p'}(\mathcal{L}^+), \Delta^+, S)$ is a locality over $\mathcal{F}$ of objective characteristic $p$. Since $\Delta^+ = \mathcal{F}^s$ by Lemma 5.4, part (b) follows. □

Note that Lemma 6.5 proves one direction of Theorem 6.2, whereas the other direction would follow from Theorem 6.3. Therefore, we will focus now on the proof of Theorem 6.3 and thus consider punctured groups which restrict to the centric linking system. If $\mathcal{L}^+$ is such a punctured group, then we will apply Lemma 6.4 to $N_{\mathcal{L}^+}(Z)$. In order to do this, we need the following two lemmas.

**Lemma 6.6.** *Let $M$ be a finite group with a Sylow $p$-subgroup $S \cong p_+^{1+2}$. Assume that $Z := Z(S)$ is normal in $M$ and $C_M(V) \leqslant V$ for every subgroup $V$ of $S$ of order at least $p^2$. Then $O_{p'}(M) = 1$.*

*Proof.* Set $U = O_{p'}(M)$. As $Z$ is normal in $M$, it centralizes $U$. So $\bar{S} = S/Z$ acts on $U$. Let $x \in S - Z(S)$. Then setting $V = \langle x, z \rangle$, the centralizer $C_M(V)$ contains the $p'$-group $C_U(\bar{x})$. So our hypothesis implies $C_U(\bar{x}) = 1$. Hence, by [KS04, 8.3.4](b), $U = \langle C_U(\bar{x}) : \bar{x} \in \bar{S}^{\#} \rangle = 1$. □

**Lemma 6.7.** *Assume Hypothesis 6.1 and let $(\mathcal{L}^+, \Delta^+, S)$ be a punctured group over $\mathcal{F}$ such that $\mathcal{L}^+|_{\mathcal{F}^c}$ is a centric linking locality over $\mathcal{F}$. If we set $M := N_{\mathcal{L}^+}(Z)$ the following conditions hold:*

(a) *$S$ is a Sylow $p$-subgroup of $M$ and $Z$ is normal in $M$,*
(b) *$\mathcal{F}_S(M) = N_{\mathcal{F}}(Z) = N_{\mathcal{F}}(S)$, and*
(c) *$C_M(V) \leqslant V$ for each subgroup $V$ of $S$ of order $p^2$.*

*Proof.* Property (a) is clearly true. Moreover, by Lemma 2.9(b) and Lemma 5.4, we have $\mathcal{F}_S(M) = N_{\mathcal{F}}(Z) = N_{\mathcal{F}}(S)$, so (b) holds. Set $\Delta = \mathcal{F}^c$. By assumption $\mathcal{L} := \mathcal{L}^+|_\Delta$ is a centric linking locality. So by [Hen19, Proposition 1(d)], we have $C_{\mathcal{L}}(V) \subseteq V$ for every $V \in \Delta$. Hence, for every subgroup $V \in \Delta$, we have $C_M(V) \subseteq C_{\mathcal{L}^+}(V) = C_{\mathcal{L}}(V) \subseteq V$, where the equality follows from the definition of $\mathcal{L} = \mathcal{L}^+|_\Delta$. As every subgroup of $S$ of order at least $p^2$ contains its centralizer in $S$, each such subgroup is $\mathcal{F}$-centric. Therefore (c) holds. □

**Lemma 6.8.** *Assume Hypothesis 6.1 and let $(\mathcal{L}^+, \Delta^+, S)$ be a punctured group over $\mathcal{F}$ such that $\mathcal{L} := \mathcal{L}^+|_{\mathcal{F}^c}$ is a centric linking locality over $\mathcal{F}$. Set $M := N_{\mathcal{L}^+}(Z)$. Then one of the following conditions holds:*

(a) *$S \trianglelefteq M$, the group $M$ is a model for $N_{\mathcal{F}}(Z) = N_{\mathcal{F}}(S)$, and $(\mathcal{L}^+, \Delta^+, S)$ is a subcentric linking locality over $\mathcal{F}$.*
(b) *$p = 3$, $\mathcal{F}$ is the 3-fusion system of the Tits group ${}^2F_4(2)'$ and $M \cong 3S_6$; or*
(c) *$p = 3$, $\mathcal{F}$ is the 3-fusion system of $Ru$ and of $J_4$, and $M \cong 3\#\mathrm{Aut}(A_6)$ or an extension of $3L_3(4)$ by a field or graph automorphism.*

*Moreover, in either of the cases (b) and (c), $N_{\mathrm{Out}(S)}(\mathrm{Out}_{\mathcal{F}}(S))$ is a Sylow 2-subgroup of $\mathrm{Out}(S) \cong GL_2(3)$, and every element of $N_{\mathrm{Aut}(S)}(\mathrm{Aut}_{\mathcal{F}}(S))$ extends to an automorphism of $M$.*

*Proof.* Set $\Delta = \mathcal{F}^c$. By Lemma 6.6 and Lemma 6.7, we have $O_{p'}(M) = 1$, $\mathcal{F}_S(M) = N_{\mathcal{F}}(Z) = N_{\mathcal{F}}(S)$ and $C_M(S) \leqslant S$. In particular, if $S \trianglelefteq M$, then $M$ is a model for $N_{\mathcal{F}}(Z) = N_{\mathcal{F}}(S)$. For any $P \in \Delta$, the group $N_{\mathcal{L}^+}(P) = N_{\mathcal{L}}(P)$ is of characteristic $p$. As $\Delta^+ = \Delta \cup Z^{\mathcal{F}}$, if $S \trianglelefteq M$, the punctured group $(\mathcal{L}^+, \Delta^+, S)$ is of objective characteristic $p$ and thus (a) holds.

So assume now that $S$ is not normal in $M$. By Lemma 6.4, we have then $p = 3$, $K := F^*(M)$ is quasisimple, $S \leqslant K$ and $Z = Z(K)$. Set $\overline{M} := M/Z$ and $G := \overline{K}$. Let $1 \neq \overline{x} \in \overline{S}$. Then the preimage $V$ of $\langle \overline{x} \rangle$ in $S$ has order at least $3^2$. Thus, by Lemma 6.7(c), we have $C_M(V) \leqslant V$. A $3'$-element in the preimage of $C_G(\overline{x}) = C_G(\overline{V})$ in $K$ acts trivially on $\overline{V}$ and $Z = Z(K)$. Thus, it is contained in $C_M(V) \leqslant V$ and therefore trivial. Hence, we have

$$C_G(\overline{x}) = \overline{S} \text{ for every } 1 \neq \overline{x} \in \overline{S}. \tag{6.4}$$

Notice also that $G$ is a simple group with Sylow 3-subgroup $\overline{S}$, which is elementary abelian of order $3^2$. Moreover, $\mathrm{Aut}_G(\overline{S})$ is contained in a Sylow 2-subgroup of $\mathrm{Aut}(\overline{S}) \cong GL_2(3)$, and such a Sylow 2-group is semidihedral of order 16. In particular, if $\mathrm{Aut}_G(\overline{S})$ has 2-rank at least 2, then $\mathrm{Aut}_G(\overline{S})$ contains a conjugate of every involution in $\mathrm{Aut}(\overline{S})$, which is impossible because of (6.4). Hence, $\mathrm{Aut}_G(\overline{S})$ has 2-rank one, and is thus either cyclic of order at most 8 or quaternion of order 8 (and certainly nontrivial by [KS04, 7.2.1]). By a result of Smith and Tyrer [ST73], $\mathrm{Aut}_G(\overline{S})$ is not cyclic of order 2. Using (6.4), it follows from [Hig68, Theorem 13.3] that $G \cong L_2(9) \cong A_6$ if $\mathrm{Aut}_G(\overline{S})$ is cyclic of order 4, and from a result of Fletcher [Fle71, Lemma 1] that $G \cong L_3(4)$ (and thus $\mathrm{Aut}_G(\overline{S})$ is quaternion) if $\mathrm{Aut}_G(\overline{S})$ is of order 8.

It follows from Lemma 6.7(b) that $\mathrm{Aut}_M(S) = \mathrm{Aut}_{\mathcal{F}}(S)$. Since $C_M(S) = Z$ and $C_G(\overline{S}) = \overline{S}$ by (6.4), we have $\mathrm{Aut}_G(\overline{S}) \cong N_G(\overline{S})/C_G(\overline{S}) = \overline{N_K(S)}/\overline{S} \cong \mathrm{Aut}_K(S)/\mathrm{Inn}(S) = \mathrm{Out}_K(S)$. Hence,

$$\mathrm{Out}_G(\overline{S}) \cong \mathrm{Aut}_G(\overline{S}) \cong \mathrm{Out}_K(S) \leqslant \mathrm{Out}_M(S) = \mathrm{Out}_{\mathcal{F}}(S).$$

As $p = 3$ and $\mathcal{F}$ has one conjugacy class of subgroups of order 3, it follows from the classification of Ruiz and Viruel that $\mathcal{F}$ is one of the two 3-fusion systems listed in [RV04, Table 1.2], i.e. the 3-fusion system of the Tits group or the 3 fusion system of $J_4$.

Consider first the case that $\mathcal{F}$ is the 3-fusion system of the Tits group ${}^2F_4(2)$, which has $\mathrm{Out}_{\mathcal{F}}(S) \cong D_8$. Then $\mathrm{Out}_G(\overline{S})$ cannot be quaternion, i.e. we have $\mathrm{Out}_G(\overline{S}) \cong C_4$ and $G = A_6$. So conclusion (b) of the lemma holds, as $S_6$ is the only two-fold extension of $A_6$ whose Sylow 3-normalizer has dihedral Sylow 2-subgroups. By [RV04, Lemma 3.1], we have $\mathrm{Out}(S) \cong GL_2(3)$. It follows from the structure of this group that $N_{\mathrm{Out}(S)}(\mathrm{Out}_{\mathcal{F}}(S)) \cong SD_{16}$ is a Sylow 2-subgroup of $\mathrm{Out}(S)$. As $M \cong 3S_6$ has an outer automorphism group of order 2, it follows that every element of $N_{\mathrm{Aut}(S)}(\mathrm{Aut}_{\mathcal{F}}(S))$ extends to an automorphism of $M$.

Assume now that $\mathcal{F}$ is the 3-fusion system of $J_4$, so that $\mathrm{Out}_{\mathcal{F}}(S) \cong SD_{16}$. An extension of $3A_6$ with this data must be $3\#\mathrm{Aut}(A_6)$. Suppose now $\mathrm{Aut}_G(\overline{S}) \cong Q_8$ and $G \cong L_3(4)$. Then $\overline{M}$ must be a two-fold extension of $L_3(4)$. However, a graph-field automorphism centralizes a Sylow 3-subgroup, and so $M$ must be an extension of $L_3(4)$ by a field or a graph automorphism. Hence, (c) holds in this case. If (c) holds, then $\mathrm{Out}_M(S) = \mathrm{Out}_{\mathcal{F}}(S) \cong SD_{16}$ is always a self-normalizing Sylow 2-subgroup in $\mathrm{Out}(S) \cong GL_2(3)$. In particular, every element of $N_{\mathrm{Aut}(S)}(\mathrm{Aut}_{\mathcal{F}}(S))$ extends to an inner automorphism of $M$. This proves the assertion. $\square$

Note that the previous lemma shows basically that, for any punctured group $(\mathcal{L}^+, \Delta^+, S)$ over $\mathcal{F}$ which restricts to a centric linking locality, one of the conclusions (a)-(c) in Theorem 6.3 holds. To give a complete proof of Theorem 6.3, we will also need to show that each of these cases actually occurs in an example. To construct the examples, we will need the following two lemmas. The reader might want to recall the definition of $\mathcal{L}_\Delta(M)$ from Example 2.6

**Lemma 6.9.** *Let $M$ be a finite group isomorphic to $3S_6$ or $3\#\mathrm{Aut}(A_6)$ or an extension of $3L_3(4)$ by a field or graph automorphism. Let $S$ be a Sylow 3-subgroup of $M$. Then $S \cong 3^{1+2}_+$ and, writing $\Delta$ for all subgroups of $S$ of order $3^2$, we have $\mathcal{L}_\Delta(M) = N_M(S)$. Moreover, $\mathcal{F}_S(M) = \mathcal{F}_S(N_M(S))$.*

*Proof.* It is well-known that $M$ has in all cases a Sylow 3-subgroup isomorphic to $3^{1+2}_+$. By definition of $\mathcal{L}_\Delta(M)$, clearly $N_M(S) \subseteq \mathcal{L}_\Delta(M)$. Moreover, if $g \in \mathcal{L}_\Delta(M)$, then there exists $P \in \Delta$ such that $P^g \leqslant S$. Note that $Z := Z(S) \trianglelefteq M$ and $\overline{M} := M/Z$ has a normal subgroup $\overline{K}$ isomorphic to $A_6$ or $L_3(4)$. Denote by $K$ the preimage of $\overline{K}$ in $M$. Then $S \leqslant K$ and by a Frattini argument, $M = KN_M(S)$. Hence we can write $g = kh$ with $k \in K$ and $h \in N_M(S)$. In order to prove that $g \in N_M(S)$ and thus $\mathcal{L}_\Delta(M) \subseteq N_M(S)$, it is sufficient to show that $k \in N_M(S)$. Note that $P^k = (P^g)^{h^{-1}} \leqslant S$. As $\overline{S}$ is abelian, fusion in $\overline{K}$ is controlled by $N_{\overline{K}}(\overline{S})$. So there exists $x \in K$ such that $\overline{kx^{-1}} \in C_{\overline{K}}(\overline{P})$. As $\overline{K} \cong A_6$ of $L_3(4)$ and $\overline{P}$ is a non-trivial 3-subgroup of $\overline{K}$, one sees that $C_{\overline{K}}(\overline{P}) = \overline{S}$. Hence $kx^{-1} \in S$ and $k \in N_M(S)$. This shows $\mathcal{L}_\Delta(M) = N_M(S)$. By Alperin's fusion theorem, we have $\mathcal{F}_S(M) = \mathcal{F}_S(\mathcal{L}_\Delta(M)) = \mathcal{F}_S(N_M(S))$. $\square$

**Lemma 6.10.** *Assume Hypothesis 6.1. If $(\mathcal{L}, \Delta, S)$ is a centric linking locality over $\mathcal{F}$, then $N_{\mathcal{L}}(Z) = N_{\mathcal{L}}(S)$. In particular, $N_{\mathcal{L}}(Z)$ is a group which is a model for $N_{\mathcal{F}}(S)$.*

*Proof.* By Lemma 5.4, we have $N_{\mathcal{F}}(Z) = N_{\mathcal{F}}(S)$. So $Z \trianglelefteq S$ is a fully $\mathcal{F}$-normalized subgroup such that every proper overgroup of $Z$ is in $\Delta$ and $O_p(N_{\mathcal{F}}(Z)) = S \in \Delta$. Hence, by [Hen19, Lemma 7.1], $N_{\mathcal{L}}(Z)$ is a subgroup of $\mathcal{L}$ which is a model for $N_{\mathcal{F}}(Z) = N_{\mathcal{F}}(S)$. Since $N_{\mathcal{L}}(S) \subseteq N_{\mathcal{L}}(Z)$ is by Lemma 2.9(b) a model for $N_{\mathcal{F}}(S)$, and a model for a constrained fusion system is by [AKO11, Theorem III.5.10] unique up to isomorphism, it follows that $N_{\mathcal{L}}(Z) = N_{\mathcal{L}}(S)$. $\square$

We are now in a position to complete the proof of Theorem 6.3.

*Proof of Theorem 6.3.* Assume Hypothesis 6.1. By Lemma 6.8, for every punctured group $(\mathcal{L}^+, \Delta^+, S)$ over $\mathcal{F}$ which restricts to a centric linking locality, one of the cases (a)-(c) of Theorem 6.3 holds. It remains to show that each of these cases actually occurs in an example and that moreover the isomorphism type of $N_{\mathcal{L}^+}(Z)$ determines $(\mathcal{L}^+, \Delta^+, S)$ uniquely up to a rigid isomorphism.

By Lemma 5.4, we have $N_{\mathcal{F}}(Z) = N_{\mathcal{F}}(S)$ and $\mathcal{F}^s$ is the set of non-trivial subgroups of $S$. Hence, the subcentric linking locality $(\mathcal{L}^s, \mathcal{F}^s, S)$ over $\mathcal{F}$ is always a punctured group over $S$. Moreover, it follows from Lemma 2.9(b) that $N_{\mathcal{L}^s}(Z)$ is a model for $N_{\mathcal{F}}(Z) = N_{\mathcal{F}}(S)$ and thus $S$ is normal in $N_{\mathcal{L}^s}(Z)$ by [AKO11, Theorem III.5.10]. So case (a) of Theorem 6.3 occurs in an example. Moreover, if $(\mathcal{L}^*, \Delta^+, S)$ is a punctured group such that $\mathcal{L}^*|_\Delta$ is a centric linking locality and $N_{\mathcal{L}^*}(Z) \cong N_{\mathcal{L}^s}(Z)$, then $N_{\mathcal{L}^*}(Z)$ has a normal Sylow $p$-subgroup and is thus by Lemma 6.8 a subcentric linking locality. Hence, by Theorem 2.20, $(\mathcal{L}^*, \Delta^+, S)$ is rigidly isomorphic to $(\mathcal{L}^s, \mathcal{F}^s, S)$.

We are now reduced to the case that $p = 3$ and we are looking at punctured groups in which the normalizer of $Z$ is not 3-closed. So assume now $p = 3$. By the classification of Ruiz and Viruel [RV04], $\mathcal{F}$ is the 3-fusion system of the Tits group or of $J_4$. Let $M$ always be a finite group containing $S$ as a Sylow 3-subgroup and assume that one of the following holds:

(b') $\mathcal{F}$ is the 3-fusion system of the Tits group ${}^2F_4(2)'$ and $M \cong 3S_6$; or
(c') $\mathcal{F}$ is the 3-fusion system of $J_4$, and $M \cong 3\#\mathrm{Aut}(A_6)$ or an extension of $3L_3(4)$ by a field or graph automorphism.

In either case, one checks that $C_M(S) \leqslant S$. Moreover, if (b') holds, then $\mathrm{Out}_{\mathcal{F}}(S) \cong D_8$ and $N_M(S) \cong 3^{1+2}_+ : D_8$. As $\mathrm{Out}(S) \cong GL_2(3)$ has Sylow 2-subgroups isomorphic to $SD_{16}$ and moreover, $SD_{16}$ has a unique subgroup isomorphic to $D_8$, it follows that $\mathrm{Out}_M(S)$ and $\mathrm{Out}_{\mathcal{F}}(S)$ are conjugate in $\mathrm{Out}(S)$. Similarly, if (c') holds, then $\mathrm{Out}_{\mathcal{F}}(S) \cong SD_{16}$ and $\mathrm{Out}_M(S)$ are both Sylow 2-sugroups of $\mathrm{Out}(S)$ and thus conjugate in $\mathrm{Out}(S)$. Hence, $N_M(S)$ is always isomorphic to a model for $N_{\mathcal{F}}(S)$ and, replacing $M$ by a suitable isomorphic group, we can and will always assume that $N_M(S)$ is a model for $N_{\mathcal{F}}(S)$. We have then in particular that $N_{\mathcal{F}}(S) = \mathcal{F}_S(N_M(S))$.

Pick now a centric linking system $(\mathcal{L}, \Delta, S)$ over $S$. By Lemma 6.10, $N_{\mathcal{L}}(Z)$ is a model for $N_{\mathcal{F}}(S)$. Hence, by the model theorem [AKO11, Theorem III.5.10(c)], there exists a group isomorphism $\lambda\colon N_{\mathcal{L}}(Z) \to N_M(S)$ which restricts to the identity on $S$. By Lemma 6.9, we have $N_M(S) = \mathcal{L}_\Delta(M)$ and $\mathcal{F}_S(M) = \mathcal{F}_S(N_M(S)) = N_{\mathcal{F}}(S) = N_{\mathcal{F}}(Z)$. Note that $N_M(S)$ and $\mathcal{L}_\Delta(M)$ are actually equal as partial groups and the group isomorphism $\lambda$ can be interpreted as a rigid isomorphism from $N_{\mathcal{L}}(Z)$ to $\mathcal{L}_\Delta(M)$. So Hypothesis 5.3 in [Che13] holds with $Z$ in place of $T$. Since $\Delta = \mathcal{F}^c$ is the set of all subgroups of $S$ of order at least $3^2$ and as all subgroups of $S$ of order 3 are $\mathcal{F}$-conjugate, the set $\Delta^+$ of non-identity subgroups of $S$ equals $\Delta \cup Z^{\mathcal{F}}$. So by [Che13, Theorem 5.14], there exists a punctured group $(\mathcal{L}^+(\lambda), \Delta^+, S)$ over $\mathcal{F}$ with $N_{\mathcal{L}^+(\lambda)}(Z) \cong M$. Thus we have shown that all the cases listed in (a)-(c) of Theorem 6.3 occur in an example.

Let now $(\mathcal{L}^*, \Delta^+, S)$ be any punctured group over $\mathcal{F}$ such that $\mathcal{L}' := \mathcal{L}^*|_\Delta$ is a centric linking locality and $N_{\mathcal{L}^*}(Z) \cong M$. Pick a group homomorphism $\varphi\colon M \to M^* := N_{\mathcal{L}^*}(Z)$ such that $S^\varphi = S$. Then $\varphi|_S$ is an automorphism of $S$ with $(\varphi|_S)^{-1}\mathrm{Aut}_M(S)\varphi|_S = \mathrm{Aut}_{M^*}(S)$. Recall that $\mathcal{F}_S(M) = N_{\mathcal{F}}(S)$, Moreover, by Lemma 2.9(b), we have $\mathcal{F}_S(M^*) = N_{\mathcal{F}}(Z) = N_{\mathcal{F}}(S)$. Hence, $\mathrm{Aut}_M(S) = \mathrm{Aut}_{\mathcal{F}}(S) = \mathrm{Aut}_{M^*}(S)$ and $\varphi|_S \in N_{\mathrm{Aut}(S)}(\mathrm{Aut}_{\mathcal{F}}(S))$. So by Lemma 6.8, there exists $\psi \in \mathrm{Aut}(M)$ such that $\psi|_S = \varphi|_S$. Then $\mu := \psi^{-1}\varphi$ is an isomorphism from $M$ to $M^* = N_{\mathcal{L}^*}(Z)$ which restricts to the identity on $S = N_S(Z)$. Moreover, by Theorem 2.19, there exists a rigid isomorphism $\beta\colon \mathcal{L} \to \mathcal{L}'$. Therefore by [Che13, Theorem 5.15(a)], there exists a rigid isomorphism from $(\mathcal{L}^+(\lambda), \Delta^+, S)$ to $(\mathcal{L}^*, \Delta^+, S)$. This shows that a punctured group $(\mathcal{L}^+, \Delta^+, S)$ over $\mathcal{F}$, which restricts to a centric linking locality, is up to a rigid isomorphism uniquely determined by the isomorphism type of $N_{\mathcal{L}^+}(Z)$. $\square$

*Proof of Theorem 6.2.* Assume Hypothesis 6.1. If $p \neq 3$, then it follows from Lemma 6.5 that $\mathcal{L}^+/O_{p'}(\mathcal{L}^+)$ is a subcentric linking locality for every every punctured group $(\mathcal{L}^+, \Delta^+, S)$ over $S$. On the other hand, if $p = 3$, then Theorem 6.3 gives the existence of a punctured group $(\mathcal{L}^+, \Delta^+, S)$ over $\mathcal{F}$ such that $O_{p'}(\mathcal{L}^+) = 1$ and $N_{\mathcal{L}^+}(Z)$ is not of characteristic $p$, i.e. such that $\mathcal{L}^+/O_{p'}(\mathcal{L}^+)$ is not a subcentric linking locality. $\square$

## Appendix A. Notation and background on groups of Lie type

We record here some generalities on algebraic groups and finite groups of Lie type which are needed in Section 4. Our main references are [Car72], [GLS98], and [BMO19], since these references contain proofs for all of the background lemmas we need.

Fix a prime $p$ and a semisimple algebraic group $\bar{G}$ over $\bar{\mathbb{F}}_p$. Let $\bar{T}$ be a maximal torus of $\bar{G}$, $W = N_{\bar{G}}(\bar{T})/\bar{T}$ the Weyl group, and let $X(\bar{T}) = \mathrm{Hom}(\bar{T}, \bar{\mathbb{F}}_p^\times)$ be the character group. Let $\bar{X}_\alpha = \{x_\alpha(\lambda) \mid \lambda \in \bar{\mathbb{F}}_p\}$ denote a root subgroup, namely a closed $\bar{T}$-invariant subgroup isomorphic $\bar{\mathbb{F}}_p$. The root subgroups are indexed by the roots of $\bar{T}$, the characters $\alpha \in X(\bar{T})$ with $x_\alpha(\lambda)^t = x_\alpha(\alpha(t)\lambda)$ for each $t \in \bar{T}$. The character group $X(\bar{T})$ is written additively: for each $\alpha, \beta \in X(\bar{T})$ and each $t \in \bar{T}$, we write $(\alpha + \beta)(t) = \alpha(t)\beta(t)$. For each $n \in N_{\bar{G}}(\bar{T})$, $\alpha \in X(\bar{T})$, and $t \in \bar{T}$ we write $({}^n\alpha)(t) = \alpha(t^n)$ for the induced action of $N_{\bar{G}}(\bar{T})$ action on $X(\bar{T})$.

Let $\Sigma(\bar{T})$ be the set of $\bar{T}$-roots $\alpha \in X(\bar{T})$, and let $V = \mathbb{R} \otimes_{\mathbb{Z}} X(\bar{T})$ be the associated real inner product space with $W$-invariant inner product $(,)$. We regard $X(\bar{T})$ as a subset of $V$, and write $w_\alpha \in W$ for the reflection in the hyperplane $\alpha^\perp$.

For each root $\alpha \in \Sigma(\bar{T})$ and each $\lambda \in \bar{\mathbb{F}}_p^\times$, let $n_\alpha(\lambda), h_\alpha(\lambda) \in \langle \bar{X}_\alpha, \bar{X}_{-\alpha} \rangle$ be the images of the elements $\left[\begin{smallmatrix} 0 & -\lambda^{-1} \\ \lambda & 0 \end{smallmatrix}\right]$, $\left[\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{smallmatrix}\right]$ under the homomorphism $SL_2(\bar{\mathbb{F}}_p) \to G$ which sends $\left[\begin{smallmatrix} 1 & 0 \\ u & 1 \end{smallmatrix}\right]$ to $x_\alpha(u)$ and $\left[\begin{smallmatrix} 1 & v \\ 0 & 1 \end{smallmatrix}\right]$ to $x_{-\alpha}(v)$. Thus

$$n_\alpha(\lambda) = x_\alpha(\lambda)x_{-\alpha}(-\lambda^{-1})x_\alpha(\lambda) \quad \text{and} \quad h_\alpha(\lambda) = n_\alpha(1)^{-1}n_\alpha(\lambda), \tag{A.1}$$

and $n_\alpha(1)$ represents $w_\alpha$ for each $\alpha \in \Sigma$. We assume throughout that parametrizations of the root groups have been chosen so that the Chevalley relations of [GLS98, 1.12.1] hold.

Although $\Sigma(\bar{T})$ is defined in terms of characters of the maximal torus $\bar{T}$, it will be convenient to identify $\Sigma(\bar{T})$ with an abstract root system $\Sigma$ inside some standard Euclidean space $\mathbb{R}^l$, $(,)$, via a $W$-equivariant bijection which preserves sums of roots [GLS98, 1.9.5]. We'll write also $V$ for this Euclidean space. The symbol $\Pi$ denotes a fixed but arbitrary base of $\Sigma$.

The maps $h_\beta \colon \bar{\mathbb{F}}_p^\times \to \bar{T}$, defined above for each $\beta \in \Sigma$, are algebraic homomorphisms lying in the group of cocharacters $X^\vee(\bar{T}) := \mathrm{Hom}(\bar{\mathbb{F}}_p^\times, \bar{T})$. Composition induces a $W$-invariant perfect pairing $X(\bar{T}) \otimes_{\mathbb{Z}} X^\vee(\bar{T}) \to \mathbb{Z}$ defined by $\alpha \otimes h \mapsto \langle \alpha, h \rangle$, where $\langle \alpha, h \rangle$ is the unique integer such that $\alpha(h(\lambda)) = \lambda^{\langle \alpha, h \rangle}$ for each $\lambda \in \bar{\mathbb{F}}_p^\times$. Since $\Sigma$ contains a basis of $V$, we can identify $V^*$ with $\mathbb{R} \otimes_{\mathbb{Z}} X^\vee(\bar{T})$, and view $X^\vee(\bar{T}) \subseteq V^*$ via this pairing. Under the identification of $V$ with $V^*$ via $v \mapsto (-, v)$, for each $\beta \in \Sigma$ there is $\beta^\vee \in V$ such that $(-, \beta^\vee) = \langle -, h_\beta \rangle$ in $V^*$, namely the unique element such that $(\beta, \beta^\vee) = 2$ and such that $w_\beta$ is reflection in the hyperplane $\ker((-, \beta^\vee))$. Thus, when viewed in $V$ in this way (as opposed to in the dual space $V^*$), $\beta^\vee = 2\beta/(\beta,\beta)$ is the abstract coroot corresponding to $\beta$. Write $\Sigma^\vee = \{\beta^\vee \mid \beta \in \Sigma\} \subseteq V$ for the dual root system of $\Sigma$.

If we set $\langle\alpha,\beta\rangle = (\alpha,\beta^\vee) = 2(\alpha,\beta)/(\beta,\beta)$ for each pair of roots $\alpha,\beta \in \Sigma$, then

$$\langle\alpha,\beta\rangle = \langle\alpha,h_\beta\rangle \tag{A.2}$$

where the first is computed in $\Sigma$, and the second is the pairing discussed above. Equivalently,

$$x_\alpha(\mu)^{h_\beta(\lambda)} = x_\alpha(\lambda^{\langle\alpha,\beta\rangle}\mu) \tag{A.3}$$

for each $\alpha,\beta \in \Sigma$, each $\mu \in \bar{\mathbb{F}}_p$, and each $\lambda \in \bar{\mathbb{F}}_p^\times$.

Additional Chevalley relations we need are

$$x_\alpha(\lambda)^{n_\beta(1)} = x_{w_\beta(\alpha)}(c_{\alpha,\beta}\lambda), \tag{A.4}$$

$$h_\alpha(\lambda)^{n_\beta(1)} = h_{w_\beta(\alpha)}(\lambda), \tag{A.5}$$

$$n_\alpha(\lambda)^{n_\beta(1)} = h_{w_\beta(\alpha)}(c_{\alpha,\beta}\lambda), \tag{A.6}$$

$$n_\alpha(1)^2 = h_\alpha(-1). \tag{A.7}$$

where

$$w_\beta(\alpha) = \alpha - \langle\alpha,\beta\rangle\beta,$$

is the usual reflection in the hyperplane $\beta^\perp$, and where the $c_{\alpha,\beta} \in \{\pm1\}$, in the notation of [GLS98, Theorem 1.12.1], are certain signs which depend on the choice of the Chevalley generators. This notation is related to the signs $\eta_{\alpha,\beta}$ in [Car72, Chapter 6] by $c_{\alpha,\beta} = \eta_{\beta,\alpha}$.

Important tools for determining the signs $c_{\alpha,\beta}$ in certain cases are proved in [Car72, Propositions 6.4.2, 6.4.3], and we record several of those results here.

**Lemma A.1.** *Let $\alpha,\beta \in \Sigma$ be linearly independent roots.*

*(1) $c_{\alpha,\alpha} = -1$ and $c_{-\alpha,\alpha} = -1$.*
*(2) $c_{-\alpha,\beta} = c_{\alpha,\beta}$.*
*(3) $c_{\alpha,\beta}c_{w_\beta(\alpha),\beta} = (-1)^{\langle\alpha,\beta\rangle}$.*
*(4) If the $\beta$-root string through $\alpha$ is of the form*

$$\alpha - s\beta, \ldots, \alpha, \ldots, \alpha + s\beta$$

*for some $s \geqslant 0$, that is, if $\alpha$ and $\beta$ are orthogonal, then $c_{\alpha,\beta} = (-1)^s$.*

*Proof.* The first three listed properties are proved in Proposition 6.4.3 of [Car72]. By the proof of that proposition, there are signs $\epsilon_i \in \{\pm1\}$ such that $c_{\alpha,\beta} = (-1)^s\frac{\epsilon_0\cdots\epsilon_{s-1}}{\epsilon_0\cdots\epsilon_{r-1}}$, whenever the $\beta$-root string through $\alpha$ is of the form $\alpha - s\beta, \ldots, \alpha, \ldots, \alpha + r\beta$. When $\alpha$ and $\beta$ are orthogonal, we have $r - s = \langle\alpha,\beta\rangle = 0$, and hence $c_{\alpha,\beta} = (-1)^s$. $\square$

**Lemma A.2.** *The following hold.*

*(1) For each $\alpha,\beta \in \Sigma$, we have*

$$\alpha(h_\beta(\lambda)) = \lambda^{\langle\alpha,\beta\rangle}.$$

*(2) The maximal torus $\bar{T}$ is generated by the $h_\alpha(\lambda)$ for $\alpha \in \Sigma$ and $\lambda \in \bar{\mathbb{F}}_p^\times$. If $\bar{G}$ is simply connected, and $\lambda_\alpha \in \mathbb{F}_p^\times$ are such that $\prod_{\alpha\in\Pi} h_\alpha(\lambda_\alpha) = 1$, then $\lambda_\alpha = 1$ for all $\alpha \in \Pi$. Thus,*

$$\bar{T} = \prod_{\alpha\in\Pi} h_\alpha(\bar{\mathbb{F}}_p^\times),$$

*and $h_\alpha$ is injective for each $\alpha$.*

*(3) If $\beta, \alpha_1, \dots, \alpha_k \in \Sigma$ and $n_1, \dots, n_k \in \mathbb{Z}$ are such that $\beta^\vee = n_1\alpha_1^\vee + \cdots + n_k\alpha_k^\vee$, then*

$$h_\beta(\lambda) = h_{\alpha_1}(\lambda^{n_1}) \cdots h_{\alpha_k}(\lambda^{n_k}).$$

*(4) Define*

$$\Phi\colon \mathbb{Z}\Sigma^\vee \times \bar{\mathbb{F}}_p^\times \longrightarrow \bar{T} \quad \text{by} \quad \Phi(\alpha^\vee, \lambda) = h_\alpha(\lambda).$$

*Then $\Phi$ is bilinear and $\mathbb{Z}[W]$-equivariant. It induces a surjective $\mathbb{Z}[W]$-module homomorphism $\mathbb{Z}\Sigma^\vee \otimes_{\mathbb{Z}} \bar{\mathbb{F}}_p^\times \to \bar{T}$ which is an isomorphism if $\bar{G}$ is of universal type.*

*Proof.* (1) is the statement in (A.2) and is part of [GLS98, Remark 1.9.6]. We refer to [BMO19, Lemma 2.4(c)] for a proof, which is based on the treatment in Carter [Car72, pp.97-100]. Part (2) is proved in [BMO19, Lemma 2.4(b)], and part (3) is [BMO19, Lemma 2.4(d)]. Finally, part (4) is proved in [BMO19, Lemma 2.6]. $\square$

**Proposition A.3.** *For each subgroup $X \leqslant \bar{T}$,*

$$C_{\bar{G}}(X) = C_{\bar{G}}(X)^\circ C_{N_{\bar{G}}(\bar{T})}(X).$$

*The connected component $C_{\bar{G}}(X)^\circ$ is generated by $\bar{T}$ and the root groups $\bar{X}_\alpha$ for those roots $\alpha \in \Sigma$ whose kernel contains $X$. In particular, if $X = \langle h_\beta(\lambda)\rangle$ for some $\beta \in \Sigma$ and some $\lambda \in \bar{\mathbb{F}}_p^\times$ having multiplicative order $r$, then*

$$C_{\bar{G}}(X)^\circ = \langle \bar{T}, \bar{X}_\alpha \mid \alpha \in \Sigma,\ r \text{ divides } \langle \alpha, \beta\rangle\rangle.$$

*Proof.* See [BMO19, Proposition 2.5], which is based on [Car85, Lemma 3.5.3]. The referenced result covers all but the last statement, which then follows from the previous parts and Lemma A.2(1), given the definition of $r$. $\square$

**Proposition A.4.** *Let $\bar{G}$ be a simply connected, simple algebraic group over $\bar{\mathbb{F}}_p$, let $\bar{T}$ be a maximal torus of $\bar{G}$, and let $T_r = \{t \in \bar{T} \mid t^r = 1\}$ with $r > 1$ prime to $p$. Then one of the following holds.*

*(a) $C_{\bar{G}}(T_r) = \bar{T}$ and $N_{\bar{G}}(T_r) = N_{\bar{G}}(\bar{T})$.*

*(b) $r = 2$, $C_{\bar{G}}(T_r) = \bar{T}\langle w_0\rangle$ for some element $w_0 \in N_{\bar{G}}(\bar{T})$ inverting $\bar{T}$, and $N_{\bar{G}}(T_r) = N_{\bar{G}}(\bar{T})$,*

*(c) $r = 2$, and $\bar{G} = Sp_{2n}(\bar{\mathbb{F}}_p)$ for some $n \geqslant 1$.*

*Proof.* By Lemma A.2(2) and since $\bar{G}$ is simply connected, the torus is direct product of the images of the coroots for fundamental roots:

$$\bar{T} = \prod_{\alpha\in\Pi} h_\alpha(\bar{\mathbb{F}}_p^\times). \tag{A.8}$$

Thus, if $\lambda \in \bar{\mathbb{F}}_p^\times$ is a fixed element of order $r$, then $T_r$ is the direct product of $\langle h_\alpha(\lambda)\rangle$ as $\alpha$ ranges over $\Pi$.

We first look at $C_{\bar{G}}(T_r)^\circ$, using Proposition A.3. By Lemma A.2(1), $T_r$ is contained in the kernel of a root $\beta$ if and only if $\beta(h_\alpha(\lambda)) = \lambda^{\langle\beta,\alpha\rangle} = 1$ for all simple roots $\alpha$, i.e., if $\langle\beta,\alpha\rangle$ is divisble by $r$ for each fundamental root $\alpha$. Let $\Sigma_r$ be the set of all such roots $\beta$. For each $\alpha \in \Pi$, the reflection $w_\alpha$ sends a root $\beta$ to $\beta - \langle\beta,\alpha\rangle\alpha$. Hence $\beta \in \Sigma_r$ if and only if $w_\alpha(\beta) \in \Sigma_r$ since $\langle -, - \rangle$ is linear in the first component. Since the Weyl group is generated by $w_\alpha$, $\alpha \in \Pi$, it follows that $\Sigma_r$ is invariant under the Weyl group. By [Hum72, Lemma 10.4C], and since $\bar{G}$ is simple, $W$ is transitive on all roots of a given length, and so either $\Sigma_r = \varnothing$, or $\Sigma_r$ contains all long roots or all short ones. Thus, by [Hum72, Table 1], we conclude that either $\Sigma_r = \varnothing$, or $r = 2$, each root in $\Pi \cap \Sigma_r$ is long, and each $\alpha \in \Pi$ not orthogonal to $\beta$ is short and has angle $\pi/4$ or $3\pi/4$ with $\beta$. Now by inspection of the Dynkin diagrams corresponding to irreducible root systems, we conclude that the latter is possible only if $\Sigma = A_1 = C_1$, $C_2$, or $C_3$. Thus, either $C_{\bar{G}}(T_r)^\circ = \bar{T}$ or (c) holds.

So we may assume that $C_{\bar{G}}(T_r)^\circ = \bar{T}$. Now $N_{\bar{G}}(\bar{T}) \leqslant N_{\bar{G}}(T_r)$ since $T_r$ is characteristic in $T$. As $C_{\bar{G}}(T_r)^\circ = \bar{T}$, also $\bar{T}$ is normalized by $N_{\bar{G}}(T_r)$, so $N_{\bar{G}}(\bar{T}) = N_{\bar{G}}(T_r)$. For $r \geqslant 3$, it follows from [BMO19, Lemma 2.7] that $C_{N_{\bar{G}}(\bar{T})}(T_r) = \bar{T}$, completing the proof of (a) in this case.

Assume now that $r = 2$ and (a) does not hold. Let $B := C_W(T_2) \leqslant W = N_{\bar{G}}(\bar{T})/\bar{T}$. To complete the proof, we need to show $B = \langle -1_V \rangle$ or else (c) holds. Here we argue as in Case 1 of the proof of [BMO19, Proposition 5.13].

Let $\Lambda = \mathbb{Z}\Sigma^\vee$ be the lattice of coroots, and fix $\lambda \in \bar{\mathbb{F}}_p^\times$ of order 4. The map $\Phi_\lambda\colon \Lambda \to \bar{T}$ defined by $\Phi_\lambda(\alpha^\vee) = h_\alpha(\lambda)$ is a $W$-equivariant homomorphism by Lemma A.2(3). Since $\bar{G}$ is simply connected, this homorphism has kernel $4\Lambda$, image $T_4$, and it identifies $\Lambda/2\Lambda$ with $T_2$, by Lemma A.2(2).

Since $B$ acts on $T_4$ and centralizes $T_2$, we have $[T_4, B] \leqslant T_2 \leqslant C_{\bar{T}}(B)$, so $B$ acts quadratically on $T_4$. Since $B$ acts faithfully on $T_4$ by (a), it follows that $B$ is a 2-group.

Assume that $B \neq \langle -1_V \rangle$. If $B$ is of 2-rank 1 with center $\langle -1_V \rangle$ then by assumption there is some $b \in B$ with $b^2 = -1_V$. In this case, $b$ endows $V$ with the structure of a complex vector space, and so $b$ does not centralize $\Lambda/2\Lambda$, a contradiction. Thus, there is an involution $b \in B$ which is not $-1_V$. Let $V = V_+ \oplus V_-$ be the decomposition of $V$ into the sum of the eigenspaces for $b$, and set $\Lambda_\pm = \Lambda \cap V_\pm$. Fix $v \in \Lambda$, and write $v = v_+ + v_-$ with $v_\pm \in V_\pm$. Then $2v_- = v - v^b = [v, b] \in V_- \cap 2\Lambda = 2\Lambda_-$. So $v_- \in \Lambda_-$, and then $v_+ \in \Lambda_+$. This shows that $\Lambda = \Lambda_+ \oplus \Lambda_-$ with $\Lambda_\pm \neq 0$. The hypotheses of [BMO19, Lemma 2.8] thus hold, and so $G = Sp_{2n}(\bar{\mathbb{F}}_p)$ for some $n \geqslant 2$ by that lemma. $\square$

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