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RESEARCH ARTICLE

ON APPROXIMATE ANALYTIC TECHNIQUES FOR THE CONSTRUCTION AND ANALYSIS OF SOLUTIONS OF MATHEMATICAL MODELS

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ABSTRACT

The study presents how to obtain the solutions of the n^{th}-order ordinary differential equations with varying delay proportional to the independent variable, where n belongs to the set of natural number, ℕ. These are equations that are often used in Mathematics to characterize real life problems such as optimizing profits, minimizing costs, and improving individuals' health. Economic models can help to understand and predict the economic behaviours of different countries. The results of this study are applied to certain economic models. Under the assumption that the market is in equilibrium, the study considers price adjustment models and proposes an adjustment model by introducing a proportional delay into the formulation, which improves the suitability of the models. The study displays the solutions of the models by using Matlab to present their graphs and compare them.

KEYWORDS

Sumudu transform, Proportional delay, Price adjustment model, Demand and supply, Equilibrium

1. INTRODUCTION

The study of the hereditary properties of linear and nonlinear systems has a lot of applications in real-life. Hereditary properties describe a situation when the rate of change of a system is considered to depend on both the state of the system at a given time and previous evolution of the process. Such studies play an important role in economics, natural science and engineering. Delay systems refer to the situation where the state of a system is determined not by its entire history, but by the current state and some in the past. Delays are inherently bonded with several dynamical systems. Consider a first-order linear Ordinary Differential Equation (ODE) that is associated with varying delay proportional to the independent variable,

$$y'(t) = f(t, y_i(t), y_i(\lambda t)), \tag{1.1}$$

where $i = 1, 2, 3, \ldots, , n$ and $0 < \lambda < 1$. This is a typical delay equation with varying delay $\tau(t) = (1 - \lambda)t$. Equation (1.1) models the dynamics of a current collection system for an electric locomotive (Ockendon and Tayler, 1551/1971). Solving delay equations with constant delay by using approximate analytical and numerical methods can be considered fairly well developed. Studies on the construction of approximate analytical solutions of delay equations with constant delay and their analysis are contained, for example, in (Cui et al., 2021; Bohner et al., 2021; Valliammal et al., 2020). Obtaining the solutions of delay equations with constant delay by using the numerical methods are carried out in (Mahmudov, 2019; Guirao et al., 2020; El-Dib, 2018). Those methods are fine if obtaining an approximate solution is the objective because they rarely give exact solutions. For an improvement in the solutions of differential equations, contemporary studies have considered the use of some new numerical and analytical techniques (Yel et al., 2022; Yavuz, 2022; Duran et al., 2023; Pak, 2009; Aibinu, 2023). The notion of delay equations with varying delay has great importance in obtaining exact optimal solutions (see, e.g., (Cai et al., 2012). The notion of delay equations with varying delay has not fared well in the literature. Studies on solutions and stability of delay equations with varying delay are an active area of research (Long and Gong, 2020; Cao et al., 2022; Ali et al., 2020; Xia et al., 2022; Aibinu and Momoniat, 2023; Aibinu et al., 2023; Aibinu et al., 2022).

In this paper, an approximate, an approximate analytic technique that is efficient in accuracy and computational time is presented for n^{th}-order ODEs. The importance of ODEs cuts across almost all fields of science, engineering and economics. This paper considers delay equations with varying delay due to their wide applications in obtaining the exact optimal solutions of mathematical models. The results are applied to a model arising in economics. Under the assumption that the market is in equilibrium, the study considers Price Adjustment Models (PAMs) and proposes an adjusted model by introducing a proportional delay into the formulation of the PAM. Using the approximate analytic technique that is presented in this study, we obtain the solution of the PAM with a proportional delay. Using Matlab, graphs of the solution of the PAM with a proportional delay are displayed and compared to the solutions of the PAMs without a delay.

2. PRELIMINARIES

In this section, we give some definitions and propositions that are essential in establishing the main results of this paper. Throughout this paper, ℕ and ℝ will denote the sets of natural and real numbers, respectively.

Consider a set of functions $\mathbb{A}$, defined as (Belgacem and Karaballi, 2006)

$$\mathbb{A} = \{y(t) : \exists Q, \tau_1\tau_2 > 0, |y(t)| < Q\mathrm{e}^{|t|/\tau_j}, \quad if\ t \in (-1)^j \times [0, \infty)\}.$$

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For all real $t > 0, y(t) \in \mathbb{A}$. The Sumudu Transform (ST) of a given function $y(t)$ is defined as

$$\mathcal{S}[y(t)] = \int_0^\infty y(tu)e^{-t}dt,\ u \in (-\tau_1, \tau_2), \tag{2.1}$$

which will be denoted by $\mathcal{S}[y(t)] := Y(u)$. The function $y(t)$ is the inverse ST of Y(u) and the relation is denoted by $y(t) = \mathcal{S}^{-1}[Y(u)]$. Recall that the Laplace transform of $y(t)$ is defined as

$$\mathcal{L}[y(t)] = \int_0^\infty y(t)e^{-st}dt,\ s > 0, \tag{2.2}$$

which can simply be denoted by $\mathcal{L}[y(t)] := L(u)$. By considering (2.1) and (2.2), one can express a relation between the Sumudu and Laplace transforms as follows:

$$Y(1/s) = sL(s), \quad L(1/u) = uY(u).$$

Like the well-known Laplace transform, the ST is an integral method. ST is a simple modified form of the Laplace transform. Using ST technique is appealing as it yields an accurate result quickly and it does not impose any restricting assumptions about the results. It is a simple, effective and universal way by which one can obtain the Lagrange multiplier. The linearity property of ST is well known (Belgacem and Karaballi, 2006; Watugala, 1993; Belgacem et al. 2003, Moltot and Deresse, 2022), that is, for any two given functions $y(t)$, $z(t) \in \mathbb{A}$, and for arbitrary real constants α and β,

$$\mathcal{S}[\alpha y(t) + \beta z(t)] = \alpha\mathcal{S}[y(t)] + \beta\mathcal{S}[z(t)].$$

The ST for the first order derivative is expressed as

$$\mathcal{S}[y'(t)] = \tfrac{1}{u}[Y(u) - y(0)]. \tag{2.3}$$

For the n^{th}-order derivative, the ST is given as

$$\mathcal{S}[y^n(t)] = \tfrac{1}{u^n}[Y(u) - \textstyle\sum_{k=0}^{n-1} u^k y^{(k)}(t)|_{t=0}], \tag{2.4}$$

where $y^{(k)}(t) = \frac{d^k y(t)}{dt^k}$. Table 1 gives some selected and frequently used Sumudu Transforms (Belgacem and Karaballi, 2006; Watugala, 1993; Belgacem et al., 2003; Moltot and Deresse, 2022).

Table 1: Selected Sumudu Transforms

$y(t)$	$Y(u) = \mathcal{S}[y(t)]$
1	1
t	u
$\frac{t^n}{n!} = \frac{t^n}{\Gamma(n+1)}$	u^n
e^{at}	$\frac{1}{1-au}$
$\sin at$	$\frac{u}{1+a^2u^2}$
$\cos at$	$\frac{1}{1+a^2u^2}$
$\frac{e^{bt}-e^{at}}{b-a}, b \neq a$	$\frac{1}{(1-bu)(1-au)}$

3. Main Results

This paper presents a blend of the variational iterative method with the ST for solving n^{th}-order ODEs with varying delay proportional to the independent variable. Then PAMs is presented as an illustration and Matlab is used to compute and display the graphs of the solutions of the models.

3.1 Approximate Analytic Technique.

An approximate analytic technique that is a blend of ST with the variational iterative method is presented in this section of the paper. When compared with other well-known methods, the flexibility, consistency and effectiveness of the variational iterative method (Wu, 2013; Wu and Baleanu, 2013) and references there in motivated its selection for amalgamation with the ST. Consider the n^{th}-ODE with a proportional delay

$$\frac{d^n y(t)}{dt^n} + R[y(t)] + N[y(\lambda t)] = \omega(t), \tag{3.1}$$

subject to the initial conditions

$$y^{(k)}(0) = a_k,$$

where $y^{(k)}(0) = \frac{d^k y(0)}{dt^k}, k = 0, 1, \ldots, n-1, R$ is a linear operator, N is a nonlinear operator, $\omega(t)$ is a given continuous function and the highest order derivative is $\frac{d^n y(t)}{dt^n}$.

Taking the ST of (3.1) transforms its linear part into an algebraic equation of the form

$$\frac{1}{u^n}Y(u) - \sum_{k=0}^{n-1}\frac{1}{u^{n-k}}y^{(k)}(0) = \mathcal{S}\big[\omega(t) - R[y(t)] - N[y(\lambda t)]\big].$$

Thus, the corresponding iteration procedure is given by

$$Y_{n+1}(u) = Y_n(u)$$

$$+\alpha(u)\left(\tfrac{1}{u^n}P_n(u) - \textstyle\sum_{k=0}^{n-1}\tfrac{1}{u^{n-k}}y^{(k)}(0) - \mathcal{S}\big[\omega(t) - R[y_n(t)] - N[y_n(\lambda t)]\big]\right) \tag{3.2}$$

where $\alpha(u)$ is the Lagrange multiplier. Taking the classical variation operator of (3.2) and considering $\mathcal{S}[R[y(t)] + N[y(\lambda t)]]$ as the restricted terms yields

$$\delta Y_{n+1}(u) = \delta Y_n(u) + \alpha(u)\frac{1}{u^n}Y_n(u),$$

which gives

$$\alpha(u) = -u^n. \tag{3.3}$$

Substituting (3.3) into (3.2) and taking the inverse of Sumudu Transform $\mathcal{S}^{-1}$ of (3.2) yields the explicit iterative procedure,

$$y_{n+1}(t) = y_n(t) + \mathcal{S}^{-1}\left[-u^n\left(\tfrac{1}{u^n}Y_n(u) - \textstyle\sum_{k=0}^{n-1}\tfrac{1}{u^{n-k}}y^{(k)}(0) - \mathcal{S}\big[\omega(t) - R[y_n(t)] - N[y_n(\lambda t)]\big]\right)\right]$$

$$= y_1(t) + \mathcal{S}^{-1}\left[u^n\mathcal{S}\big[\omega(t) - R[y_n(t)] - N[y_n(\lambda t)]\big]\right],$$

where

$$y_1(t) = \mathcal{S}^{-1}\left[\sum_{k=0}^{n-1}u^k y^{(k)}(0)\right] = y(0) + y'(0) + \cdots + \frac{y^{n-1}(0)t^{n-1}}{(n-1)!}.$$

3.2 Application to Economic Models.

Economic models can help to understand and predict the economic behaviour (Ellis et al., 2014). The economy concerning a commodity determines the trend of its price, which may increase or decrease rapidly. Through economic models, economists can predict the optimal profit to show the link between demand and supply. Mathematical models of economic processes can give insight into the interaction that exists between the price, demand and supply, dependence of supply and demand on price and how to estimate the equilibrium point on the supply and demand curves (Cohen-Vernika and Pazgal, 2017). Market equilibrium refers to a state in which the quantity demand and the quantity supply of a commodity are equivalent. Both market equilibrium and economic growth occupy important positions in the description of real world problems. Using the demand and supply functions, this paper refers to the quantity demand and supply as functions of price, respectively. These functions are respectively given as:

$$f_d(t) = d_0 - d_1 p(t) \quad and \quad f_s(t) = s_0 + s_1 p(t), \tag{3.4}$$

where $p(t)$ is the price of the commodities, while d_0, d_1, s_0 and s_1 are positive constants (see, e.g, [30]). Figure 1 shows the graph of the quantity demand and quantity supply at a given price. At equilibrium, $f_d(t) = f_s(t)$, which means that the quantity demand and quantity supply are equal and the equilibrium price is obtained as

$$p^* = \frac{d_0 + s_0}{d_1 + s_1}.$$

In Figure 1, $p^* = 5.5$ unit. The price tends to be invariant at equilibrium and there is neither a surplus nor shortage. Consider the price adjustment model which is given as

$$p'(t) = q(f_d - f_s), \tag{3.5}$$

where $q > 0$ denotes the speed of adjustment constant. This is a linear model, which indicates that the price rises when demand exceeds supply and the price falls when supply exceeds demand. Substituting (3.4) into

(3.5) gives

$$p'(t) + q(d_1 + s_1)p(t) = q(d_0 + s_0). \qquad (3.6)$$

The solution of linear differential equation (3.6) is obtained as

$$p(t) = p^* + (p(0) - p^*)e^{(d_1+s_1)},$$

where $p(0)$ denotes the price at time $t = 0$. It is possible to consider a price adjustment equation that takes the expectations of agents into account. In such a case, the demand and supply functions admit additional factors d_2 *and* s_2, respectively and take the form

$$f_d(t) = d_0 - d_1 p(t) + d_2 p'(t) \ \ and \ \ f_s(t) = s_0 + s_1 p(t) - s_2 p'(t),$$

where d_0, d_1, d_2, s_0, s_1 *and* s_2 are positive constants. Equating $f_d(t)$ *to* $f_s(t)$ gives

$$p'(t) - \frac{d_1+s_1}{d_2+s_2}p(t) = -\frac{d_0+s_0}{d_2+s_2}. \qquad (3.7)$$

The solution of linear differential equation (3.7) is obtained as

$$p(t) = p^* + (p(0) - p^*)e^{\frac{(d_1+s_1)}{(d_2+s_2)}}.$$

An increase in the price of a commodity will urge the buyers to buy more before prices increase further while the suppliers tend to offer less with the hope of earning more from higher prices in future (Nanware et al., 2022; Bas et al., 2019). In addition, when $p'(t) = 0$ for all $t > 0$, this describes equilibrium in a changing economy, which implies that the market is in dynamic equilibrium.

3.3 Price Adjustment Models with A Proportional Delay.

Consider introducing a proportional delay to formulate a new PAM

$$p'(t) - \frac{d_1+s_1}{d_2+s_2}p(\lambda t) = -\frac{d_0+s_0}{d_2+s_2}, \qquad (3.8)$$

where $p(0) = p_0$. The ST of (3.8) takes the form

$$\mathcal{S}[p'(t)] + \frac{d_1+s_1}{d_2+s_2}\mathcal{S}[p(\lambda t)] = \frac{d_0+s_0}{d_2+s_2},$$

which leads to

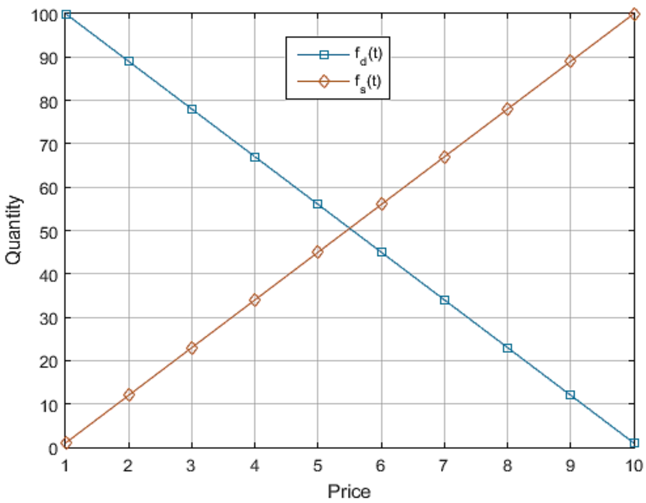

Figure 1: Demand and Supply.

$$\frac{1}{u}[P(u) - p_0] - \frac{d_1+s_1}{d_2+s_2}\mathcal{S}[p(\lambda t)] = \frac{d_0+s_0}{d_2+s_2},$$

since $p(0) = p_0$. Therefore, the Sumudu variational iteration formula is given as

$$P_{n+1}(u) = P_n(u) + \alpha(u)\left(\frac{P_n(u)-p_0}{u} - \frac{d_1+s_1}{d_2+s_2}\mathcal{S}[p_n(\lambda t)] + \frac{d_0+s_0}{d_2+s_2}\right), n \in N. \quad (3.9)$$

Taking the classical variation operator of (3.9) and considering $p_n(\lambda t)$ as the restricted term gives

$$\delta P_{n+1}(u) = \delta P_n(u) + \alpha(u)\frac{1}{u^n}Y(u),$$

which gives

$$\alpha(u) = -u.$$

Substitute for $\alpha(u)$ in (3.9) and take its inverse ST to obtain

$$p_{n+1}(t) = p_n(t) + \mathcal{S}^{-1}\left[-u\left(\frac{P_n(u)-p_0}{u} - \frac{d_1+s_1}{d_2+s_2}\mathcal{S}[p_n(\lambda t)] + \frac{d_0+s_0}{d_2+s_2}\right)\right]$$

$$= p_1(t) + \mathcal{S}^{-1}\left[u\left(\frac{d_1+s_1}{d_2+s_2}\mathcal{S}[p_n(\lambda t)] - \frac{d_0+s_0}{d_2+s_2}\right)\right]$$

$$= p_1(t) + \frac{d_1+s_1}{d_2+s_2}\mathcal{S}^{-1}\left[u\left(\mathcal{S}[p_n(\lambda t)] - \frac{d_0+s_0}{d_1+s_1}\right)\right],$$

with the initial approximation which is given as $p_1(t) = \mathcal{S}^{-1}\left[-u(\frac{-p_0}{u})\right] = p_0\mathcal{S}^{-1}[1] = p_0$. Hence, the explicit iteration formula is derived as

$$p_{n+1}(t) = p_0 + \frac{d_1+s_1}{d_2+s_2}\mathcal{S}^{-1}\left[u\left(\mathcal{S}[p_n(\lambda t)] - \frac{d_0+s_0}{d_1+s_1}\right)\right], p_1(t) = p_0. \qquad (3.10)$$

Observe that from (3.10), $p_1(\lambda t) = p_0$. Therefore

$$p_2(t) = p_0 + \frac{d_1+s_1}{d_2+s_2}\mathcal{S}^{-1}\left[u\left(\mathcal{S}[p_1(\lambda t)] - \frac{d_0+s_0}{d_1+s_1}\right)\right]$$

$$= p_0 + \frac{d_1+s_1}{d_2+s_2}\mathcal{S}^{-1}\left[u\left(p_0 - \frac{d_0+s_0}{d_1+s_1}\right)\right]$$

$$= p_0 + \frac{d_1+s_1}{d_2+s_2}\left(p_0 - \frac{d_0+s_0}{d_1+s_1}\right)t.$$

Notice that $p_2(\lambda t) = p_0 + \lambda\frac{d_1+s_1}{d_2+s_2}\left(p_0 - \frac{d_0+s_0}{d_1+s_1}\right)t$, therefore

$$p_3(t) = p_0 + \frac{d_1+s_1}{d_2+s_2}\mathcal{S}^{-1}\left[u\left(\mathcal{S}[p_2(\lambda t)] - \frac{d_0+s_0}{d_1+s_1}\right)\right]$$

$$= p_0 + \frac{d_1+s_1}{d_2+s_2}\mathcal{S}^{-1}\left[u\left(\mathcal{S}\left[p_0 + \lambda\frac{d_1+s_1}{d_2+s_2}\left(p_0 - \frac{d_0+s_0}{d_1+s_1}\right)t\right] - \frac{d_0+s_0}{d_1+s_1}\right)\right]$$

$$= p_0 + \frac{d_1+s_1}{d_2+s_2}\mathcal{S}^{-1}\left[u\left(p_0 + \lambda\frac{d_1+s_1}{d_2+s_2}\left(p_0 - \frac{d_0+s_0}{d_1+s_1}\right)u - \frac{d_0+s_0}{d_1+s_1}\right)\right]$$

$$= p_0 + \frac{d_1+s_1}{d_2+s_2}\mathcal{S}^{-1}\left[\left(p_0 - \frac{d_0+s_0}{d_1+s_1}\right)u + \lambda\frac{d_1+s_1}{d_2+s_2}\left(p_0 - \frac{d_0+s_0}{d_1+s_1}\right)u^2\right]$$

$$= p_0 + \frac{d_1+s_1}{d_2+s_2}\left(p_0 - \frac{d_0+s_0}{d_1+s_1}\right)t + \lambda\left(\frac{d_1+s_1}{d_2+s_2}\right)^2\left(p_0 - \frac{d_0+s_0}{d_1+s_1}\right)\frac{t^2}{2!}.$$

Notice that $p_3(\lambda t) = p_0 + \lambda\frac{d_1+s_1}{d_2+s_2}\left(p_0 - \frac{d_0+s_0}{d_1+s_1}\right)t + \lambda^3\left(\frac{d_1+s_1}{d_2+s_2}\right)^2\left(p_0 - \frac{d_0+s_0}{d_1+s_1}\right)\frac{t^2}{2!}$, therefore

$$p_4(t) = p_0 + \frac{d_1+s_1}{d_2+s_2}\mathcal{S}^{-1}\left[u\left(\mathcal{S}[p_3(\lambda t)] - \frac{d_0+s_0}{d_1+s_1}\right)\right]$$

$$= p_0 + \frac{d_1+s_1}{d_2+s_2}\mathcal{S}^{-1}\left[u\left(\mathcal{S}\left[\begin{array}{l} p_0 + \lambda\frac{d_1+s_1}{d_2+s_2}\left(p_0 - \frac{d_0+s_0}{d_1+s_1}\right)t \\ +\lambda^3\left(\frac{d_1+s_1}{d_2+s_2}\right)^2\left(p_0 - \frac{d_0+s_0}{d_1+s_1}\right)\frac{t^2}{2!}\end{array}\right] - \frac{d_0+s_0}{d_1-s_1}\right)\right]$$

$$= p_0 + \frac{d_1+s_1}{d_2+s_2}\mathcal{S}^{-1}\left[u\left(\begin{array}{l} p_0 + \lambda\frac{d_1+s_1}{d_2+s_2}\left(p_0 - \frac{d_0+s_0}{d_1+s_1}\right)u \\ +\lambda^3\left(\frac{d_1+s_1}{d_2+s_2}\right)^2\left(p_0 - \frac{d_0+s_0}{d_1+s_1}\right)u^2\end{array} - \frac{d_0+s_0}{d_1-s_1}\right)\right]$$

$$= p_0 + \frac{d_1+s_1}{d_2+s_2}\mathcal{S}^{-1}\left[\begin{array}{l}\left(p_0 - \frac{d_0+s_0}{d_1-s_1}\right)u + \lambda\frac{d_1+s_1}{d_2+s_2}\left(p_0 - \frac{d_0+s_0}{d_1+s_1}\right)u^2 \\ +\lambda^3\left(\frac{d_1+s_1}{d_2+s_2}\right)^2\left(p_0 - \frac{d_0+s_0}{d_1+s_1}\right)u^3\end{array}\right]$$

$$= p_0 + \frac{d_1+s_1}{d_2+s_2}\left(p_0 - \frac{d_0+s_0}{d_1-s_1}\right)t + \lambda\left(\frac{d_1+s_1}{d_2+s_2}\right)^2\left(p_0 - \frac{d_0+s_0}{d_1+s_1}\right)\frac{t^2}{2!}$$
$$+\lambda^3\left(\frac{d_1+s_1}{d_2+s_2}\right)^3\left(p_0 - \frac{d_0+s_0}{d_1+s_1}\right)\frac{t^3}{3!}.$$

Notice that

$$p(\lambda t) = p_0 + \lambda\frac{d_1+s_1}{d_2+s_2}\left(p_0 - \frac{d_0+s_0}{d_1-s_1}\right)t + \lambda^3\left(\frac{d_1+s_1}{d_2+s_2}\right)^2\left(p_0 - \frac{d_0+s_0}{d_1+s_1}\right)\frac{t^2}{2!}$$
$$+\lambda^6\left(\frac{d_1+s_1}{d_2+s_2}\right)^3\left(p_0 - \frac{d_0+s_0}{d_1+s_1}\right)\frac{t^3}{3!},$$

therefore

$$p_5(t) = p_0 + \frac{d_1+s_1}{d_2+s_2}\mathcal{S}^{-1}\left[u\left(\mathcal{S}[p_4(\lambda t)] - \frac{d_0+s_0}{d_1+s_1}\right)\right]$$

$$= p_0 + \frac{d_1+s_1}{d_2+s_2}\mathcal{S}^{-1}\left[u\left(\mathcal{S}\left[\begin{array}{c} p_0 + \lambda\frac{d_1+s_1}{d_2+s_2}\left(p_0 - \frac{d_0+s_0}{d_1-s_1}\right)t \\ +\lambda^3\left(\frac{d_1+s_1}{d_2+s_2}\right)^2\left(p_0 - \frac{d_0+s_0}{d_1+s_1}\right)\frac{t^2}{2!} \\ +\lambda^6\left(\frac{d_1+s_1}{d_2+s_2}\right)^3\left(p_0 - \frac{d_0+s_0}{d_1+s_1}\right)\frac{t^3}{3!} \end{array}\right] - \frac{d_0+s_0}{d_1+s_1}\right)\right]$$

$$= p_0 + \frac{d_1+s_1}{d_2+s_2}\mathcal{S}^{-1}\left[u\left(\begin{array}{c} p_0 + \lambda\frac{d_1+s_1}{d_2+s_2}\left(p_0 - \frac{d_0+s_0}{d_1-s_1}\right)u \\ +\lambda^3\left(\frac{d_1+s_1}{d_2+s_2}\right)^2\left(p_0 - \frac{d_0+s_0}{d_1+s_1}\right)u^2 \\ +\lambda^6\left(\frac{d_1+s_1}{d_2+s_2}\right)^3\left(p_0 - \frac{d_0+s_0}{d_1+s_1}\right)u^3 - \frac{d_0+s_0}{d_1+s_1} \end{array}\right)\right]$$

$$= p_0 + \frac{d_1+s_1}{d_2+s_2}\mathcal{S}^{-1}\left[\begin{array}{c} \left(p_0 - \frac{d_0+s_0}{d_1+s_1}\right)u + \lambda\frac{d_1+s_1}{d_2+s_2}\left(p_0 - \frac{d_0+s_0}{d_1-s_1}\right)u^2 \\ +\lambda^3\left(\frac{d_1+s_1}{d_2+s_2}\right)^2\left(p_0 - \frac{d_0+s_0}{d_1+s_1}\right)u^3 \\ +\lambda^6\left(\frac{d_1+s_1}{d_2+s_2}\right)^3\left(p_0 - \frac{d_0+s_0}{d_1+s_1}\right)u^4 \end{array}\right]$$

$$= p_0 + \tfrac{d_1+s_1}{d_2+s_2}\left(p_0 - \tfrac{d_0+s_0}{d_1+s_1}\right)t + \lambda\left(\tfrac{d_1+s_1}{d_2+s_2}\right)^2\left(p_0 - \tfrac{d_0+s_0}{d_1-s_1}\right)\tfrac{t^2}{2!}$$
$$+\lambda^3\left(\tfrac{d_1+s_1}{d_2+s_2}\right)^3\left(p_0 - \tfrac{d_0+s_0}{d_1+s_1}\right)\tfrac{t^3}{3!} + \lambda^6\left(\tfrac{d_1+s_1}{d_2+s_2}\right)^4\left(p_0 - \tfrac{d_0+s_0}{d_1+s_1}\right)\tfrac{t^4}{4!}.$$

Hence, it can be deduced that

$$\begin{cases} p_1(t) = p_0, \\ p_n(t) = p_0 + \left(p_0 + \dfrac{d_0+s_0}{d_1+s_1}\right)\displaystyle\sum_{k=1}^{n-1}\lambda^{\frac{k}{2}(k-1)}\left(\frac{d_1+s_1}{d_2+s_2}\right)^{k-1} t^{k-1}, n > 1, \\ p(t) = \lim\limits_{n\to\infty} p_n(t), \quad n \in N. \end{cases}$$

We assign the real values to the constants as follow $d_0 = 10$, $d_1 = 14$, $d_2 = 16, s_0 = 100, s_1 = 97$ *and* $s_2 = 96$, for the parameters of the PAMs. Figure 2 compares three different PAMs. It displays the graphs for the solutions of (3.5), (3.7) and (3.8) when $\lambda = 1/2$. The paper compares the iterations of (3.11) in Figure 3. It displays the iterations of PAM that involves delay and expectations of the agents. The paper shows how (3.11) varies with λ in Figure 4. It shows the trend of the PAM that involves delay and expectations of the agents as associated proportional delay $'\lambda'$ varies.

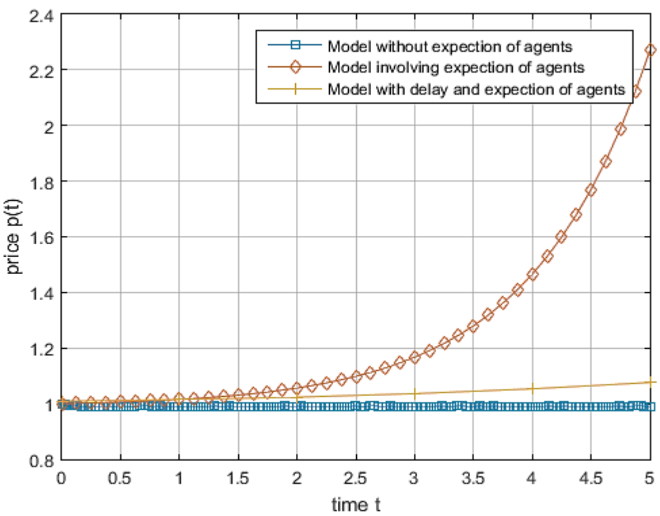

Figure 2: Comparison of three different price adjustment models.

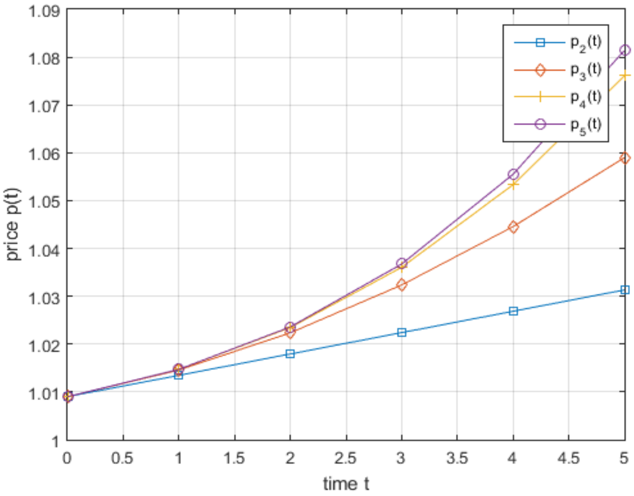

Figure 3: Iterations of price adjustment model that involves delay and expectations of the agents.

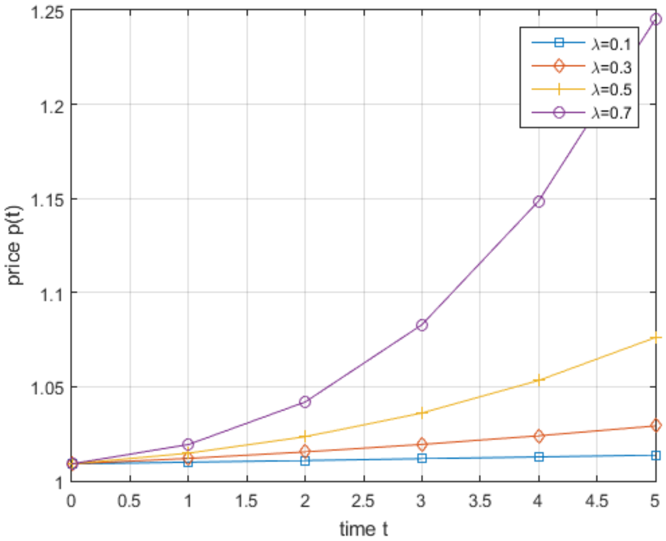

Figure 4: Effects of variation of the proportional delay $'\lambda'$ on (3.11).

4. Conclusion

This paper has presented an approximate analytic technique, which is a blend of the variational iterative method with the ST for solving linear and nonlinear problems. It is an approximate analytic technique that is efficient in computational time and accuracy. The paper considered n^{th}-order ODEs with varying delay proportional to the independent variable. The paper has presented a subtle way to obtain the Lagrange multipliers and subsequently the solutions of the mathematical models. Obtaining the optimal solutions is always the goal in mathematical modelling and the results of this study can be of great help in achieving the goal. An application is considered by applying the study to PAMs, where a new model is proposed by introducing a proportional delay into the formulation of PAM. The solution of the newly proposed PAM is obtainned and using Matlab, the paper compares the solution of the conventional PAM with the newly proposed PAM that is associated with delay by presenting their graphs.

Abbreviations

ODEs: Ordinary Differential Equations

PAMs: Price Adjustment Models

Conflicts of Interest

The authors declare no Conflicts of Interest.

Availability of Data and Materials

Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

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