DEFORMATION THEORY FOR FINITE CLUSTER COMPLEXES

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ABSTRACT. We study the deformation theory of the Stanley-Reisner rings associated to cluster complexes for skew-symmetrizable cluster algebras of geometric and finite cluster type. In particular, we show that in the skew-symmetric case, these cluster complexes are unobstructed, generalizing a result of Ilten and Christophersen in the A_n case.

We also study the connection between cluster algebras with universal coefficients and cluster complexes. We show that for a full rank positively graded cluster algebra $\mathcal{A}$ of geometric and finite cluster type, the cluster algebra $\mathcal{A}^{\mathrm{univ}}$ with universal coefficients may be recovered as the universal family over a partial closure of a torus orbit in a multigraded Hilbert scheme. Likewise, we show that under suitable hypotheses, the cluster algebra $\mathcal{A}^{\mathrm{univ}}$ may be recovered as the coordinate ring for a certain torus-invariant semiuniversal deformation of the Stanley-Reisner ring of the cluster complex.

We apply these results to show that for any cluster algebra $\mathcal{A}$ of geometric and finite cluster type, $\mathcal{A}$ is Gorenstein, and $\mathcal{A}$ is unobstructed if it is skew-symmetric. Moreover, if $\mathcal{A}$ has enough frozen variables then it has no nontrivial torus-invariant deformations. We also study the Gröbner theory of the ideal of relations among cluster and frozen variables of $\mathcal{A}$. As a byproduct we generalize previous results in this setting obtained by Bossinger, Mohammadi and Nájera Chávez for Grassmannians of planes and $\mathrm{Gr}(3,6)$.

CONTENTS

1. Introduction

1.1. **Unobstructed Simplicial Complexes.** The *dual n-associahedron* is the simplicial complex whose vertices are the diagonals in a regular n-gon, and whose faces are given by collections of pairwise non-crossing diagonals. This simplicial complex $\mathcal{K}$ is dual to the so-called Stasheff polytope. Ilten and Christophersen prove in [CI14] that the dual associahedron $\mathcal{K}$ has the remarkable property of being *unobstructed*, that is, the second cotangent cohomology $T^2(S_\mathcal{K})$ is zero, where $S_\mathcal{K}$ is the Stanley-Reisner ring of $\mathcal{K}$. See §2.1 and §2.2 for definitions.

In general, when a simplicial complex $\mathcal{K}$ is unobstructed, the projective scheme $\mathrm{Proj}\, S_\mathcal{K}$ corresponds to a smooth point of the Hilbert scheme parametrizing projective schemes with the same Hilbert polynomial as $S_\mathcal{K}$. This can be a very useful tool for studying Hilbert schemes, see for example [CI16]. The desire to better understand *why* the dual associahedron is unobstructed, and to find other natural examples of unobstructed simplicial complexes, is the starting point for this paper.

1.2. **Unobstructed Cluster Complexes.** The perspective we take in this paper is that the dual n-associahedron is the *cluster complex* for any cluster algebra of type A_n [FZ03c]. See §2.4 and §2.5 for definitions and notation for cluster algebras and the associated cluster complexes. We emphasize here that throughout the paper, all cluster algebras we consider are of *geometric type*. Our first result is the following:

Theorem 1.2.1 (Theorem 3.1.1 and Remark 3.1.2)**.** *Let $\mathcal{A}$ be a cluster algebra of finite cluster type. Then its cluster complex $\mathcal{K}$ is unobstructed if and only if $\mathcal{A}$ is skew-symmetric.*

In particular, we recover that the dual associahedron is unobstructed, making use of "cluster theoretic" arguments. While it is possible to prove this theorem using purely combinatorial arguments, we believe the most natural proof is the one we present here, which makes use of categorification.

In addition to the above unobstructedness result, we are also able to give a concrete description of the space of first order deformations $T^1(S_\mathcal{K})$ for $\mathcal{K}$ the cluster

complex of any skew-symmetrizable cluster algebra $\mathcal{A}$ of finite cluster type. See Theorem 3.2.2.

1.3. **Universal Coefficients.** The remaining results of the paper revolve around a fundamental geometric relationship that we will establish between a cluster algebra $\mathcal{A}$ of finite cluster type and the Stanley-Reisner ring $S_{\mathcal{K}}$ of its cluster complex $\mathcal{K}$. Recall that to any cluster algebra of finite cluster type $\mathcal{A}$, there is an associated cluster algebra $\mathcal{A}^{\mathrm{univ}}$ with *universal coefficients*, see Definition 4.1.9. The algebra $\mathcal{A}^{\mathrm{univ}}$ is an algebra over a polynomial ring $\mathbb{K}[t_1, \ldots, t_p]$, where p is the number of cluster variables of $\mathcal{A}$. Geometrically, this corresponds to a family

$$\mathrm{Spec}(\mathcal{A}^{\mathrm{univ}}) \to \mathbb{A}^p.$$

This leads to our second result:

Theorem 1.3.1 (Theorem 5.2.12). *Let $\mathcal{A}$ be a cluster algebra of finite cluster type. The fiber over zero of $\mathrm{Spec}\,\mathcal{A}^{\mathrm{univ}} \to \mathbb{A}^p$ is the scheme cut out by the Stanley-Reisner ideal of the cluster complex $\mathcal{K}$.*

A key ingredient for this theorem is the folklore result that for cluster algebras of finite type (with non-invertible frozen variables), the extended cluster monomials form a basis (see Theorem 4.3.1). This statement is a consequence of the machinery of [GHKK18].

Under some additional assumptions on $\mathcal{A}$, we are able to give new interpretations of the cluster algebra $\mathcal{A}^{\mathrm{univ}}$. The first shows that under appropriate hypotheses, $\mathcal{A}^{\mathrm{univ}}$ may be constructed from $\mathcal{A}$ by essentially acting on $\mathcal{A}$ by a torus:

Theorem 1.3.2 (Theorem 5.2.14). *Let $\mathcal{A}$ be a positively graded cluster algebra of finite cluster type and full rank. Then the family $\mathrm{Spec}\,\mathcal{A}^{\mathrm{univ}} \to \mathbb{A}^p$ arises as the restriction of the universal family for a multigraded Hilbert scheme to a partial closure of the torus orbit of the point corresponding to the algebra $\mathcal{A}$.*

An equivalent result was shown by Bossinger, Mohammadi, and Nájera Chávez [BMNC21] in the special cases when $\mathcal{A}$ is the coordinate ring of the Grassmannian $\mathrm{Gr}(2, n)$ or $\mathrm{Gr}(3, 6)$, see Remark 5.3.6.

In our second interpretation of $\mathcal{A}^{\mathrm{univ}}$, we show that under appropriate hypotheses, $\mathcal{A}^{\mathrm{univ}}$ can be reconstructed from the cluster complex $\mathcal{K}$ (and a little extra data) via deformation theory. We say that a cluster algebra $\mathcal{A}$ has *enough* frozen variables if properties T0* and T1 are satisfied, see Definitions 6.2.3 and 6.1.1.

Theorem 1.3.3 (Theorem 6.3.2). *Let $\mathcal{A}$ be a cluster algebra of finite cluster type with p cluster and q frozen variables, and assume it has enough frozen variables. Let $\mathbb{T}$ be the torus of the ambient space for the cluster embedding $\mathrm{Spec}\,\mathcal{A} \subseteq \mathbb{A}^{p+q}$, and $H \subseteq \mathbb{T}$ the maximal subgroup fixing $\mathrm{Spec}\,\mathcal{A}$. Then there is a canonical $\mathbb{T}$-equivariant H-invariant semiuniversal deformation $\mathcal{Y} \to \mathbb{A}^p$ of $(\mathrm{Spec}\,S_{\mathcal{K}}) \times \mathbb{A}^q$, and it may be canonically identified with the family $\mathrm{Spec}\,\mathcal{A}^{\mathrm{univ}} \to \mathbb{A}^p$.*

The slightly more general statement of Theorem 6.3.2 may be applied even when $\mathcal{A}$ does not have "enough" frozen variables. See §1.7 for an example illustrating the reconstruction of $\mathcal{A}^{\mathrm{univ}}$ from $\mathcal{K}$.

1.4. **Applications.** We may apply the theorems of the previous subsection to obtain new information about some algebraic properties of a cluster algebra of finite cluster type.

Corollary 1.4.1 (Corollary 5.2.13). *Let $\mathcal{A}$ be a cluster algebra of finite cluster type. Then $\mathcal{A}$ is Gorenstein.*

This was previously known in the case that $\mathcal{A}$ has no frozen variables, see the discussion following Corollary 5.2.13.

Corollary 1.4.2 (Corollaries 5.3.5 and 6.3.6). *Let $\mathcal{A}$ be a cluster algebra of finite cluster type.*
   (1) If $\mathcal{A}$ is skew-symmetric, then $T^2(\mathcal{A}) = 0$.
   (2) If $\mathcal{A}$ has full rank and satisfies the T1 property, then $T^1(\mathcal{A})^H = 0$, with H as in Theorem 1.3.3.

We also obtain information about the Gröbner fan for the ideal $I_{\mathcal{A}}$ of relations among the cluster and frozen variables of $\mathcal{A}$ (see §5.1).

Corollary 1.4.3 (Corollary 5.3.2). *The Gröbner fan of $I_{\mathcal{A}}$ has a maximal dimensional cone C corresponding to the Stanley-Reisner ideal $I_{\mathcal{K}}$. This cone may be described explicitly. If $\mathcal{A}$ has full rank (respectively full $\mathbb{Z}$-rank), the cone C is simplicial (respectively smooth) modulo lineality space.*

1.5. **Future Directions.** The interaction between the representation theory of associative algebras and the theory of cluster algebras is broad and fruitful. On the one hand, various fundamental concepts in cluster theory such as seeds, exchange relations, mutations, etc. can be *categorified* using module categories over associative algebras (see for example [BMR$^+$06, IY08, Kel10, Kel12, NC19]). On the other hand, various ideas and constructions within cluster theory admit a far reaching generalization within representation theory, giving rise to cluster-tilting theory and τ-tilting theory (see [AIR14, BM06, Tre21]).

Given any finite dimensional associative algebra A over a field $\mathbb{K}$ we can consider its τ-tilting complex $\Delta(A)$. This is a simplicial complex that generalizes the cluster complex arising in cluster theory. Its vertices are the indecomposable support τ-tilting A-modules; faces are given by collections of indecomposable support τ-tilting modules whose direct sum remains support τ-tilting. The τ-*tilting finite algebras* are those algebras A such that $\Delta(A)$ has only a finite number of vertices. In this case, Demonet, Iyama and Jasso showed in [DIJ19] that the geometric realization of $\Delta(A)$ is a sphere; this mirrors the situation for cluster complexes of finite type.

We believe that this striking similarity is not a coincidence. In the spirit of this paper, we thus propose studying the deformation theory of Stanley-Reisner rings of τ-tilting complexes associated to τ-tilting finite algebras. In particular, we suggest a systematic study of first order deformations, obstructions, and semiuniversal deformations. We expect to produce many more examples of unobstructed simplicial complexes in this way, explaining in particular some of the sequences of unobstructed simplicial complexes appearing in [CI14] via *unstarring*. Moreover, our hope is that we can construct via deformation theory an analogue of the "cluster algebra with universal coefficients" in the setting of τ-tilting finite algebras with unobstructed τ-tilting complex. It would be especially interesting to compare the (commutative) algebras that we obtain in this way with Caldero-Chapoton algebras [CILFS15, GLFS21], as these also provide a generalization of cluster algebras beyond the cluster setting.

Finally, we would like to extend this work to the setting of cluster algebras of infinite cluster type. However, Theorem 1.3.3 cannot be true as stated in the

case of infinite cluster type. Indeed, cluster monomials *do not* form a basis for cluster algebras of infinite cluster type, but they do still parametrize the standard monomial basis for $S_{\mathcal{K}}$. Nonetheless, one can imagine that the different approaches to the construction of canonical bases for cluster algebras might be helpful to adapt Theorem 1.3.3 to this setting [Lus00, GLS11, GLS12, GHKK18, Qin19].

1.6. **Organization.** We begin in §2 by recalling preliminaries and fixing notation. In particular, we introduce simplicial complexes, relevant notions from deformation theory, cluster algebras, and cluster complexes. None of this material is new.

In §3 we prove our first new results, showing that cluster complexes $\mathcal{K}$ for skew symmetric cluster algebras of finite cluster type are unobstructed, and giving a description of $T^1(S_{\mathcal{K}})$. Both these results rely heavily on Altmann and Christophersen's work on cotangent cohomology for Stanley-Reisner schemes [AC04, AC10]. The other essential ingredient is Lemma 3.1.5, which describes the intersection behaviour of certain links of the cluster complex $\mathcal{K}$.

In §4 we recall the notions of principal and universal coefficients for cluster algebras, as well as Grabowski's work on gradings of cluster algebras [Gra15]. We relate the degrees of cluster variables to **g**-vectors (Corollary 4.2.4), characterize positively graded cluster algebras in terms of their **g**-fan (Remark 4.2.6), and show that one can always add frozen variables to any cluster algebra of finite cluster type to obtain a positive grading (Lemma 4.2.10). We also use the results of [GHKK18] to prove the folklore result that any cluster algebra of finite type (with non-invertible frozen variables) has a basis given by extended cluster monomials (Theorem 4.3.1).

In §5 we dive into the connection between $\mathcal{A}^{\mathrm{univ}}$ and the multigraded Hilbert scheme parametrizing multigraded ideals with the same multigraded Hilbert function as the ideal of relations among the cluster and frozen variables of $\mathcal{A}$. After fixing notation in §5.1, we state and prove the main result in this direction in §5.2. The main ingredients are previous results on universal cluster algebras [FZ07, §12], the use of torus actions (or equivalently multigradings), and the basis of extended cluster monomials. In §5.3, we apply these results to study the Gröbner theory of $I_{\mathcal{A}}$.

Finally, in §6 we study the relationship between $\mathcal{A}^{\mathrm{univ}}$ and the semiuniversal deformation of the Stanley-Reisner scheme $(\mathrm{Spec}\, S_{\mathcal{K}}) \times \mathbb{A}^q$. For this, we first need to analyze the differential for the characteristic map of the family $\mathrm{Spec}\, \mathcal{A}^{\mathrm{univ}} \to \mathbb{A}^p$, and understand the behaviour of derivations on the Stanley-Reisner ideal $I_{\mathcal{K}}$.

Throughout the paper, we follow a running example of a cluster algebra of type A_2, see Examples 2.4.7, 2.5.3, 3.2.4, 4.1.10, 4.2.11, 5.1.5, 5.2.14, 5.3.3, 6.1.4, 6.2.5, and 6.3.3. We also consider an example of type G_2 immediately following in §1.7. *Macaulay2* code for both examples is found in Appendix A.

1.7. **An Example:** G_2. To conclude the introduction, we consider the example of the cluster algebra $\mathcal{A}$ of type G_2 with trivial coefficients. There are exactly eight cluster variables $v_1, \ldots, v_8$, and they are related by the exchange relations

$$v_{i-1}v_{i+1} = v_i + 1 \qquad i \text{ even};$$
$$v_{i-1}v_{i+1} = v_i^3 + 1 \qquad i \text{ odd}.$$

Here, indices are taken modulo 8. The cluster complex $\mathcal{K}$ is the boundary of an octagon. Letting z_i be the variable corresponding to the vertex v_i of the cluster

complex $\mathcal{K}$, the Stanley-Reisner ideal $I_\mathcal{K}$ is generated by

$$z_i z_{i+2}, \qquad z_i z_{i+3} \qquad i = 1, \ldots, 8$$
$$z_j z_{j+4} \qquad j = 1, \ldots, 4.$$

In this example, $\mathcal{A}$ does not have "enough" frozen variables (as it does not have any at all). We may nonetheless proceed as follows. The space $T^1(S_\mathcal{K})$ of first order deformations is infinite dimensional, but by Lemma 6.1.6, the image of the characteristic map of $\operatorname{Spec} \mathcal{A}^{\mathrm{univ}} \to \mathbb{A}^8$ in $T^1(S_\mathcal{K})$ consists of exactly the first order deformations given by the perturbation of $I_\mathcal{K}$ generated by

$$z_{i-1} z_{i+1} - t_i z_i \qquad i \text{ even};$$
$$z_{i-1} z_{i+1} - t_i z_i^3 \qquad i \text{ odd};$$
$$z_i z_{i+3}, \qquad z_j z_{j+4}$$

modulo $\langle t_1, \ldots, t_8 \rangle^2$.

One may compute a lifting of this family over $\mathbb{A}^8$ (using e.g. the *Macaulay2* package `VersalDeformations` [Ilt12]) to obtain

$z_{i-1} z_{i+1}$	$-t_i z_i$	$-t_{i+2} t_{i+3} t_{i+4}^2 t_{i+5} t_{i+6}$	i even;
$z_{i-1} z_{i+1}$	$-t_i z_i^3$	$-t_{i+2} t_{i+3}^3 t_{i+4}^2 t_{i+5}^3 t_{i+6}$	i odd;
$z_i z_{i+3}$	$-t_{i+1} t_{i+2} z_{i+2}^2$	$-t_{i+4} t_{i+5}^2 t_{i+6} t_{i+7} z_{i+4}$	i odd;
$z_i z_{i-3}$	$-t_{i-1} t_{i-2} z_{i-2}^2$	$-t_{i-4} t_{i-5}^2 t_{i-6} t_{i-7} z_{i-4}$	i odd;
$z_i z_{i+4}$	$-t_{i-1} t_{i-2} t_{i-3} z_{i-2}$	$-t_{i+1} t_{i+2} t_{i+3} z_{i+2}$	i odd;
$z_i z_{i+4}$	$-t_{i-1} t_{i-2}^3 t_{i-3} z_{i-2}$	$-t_{i+1} t_{i+2}^3 t_{i+3} z_{i+2} - 3 t_{i-2} t_{i+2} (t_1 \cdots t_8)$	i even.

See §A.1 for *Macaulay2* code computing this family. In this instance, the hypotheses of Theorem 6.3.21 are fulfilled. Furthermore, one may check that the above family is an *exchange minimal* family (see Definition 6.3.1). By Theorem 6.3.21 we may thus conclude that the above family is the universal cluster algebra $\mathcal{A}^{\mathrm{univ}}$ for $\mathcal{A}$.

Specializing all deformation parameters to 1, we obtain a presentation of the cluster algebra $\mathcal{A}$:

$z_{i-1} z_{i+1}$	$-z_i$	-1	i even;
$z_{i-1} z_{i+1}$	$-z_i^3$	-1	i odd;
$z_i z_{i+3}$	$-z_{i+2}^2$	$-z_{i+4}$	i odd;
$z_i z_{i-3}$	$-z_{i-2}^2$	$-z_{i-4}$	i odd;
$z_i z_{i+4}$	$-z_{i-2}$	$-z_{i+2}$	i odd;
$z_i z_{i+4}$	$-z_{i-2}$	$-z_{i+2} - 3$	i even.

We recover the exchange relations, but also additional relations between non-exchangeable cluster variables. In fact, the polynomials we have written down form a Gröbner basis (with initial ideal $I_\mathcal{K}$).

2. Preliminaries

Throughout, $\mathbb{K}$ is an algebraically closed field of characteristic zero. Unless otherwise stated, all tensor products are over $\mathbb{K}$.

2.1. Simplicial Complexes and Stanley-Reisner Rings.

A *simplicial complex* is a collection $\mathcal{K}$ of subsets of some fixed finite set, such that if $f \in \mathcal{K}$, then every subset of f is also in $\mathcal{K}$. Elements of $\mathcal{K}$ are called *faces* of $\mathcal{K}$. Faces of $\mathcal{K}$ consisting of a single element are called *vertices*; faces with two elements are *edges*. We denote by $\mathcal{V}(\mathcal{K})$ the union of all vertices of $\mathcal{K}$ (and will write $\mathcal{V}$ if $\mathcal{K}$ is clear from context). We note that $\mathcal{K}$ is a subset of $\mathfrak{P}(\mathcal{V}(\mathcal{K}))$, the power set of $\mathcal{V}(\mathcal{K})$.

To any simplicial complex $\mathcal{K}$, one may construct its *Stanley-Reisner ideal*

$$I_{\mathcal{K}} = \left\langle \prod_{v \in f} z_v \mid f \in \mathfrak{P}(\mathcal{V}) \setminus \mathcal{K} \right\rangle \subseteq \mathbb{K}[z_v \mid v \in \mathcal{V}]$$

and its *Stanley-Reisner ring*

$$S_{\mathcal{K}} = \mathbb{K}[z_v \mid v \in \mathcal{V}]/I_{\mathcal{K}}.$$

Since $I_{\mathcal{K}}$ is a monomial ideal, the ring $S_{\mathcal{K}}$ has a natural grading by $\mathbb{Z}^{\mathcal{V}}$. See [Sta96] for more details on Stanley-Reisner rings.

A simplicial complex $\mathcal{K}$ is a *flag complex* if all minimal non-faces have two elements, or equivalently, the ideal $I_{\mathcal{K}}$ is generated by quadrics. We call the minimal non-faces of a flag complex *non-edges*.

Given a face f of $\mathcal{K}$, the *link* of f in $\mathcal{K}$ is the simplicial complex

$$\mathrm{lk}(f, \mathcal{K}) = \{g \in \mathcal{K} \mid f \cap g = \emptyset \text{ and } f \cup g \in \mathcal{K}\}.$$

Given two simplicial complexes $\mathcal{K}, \mathcal{K}'$ with distinct vertex sets, their *join* is

$$\mathcal{K} * \mathcal{K}' = \{f \cup f' \mid f \in \mathcal{K}, f' \in \mathcal{K}'\}.$$

On the side of Stanley-Reisner rings, this corresponds to taking the tensor product:

$$S_{\mathcal{K}*\mathcal{K}'} \cong S_{\mathcal{K}} \otimes S_{\mathcal{K}'}.$$

To any simplicial complex $\mathcal{K}$ we may associate a topological space known as its *geometric realization*:

$$|\mathcal{K}| = \left\{ \alpha : \mathcal{V} \to \mathbb{R} \mid \mathrm{supp}(\alpha) \in \mathcal{K} \text{ and } \sum_{v \in V} \alpha(v) = 1 \right\}.$$

Here $\mathrm{supp}(\alpha) = \{v \in \mathcal{V} \mid \alpha(v) \neq 0\}$. We say that $\mathcal{K}$ is an *n-sphere* if $|\mathcal{K}|$ is piecewise-linear homeomorphic to the boundary of $|\mathfrak{P}(\{0, \dots, n+1\})|$.

2.2. **Cotangent Cohomology for Flag Complexes.** For any commutative $\mathbb{K}$-algebra S, there is a complex $\mathbb{L}_\bullet^{S/\mathbb{K}}$ of free S modules called the *cotangent complex*, see e.g. [And71, pp 34]. The ith *cotangent cohomology* of S is

$$T^i(S) = H^i(\mathrm{Hom}_S(\mathbb{L}_\bullet^{S/\mathbb{K}}, S)).$$

We will be particularly interested in T^i for $i \leq 2$, which may be understood in more concrete terms. For example, the module $T^0(S)$ is just the module $\mathrm{Der}_{\mathbb{K}}(S, S)$ of $\mathbb{K}$-linear derivations of S to itself.

The module $T^1(S)$ may also be understood quite concretely. Consider a presentation

$$0 \to I \to R \to S \to 0$$

of S with R a polynomial ring. We may identify the module $T^1(S)$ with the quotient of $\mathrm{Hom}_R(I, S)$ by the image of $\mathrm{Der}_{\mathbb{K}}(R, S)$ under the natural map [Har10, §1.3]. Elements of $T^1(S)$ are in bijection with isomorphism classes of *first order deformations* of S: flat $\mathbb{K}[\epsilon]/\epsilon^2$-algebras $\widetilde{S}$ whose reduction modulo ϵ is S.

When S is graded by an abelian group M, the S-modules $T^i(S)$ inherit a natural M-grading. In particular, taking a homogeneous presentation $R \to S$, we may represent a degree u element of $T^1(S)$ by an R-homomorphism $I \to S$ sending elements of degree v to degree $v + u$.

The module $T^2(S)$ also has a concrete description, for which we refer the reader to [Har10, §1.3]. Non-zero elements of $T^2(S)$ give *obstructions* to lifting deformations of S to higher order. In particular, if $T^2(S) = 0$, any deformation of S can be extended to arbitrary order. See [Har10] for a general introduction to deformation theory.

For $\mathcal{K}$ a simplicial complex, we will write $T^i(\mathcal{K}) := T^i(S_{\mathcal{K}})$.

Definition 2.2.1. A simplicial complex is *unobstructed* if $T^2(\mathcal{K}) = 0$.

The $S_{\mathcal{K}}$-modules $T^i(\mathcal{K})$ inherit the natural $\mathbb{Z}^{\mathcal{V}}$-grading of $S_{\mathcal{K}}$. Altmann and Christophersen [AC04] give a combinatorial description for the graded pieces $T^i(\mathcal{K})_{\mathbf{c}}$ for $\mathbf{c} \in \mathbb{Z}^{\mathcal{V}}$. We will not give a full account of this description here, but instead provide only the details necessary for our purposes.

We will always write $\mathbf{c} \in \mathbb{Z}^{\mathcal{V}}$ uniquely as $\mathbf{c} = \mathbf{a} - \mathbf{b}$ with $\mathbf{a}, \mathbf{b} \in \mathbb{Z}_{\geq 0}^{\mathcal{V}}$ having disjoint supports $a, b \subseteq \mathcal{V}$.

Theorem 2.2.2 ([AC04]). *For any simplicial complex $\mathcal{K}$ and $i = 1, 2$, $T^i(\mathcal{K})_{\mathbf{c}} = 0$ unless $a \in \mathcal{K}$, $\mathbf{b} \in \{0, 1\}^{\mathcal{V}}$, $b \neq \emptyset$, and b is contained in the set of vertices of $\mathrm{lk}(a, \mathcal{K})$. Furthermore, if $a \neq \emptyset$, there is a natural isomorphism*

$$T^i(\mathcal{K})_{\mathbf{c}} \cong T^i(\mathrm{lk}(a, \mathcal{K}))_{-\mathbf{b}}.$$

In this paper, we will be particularly interested in cotangent cohomology for flag simplicial complexes that are spheres.

Proposition 2.2.3. *Let $\mathcal{K}$ be a flag complex that is a sphere. Assume that for any non-edge $\{u, v\}$, $\mathrm{lk}(u, \mathcal{K}) \cap \mathrm{lk}(v, \mathcal{K})$ is either empty or contractible. Then for any $\mathbf{b} \in \{0, 1\}^{\mathcal{V}}$,*

$$T^1(\mathcal{K})_{-\mathbf{b}} \cong \begin{cases} \mathbb{K} & \textit{if } \mathcal{K} \textit{ is a 0-sphere and } b = \mathcal{V} \\ 0 & \textit{otherwise} \end{cases}$$

Proof. We begin by noting that $T^1(\mathcal{K})$ is a quotient of $\mathrm{Hom}(I_\mathcal{K}, S_\mathcal{K})$. Since $\mathcal{K}$ is a flag complex, $I_\mathcal{K}$ is generated by quadrics. It follows that if $T^1(\mathcal{K})_{-\mathbf{b}} \neq 0$, the support b satisfies $|b| \leq 2$ with $b \notin \mathcal{K}$ if $|b| = 2$.

Since $\mathcal{K}$ is a sphere, [AC10, Lemma 4.3] implies that $T^1(\mathcal{K})_{-\mathbf{b}}$ vanishes if b is a vertex, so we may reduce to the case that $b = \{u, v\}$ is a non-edge. Furthermore, by loc. cit. $\dim T^1(\mathcal{K})_{-\mathbf{b}} \leq 1$, so it only remains to characterize when $T^1(\mathcal{K})_{-\mathbf{b}}$ vanishes.

By [AC10, Theorem 4.6], $T^1(\mathcal{K})_{-\mathbf{b}} \neq 0$ if and only if $\mathcal{K}$ is the join of $\partial b := \mathfrak{P}(b) \setminus \{b\}$ with a sphere L. The complex L is necessarily $\mathrm{lk}(u, \mathcal{K}) \cap \mathrm{lk}(v, \mathcal{K})$. If this complex is contractible, then L is not a sphere, and $T^1(\mathcal{K})_{-\mathbf{b}} \neq 0$. If $\mathrm{lk}(u, \mathcal{K}) \cap \mathrm{lk}(v, \mathcal{K})$ is not contractible, then by assumption it is empty. Thus, $T^1(\mathcal{K})_{-\mathbf{b}} \neq 0$ if and only if $\mathcal{K} = \partial b$. The statement of the proposition follows. $\square$

Proposition 2.2.4. *Let $\mathcal{K}$ be a flag complex that is a sphere of dimension at least two. Then $\mathcal{K}$ is unobstructed if the following conditions are met:*

(1) For every non-empty face f of $\mathcal{K}$, $\mathrm{lk}(f, \mathcal{K})$ is unobstructed; and
(2) For any non-edge $\{u, v\}$, $\mathrm{lk}(u, \mathcal{K}) \cap \mathrm{lk}(v, \mathcal{K})$ is either empty or contractible.

Proof. This follows from [CI14, Proposition 4.3]. $\square$

Remark 2.2.5. Proposition 2.2.4 is *not* true in general if $\mathcal{K}$ is a 1-sphere. In fact, if $\mathcal{K}$ is the boundary of an m-gon, then $\mathcal{K}$ is unobstructed if and only if $m \leq 5$ [AC04, Example 17].

We will make use of the following lemma:

Lemma 2.2.6 ([CI14, Proposition 1.3]). *Let S and S' be $\mathbb{K}$-algebras. Then for all $i \geq 0$,*

$$T^i(S \otimes S') \cong T^i(S) \otimes S' \oplus T^i(S') \otimes S.$$

2.3. **Semiuniversal Deformations.** We again refer the reader to [Har10] for details on deformation theory. For invariant and equivariant deformations, see [Rim80].

Suppose that S is a quotient of a polynomial ring $\mathbb{K}[z_1, \ldots, z_n]$ by an ideal I. Let $Y = \mathrm{Spec}\, S \subseteq \mathbb{A}^n$. Let $\mathbb{T}$ be the maximal subtorus of $(\mathbb{K}^*)^n$ which sends Y to itself, and H any subgroup of $\mathbb{T}$. Recall that an *H-invariant deformation* of Y over a base Z is an H-invariant cartesian diagram

$$
\begin{array}{ccc}
Y & \longrightarrow & \mathcal{Y} \\
\downarrow & & \downarrow {\scriptstyle \pi} \\
\mathrm{Spec}\,\mathbb{K} & \longrightarrow & Z
\end{array}
$$

where π is flat, and H acts trivially on Z. Such a deformation is *embedded* if $\mathcal{Y} \subseteq \mathbb{A}^n \times Z$, π is just the projection, and the map $Y \to \mathcal{Y} \to \mathbb{A}^n \times Z$ is the natural inclusion induced by $\mathrm{Spec}\,\mathbb{K} \to Z$.

A *morphism* of H-invariant deformations from $\pi' : \mathcal{Y}' \to Z$ to $\pi : \mathcal{Y} \to Z'$ is a pair $(\phi, \widetilde{\phi})$ of H-invariant morphisms making

into a commutative diagram whose outside square is Cartesian. An isomorphism is a morphism such that $\phi, \widetilde{\phi}$ are both isomorphisms. For embedded deformations π', π, a *morphism of embedded deformations* is a morphism $(\phi, \widetilde{\phi})$ such that $\widetilde{\phi}$ is compatible with the inclusions $\mathcal{Y}' \subseteq \mathbb{A}^n \times Z'$, $\mathcal{Y} \subseteq \mathbb{A}^n \times Z$, and the map $\mathbb{A}^n \times Z' \to \mathbb{A}^n \times Z$ induced by ϕ. The map $\widetilde{\phi}$ is uniquely determined by ϕ, so we will often just refer to $\phi : Z' \to Z$ as a morphism of embedded deformations.

For any H-invariant deformation $\pi : \mathcal{Y} \to Z$ of Y with $P \in Y$ the image of $\operatorname{Spec} \mathbb{K}$, there is a natural map of $\mathbb{K}$-vector spaces

$$T_P Z \to T^1(S)^H$$

sending any tangent vector $\eta \in T_P Z$ to the isomorphism class of the first order deformation obtained by pulling back π along the morphism $\operatorname{Spec} \mathbb{K}[\epsilon]/\epsilon^2 \to Z$ corresponding to η. We call this map the *characteristic map* for the deformation π.

Definition 2.3.1. An H-invariant deformation $\pi : \mathcal{Y} \to Z$ of Y is *semiuniversal* if for any other H-invariant deformation $\pi' : \mathcal{Y}' \to Z'$ of Y with Z' the spectrum of a local Artinian ring, there is a morphism $(\phi, \widetilde{\phi})$ from π' to π, and the differential of ϕ at the distinguished point is uniquely determined.[1]

Under the condition that the H-invariant part $T^1(S)^H$ of $T^1(S)$ is finite dimensional, there is a semiuniversal H-invariant deformation $\pi : \mathcal{Y} \to Z$ of $Y = \operatorname{Spec} S$. We will call $\mathcal{Y}$ and Z respectively the *semiuniversal family* and the *semiuniversal base space*. There is a canonical identification of $T^1(S)^H$ with the tangent space of Z at P. By [Rim80], a semiuniversal deformation can be constructed so that it is also $\mathbb{T}$-equivariant. Furthermore, in this setting it can always be constructed as an embedded deformation. Such an embedded $\mathbb{T}$-equivariant deformation can be constructed algorithmically by lifting first-order perturbations of the elements of I to higher and higher order [Ste95, Ilt12] in a homogeneous fashion.

In general, there are two points of discomfort:

(1) $\mathcal{Y}$ and Z are in general formal schemes;
(2) Although any two semiuniversal deformations are isomorphic after passing to a formal completion, the isomorphism is not canonical. Likewise, the embedding of the semiuniversal family is not in general canonical.

---

[1]Technically, we have defined what it means to be *formally* semiuniversal. However, since this is the only notion of semiuniversality we are considering, we omit the word "formally".

However, in this paper we will always find ourselves to situations in which neither of these points is an issue.

Remark 2.3.2 (Algebraicity). Suppose that $\mathbb{T}$ contains a one-parameter subgroup λ acting on $\mathbb{A}^n$ with non-negative weights such that $T^1(S)^H$ has only strictly negative degrees with respect to λ. Then the above algorithmic construction will end after a finite number of steps [Ste95, pp 132], guaranteeing that we can take $\mathcal{Y}$ and Z to be algebraic, not just formal objects.

Remark 2.3.3 (Uniqueness). Let $\pi : \mathcal{Y} \to Z$ be a $\mathbb{T}$-equivariant H-invariant semi-universal deformation. Consider any $\mathbb{T}$-equivariant automorphism ϕ of this family; this induces a $\mathbb{T}$-equivariant automorphism of Z whose differential is the identity. Under appropriate hypotheses on the $\mathbb{T}$-action on $T^1(S)^H$, we may conclude that ϕ is the identity. This is the case, for example, if the degree of any homogeneous element of $T^1(S)^H$ is an indecomposable element of the semigroup generated by all degrees of $T^1(S)^H$. In such instances, any $\mathbb{T}$-equivariant semiuniversal deformation $\pi : \mathcal{Y}' \to Z'$ is induced from Z by a unique $\mathbb{T}$-equivariant map $Z' \to Z$.

We may obtain a unique $\mathbb{T}$-equivariant embedding of $\mathcal{Y}$ if we additionally require

(1) $\rho : \operatorname{Hom}(I, S)^H \to T^1(S)^H$ has a unique $\mathbb{T}$-equivariant section;
(2) The degree of any homogeneous element of $T^1(S)^H$ is an indecomposable element of the semigroup generated by all degrees of $T^1(S)^H$;
(3) Any nontrivial homogeneous element in the kernel of ρ is not in the semi-group generated by all degrees of $T^1(S)^H$.

Indeed, any two H-invariant small extensions of an embedded family $\mathcal{Y}$ over the spectrum of a local Artinian $\mathbb{K}$-algebra differ by an element of $\operatorname{Hom}(I, S)^H$. The first condition guarantees that a $\mathbb{T}$-equivariant $\mathcal{Y}$ is uniquely determined up to first-order. The latter two conditions guarantee that any $\mathbb{T}$-equivariant lifting to higher order must be unique.

Definition 2.3.4 (Property T0). The ideal $I \subset \mathbb{K}[z_1, \ldots, z_n] = R$ satisfies property T0 (with respect to $H \subseteq \mathbb{T}$) if the image of $\operatorname{Der}_\mathbb{K}(R, S)^H$ in $\operatorname{Hom}_R(I, S)$ is zero.

Concretely, I satisfies property T0 (with respect to H) if for every $1 \leq i \leq n$ and every monomial $z^\alpha \in R$ such that z^α and z_i have the same H-degree,

$$\frac{z^\alpha \partial f}{\partial z_i} \in I \text{ for all } f \in I.$$

We remark that property T0 implies the first and third conditions in Remark 2.3.3, since property T0 is equivalent to saying that the kernel of ρ is zero.

Remark 2.3.5 (Standard Action). Assume that $T^2(S)^H = 0$. Then a semiuniversal base space Z for Y is (the formal completion of) $\mathbb{A}^p$, where p is the dimension of $T^1(S)^H$. Indeed, because of the vanishing of T^2, there are no obstructions to lifting deformations to higher and higher order. In general, this identification is not canonical. See Remark 2.3.2 for a criteria to take Z and $\mathcal{Y}$ to be algebraic.

In the $\mathbb{T}$-equivariant setting above, the $\mathbb{T}$-action on Y induces a $\mathbb{T}$ action on this affine space $\mathbb{A}^p$. For a coordinate t of $\mathbb{A}^p$ corresponding to an element of $T^1(S)^H$ of degree $\mathbf{a}$, $\mathbb{T}$ acts on t with weight $-\mathbf{a}$. In particular, the action of $\mathbb{T}$ on $\mathbb{A}^p$ has a dense orbit if and only if $T^1(S)^H$ has a basis of semiinvariant functions whose degrees are linearly independent. In this case, we will say that $\mathbb{T}$ induces a *standard action* on Z.

2.4. **Cluster Algebras.** A *cluster algebra* is an integral domain equipped with a particular combinatorial structure. In the literature cluster algebras are usually defined as algebras over $\mathbb{Q}$, however, in this work we consider cluster algebras defined over $\mathbb{K}$. Since we are assuming that the characteristic of $\mathbb{K}$ is zero, a cluster algebra over $\mathbb{K}$ can be obtained from a cluster algebra over $\mathbb{Q}$ by taking the tensor product (over $\mathbb{Q}$) with $\mathbb{K}$. The reader is referred to [FZ02, FZ03a, FZ07, FWZ16, FWZ17, FWZ21] for more details on the general theory.

For positive integers n and m let $\mathrm{Mat}_{n \times m}(\mathbb{Z})$ denote the set of $n \times m$ matrices with integer entires. A square matrix $B = (b_{ij}) \in \mathrm{Mat}_{n \times n}(\mathbb{Z})$ is *skew-symmetrizable* if there exists $d_1, \ldots, d_n \in \mathbb{Z}_{>0}$ such that $d_i b_{ij} = -d_j b_{ji}$ for all i, j. If we can take $d_1, \ldots, d_n$ to be 1 we say that B is *skew-symmetric*.

From now on, we fix positive integers n and m such that $n \leq m$. We further fix a field $\mathcal{F}$ isomorphic to the field of rational functions over $\mathbb{K}$ in m variables.

Definition 2.4.1. A *labeled seed of geometric type* in $\mathcal{F}$ is a pair $(\widetilde{\mathbf{x}}, \widetilde{B})$, where

- $\widetilde{\mathbf{x}} = (x_1, \ldots, x_m)$ is an m-tuple of algebraically independent elements of $\mathcal{F}$ such that $\mathcal{F} = \mathbb{K}(x_1, \ldots, x_m)$;
- $\widetilde{B} \in \mathrm{Mat}_{m \times n}(\mathbb{Z})$ is such that its top $n \times n$ submatrix B is skew-symmetrizable.

We adopt the following nomenclature: n is the *rank* of $(\widetilde{\mathbf{x}}, \widetilde{B})^2$, $\widetilde{B}$ is an *extended exchange matrix*, B is an *exchange matrix*, $\widetilde{\mathbf{x}}$ is an *extended cluster*, the n-tuple $\mathbf{x} := (x_1, \ldots, x_n)$ is a *cluster* and $x_{n+1}, \ldots, x_m$ are the *frozen variables*.

In this paper we will only consider labeled seeds of geometric type. Hence, we will simply refer to these as seeds.

Remark 2.4.2. To any exchange matrix B we can associate a weighted directed graph $\Gamma(B)$ as follows. We take the graph $\Gamma(B)$ to be the graph whose vertices are $\{1, \ldots, n\}$, with an edge from i to j if and only if $b_{ij} > 0$. The *weight* of such an edge is $c_{ij} = |b_{ij}| \cdot |b_{ji}|$. In case $\widetilde{B}$ is skew-symmetric $\Gamma(\widetilde{B})$ can be thought of (and will be referred to) as a quiver (the weights correspond to the square of the number of arrows between the vertices).

Definition 2.4.3. Consider any exchange matrix B. The graph $\Gamma(B)$ decomposes into connected components $\Gamma_1, \ldots, \Gamma_k$. We let B_i be the submatrix of B whose row and column indices are among the vertices of Γ_i. We call $B_1, \ldots, B_k$ the *connected components* of B and say that B is indecomposable if $\Gamma(B)$ is connected.

For $k \in \{1, \ldots, n\}$, the *mutation in direction k* of a seed $(\widetilde{\mathbf{x}}, \widetilde{B})$ is the new seed $\mu_k(\widetilde{\mathbf{x}}, \widetilde{B}) = (\widetilde{\mathbf{x}}', \widetilde{B}')$, defined as follows:

- The new extended cluster is $\widetilde{\mathbf{x}}' = (x_1, \ldots, x_{k-1}, x_k', x_{k+1}, \ldots, x_m)$ where

$$(2.4.4) \qquad x_k x_k' = \prod_{i:b_{ik}>0} x_i^{b_{ik}} + \prod_{i:b_{ik}<0} x_i^{-b_{ik}}.$$

  An expression of the form (2.4.4) is called an *exchange relation* and the monomial $x_k x_k'$ is the associated *exchange monomial*.

- The new extended exchange matrix $\widetilde{B}' = (b_{ij}')$ is defined by the following rule

$$(2.4.5) \qquad b_{ij}' := \begin{cases} -b_{ij} & \text{if } i = k \text{ or } j = k, \\ b_{ij} + \mathrm{sgn}(b_{ik})\max(b_{ik}b_{kj}, 0) & \text{else;} \end{cases}$$

---

[2]Observe that this notion of rank has nothing to do with the rank of B or $\widetilde{B}$.

$$\text{where, for } x \in \mathbb{R} \text{ we let } \operatorname{sgn}(x) := \begin{cases} -1 & \text{if } x < 0, \\ 0 & \text{if } x = 0, \\ 1 & \text{if } x > 0. \end{cases}$$

One can show that $(\widetilde{\mathbf{x}}', \widetilde{B}')$ as constructed above is again a seed.

Two seeds are *mutation equivalent* if they can be linked to each other by a finite number of mutations. This is an equivalence relation in the set of seeds in $\mathcal{F}$ since μ_k is an involution, that is, we have

$$\mu_k(\mu_k(\widetilde{\mathbf{x}}, \widetilde{B})) = (\widetilde{\mathbf{x}}, \widetilde{B}).$$

Moreover, if $(\widetilde{\mathbf{x}}', \widetilde{B}')$ is a seed mutation equivalent to $(\widetilde{\mathbf{x}}, \widetilde{B})$ then $\widetilde{\mathbf{x}}'$ is of the form $(x_1', \ldots, x_n', x_{n+1}, \ldots, x_m)$ for some $x_i' \in \mathcal{F}$.

In order to parametrize the cluster variables and the (extended) exchange matrices associated to $\mathcal{A}(\widetilde{\mathbf{x}}, \widetilde{B})$ we label the seeds mutation equivalent to $(\widetilde{\mathbf{x}}, \widetilde{B})$ as follows. We consider the n-regular tree $\mathfrak{T}^n$ whose edges are labeled by the mutable directions $1, \ldots, n$, so that the n edges incident to each vertex carry different labels. Since μ is an involution we can assign (in different ways) a seed $(\widetilde{\mathbf{x}}_v, \widetilde{B}_v)$ to every vertex v of $\mathfrak{T}^n$ in such a way that if v and v' are vertices of $\mathfrak{T}^n$ joined by an edge labeled with k then $(\widetilde{\mathbf{x}}_{v'}, \widetilde{B}_{v'}) = \mu_k(\widetilde{\mathbf{x}}_v, \widetilde{B}_v)$. Any such labeling is completely determined by choosing an initial vertex v_0 of $\mathfrak{T}^n$ and setting $(\widetilde{\mathbf{x}}_{v_0}, \widetilde{B}_{v_0}) = (\widetilde{\mathbf{x}}, \widetilde{B})$. We fix once and for all the choice of v_0. Moreover, we let

$$\widetilde{\mathbf{x}}_v := (x_{1;v}, \ldots, x_{m;v}).$$

Observe that $x_{i;v} = x_i$ for all $n < i \le m$ and every vertex v of $\mathfrak{T}^n$.

Definition 2.4.6. Let $(\widetilde{\mathbf{x}}, \widetilde{B})$ be a seed in $\mathcal{F}$. The *cluster algebra* $\mathcal{A}(\widetilde{\mathbf{x}}, \widetilde{B})$ (of geometric type, over $\mathbb{K}$) is the $\mathbb{K}$-subalgebra of $\mathcal{F}$ generated by the elements of *all* the extended clusters $\widetilde{\mathbf{x}}_v$ in the seeds mutation equivalent to $(\widetilde{\mathbf{x}}, \widetilde{B})$. The elements $x_{i;v}$ for $i \le n$ of the clusters constructed in this way are the *cluster variables* of $\mathcal{A}(\widetilde{\mathbf{x}}, \widetilde{B})$. We call $(\widetilde{\mathbf{x}}, \widetilde{B})$ the *initial seed* and the elements of the cluster $\mathbf{x}$ the *initial cluster variables*. If $m > n$ we say that the cluster algebra $\mathcal{A}(\widetilde{\mathbf{x}}, \widetilde{B})$ has *coefficients* and $x_{n+1}, \ldots, x_m$ will be called the *frozen variables*.

Example 2.4.7 (Running Example)**.** We consider the cluster algebra $\mathcal{A}$ associated to the seed

$$\widetilde{B} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \\ 1 & 1 \\ -1 & -1 \\ 1 & -1 \end{pmatrix} \qquad \widetilde{\mathbf{x}} = (x_{13}, x_{14}, s_1, s_2, s_3).$$

The frozen variables are s_1, s_2, s_3. The cluster variables of $\mathcal{A}$ are

$$x_{13}, \qquad x_{14}, \qquad x_{24} := \frac{s_2 x_{14} + s_1 s_3}{x_{13}}$$

$$x_{35} := \frac{s_1 x_{13} + s_2 s_3}{x_{14}} \qquad x_{25} := \frac{x_{13} s_1^2 + x_{14} s_2^2 + s_1 s_2 s_3}{x_{13} x_{14}}$$

This cluster algebra is obtained from the coordinate ring of the Grassmannian $\mathrm{Gr}(2,5)$ in its Plücker embedding by setting x_{12} and x_{45} to be s_1, x_{23} and x_{15} to be s_2, and $x_{34} = s_3$.
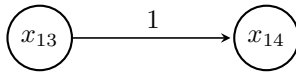
FIGURE 1. The weighted graph $\Gamma(B)$ for Example 2.4.7

The weighted directed graph $\Gamma(B)$ associated to the exchange matrix B is pictured in Figure 1. We will follow this example throughout the entire paper. □

Remark 2.4.8. In the literature, cluster algebras as introduced in Definition 2.4.6 are called skew-symmetrizable cluster algebras of geometric type over $\mathbb{K}$. We call a cluster algebra *skew-symmetric* if one (or equivalently all) of its seeds $(\widetilde{\mathbf{x}}, \widetilde{B})$ has skew-symmetric exchange matrix.

Remark 2.4.9. In the literature, cluster algebras with frozen variables $x_{n+1}, \ldots, x_m$ are often defined as algebras over the ring $\mathbb{K}[x_{n+1}^{\pm 1}, \ldots, x_m^{\pm 1}]$, that is, the cluster algebra also contains the inverse of each of the frozen variables. For our purposes, it is important that we do not follow this convention.

Remark 2.4.10. Let $\widetilde{B}$ be an extended exchange matrix (of size $m \times n$). It is straightforward to show that $\widetilde{B}$ can be extended (non-uniquely) to an $m \times m$ skew-symmetrizable matrix $\widetilde{B}'$. There is a canonical inclusion of $\mathcal{A}(\widetilde{B})$ into $\mathcal{A}(\widetilde{B}')$. Thus, any cluster algebra with frozen variables may be thought of as a subalgebra of a cluster algebra with no frozen variables, with the subalgebra generated by a subset of the cluster variables.

Up to a canonical isomorphism, $\mathcal{A}(\widetilde{\mathbf{x}}, \widetilde{B})$ is independent of $\widetilde{\mathbf{x}}$. Therefore, we will frequently write $\mathcal{A}(\widetilde{B})$ instead of $\mathcal{A}(\widetilde{\mathbf{x}}, \widetilde{B})$. The set of cluster variables associated to $\mathcal{A}(\widetilde{\mathbf{x}}, \widetilde{B})$ will be denoted by $\mathcal{V}(\widetilde{\mathbf{x}}, \widetilde{B})$. We will write $\mathcal{V}(\widetilde{B})$ or simply $\mathcal{V}$ to denote this set if the initial seed is clear from the context.

Theorem 2.4.11 (The strong Laurent phenomenon [FWZ16, Theorem 3.3.6])**.** *Let $(\widetilde{\mathbf{x}}', \widetilde{B}')$ be a seed mutation equivalent to $(\widetilde{\mathbf{x}}, \widetilde{B})$ and let $\mathbf{x}' = (x_1', \ldots, x_n')$ be the corresponding cluster. Then the cluster algebra $\mathcal{A}(\widetilde{\mathbf{x}}, \widetilde{B})$ is a $\mathbb{K}[x_{n+1}, \ldots, x_m]$-subalegbra of $\mathbb{K}[x_{n+1}, \ldots, x_m][x_1'^{\pm 1}, \ldots x_n'^{\pm 1}]$.*

Definition 2.4.12. A cluster algebra is of *finite cluster type* if its set of cluster variables is a finite set. Otherwise it is of *infinite cluster type.*

Remark 2.4.13. Observe that the strong Laurent phenomenon has the following geometric interpretation. If $\mathcal{A}(\widetilde{B})$ is of finite cluster type then we can consider $\mathrm{Spec}(\mathcal{A}(\widetilde{B}))$. By definition this an irreducible scheme over $\mathrm{Spec}(\mathbb{K})$. By Theorem 2.4.11 we can consider $\mathrm{Spec}(\mathcal{A}(\widetilde{B}))$ as a scheme over $\mathrm{Spec}(\mathbb{K}[\mathbf{t}])$, for any $\mathbf{t} \subseteq \{x_{n+1}, \ldots x_m\}$.

The *Cartan counterpart* of an exchange matrix B is the square matrix $A(B) = (a_{ij})$ of the same size given by $a_{ij} = 2$ if $i = j$ and $a_{ij} = -|b_{ij}|$ if $i \neq j$. A Cartan matrix $A(B)$ is *indecomposable* if $\Gamma(B)$ is connected; an indecomposable Cartan matrix is of *finite type* if all the principal minors of $A(B)$ are positive. It is a classic fact in the theory of Lie algebras that indecomposable Cartan matrices of finite type are classified by Dynkin diagrams.

Theorem 2.4.14 (Finite type classification [FZ03a]). *A cluster algebra $\mathcal{A}$ is of finite cluster type if and only if $\mathcal{A}$ has an extended exchange matrix $\widetilde{B}$ such that for each connected component B_i of B the Cartan matrix $A(B_i)$ is of finite type.*

2.5. **The Cluster Complex.** To any cluster algebra $\mathcal{A} = \mathcal{A}(\widetilde{B})$ we can associate a simplicial complex $\mathcal{K}(\mathcal{A}) = \mathcal{K}(\widetilde{B})$ called the *cluster complex*.

- The set vertices of $\mathcal{K}(\widetilde{B})$ is the set of cluster variables $\mathcal{V}$ associated to $\mathcal{A}(\widetilde{B})$;
- A subset of cluster variables $\{u_1, \ldots, u_i\}$ is a face of $\mathcal{K}(\widetilde{B})$ if and only if there exists a cluster $\mathbf{u}$ containing $u_1, \ldots, u_i$ as entries.

Cluster variables belonging to a common face of $\mathcal{K}(\widetilde{B})$ are called *compatible*.

It follows directly from the above definition that this complex is a flag complex. Likewise, it is straightforward to verify that if $\widetilde{B}$ and $\widetilde{B}'$ are extended exchange matrices that can be related by iterated mutation, then $\mathcal{K}(\widetilde{B})$ and $\mathcal{K}(\widetilde{B}')$ are canonically isomorphic. A much deeper fact is that the cluster complex only depends on the mutable part of an extended exchange matrix. In other words, we have that $\mathcal{K}(\widetilde{B})$ is canonically isomorphic to $\mathcal{K}(B)$. This is an instance of the synchronicity phenomenon in cluster patterns studied in [Nak21b] and follows directly from [CL20, Proposition 6.1].

Theorem 2.5.1. [CL20] *Let $\mathcal{V}(B) = \{x_{i;v} \mid 1 \leq i \leq n, v \in (\mathfrak{T}^n)_0\}$ and $\mathcal{V}(\widetilde{B}) = \{\tilde{x}_{i;v} \mid 1 \leq i \leq n, v \in (\mathfrak{T}^n)_0\}$ be the sets of cluster variables of $\mathcal{A}(B)$ and $\mathcal{A}(\widetilde{B})$, respectively. Then the assignment $x_{i;v} \mapsto \tilde{x}_{i;v}$ establishes a bijection from $\mathcal{V}(B)$ to $\mathcal{V}(\widetilde{B})$. In particular, two cluster variables $x_{i;v}$ and $x_{j;v'}$ are compatible if and only if $\tilde{x}_{i;v}$ and $\tilde{x}_{j;v'}$ are compatible.*

In [FZ03c] the authors study cluster complexes associated to cluster algebras of finite cluster type. In particular, they introduced a beautiful approach to understand the compatibility of cluster variables in terms of the associated root system. For cluster algebras of type A the associated cluster complex is the simplicial complex dual to the boundary of the classical associahedron, also known as the Stasheff Polytope. More generally, we have the following:

Theorem 2.5.2 ([FZ03c, Corollary 1.11]). *Let $\mathcal{A}(\widetilde{B})$ be a cluster algebra of rank n and finite cluster type. Then the cluster complex $\mathcal{K}(\widetilde{B})$ is an $(n-1)$-sphere.*

Example 2.5.3 (Continuation of Running Example 2.4.7). We consider the cluster algebra $\mathcal{A}$ from Example 2.4.7. This is a cluster algebra of type A_2. The cluster complex $\mathcal{K}$ for $\mathcal{A}$ is the boundary of a pentagon, with vertices labeled as pictured in Figure 2. $\qquad\square$

3. Cotangent Cohomology for Cluster Complexes

3.1. **Unobstructedness.**

Theorem 3.1.1. *Let $\mathcal{K}$ be the cluster complex of a skew-symmetric cluster algebra $\mathcal{A}$ of finite cluster type. Then $\mathcal{K}$ is unobstructed, that is, $T^2(\mathcal{K}) = 0$.*

Remark 3.1.2. The skew-symmetric cluster algebras of finite cluster type are exactly those with a seed $(\widetilde{\mathbf{x}}, \widetilde{B})$ such that the Cartan counterpart of each connected component of B corresponds to a simply laced Dynkin diagram, that is, of type ADE. However, for cluster algebras having a seed whose exchange matrix has a
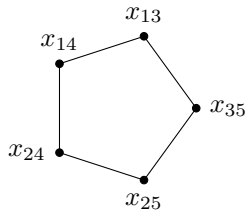
FIGURE 2. The cluster complex $\mathcal{K}$ from Example 2.4.7

connected component whose Cartan counterpart corresponds to a non-simply laced Dynkin diagram, one can see that $T^2(\mathcal{K}) \neq 0$. In the rank 2 case this follows from Remark 2.2.5 and the fact that a cluster complex of type B_2 or C_2 is a hexagon (see Example 6.3.7) and of type G_2 is an octagon (see Example 1.7). Cluster complexes of rank 2 arise as links of certain faces of cluster complexes of higher rank (see Lemma 3.1.4 below). Combining this with Theorem 2.2.2 shows that skew-symmetrizable but non-skew-symmetric cluster complexes of arbitrary rank are obstructed. In fact, we will see that there is a "geometric" reason for this, see Example 6.3.7.

Although we are proving the theorem for skew-symmetric cluster algebras, we will later use several of the following lemmas without this restriction.

Lemma 3.1.3. *Let $B_1, \ldots, B_k$ be the connected components of a skew-symmetrizable matrix B. Then*

$$\mathcal{K}(B) = \mathcal{K}(B_1) * \cdots * \mathcal{K}(B_k).$$

Proof. This follows from the definition of the cluster complex. Any sequence of mutations at indices in B_i do not change the B_j for $j \neq i$. Likewise, none of the associated cluster variables are changed. Hence, any cluster variable created by iterated mutation in one component is compatible with any other created by iterated mutation in some other component. □

Lemma 3.1.4. *Let $(\widetilde{\mathbf{x}}, \widetilde{B})$ be a seed and $f \in \mathcal{K}(B)$ a subset of the elements of $\mathbf{x}$. Let B' be the matrix obtained from $Q(B)$ by removing the rows and columns indexed by $\{i \mid x_i \in f\}$. Then*

$$\mathrm{lk}(f, \mathcal{K}(B)) \cong \mathcal{K}(B').$$

Proof. It is clear from the definition of the cluster complex that $\mathcal{K}(B')$ can be identified with a subcomplex $\mathcal{K}'$ of $\mathrm{lk}(f, \mathcal{K}(B))$. By [CL20, Theorem 6.2], any two clusters containing a given fixed subset of cluster variables can be linked to each other via mutations that do not involve any exchange of the fixed cluster variables. It follows that $\mathcal{K}'$ is the entire link. □

The following lemma is key:

Lemma 3.1.5. *Let $\mathcal{A}$ be a cluster algebra of finite cluster type such that some (or equivalently all) seeds $(\widetilde{\mathbf{x}}, \widetilde{B})$ satisfy that $\Gamma(B)$ is connected. For any two vertices u, v of $\mathcal{K}(\mathcal{A})$ that do not form an edge,*

$$\mathrm{lk}(u, \mathcal{K}(\mathcal{A})) \cap \mathrm{lk}(v, \mathcal{K}(\mathcal{A}))$$

is either contractible or empty.

In the A_n case, this lemma is proven combinatorially in [CI14, Lemma 5.2]. For the B_n, C_n, and D_n cases, one may prove the lemma similarly using a combinatorial model for the type B, C, and D associahedra [FZ03c, Propositions 3.15 and 3.16]. It remains to check the E_6, E_7, E_8, F_4 and G_2 cases. Here, we will instead provide a uniform proof using categorification. We refer the reader to [BMR$^+$06, Dem11, FK10, Kel10] for background and notation.

Proof of Lemma 3.1.5. In view of [Dem11, Theorem 4.47] and [GLS10, Theorem 2.1] there is a triple $(\mathcal{C}, G, \mathcal{T})$ that categorifies a cluster algebra of geometric type $\mathcal{A}(\mathcal{C}, G, \mathcal{T})$ whose cluster type is the same as $\mathcal{A}$ (see [Dem11, §3.5 and §4.3]). Here the category $\mathcal{C}$ is a Frobenius categorification of a skew-symmetric cluster algebra $\mathcal{A}(\mathcal{C})$ of geometric and finite cluster type, G is a finite group acting on $\mathcal{C}$, the action is 2-Calabi-Yau and $\mathcal{T}$ is a G-stable cluster-tilting subcategory of $\mathcal{C}$. In the skew-symmetric case G is the trivial group and $\mathcal{A}(\mathcal{C}) = \mathcal{A}(\mathcal{C}, G, \mathcal{T})$. Since the cluster complex is independent of the choice of coefficients it is enough to show the statement for $\mathcal{A}(\mathcal{C}, G, \mathcal{T})$. We now consider the equivariant category $\mathcal{C}G$ in the sense of [Dem11, §2]. The properties of $\mathcal{C}G$ that we will use are the following.

- The category $\mathcal{C}G$ is Hom-finite, Krull-Schmidt, Frobenius and stably 2-Calabi-Yau. In particular, $\mathrm{Ext}^1_{\mathcal{C}G}(X, Y) \cong \mathrm{Hom}_{\mathbb{K}}(\mathrm{Ext}^1_{\mathcal{C}G}(Y, X), \mathbb{K})$ for all objects X and Y of $\mathcal{C}G$;
- there is a bijection $v \mapsto T_v$ between the set of cluster and frozen variables associated to $\mathcal{A}(\mathcal{C}, G, \mathcal{T})$ and the isomorphism classes of indecomposable objects of $\mathcal{C}G$. Under this bijection frozen variables correspond to the classes of the indecomposable projective-injective objects;
- two cluster variables u and v of $\mathcal{A}(\mathcal{C}, G, \mathcal{T})$ are compatible if and only if $\mathrm{Ext}^1_{\mathcal{C}G}(T_u, T_v) = 0$. Moreover, if $\mathrm{Hom}_{\underline{\mathcal{C}G}}(T_u, T_v) \neq 0$ then u and v are compatible, where $\underline{\mathcal{C}G}$ is the triangulated category obtained from $\mathcal{C}G$ by quotienting out the morphisms that factor through projective objects.

The first two points are contained in [Dem11]. The last one is well-known in the skew-symmetric case and can be extended to the skew-symmetrizable one using the fact that $\mathrm{Hom}_{\underline{\mathcal{C}G}}(T_u, T_v) \subseteq \mathrm{Hom}_{\underline{\mathcal{C}}}(F(T_u), F(T_v))$ and $\mathrm{Ext}^1_{\mathcal{C}G}(T_u, T_v) \subseteq \mathrm{Ext}^1_{\mathcal{C}}(F(T_u), F(T_v))$, where $F : \mathcal{C}G \to \mathcal{C}$ is the forgetful functor.

Consider non-compatible cluster variables u and v of $\mathcal{A}(\mathcal{C}, G, \mathcal{T})$ and set $\mathcal{K} = \mathcal{K}(\mathcal{A}(\mathcal{C}, G, \mathcal{T}))$. Then

$$\mathrm{Ext}^1_{\mathcal{C}G}(T_v, T_u) \neq 0 \qquad \text{and} \qquad \mathrm{Ext}^1_{\mathcal{C}G}(T_u, T_v) \neq 0.$$

Let $0 \to T_v \to E \to T_u \to 0$ and $0 \to T_u \to E' \to T_v \to 0$ be non-zero elements of $\mathrm{Ext}^1_{\mathcal{C}G}(T_u, T_v)$ and $\mathrm{Ext}^1_{\mathcal{C}G}(T_v, T_u)$, respectively.

Assume first that both E and E' are projective. Then $T_u \cong T_v[1]$ and $T_v \cong T_u[1]$ in $\underline{\mathcal{C}G}$, where $[1] : \underline{\mathcal{C}G} \to \underline{\mathcal{C}G}$ is the suspension functor. This implies that T_u is isomorphic to $T_u[2]$. We claim this can only happen if $\Gamma(B)$ has a connected component that consists of a single vertex, in which case it is easily verified that $\mathrm{lk}(u, \mathcal{K}(\mathcal{A})) \cap \mathrm{lk}(v, \mathcal{K}(\mathcal{A})) = \emptyset$.

Indeed, we can identify $\underline{\mathcal{C}}$ with a cluster category associated to a connected Dynkin quiver. Namely, $\underline{\mathcal{C}} \cong \mathcal{D}^b_Q/[1] \circ \tau^{-1}$, where $\mathcal{D}^b_Q$ is the derived category of a Dynkin quiver and τ is its Auslander-Reiten translation. Since the action of G preserves bipartite slices of $\mathcal{C}$, we have that $F(T_u)$ is a sum of indecomposable objects lying in the same bipartite slice of $\underline{\mathcal{C}}$. In particular, if $F(T_u)$ is invariant under $[2]$ then an indecomposable summand W of $F(T_u)$ corresponds to an indecomposable

object of $\mathcal{D}_Q^b$ such that $\tau^2 \circ [1] \circ \tau^{-1}(W) = \tau W[1]$ lies in the same bipartite slice of $\mathcal{D}_Q^b$ as W. This is only possible if the Coxeter number of the Dynkin diagram underlying Q is 2. Therefore Q is of type A_1, and $\Gamma(B)$ must consist of a single vertex.

We may thus assume that at least one of E and E' has non-projective indecomposable summands. So without loss of generality let $E = P \oplus Z$, where P is projective and Z is non-zero and has no projective direct summands. Let $T_{z_1}, \ldots, T_{z_s}$ be the indecomposable direct summands of Z satisfying that $\mathrm{Hom}_{\underline{\mathcal{C}G}}(T_v, T_{z_i})$ and $\mathrm{Hom}_{\underline{\mathcal{C}G}}(T_{z_i}, T_u)$ are non-zero. Observe that there is at least one such summand since $T_v \to Z \to T_u \to T_v[1]$ is a non-zero element of $\mathrm{Ext}^1_{\underline{\mathcal{C}G}}(T_u, T_v) \cong \mathrm{Ext}^1_{\mathcal{C}G}(T_u, T_v)$ (recall that projectives vanish in $\underline{\mathcal{C}G}$). Therefore, $\mathrm{Ext}^1_{\mathcal{C}G}(T_v, T_{z_i})$ and $\mathrm{Ext}^1_{\mathcal{C}G}(T_{z_i}, T_u)$ vanish. In particular, every z_i is compatible with both u and v, so $z_i \in \mathrm{lk}(u, \mathcal{K}) \cap \mathrm{lk}(v, \mathcal{K})$. We now claim that

$$\mathrm{lk}(u, \mathcal{K}) \cap \mathrm{lk}(v, \mathcal{K}) = \mathfrak{P}(\{z_1, \ldots, z_s\}) * \mathcal{K}',$$

where $\mathcal{K}'$ is a sub-complex of $\mathcal{K}$ and $\mathfrak{P}(\{z_1, \ldots, z_s\})$ is the standard simplex with vertices $z_1, \ldots, z_s$. For this, it is sufficient to show that every z_i is compatible with every element $y \in \mathrm{lk}(u, \mathcal{K}) \cap \mathrm{lk}(v, \mathcal{K})$. We apply the functor $\mathrm{Hom}_{\mathcal{C}G}(T_y, -)$ to $0 \to T_v \to E \to T_u \to 0$ to obtain the exact sequence

$$\mathrm{Ext}^1_{\mathcal{C}G}(T_y, T_v) \to \mathrm{Ext}^1_{\mathcal{C}G}(T_y, E) \to \mathrm{Ext}^1_{\mathcal{C}G}(T_y, T_u)$$

Since y is compatible with both v and u we have that

$$\mathrm{Ext}^1_{\mathcal{C}G}(T_y, T_v) = 0 = \mathrm{Ext}^1_{\mathcal{C}G}(T_y, T_u).$$

Therefore, $\mathrm{Ext}^1_{\mathcal{C}G}(T_y, E) = 0$ which implies that $\mathrm{Ext}^1_{\mathcal{C}G}(T_y, T_{z_i}) = 0$ for all i. The result follows.

$\square$

Proof of Theorem 3.1.1. We will proceed by induction on the rank of the cluster algebra $\mathcal{A}$. In the rank one case, $\mathcal{K}$ is a 0-sphere. One checks directly that $T^2(\mathcal{K}) = 0$. In the rank two case, $\mathcal{K}$ is the boundary of a square (type $A_1 \times A_1$) or of a pentagon (type A_2). Here, we are using that $\mathcal{A}$ is skew-symmetric. In both cases, one again checks directly that $T^2(\mathcal{K}) = 0$.

For the induction step, suppose first that $\mathcal{A}$ has a seed $(\widetilde{\mathbf{x}}, \widetilde{B})$ with $\Gamma(B)$ disconnected. By the induction hypothesis along with Lemmas 2.2.6 and 3.1.3, we may conclude that $\mathcal{K}$ is unobstructed.

Suppose instead that every (or equivalently, some) seed $(\widetilde{\mathbf{x}}, \widetilde{B})$ has $\Gamma(B)$ connected. It follows by Theorem 2.4.14 that $\Gamma(B)$ is a finite Dynkin diagram for some exchange matrix B. In this case, we will apply Proposition 2.2.4 to show the induction step. To check hypothesis (1) of the proposition, we note that Lemma 3.1.4 shows that any link of $\mathcal{K}$ is a cluster complex for a cluster algebra $\mathcal{A}'$ of smaller rank. Hypothesis (1) is then fulfilled by the induction hypothesis. To check hypothesis (2) of Proposition 2.2.4, we simply apply Lemma 3.1.5.

The theorem follows by induction. $\square$

3.2. **First Order Deformations.** Let $\mathcal{K}$ be the cluster complex of a cluster algebra $\mathcal{A}$ of finite cluster type. We will now give a description of $T^1(\mathcal{K})$ using the language of seeds. Consider any seed $(\widetilde{\mathbf{x}}, \widetilde{B})$ be a seed of $\mathcal{A}$. Recall that the elements of the cluster $\mathbf{x}$ all appear as vertices of some face of the complex $\mathcal{K}$.

Definition 3.2.1. Let $(\widetilde{\mathbf{x}}, \widetilde{B})$ be a seed of $\mathcal{A}$, and Ω a subset of the elements of $\mathbf{x} = (x_1, \ldots, x_n)$. A cluster variable x_k in the cluster $\mathbf{x}$ is Ω-*isolated* if

- (1) x_k is not in Ω; and
- (2) after removing the vertices of $\Gamma(B)$ corresponding to elements of Ω, the vertex k of $\Gamma(B)$ has no neighbors.

Theorem 3.2.2. *Every graded piece of $T^1(\mathcal{K})$ is either 0- or 1-dimensional. The degrees $\mathbf{c} = \mathbf{a} - \mathbf{b}$ with non-zero T^1 may be constructed exactly as follows: given a seed $(\widetilde{\mathbf{x}}, \widetilde{B})$, a subset Ω of the elements of $\mathbf{x}$, and an Ω-isolated cluster variable x_k, let $\mathbf{a}$ be any non-negative vector with support Ω, and let $\mathbf{b} \in \{0,1\}^{\mathcal{V}}$ have support $\{x_k, x'_k\}$. Here x'_k is defined as in (2.4.4).*

Proof. By Theorem 2.2.2, $T^1(\mathcal{K})_{\mathbf{a}-\mathbf{b}} \neq 0$ implies that $a \in \mathcal{K}$ and b is contained in the set of vertices of $\mathrm{lk}(a, \mathcal{K})$. In such situations,

$$T^1(\mathcal{K})_{\mathbf{a}-\mathbf{b}} = T^1(\mathrm{lk}(a, \mathcal{K}))_{-\mathbf{b}}.$$

Since $a \in \mathcal{K}$, there is a seed $(\widetilde{\mathbf{x}}, \widetilde{B})$ with $\Omega := a$ a subset of the elements of $\mathbf{x}$.

By Lemma 3.1.4, we may identify $\mathrm{lk}(\Omega, \mathcal{K})$ with $\mathcal{K}(B')$ where B' is the matrix obtained from B by removing the rows and columns corresponding to elements of Ω. Decomposing B' into connected components $B'_1, \ldots, B'_k$, we further obtain

$$\mathrm{lk}(\Omega, \mathcal{K}) \cong \mathcal{K}(B'_1) * \cdots * \mathcal{K}(B'_k)$$

by Lemma 3.1.3. By Lemma 2.2.6, we conclude that

$$T^1(\mathcal{K})_{\mathbf{a}-\mathbf{b}} \cong \bigoplus_i T^1(\mathcal{K}(B'_i))_{-\mathbf{b}}.$$

Note that at most one term $T^1(\mathcal{K}(B'_i))_{-\mathbf{b}}$ is non-zero, as b is contained in the vertices of at most one of the $\mathcal{K}(B'_i)$.

Suppose that b is contained in the vertices of $\mathcal{K}(B'_i)$. Since $\Gamma(B'_i)$ is connected and $\mathcal{A}(B'_i)$ is of finite cluster type, we may apply Lemma 3.1.5. Hence, for any non-edge $\{u, v\}$ of $\mathcal{K}' = \mathcal{K}(B'_i)$, the intersections of $\mathrm{lk}(u, \mathcal{K}')$ and $\mathrm{lk}(v, \mathcal{K}')$ is either empty or contractible. Since $\mathcal{K}'$ is also a spherical flag complex, we may use Proposition 2.2.3 to determine $T^1(\mathcal{K}')_{-\mathbf{b}}$.

To have $T^1(\mathcal{K}')_{-\mathbf{b}} \cong \mathbb{K}$, we must have that $\mathcal{K}'$ is a zero-sphere, and b consists of the two vertices of this sphere. The condition that $\mathcal{K}'$ is a zero-sphere is fulfilled exactly when B'_i has rank one, that is consists of a single row and column k. This is the same thing as saying that the cluster variable x_k is Ω-isolated. The condition that b consists of the two vertices of the sphere is the same as saying that $b = \{x_k, x'_k\}$.

In all other cases, $T^1(\mathcal{K}')_{-\mathbf{b}} = 0$. The claims in the statement of the theorem follow. □

Remark 3.2.3. Suppose that $T^1(\mathcal{K})_{\mathbf{a}-\mathbf{b}}$ is non-zero. Then the generator for this graded piece is the image of the homomorphism $\psi \in \mathrm{Hom}(I_{\mathcal{K}}, S_{\mathcal{K}})$ that sends $z^{\mathbf{b}}$ to $z^{\mathbf{a}}$, and every other minimal generator of $I_{\mathcal{K}}$ to zero. Indeed, the degree forces the condition that every other generator of $I_{\mathcal{K}}$ be sent to zero.

Example 3.2.4 (Continuation of Running Example 2.5.3). Let $\mathcal{A}$ be the cluster algebra from Example 2.4.7. Its cluster complex $\mathcal{K}$ is described in Example 2.5.3. By Theorem 3.1.1, $\mathcal{K}$ is unobstructed. Furthermore, we may apply $T^1(\mathcal{K})$ to compute the non-zero degrees of $\mathcal{T}^1(\mathcal{K})$. Each cluster of $\mathcal{A}$ just consists of two variables, so

supp($\mathbf{a}$)	supp($\mathbf{b}$)
x_{14}	x_{13}, x_{24}
x_{24}	x_{14}, x_{25}
x_{25}	x_{24}, x_{35}
x_{35}	x_{25}, x_{13}
x_{13}	x_{35}, x_{14}

TABLE 1. Support of $T^1(\mathcal{K})$ for Example 3.2.4

for any cluster $\{x_1, x_2\}$, x_1 is $\Omega = \{x_2\}$-isolated (and vice versa). Making use of $\mathcal{K}$, we see that $T^1(\mathcal{K})_{\mathbf{a}-\mathbf{b}} \neq 0$ if and only if $\mathbf{a}, \mathbf{b}$ have support as in Table 1, and $\mathbf{b} \in \{0,1\}^{\mathcal{V}}$. $\qquad\square$

4. Coefficients, Gradings, and Bases of Cluster Monomials

4.1. **Principal and Universal Coefficients.** Let $\widetilde{B} \in \mathrm{Mat}_{m \times n}(\mathbb{Z})$ be an extended exchange matrix. Its *principal extension* is the extended exchange matrix $\widetilde{B}^{\mathrm{prin}}$ obtained from $\widetilde{B}$ by attaching a copy of the $n \times n$ identity matrix I_n at its bottom. This can be represented as follows:

$$\widetilde{B}^{\mathrm{prin}} = \left(\begin{array}{c} \widetilde{B} \\ I_n \end{array} \right) \in \mathrm{Mat}_{2n \times n}(\mathbb{Z}).$$

Definition 4.1.1. The *cluster algebra with principal coefficients* associated to $\widetilde{B}$ is $\mathcal{A}(\widetilde{B}^{\mathrm{prin}})$.

In order to distinguish this case from others, we denote the new frozen variables of $\mathcal{A}(\widetilde{B}^{\mathrm{prin}})$ by $y_i = x_{m+i}$ for $1 \leq i \leq n$. By the strong Laurent phenomenon 2.4.11 $\mathcal{A}(\widetilde{B}^{\mathrm{prin}})$ is a $\mathbb{K}[y_1, \ldots, y_n]$-subalgebra of $\mathbb{K}[y_1, \ldots, y_n][x_{n+1}, \ldots, x_m][x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$. Any cluster variable of $\mathcal{A}(\widetilde{B}^{\mathrm{prin}})$ can be written as $x_{i;v}$ for some $1 \leq i \leq n$ and a vertex v of $\mathfrak{T}^n$.

Remark 4.1.2. Cluster algebras with principal coefficients were initially defined [FZ07, Definition 3.1] in the case of $m = n$, that is, no frozen variables. However, as mentioned in 2.4.10, we may view $\mathcal{A}(\widetilde{B})$ as a subalgebra of a rank m cluster algebra $\mathcal{A}(\widetilde{B}')$ with no frozen variables. Then $\mathcal{A}(\widetilde{B}^{\mathrm{prin}})$ is a subalgebra of $\mathcal{A}((\widetilde{B}')^{\mathrm{prin}})$, and we may apply structural results on $\mathcal{A}((\widetilde{B}')^{\mathrm{prin}})$ (and the related concepts of F-polynomials and $\mathbf{g}$-vectors) to deduce similar results for $\mathcal{A}(\widetilde{B}^{\mathrm{prin}})$. We tacitly do this in the following.

We now recall some of the fundamental results about cluster algebras with principal coefficients. It follows from [FZ07, Corollary 6.3, Proposition 5.1] and [GHKK18, Theorem 5.11] that a cluster variable (or a frozen variable) $x_{i;v}^{\mathrm{prin}}$ of $\mathcal{A}(\widetilde{B}^{\mathrm{prin}})$ has the following shape

$$(4.1.3) \qquad x_{i;v}^{\mathrm{prin}} = x_1^{g_1} \cdots x_m^{g_m} F_{i;v}(\hat{y}_1, \ldots, \hat{y}_n),$$

where,

- $g_1, \ldots, g_m \in \mathbb{Z}$ and $g_j \geq 0$ for all $n < j \leq m$;
- $F_{i;v}$ is a polynomial in $\mathbb{Z}[y_1, \ldots, y_n]$ with constant term equal to 1;

- for $1 \leq j \leq n$ we take $\hat{y}_j = y_j \prod_{i=1}^{m} x_i^{b_{ij}}$.

Definition 4.1.4. Let $x_{i;v}$ be a cluster or frozen variable of $\mathcal{A}(\widetilde{B})$. The **g**-*vector* of $x_{i;v}$ (with respect to the initial seed of $\mathcal{A}(\widetilde{B}^{\mathrm{prin}})$) is the vector

$$\mathbf{g}(x_{i;v}) := (g_1, \ldots, g_m) \in \mathbb{Z}^n \times \mathbb{Z}_{\geq 0}^{m-n},$$

where g_j is as in (4.1.3) for $x_{i;v}^{\mathrm{prin}}$. For each vertex v of $\mathfrak{T}^n$ the **g**-matrix G_v is the $m \times m$ square matrix whose ith column is $\mathbf{g}(x_{i;v})$. Polynomials $F_{i;v}$ as in (4.1.3) are called F-*polynomials*.

Since by Theorem 2.5.1 there is canonical bijection $x_{i;v} \mapsto x_{i;v}^{\mathrm{prin}}$ between the sets of cluster variables of $\mathcal{A}(\widetilde{B})$ and $\mathcal{A}(\widetilde{B}^{\mathrm{prin}})$, $\mathbf{g}(x_{i;v})$ only depends on the cluster or frozen variable $x_{i;v}$ and not directly on i, v.

Remark 4.1.5. In the literature, $\mathbf{g}(x_{i;v})$ as we have defined it is often called the (extended) **g**-*vector* of $x_{i;v}^{\mathrm{prin}}$ (as opposed to $x_{i;v}$). On the other hand, with the conventions adopted in Definition 4.1.4, the **g**-*vector* of $x_{i;v}^{\mathrm{prin}}$ is $\mathbf{g}(x_{i;v}^{\mathrm{prin}})$, which lives in $\mathbb{Z}^{m+n}$ (as opposed to $\mathbb{Z}^m$). However, notice that if $x_{i;v}^{\mathrm{prin}}$ is mutable then the last n entries of $\mathbf{g}(x_{i;v}^{\mathrm{prin}})$ are zero.

Remark 4.1.6. We observe that the F-polynomials and the **g**-vector of a cluster variable depends on the choice of an initial seed and the vertex v_0 of $\mathfrak{T}^n$ labeling the initial seed of $\mathcal{A}(\widetilde{B}^{\mathrm{prin}})$. In this paper, whenever we talk about F-polynomials or **g**-vectors we assume that the aforementioned choices have been taken once and for all.

Remark 4.1.7. The fact that the last $m - n$ entries of a **g**-vector are non-negative follows from the sign-coherence of **g**-vectors proved in the skew-symmetric case in [DWZ10] and in full generality in [GHKK18]. Indeed, any **g**-matrix of $\widetilde{B}$ obtained by iterated mutation in the indices $1, \ldots, n$ must contain the **g**-vectors of the frozen variables $x_{n+1}, \ldots, x_m$. We have that $\mathbf{g}(x_k) = (\delta_{k,j})_{j=1}^{m}$. In particular, given $n < j \leq m$ we have that the jth entry of $\mathbf{g}(x_j)$ is positive and by sign-coherence the jth entry of all the **g**-vectors of $\mathcal{A}(\widetilde{B}^{\mathrm{prin}})$ are non-negative.

Corollary 6.3 of [FZ07] asserts that $F_{i;v} \in \mathbb{Z}[y_1, \ldots, y_m]$ (working in $\mathcal{A}((\widetilde{B}')^{\mathrm{prin}})$, see Remark 4.1.2. However, using the mutation rule of F-polynomials appearing in [FZ07, Proposition 5.1] and the fact that to reach v we never mutated in the directions $n + 1, \ldots, m$, we can conclude that the variables $y_{n+1}, \ldots, y_m$ do not appear in $F_{i;v}$.

In many cases, the F-polynomial $F_{i;v}$ can also be used to compute $\mathbf{g}(x_{i;v})$.

Conjecture 4.1.8. [FZ07, Conjecture 6.11] Suppose that $F_{i;v}$ is not identically equal to 1. Then the **g**-vector $\mathbf{g}(x_{i;v}) = (g_1, \ldots, g_m)$ is given by

$$\prod_{i=1}^{m} u_i^{\tilde{g}_i} = \frac{F_{i;v}|_{\mathrm{Trop}(u_1, \ldots, u_m)}(u_1^{-1}, \ldots, u_n^{-1})}{F_{i;v}|_{\mathrm{Trop}(u_1, \ldots, u_m)}(\prod_{s=1}^{m} u_s^{b_{s1}}, \ldots, \prod_{s=1}^{m} u_s^{b_{sn}})}.$$

Here the specialization to $\mathrm{Trop}(u_1, \ldots, u_m)$ means we are working in the *tropical semifield*, see e.g. [FZ07, Definition 2.2].

Conjecture 4.1.8 is true whenever $\mathcal{A}(\widetilde{B})$ is of finite cluster type [FZ07, Corollary 10.10].

Although $\mathcal{A}(\widetilde{B}^{\mathrm{prin}})$ depends on our choice of seed, we may also add coefficients in a way that is *independent* of the seed choice:

Definition 4.1.9 ([FZ07, §12], [Rea14])**.** Let $\widetilde{B}$ be an extended exchange matrix of finite cluster type. The *cluster algebra with universal coefficients* $\mathcal{A}^{\mathrm{univ}}(\widetilde{B})$ is the cluster algebra associated to $\widetilde{B}^{\mathrm{univ}}$, where

$$\widetilde{B}^{\mathrm{univ}} = \left(\begin{array}{c} \widetilde{B} \\ U_B \end{array} \right)$$

and U_B is the matrix whose rows are the **g**-vectors of the cluster variables of $\mathcal{A}(B^{\mathrm{tr}})$.

Up to canonical isomorphism, $\mathcal{A}^{\mathrm{univ}}(\widetilde{B})$ is independent of the choice of seed, and we will simply write $\mathcal{A}^{\mathrm{univ}}$. This definition agrees with the original definition of a universal cluster algebra [FZ07, §12], see [Rea14]. We see from the above that $\mathcal{A}^{\mathrm{univ}}$ is obtained from $\mathcal{A}$ by adding in p frozen variables $\mathbf{t} = t_1, \ldots, t_p$, where p is the number of cluster variables of $\mathcal{A}$.

Example 4.1.10 (Continuation of Running Example 3.2.4)**.** Let $\mathcal{A}$ be the cluster algebra from Example 2.4.7. One computes its **g**-vectors to be

$$\mathbf{g}(x_{25}) = (-1, 0, 0, 2, 0) \qquad \mathbf{g}(x_{13}) = (1, 0, 0, 0, 0)$$
$$\mathbf{g}(x_{24}) = (-1, 1, 0, 1, 0) \qquad \mathbf{g}(x_{35}) = (0, -1, 0, 1, 1)$$
$$\mathbf{g}(x_{14}) = (0, 1, 0, 0, 0) \qquad \mathbf{g}(s_1) = (0, 0, 1, 0, 0)$$
$$\mathbf{g}(s_2) = (0, 0, 0, 1, 0) \qquad \mathbf{g}(s_3) = (0, 0, 0, 0, 1).$$

The projection of these **g**-vectors to the first two coordinates is pictured in Figure 3.

One computes that, up to permutation of the rows, the matrix U_B used in the extended exchange matrix for the cluster algebra with universal coefficients is

$$\left(\begin{array}{cc} 0 & -1 \\ 0 & 1 \\ 1 & -1 \\ 1 & 0 \\ -1 & 0 \end{array} \right)$$

Taking frozen variables $t_1, \ldots, t_5$, we obtain the following exchange relations for $\mathcal{A}^{\mathrm{univ}}$:

$$x_{24}x_{35} = t_1 s_3 x_{25} + t_2 t_5 s_1 s_2$$
$$x_{14}x_{35} = t_2 s_1 x_{13} + t_1 t_3 s_2 s_3$$
$$x_{14}x_{25} = t_3 s_2 x_{24} + t_2 t_4 s_1^2$$
$$x_{13}x_{25} = t_4 s_1 x_{35} + t_3 t_5 s_2^2$$
$$x_{13}x_{24} = t_5 s_2 x_{14} + t_1 t_4 s_1 s_3.$$

$\square$

4.2. **Positively Graded Cluster Algebras.** We will consider $\mathbb{Z}^d$-gradings on $\mathcal{A}(\widetilde{B})$ such that every cluster variable is homogeneous. Observe that this is equivalent to the condition that all the exchange relations are homogeneous. A systematic
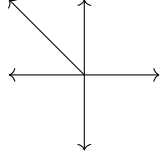
FIGURE 3. Projection of **g**-vectors for Example 4.1.10

study of these gradings was initiated in [Gra15] and we follow this approach. Although we allow frozen variables, our discussion essentially follows from that of [Gra15, §3].

Definition 4.2.1. A *graded seed* is a triple $(\widetilde{\mathbf{x}}, \widetilde{B}, D)$, where $(\widetilde{\mathbf{x}}, \widetilde{B})$ is a seed in $\mathcal{F}$ and $D \in \mathrm{Mat}_{m \times d}(\mathbb{Z})$ such that $\widetilde{B}^{\mathrm{tr}} D = 0$.

Assume that $(\widetilde{\mathbf{x}}, \widetilde{B}, D)$ is a graded seed and let $D_i \in \mathbb{Z}^d$ be the ith row of D. For $1 \leq i \leq m$ we can declare $\deg_D(x_i) := D_i$ to be the degree of the variable x_i of $\widetilde{\mathbf{x}}$. Since $(\widetilde{\mathbf{x}}, \widetilde{B}, D)$ is a graded seed the right hand side of the exchange relation (2.4.4) is homogeneous with respect to this grading. Therefore, we can use the exchange relation to determine the degree of the cluster variable x_k' and obtain a graded seed $(\widetilde{\mathbf{x}}', \widetilde{B}', D')$, where $(\widetilde{\mathbf{x}}', \widetilde{B}') = \mu_k(\widetilde{\mathbf{x}}, \widetilde{B})$. We can iterate this process to obtain a well defined[3] degree $\deg_D(x) \in \mathbb{Z}^d$ for every cluster variable x of $\mathcal{A}(\widetilde{\mathbf{x}}, \widetilde{B})$. The way the grading changes under mutation is encoded by the following class of matrices.

Definition 4.2.2. Let $\widetilde{B}$ be an extended exchange matrix. Consider a sign $\varepsilon \in \{+, -\}$ and $k \in \{1, \ldots, n\}$. Let $E_{k,\varepsilon} = (e_{ij})$ be the $m \times m$ matrix given by

$$e_{ij} := \begin{cases} \delta_{ij} & \text{if } k \neq j \\ -1 & \text{if } i = j = k, \\ \max(0, -\varepsilon b_{ik}) & \text{if } i \neq j = k. \end{cases}$$

Observe that e_{ij} depends on $\widetilde{B}$ although we are not incorporating this dependence into the notation.

For a mutable index $k \in \{1, \ldots, n\}$ the graded seed $\mu_k(\widetilde{\mathbf{x}}, \widetilde{B}, D)$ is $(\widetilde{\mathbf{x}}', \widetilde{B}', D')$, where $(\widetilde{\mathbf{x}}', \widetilde{B}') = \mu_k(\widetilde{\mathbf{x}}, \widetilde{B})$ and

$$D' = (E_{k,+})^T D = (E_{k,-})^T D.$$

Observe that it is indeed the case that $(E_{k,+})^T D = E_{k,-} D$ since the kth column of $\widetilde{B}^T$ coincides with the kth column of $E_{k,+} - E_{k,-}$.

As before, given an initial assignment $(\widetilde{\mathbf{x}}_{v_0}, \widetilde{B}_{v_0}, D_{v_0})$ of a graded seed to a fixed vertex v_0 of $\mathfrak{T}^n$, by applying iterated mutations we have a graded seed $(\widetilde{\mathbf{x}}_v, \widetilde{B}_v, D_v)$ for every vertex v of $\mathfrak{T}^n$. We assume that an initial assignment is fixed once and for all. The following result follows from [GHKK18, Theorem 5.11], see also [NZ12] and [Kel12, §5.6].

Lemma 4.2.3. *Suppose that v is a vertex of $\mathfrak{T}^n$ connected to v_0 by edges carrying the labels $k_0, \ldots, k_r$, where the edge labeled by k_i connects the vertices v_i with v_{i+1} and $v_{r+1} = v$. Recall that for a vertex $v \in \mathfrak{T}^n$, G_v is the matrix whose columns are*

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[3]It is a nontrivial fact that this is indeed well defined since we can reach the same cluster variable by different mutation sequences.

$\mathbf{g}$-vectors for the extended cluster at v. Then there exists a choice of signs $\varepsilon_0, \ldots, \varepsilon_r$ such that

$$G_v = E_{k_0,\varepsilon_0} \cdots E_{i_r,\varepsilon_r}.$$

Corollary 4.2.4. *Let* $(\widetilde{\mathbf{x}}, \widetilde{B}, D)$ *be the initial graded seed associated to a vertex* v_0 *of* $\mathfrak{T}^n$ *and* $(\widetilde{\mathbf{x}}_v, \widetilde{B}_v, D_v)$ *the graded seed associated to a vertex* v *of* $\mathfrak{T}^n$. *Then*

$$\deg(x_{i;v}) = \mathbf{g}_{i;v} \cdot D.$$

Proof. By definition, the vectors $\deg(x_{1;v}), \ldots, \deg(x_{m;v}) \in \mathbb{Z}^d$ are the rows of D_v. By Lemma 4.2.3 we have that $G_v = E_{k_0,\varepsilon_0} \cdots E_{k_r,\varepsilon_r}$ for a choice of signs $\varepsilon_0, \ldots, \varepsilon_r$. In particular, we have that

$$\begin{aligned}
D_v &= E_{k_r,\varepsilon_r}^T \cdots E_{k_0,\varepsilon_0}^T D \\
&= (E_{k_0,\varepsilon_0} \cdots E_{k_r,\varepsilon_r})^T D \\
&= G_v^T D.
\end{aligned}$$

The result follows. □

Definition 4.2.5. A cluster algebra $\mathcal{A}(\widetilde{B})$ is *positively graded* if it admits a $\mathbb{Z}$-grading such that every cluster variable is homogeneous of positive degree.

Remark 4.2.6. By [GHKK18, Theorem 2.13, Corollary 5.9] the cluster complex $\mathcal{K}(\widetilde{B})$ can be realized as a simplicial fan $\Delta^+(\widetilde{B})$ in $\mathbb{Z}^m$ (every choice of initial seed of $\mathcal{A} = \mathcal{A}(\widetilde{B})$ gives rise to a realization). We call $\Delta^+(\widetilde{B})$ the $\mathbf{g}$-fan since the maximal cones of $\Delta^+(\widetilde{B})$ are precisely the cones spanned by the columns of the $\mathbf{g}$-matrices G_v, where v ranges on all the vertices of $v \in \mathfrak{T}^n$. From this perspective, Corollary 4.2.4 implies the following: A cluster algebra $\mathcal{A}(\widetilde{B})$ is positively graded if and only if there exists a vector $D \in \mathbb{Z}^m$ such that $\widetilde{B}^T D = 0$ and the $\mathbf{g}$-fan $\Delta^+(\widetilde{B})$ is contained in the open half space $\{v \in \mathbb{R}^m \mid v \cdot D > 0\}$.

Observe that if a cluster algebra of finite cluster type $\mathcal{A}(\widetilde{B})$ is positively graded, then $n < m$ [Gra15, §4].

Lemma 4.2.7. *Let* w *be a cluster variable of* $\mathcal{A}(\mathbf{x}, B)$ *and* $\tilde{w}$ *be the corresponding cluster variable of* $\mathcal{A}(\widetilde{\mathbf{x}}, \widetilde{B})$ *under the bijection of Theorem 2.5.1. Denote by* $\mathbf{g}(w) = (g_1(w), \ldots, g_n(w))$ *the* $\mathbf{g}$*-vector of* w *with respect to* $(\mathbf{x}, B)$ *and by* $\mathbf{g}(\tilde{w}) = (g_1(\tilde{w}), \ldots, g_m(\tilde{w}))$ *the* $\mathbf{g}$*-vector of* $\tilde{w}$ *with respect to* $(\widetilde{\mathbf{x}}, \widetilde{B})$.

 (1) *Then* $g_j(w) = g_j(\tilde{w})$ *for all* $1 \leq j \leq n$.
 (2) *Assume Conjecture 4.1.8 holds and* $\tilde{w}$ *does not belong to the initial cluster. Then there exists a mutable index* $k \leq n$ *such that for any frozen index* $n < j \leq m$, $-b_{jk} \leq g_j(\tilde{w})$.

Proof. Item 1 follows from the mutation formula for $\mathbf{g}$-vectors contained in [FZ07, Proposition 6.6].

We proceed to show item 2. Assume that $w = x_{i;v}$ and $\tilde{w} = \tilde{x}_{i;v}$ for some mutable index i and some vertex v of $\mathfrak{T}^n$. Since we are assuming that $x_{i;v}$ is non-initial the F-polynomial $F_{i;v}$ is not identically equal to 1. This follows from the combination of [GHKK18, Theorem 1.12, Theorem 1.3] and [Nak21a, Theorem 6.16]. Since we are assuming that Conjecture 4.1.8 holds then we can compute $\mathbf{g}(\tilde{x}_{i;v})$ as the exponent

of the monomial

$$(4.2.8) \qquad \prod_{i=1}^{m} u_i^{g_i(\tilde{w})} = \frac{F_{i;v}|_{\mathrm{Trop}(u_1,\ldots,u_m)}(u_1^{-1},\ldots,u_n^{-1})}{F_{i;v}|_{\mathrm{Trop}(u_1,\ldots,u_m)}(\prod_{s=1}^{m} u_s^{b_{s1}},\ldots,\prod_{s=1}^{m} u_s^{b_{sn}})}.$$

Recall that $F_{i;v} \in \mathbb{Z}[y_1,\ldots,y_n]$. Since $F_{i;v}$ is not identically equal to 1, the variable y_k appears in $F_{i;v}$ for some $1 \leq k \leq n$. Consider any $n < j \leq m$. If $b_{jk} \geq 0$, we are done, since we know that $g_j(\tilde{w}) \geq 0$ for all $j > n$.

Suppose instead that $b_{jk} < 0$. By our assumption that y_k appears in $F_{i;v}$, a power of the monomial $\prod_{s=1}^{m} u_s^{b_{sk}} = u_j^{b_{jk}} \prod_{s \neq j} u_s^{b_{sk}}$ appears in $F_{i;v}$. Since $b_{jk} < 0$, the exponent of u_j in

$$F_{i;v}|_{\mathrm{Trop}(u_1,\ldots,u_m)}(\prod_{s=1}^{m} u_s^{b_{s1}},\ldots,\prod_{s=1}^{m} u_s^{b_{sn}})$$

is less than or equal to b_{jk}.

On the other hand, as noticed earlier, u_j does not appear in $F_{i;v}$. Hence u_j does not divide $F_{i;v}|_{\mathrm{Trop}(u_1,\ldots,u_m)}(u_1^{-1},\ldots,u_m^{-1})$. From these facts we conclude that the exponent of u_j in $\prod_{i=1}^{m} u_i^{g_i(\tilde{w})}$ is at least $-b_{jk}$, and thus $-b_{jk} \leq g_j(\tilde{w})$. $\qquad\square$

Remark 4.2.9. Conjecture 6.11 of [FZ07] holds for any cluster algebra of finite cluster type, see [FZ07, Corollary 10.10]. Hence we can apply item (2) of Lemma 4.2.7 to any cluster algebra of finite cluster type.

Lemma 4.2.10. *Let $\mathcal{A}(\widetilde{B})$ be a cluster algebra of finite cluster type. Then we can add rows to $\widetilde{B}$ to obtain a larger exchange matrix $\widetilde{B}^+$ such that $\mathcal{A}(\widetilde{B}^+)$ admits a positive grading. Moreover, we can do this in a way such that all the frozen variables and the initial cluster variables have degree 1.*

Proof. Consider the **g**-vectors associated to $\mathcal{A}(\widetilde{B})$. All these are vectors in $\mathbb{Z}^m$. Let $\mathbf{g}(w) = (g_1(w),\ldots,g_m(w))$ denote the **g**-vector of a cluster variable w with respect to a fixed choice of initial seed $(\widetilde{\mathbf{x}}, \widetilde{B})$. Let $c = \min_{w \in \mathcal{V}}\{\sum_{i=1}^{m} g_i(w)\} - 1$. We proceed to construct $\widetilde{B}'$ by adding n rows (corresponding to new frozen indices) to $\widetilde{B}$. For every mutable index $k \in \{1,\ldots,n\}$ we add one new frozen index $k' = m+k$ and a new row $(b_{k'1},\ldots,b_{k'n})$ given by

$$b_{k'j} = \begin{cases} 0 & \text{if } j \neq k \\ -|c| & \text{if } j = k. \end{cases}$$

Observe that the triple $(\widetilde{\mathbf{x}}', \widetilde{B}', (1,\ldots,1))$ might not be a graded seed since the product $(1,\ldots,1) \cdot \widetilde{B}'$ might not be zero. However, we can always consider a graded seed $(\widetilde{\mathbf{x}}^+, \widetilde{B}^+, (1,\ldots,1))$ obtained from $(\widetilde{\mathbf{x}}', \widetilde{B}', (1,\ldots,1))$ by adding still yet another row to $\widetilde{B}'$, corresponding to a new frozen index $m+n+1$. Denote by $\mathbf{1} := (1,\ldots,1) \in \mathbb{Z}^{m+n+1}$ the grading vector of this graded seed. By Lemma 4.2.7[1], we know that for every cluster variable w^+ of $\mathcal{A}(\widetilde{B}^+)$, its **g**-vector (computed with respect to $(\widetilde{\mathbf{x}}^+, \widetilde{B}^+)$) is of the form $(g_1(w^+),\ldots,g_{m+n+1}(w^+))$, where the first m entries coincide with the **g**-vector of a cluster variable of $\mathcal{A}(\widetilde{B})$, and the last entries are non-negative. Moreover, if w^+ is non-initial then by Lemma 4.2.7[2] we know that $g_{k'}(w^+) \geq -b_{k'k} = |c|$, where k is a mutable index and $k' = m+k$ as above. Now recall that $g_j(w^+) \geq 0$ for all frozen indices j. Putting these facts together we

obtain the following inequalities for non-initial w^+:

$$\mathbf{g}(w^+) \cdot \mathbf{1} \geq g_{k'}(w^+) + \sum_{i=1}^{m} g_i(w^+) \geq |c| + \sum_{i=1}^{m} g_i(w^+) \geq 1.$$

The result now follows from Corollary 4.2.4. □

Example 4.2.11 (Continuation of Running Example 4.1.10)**.** We continue our discussion of the cluster algebra $\mathcal{A}$ from Example 2.4.7. Let D be the matrix whose transpose is $(1, 1, 1, 1, 1)$. Then clearly $\widetilde{B}^{\mathrm{tr}} \cdot D = 0$, and we thus obtain a $\mathbb{Z}$-grading on $\mathcal{A}$. Multiplying D by the $\mathbf{g}$-vectors of the cluster variables of $\mathcal{A}$ (see Example 4.1.10), we obtain that all cluster variables have degree $1 \in \mathbb{Z}$. In particular, the cluster algebra $\mathcal{A}$ has a positive grading. □

Remark 4.2.12. The above discussion of gradings on cluster algebras via graded seeds can be extended to include gradings by any abelian group, not just $\mathbb{Z}^d$, see [GP18, §2.1]. When the grading group is finitely generated, this corresponds to considering actions on $\operatorname{Spec} \mathcal{A}$ of general diagonalizable groups as opposed to just algebraic tori. For this, it is helpful to slightly rephrase things. We may interpret the grading matrices D from above as encoding a homomorphism

$$D^{\mathrm{tr}} : \mathbb{Z}^m \to \mathbb{Z}^d.$$

More generally, we may consider homomorphisms

$$\gamma : \mathbb{Z}^m \to M$$

where M is any abelian group. The condition that γ defines a grading on $\mathcal{A}$ now becomes that γ vanishes on the $\mathbb{Z}$-linear rowspan of $\widetilde{B}^{\mathrm{tr}}$, or equivalently, that $\gamma \circ \widetilde{B}(\mathbb{Z}^n) = 0$.

The map γ tells us the degrees of any cluster or frozen variable in the initial extended cluster $\widetilde{\mathbf{x}}$. After mutating at some mutable index $k \in \{1, \ldots, n\}$, the mutated version of γ is given by

$$\gamma' = \gamma \circ E_{k,+} = \gamma \circ E_{k,-}.$$

Example 4.2.13. Consider a cluster algebra $\mathcal{A}$ of type B_2, with exchange matrix

$$\widetilde{B} = B = \begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix}.$$

A possible grading for $\mathcal{A}$ is given by the homomorphism $\gamma : \mathbb{Z}^2 \to \mathbb{Z}/2\mathbb{Z}$ with $\gamma(e_1) = \overline{1}$ and $\gamma(e_2) = 0$.

The cluster algebra $\mathcal{A}$ has six cluster variables $x_1, \ldots, x_6$ which are related via the exchange relations

$$x_{i-1}x_{i+1} = x_i^2 + 1 \qquad i \text{ even}$$
$$x_{i-1}x_{i+1} = x_i + 1 \qquad i \text{ odd}.$$

Here indices are taken modulo 6. The variables x_1, x_2 are in the cluster with exchange matrix $\widetilde{B}$. The grading constructed via γ assigns degree $\overline{1} \in \mathbb{Z}/2\mathbb{Z}$ to all x_i with i odd, and 0 to all x_i to i even. It is clear that the exchange relations are homogeneous with respect to this grading. □

4.3. **Cluster Monomials.** Let $\mathcal{A}$ be a cluster algebra. A *cluster monomial* is any monomial in the cluster variables of $\mathcal{A}$ such that all cluster variables appearing as a factor belong to a common cluster. An *extended cluster monomial* is the product of a cluster monomial with any monomial in the frozen variables. A *Laurent-extended cluster monomial* is the product of a cluster monomial with any Laurent monomial in the frozen variables.

We can extend the definition of the **g**-vector of a cluster variable for Laurent-extended cluster monomials as follows. Let $\mathcal{V}$ and $\mathcal{W}$ be the cluster and frozen variables of $\mathcal{A}$. For a Laurent-extended cluster monomial $\prod_{x \in \mathcal{V} \cup \mathcal{W}} x^{\alpha_x}$ we set

$$\mathbf{g}\left(\prod_{x \in \mathcal{V} \cup \mathcal{W}} x^{\alpha_x}\right) := \sum_x \alpha_x \mathbf{g}(x) \in \mathbb{Z}^m.$$

In this subsection, we prove the following folklore result:

Theorem 4.3.1. *Let $\mathcal{A}$ be a cluster algebra of finite cluster type. The cluster monomials of $\mathcal{A}$ form a basis over the polynomial ring in the frozen variables. Equivalently, the extended cluster monomials form a basis over $\mathbb{K}$.*

It was stated without proof that for cluster algebras of classical type, the Laurent-extended cluster monomials form a $\mathbb{K}$-basis after tensoring with the ring of Laurent polynomials in the frozen coefficients [FZ03b, Theorem 4.27]. Theorem 4.3.1 was proven for cluster algebras of finite cluster type with trivial coefficients in [CK08, Corollary 3]. Theorem 4.3.1 implies that any cluster algebra of finite cluster type may be obtained from the universal one via base change, see [FZ07, Remark 12.11]

The main ingredient is the following lemma, which follows from [GHKK18]:

Lemma 4.3.2. *Let $\mathcal{A}(B)$ be a cluster algebra of finite cluster type with trivial coefficients. Let $\mathcal{A}'$ be the algebra obtained from $\mathcal{A}(B^{\mathrm{univ}})$ by adding principal coefficients $s_1, \ldots, s_n$, and adjoining $s_1^{-1}, \ldots, s_n^{-1}$. Then the cluster monomials of $\mathcal{A}'$ form a $\mathbb{K}[\mathbf{t}][s_1^{\pm 1}, \ldots, s_n^{\pm 1}]$-basis of $\mathcal{A}'$.*

Proof. To prove this statement we rely on the results of [GHKK18, §8 and §9]. The reader is referred to [GHK15, GHKK18] for a thorough treatment of the objects we consider in this proof. Throughout we use the following notation. If $\widetilde{B}$ is an extended exchange matrix we let $\mathfrak{A}(\widetilde{B})$ be the cluster variety associated to (the fixed data corresponding to) $\widetilde{B}$ in the sense of [GHK15, §2], and $\mathfrak{A}(\widetilde{B})_{\mathrm{prin}}$ be the corresponding cluster variety with principal coefficients. We note that in loc. cit. the authors add principal coefficients for not only cluster variables in an initial seed (as we do in Definition 4.1.1) but also the frozen variables. These principal coefficients corresponding to frozen variables are irrelevant for our discussion, as they do not appear in any exchange relations. We will thus stay consistent with Definition 4.1.1 and only add principal coefficients for non-frozen variables. We also note that in [GHKK18], the **g**-fan (as introduced in our Remark 4.2.6) also contains directions coming from the **g**-vectors of the multiplicative inverses of frozen variables. More precisely, each chamber of the **g**-fan is the cone generated by a maximal collection of compatible Laurent-extended cluster monomials.

We let $\overline{\mathfrak{A}}(\widetilde{B})$ be the partial compactification of $\overline{\mathfrak{A}}(\widetilde{B})$ given by setting the frozen variables to zero and denote by $\mathfrak{A}^{\vee}(\widetilde{B})$ the Langlands dual cluster variety. Using the above notation, $\mathcal{A}'$ is the *ordinary* cluster algebra of $\overline{\mathfrak{A}}(B^{\mathrm{univ}})_{\mathrm{prin}}$. We will show using [GHKK18, Corollary 9.18] that the *upper* cluster algebra $\mathcal{A}''$ of $\overline{\mathfrak{A}}(B^{\mathrm{univ}})_{\mathrm{prin}}$

has a $\mathbb{K}[\mathbf{t}][s_1^{\pm 1}, \ldots, s_n^{\pm 1}]$-basis given by the cluster monomials. Since the upper cluster algebra contains $\mathcal{A}'$, and all cluster monomials are contained in $\mathcal{A}'$, the claim of the lemma will follow.

First we show that the two assumptions of the corollary are satisfied: the variety $\mathfrak{A}^\vee(B^{\mathrm{univ}})_{\mathrm{prin}}$ has enough global monomials, and every frozen variable of $\mathcal{A}(B^{\mathrm{univ}})$ has an optimized seed. The first condition follows at once from Propositions 8.16 and 8.25 of [GHKK18]. Indeed, since $\mathcal{A}(B)$ has finite cluster type, the $\mathbf{g}$-fan for $\mathfrak{A}(B^{\mathrm{univ}})_{\mathrm{prin}}$ is the product of a complete fan (the $\mathbf{g}$-fan for $\mathcal{A}(B)$) with $\mathbb{R}^{p+n}$, where p is the number of cluster variables of $\mathcal{A}(B)$. In particular, the $\mathbf{g}$-fan for $\mathfrak{A}(B^{\mathrm{univ}})_{\mathrm{prin}}$ is large.

The property that every frozen variable of $\mathcal{A}(B^{\mathrm{univ}})$ has an optimized seed is equivalent to the property that for every frozen index j there is a matrix in the mutation class of B^{univ} such that its jth row has non-negative entries (see [GHKK18, Definition 9.1]). This existence is clear since the jth row of the matrices in this mutation class can be identified with the set of the different $\mathbf{g}$-vectors of a fixed cluster variable of $\mathcal{A}(B^{\mathrm{tr}})$ with respect to all the different (labeled) seeds in $\mathcal{A}(B^{\mathrm{tr}})$. See Definition 4.1.9. In particular, this set contains a canonical basis vector of $\mathbb{Z}^n$.

In order to apply [GHKK18, Corollary 9.18], we must also show that the cone Ξ of [GHKK18, Corollary 9.17] is contained in the convex hull of the set Θ of loc. cit. But the set Θ contains all integral points of the corresponding $\mathbf{g}$-fan, so as noted above, this will contain all of $\mathbb{R}^{n+p} \times \mathbb{R}^n$. In fact, since this $\mathbf{g}$-fan covers the entire ambient space, Θ consists of exactly the integral points of the $\mathbf{g}$-fan.

We now apply [GHKK18, Corollary 9.18] to $\overline{\mathfrak{A}}(B^{\mathrm{univ}})_{\mathrm{prin}}$. Since Θ consists of the integral points of the $\mathbf{g}$-fan, we can conclude that $\mathcal{A}''$ has a $\mathbb{K}$-basis given by the Laurent-extended cluster monomials whose $\mathbf{g}$-vectors lie in a rational polyhedral cone $\Xi \subset \mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^n$. Here, the factor $\mathbb{R}^p$ corresponds to the universal coefficients, and the latter $\mathbb{R}^n$ comes from the principal coefficients. Moreover, from the description of Ξ in [GHKK18, Corollary 9.17] and our choice of partial compactification, it follows that $\Xi \subset \mathbb{R}^n \times \mathbb{R}_{\geq 0}^p \times \mathbb{R}^n$, so the basis of $\mathcal{A}''$ only contains products of cluster monomials with monomials in $\mathbb{K}[\mathbf{t}][s_1^{\pm 1}, \ldots, s_n^{\pm 1}]$. Since Laurent-extended cluster monomials are linearly independent ([FWZ21, Theorem 6.8.10] in the finite cluster type case), and $\mathcal{A}''$ contains every product of a cluster monomial with $\mathbb{K}[\mathbf{t}][s_1^{\pm 1}, \ldots, s_n^{\pm 1}]$, we obtain that $\mathcal{A}''$ has the desired basis. $\qquad\square$

We may now complete the proof of the theorem:

Proof of Theorem 4.3.1. The linear independence of the cluster monomials is [FZ07, Theorem 11.2]. Thus, it remains to show that they span the cluster algebra. To that end, let $\mathcal{A}'$ be as in Lemma 4.3.2 for $\mathcal{A}(B)$, where B is an exchange matrix for $\mathcal{A}$. By Lemma 4.3.2, the cluster monomials of $\mathcal{A}'$ form a basis over $\mathbb{K}[\mathbf{t}][s_1^{\pm 1}, \ldots, s_n^{\pm 1}] = \mathbb{K}[\mathbf{t}][\mathbf{s}^{\pm 1}]$.

Let $\mathcal{V}$ be the set of cluster variables for $\mathcal{A}$ and $\mathcal{W}$ the set of frozen variables. Let $I_{\mathcal{A}}^{\mathrm{ex}}$ and $I_{\mathcal{A}'}^{\mathrm{ex}}$ denote the ideals of respectively $\mathbb{K}[z_v \mid v \in \mathcal{V} \cup \mathcal{W}]$ and $\mathbb{K}[z_v \mid v \in \mathcal{V}][\mathbf{t}][\mathbf{s}^{\pm 1}]$ generated by the expressions obtained from the exchange relations (2.4.4) by substituting z_v for the cluster or frozen variable v. Here we are using the bijection between cluster variables of $\mathcal{A}$ and $\mathcal{A}'$ of Theorem 2.5.1.

By e.g. [FWZ21, Theorem 6.8.10], we may obtain $\mathcal{A}$ (or $\mathcal{A}'$) by taking the quotient of $\mathbb{K}[z_v \mid v \in \mathcal{V} \cup \mathcal{W}]$ (or $\mathbb{K}[z_v \mid v \in \mathcal{V}][\mathbf{t}][\mathbf{s}^{\pm 1}]$) by the saturation of $I_{\mathcal{A}}^{\mathrm{ex}}$ (respectively $I_{\mathcal{A}'}^{\mathrm{ex}}$) with respect to the product $\prod_{v \in \mathcal{V}} z_v$. Furthermore, by the universal

property of $\mathcal{A}(B^{\mathrm{univ}})$ (cf. [FZ07, Definition 12.3]) there is a ring homomorphism $\mathbb{K}[\mathbf{t}] \to \mathbb{K}[z_w \mid w \in \mathcal{W}]$ such that for the induced map

$$\rho : \mathbb{K}[z_v \mid v \in \mathcal{V}][\mathbf{t}][\mathbf{s}^{\pm 1}] \to \mathbb{K}[z_v \mid v \in \mathcal{V} \cup \mathcal{W}]$$

sending s_i to 1 we have

$$I_{\mathcal{A}}^{\mathrm{ex}} = \rho(I_{\mathcal{A}'}^{\mathrm{ex}}).$$

By an abuse of notation, we will call a monomial of $\mathbb{K}[z_v \mid v \in \mathcal{V}]$ a cluster monomial if the cluster variables corresponding to its factors form a cluster monomial in $\mathcal{A}$ (or equivalently $\mathcal{A}'$).

To prove the claim, we must show that any monomial $\mathbf{z}^{\alpha} \in \mathbb{K}[z_v \mid v \in \mathcal{V}]$ can be written as a $\mathbb{K}[z_w \mid w \in \mathcal{W}]$-linear combination of cluster monomials, modulo the saturation $I_{\mathcal{A}}$ of $I_{\mathcal{A}}^{\mathrm{ex}}$. By Lemma 4.3.2, we can write $\mathbf{z}^{\alpha}$ as a $\mathbb{K}[\mathbf{t}][\mathbf{s}^{\pm 1}]$-linear combination of cluster monomials, modulo the saturation $I_{\mathcal{A}}'$ of $I_{\mathcal{A}'}^{\mathrm{ex}}$. That is,

$$\left(\prod_{v \in \mathcal{V}} z_v^k \right) \left(\mathbf{z}^{\alpha} - \sum_i f_i \mathbf{z}^{\beta_i} \right) \in I_{\mathcal{A}'}^{\mathrm{ex}}$$

for some $k \geq 0$, where $\mathbf{z}^{\beta_i}$ are cluster monomials and $f_i \in \mathbb{K}[\mathbf{t}][\mathbf{s}^{\pm 1}]$. Applying ρ, we obtain

$$\left(\prod_{v \in \mathcal{V}} z_v^k \right) \left(\mathbf{z}^{\alpha} - \sum_i \rho(f_i) \mathbf{z}^{\beta_i} \right) \in I_{\mathcal{A}}^{\mathrm{ex}}$$

and we see that $\mathbf{z}^{\alpha}$ is indeed a $\mathbb{K}[z_w \mid w \in \mathcal{W}]$-linear combination of cluster monomials, modulo $I_{\mathcal{A}}$.

$\square$

5. Universal Coefficients and Hilbert Schemes

5.1. **The Cluster Embedding.** Let $\mathcal{A}$ be a cluster algebra of finite cluster type, with $\mathcal{V}$ the set of cluster variables and $\mathcal{W}$ the set of frozen variables. Since $\mathcal{A}$ is generated by the cluster and frozen variables, we obtain a presentation

$$0 \to I_{\mathcal{A}} \to \mathbb{K}[z_v \mid v \in \mathcal{V} \cup \mathcal{W}] \to \mathcal{A} \to 0$$

by sending the z_v to the corresponding cluster and frozen variables. The ideal $I_{\mathcal{A}}$ includes the exchange relations (2.4.4), but in general has additional generators.

Geometrically, the above presentation gives rise to the so-called *cluster embedding*

$$\operatorname{Spec} \mathcal{A} \subseteq \mathbb{A}^{p+q}$$

where $p = \#\mathcal{V}$ and $q = m - n = \#\mathcal{W}$. Set $\mathbb{T} = (\mathbb{K}^*)^{p+q}$ and let $H \subseteq \mathbb{T}$ be the maximal subgroup of $\mathbb{T}$ whose elements send $\operatorname{Spec} \mathcal{A}$ to itself in this embedding. This inclusion of tori induces a surjection $\deg_H : \mathbb{Z}^{p+q} \to M$ to the character group M of H. Since H preserves $\operatorname{Spec} \mathcal{A}$, the map $\deg_H$ gives an M-grading of the cluster algebra $\mathcal{A}$. In general, H could be disconnected; equivalently, M could contain torsion.

Proposition 5.1.1. *Let $(\widetilde{\mathbf{x}}, \widetilde{B})$ be a seed of $\mathcal{A}$. This induces an isomorphism of the cokernel of $\widetilde{B}$ with M, the character group of H.*

Proof. Let M' be the cokernel of $\widetilde{B}$. We thus obtain a map $\mathbb{Z}^m \to M'$. This gives a grading of $\mathbb{K}[\mathbf{z}]$ as follows. The degree of the variable z_{x_i} corresponding to the initial cluster element x_i is the image of the ith basis vector. Using the procedure of §4.2

(in particular Remark 4.2.12), this determines the degrees of all the variables z_v, and by construction, $\mathcal{A}$ is homogeneous with respect to this grading. It is clear from construction that any other grading must factor through this one, so we conclude that M' is indeed isomorphic to M, the character group of H. $\qquad\square$

Definition 5.1.2 (cf. [FWZ17, §4.3.7]). A cluster algebra $\mathcal{A}$ has *full $\mathbb{Z}$-rank* if for some (or equivalently for every) seed $(\widetilde{\mathbf{x}}, \widetilde{B})$, the rows of $\widetilde{B}$ span $\mathbb{Z}^m$. We say it has *full rank* if for some (or equivalently for every) seed $(\widetilde{\mathbf{x}}, \widetilde{B})$, $\widetilde{B}$ has full rank.

Corollary 5.1.3. *Suppose that $\mathcal{A}$ has full rank. Then H is a torus if and only if $\mathcal{A}$ has full $\mathbb{Z}$-rank.*

Proof. The property of H being a torus is equivalent to M being torsion-free. By Proposition 5.1.1, this is equivalent to $\widetilde{B}$ having torsion-free cokernel. Under the assumption that $\mathcal{A}$ has full rank, this is clearly equivalent to $\mathcal{A}$ having full $\mathbb{Z}$-rank. $\qquad\square$

The presentation of $\mathcal{A}$ above lifts canonically to a presentation of $\mathcal{A}^{\mathrm{univ}}$:

$$(5.1.4) \qquad 0 \to I_{\mathcal{A}^{\mathrm{univ}}} \to \mathbb{K}[z_v, \mathbf{t} \mid v \in \mathcal{V} \cup \mathcal{W}] \to \mathcal{A}^{\mathrm{univ}} \to 0.$$

It follows from the discussion of §4.2 that the map $\deg_H : \mathbb{Z}^{p+q} \to M$ also makes $\mathcal{A}^{\mathrm{univ}}$ M-graded. In this grading, each coefficient t_i has degree 0, and the degree of a cluster or frozen variable of $\mathcal{A}^{\mathrm{univ}}$ is the same as the degree of the variable of $\mathcal{A}$ to which it specializes.

Geometrically, we have

$$\operatorname{Spec}\mathcal{A}^{\mathrm{univ}} \longhookrightarrow \mathbb{A}^{p+q} \times \mathbb{A}^p$$
$$\searrow \qquad \downarrow$$
$$\mathbb{A}^p$$

with the vertical arrow a projection, and $\operatorname{Spec}\mathcal{A}^{\mathrm{univ}}$ a H-invariant subvariety of $\mathbb{A}^{p+q} \times \mathbb{A}^p$.

Example 5.1.5 (Continuation of Running Example 4.2.11). Let $\mathcal{A}$ be the cluster algebra from Example 2.4.7. We have $p = 5$ and $q = 3$, and thus obtain an embedding of $\operatorname{Spec}\mathcal{A}$ in $\mathbb{A}^8$. The exchange matrix $\widetilde{B}$ has full $\mathbb{Z}$-rank. Identifying M with the cokernel of $\widetilde{B}$ and choosing as a basis the images of the third, fourth, and fifth standard basis vectors, we obtain an isomorphism of M with $\mathbb{Z}^3$. Under this identification, the grading matrix D is given by

$$D^{\mathrm{tr}} = \begin{pmatrix} -1 & 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{pmatrix}$$

and the degrees of the variables z_v are as follows:

$$\deg_H(x_{25}) = (1, 1, -1) \qquad\qquad \deg_H(x_{13}) = (-1, 1, 1)$$
$$\deg_H(x_{24}) = (2, -1, 0) \qquad\qquad \deg_H(x_{35}) = (-1, 2, 0)$$
$$\deg_H(x_{14}) = (1, -1, 1) \qquad\qquad \deg_H(s_1) = (1, 0, 0)$$
$$\deg_H(s_2) = (0, 1, 0) \qquad\qquad \deg_H(s_3) = (0, 0, 1).$$

We observe that the map $\mathbb{Z}^3 \to \mathbb{Z}$ obtained by summing the three coordinates recovers the positive $\mathbb{Z}$-grading of Example 4.2.11. $\qquad\square$

5.2. **The Hilbert Scheme.** Assume that $\mathcal{A}$ is positively graded. Let $\mathcal{H}_{\mathcal{A}}^{H}$ denote the multigraded Hilbert scheme whose closed points parametrize H-invariant subschemes of $\mathbb{A}^{p+q}$ with the same M-graded Hilbert function as $\operatorname{Spec} \mathcal{A}$. Equivalently, $\mathcal{H}_{\mathcal{A}}^{H}$ parametrizes M-graded ideals J of $\mathbb{K}[\mathbf{z}]$ such that $\mathbb{K}[\mathbf{z}]/J$ and $\mathbb{K}[\mathbf{z}]/I_{\mathcal{A}}$ have the same M-graded Hilbert function. This is a projective scheme; see [HS04] for details. Clearly the cluster algebra $\mathcal{A}$ corresponds to a point of $\mathcal{H}_{\mathcal{A}}^{H}$, which we denote by $[\mathcal{A}]$.

The multigraded Hilbert scheme comes equipped with a *universal family* $\mathcal{U} \to \mathcal{H}_{\mathcal{A}}^{H}$, where the fiber of $\mathcal{U}$ over a geometric point $P \in \mathcal{H}_{\mathcal{A}}^{H}$ is exactly the subscheme of $\mathbb{A}^{p+q}$ corresponding to P. We note that the action of $\mathbb{T} = (\mathbb{K}^{*})^{p+q}$ on $\mathbb{A}^{p+q}$ induces an action of $\mathbb{T}$ on $\mathcal{H}_{\mathcal{A}}^{H}$.

We now relate $\mathcal{A}^{\mathrm{univ}}$ to the Hilbert scheme $\mathcal{H}_{\mathcal{A}}^{H}$:

Theorem 5.2.1. *Let $\mathcal{A}$ be a cluster algebra of finite cluster type, and let $\mathcal{K}$ be the cluster complex of $\mathcal{A}$.*

(1) *There is an action of $\mathbb{T}$ on $\mathbb{A}^{p} = \operatorname{Spec} \mathbb{K}[\mathbf{t}]$ so that $\operatorname{Spec} \mathcal{A}^{\mathrm{univ}}$ is a $\mathbb{T}$-invariant subvariety of $\mathbb{A}^{p+q} \times \mathbb{A}^{p}$.*

(2) *The specialization of $\mathcal{A}^{\mathrm{univ}}$ at $\mathbf{t} = 0$ is the Stanley-Reisner ring $S_{\mathcal{K}*\mathfrak{P}(\mathcal{W})}$.*

(3) *Assume that $\mathcal{A}$ is positively graded. Then there is a $\mathbb{T}$-equivariant morphism*

$$\phi : \operatorname{Spec} \mathbb{K}[\mathbf{t}] \to \mathcal{H}_{\mathcal{A}}^{H}$$

such that $\operatorname{Spec} \mathcal{A}^{\mathrm{univ}}$ is the pullback of the universal family $\mathcal{U}$ along ϕ. Moreover, ϕ is injective with injective differential.

(4) *Suppose that $\mathcal{A}$ is positively graded and of full rank. Let W be the unique torus invariant chart of $\overline{\mathbb{T}.[\mathcal{A}]} \subseteq \mathcal{H}_{\mathcal{A}}^{H}$ containing the torus fixed point corresponding to $I_{\mathcal{K}*\mathfrak{P}(\mathcal{W})}$. Then ϕ is an immersion with image W. In particular, the family*

$$\operatorname{Spec} \mathcal{A}^{\mathrm{univ}} \to \operatorname{Spec} \mathbb{K}[\mathbf{t}]$$

is canonically isomorphic to the restriction of $\mathcal{U} \to \mathcal{H}_{\mathcal{A}}^{H}$ to W.

We proceed by first proving a number of lemmas. Throughout, we use the assumption of a positive grading on $\mathcal{A}$ primarily to ensure the existence of $\mathcal{H}_{\mathcal{A}}^{H}$.

Lemma 5.2.2. *Assume that $\mathcal{A}$ is positively graded. There is a natural morphism $\phi : \operatorname{Spec} \mathbb{K}[\mathbf{t}] \to \mathcal{H}_{\mathcal{A}}^{H}$ such that $\operatorname{Spec} \mathcal{A}^{\mathrm{univ}}$ is the pullback for the universal family $\mathcal{U}$.*

Proof. By Theorem 4.3.1, $\mathcal{A}^{\mathrm{univ}}$ is a free module over $\mathbb{K}[\mathbf{t}]$ with basis given by certain monomials in the cluster and frozen variables. It follows that for any $P \in \operatorname{Spec} \mathbb{K}[\mathbf{t}]$, the fiber of $\operatorname{Spec} \mathcal{A}^{\mathrm{univ}}$ at P has the same M-graded Hilbert function as M. By the universal property of $\mathcal{H}_{\mathcal{A}}^{H}$, there is thus a unique morphism $\phi : \operatorname{Spec} \mathbb{K}[\mathbf{t}] \to \mathcal{H}_{\mathcal{A}}^{H}$ such that $\operatorname{Spec} \mathcal{A}^{\mathrm{univ}}$ is the pullback for the universal family $\mathcal{U}$. $\square$

Lemma 5.2.3. *Assume that $\mathcal{A}$ is positively graded. Then the morphism ϕ is injective.*

Proof. Consider geometric points $P \neq P' \in \mathbb{A}^{p}$, and suppose $\phi(P) = \phi(P')$. Let $1 \leq i \leq p$ be such that $t_i(P) \neq t_i(P')$. By [FZ07, Lemma 12.7 and Lemma 12.8], there is a pair of exchangeable cluster variables x, x' such that in the universal cluster algebra,

$$(5.2.4) \qquad x \cdot x' = t_i \cdot \alpha_1 + g \cdot \alpha_2$$

where $g \in \langle t_j \mid j \neq i \rangle$, α_1 is an extended cluster monomial, and α_2 is a different monomial in the frozen variables. Indeed, the only way to have $\alpha_1 = \alpha_2$ is if $\alpha_1 = \alpha_2 = 1$. This is impossible since $\mathcal{A}$ is positively graded.

Equation (5.2.4) translates into an element

$$(5.2.5) \qquad f_i := z_x \cdot z_{x'} - t_i \cdot z_{\alpha_1} - g \cdot z_{\alpha_2} \in I_{\mathcal{A}^{\mathrm{univ}}}.$$

Here z_{α_1} and z_{α_2} are the monomials in the z_v mapping to α_1 and α_2. The fact that $\phi(P) = \phi(P')$ means that $f_i(P')$ must be in the specialization $I(P)$ of $I_{\mathcal{A}^{\mathrm{univ}}}$ to P. Thus, we obtain that $f_i(P) - f_i(P')$ is in $I(P)$. This implies that in the specialization of $\mathcal{A}^{\mathrm{univ}}$ to P, a non-zero multiple of α_1 can be expressed in terms of α_2. This contradicts the fact that a basis of $\mathcal{A}^{\mathrm{univ}}$ is given by extended cluster monomials (Theorem 4.3.1).

We conclude that instead $\phi(P) \neq \phi(P')$. $\qquad \square$

Lemma 5.2.6. *Assume that $\mathcal{A}$ is positively graded. Then the differential of ϕ is injective everywhere.*

Proof. Consider a geometric point $P \in \operatorname{Spec} \mathbb{K}[\mathbf{t}]$ with coordinates $(s_1, \ldots, s_p)$. The corresponding maximal ideal $\mathfrak{m} \subset \mathbb{K}[\mathbf{t}]$ is $\mathfrak{m} = \langle t_i - s_i \rangle$. The point $\phi(P)$ corresponds to an ideal $I(P) \subset \mathbb{K}[\mathbf{z}]$. This is obtained by specializing the ideal $I_{\mathcal{A}^{\mathrm{univ}}}$ to P.

We will use the identification of $T_{\phi(P)} \mathcal{H}_{\mathcal{A}}^{H}$ with $\operatorname{Hom}(I(P), \mathbb{K}[\mathbf{z}]/I(P))_0$, see e.g. [HS04, Proposition 1.6]. Here we are taking degree 0 elements with respect to the M-grading. Consider the tangent vector η_i at P in direction t_i. To describe the element $\psi_i \in \operatorname{Hom}(I(P), \mathbb{K}[\mathbf{z}]/I(P))_0$ corresponding to $d\phi(\eta_i)$, consider any $f \in I(P)$. Then there exists $\widetilde{f} \in \widetilde{I}_{\mathcal{A}^{\mathrm{univ}}}$ such that $\widetilde{f}$ reduces modulo $\mathfrak{m}$ to f. We may write

$$\widetilde{f} = f + (t_i - s_i)f' + g$$

for some $f' \in \mathbb{K}[\mathbf{z}]$ and $g \in \langle (t_i - s_i)^2, t_j - s_j \mid j \neq i \rangle$. Then $\psi_i(f)$ is the image of f' in $\mathbb{K}[\mathbf{z}]/I(P)$.

The polynomial f_i from (5.2.5) belongs to $I_{\mathcal{A}^{\mathrm{univ}}}$. Let $f_i(P)$ be its reduction modulo $\mathfrak{m}$, viewed as an element of $\mathbb{K}[\mathbf{z}]$. We see

$$f_i = f_i(P) - (t_i - s_i)z_{\alpha_1} - g' \cdot z_{\alpha_2}$$

for some $g' \in \langle t_j - s_j \mid j \neq i \rangle$. With notation as in the preceding paragraph, we thus have $\psi_i(f_i) \equiv z_{\alpha_1}$. Furthermore, for $j \neq i$ we have that $\psi_j(f_i)$ is a multiple of the image of z_{α_2} modulo $I(P)$ (and possibly zero). Since the extended cluster monomials form a basis (Theorem 4.3.1), there is no way to express ψ_i as a linear combination of the $\{\psi_j\}_{j \neq i}$. The claim of the lemma follows. $\qquad \square$

Lemma 5.2.7. *Assume that $\mathcal{A}$ is positively graded and has full $\mathbb{Z}$-rank. There is a morphism of tori $\rho : \operatorname{Spec} \mathbb{K}[\mathbf{t}^{\pm 1}] \to \mathbb{T}$ making the morphism $\phi : \operatorname{Spec} \mathbb{K}[\mathbf{t}] \to \mathcal{H}_{\mathcal{A}}^{H}$ into a torus equivariant morphism.*

Proof. Let $(\widetilde{\mathbf{x}}, \widetilde{B})$ be a seed of $\mathcal{A}$. There is a corresponding seed of $\mathcal{A}^{\mathrm{univ}}$ with extended exchange matrix $\widetilde{B}^{\mathrm{univ}}$; the first m rows of $\widetilde{B}^{\mathrm{univ}}$ agree with those of $\widetilde{B}$, but there are additional rows corresponding to the universal coefficients $\mathbf{t}$. In fact, there are exactly p additional rows, since the number of universal coefficients agrees with the number of almost positive roots of the corresponding root system [FZ07, Lemma 12.8], and this in turn agrees with the number of cluster variables [FZ07, Lemma 11.1].

We now consider any $D \in \mathrm{Mat}_{m \times p}$ such that

$$(\widetilde{B}^{\mathrm{univ}})^{\mathrm{tr}} \cdot \left(\begin{array}{c} D \\ -I_p \end{array} \right) = 0.$$

Note that D necessarily exists since we are assuming that the rows of $\widetilde{B}$ generate $\mathbb{Z}^n$.

Following §4.2, the matrix

$$D' = \left(\begin{array}{c} D \\ -I_p \end{array} \right)$$

defines a grading on $\mathcal{A}^{\mathrm{univ}}$. In particular, every cluster and frozen variable has a degree in $\mathbb{Z}^p$. We now assign these same degrees to the variables z_v and z_w, forgetting about the universal coefficients. Note that the ideal $I_{\mathcal{A}}$ is *not* homogeneous with respect to this grading.

Let $\widetilde{\mathbb{T}}$ be the torus $\mathrm{Spec}\,\mathbb{K}[\mathbf{t}^{\pm 1}]$. Using the above grading, we obtain a map of tori $\rho : \widetilde{\mathbb{T}} \to \mathbb{T}$. We now consider the action of $\widetilde{\mathbb{T}}$ on $[\mathcal{A}] \in \mathcal{H}_{\mathcal{A}}^H$ under ρ. By construction, the one-parameter subgroup λ of $\widetilde{\mathbb{T}}$ dual to the ith universal coefficient t_i acts under ρ in the exchange relations for $\mathcal{A}$ exactly as t_i does. More specifically, if s is the canonical coordinate for λ, after acting by λ (via ρ) on any exchange relation of $\mathcal{A}$ and rescaling the exchange relation uniformly by a power of s, the power of s in each monomial agrees with the power of t_i in the corresponding monomial of the exchange relation of $\mathcal{A}^{\mathrm{univ}}$.

Any fiber of $\mathrm{Spec}\,\mathcal{A}^{\mathrm{univ}}$ over $P \in \widetilde{\mathbb{T}} \subset \mathrm{Spec}\,\mathbb{K}[\mathbf{t}]$ is determined by the specialization of the exchange relations to the point P. It follows that the restriction of ϕ to $\widetilde{\mathbb{T}}$ is torus equivariant. It is a straightforward exercise to show that any morphism of varieties with torus action $X \to Y$ that is equivariant on a dense open subset is equivariant on all of X; the claim follows. $\qquad\square$

Proof of Theorem 5.2.1. Assume first that $\mathcal{A}$ is positively graded. By Lemma 5.2.2 we have seen that there is a natural morphism $\phi : \mathrm{Spec}\,\mathbb{K}[\mathbf{t}] \to \mathcal{H}_{\mathcal{A}}^H$. Moreover, it is injective with injective differential by Lemma 5.2.3 and Lemma 5.2.6.

Before proving the claims of the theorem in general, we continue with the assumption that $\mathcal{A}$ is positively graded and additionally assume that is has full $\mathbb{Z}$-rank. Then ϕ is torus equivariant by Lemma 5.2.7. Since ϕ is equivariant and the torus is acting transitively on $\mathbb{A}^p$, $\phi(\mathbb{A}^p)$ is contained in the closure of the torus orbit $\mathbb{T}.[\mathcal{A}]$.

Since ϕ is injective, we obtain that $\dim \overline{\phi(\mathbb{A}^p)} = \dim \mathrm{Spec}\,\mathbb{K}[\mathbf{t}]$. As noted in the proof of Lemma 5.2.7, $\dim \mathrm{Spec}\,\mathbb{K}[\mathbf{t}] = p$. On the other hand, the dimension of the torus orbit $\mathbb{T}.[\mathcal{A}]$ is

$$\dim \mathbb{T} - \dim \mathbb{T}_{[\mathcal{A}]} = (p + q) - \dim H = (p + q) - q = p.$$

Indeed, the dimension of H is equal to the rank of the cokernel of $\widetilde{B}$ for an extended seed $\widetilde{B}$ of $\mathcal{A}$; since $\widetilde{B}$ has full rank this is $m - n = q$. We conclude that $\overline{\phi(\mathbb{A}^p)}$ is equal to the closure of the torus orbit $\mathbb{T}.[\mathcal{A}]$. Since $\mathcal{A}$ is positively graded, $\mathcal{H}_{\mathcal{A}}^H$ is projective, so $\overline{\phi(\mathbb{A}^p)}$ is a projective (potentially non-normal) toric variety, and ϕ is a toric morphism.

It follows that the image of ϕ is an affine $\mathbb{T}$-invariant chart of the orbit closure of $[\mathcal{A}]$. Since ϕ has injective differential at its torus fixed point by Lemma 5.2.6, it follows by Lemma 5.2.8 below that ϕ is an isomorphism onto its image.

We may thus identify $\operatorname{Spec} \mathcal{A}^{\mathrm{univ}} \to \mathbb{A}^p$ with the restriction of $\mathcal{U} \to \mathcal{H}_{\mathcal{A}}^H$ to W. This latter family is naturally $\mathbb{T}$-equivariant. Since W is a $\mathbb{T}$-invariant subvariety of the Hilbert scheme, we obtain a $\mathbb{T}$-equivariant structure on the family

$$\operatorname{Spec} \mathcal{A}^{\mathrm{univ}} \to \mathbb{A}^p$$

as well.

Continuing in this special case of full $\mathbb{Z}$-rank, we now show that $P = \phi(0)$ is the point of $\mathcal{H}_{\mathcal{A}}^H$ corresponding to $I_{\mathcal{K}*\mathfrak{P}(\mathcal{W})}$. Let $I(P)$ be the ideal corresponding to P. We notice that P is a torus fixed point of $\mathcal{H}_{\mathcal{A}}^H$, so $I(P)$ must be a monomial ideal. Again by Theorem 4.3.1, the monomials in z_v from compatible cluster and frozen variables form a $\mathbb{K}$-basis for $\mathbb{K}[\mathbf{z}]/I(P)$. It follows that $I(P) = I_{\mathcal{K}*\mathfrak{P}(\mathcal{W})}$ as desired. We have thus proven all claims of the theorem when $\mathcal{A}$ is positively graded of full $\mathbb{Z}$-rank.

We now drop the assumption that $\mathcal{A}$ is positively graded of full $\mathbb{Z}$-rank. We can certainly add frozen variables to $\mathcal{A}$ to obtain a new cluster algebra $\mathcal{A}'$ of full $\mathbb{Z}$-rank. By Lemma 4.2.10, we can assume that we have done this in a way so that $\mathcal{A}'$ is also positively graded. We have shown that the statements of Theorem 5.2.1 thus apply to $\mathcal{A}'$. In particular, the family

$$\operatorname{Spec}(\mathcal{A}')^{\mathrm{univ}} \to \mathbb{A}^p$$

is $\mathbb{T}'$-equivariant, where $\mathbb{T}'$ is the diagonal torus on $\mathbb{A}^{p+q'}$. Here, $q' - q$ is the number of frozen variables we added. Furthermore, the fiber over 0 in this family is $\operatorname{Spec} S_{\mathcal{K}*\mathfrak{P}(\mathcal{W}')}$. Here $\mathcal{W}'$ is the set of frozen variables for $\mathcal{A}'$.

We may obtain $\operatorname{Spec} \mathcal{A}^{\mathrm{univ}}$ by intersecting $\operatorname{Spec}(\mathcal{A}')^{\mathrm{univ}} \subseteq \mathbb{A}^{p+q'}$ with the $q' - q$ hyperplanes $z_v = 1$ for those $v \in \mathcal{W}' \setminus \mathcal{W}$, that is, setting the additional frozen variables to 1. The torus $\mathbb{T}$ embeds as a subtorus of $\mathbb{T}'$, and since it acts trivially on z_v for $v \in \mathcal{W}' \setminus \mathcal{W}$, we obtain a $\mathbb{T}$-action on $\operatorname{Spec} \mathcal{A}^{\mathrm{univ}}$. With regards to this action, the family $\operatorname{Spec} \mathcal{A}^{\mathrm{univ}} \to \mathbb{A}^p$ is $\mathbb{T}$-equivariant by construction. This implies that in the positively graded case, the morphism ϕ is $\mathbb{T}$-equivariant.

Finally, we note that the fiber over 0 of this family is obtained from the fiber over 0 of $\operatorname{Spec}(\mathcal{A}')^{\mathrm{univ}} \to \mathbb{A}^p$ by setting $z_v = 1$ for those $v \in \mathcal{W}' - \mathcal{W}$. This is clearly $\operatorname{Spec} S_{\mathcal{K}*\mathfrak{P}(\mathcal{W})}$ as desired.

It remains to consider claim 4 in the cases where $\mathcal{A}$ has full rank, but not full $\mathbb{Z}$-rank. Similar to the case of full $\mathbb{Z}$-rank, we obtain that the dimension of $\mathbb{T}.[\mathcal{A}]$ is p, so again $\overline{\phi(\mathbb{A}^p)}$ is the closure of $\mathbb{T}.\mathcal{A}$. Since by the above ϕ is $\mathbb{T}$-equivariant, we see that it acts on $\mathbb{A}^p$ with a dense orbit. We may again appeal to Lemma 5.2.8 to conclude that ϕ is an isomorphism. This shows claim 4. $\qquad\square$

Lemma 5.2.8. *Let W be an affine variety. Suppose an algebraic torus $\widetilde{\mathbb{T}}$ acts on $\mathbb{A}^p$ and W, both with dense orbits. Let $f : \mathbb{A}^p \to W$ be a dominant $\widetilde{\mathbb{T}}$-equivariant morphism, with the differential df injective at 0. Then f is an isomorphism.*

Proof. See [CLS11] for details on toric varieties. The variety W may be constructed as $\operatorname{Spec} \mathbb{K}[S]$ for some affine semigroup S, and the morphism f is induced by a map of semigroups $g : S \to \mathbb{Z}_{\geq 0}^p$. Since f is dominant, g must be injective, so we may identify S with a submonoid of $\mathbb{Z}_{\geq 0}^p$.

The differential being injective at 0 is equivalent to the induced map of cotangent spaces $\mathfrak{m}_{f(0)}/\mathfrak{m}_{f(0)}^2 \to \mathfrak{m}_0/\mathfrak{m}_0^2$ being surjective, where $\mathfrak{m}_0$ is the maximal ideal of $0 \in \mathbb{A}^p$, etc. Notice that $\mathfrak{m}_0/\mathfrak{m}_0^2$ has a basis given by the images of the characters

corresponding to the standard basis of $\mathbb{Z}_{\geq 0}^p$. Since the map of cotangent spaces takes monomials to monomials, its surjectivity implies that S must contain the standard basis of $\mathbb{Z}_{\geq 0}^p$. But then $S = \mathbb{Z}_{\geq 0}^p$ and the claim follows. $\square$

Remark 5.2.9. Instead of considering $\mathcal{A}^{\mathrm{univ}}$, we may consider the algebra

$$\mathcal{A}_{\circ}^{\mathrm{univ}} := \mathcal{A}^{\mathrm{univ}} \otimes_{\mathbb{K}[\mathbf{t}]} \mathbb{K}[\mathbf{t}^{\pm 1}]$$

where we have inverted all the universal coefficients. Then $\operatorname{Spec}\mathcal{A}_{\circ}^{\mathrm{univ}}$ fibers over $(\mathbb{K}^*)^p$ instead of $\mathbb{A}^p$. When $\mathcal{A}$ is positively graded and has full rank, $\operatorname{Spec}\mathcal{A}_{\circ}^{\mathrm{univ}}$ is the restriction of the universal family $\mathcal{U}$ to the torus orbit $\mathbb{T}.[\mathcal{A}]$. In fact, one is able to prove this much simpler statement without making use of the machinery of [GHKK18] (in the form of Theorem 4.3.1). Indeed, using the full rank of $\mathcal{A}$ and an argument similar to that of Lemma 5.2.7, the family $\operatorname{Spec}\mathcal{A}_{\circ}^{\mathrm{univ}} \to (\mathbb{K}^*)^p$ is equivariant with a transitive action on the base. As such, the map must be flat, and all fibers must be irreducible. It follows by dimension reasons that the fiber over $(1,\ldots,1)$ is $\operatorname{Spec}\mathcal{A}$, and we thus obtain obtain a corresponding morphism $(\mathbb{K}^*)^p \to \mathbb{T}.[\mathcal{A}]$. The rest of the argument is similar to the proof of Theorem 4.3.1.

In the proof of Theorem 5.2.1, we used the existence of $\mathcal{A}^{\mathrm{univ}}$, along with some of its specific properties. However, we suspect that one should be able to show directly that the universal family over the orbit $\mathbb{T}.[\mathcal{A}]$ satisfies the universal properties required of $\mathcal{A}^{\mathrm{univ}}$. Going a step further, we believe that one should be able to obtain a similar identification in the infinite cluster type case. Here, the torus acting would be infinite dimensional, since there are infinitely many cluster variables. This is a necessary feature of the infinite cluster type case, since we know from [Rea14] that there are infinitely many universal coefficients in this case.

In many infinite cluster type cases, $\mathcal{A}$ is nonetheless generated by a finite number of cluster variables. Instead of considering the cluster embedding, we could consider one coming from a finite number of cluster variables. Let $\mathbb{T}'$ be the torus of the ambient space. As noted above, we cannot recover $\mathcal{A}_{\circ}^{\mathrm{univ}}$ as the universal family over $\mathbb{T}'.[\mathcal{A}]$. It would be interesting to identify the subfamily of $\mathcal{A}_{\circ}^{\mathrm{univ}}$ obtained in this way.

Extending from $\operatorname{Spec}\mathcal{A}_{\circ}^{\mathrm{univ}}$ to $\operatorname{Spec}\mathcal{A}^{\mathrm{univ}}$ presents significantly more difficulty in the infinite cluster type case, since Theorem 4.3.1 will no longer be true in general. However, in good situations, $\mathcal{A}^{\mathrm{univ}}$ will still have a theta basis in the sense of [GHKK18]. We expect that the fiber over zero of the family $\operatorname{Spec}\mathcal{A}^{\mathrm{univ}} \to \operatorname{Spec}\mathbb{K}[\mathbf{t}]$ will no longer come directly from the cluster complex, but will instead be related to the scattering diagram for $\mathcal{A}$.

Remark 5.2.10. Consider any cluster algebra $\mathcal{A}$ of finite cluster type. As noted in the proof of Theorem 5.2.1, it is always possible to add additional frozen variables so that the new cluster algebra has full $\mathbb{Z}$-rank. Furthermore, by Lemma 4.2.10, it is always possible to do this in a way so that the new cluster algebra $\mathcal{A}'$ admits a positive grading.

Example 5.2.11. It is straightforward to check that many classical examples of cluster algebras of finite cluster type are positively graded and of full $\mathbb{Z}$-rank:

(1) The polynomial ring in $2(n + 1)$ variables (type A_n) [FWZ21, Example 6.1.2];

(2) The coordinate ring of the Grassmannian $\operatorname{Gr}(2, n + 3)$ of 2-planes in $\mathbb{K}^{n+3}$ in its Plücker embedding (type A_n) [FWZ21, Example 6.3.1];

(3) The coordinate rings of the Grassmannians $\mathrm{Gr}(3,6)$ (type D_4), $\mathrm{Gr}(3,7)$ (type E_6), and $\mathrm{Gr}(3,8)$ (type E_8) in their Plücker embeddings [Sco06] [FWZ21, Theorem 6.7.8].

A nice example of a cluster algebra of type D_n is the coordinate ring of the Schubert divisor of $\mathrm{Gr}(2, n+2)$ in its Plücker embedding (type D_n) [FWZ21, Example 6.3.5]. However, this example is not of full $\mathbb{Z}$-rank. Instead, we may consider

(4) The cluster algebra of type D_n described in [FWZ17, Example 5.4.11] with $n+2$ frozen variables.

This cluster algebra does satisfy the hypotheses of Theorem 5.2.1. The coordinate ring of the Schubert divisor may be obtained by specializing two of the frozen variables to 1. We may also consider non-simply laced examples:

(5) The coordinate ring of $\mathrm{Gr}(2, n+2)$ in its Plücker embedding with type B_n cluster structure [FZ03a, Proposition 12.11];
(6) The coordinate ring of the Segre embedding $\mathbb{P}^n \times \mathbb{P}^n \subset \mathbb{P}^{n^2+2n}$ with type C_n cluster structure [FWZ21, Example 6.3.4].

Both of these examples are also positively graded and of full $\mathbb{Z}$-rank.   □

Remark 5.2.12. Assume that $\mathcal{A}$ is a positively graded cluster algebra of finite cluster type, but for some (or equivalently all) seeds, the extended exchange matrix $\widetilde{B}$ is rank deficient. Then the conclusion of Theorem 5.2.1 that $\operatorname{Spec} \mathcal{A}^{\mathrm{univ}} \to \operatorname{Spec} \mathbb{K}[\mathbf{t}]$ may be identified with the universal family over a partial closure of $\mathbb{T}.[\mathcal{A}]$ cannot be true. Indeed, since $\widetilde{B}$ is rank deficient, the dimension of H is strictly larger than $m - n = q$. We thus have

$$\dim \mathbb{T}.[\mathcal{A}] = p + q - \dim H < p.$$

Since $\dim \operatorname{Spec} \mathbb{K}[\mathbf{t}] = p$, the claim follows.

We record the following algebraic consequence of Theorem 5.2.1.

Corollary 5.2.13. *Let $\mathcal{A}$ be a cluster algebra of finite cluster type. Then $\mathcal{A}$ is Gorenstein.*

See [MRZ18] for a related result for lower bound algebras. We also note that in the special case of $\mathcal{A}$ having no frozen variables, the possible singularities of $\mathcal{A}$ have been classified in [BFMS21]. It follows from loc. cit. that cluster algebras of finite cluster type *with no frozen variables* are Gorenstein.

Proof of Corollary 5.2.13. We may add frozen variables to obtain a positively graded cluster algebra $\mathcal{A}'$ of full $\mathbb{Z}$-rank. If we can show that $\mathcal{A}'$ is Gorenstein, then it follows that $\mathcal{A}$ is also Gorenstein, since $\mathcal{A}$ is a complete intersection in $\mathcal{A}'$ (obtained by setting additional frozen variables equal to 1). Hence, in the following we may assume that $\mathcal{A}$ is positively graded of full $\mathbb{Z}$-rank.

The ring $S_{\mathcal{K}*\mathfrak{P}(\mathcal{W})}$ is Gorenstein since $\mathcal{K}$ is a sphere, see [Sta96, §II.5]. By Theorem 5.2.12 $\operatorname{Spec} S_{\mathcal{K}*\mathfrak{P}(\mathcal{W})}$ is a complete intersection in $\operatorname{Spec} \mathcal{A}^{\mathrm{univ}}$, so it follows from the definition of Gorenstein that $\mathcal{A}^{\mathrm{univ}}$ is also Gorenstein along $\mathbf{t} = 0$. By Theorem 5.2.14 every $\mathbb{T}$-orbit closure of $\operatorname{Spec} \mathcal{A}^{\mathrm{univ}}$ intersects the special fiber over $\mathbf{t} = 0$. Since the non-Gorenstein locus of $\operatorname{Spec} \mathcal{A}^{\mathrm{univ}}$ is closed, $\mathbb{T}$-invariant, and avoids the fiber over $\mathbf{t} = 0$, it follows that the non-Gorenstein locus of $\operatorname{Spec} \mathcal{A}^{\mathrm{univ}}$ is empty. Thus, $\mathcal{A}^{\mathrm{univ}}$ is Gorenstein.

To conclude, we observe that since $\mathcal{A}$ is a complete intersection in $\mathcal{A}^{\mathrm{univ}}$ (obtained by setting $t_1 = \ldots = t_p = 1$), $\mathcal{A}$ inherits the Gorenstein property from $\mathcal{A}^{\mathrm{univ}}$. $\qquad\square$

Example 5.2.14 (Continuation of Running Example 5.1.5). Let $\mathcal{A}$ be the cluster algebra from Example 2.4.7. Since $\mathcal{A}$ has full $\mathbb{Z}$-rank, we obtain that $\mathcal{A}^{\mathrm{univ}}$ (cf. Example 4.1.10) is the coordinate ring of the universal family over an open subset of $\overline{\mathbb{T}.[\mathcal{A}]}$. Specializing to $\mathbf{t} = 0$, we obtain the Stanley-Reisner ring $S_{\mathcal{K}}$ for the cluster complex $\mathcal{K}$ of Example 2.5.3. The torus $\mathbb{T}$ is acting on the parameters t_i with the following weights:

$$\deg_{\mathbb{T}}(t_1) = (-1, 0, 1, 1, 0, 0, 0, -1)$$
$$\deg_{\mathbb{T}}(t_2) = (0, -1, 0, 1, 1, -1, 0, 0)$$
$$\deg_{\mathbb{T}}(t_3) = (1, 0, -1, 0, 1, 0, -1, 0)$$
$$\deg_{\mathbb{T}}(t_4) = (1, 1, 0, -1, 0, -1, 0, 0)$$
$$\deg_{\mathbb{T}}(t_5) = (0, 1, 1, 0, -1, 0, -1, 0).$$

Here, we have ordered the cluster and frozen variables as x_{25} ,x_{13}, x_{24}, x_{35}, x_{14}, s_1, s_2, s_3. $\qquad\square$

5.3. **Connection to Gröbner Theory.** Our Theorem 5.2.1 is connected to Gröbner theory for cluster algebras as we now explain. See [Stu96] for an introduction to Gröbner theory.

Let $J \subset \mathbb{K}[z_1, \ldots, z_s]$ be an M-graded ideal with each graded piece a finite dimensional $\mathbb{K}$-vector space. We may thus consider the Hilbert scheme $\mathcal{H}_J$ parametrizing M-graded ideals with the same multigraded Hilbert function as J. In particular, J gives a point $[J]$ of this Hilbert scheme. Let $\mathbb{T}$ be the diagonal torus acting on $\operatorname{Spec} \mathbb{K}[\mathbf{z}] = \mathbb{A}^s$ and let $N_{\mathbb{T}} = \mathbb{Z}^s$ be its cocharacter lattice. The action of $\mathbb{T}$ on $\operatorname{Spec} \mathbb{K}[\mathbf{z}]$ induces an action on $\mathcal{H}_J$. The setting of Theorem 5.2.1 is exactly the special case where $J = I_{\mathcal{A}}$ and M is the grading induced by the stabilizer of $\mathbb{T}$ on $\operatorname{Spec} \mathcal{A}$.

Returning to the general setting, consider the orbit closure $Y = \overline{\mathbb{T}.[J]} \subset \mathcal{H}_J$. This is a quasi-projective, potentially non-normal toric variety. Let $\widetilde{\mathbb{T}}$ be the image of $\mathbb{T}$ in $\operatorname{Aut}(Y)$; we denote the character lattice of this torus by $N_{\widetilde{\mathbb{T}}}$. The normalization of Y corresponds to a fan $\widetilde{\Sigma}$ in the $\mathbb{R}$-vector space associated to $N_{\widetilde{\mathbb{T}}}$, see e.g. [CLS11].

The map $\mathbb{T} \to \widetilde{\mathbb{T}}$ induces a map

$$\eta : N_{\mathbb{T}} \to N_{\widetilde{\mathbb{T}}}.$$

Let Σ be the preimage in $\mathbb{R}^s$ of $\widetilde{\Sigma}$ under the map of real vector spaces associated to η. The following is well-known to experts:

Proposition 5.3.1. *The fan Σ is the Gröbner fan of J.*

Proof. Two weight vectors in $\mathbb{Z}^s$ lie in the interior of the same maximal dimensional cone of the Gröbner fan of J if and only if the initial ideals of J with respect to these weight vectors is the same monomial ideal. This is equivalent to requiring that the limits in $\mathcal{H}_J$ of the images under η of the corresponding one-parameter subgroups are equal. But this is equivalent to requiring that the images under η of these one-parameter subgroups lie in the interior of the same maximal cone of Σ. $\qquad\square$

We now return to the special situation of Theorem 5.2.1. Let $\deg_{\mathbb{T}}$ be the map assigning to any cluster or frozen variable $v \in \mathcal{V} \cup \mathcal{W}$ the degree in $\mathbb{Z}^{p+q}$ of z_v. We extend this in the obvious way to a map on the extended cluster monomials of $\mathcal{A}$.

Corollary 5.3.2. *Let $\mathcal{A}$ be a cluster algebra of finite cluster type, with ideal $I_\mathcal{A}$ as in §5.1.*

(1) *The Gröbner fan of $I_\mathcal{A}$ contains a maximal cone C corresponding to the monomial ideal $I_{\mathcal{K}*\mathfrak{P}(\mathcal{W})}$.*

(2) *The cone C is dual to the cone of $\mathbb{Z}^{p+q}$ generated by all*

$$\deg_{\mathbb{T}}(x \cdot x') - \deg_{\mathbb{T}} \alpha_1 \in \mathbb{Z}^{p+q}$$

where x, x', α_1 appear in an exchange relation

$$x \cdot x' = \alpha_1 + \alpha_2$$

such that α_1, α_2 are extended cluster monomials, and α_2 only depends on the frozen variables.

(3) *Assume that $\mathcal{A}$ is positively graded of full rank. Modulo the lineality space, the cone C is a simplicial cone. Moreover, if $\mathcal{A}$ has full $\mathbb{Z}$-rank, C is smooth modulo lineality space.*

Proof. We first prove all claims of the theorem in the special case that $\mathcal{A}$ is positively graded and of full rank. Let $\widetilde{\Sigma}$ be the fan describing the normalization of the closure of $\mathbb{T}.[\mathcal{A}]$ in $\mathcal{H}_\mathcal{A}^H$. By Theorem 5.2.14, there is a maximal cone $\widetilde{C}$ of $\widetilde{\Sigma}$ corresponding to $I_{\mathcal{K}*\mathfrak{P}(\mathcal{W})}$. Applying Proposition 5.3.1, the preimage under η of $\widetilde{C}$ in the Gröbner fan Σ is the desired cone C. This shows the first claim.

Again by Theorem 5.2.14, we know that the cone $\widetilde{C}$ is smooth, since it corresponds to a smooth chart of $\overline{\mathbb{T}.[\mathcal{A}]}$. In particular, the generators of the rays of $\widetilde{C}$ are linearly independent. It follows that the cone C, modulo lineality space, is simplicial.

Let us now additionally assume that $\mathcal{A}$ has full $\mathbb{Z}$-rank. By Corollary 5.1.3, this implies that H is a torus, or equivalently, that M is torsion free. The torus $\widetilde{\mathbb{T}}$ may be identified with $\mathbb{T}/H$, so we have an exact sequence of character groups

$$0 \to M_{\widetilde{\mathbb{T}}} \to \mathbb{Z}^{p+q} \to M \to 0$$

where $M_{\widetilde{\mathbb{T}}}$ is the character lattice of $\widetilde{\mathbb{T}}$. Since M is torsion free, this sequence splits, and we dually obtain a section of the map $\eta : N_{\mathbb{T}} \to N_{\widetilde{\mathbb{T}}}$. This implies that C is smooth modulo lineality space.

For the second claim of the corollary, we consider the cone $\widetilde{C}^\vee$ dual to $\widetilde{C}$. The generators of $\widetilde{C}^\vee$ are the degrees (in $M_{\widetilde{\mathbb{T}}}$) of the universal coefficients $t_1, \ldots, t_p$. Since $M_{\widetilde{\mathbb{T}}}$ is a sublattice of $\mathbb{Z}^{p+q}$, we will view these degrees as elements of the latter lattice. By [FZ07, Lemma 12.7 and Lemma 12.8], these degrees are exactly of the form $\deg_{\mathbb{T}}(x \cdot x') - \deg_{\mathbb{T}} \alpha_1$, subject to the conditions in the statement of the corollary. Since we obtain C by taking the dual of the image of $\widetilde{C}^\vee$ in $\mathbb{R}^{p+q}$, the final claim follows.

We now return to the general setting of $\mathcal{A}$ being an arbitrary cluster algebra of finite cluster type. By Lemma 4.2.10, we can add frozen variables to obtain a positively graded full rank cluster algebra $\mathcal{A}'$. Since $\mathcal{A}'$ is positively graded, the Gröbner cone C' for $\mathcal{A}'$ contains in its interior a weight $w' \in \mathbb{Z}_{\geq 0}^{p+q'}$. The initial term with respect to w' of each element of $I_{\mathcal{A}'}$ is in $I_{\mathcal{K}*\mathfrak{P}(\mathcal{W}')}$, and does not involve

any of the extra frozen variables. Projecting w' to $w \in \mathbb{Z}_{\geq 0}^{p+q}$, it follows that the initial terms with respect to w of each element of $I_{\mathcal{A}}$ is in $I_{\mathcal{K}*\mathfrak{P}(\mathcal{W})}$. It follows that $I_{\mathcal{K}*\mathfrak{P}(\mathcal{W})}$ is an initial ideal of $I_{\mathcal{A}}$, and the first claim of the corollary follows.

For the second claim of the corollary, it is straightforward to verify that the elements listed in the claim must lie in $C^{\vee}$, the dual cone of C. We now show that they generate $C^{\vee}$. To that end, let σ be the cone generated by these elements. Let $w \in \mathbb{R}^{p+q} \cap \sigma^{\vee}$. Extend w to the unique element $w' \in \mathbb{R}^{p+q'}$ by giving all coordinates corresponding to a new frozen variable value 0. Then from the description of the Gröbner cone C' for $\mathcal{A}'$, we see that $w' \in C'$. It follows by the discussion of the preceding paragraph that $w \in C$. Hence $\sigma^{\vee} \subseteq C$, and we conclude that $C^{\vee} \subseteq \sigma$. The claim now follows. $\square$

Example 5.3.3 (Continuation of Running Example 5.2.14)**.** For the cluster algebra of Example 2.4.7, the Gröbner cone C corresponding to $I_{K*\mathfrak{P}(\mathcal{W})}$ is the cone dual to the degrees of the t_i from Example 5.3.3. One computes that C is the cone whose lineality space is spanned by the rows of

$$\begin{pmatrix} -1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 2 & -1 & 2 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

and has positive directions generated by the rows of

$$\begin{pmatrix} -1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \end{pmatrix}.$$

We are again ordering variables as x_{25} ,x_{13}, x_{24}, x_{35}, x_{14}, s_1, s_2, s_3. We see immediately that this cone is in fact smooth modulo lineality space, as Corollary 5.3.2 predicts. $\square$

In general, if $\mathcal{A}$ has full rank but not full $\mathbb{Z}$-rank, the cone C of Corollary 5.3.2 may not be smooth, as the following example shows.

Example 5.3.4 (Pullback of $\mathrm{Gr}(2,6)$)**.** Consider the cluster algebra $\mathcal{A}$ associated to the seed

$$\widetilde{B}^{\mathrm{tr}} = \begin{pmatrix} 0 & 1 & 0 & -2 & 2 & -2 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 2 & -2 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 2 & -2 & 2 \end{pmatrix}.$$

$$\widetilde{\mathbf{x}} = (\quad x_{13}, \quad x_{14}, \quad x_{15}, \quad y_{12}, \quad y_{23}, \quad y_{34}, \quad y_{45}, \quad y_{56}, \quad y_{16} \quad)$$

Note that $\mathcal{A}$ has full rank, but not full $\mathbb{Z}$-rank. It is straightforward to check that $\mathcal{A}$ may be obtained from the coordinate ring of $\mathrm{Gr}(2,6)$ by replacing each frozen Plücker variable $x_{i(i+1)}$ with $y_{i(i+1)}^2$.

Although the Gröbner cone C for $I_{\mathcal{A}}$ is simplicial modulo lineality space, we claim it is not smooth. Indeed, this equivalent to seeing that the dual cone $C^{\vee}$ is not smooth. We may describe primitive generators for the rays of $C^{\vee}$ as in Corollary 5.3.2. This gives 9 rays in $\mathbb{Z}^{15}$, which we may encode as the columns of a 15×9 matrix. The first 9 rows correspond to the cluster variables. Since omitting the frozen variables from $\mathcal{A}$ leads to a cluster algebra of non-full rank, it follows

that the determinant of the top 9×9 submatrix vanishes (cf. Remark 5.2.12). On the other hand, since the variables y_{ij} are always appearing in any exchange relation with a power divisible by two, it follows that any other maximal minor of this matrix is divisible by 2. Hence $C^\vee$ is not smooth. □

We record another algebraic consequence of our results:

Corollary 5.3.5. *Let $\mathcal{A}$ be a skew-symmetric cluster algebra of finite cluster type. Then $T^2(\mathcal{A}) = 0$.*

Proof. By Corollary 5.3.2, there is a $\mathbb{K}^*$-equivariant family over $\mathbb{A}^1$ with special fiber $\mathrm{Spec}(S_{\mathcal{K}*\mathfrak{P}(\mathcal{W})})$ and all other fibers isomorphic to $\mathrm{Spec}(\mathcal{A})$. The claim now follows from semi-continuity of T^2 and Theorem 3.1.1. □

Remark 5.3.6. In [BMNC21], Bossinger, Mohammadi, and Nájera Chávez show that for the coordinate rings of $\mathrm{Gr}(2, n+3)$ and $\mathrm{Gr}(3,6)$, one may obtain the corresponding cluster algebras with universal coefficients via a construction from Gröbner theory. In light of Example 5.2.11, our Theorem 5.2.1 is a generalization of this, as we now explain.

Given a weighted homogeneous ideal J of a polynomial ring $\mathbb{K}[\mathbf{z}]$ and a maximal cone C in the Gröbner fan of J, the authors of [BMNC21] construct a flat family over $\mathbb{A}^s$ whose generic fiber is $V(J)$ [BMNC21, Theorem 3.14]. The cluster algebras with universal coefficients $\mathcal{A}^{\mathrm{univ}}$ are then realized for $\mathrm{Gr}(2, n+3)$ and $\mathrm{Gr}(3,6)$ by applying this construction to a well-chosen cone of the Gröbner fan for the ideal $I_{\mathcal{A}}$ of the appropriate cluster algebra $\mathcal{A}$ [BMNC21, Theorem 1.3, Theorem 1.4]. That this construction agrees with $\mathcal{A}^{\mathrm{univ}}$ is verified by an explicit combinatorial computation.

A careful study of the flat family of [BMNC21] reveals that it may be constructed as follows. With notation as in the beginning of this section, the choice of C determines a torus fixed point P of $\mathcal{H}_J$. Consider the $\mathbb{T}$-orbit closure in $\mathcal{H}_J$ of $[J]$. There is a unique affine torus-invariant chart W of this orbit closure that contains P. Let d be the number of connected components of the stabilizer $\mathbb{T}_{[J]}$; this is equal to the number of connected components in a fiber of the map of tori $\mathbb{T} \to \widetilde{\mathbb{T}}$.

The flat family of [BMNC21] may be identified with the pullback of the universal family over $\mathcal{H}_J$ to the Cox torsor over a degree-d cover of the normalization of W. To check this, it suffices to show that the restriction of the family from [BMNC21] to $(\mathbb{K}^*)^s \subset \mathbb{A}^s$ has the desired form. But on this open set, the family of [BMNC21] is relatively easy to describe; we leave it to the reader to check the details.

In the instances where $\mathcal{A}$ is a positively graded cluster algebra of full $\mathbb{Z}$-rank and finite cluster type, we showed in the course of the proof of Theorem 5.2.1 that the chart W is just a copy of affine space $\mathbb{A}^p$. Hence, it agrees with the Cox torsor of its normalization. Furthermore, by Corollary 5.1.3 it follows that $d = 1$. We thus recover [BMNC21, Theorem 1.3, Theorem 1.4].

Suppose instead that $\mathcal{A}$ is only of full rank, but not full $\mathbb{Z}$-rank. In such cases, the construction of [BMNC21] cannot produce the universal cluster algebra, since their family arises by pulling back along (among other things) a degree d cover for some $d > 1$.

6. Universal Coefficients and Deformations

6.1. **The Characteristic Map.** By Theorem 4.3.1, the family $\operatorname{Spec}\mathcal{A}^{\mathrm{univ}} \to \mathbb{A}^p$ is flat, and by Theorem 5.2.1, the fiber over 0 is $\operatorname{Spec}S_{\mathcal{K}*\mathfrak{P}(\mathcal{W})}$. Thus, this family is a *deformation* of $Y = \operatorname{Spec}S_{\mathcal{K}*\mathfrak{P}(\mathcal{W})}$. Our next goal will be to relate $\mathcal{A}^{\mathrm{univ}}$ to the semiuniversal deformation of Y. Before doing this, we need to analyze the first order deformations induced by $\mathcal{A}^{\mathrm{univ}}$.

Definition 6.1.1 (Property T1). We say a matrix $\widetilde{B} = (b_{ij}) \in \operatorname{Mat}_{m\times n}(\mathbb{Z})$ fulfills property T1 if the following holds:

For every $1 \leq j \leq n$ and every w in the $\mathbb{Z}$-span of the columns of $\widetilde{B}$ satisfying $w_j = 0$ and

$$(6.1.2) \qquad\qquad w_i \geq \begin{cases} 1 - \max(0, b_{ij}) & i \leq n, \ b_{ij} \neq 0 \\ -\max(0, b_{ij}) & \text{else} \end{cases}$$

for all $i \neq j$, it follows that either $w = 0$ or $w = -B_j := (-b_{1j}, \ldots, -b_{mj})$.

We say that a cluster algebra $\mathcal{A}$ fulfills property T1 if every extended exchange matrix fulfills property T1.

Remark 6.1.3. Suppose that $w = 0$ satisfies (6.1.2) for some fixed j. It follows that b_{ij} is non-negative for $i \leq n$. Likewise, suppose that $w = -B_j$ satisfies (6.1.2) for some fixed j. It follows that b_{ij} is non-positive for $i \leq n$. In these situations, the vertex j of the graph $\Gamma(B)$ is respectively a sink or a source.

Example 6.1.4 (Continuation of Running Example 5.3.3). We check the T1 property for the extended exchange matrix $\widetilde{B}$ from Example 2.4.7. Consider first $j = 1$. For w to be in the $\mathbb{Z}$-span of the columns of $\widetilde{B}$ and satisfy $w_1 = 0$, w must be a multiple of the first column of $\widetilde{B}$. Imposing (6.1.2) for $j = 1$ imposes the (in)equalities

$$w_2 \geq 1, \ w_3 \geq -1, w_4 \geq 0, w_5 \geq -1.$$

The only way to satisfy this is by taking w as the -1 times the first column of $\widetilde{B}$.

Consider instead $j = 2$. For w to be in the $\mathbb{Z}$-span of the columns of $\widetilde{B}$ and satisfy $w_2 = 0$, w must be a multiple of the second column of $\widetilde{B}$. Imposing (6.1.2) for $j = 0$ imposes the (in)equalities

$$w_1 \geq 0, \ w_3 \geq -1, w_4 \geq 0, w_5 \geq 0.$$

The only way to satisfy this is by taking $w = 0$.

We thus see that property T1 is satisfied for $\widetilde{B}$. One may similarly check that the four other extended exchange matrices for $\mathcal{A}$ also satisfy property T1. Hence, the cluster algebra $\mathcal{A}$ satisfies property T1. $\qquad\qquad\square$

Let $\mathcal{A}$ be a cluster algebra of finite cluster type. We say $\mathcal{A}$ has an *isolated vertex* if for some (or equivalently every) extended exchange matrix $\widetilde{B} = (b_{ij})$ of $\mathcal{A}$, there is an index $1 \leq k \leq n$ such that $b_{ij} = 0$ whenever $i = k$ or $j = k$. Note that if $\mathcal{A}$ is positively graded, it cannot have an isolated vertex.

Let $J = I_{\mathcal{K}*\mathfrak{P}(\mathcal{W})}$. As in the proof of Lemma 5.2.6, there is a natural map $T_0 \operatorname{Spec}\mathbb{K}[\mathbf{t}] \to \operatorname{Hom}(J, \mathbb{K}[\mathbf{z}]/J)_0$. Indeed, using the notation in the proof of Lemma 5.2.6, the ideal J is just the ideal $I(0)$. Here the degree 0 part of $\operatorname{Hom}(J, \mathbb{K}[\mathbf{z}]/J)$ is taken with respect to the M grading. Composing this with the surjection $\operatorname{Hom}(J, \mathbb{K}[\mathbf{z}]/J) \to T^1(\mathbb{K}[\mathbf{z}]/J)$, we obtain the *characteristic map*

$$T_0 \operatorname{Spec}\mathbb{K}[\mathbf{t}] \to T^1(\mathcal{K}*\mathfrak{P}(\mathcal{W}))^H.$$

Here $T^1(\mathcal{K} * \mathfrak{P}(\mathcal{W}))^H$ denotes the H-invariant elements of T^1.

The main proposition of this subsection shows the importance of the T1 property:

Proposition 6.1.5. *Let $\mathcal{A}$ be a cluster algebra of finite cluster type with no isolated vertices. Then the characteristic map $T_0 \operatorname{Spec} \mathbb{K}[\mathbf{t}] \to T^1(\mathcal{K} * \mathfrak{P}(\mathcal{W}))^H$ is an isomorphism if and only if $\mathcal{A}$ satisfies property T1.*

To prove the proposition, we will first consider several lemmas:

Lemma 6.1.6. *Let $\mathcal{A}$ be a cluster algebra of finite cluster type with no isolated vertex.*

(1) The characteristic map $T_0 \operatorname{Spec} \mathbb{K}[\mathbf{t}] \to T^1(\mathcal{K} * \mathfrak{P}(\mathcal{W}))^H$ is injective.

(2) The image of the characteristic map consists precisely of the graded pieces $T^1(\mathcal{K} * \mathfrak{P}(\mathcal{W}))_{\mathbf{c}}$ for $\mathbf{c} = \mathbf{a} - \mathbf{b}$, where $z^{\mathbf{b}} = z_x \cdot z_{x'}$ for exchangeable cluster variables x, x', and the exchange relation for x, x' has the form

$$x \cdot x' = \alpha_1 + \alpha_2$$

for extended cluster monomials α_1, α_2 such that $\mathbf{a}$ is the $\mathbb{Z}^{\mathcal{V} \cup \mathcal{W}}$ degree of α_1, and α_2 only involves frozen variables.

Proof. Even though $\mathcal{A}$ is not necessarily positively graded, we will use the discussion and notation from (5.2.4) and the proof of Lemma 5.2.6 (specializing to $P = 0$). Using Theorem 5.2.1, we now know that $I(0) = J = I_{\mathcal{K}*\mathfrak{P}(\mathcal{W})}$. In particular, $\operatorname{Hom}(J, \mathbb{K}[\mathbf{z}]/J)$ has a "fine" grading by $\mathbb{Z}^{\mathcal{V} \cup \mathcal{W}}$. In the proof of Lemma 5.2.6, we saw that the image of the tangent vector $\eta_i : \operatorname{Spec} \mathbb{K}[t_i]/t_i^2 \to \operatorname{Spec} \mathbb{K}[\mathbf{t}]$ is a particular element $\psi_i \in \operatorname{Hom}(J, \mathbb{K}[\mathbf{z}]/J)_0$ which sends $z_x \cdot z_{x'}$ to α_1 for a certain exchangeable pair $\{x, x'\}$. Let $\mathbf{a}$ be the degree of the monomial in $\mathbf{z}$ corresponding to α_1, and $\mathbf{b}$ the degree of $z_x \cdot z_{x'}$. Then the graded piece of ψ_i of degree $\mathbf{a} - \mathbf{b}$ is the homomorphism sending $z_x \cdot z_{x'}$ to α_1, and all other exchange pair generators of J to 0. Here we are using that α_1 and α_2 from (5.2.4) are not equal; this follows from the assumption that $\mathcal{A}$ has no isolated vertex.

We claim that the image of $(\psi_i)_{\mathbf{a}-\mathbf{b}}$ in $T^1(\mathcal{K}*\mathfrak{P}(\mathcal{W}))$ is non-zero. Indeed, consider any seed $(\widetilde{\mathbf{x}}, \widetilde{B})$ with $x = x_k$ in the cluster $\widetilde{\mathbf{x}}$ and $x' = x'_k$ obtained via mutation of this seed at k. Let Ω be the set of cluster variables appearing in α_1. It follows from the exchange relation (5.2.4) that k is Ω-isolated. By Lemma 2.2.6 and the fact that $T^1(\mathbb{K}[z_w \mid w \in \mathcal{W}]) = 0$, $T^1(\mathcal{K} * \mathfrak{P}(\mathcal{W})) \cong T^1(\mathcal{K}) \otimes \mathbb{K}[z_w \mid w \in \mathcal{W}]$. By Theorem 3.2.2 $T^1(\mathcal{K} * \mathfrak{P}(\mathcal{W}))$ thus has a non-zero element of degree $\mathbf{a} - \mathbf{b}$, and by Remark 3.2.3, that element is exactly $(\psi_i)_{\mathbf{a}-\mathbf{b}}$.

We have thus seen that each of the image of each of the ψ_i in $T^1(\mathcal{K} * \mathfrak{P}(\mathcal{W}))^H$ has a non-zero homogeneous component, and the degrees of these components are pairwise disjoint. It follows that the map $T_0 \operatorname{Spec} \mathbb{K}[t] \to T^1(\mathcal{K})^H$ is injective.

Moreover, since the family $\operatorname{Spec} \mathcal{A}^{\mathrm{univ}} \to \operatorname{Spec} \mathbb{K}[\mathbf{t}]$ is $\mathbb{T}$-equivariant, each ψ_i is necessarily homogeneous. It follows that $\psi_i = (\psi_i)_{\mathbf{a}-\mathbf{b}}$. The description of the degrees of the image now follows from (5.2.4) and the above discussion. $\square$

Consider some degree $\mathbf{c} = \mathbf{a} - \mathbf{b}$ for which $T^1(\mathcal{K} * \mathfrak{P}(\mathcal{W})) \neq 0$. The degree $\mathbf{c}$ is constructed exactly as in Theorem 3.2.2, except that $\mathbf{a}$ may also have support involving the frozen variables $\mathcal{W}$. We wish to understand what it means for this degree to be H-invariant. This is the same thing as requiring that it be of degree zero with respect to the M-grading. We write $\mathbf{c} \equiv 0$ in the M-grading when this is the case.

Lemma 6.1.7. *The degrees* $\mathbf{c} = \mathbf{a} - \mathbf{b}$ *arising as in Theorem 3.2.2 from the seed* $(\widetilde{\mathbf{x}}, \widetilde{B})$ *with* $T^1(\mathcal{K} * \mathfrak{P}(\mathcal{W}))_{\mathbf{c}} \neq 0$, $\mathbf{c} \equiv 0$ *in the M-grading, and* $b = \{x_k, x'_k\}$ *are in bijection with those* $w \in \mathbb{Z}^m$ *such that w is in the $\mathbb{Z}$-span of the columns of $\widetilde{B}$, $w_k = 0$, and w satisfies (6.1.2) for $j = k$ and all $i \neq j$. The positive part $\mathbf{a}$ is given by* $\mathbf{a}_{x_i} = w_i + \max(0, b_{ik})$.

Proof. Let $(\widetilde{\mathbf{x}}, \widetilde{B})$ be a seed giving rise to the degree $\mathbf{c}$ as in Theorem 3.2.2, that is, $\mathbf{b}$ has support $\{x_k, x'_k\}$ for some cluster variable x_k of $\mathbf{x}$, x_k is Ω-isolated for some subset Ω of the elements of $\mathbf{x}$, and the support of $\mathbf{a}$ is the union of Ω with a subset of $\mathcal{W}$. Recall from §5.1 that we can identify M with the cokernel of $\widetilde{B}$. The M-degree of $z_{x_k} \cdot z_{x'_k}$ is the same as the degree of

$$\prod_{i=1}^m z_{x_i}^{\max(0, b_{ik})}.$$

On the other hand, a monomial

$$\prod_{i=1}^m z_{x_i}^{d_i}$$

having M-degree zero is equivalent to requiring that $(d_1, \ldots, d_m)^{\mathrm{tr}}$ is in the $\mathbb{Z}$-span of the columns of $\widetilde{B}$. Putting this together, we conclude that $\mathbf{c} = \mathbf{a} - \mathbf{b}$ has degree zero if and only if the element $w \in \mathbb{Z}^m$ with

$$w_i = \mathbf{a}_{x_i} - \max(0, b_{ik})$$

is in the $\mathbb{Z}$-linear column span of $\widetilde{B}$.

Let Ω' consist of those x_i such that $i \leq n$ and $b_{ik} \neq 0$. The condition that x_k be Ω-isolated is exactly the condition that $\Omega' \subseteq \Omega$. This translates to the condition that $\mathbf{a}_{x_i} \geq 1$ for those i such that $i \leq n$ and $b_{ik} \neq 0$. Combining this with the above discussion, we arrive at the statement of the lemma. $\square$

Lemma 6.1.8. *Let $\mathcal{A}$ be a cluster algebra of finite cluster type with extended exchange matrix $\widetilde{B}$. Fix a vertex j that is a sink or a source of $\Gamma(B)$. Then there exists a sequence of mutations avoiding j and its neighbors leading to an exchange matrix B' such that every vertex of $\Gamma(B')$ is a sink or a source.*

Proof. It is enough to treat the case where $\mathcal{A}$ has no frozen variables and is connected. We may assume moreover that j is a source, the sink case being analogous. Let c be the number of neighbors of j.

We claim that if we remove j from $\Gamma(B)$ we obtain a graph with c connected components. Assume to the contrary that there are fewer connected components. Then there would be at least two neighbors of j, say i and k, that belong to the same component. Consider a minimal subgraph in this component containing both i and k. By adding back the vertex j (and the corresponding edges), we obtain a non-oriented cycle as a full graph of $\Gamma(B)$. However, this contradicts $\mathcal{A}$ being of finite cluster type by [FZ03a, Proposition 9.7]. Hence, $\Gamma(B)$ has exactly c connected components after removing j.

To prove the lemma, it remains to check that while working on a component of $\Gamma(B) \backslash \{j\}$, we can perform mutations avoiding the unique neighbor k of j contained in that component in order to make this neighbor k a sink. This follows at once from [FZ07, Proposition 11.1 (1)] and [CL20, Theorem 6.2]. $\square$

Proof of Proposition 6.1.5. First, suppose that there exists a seed $(\widetilde{\mathbf{x}}, \widetilde{B})$, an index $1 \le j \le n$, and $w \in \mathbb{Z}^n$ in the $\mathbb{Z}$-column span of $\widetilde{B}$ satisfying $w_j = 0$ and (6.1.2) for all $i \ne j$ but such that $w \ne 0$ and $w \ne -B_j$. By Lemma 6.1.7, there is an H-invariant element of $T^1(\mathcal{K} * \mathfrak{P}(\mathcal{W}))$ which is the image of the homomorphism sending $z_{x_j} \cdot z_{x'_j}$ to

$$\prod_{i=1}^{m} z_{x_i}^{w_i + \max(0, b_{ij})}.$$

However, since $w \ne 0$ and $w \ne -B_j$, this monomial is not one of the monomials appearing in the exchange relations for $\{x_k, x'_k\}$, and as such, this element of $T^1(\mathcal{K} * \mathfrak{P}(\mathcal{W}))$ cannot be in the image of the map in question.

Conversely, let us suppose that for every seed $(\widetilde{\mathbf{x}}, \widetilde{B})$, $\widetilde{B}$ satisfies property T1. By Lemma 6.1.6, the map $T_0 \operatorname{Spec} \mathbb{K}[\mathbf{t}] \to T^1(\mathcal{K} * \mathfrak{P}(\mathcal{W}))^H$ being an isomorphism is equivalent to showing that $\dim T^1(\mathcal{K} * \mathfrak{P}(\mathcal{W}))^H \le p$. We will do this by showing that to every element in a basis of $T^1(\mathcal{K} * \mathfrak{P}(\mathcal{W}))^H$, we may associate a distinct universal coefficient t_i.

To that end, consider any degree $\mathbf{c}$ with $T^1(\mathcal{K} * \mathfrak{P}(\mathcal{W}))_{\mathbf{c}} \ne 0$ and $\mathbf{c} \equiv 0$ with respect to the M grading. Suppose that this T^1 element comes from the seed $(\widetilde{\mathbf{x}}, \widetilde{B})$, with $b = \{x_k, x'_k\}$. This degree $\mathbf{c}$ corresponds to $w \in \mathbb{Z}^n$ as in the statement of Lemma 6.1.7, with $\mathbf{a}_{x_i} = w_i + \max(0, b_{ik})$. Since we are assuming that the T1 property is satisfied, either $w = 0$ or $w = -B_j$.

By Remark 6.1.3, the vertex k of $\Gamma(B)$ is a sink or a source. By Lemma 6.1.8, we may find a sequence of mutations in vertices distinct from k and its neighbors that allow us to replace the seed $(\widetilde{\mathbf{x}}, \widetilde{B})$ with a new seed $(\widetilde{\mathbf{x}}', \widetilde{B}')$ where every vertex of $\Gamma(B')$ is a sink or a source. We may thus assume from the start that without loss of generality, $\Gamma(B)$ has every vertex a sink or a source. In the language of [FZ07], this seed lies on the *bipartite belt* of the cluster complex.

Thus, to the degree $\mathbf{c}$, we may associate a universal coefficient coming from the pair $\{x_k, x'_k\}$ as described in [FZ07, Lemma 12.7 and Lemma 12.8]. There is a single such coefficient, unless k is both a sink and a source in $\Gamma(B)$, in which case there are two. Similarly, by the above discussion, the T1 property guarantees that for any pair $\{x_k, x'_k\}$, there is only one H-invariant element of T^1 of with $b = \{x_k, x'_k\}$, unless k is both a sink and source of $\Gamma(B)$, in which case there are two. This proves the proposition. $\qquad \square$

Remark 6.1.9. Let $\mathcal{A}$ be a full rank cluster algebra, and suppose that one of its extended exchange matrices $\widetilde{B}$ satisfies the T1 property. If we add additional frozen variables to $\mathcal{A}$, then the new extended exchange matrix $\widetilde{B}'$ corresponding to $\widetilde{B}$ will still satisfy the T1 property. Indeed, we can view $\widetilde{B}'$ as having the same first m rows as $\widetilde{B}$, with $m' - m$ additional rows. Fix $1 \le j \le n$. Consider any element w of $\mathbb{Z}^{m'}$ in the $\mathbb{Z}$-linear column span of $\widetilde{B}'$ satisfying (6.1.2) for the matrix $\widetilde{B}'$ to $\mathbb{Z}^m$. Projecting to $\mathbb{Z}^m$, we see that the first m coordinates of w are either all 0, or all $-b_{ij}$. Since $\widetilde{B}$ has full rank, there is a unique way to obtain the projection of w as a linear combination of the columns of $\widetilde{B}$; w itself must be that same linear combination of the columns of $\widetilde{B}'$. It follows that either $w = 0$ or $w_i = -b_{ij}$ for all i. Hence, $\widetilde{B}'$ fulfills property T1.

Proposition 6.1.10. *Let $\mathcal{A}$ be a cluster algebra of finite cluster type. Then we can always add additional frozen variables to $\mathcal{A}$ so that the new cluster algebra $\mathcal{A}'$ is positively graded and satisfies property T1.*

Proof. By adding frozen variables, we can without loss of generality assume that $\mathcal{A}$ has full rank and is positively graded, see Lemma 4.2.10.

Let $\widetilde{B}$ be an extended exchange matrix of $\mathcal{A}$ such that property T1 is not satisfied. Let j be an index for which there exists an element w in the $\mathbb{Z}$-linear column span of $\widetilde{B}$ satisfying (6.1.2) but such that $w \neq 0, -B_j$. We note that since $\mathcal{A}$ is positively graded, only finitely many such w exist.

We may write w uniquely as a $\mathbb{Z}$-linear combination $w = \sum_i \lambda_i \cdot B_i$ of the columns of $\widetilde{B}$, and by assumption, we must have $\lambda_k \neq 0$ for some k with $k \neq j$. We add a new coefficient x_{m+1} by adding a row to the matrix $\widetilde{B}$ and setting

$$b_{(m+1)i} = \begin{cases} 1 & i = k, \lambda_k < 0 \\ -1 & i = k, \lambda_k > 0 \\ 0 & i \neq k \end{cases}.$$

Taking $w' = \sum_i \lambda_i \cdot B_i'$ where B_i' are the columns of this new matrix $\widetilde{B}'$, we see that w' does not satisfy (6.1.2). It follows that there is one less obstruction to $\widetilde{B}'$ satisfying the T1 property as there is for $\widetilde{B}$. Thus, we can continue adding coefficients until we arrive at an exchange matrix $\widetilde{B}'$ that does satisfy the T1 property.

We may now do the same thing at all the other (finitely many) extended exchange matrices of $\mathcal{A}$. By Remark 6.1.9, once we have added enough frozen variables so that one exchange matrix satisfies T1, adding further coefficients doesn't destroy this property. Finally, we may add more frozen variables if necessary so that $\mathcal{A}'$ again becomes positively graded (Lemma 4.2.10). Again, by Remark 6.1.9, this will not destroy the T1 property. $\square$

Example 6.1.11. It is straightforward to show that any positively graded cluster algebra of type A_1 or A_2 will always satisfy the T1 property for every extended exchange matrix. This is no longer the case for type A_3, as the following example demonstrates.

Consider the matrix

$$\widetilde{B} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 0 & -2 \\ 0 & 0 & 1 \end{pmatrix}$$

Taking $j = 1$, the element $w = (0,0,0,0,-1,1)^{\mathrm{tr}}$ is in the column span of $\widetilde{B}$ and satisfies (6.1.2), showing that the T1 property is not fulfilled for this matrix. Furthermore, one may check that it is possible to endow the cluster algebra $\mathcal{A}(\widetilde{B})$ with a positive grading. $\square$

Remark 6.1.12. Instead of looking at exchange matrices directly, one may verify property T1 by looking at all exchange monomials $x_k \cdot x_k'$ for a cluster algebra $\mathcal{A}$, as we now describe. Consider a homogeneous element of $T^1(\mathcal{K} * \mathfrak{P}(\mathcal{W}))^H$. By Theorem 3.2.2, its negative part corresponds to some exchange monomial $x_k \cdot x_k'$. Its positive part corresponds to an extended cluster monomial x^α. Again by Theorem

3.2.2, each cluster variable appearing in x^α must be compatible with both x_k and x'_k. Furthermore, considering the exchange relation for x_k, x'_k, any cluster (but not frozen) variable appearing on the right hand side of (2.4.4) must be a factor of x^α: this corresponds to k being Ω-isolated. Here we are using the fact that for every exchange pair, there is a unique exchange relation [FZ03a, Theorem 1.11]. Finally, the M-degrees of x^α and $x_k \cdot x'_k$ must agree in order for this element to be H-invariant.

In light of Lemma 6.1.7, verifying the T1 property for every extended exchange matrix of $\mathcal{A}$ is equivalent to showing that for every exchange monomial $x_k \cdot x'_k$, the only extended cluster monomials x^α satisfying the conditions of the previous paragraph are exactly one of the two monomials on the right hand side of the exchange relation (2.4.4). See Example 6.1.13 for a demonstration of this criterion.

Example 6.1.13. One may show that for each of the cluster algebras (1)–(6) listed in Example 5.2.11, the T1 property holds. Furthermore, property T1 also holds for the D_n case of the Schubert divisor of $\mathrm{Gr}(2, n+2)$.

For the individual cases $\mathrm{Gr}(3,6)$, $\mathrm{Gr}(3,7)$, and $\mathrm{Gr}(3,8)$ we checked this using a computer calculation. For the other families of examples, one may use the approach of Remark 6.1.12. We show how this works in the D_n case of the Schubert divisor of $\mathrm{Gr}(2, n+2)$ [FWZ21, Example 6.3.5]. The other cases are similar.

Following the notation of [FZ03a, §12.4], the cluster and frozen variables for the Schubert divisor are

$$x_{ab} = x_{\overline{a}\overline{b}} \qquad 1 \le a, b \le 2n, \ a \ne b;$$
$$x_{a\overline{a}} = x_{\overline{a}a} \qquad 1 \le a \le n;$$
$$\overline{x}_{a\overline{a}} = \overline{x}_{\overline{a}a} \qquad 1 \le a \le n$$

where $\overline{a}$ is the remainder of $a + n$ modulo $2n$. The frozen variables are exactly the x_{ab} where $|a - b| \equiv 1$ modulo $2n$. Taking $e_0, \ldots, e_n$ as the standard basis for $\mathbb{Z}^{n+1}$, we may define a grading via

$$\deg_H x_{ab} = \deg x_{a\overline{b}} = \deg_H x_{\overline{a}b} = \deg_H x_{\overline{a}\overline{b}} = e_a + e_b \qquad 1 \le a, b \le n, \ a \ne b;$$
$$\deg_H x_{a\overline{a}} = e_0 + e_a \qquad 1 \le a \le n;$$
$$\deg_H \overline{x}_{a\overline{a}} = -e_0 + e_a \qquad 1 \le a \le n.$$

There are five types of exchange relations, which we consider in turn. Each of these is homogeneous with respect to the above grading. We will view the numbers $1, \ldots, 2n$ as the labels for the vertices of a $2n$-gon, arranged in counter-clockwise order. We will always work modulo $2n$.

(1) For $a, b, c, d, \overline{a}$ in counterclockwise order,

$$x_{ac} x_{bd} = x_{ab} x_{cd} + x_{ad} x_{bc}.$$

Consider any corresponding T^1 element. Since any x_{ad} is not frozen, any perturbation of $x_{ac} x_{bd}$ must involve it. The product of the remaining variables in the perturbation must have degree $e_b + e_c$. The monomials of this degree are x_{bc}, $x_{b\overline{c}}$, $x_{b\overline{b}}\overline{x}_{c\overline{c}}$, and $\overline{x}_{b\overline{b}}\overline{x}_{c\overline{c}}$. The latter two monomials are incompatible, and $x_{b\overline{c}}$ is incompatible with x_{ad}. Thus, the only possible perturbation is $x_{ad} x_{bc}$ as desired.

(2) For $a, b, c, \overline{a}$ in counterclockwise order,

$$x_{ac} x_{a\overline{b}} = x_{ab} x_{a\overline{c}} + x_{a\overline{a}} \overline{x}_{a\overline{a}} x_{bc}.$$

Consider any corresponding T^1 element. The perturbation must involve $x_{a\bar{a}}\overline{x}_{a\overline{a}}$; the product of the remaining variables in the perturbation must have degree $e_b + e_c$. The monomials of this degree were listed in case 1 above. As noted, the latter two are incompatible, and $x_{b\overline{c}}$ is incompatible with $x_{a\overline{a}}$. Thus, the only possible perturbation is $x_{a\overline{a}}\overline{x}_{a\overline{a}}x_{bc}$ as desired.

(3) For $a, b, \overline{a}$ in counterclockwise order,

$$x_{a\overline{a}}\overline{x}_{b\overline{b}} = x_{ab} + x_{a\overline{b}}.$$

Consider any corresponding T^1 element. The perturbation must be x_{ab}, $x_{a\overline{b}}$, or $\overline{x}_{a\overline{a}}x_{b\overline{b}}$, but the latter is incompatible.

(4) For $a, b, c, \overline{a}$ in counterclockwise order,

$$x_{a\overline{a}}x_{b\overline{c}} = x_{ab}x_{c\overline{c}} + x_{a\overline{c}}x_{b\overline{b}}$$

Consider any corresponding T^1 element. The perturbation must involve both $x_{c\overline{c}}$ and $x_{b\overline{b}}$. The product of the remaining variables in the perturbation must have degree $e_a - e_0$. The only possibility is $\overline{x}_{a\overline{a}}$. But this is incompatible with $x_{c\overline{c}}$, so there can be no perturbation at all.

(5) For $a, b, c, \overline{a}$ in counterclockwise order,

$$\overline{x}_{a\overline{a}}x_{b\overline{c}} = x_{ab}\overline{x}_{c\overline{c}} + x_{a\overline{c}}\overline{x}_{b\overline{b}}$$

This is similar to case 4.

We have thus verified for each exchange relation that any corresponding element of $T^1(\mathcal{K}*\mathfrak{P}(\mathcal{W}))^H$ must have the form described in Remark 6.1.12. It follows that the T1 property holds for this example. $\qquad\square$

6.2. **Derivations.** To construct a *canonical* $\mathbb{T}$-equivariant family with prescribed characteristic map, we will need control over the embedded deformations induced by derivations. We again consider a cluster algebra $\mathcal{A}$ of finite cluster type. Let H be the quasitorus fixing $\operatorname{Spec}\mathcal{A}$ in its cluster embedding, with character group M. We will use other notation as in §5.1. In particular, we have the cluster complex $\mathcal{K}$ of $\mathcal{A}$.

Consider the Stanley-Reisner ideal $J = I_{\mathcal{K}*\mathfrak{P}(\mathcal{W})}$. For a cluster variable $v \in \mathcal{V}$ and a monomial $z^\alpha \in \mathbb{K}[\mathbf{z}]$, we say that the derivation

$$\partial(v, \alpha) := z^\alpha \cdot \frac{\partial}{\partial z_v} \in \operatorname{Der}_{\mathbb{K}}(\mathbb{K}[\mathbf{z}], \mathbb{K}[\mathbf{z}])$$

is J-non-trivial if its image in $\operatorname{Hom}_{\mathbb{K}[\mathbf{z}]}(J, S_{\mathcal{K}*\mathfrak{P}(\mathcal{W})})$ is non-zero. We say that $\partial(v, \alpha)$ is *exchangeable* if its image in $\operatorname{Hom}_{\mathbb{K}[\mathbf{z}]}(J, S_{\mathcal{K}*\mathfrak{P}(\mathcal{W})})$ acts non-trivially on some monomial $z_v \cdot z_w \in J$, where w is exchangeable with v.

More concretely, $\partial(v, \alpha)$ is J-non-trivial if there exists a cluster variable w such that

(6.2.1) $$z_v \cdot z_w \in J \qquad \text{and} \qquad z_w \cdot z^\alpha \notin J.$$

It is exchangeable if (6.2.1) is satisfied for some cluster variable w that is exchangeable with v.

Remark 6.2.2. It is possible to see which cluster variables v and w are exchangeable by only inspecting the cluster complex $\mathcal{K}$. Indeed, v and w are exchangeable if and only if $\{v, w\} \notin \mathcal{K}$ and there exists a face $f \in \mathcal{K}$ with $f \cup \{v\}$ and $f \cup \{w\}$ being maximal faces of $\mathcal{K}$.

Let $\mathfrak{S} \subset \mathbb{Z}^{\mathcal{V} \cup \mathcal{W}}$ be the semigroup generated by the degrees of the non-trivial pieces of the image of the characteristic map

$$T_0 \operatorname{Spec} \mathbb{K}[\mathbf{t}] \to T^1(\mathcal{K} * \mathfrak{P}(\mathcal{W}))^H.$$

Here we are taking degrees with respect to the $\mathbb{T}$-action, and denote degree by $\deg_{\mathbb{T}}$.

Definition 6.2.3 (Property T0*)**.** We say that $\mathcal{A}$ satisfies property T0* if for every cluster variable v and every monomial z^α such that

(6.2.4) $$\deg_{\mathbb{T}} z^\alpha - \deg_{\mathbb{T}} z_v \in \mathfrak{S},$$

if $\partial(v, z^\alpha)$ is J-non-trivial, then it is exchangeable.

We will say that $\mathcal{A}$ satisfies property T0 if J satisfies property T0 with respect to H (cf. Definition 2.3.4). Clearly, if $\mathcal{A}$ satisfies property T0, then it satisfies property T0*. Indeed, in this context, property T0 says that if $\deg_H z^\alpha = \deg_H z_v$ and $z_v \cdot z_w \in J$, then $z_w \cdot z^\alpha \in J$ for any w with $z_v \cdot z_w \in J$. Since the condition (6.2.4) implies that $\deg_H z^\alpha = \deg_H z_v$, it follows that such $\partial(v, z^\alpha)$ cannot be J-non-trivial.

Example 6.2.5 (Continuation of Running Example 6.1.4)**.** We continue our discussion of the cluster algebra $\mathcal{A}$ from Example 2.4.7. Property T0* is automatically satisfied, since for this example if $z_v \cdot z_w \in J$, then v and w are exchangeable.

Nonetheless, let us analyze this example further. The semigroup $\mathfrak{S}$ is generated by the opposites of the degrees of the t_i listed in Example 5.2.14. On the other hand, up to cyclic permutations of the first five entries, the degrees of J-non-trivial derivations are of the form

$$(-1, *, 0, *, 0, *, *, *), \qquad (-1, *, 0, 0, *, *, *, *), \qquad (-1, 0, *, 0, *, *, *, *).$$

Here, $*$ means any non-negative integer. It is straightforward to check that none of these degrees can lie in $\mathfrak{S}$, that is, Equation (6.2.4) is never satisfied.

So in this example, property T0* is satisfied for two reasons: all non-comparable cluster variables are exchangeable, and no J-non-trivial derivations have degree in $\mathfrak{S}$. $\qquad\square$

Proposition 6.2.6. *Let $\mathcal{A}$ be a cluster algebra of finite cluster type.*

(1) If $\mathcal{A}'$ is obtained from $\mathcal{A}$ by adding frozen variables and $\mathcal{A}$ satisfies property T0 or T0, then $\mathcal{A}'$ satisfies the same property.*

(2) It is possible to add frozen variables to $\mathcal{A}$ so that the resulting cluster algebra satisfies property T0 (and in particular, property T0).*

Proof. Using Theorem 2.5.1, we may and will identify the cluster variables of $\mathcal{A}$ and $\mathcal{A}'$. Likewise, letting $\mathcal{W}'$ be the frozen variables for $\mathcal{A}'$, we may view $\mathcal{W}$ as a subset of $\mathcal{W}'$. We have a natural projection $\mathbb{Z}^{\mathcal{V} \cup \mathcal{W}'} \to \mathbb{Z}^{\mathcal{V} \cup \mathcal{W}}$.

Fix a cluster variable $v \in \mathcal{V}$. Consider a monomial $z^{\alpha'}$ for $\mathcal{A}'$. We may write this as $z^\alpha \cdot h$, where z^α is a monomial whose terms correspond to frozen and cluster variables of $\mathcal{A}$, and h only involves the additional frozen variables of $\mathcal{A}'$. Let $\mathfrak{S}' \subset \mathbb{Z}^{\mathcal{V} \cup \mathcal{W}'}$ be the semigroup generated by the degrees of the image of the characteristic map for $\mathcal{A}'$. This surjects onto $\mathfrak{S}$. If $\deg_{\mathbb{T}} z^{\alpha'} - \deg_{\mathbb{T}} z_v \in \mathfrak{S}'$, then $\deg_{\mathbb{T}} z^\alpha - \deg_{\mathbb{T}} z_v \in \mathfrak{S}$.

Suppose that there exists a cluster variable w such that $z^{\alpha'}$ satisfies (6.2.1). Then so does z^α, and so there exists an exchangeable variable w' such that z^α satisfies (6.2.1) with respect to w'. But then clearly $z^{\alpha'}$ does as well, since h involves only

frozen variables. This proves the first claim for the T0* case. The T0 case is similar and left to the reader.

We now consider the second claim. Since $\mathbb{T}$ acts on $\operatorname{Spec}\mathcal{A}^{\mathrm{univ}}$ by Theorem 5.2.1, we know that after adding enough frozen variables, we may assume that all cluster variables of $\mathcal{A}$ have distinct M-degrees. Furthermore, we may add frozen variables such that for any exchange matrix $\widetilde{B}$, $\widetilde{B}$ still has full rank after omitting any two rows. Thus we may assume that the M-degree of any cluster variable is also distinct from the M-degree of any frozen variable.

By iterated application of Lemma 4.2.10, we may arrive at a situation such that the M-degree of a cluster variable cannot be written as a sum of the M-degrees of more than one cluster or frozen variable. Thus, we may assume that if $\deg_H z_v = \deg_H z^\alpha$, then $z^\alpha = z_w$ for w some cluster or frozen variable. But by the previous paragraph, we conclude that $w = v$. Since the derivation $z_v \cdot \partial/\partial z_v$ is not J-non-trivial, we conclude that property T0 holds. $\qquad\square$

Example 6.2.7. For cluster algebras of type A_n, B_n, C_n, and D_n, $\mathcal{A}$ *always* satisfies property T0*. For A_n this is obvious, since every pair of incomparable cluster variables is exchangeable. For the other classical types, it is straightforward to verify that every J-non-trivial derivation $\partial(v, z^\alpha)$ is also exchangeable by using the combinatorial models and exchange relations of [FZ03a, §12]. This holds for all derivations $\partial(v, z^\alpha)$, not just those satisfying Equation (6.2.4).

On the other hand, let $\mathcal{A}$ be the cluster algebra of type G_2 with trivial coefficients. It is no longer true that every J-non-trivial derivation is exchangeable. Indeed, in this case, the cluster complex $\mathcal{K}$ is an octagon. Label the vertices $1, \ldots, 8$ in counterclockwise order. Thus, 1 is exchangeable with 3 and 7, 2 with 4 and 8, etc. Taking $z^\alpha = z_5$, we note that $z_1 \cdot z_5 \in I_\mathcal{K}$, $z_5 \cdot z_5 \notin I_\mathcal{K}$. However, we also have $z_3 \cdot z_5 \in I_\mathcal{K}$ and $z_7 \cdot z_5 \in I_\mathcal{K}$, so $\partial(z_1, z_5)$ is J-non-trivial but not exchangeable.

Nonetheless, any cluster algebra of type G_2 still satisfies property T0*. Indeed, in the case of no frozen variables, it is straightforward to check that if $\partial(v, z^\alpha)$ is J-non-trivial, then Equation (6.2.4) cannot be satisfied.

$\qquad\square$

Based on the above example, we make the following conjecture:

Conjecture 6.2.8. Property T0* is satisfied for any cluster algebra of finite cluster type.

In fact, we suspect that something stronger is true:

Conjecture 6.2.9. Let $\mathcal{A}$ be a cluster algebra of finite cluster type. For any J-non-trivial derivation $\partial(v, z^\alpha)$, Equation (6.2.4) cannot be satisfied.

Clearly Conjecture 6.2.9 implies Conjecture 6.2.8. To verify Conjecture 6.2.8, it suffices by Example 6.2.7 to check the cases of E_6, E_7, E_8 and F_4. See Remark 6.3.5 for consequences of Conjecture 6.2.9.

6.3. **Exchange Minimal Deformations.** We continue with the notation of the previous section. We wish to identify a canonical deformation of

$$Y = \operatorname{Spec} S_{\mathcal{K}*\mathfrak{P}(\mathcal{W})} \subseteq \mathbb{A}^{p+q}$$

whose characteristic map has the same image as the image of the characteristic map for the family $\operatorname{Spec}\mathcal{A}^{\mathrm{univ}} \to \mathbb{A}^p$.

To that end, we will introduce the notion of *exchange minimality*. Let $\mathfrak{m}$ be the maximal ideal of $\mathbb{K}[\mathbf{t}]$. Fix a homogeneous subspace V of $T^1(\mathcal{K} * \mathfrak{P}(\mathcal{W}))^H$.

Definition 6.3.1 (Exchange Minimality). Let $\pi : \mathcal{Y} \to \operatorname{Spec} \mathbb{K}[\mathbf{t}]/\mathfrak{m}^{k+1}$ be a $\mathbb{T}$-equivariant H-invariant embedded deformation of Y. Let $\widetilde{J}$ be the ideal of $\mathcal{Y}$ in $\mathbb{A}^{p+q} \times \operatorname{Spec} \mathbb{K}[\mathbf{t}]/\mathfrak{m}^{k+1}$.

- Suppose $k = 1$. We say the deformation π is 1-exchange minimal (with respect to V) if the characteristic map of π is injective with image V.
- Suppose $k > 1$. We say the deformation π is k-exchange minimal (with respect to V) if the reduction of π modulo $\mathfrak{m}^k$ is $(k-1)$-exchange minimal, and for every exchangeable pair v, w, $z_v \cdot z_w$ has a lift to $\widetilde{J}/\mathfrak{m}^{k+1}$ with the fewest number of non-zero terms among all possible lifts in all $(k-1)$-exchange-minimal families.
- We say that a $\mathbb{T}$-equivariant H-invariant embedded deformation $\pi : \mathcal{Y} \to \mathbb{A}^p$ of Y is exchange minimal (with respect to V) if for every $k > 0$, its reduction modulo $\mathfrak{m}^{k+1}$ is k-exchange minimal.

We now state our main result of this section:

Theorem 6.3.2. *Let $\mathcal{A}$ be a cluster algebra of finite cluster type with no isolated vertex. Let $V \subseteq T^1(\mathcal{K} * \mathfrak{P}(\mathcal{W}))^H$ be the image of the characteristic map of the family $\operatorname{Spec} \mathcal{A}^{\mathrm{univ}} \to \mathbb{A}^p$.*

(1) If $\mathcal{A}$ satisfies property T0*, there is a unique (up to canonical embedded isomorphism) exchange-minimal deformation $\pi : \mathcal{Y} \to \mathbb{A}^p$ of Y. There is a canonical $\mathbb{T}$-equivariant isomorphism $\phi : \mathbb{A}^p \to \mathbb{A}^p$ under which the family $\operatorname{Spec} \mathcal{A}^{\mathrm{univ}} \to \mathbb{A}^p$ is canonically identified with π.

(2) If additionally $\mathcal{A}$ satisfies property T1, the above family π is semiuniversal. In particular, the family $\operatorname{Spec} \mathcal{A}^{\mathrm{univ}} \to \mathbb{A}^p$ may be identified with a canonical H-invariant semiuniversal deformation of Y.

Proof. By Theorem 3.2.2 and Remark 3.2.3, there is a canonical $\mathbb{K}$-vector space decomposition

$$\operatorname{Hom}(J, S_{\mathcal{K}*\mathfrak{P}(\mathcal{W})})^H = T^1(\mathcal{K} * \mathfrak{P}(\mathcal{W}))^H \oplus D$$

where D is the image of $\operatorname{Der}_{\mathbb{K}}(\mathbb{K}[\mathbf{z}], S_{\mathcal{K}*\mathfrak{P}(\mathcal{W})})^H$. Under this decomposition, any homogeneous T^1 element only affects a single minimal generator of J, and that generator is always coming from an exchangeable pair of cluster variables. The module D further decomposes as

$$D = D_{\mathfrak{S}} \oplus D',$$

where $D_{\mathfrak{S}}$ consists of the graded pieces of D whose degrees lie in $\mathfrak{S}$. By property T0*, any non-trivial homogeneous element of $D_{\mathfrak{S}}$ necessarily affects at least one minimal generator of J coming from an exchangeable pair of cluster variables. Furthermore, each homogeneous piece of $\operatorname{Hom}(J, S_{\mathcal{K}*\mathfrak{P}(\mathcal{W})})$ (with respect to the $\mathbb{Z}^{\mathcal{V} \cup \mathcal{W}}$-grading) is at most one-dimensional.

Let $\mathcal{Y}$ be any k-exchange-minimal deformation, and let $\mathcal{Y}' = \operatorname{Spec} \mathcal{A}^{\mathrm{univ}}$. Consider their ideals $\widetilde{J}$ and $\widetilde{J}'$. Since exchange-minimal deformations have injective characteristic maps with image V, and $\mathcal{Y}$ also has injective characteristic map by Lemma 5.2.6, $\mathcal{Y}$ and $\mathcal{Y}'$ modulo $\mathfrak{m}^2$ are isomorphic as abstract deformations, differing by a unique linear automorphism ϕ of $\mathbb{A}^p$. By applying the automorphism ϕ, we

may reduce to the case that the characteristic maps of $\mathcal{Y}$ and $\mathcal{Y}'$ are equal. After this reduction, we claim that modulo $\mathfrak{m}^{k+1}$, $\mathcal{Y} = \mathcal{Y}'$ as embedded deformations.

Suppose that we have shown that $\mathcal{Y}$ and $\mathcal{Y}'$ agree after reducing by $\mathfrak{m}^\ell$ for some $1 \leq \ell < k$. Then there are some $\mathbf{t}^{\alpha_i} \in \mathfrak{m}^\ell$ and homogeneous $\psi_i \in \mathrm{Hom}(J, S_{\mathcal{K}*\mathfrak{P}(\mathcal{W})})^H$ such that the minimal $\mathbb{T}$-equivariant lifts of any $z_v \cdot z_w \in J$ to $\widetilde{J}$ and $\widetilde{J}'$ differ modulo $\mathfrak{m}^{\ell+1}$ by

$$\sum_i \mathbf{t}^{\alpha_i} \psi_i(z_v \cdot z_w)$$

where $\alpha_i = (\alpha_{i1}, \ldots, \alpha_{ip}) \in \mathbb{Z}^p$ satisfies $\sum_j \alpha_{ij} = \ell$. The $\mathbb{T}$-equivariance imposes the condition that each ψ_i has degree contained in $\mathfrak{S}$, and thus either belongs to $T^1(\mathcal{K}*\mathfrak{P}(\mathcal{W}))^H$ or to $D_\mathfrak{S}$.

We claim that all ψ_i must be zero. In the case $\ell = 1$, the $\mathbb{T}$-equivariance imposes that any non-zero ψ_i must belong to $T^1(\mathcal{K}*\mathfrak{P}(\mathcal{W}))^H$. But these ψ_i must also be zero, since otherwise the characteristic maps would not agree.

We may thus assume that $\ell \geq 2$. Consider first some $0 \neq \psi \in T^1(\mathcal{K}*\mathfrak{P}(\mathcal{W}))$. Then $\psi(z_v \cdot z_w) \neq 0$ for exactly one pair exchangeable pair $\{v, w\}$, and adding $\mathbf{t}^\alpha \cdot \psi(z_v \cdot z_w)$ to the exchange relation will introduce some non-zero multiple of a monomial $\mathbf{t}^\alpha \cdot z^{\mathbf{a}}$. Moreover, it follows from Theorem 3.2.2 and the assumption $\ell \geq 2$ that $\mathbf{t}^\alpha \cdot z^{\mathbf{a}}$ cannot be one of the monomials appearing in the exchange relation of $\mathcal{A}^{\mathrm{univ}}$ for v, w. Taking any $\psi_i = \psi$ would thus violate the assumption that $\mathcal{Y}$ is k-exchange minimal.

Hence, we may assume that each $\psi_i \in D_\mathfrak{S}$. Consider thus some non-zero $\psi \in D_\mathfrak{S}$. By property T0*, we find at least one exchangeable pair $\{v, w\}$ for which adding $\mathbf{t}^\alpha \cdot \psi(z_v \cdot z_w)$ to the exchange relation will introduce some non-zero multiple of a monomial $\mathbf{t}^\alpha \cdot z^{\mathbf{a}}$. This monomial cannot already appear in the corresponding exchange relation of $\mathcal{A}^{\mathrm{univ}}$, since it will necessarily involve z_v or z_w. The same applies to all other exchangeable pairs $\{v', w'\}$ on which ψ acts non-trivially. Again, taking $\psi_i = \psi$ would thus violate the assumption that $\mathcal{Y}$ is $(k+1)$-exchange minimal. We conclude that $\mathcal{Y}$ and $\mathcal{Y}'$ agree modulo $\mathfrak{m}^{\ell+1}$, and thus by induction that $\mathcal{Y} = \mathcal{Y}'$ modulo $\mathfrak{m}^{k+1}$.

It follows that $\mathcal{Y}'$ is exchange minimal, and any other exchange minimal deformation is canonically isomorphic to it. The claims in the first statement of the theorem follow. The claims in the second statement follow from Proposition 6.1.5, and the fact that a deformation over a smooth base whose characteristic map is an isomorphism is semiuniversal. $\square$

Example 6.3.3 (Continuation of Running Example 6.2.5)**.** We consider the cluster algebra $\mathcal{A}$ from Example 2.4.7 for the final time. We have verified that properties T1 (Example 6.1.4) and T0* (Example 6.2.5) are satisfied. We may thus apply Theorem 6.3.2 and reconstruct the universal cluster algebra $\mathcal{A}^{\mathrm{univ}}$ (Example 4.1.10) as the canonical exchange minimal H-invariant semiuniversal deformation of $\mathrm{Spec}\, S_{\mathcal{K}*\mathfrak{P}(\mathcal{W})}$. See §A.2 for an explicit *Macaulay2* computation doing this. $\square$

Remark 6.3.4. In the context of Theorem 6.3.2, if $\mathcal{A}$ satisfies property T1 but not necessarily property T0*, it is still true that $\mathrm{Spec}\, \mathcal{A}^{\mathrm{univ}} \to \mathbb{A}^p$ is a $\mathbb{T}$-equivariant H-invariant semiuniversal deformation of Y. However, without additional assumptions, we only know how reconstruct this family via deformation theory up to a non-canonical isomorphism.

More generally, if we do not even assume that $\mathcal{A}$ satisfies property T1, it is still true that in a formal neighborhood of the origin, the family $\operatorname{Spec}\mathcal{A}^{\mathrm{univ}} \to \mathbb{A}^p$ is formally (non-canonically) isomorphic to a subfamily of any $\mathbb{T}$-equivariant H-invariant semiuniversal deformation of Y. Indeed, $\operatorname{Spec}\mathcal{A}^{\mathrm{univ}} \to \mathbb{A}^p$ is $\mathbb{T}$-equivariant, flat, and the characteristic map is injective.

Remark 6.3.5. Assume that $\mathcal{A}$ is a positively graded cluster algebra of full rank and finite cluster type. By Theorem 5.2.1, it follows that the $\mathbb{T}$-action on the family $\operatorname{Spec}\mathcal{A}^{\mathrm{univ}} \to \mathbb{A}^p$ has a standard action on the base $\mathbb{A}^p$ (cf. Remark 2.3.5). This implies

(1) The non-zero degrees of the image V of the characteristic map
$$\mathbb{A}^p \to T^1(\mathcal{K} * \mathfrak{P}(\mathcal{W}))^H$$
are linearly independent;

(2) There exists a one-parameter subgroup λ of $\mathbb{T}$ acting non-negatively on the cluster variables, and on V with strictly negative weights.

If we assume that property T1 holds, then by Proposition 6.1.5, V can be identified with all of $T^1(\mathcal{K} * \mathfrak{P}(\mathcal{W}))^H$. By Remark 2.3.3, this guarantees a canonical $\mathbb{T}$-equivariant map from $\mathbb{A}^p$ to any $\mathbb{T}$-equivariant semiuniversal base space.

If we additionally impose the condition that any J-non-trivial derivation has degree lying outside of $\mathfrak{S}$, then by the same remark, we obtain a canonical *embedded* isomorphism between the family $\operatorname{Spec}\mathcal{A}^{\mathrm{univ}} \to \mathbb{A}^p$ and any embedded $\mathbb{T}$-equivariant H-invariant semiuniversal deformation π of Y. In such situations, we don't need to concern ourselves with the notion of exchange minimality; the $\mathbb{T}$-equivariant structure is already enough to deliver a canonical embedded isomorphism. We note that it is always possible to arrive in this situation by adding additional frozen variables, see Propositions 6.2.6 and 6.1.10, noting that condition T0 implies there are no J-non-trivial derivations at all.

We obtain the following Corollary of Theorem 6.3.2:

Corollary 6.3.6. *Let $\mathcal{A}$ be a positively graded cluster algebra of full rank and finite cluster type. Assume further that $\mathcal{A}$ satisfies the T1 property. Then $T^1(\mathcal{A})^H = 0$, that is, $\operatorname{Spec}\mathcal{A}$ has no H-invariant deformations.*

Proof. By Theorem 6.3.2, the family $\operatorname{Spec}\mathcal{A}^{\mathrm{univ}} \to \mathbb{A}^p$ is an H-invariant semiuniversal deformation for $\mathcal{A}$. By Theorem 5.2.14, the $\mathbb{T}$-action on $\mathbb{A}^p$ has a dense orbit, so the family $\operatorname{Spec}\mathcal{A}^{\mathrm{univ}} \to \mathbb{A}^p$ is isomorphic to the trivial family on a dense open set. Any of the corresponding fibers is just isomorphic to $\operatorname{Spec}\mathcal{A}$.

By openness of versality [Art74], it follows that the family $\operatorname{Spec}\mathcal{A}^{\mathrm{univ}} \to \mathbb{A}^p$ contains a semiuniversal family for the fiber $\operatorname{Spec}\mathcal{A}$ over $\mathbf{t} = 1$. But since the family is isomorphic to the trivial family over $(\mathbb{K}^*)^p \subseteq \mathbb{A}^p$, it follows that $T^1(\mathcal{A})^H = 0$. $\square$

Example 6.3.7 (Obstructedness for B_n and C_n)**.** We consider the cluster complex $\mathcal{K}$ of a cluster algebra of type B_2 or C_2. Following [FZ03a, §12.3], vertices of $\mathcal{K}$ correspond to orbits of diameters of a 6-gon under rotation by 180 degrees. Cluster variables are compatible if the corresponding orbits of diameters do not cross. The cluster complex is thus the hexagon pictured in Figure 4, where we have labeled cluster variables by choosing an element of the corresponding orbit of diameters.

The space $T^1(\mathcal{K})$ is infinite dimensional. Using Theorem 3.2.2, one may compute that the first order deformations of $S_{\mathcal{K}}$, viewed as perturbations of the generators
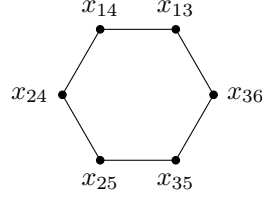
FIGURE 4. The cluster complex for B_2 and C_2

of $I_\mathcal{K}$, are given by

$$x_{13}x_{35} + \sum_{k=1}^{\infty} t_{36}^{(k)}x_{36}^k, \qquad x_{35}x_{24} + \sum_{k=1}^{\infty} t_{25}^{(k)}x_{25}^k, \qquad x_{24}x_{13} + \sum_{k=1}^{\infty} t_{14}^{(k)}x_{14}^k,$$

$$x_{14}x_{36} + \sum_{k=1}^{\infty} t_{13}^{(k)}x_{13}^k, \qquad x_{36}x_{25} + \sum_{k=1}^{\infty} t_{35}^{(k)}x_{35}^k, \qquad x_{25}x_{14} + \sum_{k=1}^{\infty} t_{24}^{(k)}x_{24}^k.$$

Here, the $t_{ij}^{(k)}$ are the deformation parameters.

Consider now the images V_B and V_C of the characteristic maps for the cluster algebras $\mathcal{A}_B$ and $\mathcal{A}_C$ of types B_2 and C_2 with no frozen variables. Using the exchange relations of [FZ03a, §12.3] and Lemma 6.1.6, we obtain that V_B and V_C are respectively spanned by the first order deformations corresponding to the parameters

$$V_B: \qquad t_{36}^{(2)}, t_{25}^{(2)}, t_{14}^{(2)}, t_{13}^{(1)}, t_{35}^{(1)}, t_{24}^{(1)}$$
$$V_C: \qquad t_{36}^{(1)}, t_{25}^{(1)}, t_{14}^{(1)}, t_{13}^{(2)}, t_{35}^{(2)}, t_{24}^{(2)}.$$

Since Property T0* is satisfied for $\mathcal{A}_B$ and $\mathcal{A}_C$, we may apply Theorem 6.3.2 and recover $\mathcal{A}_B^{\mathrm{univ}}$ and $\mathcal{A}_C^{\mathrm{univ}}$ from the total spaces $\mathcal{Y}_B$ and $\mathcal{Y}_C$ of canonical exchange minimal deformations (with respect to V_B and V_C).

Suppose that the cluster complex $\mathcal{K}$ were unobstructed. Then the semiuniversal deformation of $Y = \operatorname{Spec} S_\mathcal{K}$ would have smooth base space, and we would be able to find a (formal) family over a smooth base which would specialize to both $\operatorname{Spec} \mathcal{A}_B$ and $\operatorname{Spec} \mathcal{A}_C$. Given the stark distinction between the exchange relations for types B_2 and C_2, this seems highly improbable. Thus, we expect the cluster complex $\mathcal{K}$ to be obstructed, and indeed, this is the case.

A semiuniversal deformation $\mathcal{Y} \to Z$ of $S_\mathcal{K}$ has been described in [AC10, Proposition 6.6]. For the choices made in loc. cit. the equations for the base Z are given by the vanishing of the 2×2 minors of

$$\begin{pmatrix} t_{36}^{(1)} & t_{25}^{(1)} & t_{14}^{(1)} \\ t_{24}^{(1)} & t_{13}^{(1)} & t_{35}^{(1)} \end{pmatrix}.$$

Note in particular that the base space is not smooth; this is possible only since $T^2(\mathcal{K}) \neq 0$. Specializing to the subspaces V_A or V_B of $T^1(\mathcal{K})$, the obstructions vanish (as we know they must by Theorem 6.3.2). Somewhat remarkably, although the semiuniversal base space is not smooth, it is still irreducible.

This example might seem somewhat artificial: the Dynkin diagrams of type B_2 and C_2 are after all isomorphic. Indeed, the obstruction to having $\operatorname{Spec} \mathcal{A}_B$ and $\operatorname{Spec} \mathcal{A}_C$ appear as fibers in a family over a smooth base arose from our insistence

on labeling the vertices for $\mathcal{K}$ as we did. However, for $n \geq 3$, B_n and C_n are no longer isomorphic. Similar to the case $n = 2$, we view the obstructedness of the cluster complex for $n \geq 3$ as arising due to the "incompatibility" of types B_n and C_n. $\qquad \square$

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Appendix A. Macaulay2 Code for Examples

A.1. **Example from §1.7.**

```
i1 : loadPackage "VersalDeformations"

o1 = VersalDeformations

o1 : Package

i2 : R=QQ[x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_8,Degrees=>entries id_(ZZ^8)];

i3 : J=ideal {x_1*x_3,x_3*x_5,x_5*x_7,x_7*x_1,x_2*x_4,x_4*x_6,x_6*x_8,
         x_8*x_2,x_1*x_4,x_2*x_5,x_3*x_6, x_4*x_7,x_5*x_8,x_6*x_1,x_7*x_2,
         x_8*x_3,x_1*x_5,x_2*x_6,x_3*x_7,x_4*x_8}; --SR ideal for cluster complex

o3 : Ideal of R

i4 : L={{-1,1,-1,0,0,0,0,0},{0,0,-1,1,-1,0,0,0},
     {0,0,0,0,-1,1,-1,0},{-1,0,0,0,0,0,-1,1},
     {0,-1,3,-1,0,0,0,0},{0,0,0,-1,3,-1,0,0},
     {0,0,0,0,0,-1,3,-1},{3,-1,0,0,0,0,0,-1}}; -- degrees of characteristic map

i5 : t1=matrix {apply(L,i->CT^1(i,J))}; -- image of characteristic map

              20        8
o5 : Matrix R    <--- R

i6 : t2=CT^2(J); -- obstruction space

              64        12
o6 : Matrix R    <--- R

i7 : (F,R,G,C)=versalDeformation(gens J,-t1,t2);

i8 : matrix entries transpose sum F -- generators of lifted ideal

o8 = | -t_2t_3^2t_4t_6t_7-x_2t_1+x_1x_3                          |
     | -t_1t_3t_4^2t_7t_8-x_4t_2+x_3x_5                          |
     | -t_1^2t_2t_4t_5t_8-x_6t_3+x_5x_7                          |
     | -t_1t_2^2t_3t_5t_6-x_8t_4+x_1x_7                          |
     | -t_3^3t_4^3t_6t_7^2t_8-x_3^3t_5+x_2x_4                    |
     | -t_1^3t_4^3t_5t_7t_8^2-x_5^3t_6+x_4x_6                    |
     | -t_1^3t_2^3t_5^2t_6t_8-x_7^3t_7+x_6x_8                    |
     | -t_2^3t_3^3t_5t_6^2t_7-x_1^3t_8+x_2x_8                    |
     | -x_5t_3^2t_4t_6t_7-x_3^2t_1t_5+x_1x_4                     |
     | -x_1t_3t_4^2t_7t_8-x_3^2t_2t_5+x_2x_5                     |
     | -x_7t_1t_4^2t_7t_8-x_5^2t_2t_6+x_3x_6                     |
     | -x_3t_1^2t_4t_5t_8-x_5^2t_3t_6+x_4x_7                     |
     | -x_1t_1^2t_2t_5t_8-x_7^2t_3t_7+x_5x_8                     |
     | -x_5t_1t_2^2t_5t_6-x_7^2t_4t_7+x_1x_6                     |
     | -x_3t_2^2t_3t_5t_6-x_1^2t_4t_8+x_2x_7                     |
```

```
| -x_7t_2t_3^2t_6t_7-x_1^2t_1t_8+x_3x_8                                    |
| -x_3t_1t_2t_5-x_7t_3t_4t_7+x_1x_5                                        |
| -3t_1t_2^2t_3t_4^2t_5t_6t_7t_8-x_4t_2^3t_5t_6-x_8t_4^3t_7t_8+x_2x_6 |
| -x_5t_2t_3t_6-x_1t_1t_4t_8+x_3x_7                                        |
| -3t_1^2t_2t_3^2t_4t_5t_6t_7t_8-x_6t_3^3t_6t_7-x_2t_1^3t_5t_8+x_4x_8 |
```

A.2. Running Example 6.3.3 of type A_2.

```
i1 : loadPackage "VersalDeformations"

o1 = VersalDeformations

o1 : Package

i2 : R=QQ[x_25,x_13,x_24,x_35,x_14,s_1,s_2,s_3,
        Degrees=>{{1,1,-1},{-1,1,1},{2,-1,0},{-1,2,0},{1,-1,1},{1,0,0},{0,1,0},{0,0,1}}];

i3 : J=ideal{x_24*x_35,x_35*x_14,x_14*x_25,x_25*x_13,x_13*x_24}; -- SR ideal

o3 : Ideal of R

i4 : (F,R,G,C)=versalDeformation(gens J,-CT^1({0,0,0},J),CT^2({0,0,0},J));

i5 : matrix entries transpose sum F -- generators of lifted ideal

o5 = | -s_1s_2t_2t_5-x_25s_3t_1+x_24x_35 |
     | -s_2s_3t_1t_3-x_13s_1t_2+x_35x_14 |
     | -s_1^2t_2t_4-x_24s_2t_3+x_25x_14  |
     | -s_2^2t_3t_5-x_35s_1t_4+x_25x_13  |
     | -s_1s_3t_1t_4-x_14s_2t_5+x_13x_24 |
```

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