

A GEOMETRIC MODEL FOR SYZYGIES OVER 2-CALABI-YAU TILTED ALGEBRAS

RALF SCHIFFLER AND KHRYSTYNA SERHIYENKO

ABSTRACT. In this article, we consider the class of 2-Calabi-Yau tilted algebras that are defined by a quiver with potential whose dual graph is a tree. We call these algebras *dimer tree algebras* because they can also be realized as quotients of dimer algebras on a disc. These algebras are wild in general. For every such algebra B , we construct a polygon \mathcal{S} with a checkerboard pattern in its interior that gives rise to a category $\text{Diag}(\mathcal{S})$. The indecomposable objects of $\text{Diag}(\mathcal{S})$ are the 2-diagonals in \mathcal{S} , and its morphisms are given by certain pivoting moves between the 2-diagonals. We conjecture that the category $\text{Diag}(\mathcal{S})$ is equivalent to the stable syzygy category over the algebra B , such that the rotation of the polygon corresponds to the shift functor on the syzygies. In particular, the number of indecomposable syzygies is finite and the projective resolutions are periodic. We prove the conjecture in the special case where every chordless cycle in the quiver is of length three.

As a consequence, we obtain an explicit description of the projective resolutions. Moreover, we show that the syzygy category is equivalent to the 2-cluster category of type \mathbb{A} , and we introduce a new derived invariant for the algebra B that can be read off easily from the quiver.

CONTENTS

1. Introduction	2
Acknowledgements:	5
2. Preliminaries	5
2.1. 2-Calabi-Yau tilted algebras	6
2.2. Cohen-Macaulay modules over 2-Calabi-Yau tilted algebras	6
2.3. Translation quivers and mesh categories	7
2.4. The category of 2-diagonals of a polygon	7
2.5. The 2-cluster category of type \mathbb{A}	8
3. Construction and properties of the checkerboard polygon	8
3.1. The construction	8
3.2. Example	11
3.3. Properties of \mathcal{S}	14
3.4. Properties of dimer tree algebras	20
3.5. Some results toward the main conjecture	22
4. Objects of the syzygy category	24
4.1. Definition of f_γ	24
4.2. Exactness	26
4.3. Homotopy	27
4.4. Indecomposibility	30
4.5. Syzygy M_γ	31
5. Morphisms of the syzygy category	32
6. Auslander-Reiten triangles of the syzygy category	36
6.1. Nilpotent endomorphisms of M_γ	36
6.2. Mesh relations	37

2010 *Mathematics Subject Classification.* Primary 16G20, Secondary 13F60.

The first author was supported by the NSF grants DMS-1800860, DMS-205461, and by the University of Connecticut. The second author was supported in part by NSF grant DMS-2054255 and NSF Postdoctoral Fellowship MSPRF-1502881. This work was partially supported by a grant from the Simons Foundation. The authors would like to thank the Isaac Newton Institute for Mathematical Sciences for support and hospitality during the programme Cluster Algebras and Representation Theory when work on this paper was undertaken. This work was supported by: EPSRC Grant Number EP/R014604/1.

6.3. Auslander-Reiten triangles for M_γ	37
7. Main Result	39
8. Consequences of the main result	40
8.1. Rigidity and τ -rigidity of syzygies	40
8.2. Syzygies and cosyzygies	41
9. Examples	41
References	45
Appendix A. Proofs of section 4	47
A.1. Properties of f_γ	47
A.2. Exactness	54
A.3. Indecomposibility	65
Appendix B. Proofs of section 5	72
B.1. The induced morphism of complexes on the projective resolutions	76
Appendix C. Proofs of section 6	80
C.1. The staircase shape of f_γ	80
C.2. Nilpotent endomorphisms of M_γ	84
C.3. Proof of Proposition 6.3	98
C.4. Proof of Proposition 6.4	102

1. INTRODUCTION

Overview. In this paper, we study the syzygy categories of certain 2-Calabi-Yau tilted algebras. Syzygies are the building blocks for free resolutions in commutative algebra as well as for projective resolutions of modules in representation theory of algebras. Every module has a projective resolution which may be thought of as an approximation of the module by projectives. Conversely, any morphism between projectives defines a module via its cokernel. For any finite dimensional algebra, there are only finitely many indecomposable projectives but, in general, there is no hope for a classification of all indecomposable modules.

It is therefore natural to study the subcategory of all syzygy modules over the algebra. In the extreme cases, it is possible that every module is a syzygy (if the algebra is self-injective) or that only the projective modules are syzygies (if the algebra is hereditary), and in general the behavior can be anywhere in between these two extremes. A particularly nice situation is when the number of indecomposable syzygies is finite. It is an open problem to classify all syzygy-finite algebras.

We are interested in 2-Calabi-Yau tilted algebras, a class of finite dimensional non-commutative algebras over a field, introduced in [R] in the context of the categorification of cluster algebras by [BMRRT, Am]. These algebras are far-reaching generalizations of cluster-tilted algebras introduced in [CCS, BMR] and studied extensively by a large number of authors, see for example the lecture notes [As, BuMa] and references therein. For example, every (finite dimensional) Jacobian algebra of a quiver with potential is 2-Calabi-Yau tilted [Am]. These algebras also arise naturally in mathematical physics, for example, in relation to Postnikov diagrams and dimer models [HK, Po, JKS, BKM, Pr].

Keller and Reiten showed in [KR] that 2-Calabi-Yau tilted algebras are Gorenstein and that their stable syzygy categories are 3-Calabi-Yau. The Gorenstein property implies that the syzygy category is equivalent to the category of Cohen-Macaulay modules over the algebra, which in turn is equivalent to the singularity category [Bu, O]. Cohen-Macaulay modules play a central role in commutative algebra and algebraic geometry, in particular in the McKay correspondence and resolutions of singularities, see for example the textbook [LW], and more recently in matrix factorization [M, PV].

In this paper, we restrict to a certain class of 2-Calabi-Yau tilted algebras B , which we call *dimer tree algebras*, because they can be realized as a quotient of a dimer algebra on a disc. For example, algebras arising from the coordinate rings of the Grassmannians $\text{Gr}(3, n)$ are dimer tree algebras.

We give a complete description of the category of all syzygies $\underline{\text{CMP}} B$, including a description of objects, morphisms, shift, exact structure, and the Auslander-Reiten quiver. Furthermore, we construct explicit projective resolutions for the syzygies. In particular, we show that these algebras, which are generally of wild representation type, only admit a finite number of indecomposable syzygies.

Main results. Let B be a dimer tree algebra, that is, B is a Jacobian algebra given by a quiver with potential such that the dual graph of the quiver is a tree, see Definition 3.3. We construct a combinatorial model for the syzygy category inside a polygon \mathcal{S} with $2N$ vertices. Our polygon is equipped with a checkerboard pattern defined by a set of radical lines $\rho(i)$ corresponding to the vertices i of the quiver. Then each 2-diagonal γ in \mathcal{S} corresponds to an indecomposable syzygy M_γ and its intersection with the radical lines gives a projective presentation of M_γ .

We consider the category $\text{Diag}(\mathcal{S})$ of 2-diagonals. The morphisms in $\text{Diag}(\mathcal{S})$ are defined combinatorially via certain pivoting moves between the 2-diagonals. Moreover, $\text{Diag}(\mathcal{S})$ is a triangulated category with shift functor given by the rotation by π/N . If we omit the checkerboard pattern, this category has already appeared in [BM], where it was used to give a combinatorial model for the 2-cluster category of type \mathbb{A} .

Before stating our main result, we introduce the following notation. Let γ be a 2-diagonal in \mathcal{S} . Then γ crosses several of the radical lines $\rho(i)$ of the checkerboard pattern of \mathcal{S} . Each of these crossings has degree 0 or 1 according to Definition 3.27. We define two projective B -modules $P_0(\gamma)$ and $P_1(\gamma)$ as follows. Let $P_0(\gamma) = \bigoplus_i P(i)$, where the sum is over all vertices i of Q such that the radical line $\rho(i)$ crosses γ in degree 0. Similarly, let $P_1(\gamma) = \bigoplus_j P(j)$, where the sum is over all vertices j of Q such that the radical line $\rho(j)$ crosses γ in degree 1. Let Ω denote the syzygy functor.

We are now ready to state our main conjecture and our main result.

Conjecture 1.1. *Let B be a dimer tree algebra. For each 2-diagonal γ in \mathcal{S} there exists a morphism $f_\gamma: P_1(\gamma) \rightarrow P_0(\gamma)$ such that the mapping $\gamma \mapsto \text{coker } f_\gamma$ induces an equivalence of categories*

$$F: \text{Diag}(\mathcal{S}) \rightarrow \underline{\text{CMP}} B.$$

Under this equivalence, the radical line $\rho(i)$ corresponds to the radical of the indecomposable projective $P(i)$ for all $i \in Q_0$. The clockwise rotation R of \mathcal{S} corresponds to the shift Ω in $\underline{\text{CMP}} B$ and R^2 corresponds to the inverse Auslander-Reiten translation $\tau^{-1} = \Omega^2$. Thus

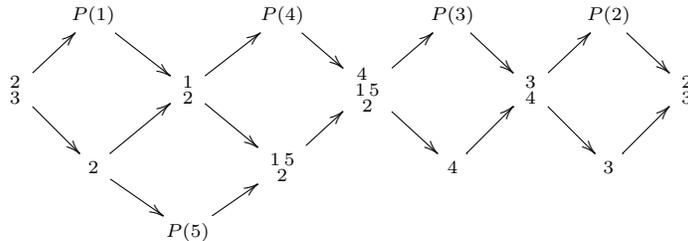
$$\begin{aligned} F(\rho(i)) &= \text{rad } P(i) \\ F(R(\gamma)) &= \Omega F(\gamma) \\ F(R^2(\gamma)) &= \tau^{-1} F(\gamma) \end{aligned}$$

Furthermore, F maps the 2-pivots in $\text{Diag}(\mathcal{S})$ to the irreducible morphisms in $\underline{\text{CMP}} B$, and the meshes in $\text{Diag}(\mathcal{S})$ to the Auslander-Reiten triangles in $\underline{\text{CMP}} B$.

In this paper, we prove the conjecture in the following special case. We shall prove the full conjecture in a forthcoming paper using the model developed here. Recall that a cycle in a graph is said to be chordless if it is equal to the induced subgraph on its vertices.

Theorem 1.2. *(Theorem 7.1) The conjecture holds in the special case where every chordless cycle in Q is of length 3.*

A small example. Below we provide a small example; a bigger and more detailed one is given in section 3.2. Let B be the algebra given by the quiver in Figure 1 with potential the sum of the two chordless cycles. The Auslander-Reiten quiver of the syzygy category $\text{CMP } B$ is given below, and the Auslander-Reiten quiver of the stable syzygy category $\underline{\text{CMP}} B$ is obtained from this one by removing the projective modules.



The polygon \mathcal{S} is shown in Figure 1. The arc γ as in the figure crosses radical lines 1,2, and 5. This gives a morphism $f_\gamma: P(2) \rightarrow P(1) \oplus P(5)$ between the projectives, whose cokernel is the corresponding syzygy $M_\gamma = \begin{smallmatrix} 1 & 5 \\ 2 \end{smallmatrix}$. Moreover, the projective resolution of M_γ can be constructed by applying the clockwise rotation R . Thus, $R\gamma$ is the radical line $\rho(1)$ and its associated syzygy $M_{R\gamma} = \text{coker } f_{R\gamma}: P(4) \rightarrow P(2)$ is the radical

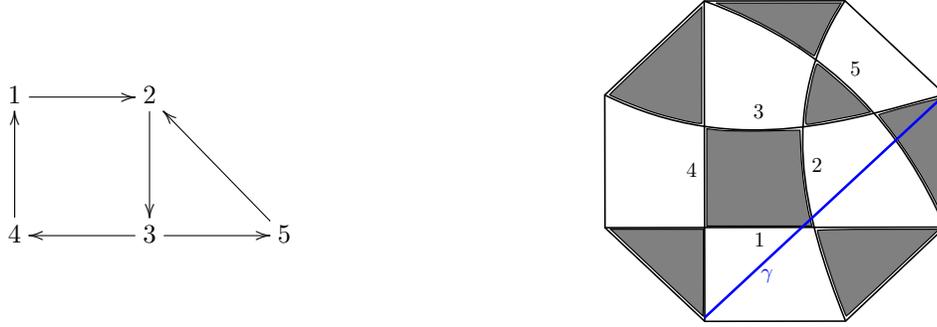


FIGURE 1. A quiver Q together with its checkerboard polygon \mathcal{S} . The radical lines are labeled by the vertices of Q . The crossing points of two radical lines represent arrows in the quiver, and the interior shaded regions represent chordless cycles in Q .

of the projective $P(1)$. Hence, $\Omega M_\gamma = M_{R\gamma} = \text{rad } P(1)$ and the first few terms of the projective resolution of M_γ are as follows.

$$\dots \longrightarrow P(4) \xrightarrow{f_{R\gamma}} P(2) \xrightarrow{f_\gamma} P(1) \oplus P(5) \longrightarrow M_\gamma \longrightarrow 0$$

Applications. As an application, we obtain the following corollaries. The first one uses the results of [BM].

Corollary 1.3. *The category $\underline{\text{CMP}}B$ is equivalent to the 2-cluster category of type \mathbb{A}_{N-2} . In particular, the number of indecomposable syzygies is $N(N-2)$.*

Equivalently this category can also be described as the stable module category over the self-injective algebra Λ_n defined as the quotient of the path algebra of an oriented n -cycle by the rad^{n-1} .

The corollary shows that the size of the polygon $2N$ determines the syzygy category up to equivalence. In Corollary 3.19, we provide a simple formula for $2N$ as a weighted sum over the boundary arrows of Q , where each arrow has weight 1 or 2. Since derived equivalent algebras have equivalent singularity categories, we obtain a new derived invariant that is easy to compute.

Corollary 1.4. *The size of the polygon is a derived invariant for the algebra B .*

The checkerboard pattern on \mathcal{S} determines the algebra B completely. On the other hand, two algebras may have the same size polygon while having non-equivalent derived categories. It is an interesting problem to determine when two checkerboard patterns are derived equivalent.

Corollary 1.5. *(Corollary 4.21) The projective resolution of any syzygy is periodic of period N or $2N$. An indecomposable syzygy M_γ has period N if and only if the corresponding 2-diagonal γ is a diameter in \mathcal{S} .*

Recall that a module is rigid if it has no nontrivial self-extensions.

Corollary 1.6. *(Corollary 8.1) The indecomposable syzygies over B are rigid B -modules.*

Moreover, we conjecture that the indecomposable syzygies are τ -rigid in $\text{mod } B$. This is of interest, since the τ -rigid indecomposable modules correspond to cluster variables in the cluster algebra of Q provided they are reachable by mutation.

Corollary 1.7. *(Corollary 8.2) Let M, N be indecomposable syzygies over B . Then the dimension of $\text{Ext}_B^1(M, N) \oplus \text{Ext}_B^1(N, M)$ is equal to the number of crossing points between the corresponding 2-diagonals. In particular, the dimension is either 1 or 0.*

It was shown in [BR] that the Auslander-Reiten translation τ in $\text{mod } B$ induces an equivalence from the stable syzygy category $\underline{\text{CMP}}B$ to the stable cosyzygy category $\underline{\text{CMI}}B$. In our geometric model the same checkerboard polygon describes both categories. To switch to the cosyzygies, it suffices to apply the Nakayama functor ν which replaces the projectives with injectives.

Corollary 1.8. (Corollary 8.5) *The following diagram commutes.*

$$\begin{array}{ccc}
 \underline{\text{CMP}} B & \xrightarrow{\tau} & \underline{\text{CMI}} B \\
 \text{cok } f_\gamma \uparrow & & \uparrow \ker \nu f_\gamma \\
 \text{Diag}(\mathcal{S}) & \xrightarrow{1} & \text{Diag}(\mathcal{S})
 \end{array}$$

The proofs of the above corollaries do not depend on the additional assumption of Theorem 1.2 and will therefore generalize to the setting of Conjecture 1.1.

We also classify all checkerboard patterns up to the decagon in Example 9.2. The decagon admits 17 different checkerboard patterns up to symmetry.

A few words about the proof. The key ingredient in most of our arguments is an explicit definition of the morphism f_γ in terms of certain paths in the quiver, which we call *valid paths* and whose definition involves the checkerboard pattern. This provides a complete description of not just the modules but also the morphisms in the projective resolution of the syzygies. It would also be useful to have a more conceptual definition of f_γ , but we have not found one so far. Let us point out that, in general, $f_\gamma: P_1(\gamma) \rightarrow P_0(\gamma)$ is not simply a generic map, and the knowledge of $P_0(\gamma)$ and $P_1(\gamma)$ does not determine f_γ in this way.

The first part of the proof is to show that our category of 2-diagonals $\text{Diag}(\mathcal{S})$ gives a subcategory of the syzygy category. The second step is then to show that we obtain every syzygy in this way, which may actually come as a surprise, since the algebra is wild in general. For this we show that $\text{Diag}(\mathcal{S})$ gives a finite component of the Auslander-Reiten quiver of $\underline{\text{CMP}} B$ by explicitly constructing the Auslander-Reiten triangles. In addition, the argument requires a detailed analysis of the endomorphisms of the indecomposable syzygies M_γ .

Related work. For the very special class of cluster-tilted algebras of finite representation type, the syzygy categories were studied before by Chen, Geng and Lu in [CGL], where they gave a classification of the syzygy categories of these algebras. In particular, they show that the components of $\underline{\text{CMP}} B$ are equivalent to the stable categories of the self-injective algebras Λ_n . Their procedure involves a case by case analysis that uses a classification of the derived equivalence classes of cluster-tilted algebras of Dynkin type in [BHL, BHL2]. Later Lu extended these results to simple polygon-tree algebras [L]. One of the ingredients of the proof is successive mutation at vertices of the exterior cycles and reduction to a cluster-tilted algebra of Dynkin type \mathbb{D} . These algebras are special cases of those included in Conjecture 1.1 and the results provide evidence for the conjecture. The above results determine only the type of the syzygy category but do not describe the objects or the morphisms.

Garcia-Elsener and the first author have described the syzygy category of cluster-tilted algebras of type \mathbb{D} in terms of arcs in a once-punctured polygon in [GES].

For gentle algebras, the singularity categories have been described by Kalck in [K] using m -cluster categories of type \mathbb{A}_1 . In our setting the algebra is gentle if and only if the quiver has a unique chordless cycle. This has been extended to skew-gentle algebras by Chen and Lu in [CL]. For further results on singularity categories of finite dimensional algebras see [C, C2, CDZ, LZ, Sh].

Future directions. In a forthcoming paper [SS], we prove Conjecture 1.1 in full generality. In a different paper, we will describe the connection to dimer algebras, where we will show how to embed our checkerboard polygon in an alternating strand diagram of the dimer model, see Example 3.8.

Furthermore, it will be interesting to see if we can relax the conditions on the quiver such as allowing the dual graph to be disconnected or to contain cycles, see Remark 3.5. Other future directions include the behavior of the checkerboard polygon under mutations, a description of the syzygies in terms of their composition factors, and the question of τ -rigidity of the indecomposable syzygies.

Acknowledgements: We thank Alastair King and Matthew Pressland for interesting discussions on the connection to dimer algebras, as well as an anonymous referee for pointing us to the paper [Bo].

2. PRELIMINARIES

Let \mathbb{k} be an algebraically closed field. If A is a finite-dimensional \mathbb{k} -algebra, we denote by $\text{mod } A$ the category of finitely generated right A -modules. Let D denote the standard duality $D = \text{Hom}(-, \mathbb{k})$. If Q_A is

the ordinary quiver of the algebra A , and i is a vertex of Q_A , we denote by $P(i), I(i), S(i)$ the corresponding indecomposable projective, injective, simple A -module, respectively. We let $\text{pd}M, \text{id}M$ denote the projective dimension, respectively the injective dimension of the A -module M .

A *loop* in a quiver Q is an arrow that starts and ends at the same vertex. Two arrows are called *parallel* if they share the same starting point and the same endpoint. An *oriented cycle* in Q is a path that starts and ends at the same vertex. Thus a loop is an oriented cycle of length 1. A *2-cycle* is an oriented cycle of length 2.

For further information about representation theory and quivers we refer to [ASS, S2].

2.1. 2-Calabi-Yau tilted algebras. A \mathbb{k} -linear triangulated category \mathcal{C} with split idempotent and finite-dimensional Hom spaces is said to be *2-Calabi-Yau* if $D\text{Ext}_{\mathcal{C}}^1(X, Y) \cong \text{Ext}_{\mathcal{C}}^1(Y, X)$, for all objects $X, Y \in \mathcal{C}$. Let \mathcal{C} be a 2CY category. A basic object T in \mathcal{C} is called a *cluster-tilting object* if $\text{add}T = \{X \in \mathcal{C} \mid \text{Ext}_{\mathcal{C}}^1(X, T) = 0\}$. The endomorphism algebra $B = \text{End}_{\mathcal{C}}T$ of a cluster-tilting object T is called a *2-Calabi-Yau tilted algebra*. These algebras are a far reaching generalization of cluster-tilted algebra and have been studied extensively. For example, every finite dimensional Jacobian \mathbb{k} -algebra B in the sense of [DWZ] is 2-Calabi-Yau tilted, because the associated generalized cluster category \mathcal{C}_B contains a cluster-tilting object T whose endomorphism algebra is B , see [Am]. On the other hand, not every 2-Calabi-Yau tilted algebra is a Jacobian algebra, see [L].

Let us highlight the following result by Keller and Reiten. Recall that a \mathbb{k} -algebra A is said to be *Gorenstein of dimension d* if $\text{pd}DA = \text{id}A = d < \infty$.

Theorem 2.1. *Every 2-Calabi-Yau tilted algebra is Gorenstein of dimension at most 1.*

For further results about 2-Calabi-Yau tilted algebras, we refer to the surveys [As, R].

2.2. Cohen-Macaulay modules over 2-Calabi-Yau tilted algebras. From now on, let B be a 2-Calabi-Yau tilted algebra. A B -module M is said to be *projectively Cohen-Macaulay* if $\text{Ext}_B^i(M, B) = 0$ for all $i > 0$. In other words, M has no extensions with projective modules. Dually, a B -module N is said to be *injectively Cohen-Macaulay* if $\text{Ext}_B^i(B, N) = 0$ for all $i > 0$.

We denote by $\text{CMP } B$ and $\text{CMI } B$ the full subcategories of $\text{mod } B$ whose objects are the projectively Cohen-Macaulay modules or the injectively Cohen-Macaulay modules, respectively. Both categories are Frobenius categories. The projective-injective objects in $\text{CMP } B$ are precisely the projective B -modules, whereas the projective-injective objects in $\text{CMI } B$ are precisely the injective B -modules. The corresponding stable categories $\underline{\text{CMP}} B$ and $\underline{\text{CMI}} B$ are triangulated categories. The syzygy operator Ω in $\text{mod } B$ is the inverse shift in $\underline{\text{CMP}} B$, and the cosyzygy operator Ω^{-1} in $\text{mod } B$ is the shift in $\underline{\text{CMI}} B$.

The Auslander-Reiten translations induce quasi-inverse triangle equivalences [BR, Chapter X]

$$(2.1) \quad \tau: \underline{\text{CMP}} B \rightarrow \underline{\text{CMI}} B \quad \tau^{-1}: \underline{\text{CMI}} B \rightarrow \underline{\text{CMP}} B.$$

Moreover, by Buchweitz's theorem [Bu, Theorem 4.4.1], there exists a triangle equivalence between $\underline{\text{CMP}} B$ and the singularity category $\mathcal{D}^b(B)/\mathcal{D}_{\text{perf}}^b(B)$ of B . Keller and Reiten showed in [KR] that the category $\underline{\text{CMP}} B$ is 3-Calabi-Yau. The following result was proved in [GES].

Theorem 2.2. *Let M be an indecomposable module over a 2-Calabi-Yau tilted algebra B . Then the following are equivalent.*

- (a) M is a non-projective syzygy;
- (b) $M \in \text{ind } \underline{\text{CMP}} B$;
- (c) $\Omega_B^2 \tau_B M \cong M$.

We may therefore use the terminology “syzygy” and “Cohen-Macaulay module” interchangeably.

For convenience of the reader, we give a proof of the following fact.

Proposition 2.3. *The category $\text{CMP } B$ is closed under extensions.*

Proof. Let $0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$ be a short exact sequence in $\text{mod } B$ with $L, N \in \text{CMP } B$. Applying the functor $\text{Hom}_B(-, B)$ yields an exact sequence

$$\cdots \longrightarrow \text{Ext}_B^i(N, B) \longrightarrow \text{Ext}_B^i(M, B) \longrightarrow \text{Ext}_B^i(L, B) \longrightarrow \cdots$$

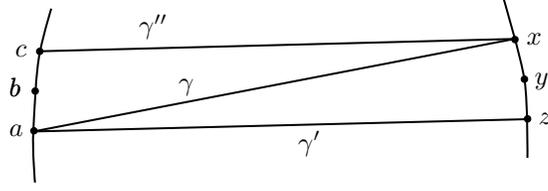


FIGURE 2. γ' is the 2-pivot of γ fixing the endpoint a and γ'' is the 2-pivot of γ fixing the endpoint b .

for every $i \geq 1$. Now $\text{Ext}_B^i(N, B) = \text{Ext}_B^i(L, B) = 0$, because N and L are Cohen-Macaulay modules. Thus $\text{Ext}_B^i(M, B) = 0$, and hence $M \in \text{CMP } B$. \square

Corollary 2.4. *Let $M \in \text{CMP } B$. Then M is rigid in $\text{CMP } B$ if and only if M is rigid in $\text{mod } B$.* \square

2.3. Translation quivers and mesh categories. We review here the notions of translation quiver and mesh category from [Ri, Ha]. These notions are often used in order to define a category from combinatorial data. Examples of such constructions are the combinatorial constructions of cluster categories of finite type in [BM, BM2, CCS, S1].

A *translation quiver* (Γ, τ) is a quiver $\Gamma = (\Gamma_0, \Gamma_1)$ without loops together with an injective map $\tau: \Gamma'_0 \rightarrow \Gamma_0$ (the *translation*) from a subset Γ'_0 of Γ_0 to Γ_0 such that, for all vertices $x \in \Gamma'_0$, $y \in \Gamma_0$, the number of arrows from $y \rightarrow x$ is equal to the number of arrows from $\tau x \rightarrow y$. Given a translation quiver (Γ, τ) , a *polarization* of Γ is an injective map $\sigma: \Gamma'_1 \rightarrow \Gamma_1$, where Γ'_1 is the set of all arrows $\alpha: y \rightarrow x$ with $x \in \Gamma'_0$, such that $\sigma(\alpha): \tau x \rightarrow y$ for every arrow $\alpha: y \rightarrow x \in \Gamma_1$. From now on we assume that Γ has no multiple arrows. In that case, there is a unique polarization of Γ .

The *path category* of a translation quiver (Γ, τ) is the category whose objects are the vertices Γ_0 of Γ , and, given $x, y \in \Gamma_0$, the \mathbb{k} -vector space of morphisms from x to y is given by the \mathbb{k} -vector space with basis the set of all paths from x to y . The composition of morphisms is induced from the usual composition of paths. The *mesh ideal* in the path category of Γ is the ideal generated by the *mesh relations*

$$m_x = \sum_{\alpha: y \rightarrow x} \sigma(\alpha)\alpha$$

for all $x \in \Gamma'_0$.

The *mesh category* of the translation quiver (Γ, τ) is the quotient of its path category by the mesh ideal.

2.4. The category of 2-diagonals of a polygon. Let \mathcal{S} be a regular polygon with an even number of vertices, say $2N$. Let R be the automorphism of \mathcal{S} given by a clockwise rotation about $180/N$ degrees. Thus R^{2N} is the identity.

Following Baur and Marsh, we define the category $\text{Diag}(\mathcal{S})$ of 2-diagonals of \mathcal{S} as follows. The indecomposable objects of $\text{Diag}(\mathcal{S})$ are the 2-diagonals in \mathcal{S} . Recall that a 2-diagonal is a diagonal of \mathcal{S} connecting two vertices such that the two polygons obtained by cutting \mathcal{S} along the diagonal both have an even number of vertices and at least 4. In particular, boundary segments are not 2-diagonals.

The irreducible morphisms of \mathcal{S} are given by 2-pivots. We recall the definition below. An illustration is given in Figure 2.

Definition 2.5. Let γ be a 2-diagonal in the checkerboard polygon \mathcal{S} and denote its endpoints by a and x . Denote by b the clockwise neighbor of a , and by c the clockwise neighbor of b on the the boundary of \mathcal{S} . At the other end, denote by y the clockwise neighbor of x , and by z the clockwise neighbor of y on the the boundary of \mathcal{S} .

Unless a and z are neighbors on the boundary, the 2-diagonal γ' connecting a and z is called the *2-pivot of γ fixing the endpoint a* .

Unless c and x are neighbors on the boundary, the 2-diagonal γ'' connecting c and x is called the *2-pivot of γ fixing the endpoint x* .

Let Γ be the quiver whose vertices is the set of 2-diagonals in \mathcal{S} , and there is an arrow from the 2-diagonal γ to the 2-diagonal γ' precisely if γ' is obtained from γ by a 2-pivot. Then the pair (Γ, R^{-2}) is a translation quiver.

Definition 2.6. [BM] The category $\text{Diag}(\mathcal{S})$ of 2-diagonals in the polygon \mathcal{S} is the mesh category of the translation quiver (Γ, R^{-2}) .

2.5. The 2-cluster category of type \mathbb{A} . In this subsection, let H be the path algebra of a Dynkin quiver of type \mathbb{A}_r . Let \mathcal{C}^2 denote the 2-cluster category of type \mathbb{A}_r . This category is defined as the orbit category of the bounded derived category $\mathcal{D}^b(\text{mod } H)$ by the functor $\tau_{\mathcal{D}}^{-1}[2]$. Here $\tau_{\mathcal{D}}$ is the Auslander-Reiten translation and $[2] = [1] \circ [1]$ is the second power of the shift functor in the derived category. Thus

$$\mathcal{C}^2 = \mathcal{D}^b(\text{mod } H) / \tau_{\mathcal{D}}^{-1}[2].$$

This category was introduced in [K, T], and was studied in [BRT, IY, Tor].

Theorem 2.7. [BM] *Let \mathcal{S} be a polygon with $2N$ vertices. Then the category $\text{Diag}(\mathcal{S})$ is equivalent to the 2-cluster category of type \mathbb{A}_{N-2} .*

Under this equivalence each 2-diagonal of \mathcal{S} corresponds to an indecomposable object in \mathcal{C}^2 . Moreover, there exists a nontrivial extension between two indecomposable objects in \mathcal{C}^2 if and only if the corresponding 2-diagonals cross. And the maximal sets of non-crossing 2-diagonals, or quadrangulations, correspond to the cluster-tilting objects in \mathcal{C}^2 .

In particular, since every indecomposable object in \mathcal{C}^2 is rigid, we have the following.

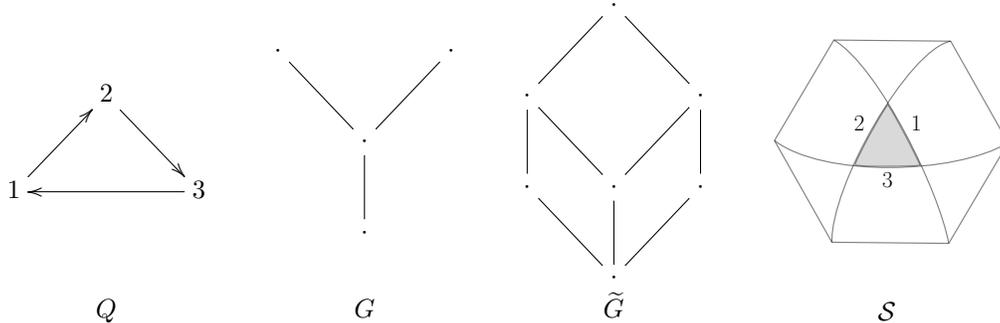
Corollary 2.8. *Every 2-diagonal is a rigid object in the category $\text{Diag}(\mathcal{S})$.*

Remark 2.9. Baur and Marsh actually proved more generally that the category of m -diagonals is equivalent to the m -cluster category of type \mathbb{A} .

3. CONSTRUCTION AND PROPERTIES OF THE CHECKERBOARD POLYGON

3.1. The construction. In this section, we define dimer tree algebras as a class of 2-Cababi-Yau tilted algebras by imposing restrictions on the quiver and specifying the potential. We start by constructing a checkerboard polygon \mathcal{S} from the quiver Q in three steps. First, we associate the dual graph G to Q , then we construct the twisted completed dual graph \tilde{G} from G , and finally we obtain the checkerboard polygon \mathcal{S} as the medial graph of \tilde{G} . The smallest example of the construction is given below. The definition of the algebra B is given at the end of this subsection, because in order to define the signs of the terms of the potential, we first need to introduce a distance function on the dual graph.

Example 3.1.



A larger example is given in subsection 3.2.

3.1.1. *The quiver Q .* A *chordless cycle* in a quiver Q is a cyclic path $C = x_0 \rightarrow x_1 \rightarrow \cdots \rightarrow x_t \rightarrow x_0$ such that $x_i \neq x_j$ if $i \neq j$ and the full subquiver on vertices x_0, x_1, \dots, x_t is equal to C . The arrows that lie in exactly one chordless cycle will be called *boundary arrows* and those that lie in two or more chordless cycles *interior arrows* of Q .

Definition 3.2. The *dual graph* G of Q is defined as follows. The set of vertices G_0 is the union of the set of chordless cycles of Q and the set of boundary arrows of Q . The set of edges G_1 is the union of two sets called the set of *trunk edges* and the set of *leaf branches*. A trunk edge $C \overset{\alpha}{-} C'$ is drawn between any pair of chordless cycles (C, C') that share an arrow α . A leaf branch $C \overset{\alpha}{-} \alpha$ is drawn between any pair (C, α) where C is a chordless cycle and α is a boundary arrow such that α is contained in C .

Definition 3.3 (The quiver). Throughout the paper, we let Q be a finite connected quiver without loops and 2-cycles satisfying the following conditions.

- (Q1) Every arrow of Q lies in at least one chordless cycle.
- (Q2) The dual graph of Q is a tree.

The following properties follow easily from the definition.

Proposition 3.4. *Let Q be a quiver satisfying Definition 3.3.*

- (1) Q has no parallel arrows.
- (2) Q is planar.
- (3) For all arrows α of Q ,
 - (a) either α lies in exactly one chordless cycle,
 - (b) or α lies in exactly two chordless cycles.
- (4) Any two chordless cycles in Q share at most one arrow.

Proof. (1) Suppose α, β are parallel arrows in Q and let C be a chordless cycle containing α . Then the full subquiver on the vertices of C also contains β , so it does not equal C , a contradiction.

(2) Let G be the dual graph of Q . Since G is a tree, there is a planar embedding of G . With respect to this planar embedding, we can reconstruct the quiver as follows. We get a subdivision of the plane into unbounded regions by extending the leaves of G to infinity. Each region corresponds to a vertex in Q . Each edge in G corresponds to an arrow connecting the vertices of the two adjacent regions. This gives a planar embedding of Q .

(3) If an arrow α lies in three (or more) chordless cycles C_1, C_2, C_3 then α gives rise to the following three edges $C_1 - C_2 - C_3 - C_1$ in G , contradicting that G is a tree.

(4) If two chordless cycles share two arrows, then there would be two edges between the corresponding vertices in G contradicting (Q2). \square

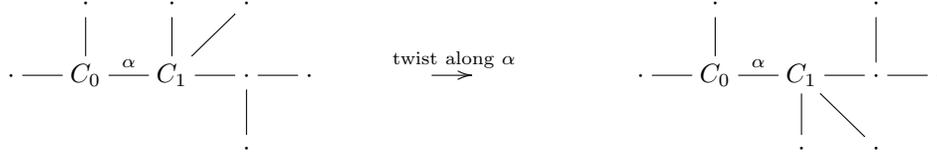
Remark 3.5. We could relax Condition (Q2) and allow $G = G' \amalg G''$ to be a disjoint union of trees that correspond to disjoint subquivers Q', Q'' each satisfying Definition 3.3, and there is exactly one arrow from one to the other and this arrow is not contained in a relation. In that case the algebra is a triangular matrix algebra and using [Z, Theorem 3.3] together with [CGL, Lemma 3.7] we get that $\text{CMP } B = \text{CMP } B' \amalg \text{CMP } B''$. In particular, we can relax conditions on the quiver by allowing arrows α on which there are no relations such that $Q \setminus \{\alpha\}$ is disconnected.

The condition that the tree G has no cycles seems more serious. If G were allowed to have cycles our construction would not yield a polygon but a more complicated surface. It is an interesting question if the results and conjectures of this paper generalize to that setting.

3.1.2. *The twisted completed dual graph \tilde{G} .* If Q has exactly one chordless cycle the twisted dual graph \tilde{G} is defined to be the dual graph G . Suppose now that Q has at least two chordless cycles. Then there exists a chordless cycle C_0 that is connected to exactly one other chordless cycle C_1 . We are going to twist the graph G at every trunk edge in the following way.

We label the chordless cycles $C_0, C_1, C_2, \dots, C_s$ in such a way that the length $d(i)$ of the unique path from C_0 to C_i in G satisfies the condition $d(i) < d(j)$ if $i < j$.

To construct \tilde{G} , we start with G and twist at the edge $C_0 \overset{\alpha}{-} C_1$, meaning that we keep the connected component of $G \setminus \alpha$ that contains C_0 , flip the other component and reconnect the two components via α . Call the result G_1 . An example is given below.

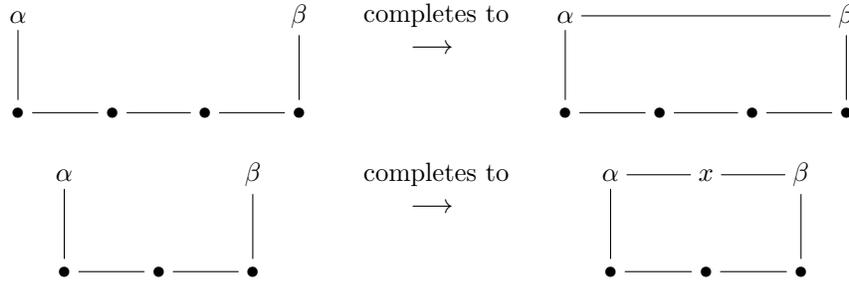


Recursively, the graph G_i is the twist of the graph G_{i-1} along the last edge on the unique path from C_0 to C_i . After s steps, we have twisted once along every trunk edge of G and we obtain the graph G_s .

To define the graph \tilde{G} , we complete the graph G_s as follows. For every pair of neighboring leaf vertices α, β in G_s

- we add an edge $\alpha - \beta$, if the resulting face has an even number of edges; or
- we add a new vertex x and two edges $\alpha - x$, $x - \beta$, otherwise.

An example is shown below.



Note that in both cases the vertices α, β of \tilde{G} lie in a face that has an even number of vertices. Moreover

$$\tilde{G}_0 = G_0 \cup \{\text{completion vertices}\} \quad \text{and} \quad \tilde{G}_1 = G_1 \cup \{\text{completion edges}\}.$$

We denote by \tilde{G}_2 the set of bounded faces of \tilde{G} . By construction, every face in \tilde{G}_2 has an even number (≥ 4) of edges and at least one and at most two of them are completion edges.

3.1.3. The checkerboard polygon \mathcal{S} . Recall that the *medial graph* $M(H)$ of a planar graph H is defined as the graph that has one vertex for each edge of H , and two vertices are connected in $M(H)$ if the corresponding edges in H are consecutive in one of the faces of H .

The checkerboard polygon \mathcal{S} is obtained from the medial graph of \tilde{G} by adding one edge for every leaf vertex of G . The faces of \mathcal{S} come in two types: faces that surround a vertex of G and faces that sit inside a face of \tilde{G} . We think of \mathcal{S} as a polygon with checkerboard pattern where the shaded regions are those coming from vertices in G , thus from the chordless cycles and the boundary arrows in Q .

Remark 3.6. If we started with the opposite quiver we would obtain the same checkerboard polygon. To fix orientations, we shall always assume that the first chordless cycle C_0 of the quiver Q corresponds to running around the associated shaded region of the polygon \mathcal{S} in counterclockwise direction. Then the other chordless cycles in the quiver also correspond to the counterclockwise direction in their respective shaded regions, since we use the twisted dual graph. Indeed, in the quiver, the orientation of the chordless cycles alternates between clockwise and counterclockwise, and when twisting the dual graph all chordless cycles acquire the counterclockwise orientation.

The interior vertices of \mathcal{S} correspond to arrows in Q , and every interior vertex has degree four. An edge in \mathcal{S} may connect two interior vertices, or an interior vertex and a boundary vertex, or two boundary vertices.

An edge in \mathcal{S} that connects two interior vertices α, β carries the label i , where i is the unique vertex in Q that is shared by the two arrows α and β .

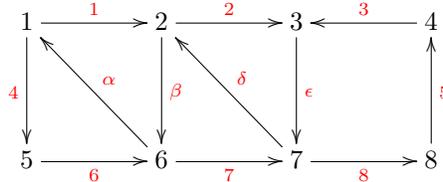
Every shaded boundary region in \mathcal{S} contains precisely one boundary edge as well as two interior edges that meet at an interior vertex α . The two interior edges are labeled by the starting vertex $s(\alpha)$ and the terminal vertex $t(\alpha)$ of the arrow α in Q such that, when running around the shaded region in counterclockwise order, we go along $s(\alpha)$ towards α and along $t(\alpha)$ away from α . The boundary edges in \mathcal{S} are not labeled.

3.1.4. *The algebra B.* We are now ready to define the algebra we are going to study.

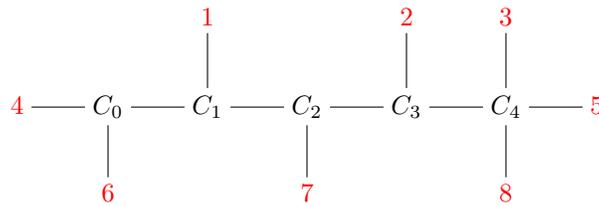
Definition 3.7. Let Q be quiver satisfying the conditions of Definition 3.3 and let W be the potential $W = \sum_{i=0}^t (-1)^{d(i)} C_i$. Then the Jacobian algebra over \mathbb{k} of (Q, W) is called a *dimer tree algebra*.

We are going to show in subsection 3.4 that every cyclic path in Q is zero in B and that any two nonzero parallel paths in Q are equal. In particular, this implies that dimer tree algebras are finite-dimensional and thus 2-Calabi-Yau tilted.

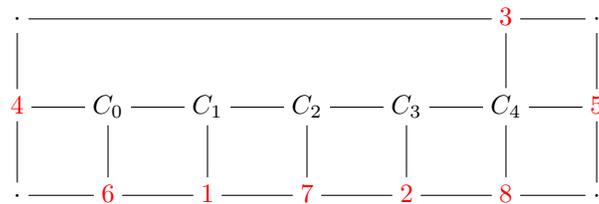
3.2. **Example.** Let Q be the quiver below. It has 5 chordless cycles and 8 boundary arrows. We have labeled the arrows in red.



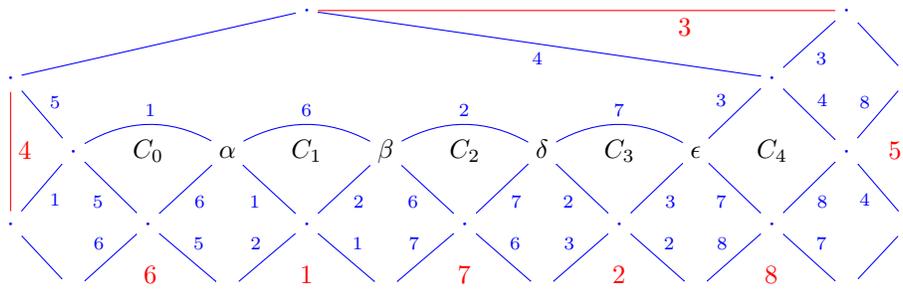
Its dual graph G below has 5 trunk vertices labeled C_0, \dots, C_4 which correspond to the 5 chordless cycles in Q . It also has 8 leaf vertices labeled in red which correspond to the boundary arrows in Q .



The twisted completed dual graph \tilde{G} is shown below. Note that the vertices 1 and 2 are now at the bottom because they have been involved in an odd number twist operations, while the vertices 3,7, and 8 have not changed position, since they have been involved in an even number of twists. The completion step has introduced 4 completion vertices marked by a dot in the figure below and 12 completion edges which create 8 bounded faces.



The polygon \mathcal{S} is shown below. The medial graph of \tilde{G} is drawn in blue and the 8 extra edges that come from the 8 leaf vertices of G are drawn in red. The faces that correspond to a vertex of G are labeled by that vertex as follows: the faces C_0, \dots, C_4 are labeled in black and the faces corresponding to the leaf vertices 1, 2, ... 8 are labeled in red. The interior vertices of \mathcal{S} correspond to arrows in Q . We have labeled only those vertices that correspond to the interior arrows $\alpha, \beta, \delta, \epsilon$, since the labels of the remaining interior vertices of \mathcal{S} are the same as the red labels of the adjacent boundary regions.



Each interior edge of \mathcal{S} carries a blue label that corresponds to a vertex of Q . For example, the edge connecting the vertices α and β in \mathcal{S} is labeled by the vertex 6 of Q , since 6 is the common vertex of the arrows α and β .

Moreover, the vertex shared by the faces labeled C_0 and the red 6 corresponds to the arrow labeled 6 in the quiver. This arrow shares the vertex 6 with the arrow α , so the label on the southeast edge of the face C_0 is labeled by a blue 6.

The polygon \mathcal{S} with its checkerboard pattern is redrawn in Figure 3. The shaded regions are those labeled by a vertex of G .

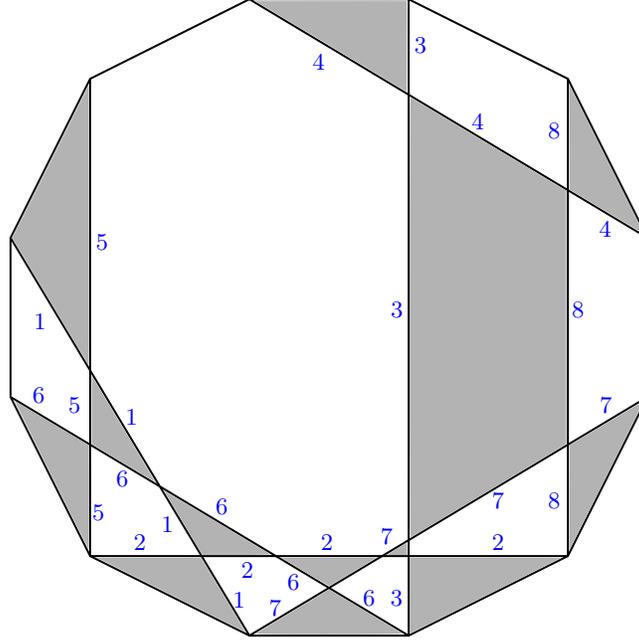


FIGURE 3. The checkerboard polygon in the example of Section 3.2. The diagonals shown are called radical lines, they are labeled by the vertices of the quiver Q . The crossing points of two radical lines represent an arrow in the quiver and the interior shaded regions represent chordless cycles in Q . The boundary shaded regions correspond to the boundary arrows of Q .

Conjecture 1.1 claims that the syzygy category $\text{CMP } B$ is equivalent to the category of 2-diagonals. To illustrate this, we compute the projective resolutions of the syzygies below.

Let γ denote the radical line $\rho(4)$ in Figure 3. Note that γ crosses the radical lines $\rho(3)$ and $\rho(8)$. Then γ represents the projective presentation

$$P(8) \xrightarrow{f_\gamma} P(3) \longrightarrow M_\gamma \longrightarrow 0$$

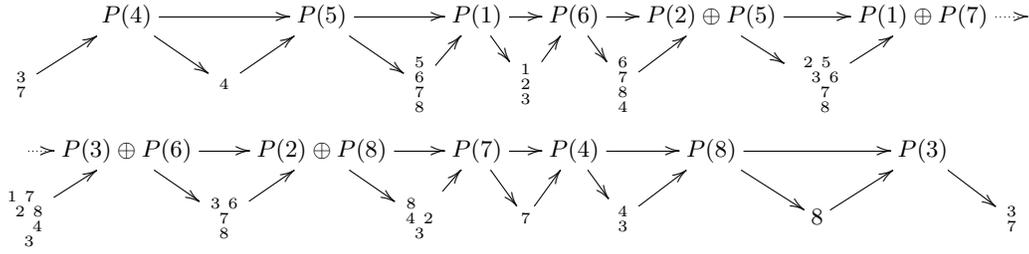
and we see that $M_\gamma = \text{rad } P(4) = \frac{3}{7}$. Thus the radical line $\gamma = \rho(4)$ represents the projective presentation of the radical of $P(4)$. The map f_γ factors through the syzygy $\Omega M_\gamma = S(8)$ of M_γ whose projective presentation

$$P(4) \xrightarrow{f_{R\gamma}} P(8) \longrightarrow \Omega M_\gamma \longrightarrow 0$$

is represented by the 2-diagonal $R\gamma$ where R is the clockwise rotation about the angle $2\pi/12$. Notice that $R\gamma$ crosses the radical lines $\rho(8)$ and $\rho(4)$ corresponding to the projective modules in the projective presentation.

We can continue this procedure and represent the projective presentation of the higher syzygies $\Omega^i M_\gamma$ by the rotations $R^i \gamma$ of the diagonal γ . In particular, after 12 steps we come back to where we started because

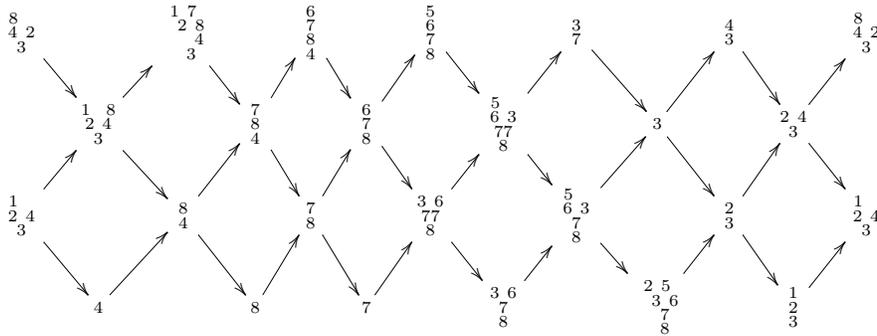
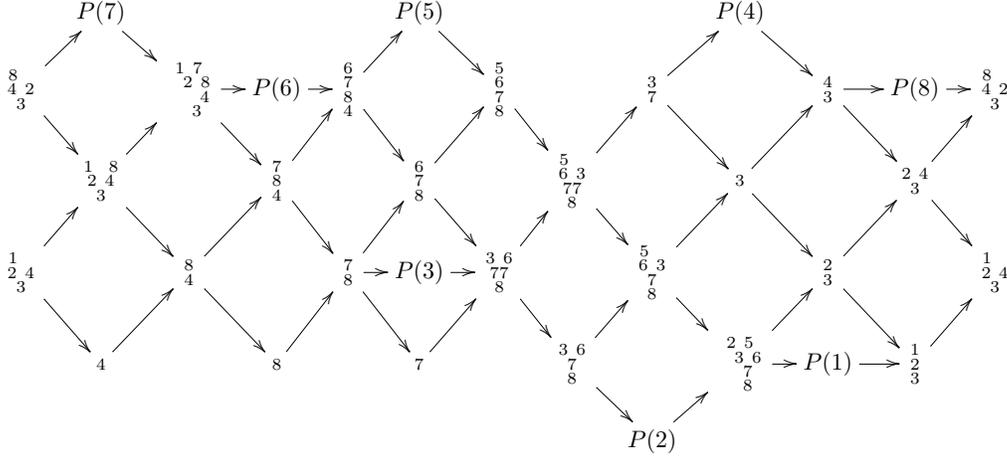
the polygon has 12 vertices. The complete projective resolution is shown below over two lines.



This projective resolution corresponds to the rotation orbit of the diagonal $\rho(4)$ in the polygon. This orbit contains all the 2-diagonals that cut the polygon into a quadrilateral and a decagon. In the module category of B this yields 12 indecomposable non-projective syzygies.

In the polygon, there is precisely one other orbit under the rotation and it contains all the 2-diagonals that cut the polygon into a hexagon and an octagon. This orbit also yields 12 indecomposable syzygies.

The Auslander-Reiten quiver of the syzygy categories $\text{CMP}B$ and $\underline{\text{CMP}}B$ are given below.



Notice that the second syzygy Ω^2 corresponds to the inverse Auslander-Reiten translation τ^{-1} in $\text{CMP}B$. For example, $\Omega^2 \frac{2}{7} = \frac{4}{3} = \tau^{-1} \frac{3}{7}$.

Example 3.8. The checkerboard polygon \mathcal{S} from Figure 3 can be embedded into the alternating strand diagram on a disc with the same number of boundary vertices shown in the left picture in Figure 4. The orientation of the strands is such that the shaded regions are oriented while the white regions are alternating. The corresponding dimer algebra on the disc is given by the quiver on the right in the same figure. Each vertex represents a white region in the alternating strand diagram and two regions are connected by an arrow if they share a crossing point.

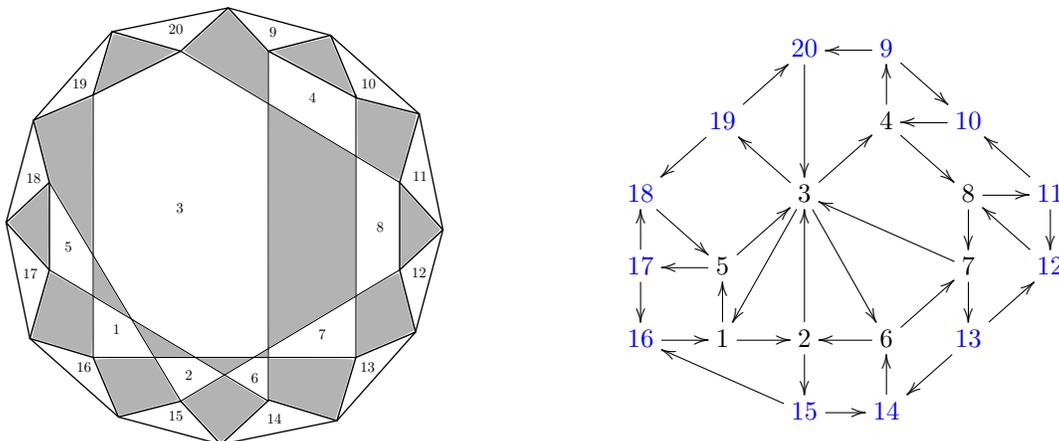


FIGURE 4. An alternating strand diagram that contains the checkerboard polygon of Figure 3 and its corresponding quiver.

The full subquiver on the vertices $1, 2, \dots, 8$ is equal to the twisted quiver \tilde{Q} of Q in the sense of Bocklandt [Bo]. The vertices $9, 10, \dots, 20$ are frozen vertices. This quiver gives a seed for a cluster algebra associated to a certain positroid variety in the Grassmannian $\text{Gr}(5, 12)$. The dual graph of \tilde{Q} is equal to the twist of the dual graph of Q .

Thus the checkerboard polygon models the syzygy category of the Jacobian algebra of Q and at the same time it models a seed in the cluster algebra of the twisted quiver \tilde{Q} .

3.3. Properties of \mathcal{S} . We have already observed that the interior vertices of the checkerboard polygon \mathcal{S} correspond to arrows in the quiver Q , and the shaded regions in \mathcal{S} correspond to chordless cycles and boundary arrows in Q .

Lemma 3.9. *Let α be an interior vertex of \mathcal{S} and let $\beta_i, i = 1, 2, 3, 4$ be the four vertices connected to α in clockwise order. Denote by x_i the label of the edge $\alpha - \beta_i$ in \mathcal{S} . Then $x_i = x_{i+2}$ and $x_i \neq x_{i+1}$, where the addition in the indices is modulo 4. Thus the opposite edges carry the same label.*

Proof. The vertex α corresponds to an arrow in Q and the four labels x_i must be taken from the endpoints $s(\alpha), t(\alpha)$ of α in Q . Consider a pair of consecutive edges $\alpha - \beta_i, \alpha - \beta_{i+1}$ that are part of the same shaded region in \mathcal{S} . Assume first that this shaded region has no side on the boundary of \mathcal{S} . Then the region corresponds to a chordless cycle C of Q and the three arrows $\beta_i, \alpha, \beta_{i+1}$ form a subpath of C . In particular, the arrows β_i, α have the starting vertex $s(\alpha)$ of α in common, while the arrows α, β_{i+1} share the terminal vertex $t(\alpha)$ of α . Thus $x_i = s(\alpha)$ and $x_{i+1} = t(\alpha)$. Now assume that the shaded region is labeled by a boundary vertex of Q . Then we also have $x_i = s(\alpha)$ and $x_{i+1} = t(\alpha)$, by definition of the labeling. Because of the checkerboard pattern, the same argument shows that $x_{i+2} = s(\alpha)$ and $x_{i+3} = t(\alpha)$. Moreover, since the quiver Q does not contain any loops, we have $x_i \neq x_{i+1}$. \square

Definition 3.10. The piecewise linear curve $\rho(i)$ given as the union of edges in \mathcal{S} that carry the label i will be called the *radical line* of the vertex $i \in Q_0$. The polygon \mathcal{S} together with the system of radical lines $\{\rho(i) \mid i \in Q_0\}$ will be called the *checkerboard polygon* of Q .

Remark 3.11. We shall show in Proposition 4.22 that $\rho(i)$ corresponds to the radical of the projective $P(i)$ at vertex i .

It follows from Lemma 3.9 that $\rho(i)$ is connected and starts and ends at the boundary of \mathcal{S} . Since G is a tree, we also know that $\rho(i)$ does not intersect itself. It is always possible and often convenient to draw $\rho(i)$ as a smooth curve. Sometimes, we can draw $\rho(i)$ as a straight line segment, see for example Figure 3.

3.3.1. *White regions.*

- Lemma 3.12.** (a) *Every white region in \mathcal{S} has an even number of edges with distinct labels, and either exactly one boundary vertex or two boundary vertices connected by a boundary edge.*
 (b) *Every boundary vertex in \mathcal{S} is incident to exactly one white region and one or two shaded regions.*
 (c) *Every boundary vertex in \mathcal{S} is incident to at least one and at most two radical lines.*

Proof. (a) By construction, the faces in \tilde{G} have an even number of edges, hence the white regions have an even number of vertices, thus edges. The edge labels are distinct because of Lemma 3.9 and since G is a tree. The fact that G is a tree also implies that the white region has at least one vertex on the boundary. Moreover, it has exactly two boundary vertices connected by a boundary edge precisely if it comes from a face of \tilde{G} which has a completion vertex.

(b) The shaded boundary regions correspond to boundary arrows in Q and the two boundary vertices of a shaded boundary region correspond to the endpoint of that arrow. In particular, every shaded region has exactly two boundary vertices. This implies that every boundary vertex x is incident to exactly one white region W . Moreover, if x is the only boundary vertex of W then x is incident to two shaded regions and if W contains a second boundary vertex then x is incident to only one shaded region.

Part (c) follows directly from (b). \square

Lemma 3.13. *Let W be a white region in \mathcal{S} , with interior edges labeled i_1, i_2, \dots, i_t starting at the boundary and going around W in the clockwise direction.*

- (a) *There is a unique path $\mathfrak{c}(W) = \alpha_{i_1}\alpha_{i_2} \cdots \alpha_{i_{t-1}}$ in Q and a corresponding sequence of distinct chordless cycle C_1, C_2, \dots, C_{t-2} in Q such that α_{i_j} is an arrow $i_j \rightarrow i_{j+1}$ and $\alpha_{i_j}\alpha_{i_{j+1}}$ is a subpath of $C_j = \alpha_{i_j}\alpha_{i_{j+1}}C'_j$.*
 (b) *There are two unique paths $\mathfrak{v}_1(W) = C'_{t-2}C'_{t-4} \cdots C'_a$ and $\mathfrak{v}_2(W) = C'_{t-3}C'_{t-5} \cdots C'_b$ with $a = 1, b = 2$ if t is odd, and $a = 2, b = 1$ if t is even.*
 (c) *The full subquiver $Q(W)$ of Q whose vertices are those visited by $\mathfrak{c}(W)$, $\mathfrak{v}_1(W)$, or $\mathfrak{v}_2(W)$ is the quiver given by the union of the cycles C_1, \dots, C_{t-2} , and this quiver is equal to the full subquiver generated by $\mathfrak{c}(W)$ if and only if all C_i are 3-cycles.*

Proof. (a) Every edge i_j in the sequence bounds W on one side and a shaded region, hence a chordless cycle C , on the other side. The vertices $\alpha_{i_2}, \dots, \alpha_{i_{t-2}}$ of W correspond to arrows in Q that lie in exactly two chordless cycles in Q . Say α_{i_j} lies in the chordless cycles C_{j-1} and C_j . Then the sequence $C_1, \alpha_{i_2}, C_2, \alpha_{i_3}, \dots, \alpha_{i_{t-2}}, C_{t-2}$ defines a path in the trunk of G . In particular, $C_j \neq C_\ell$, since G is a tree.

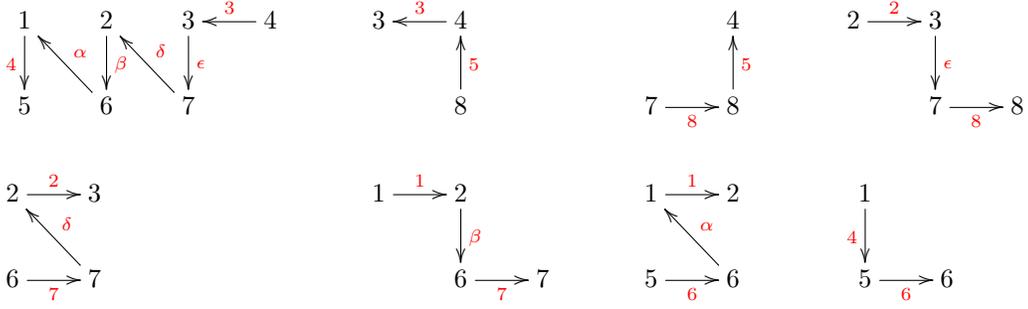
(b) We need to show that the paths are well-defined. Since $C_j = \alpha_{i_j}\alpha_{i_{j+1}}C'_j$ is a cycle starting at i_j , and the arrow $\alpha_{i_{j+1}}$ ends at i_{j+2} , we see that C'_j is a path from i_{j+2} to i_j . Therefore the terminal point C'_{j+2} is the starting point of C'_j .

(c) By definition the vertices of $Q(W)$ are precisely those in the cycles, and the path $\mathfrak{c}(W)$ visits all of them if and only if every C'_j is a single arrow, or, equivalently, if C_j is a three cycle. Moreover, each arrow contained in the cycles also lies in one of the paths $\mathfrak{c}(W)$, $\mathfrak{v}_1(W)$ or $\mathfrak{v}_2(W)$. There are no other arrows in $Q(W)$ because the dual graph of Q is a tree. \square

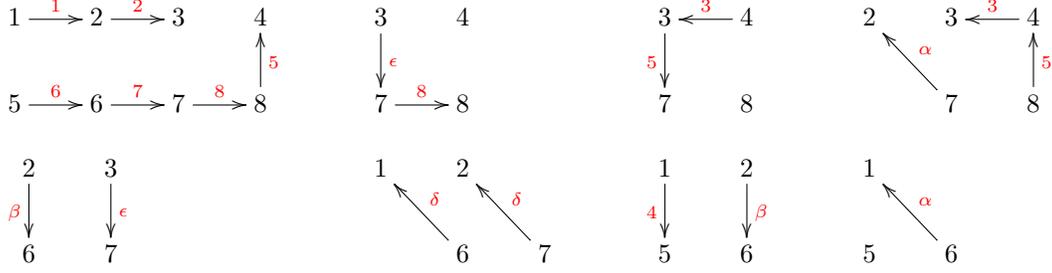
Definition 3.14. Let W be a white region. The path $\mathfrak{c}(W)$ is called the *cycle path* of W and the paths $\mathfrak{v}_i(W)$ are called the *maximal valid paths* of W .

Remark 3.15. The cycle path $\mathfrak{c}(W)$ goes clockwise around the white region W while the valid paths $\mathfrak{v}_i(W)$ go counterclockwise.

Example 3.16. In the example in Section 3.2, there are 8 white regions. The corresponding paths are listed below, starting with the large white region and going clockwise around the polygon shown in Figure 3.



The corresponding maximal valid paths are



Proposition 3.17. *There is a bijection*

$$\begin{aligned} \varphi: \{\text{white regions in } \mathcal{S}\} &\longrightarrow \{\text{boundary arrows in } Q\} \\ W &\longmapsto \text{first arrow in } \mathfrak{c}(W) \end{aligned}$$

given by mapping the white region W to the first arrow of its cycle path $\mathfrak{c}(W)$. Moreover, W has exactly two vertices on the boundary of \mathcal{S} , if $\mathfrak{c}(W)$ is of even length, and W has exactly one vertex on the boundary, otherwise.

Proof. Notice first that the map φ is well-defined by Lemma 3.13.

Every boundary arrow α of Q determines a unique shaded region at the boundary of \mathcal{S} which in turn is adjacent to precisely two white regions W_1 and W_2 . For one of them, say W_1 , the arrow α is the initial arrow in the path $\mathfrak{c}(W_1)$, thus $\alpha = \varphi(W_1)$ and ϕ is surjective. On the other hand, α is the terminal arrow in $\mathfrak{c}(W_2)$ which shows that φ is injective.

It remains to show the statement on the number of boundary vertices in W . If the path $\mathfrak{c}(W)$ has even length then it goes through an odd number of vertices, and thus W has an odd number of edges in the interior of \mathcal{S} . Similarly, if $\mathfrak{c}(W)$ has odd length then W has an even number of edges in the interior of \mathcal{S} . Now the statement follows from Lemma 3.12. \square

Thanks to Proposition 3.17 we can label the cycle paths by the boundary arrows of the quiver. Part (a) of the following definition is a reformulation adapted to this point of view.

Definition 3.18. Let α be a boundary arrow in Q .

- (a) The *cycle path* of α is the unique path $\mathfrak{c}(\alpha) = \alpha_1 \alpha_2 \cdots \alpha_{\ell(\alpha)}$ such that
 - (i) $\alpha_1 = \alpha$ and $\alpha_{\ell(\alpha)}$ are boundary arrows, and $\alpha_2, \dots, \alpha_{\ell(\alpha)-1}$ are interior arrows,
 - (ii) every subpath of length two $\alpha_i \alpha_{i+1}$, is a subpath of a chordless cycle C_i , and $C_i \neq C_j$ if $i \neq j$.
- (b) The *weight* $\text{wt}(\alpha)$ of α is defined as

$$\text{wt}(\alpha) = \begin{cases} 1 & \text{if the length of } \mathfrak{c}(\alpha) \text{ is odd;} \\ 2 & \text{if the length of } \mathfrak{c}(\alpha) \text{ is even.} \end{cases}$$

The proposition yields the following important formula for the size of the polygon.

Corollary 3.19. *The number of boundary edges in \mathcal{S} is equal to*

$$\sum_{\alpha} \text{wt}(\alpha),$$

where the sum is over all boundary arrows of Q .

Proof. Each of the boundary edges in the shaded regions of \mathcal{S} corresponds to a unique boundary arrow in Q . According to the proposition, each boundary in a white region corresponds to a unique boundary arrow α in Q whose cycle path $\mathfrak{c}(\alpha)$ is of even length. Now the result follows from the definition of the weight $\text{wt}(\alpha)$ of α . \square

Example 3.20. In our running example, there are 8 boundary arrows and their white region paths are listed in Example 3.16. We have $\text{wt}(\alpha) = 2$ for the arrows $\alpha = 3, 5, 8, 4$, and $\text{wt}(\alpha) = 1$ for the arrows $\alpha = 2, 7, 1, 6$. Thus Corollary 3.19 implies that the polygon has 12 sides.

Lemma 3.21. *The number of boundary vertices of \mathcal{S} is even and at least 6.*

Proof. Let m be the number of boundary vertices of \mathcal{S} . By construction, m is the number of boundary edges in \tilde{G} , and we use induction on the number of faces in \tilde{G} . Since Q has at least one chordless cycle, which is at least of length 3, the smallest possible case is the one shown in Example 3.1. Thus $m \geq 6$ and \tilde{G} has at least 3 faces.

Choose a face of \tilde{G} . Let $2a$ be the number of edges of the chosen face, and denote by b the number of its edges that lie on the boundary of \tilde{G} . When we remove that face from \tilde{G} , the difference in the number of boundary edges between \tilde{G} and the resulting graph is $-b + (2a - b) = 2(a - b)$ which is even, so the parity is preserved. This shows that m is even. \square

3.3.2. Radical lines and 2-diagonals.

Lemma 3.22. *The radical lines $\rho(i)$ and $\rho(j)$ cross if and only if there is an arrow $i \rightarrow j$ or $j \rightarrow i$ in Q .*

Proof. If $\rho(i)$ and $\rho(j)$ cross then there is an interior vertex α in \mathcal{S} that lies on both $\rho(i)$ and $\rho(j)$. Thus α corresponds to an arrow in Q that connects i and j . Conversely, if $i \xrightarrow{\alpha} j$ is an arrow in Q then there is an interior vertex α in \mathcal{S} which is of degree 4. By Lemma 3.9, α lies on both $\rho(i)$ and $\rho(j)$. \square

Corollary 3.23. *There is an even number of boundary arrows α in Q whose white region path $w(\alpha)$ is of odd length.*

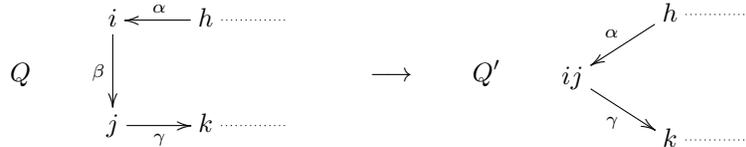
Proof. This follows from Corollary 3.19 and Lemma 3.22. \square

Recall that a *2-diagonal* in a polygon is (the homotopy class of) a diagonal such that the dissection of the polygon along this diagonal consists of two polygons each with an even number of sides.

Lemma 3.24. *Each radical line $\rho(x)$ is a 2-diagonal in \mathcal{S} .*

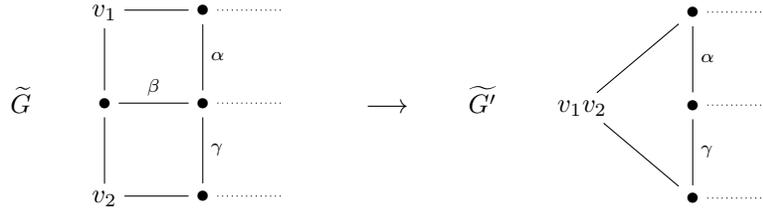
Proof. We proceed by induction on n , the number of vertices in Q . If $n = 3$, we are done by Example 3.1. Suppose that $n \geq 4$. We consider two cases.

(a) Suppose first there exists a boundary arrow $i \xrightarrow{\beta} j$ and such that the unique chordless cycle containing β is of length at least 4 and there are no interior arrows at i or j . In this situation, we define a new quiver Q' by contracting the arrow β , meaning that we identify the vertices i and j and remove β , see below.

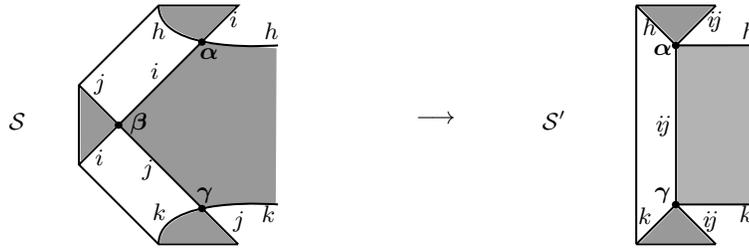


At the level of \tilde{G} , the edge β bounds two faces F_1, F_2 . Since α and γ are boundary arrows in Q , both F_i have 4 vertices one of which, say v_i , is a completion vertex, $i = 1, 2$. The graph \tilde{G}' can be obtained from \tilde{G} by removing the edge β together with the two boundary edges incident to it, and identifying the vertices v_1

and v_2 , see below.



At the level of \mathcal{S} , the vertex β is the crossing point of the radical lines $\rho(i)$ and $\rho(j)$. The two shaded regions at this crossing are a triangular boundary region on one side and an interior region with at least 4 sides corresponding to the chordless cycle in Q that contains the arrow β . The two white regions at the crossing are quadrilaterals coming from the faces F_1, F_2 of \tilde{G} . The vertex β is connected to the two vertices α and γ because there are no interior arrows at i and j . Moreover α and γ are each a vertex of a triangular boundary region, because they are boundary arrows in Q . The polygon \mathcal{S}' can be obtained from \mathcal{S} by removing the shaded boundary region at β together with the two adjacent boundary edges, and replacing the two edges $\beta-\alpha, \beta-\gamma$ with one edge $\alpha-\gamma$ and closing the white region with one edge on the boundary, see below.



Now consider the two polygons obtained by cutting \mathcal{S} along the radical line $\rho(x)$. If $i = x$ or $i = y$ then $\rho(x)$ is clearly a 2-diagonal and we are done. If $x \neq i, j$, then one of the two polygons is also obtained by cutting \mathcal{S}' along $\rho(x)$. By induction, this polygon has an even number of boundary edges, and by Lemma 3.21, the other polygon as well. This completes the proof in case (a).

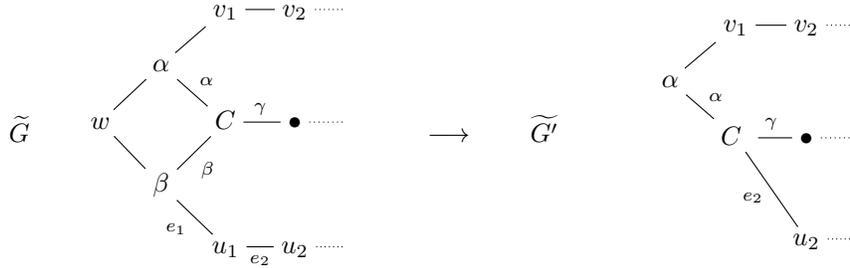
(b) Suppose we are not in the situation of case (a). Then there exists a boundary arrow $\alpha: i \rightarrow j$ that lies in a unique 3-cycle $C = i \xrightarrow{\alpha} j \xrightarrow{\beta} k \xrightarrow{\gamma} i$ such that β is also a boundary arrow, and γ is an interior arrow. We define a new quiver Q' by removing the vertex j and its adjacent arrows α, β , see below.



At the level of \tilde{G} , the cycle C is a vertex with 3 edges α, β, γ . The two edges α and β connect C to the boundary vertices α and β , forming a quadrilateral with vertices C, α, β and a completion vertex w . The face containing the edges α and γ has vertices $C, \alpha, v_1, v_2, \dots$, where v_1 may be a completion vertex. Similarly, the face containing the edges β and γ has vertices $C, \beta, u_1, u_2, \dots$ where u_1 may be a completion vertex. In order to illustrate both cases, assume that v_1 is not a completion vertex and u_1 is a completion vertex. Label the two edges at u_1 by e_1 and e_2 .

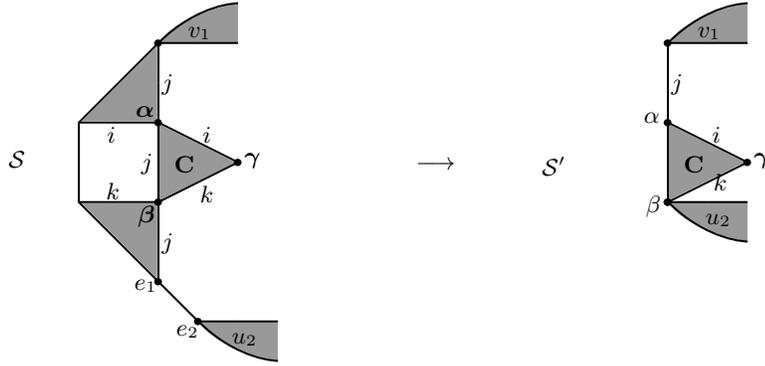
\tilde{G}' is obtained from \tilde{G} by removing the vertices w, β and their adjacent edges, as well as identifying the vertex C with the vertex u_1 , see below. Note that in \tilde{G}' , the vertex α is a completion vertex and the vertex

u_2 is not.



Now consider \mathcal{S} . The cycle C becomes a triangular shaded region with sides labeled i, j, k and vertices α, β, γ . The adjacent white region towards the boundary is a quadrilateral with edge labels i, j, k and a boundary edge that corresponds to the completion vertex w in \tilde{G} . The adjacent triangular shaded boundary regions in \mathcal{S} correspond to the arrows α and β , and the edge labels are shown in the figure below. The vertices v_1 and u_2 are not completion vertices in \tilde{G} and therefore give rise to shaded regions in \mathcal{S} . On the other hand, the vertex u_1 is a completion vertex in \tilde{G} and it does not yield a shaded region in \mathcal{S} .

The polygon \mathcal{S}' can be obtained from \mathcal{S} by removing the three regions to the left of the radical line $\rho(j)$ together with the two edges adjacent to the vertex e_1 , as well as identifying the vertices β and e_2 , see below.



Now consider the two polygons obtained by cutting \mathcal{S} along the radical line $\rho(x)$ with $x \neq j$. If $x \neq i, k$ then one of these two polygons is also obtained by cutting \mathcal{S}' along $\rho(x)$. Using induction this completes the proof as in case (a).

Now suppose $x = i$ or $x = k$. Let A and B denote the two polygons in \mathcal{S} obtained by cutting along $\rho(x)$ such that the shaded triangle C lies inside B . Similarly, let A' and B' denote the two polygons in \mathcal{S}' obtained by cutting along $\rho(x)$ such that the shaded triangle C lies inside B' . Then, if $x = i$, the polygon A' is obtained from A by removing the triangular shaded region on the boundary of \mathcal{S} with labels α, i, j . In particular, A and A' have the same number of boundary edges. On the other hand, if $x = k$ then A' is obtained from A by removing the triangular region at the boundary of \mathcal{S} with labels β, k, j together with the two edges incident to e_1 and identifying the vertices β and e_2 . In particular, A has exactly two more boundary edges than A' , so both have the same parity. By induction, the result follows as in case (a). Finally, if $x = j$ the result is obvious. \square

3.3.3. *Orientation of 2-diagonals.* We now define an orientation on each 2-diagonal in the polygon \mathcal{S} , which will allow us later to give a direction to the crossing of two 2-diagonals.

The polygon \mathcal{S} has an even number of boundary vertices, by Lemma 3.21. We give a sign to each boundary vertex in such a way that the signs alternate along the boundary of \mathcal{S} . There are exactly two ways of doing this, and it does not matter which one we choose.

Then a 2-diagonal in \mathcal{S} is the homotopy class of a line segment that connects two boundary vertices of opposite sign. We orient each 2-diagonal in the direction from $-$ to $+$. In particular, this defines an orientation on the radical lines $\rho(i)$ in the checkerboard pattern of \mathcal{S} .

Lemma 3.25. *Each region in the checkerboard pattern in \mathcal{S} is bounded by a sequence of segments of oriented radical lines, and possibly one boundary segment. For a shaded region these oriented segments form an*

oriented path bounding the region. For a white region, the orientation of the segments is alternating around the region.

Proof. Every crossing point between two radical lines is incident to four regions and, locally, the opposite regions have the same type of orientation (same direction or alternating) on their boundary segments. Opposite regions also have the same color. Thus it suffices to show that shaded regions at the boundary of \mathcal{S} satisfy the statement. This however, is easy to see, since such a region has exactly three vertices, one that is interior and two on the boundary of \mathcal{S} , and the two boundary vertices have opposite signs. This implies that the two segments of the radical lines that bound the region form an oriented path. \square

Remark 3.26. The orientation of the radical lines in \mathcal{S} is not related to and should not be confused with the orientation of the arrows in the quiver. Every arrow in Q corresponds to an interior vertex in the checkerboard pattern of \mathcal{S} which is incident to exactly two shaded regions. The segments of the radical lines go clockwise around one of these regions and counterclockwise around the other. Similarly, the segments around a white region W have alternating orientation, while the path $c(W)$ associated to W in Lemma 3.13 is an oriented path in Q .

The orientation of the 2-diagonals allows us to define the degree of a crossing between a 2-diagonal and a radical line as follows.

Definition 3.27. Let $\rho(i)$ be a radical line in the checkerboard polygon \mathcal{S} , and let γ be an arbitrary 2-diagonal that has a crossing with $\rho(i)$. We define the *degree* of the crossing to be

$$\begin{aligned} 0 & \text{ if } \rho(i) \text{ crosses } \gamma \text{ from right to left;} \\ 1 & \text{ if } \rho(i) \text{ crosses } \gamma \text{ from left to right.} \end{aligned}$$

3.4. Properties of dimer tree algebras. Let B be a dimer tree algebra. In this subsection, we give some basic properties of the algebra. Throughout we use the notation $w \sim w'$ to indicate that two parallel paths w, w' in Q are equal in B .

We start with a property of the quiver. Recall that Q comes with a fixed embedding in the plane. A vertex of Q is called a *boundary vertex* if it is incident to the unique unbounded face Q .

Lemma 3.28. *All vertices of Q are boundary vertices.*

Proof. Suppose $x \in Q_0$ is a vertex that is not a boundary vertex. Denote by C_1, C_2, \dots, C_s the faces of Q that are incident to x in clockwise order. Then every C_i is a bounded face, hence a chordless cycle in Q . Moreover every two consecutive faces C_i, C_{i+1} share an arrow α_i that is incident to x . This configuration gives rise to a cycle $C_1 \xrightarrow{\alpha_1} C_2 \xrightarrow{\alpha_2} \dots \xrightarrow{\alpha_{s-1}} C_s \xrightarrow{\alpha_s} C_1$ in the dual graph G , which is a contradiction to condition (Q2). \square

Lemma 3.29. (a) *If α_1 is a boundary arrow and $\alpha_1\alpha_2 \cdots \alpha_s$ is the unique chordless cycle containing α_1 then the path $\alpha_2 \cdots \alpha_s$ is zero in B .*

(b) *If α_1 is an interior arrow and $\alpha_1\alpha_2 \cdots \alpha_s$ and $\alpha_1\alpha'_2 \cdots \alpha'_t$ are the two chordless cycles containing α_1 then the paths $\alpha_2 \cdots \alpha_s$ and $\alpha'_2 \cdots \alpha'_t$ are equal in B .*

Proof. This follows directly from the definition of the Jacobian ideal. Indeed, in case (a), the arrow α_1 lies in a unique chordless cycle, hence in a unique summand of the potential W , and thus the partial derivative $\partial_{\alpha_1} W$ is exactly equal to the path $\alpha_2 \cdots \alpha_s$. In case (b), the arrow α_1 lies in exactly two summands of W , and thus $\partial_{\alpha_1} W$ is the difference of the two paths $\alpha_2 \cdots \alpha_s$ and $\alpha'_2 \cdots \alpha'_t$. \square

Proposition 3.30. *Every cyclic path in Q is zero in B .*

Proof. Let w be a cyclic path in Q . We proceed by induction on the length $\ell(w)$ of w . Since Q is without loops and 2-cycles, the base case is $\ell(w) = 3$. Thus suppose $w = \alpha\beta\gamma$ is a 3-cycle at vertex x in Q . If α is a boundary arrow then Lemma 3.29 yields $\beta\gamma = 0$ and thus $w = 0$. Otherwise, α is contained in a unique other 3-cycle $w' = \alpha\beta'\gamma'$. Again using Lemma 3.29, we have $\beta\gamma \sim \beta'\gamma'$ and thus $w \sim w'$. Moreover, $\gamma \neq \gamma'$, because x is a boundary vertex, by Lemma 3.24. Similarly, if γ' is a boundary arrow, then $\alpha\beta' = 0$ and hence $w \sim w' = 0$ in B . Otherwise, γ' is contained in a unique other 3-cycle $w'' = \alpha'\beta''\gamma'$ and Lemma 3.29 implies $w'' \sim w' \sim w$. Moreover $\alpha' \neq \alpha$ because x is a boundary vertex of Q .

Continuing this way, we either obtain a 3-cycle $w^{(t)} \sim w$ such that $w^{(t)}$ contains a boundary arrow, and in this case $w = 0$, or we obtain an unbounded number of arrows incident to the vertex x , which is impossible, since Q is finite. This completes the proof for 3-cycles.

Now let w be a cycle of length $\ell(w)$ greater than 3 and denote its start and terminal vertex by x . We may assume that w does not contain a proper subcycle, that is, w does not visit the same vertex twice except for x . In other words, the set of all vertices $V(w)$ that lie on w has cardinality $\ell(w)$.

Let G be the dual graph of Q , and G° be the graph obtained from G by removing the leaf vertices (and their edges). Thus $G^\circ = (G_0^\circ, G_1^\circ)$ with $G_0^\circ = \{\text{chordless cycles in } Q\}$ and $G_1^\circ = \{C \xrightarrow{\alpha} C' \mid \text{the chordless cycles } C, C' \text{ share the arrow } \alpha\}$.

Let R be the full subquiver of Q whose vertex set is $V(w)$. In particular w is a cyclic path consisting of boundary arrows in R . Note however, since we do not assume that w is chordless, the quiver R may have arrows that do not lie on w . Let H° be the subgraph of G° associated to R . Since G is a tree, H° is a tree as well, and we may choose a leaf vertex C in H° . Thus $C = \alpha_1 \alpha_2 \cdots \alpha_t$ is a chordless cycle in R which has at most one interior arrow, say α_1 , in R . We may choose C such that the starting point x of w does not lie on the path $\alpha_3 \cdots \alpha_{t-1}$. Then the path $\alpha_2 \cdots \alpha_t$ must be a subpath of w . Since α_1 is an interior arrow in R , there exists a unique other chordless cycle $C' = \alpha_1 \alpha'_2 \cdots \alpha'_s$ in R containing α_1 , and we have $\alpha_2 \cdots \alpha_t \sim \alpha'_2 \cdots \alpha'_s$. Let us point out here that the two paths have the same starting vertex and the same terminal vertex, but they do not have any other vertex in common.

Now define w' to be the path obtained from w by replacing $\alpha_2 \cdots \alpha_t$ with $\alpha'_2 \cdots \alpha'_s$, and let $V(w')$ be the set of all vertices on w' . Let $y = t(\alpha'_2) \in V(w')$. Since w' is a path in R , and R is the subquiver of Q whose vertices are $V(w)$, we must have $V(w') \subset V(w)$. In particular, the vertex y is also a vertex on the path w . Therefore, the path w' must visit the vertex y at least twice, and thus w' contains a proper subcycle w'' which must be zero by induction. Hence $w \sim w' = 0$, and the proof is complete. \square

Proposition 3.31. *Any two nonzero parallel paths in Q are equal in B .*

Proof. Let $x, y \in Q_0$ and let $v = \alpha_1 \alpha_2 \cdots \alpha_s$, $w = \beta_1 \beta_2 \cdots \beta_t$ be two nonzero paths from x to y . Without loss of generality, we may assume that v and w do not cross each other, because otherwise we would work with the shorter parallel subpaths instead. We proceed by induction on $\max\{\ell(v), \ell(w)\} = \max\{s, t\}$. Since there are no parallel arrows in Q , the base case for the induction is when $s = t = 2$. Then there exists a path u from y to x such that uv is a chordless cycle. Since Q has no interior vertices, by Lemma 3.24, the path u must be a single arrow. Similarly, there exists an arrow u' from y to x such that uw is a chordless cycle, and since Q has no parallel arrows we must have $u = u'$. Therefore $v = w$ by Lemma 3.29.

Now suppose that one of s, t is at least 3. Let R be the full subquiver of Q whose vertex set is the set of vertices visited by v or w . From our conditions of Q , we conclude that R is a planar graph, and each of its faces is a chordless cycle. Moreover, since the paths v, w do not share a vertex other than x and y , the arrows in the two paths v or w are boundary arrows in R and all other arrows in R are interior arrows. Moreover, all vertices of R are boundary vertices. In particular, the vertex x is the starting point of two boundary arrows α_1 and β_1 .

We are going to show first that the number of chordless cycles in R that contain the vertex x is exactly two. Indeed, since α_1 and β_1 both start at x , there must be at least two such chordless cycles. Suppose there are more than two. Then there must be at least four and we denote the first three of them by

$$C_1 = \beta_1 C'_1 \gamma_1, \quad C_2 = \epsilon_1 C'_2 \gamma_1 \quad C_3 = \epsilon_1 C'_3 \gamma_2$$

see Figure 5. Let $z_1 = s(\gamma_1)$, $z_2 = t(\epsilon_1)$ and $z_3 = s(\gamma_2)$. Then all three vertices z_1, z_2, z_3 are vertices of the subquiver R and thus must lie on one of the two paths v, w . There are two possibilities; either all three vertices lie on one of the paths or two vertices lie on one path and the third lies on the other. To reach a contradiction, it suffices to show that neither v nor w can contain two of the vertices z_1, z_2, z_3 , and, by symmetry and because the paths v, w do not cross each other, it suffices to show that w cannot contain z_1 and z_2 .

Suppose the contrary. Then w must visit z_1 before z_2 , because otherwise w would cross itself. Let w_1 be the subpath of w from $t(\beta_1)$ to z_1 , let w_2 be the subpath from z_1 to z_2 and w_3 the subpath from z_2 to y , see the left picture in Figure 5.

Then the path C'_2 must be an arrow, because otherwise it would run through an interior vertex, which is impossible. In particular, the path $\epsilon_1 C'_2$ is shorter than w , and, by induction, we conclude that $\epsilon_1 C'_2 \sim \beta_1 w_1$,

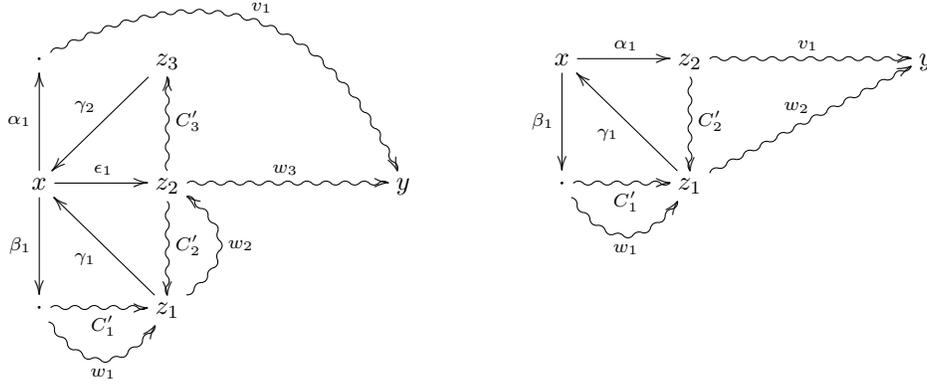


FIGURE 5. Proof of Proposition 3.31

since both paths are from x to z_1 . Therefore we have $w \sim \epsilon_1 C'_2 w_2 w_3 = 0$, by Proposition 3.30, since $C'_2 w_2$ is a cyclic path. This contradiction shows that the number of chordless cycles that contain the vertex x is exactly two.

We now denote these two chordless cycles by

$$C_1 = x \xrightarrow{\beta_1} \cdot \rightsquigarrow C'_1 \rightsquigarrow z_1 \xrightarrow{\gamma_1} x \quad \text{and} \quad C_2 = x \xrightarrow{\alpha_1} \cdot \rightsquigarrow C'_2 \rightsquigarrow z_1 \xrightarrow{\gamma_1} x,$$

see the right hand side of Figure 5. The vertex z_1 must lie on one of the paths v or w . Without loss of generality, we assume that z_1 lies on w , and we write

$$w = x \xrightarrow{\beta_1} \cdot \rightsquigarrow w_1 \rightsquigarrow z_1 \rightsquigarrow w_2 \rightsquigarrow y \quad \text{and} \quad v = x \xrightarrow{\alpha_1} \cdot \rightsquigarrow v_1 \rightsquigarrow y.$$

Then the path C'_2 must be an arrow, because otherwise it would go through an interior vertex, which is impossible. In particular, we have the following inequalities on the lengths of subpaths $\ell(\alpha_1 C'_2) < \ell(w)$, $\ell(C'_2 w_2) < \ell(w)$, $\ell(v_1) < \ell(v)$, and, by induction, we conclude $\alpha_1 C'_2 \sim \beta_1 w_1$ and $C'_2 w_2 \sim v_1$. Consequently, $v = \alpha_1 v_1 \sim \alpha_1 C'_2 w_2 \sim \beta_1 w_1 w_2 = w$. \square

An algebra is called *schurian* if $\dim \text{Hom}(P(i), P(j)) \leq 1$ for all vertices i, j of Q . With this terminology, we have the following reformulation of Proposition 3.31.

Corollary 3.32. *Every dimer tree algebra is schurian.*

3.5. Some results toward the main conjecture. In this section we show a few results that hold in the setting of Conjecture 1.1. We start by showing that two radical lines cross if and only if there is an extension between the corresponding radicals.

Proposition 3.33. *Let $\rho(i)$ and $\rho(j)$ be two radical lines in \mathcal{S} . Then the following are equivalent.*

- (a) $\rho(i)$ and $\rho(j)$ cross;
- (b) there is an arrow $i \rightarrow j$ or and arrow $j \rightarrow i$ in the quiver Q ;
- (c) $\text{Ext}^1(\text{rad } P(i), \text{rad } P(j)) \oplus \text{Ext}^1(\text{rad } P(j), \text{rad } P(i)) \neq 0$.

Proof. The equivalence of (a) and (b) was proved in Lemma 3.22. Suppose now (b) holds and let us assume without loss of generality the arrow is $j \rightarrow i$. Then we have the following commutative diagram with exact

rows and columns,

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & \text{rad } P(i) & \longrightarrow & X & \longrightarrow & \text{rad } P(j) \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & P(i) & \longrightarrow & P(i) \oplus P(j) & \longrightarrow & P(j) \longrightarrow 0 \\
 & & \downarrow f & & \downarrow [g_i \ g_j] & & \downarrow h \\
 0 & \longrightarrow & S(i) & \longrightarrow & j_i & \longrightarrow & S(j) \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0
 \end{array}$$

where f, g_j and h are projective covers, the morphism $[g_i \ g_j]$ is given by the horseshoe lemma and $X = \ker [g_i \ g_j]$. Since g_j is a projective cover, we have $X \cong P(i) \oplus \ker g_j$. Therefore the top row of the diagram is a non-split short exact sequence, and thus $\text{Ext}^1(\text{rad } P(j), \text{rad } P(i))$ is nonzero. Hence (c) holds.

Conversely, suppose that (c) holds and assume without loss of generality that the nonzero space is $\text{Ext}^1(\text{rad } P(j), \text{rad } P(i))$. Applying the functor $\text{Hom}(\text{rad } P(j), -)$ to the short exact sequence

$$0 \longrightarrow \text{rad } P(i) \longrightarrow P(i) \longrightarrow S(i) \longrightarrow 0$$

yields an exact sequence

$$\text{Hom}(\text{rad } P(j), S(i)) \longrightarrow \text{Ext}^1(\text{rad } P(j), \text{rad } P(i)) \longrightarrow \text{Ext}^1(\text{rad } P(j), P(i)) = 0$$

where the last term is zero, since $\text{rad } P(j)$ is a Cohen-Macaulay module. By assumption, the middle term is nonzero, and we can conclude the existence of a nonzero morphism $f \in \text{Hom}(\text{rad } P(j), S(i))$. It follows that $P(j)$ is supported at vertex i , thus there is a path $j = x_0 \rightarrow x_1 \rightarrow \dots \rightarrow x_s = i$, and this path is unique in B , by Proposition 3.31. If $s > 1$ then the morphism f would produce the following commutative diagram

$$\begin{array}{ccc}
 k = \text{rad } P(j)_{x_{s-1}} & \longrightarrow & \text{rad } P(j)_i = k \\
 \downarrow f_{x_{s-1}} & & \downarrow f_i \neq 0 \\
 0 = S(i)_{x_{s-1}} & \longrightarrow & S(i)_i = k
 \end{array}$$

which is impossible. Thus $s = 1$ and our path is an arrow $j \rightarrow i$ and (b) holds. \square

Proposition 3.34. *Let $\gamma, R(\gamma)$ be 2-diagonals in \mathcal{S} . Then*

- (a) $\text{add } P_0(\gamma) \cap \text{add } P_1(\gamma) = \{0\}$ and $\text{add } P_0(R(\gamma)) \cap \text{add } P_1(R(\gamma)) = \{0\}$.
- (b) $P_1(\gamma) \cong P_0(R(\gamma))$.

Proof. Part (a) follows from the assumption that $\gamma, R(\gamma)$ are 2-diagonals in \mathcal{S} , so they have a minimal number of crossings with radical lines of \mathcal{S} . Then each of them crosses a radical line of \mathcal{S} at most once, and the direction of the crossing then determines whether this radical line gives rise to a summand of P_0 or P_1 .

To show part (b), suppose $P(j) \in \text{add } P_1(\gamma)$. Then the radical line $\rho(j)$ crosses γ from left to right, see Figure 6. Because the endpoints of γ and $R(\gamma)$ are one apart, it follows that the arcs have opposite directions. Then if $\rho(j)$ also crosses $R(\gamma)$ then it crosses $R(\gamma)$ from right to left and $P(j) \in \text{add } P_0(R(\gamma))$. Now, suppose $\rho(j)$ does not cross $R(\gamma)$. Then $\rho(j)$ has a common endpoint with $R(\gamma)$, which we call x . Without loss of generality suppose the direction of $R(\gamma)$ and $\rho(j)$ is such that they both end in x . Then $\rho(j)$ crosses γ from right to left, so $P(j) \in \text{add } P_0(\gamma)$. Then $P(j) \in \text{add } P_0(\gamma) \cap \text{add } P_1(\gamma)$, which contradicts part (a) of the proposition. This shows that $P(j) \in \text{add } P_0(R(\gamma))$, and more generally that $\text{add } P_1(\gamma) \subset \text{add } P_0(R(\gamma))$. The reverse inclusion follows similarly. \square

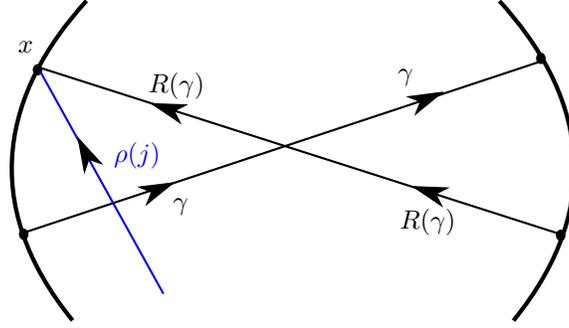


FIGURE 6. Proof of Proposition 3.34.

Remark 3.35. In particular, the composition $P_1(R(\gamma)) \xrightarrow{f_{R(\gamma)}} P_1(\gamma) \xrightarrow{f_\gamma} P_0(\gamma)$ is defined. We will show in Proposition 4.12 that up to an automorphism of $P_1(\gamma)$ this sequence is exact in the case where every chordless cycle in Q has length three.

4. OBJECTS OF THE SYZYGY CATEGORY

For the remainder of the paper, we let Q be a quiver satisfying Definition 3.3 and such that every chordless cycle in Q has length three, and let B denote the corresponding dimer tree algebra. In this section we define and study the map $f_\gamma : P_1(\gamma) \rightarrow P_0(\gamma)$ for each 2-diagonal γ in the associated checkerboard polygon \mathcal{S} of size $2N$. The main result of this section says that the cokernel of f_γ is an indecomposable syzygy in $\text{mod } B$, see Corollary 4.21.

Below we provide an overview of this section.

In Section 4.1 we introduce the necessary terminology and then present the map f_γ in Definition 4.5. Then we compare the two maps f_γ and $f_{R(\gamma)}$, where R denotes the clockwise rotation of the arc γ by $180/N$ degrees. The main result of Section 4.2 is Proposition 4.12 which shows that the sequence

$$P_1(R(\gamma)) \xrightarrow{\bar{f}_{R(\gamma)}} P_1(\gamma) \xrightarrow{f_\gamma} P_0(\gamma)$$

is exact, where $\bar{f}_{R(\gamma)}$ is obtained from $f_{R(\gamma)}$ by applying certain isomorphisms. Afterwards we study the properties of the cokernel of f_γ which we denote by M_γ . By definition, the map f_γ depends on the choice of a representative in the homotopy class of the arc γ , and in Proposition 4.13 of Section 4.3 we show that the associated cokernel M_γ is independent of the choice of γ . This gives a well-defined map from $\text{Diag}(\mathcal{S})$ to $\text{CMP } B$. The goal of the next subsection is to prove Proposition 4.18 that M_γ is an indecomposable module. Finally, in the last subsection Section 4.5 we combine all these results to conclude that M_γ , the cokernel of f_γ , is an indecomposable object in $\text{CMP } B$ whose syzygy ΩM_γ is isomorphic to $M_{R(\gamma)}$, see Corollary 4.21 and Theorem 4.20.

4.1. Definition of f_γ . Let γ be a fixed representation of a 2-diagonal in $\text{Diag}(\mathcal{S})$ such that it has a minimal number of crossings with radical lines in \mathcal{S} and except for the endpoints γ crosses one radical line at a time. Then as we move along γ , it crosses a subset of the radical lines in \mathcal{S} in a certain order. Between any two consecutive crossings of γ with radical lines $\rho(i), \rho(j)$ the arc γ traverses either a shaded region or a white region in \mathcal{S} bounded by $\rho(i), \rho(j)$. A pair (i, j) is called a *crossing pair* for γ if γ crosses a shaded region and it crosses two radical lines $\rho(i), \rho(j)$ that bound this region consecutively. Moreover, we always use the letter i for the degree 0 crossing and letter j for the degree 1 crossing. Thus, $P(i) \in \text{add } P_0(\gamma)$ and $P(j) \in \text{add } P_1(\gamma)$.

Note that every shaded region in \mathcal{S} is a triangle, since by assumption every chordless cycles in Q has length three. This implies that $\rho(i), \rho(j)$ always cross γ in opposite directions so $P(i)$ and $P(j)$ cannot both belong to $P_0(\gamma)$ or $P_1(\gamma)$. In addition, the vertices i, j in a crossing pair (i, j) belong to the same 3-cycle in Q , so they are connected by an arrow $i \rightarrow j$ or $j \rightarrow i$ which we call α . In this case, we will sometimes use $(i \xrightarrow{\alpha} j)$ to denote the crossing pair (i, j) .

Choosing a representative of γ determines the corresponding order on the crossing pairs. At the endpoints γ may cross only one side of a shaded region which lies on the boundary of \mathcal{S} . In what follows, this crossing will simply be denoted by (i) or (j) depending on whether it gives a summand of $P_0(\gamma)$ or $P_1(\gamma)$ respectively.

Given γ we define a *crossing sequence* for γ

$$(i_0) \text{ or } (j_0), (i_1, j_1), (i_2, j_2), \dots, (i_n, j_n), (i_{n+1}) \text{ or } (j_{n+1})$$

to be a complete ordered set of crossing pairs for γ for some $n \geq 0$ and where i_k denotes the degree 0 crossing and j_k denotes the degree 1 crossing of the pair (i_k, j_k) . Note that depending on the end behavior of γ the elements $i_0, j_0, i_{n+1}, j_{n+1}$ may or may not appear in the crossing sequence for γ . Here, we depict the most general case.

Remark 4.1. The crossing sequence for a fixed γ is unique up to reversing the order. However, different representatives γ, γ' of a 2-diagonal in $\text{Diag}(\mathcal{S})$ can give different crossing sequences.

Next we introduce two important notions needed to define the map f_γ .

Definition 4.2. We say that a path w in the quiver Q is *valid* if no two arrows in w lie in the same 3-cycle.

Remark 4.3. A valid path in Q is nonzero in the algebra B because such a path does not contain any relations. Therefore, there is at most one valid path between any two vertices of Q .

We will often use $x \rightsquigarrow y$ to denote a path in the quiver starting at vertex x and ending in vertex y .

Definition 4.4. Let the crossing sequence for γ be as follows

$$(i_0) \text{ or } (j_0), (i_1, j_1), (i_2, j_2), \dots, (i_n, j_n), (i_{n+1}) \text{ or } (j_{n+1})$$

and let $s, t \in \{0, \dots, n+1\}$. For $|s-t|=1$, we say the step from s to t in the crossing sequence is *forward* if there exists a valid path $i_s \rightsquigarrow j_t$ or a valid path $j_s \rightsquigarrow i_t$. For general $s, t \in \{0, \dots, n+1\}$ and $s \neq t$, the subsequence of the crossing sequence from s to t is *forward* if each of its steps is forward.

Definition 4.5. Let γ be a fixed representative of a 2-diagonal in $\text{Diag}(\mathcal{S})$ with crossing sequence

$$(i_0) \text{ or } (j_0), (i_1, j_1), (i_2, j_2), \dots, (i_n, j_n), (i_{n+1}) \text{ or } (j_{n+1}).$$

Let $P_1(\gamma) = \bigoplus_l P(j_l)$ and $P_0(\gamma) = \bigoplus_l P(i_l)$ and define

$$f_\gamma: P_1(\gamma) \longrightarrow P_0(\gamma)$$

by setting

$$f_{\gamma_{s,t}}: P(j_t) \longrightarrow P(i_s)$$

the multiplication by the path $w: i_s \rightsquigarrow j_t$ in Q unless $s \neq t$ and

- (i) the subsequence of the crossing sequence from s to t is not forward, or
- (ii) the path w is not valid

and in these cases we define $f_{\gamma_{i_s, j_t}} = 0$.

Remark 4.6. In the definition of the map f_γ condition (i) does not imply condition (ii). For example, suppose that there is a 2-diagonal γ such that $(i_s, j_s), \dots, (i_{s+3}, j_{s+3})$ is a subsequence of its crossing sequence and that these vertices form a full subquiver of Q given below.

$$\begin{array}{ccccc} & & j_{s+3} & \rightarrow & i_{s+3} \\ & & \uparrow & \swarrow & \uparrow \\ j_s & \leftarrow & i_{s+1} & \rightarrow & j_{s+2} \\ \downarrow & \nearrow & \downarrow & \swarrow & \downarrow \\ i_s & \leftarrow & j_{s+1} & \rightarrow & i_{s+2} \end{array}$$

Then the step from s to $s+1$ is not forward, so the map f_γ will not contain the path $i_s \rightarrow i_{s+1} \rightarrow j_{s+3}$ even though this path is valid.

Note that, for $s \neq t$, if $f_{\gamma_{i_s, j_t}} \neq 0$ then the path w is valid and all steps from s to t are forward. Also, if $s = t$ then i_s, j_s are connected by an arrow and lie in a common 3-cycle, so there is a path $w: i_s \rightsquigarrow j_s$ in Q . If there is an arrow $\alpha: i_s \rightarrow j_s$ in Q then $w = \alpha$, and if there is an arrow $\alpha: j_s \rightarrow i_s$ then the path is given the composition of two other arrows $i_s \rightarrow \bullet \rightarrow j_s$ in this 3-cycle.

Remark 4.7. By Proposition 3.31 if there is a path from i_s to j_t in Q then it is unique in the algebra B , so the map f_γ is well-defined. It is not a generic map in general, as there may be nonzero paths in Q from i_s to j_t that contain two arrows in the same 3-cycle. We also note that f_γ depends on the choice of the representative γ of the 2-diagonal, however we will show later in Proposition 4.13 that the cokernel of f_γ is independent of the representative and gives an indecomposable object in $\underline{\text{CMP}} B$.

4.2. Exactness. In this section, given two 2-diagonals in \mathcal{S} related by a clockwise rotation R , we define particular representatives $\gamma, R(\gamma)$ of these 2-diagonals that satisfy certain compatibility criteria. Then we show that the corresponding maps $f_\gamma, f_{R(\gamma)}$ are related as follows $\ker f_\gamma = \text{Im } \bar{f}_{R(\gamma)}$, where $\bar{f}_{R(\gamma)}$ is obtained from $f_{R(\gamma)}$ by introducing some negative signs. As a corollary we conclude that the syzygy functor Ω in $\underline{\text{CMP}} B$ is given by the clockwise rotation R in \mathcal{S} .

Let γ be a representative of a 2-diagonal in \mathcal{S} , and let $f_\gamma : P_1(\gamma) \rightarrow P_0(\gamma)$ be the corresponding map given in Definition 4.5. For a positive inter k , let $J_k = \begin{bmatrix} 1 & & & & \\ & -1 & & & \\ & & 1 & & \\ & & & \ddots & \\ 0 & & & & (-1)^{k+1} \end{bmatrix}$ denote a $k \times k$ diagonal matrix with diagonal entries alternating between 1 and -1 . Given a 2-diagonal represented by an arc γ we define a new map $\bar{f}_\gamma : P_1(\gamma) \rightarrow P_0(\gamma)$ by $\bar{f}_\gamma := J_{|P_0(\gamma)|} f_\gamma J_{|P_1(\gamma)|}$.

Remark 4.8. (a) The signs in \bar{f}_γ are $\begin{bmatrix} + & - & + & \cdots \\ - & + & - & \cdots \\ + & - & + & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$.

(b) The cokernels of f_γ and \bar{f}_γ are isomorphic. Indeed, this follows from the 5-Lemma applied to the following commutative diagram with exact rows.

$$\begin{array}{ccccccccc} P_1(\gamma) & \xrightarrow{f_\gamma} & P_0(\gamma) & \longrightarrow & \text{coker } f_\gamma & \longrightarrow & 0 & \longrightarrow & 0 \\ \cong \downarrow J_{|P_1(\gamma)|} & & \cong \downarrow J_{|P_0(\gamma)|} & & \cong \downarrow & & \cong \downarrow & & \cong \downarrow \\ P_1(\gamma) & \xrightarrow{\bar{f}_\gamma} & P_0(\gamma) & \longrightarrow & \text{coker } \bar{f}_\gamma & \longrightarrow & 0 & \longrightarrow & 0 \end{array}$$

Definition 4.9. Let γ, γ' be representatives of 2-diagonals. We say that γ and γ' are *compatible* if

- (a) whenever (i, j) is a crossing pair for γ , and γ' crosses both $\rho(i)$ and $\rho(j)$ then either (i, j) or (j, i) is a crossing pair for γ' , or
- (b) $\gamma' = R\gamma$ is a representative of the rotation of γ and both γ, γ' are radical lines.

Remark 4.10. (a) In part (a) of the definition the order of vertices in a pair changes whenever the degree of the crossings of γ and $\rho(i), \rho(j)$ is opposite to the degree of the crossings between γ' and $\rho(i), \rho(j)$.
(b) A radical line has essentially two representatives, one running parallel on the left of the radical line and the other running parallel on the right. These two representatives will have different crossing sequences. However, as we shall see in Proposition 4.13, both representatives will induce the same morphism on projectives.

In this section, we will only be interested in the compatibility of the representatives $\gamma, R(\gamma)$. The notion of compatibility means that that the two arcs $\gamma, R(\gamma)$ follow each other closely and have the same crossing sequences except possibly at the ends.

Remark 4.11. For every pair of 2-diagonals in \mathcal{S} related by the rotation R there exist representatives $\gamma, R(\gamma)$ that are compatible.

Now we take a pair of compatible 2-diagonals $\gamma, R(\gamma)$ and consider the sequence of morphisms

$$P_1(R(\gamma)) \xrightarrow{\bar{f}_{R(\gamma)}} P_0(R(\gamma)) = P_1(\gamma) \xrightarrow{f_\gamma} P_0(\gamma)$$

where the equality in the middle follows from Proposition 3.34 (b).

Proposition 4.12. *Let $\gamma, R(\gamma)$ be a pair of compatible arcs in \mathcal{S} . Then $\ker f_\gamma = \text{Im } \bar{f}_{R(\gamma)}$.*

Proof. See section A.2. □

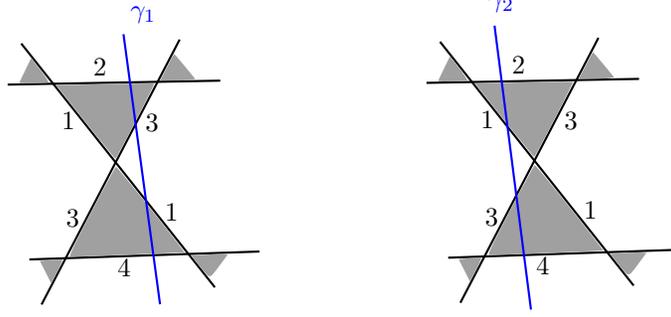


FIGURE 7. Proof of Proposition 4.13. The arc γ_2 is obtained from the arc γ_1 by an elementary homotopy.

4.3. Homotopy. Recall that the map f_γ depends on the crossing sequence of a particular representative γ of a given 2-diagonal. In this subsection, we show that the cokernel of f_γ is independent of the choice of the representative.

Let γ be a representative of a 2-diagonal in \mathcal{S} . We shall say that a homotopy of γ is *trivial* if it fixes the crossing sequence of γ , and it is *elementary* if it moves γ across the meeting point p of two shaded triangles of which γ crosses all four sides and is trivial elsewhere. For example, the two arcs γ_1 and γ_2 in Figure 26 differ by an elementary homotopy whose point p is the crossing points of the radical lines labeled 1 and 3.

We define the *degree* of an elementary homotopy to be the degree of the crossing between γ and any of the two radical lines that meet at the point p . We remark that the degree is well-defined because both radical lines at p cross γ in the same direction. Indeed, this follows immediately from the fact that the directions of the radical lines alternate along white regions.

An elementary homotopy of γ acts on the crossing sequence of γ by replacing two consecutive pairs

$$\begin{cases} (x, w), (y, z) \text{ by } (y, w), (x, z), & \text{if the degree is zero;} \\ (w, x), (z, y) \text{ by } (w, y), (z, x), & \text{if the degree is one.} \end{cases}$$

We are now ready for the main result of this subsection.

Proposition 4.13. *Let γ_1, γ_2 be two representatives of the same 2-diagonal in \mathcal{S} . Then $\text{coker } f_{\gamma_1} \cong \text{coker } f_{\gamma_2}$. Moreover, if the 2-diagonal is a radical line then $f_{\gamma_1} = f_{\gamma_2}$.*

Proof. Suppose first that the arc γ_1 is not a radical line. It suffices to show that the cokernel does not change under an elementary homotopy. So suppose γ_1, γ_2 are as in Figure 7.

The crossing sequences for γ_1, γ_2 only differ by two consecutive crossing pairs $\{2, 3\}, \{1, 4\}$ for γ_1 and $\{1, 2\}, \{4, 3\}$ for γ_2 . Note that the step between these pairs is forward for γ_1 but it is not forward for γ_2 , because by Remark A.5 forward steps correspond to moving counterclockwise around the boundary of a white region. Moreover, 1 and 3 cross γ_1, γ_2 in the same degree because the direction of the boundary edges around a white region is alternating. The degree of the crossing of 2 and 4 is opposite to that of 1 and 3 because the orientation is directed along the boundary edges of a shaded region. We suppose that the direction of γ_1, γ_2 is such that 1, 3 are the degree 0 crossings and 2, 4 are the degree 1 crossings. The case when 1, 3 are of degree 1 and 2, 4 are of degree 0 follows in the same way. Then let

$$(i_0) \text{ or } (j_0), (i_1, j_1), \dots, (i_t, j_t), (3, 2), (1, 4), (i_{t+3}, j_{t+3}), \dots, (i_n, j_n), (i_{n+1}) \text{ or } (j_{n+1})$$

$$(i_0) \text{ or } (j_0), (i_1, j_1), \dots, (i_t, j_t), (1, 2), (3, 4), (i_{t+3}, j_{t+3}), \dots, (i_n, j_n), (i_{n+1}) \text{ or } (j_{n+1})$$

be the crossing sequences for γ_1, γ_2 respectively.

Define

$$P_a = \bigoplus_{h=0}^t P(j_h) \quad P'_a = \bigoplus_{h=0}^t P(i_h) \quad P_b = \bigoplus_{h=t+3}^{n+1} P(j_h) \quad P'_b = \bigoplus_{h=t+3}^{n+1} P(i_h)$$

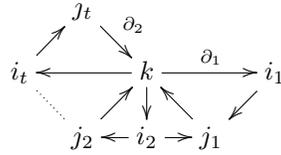
Then

$$P_0(\gamma_1) = P_0(\gamma_2) = P'_a \oplus P(1) \oplus P(3) \oplus P'_b \quad P_1(\gamma_1) = P_1(\gamma_2) = P_a \oplus P(2) \oplus P(4) \oplus P_b$$

We claim that there exists an isomorphism φ such that the following diagram commutes.

Now, suppose that γ_1, γ_2 are homotopic to a radical line $\rho(k)$. Then γ_1, γ_2 cross the same set of radical lines as $\rho(k)$. Note that γ_1, γ_2 do not cross $\rho(k)$. By Lemma 3.22 we have $\rho(k)$ crosses $\rho(h)$ if and only if there is an arrow between k and h in the quiver Q . Moreover, γ_1, γ_2 have the same orientation as $\rho(k)$, which means that if there is an arrow $k \rightarrow i$ in Q then $P(i) \in \text{add } P_0(\gamma_1)$ and if there is an arrow $j \rightarrow k$ then $P(j) \in \text{add } P_1(\gamma_1)$.

Label the two boundary arrows in Q adjacent to k by ∂_1, ∂_2 . The subquiver of Q determined by k and the vertices adjacent to k is as follows, when one of the boundary arrows starts in k and another ends in k . The cases when both boundary arrows start in k or both boundary arrows end in k follow in the exact same way. Thus, for simplicity we may assume that we have the following situation.



Because the only arrows between these vertices are $i_{s+1} \rightarrow j_s \leftarrow i_s$ for $s \in \{1, \dots, t-1\}$, it follows that in the crossing sequence for γ_1, γ_2 the vertex j_s is in a pair with i_s or i_{s+1} . Moreover, one crossing pair determines the pairing for the remaining vertices in the crossing sequence uniquely. This means that there are two possible crossing sequences

$$(i_1, j_1), (i_2, j_2), \dots, (i_t, j_t) \quad \text{or} \quad (i_1), (i_2, j_1), \dots, (i_t, j_{t-1}), (j_t)$$

for any diagonal homotopic to $\rho(k)$. Let the crossing sequences for γ_1, γ_2 be as above. For γ_1 no step in its crossing sequence is forward and all steps are trapezoidal with the path from i_{l+3} to j_l for all $l = 1, \dots, t-3$ being invalid. Hence f_γ is zero except on the main diagonal and the diagonal below the main diagonal. For γ_2 all steps in its crossing sequence are forward, and all of them except for the first and the last are trapezoidal with the path from i_t to j_1 being invalid. Then the two maps $f_{\gamma_1}, f_{\gamma_2}$ remain the same.

$$f_{\gamma_1} = f_{\gamma_2} = \begin{bmatrix} i_1 \rightarrow j_1 & 0 & \cdots & \cdots & \cdots \\ i_2 \rightarrow j_1 & i_2 \rightarrow j_2 & 0 & \cdots & \cdots \\ 0 & i_3 \rightarrow j_2 & i_3 \rightarrow j_3 & 0 & \cdots \\ \vdots & & & \ddots & \ddots \\ & & 0 & & \ddots \\ & & & & & \ddots \end{bmatrix}$$

This shows that the lemma holds in the case when γ_1, γ_2 are homotopic to a radical line, and the proof is complete. \square

The following corollary, which will be important in the next sections, is a direct consequence of the above proof.

Corollary 4.14. *Let γ_1, γ_2 be two representatives of the same 2-diagonal γ and $f_{\gamma_1}, f_{\gamma_2}$ the corresponding morphisms. Then there is a commutative diagram*

$$\begin{array}{ccc} P_1(\gamma) & \xrightarrow{f_{\gamma_1}} & P_0(\gamma) \\ \varphi_1 \downarrow & & \downarrow \varphi_0 \\ P_1(\gamma) & \xrightarrow{f_{\gamma_2}} & P_0(\gamma) \end{array}$$

with φ_0, φ_1 the isomorphisms induced by the homotopy between γ_1 and γ_2 . \square

Definition 4.15. We call the pair $(\varphi_1, \varphi_0) \in \text{Aut}(P_1(\gamma)) \oplus \text{Aut}(P_0(\gamma))$ the *automorphism of the homotopy* that transforms γ_1 into γ_2 , and we say that

$$f_{\gamma_2} = \varphi_0 f_{\gamma_1} \varphi_1^{-1}$$

is obtained from f_{γ_1} by *conjugation with the homotopy automorphism*.

4.4. Indecomposability. In this section we establish that the cokernel of f_γ is indecomposable. First we need a few preparatory lemmas.

Lemma 4.16. *Let γ be a representative of a 2-diagonal in \mathcal{S} , and let $f_\gamma : \bigoplus_{h=1}^n P(j_h) \rightarrow \bigoplus_{h=1}^m P(i_h)$ be the associated morphism. Suppose we have a commutative diagram*

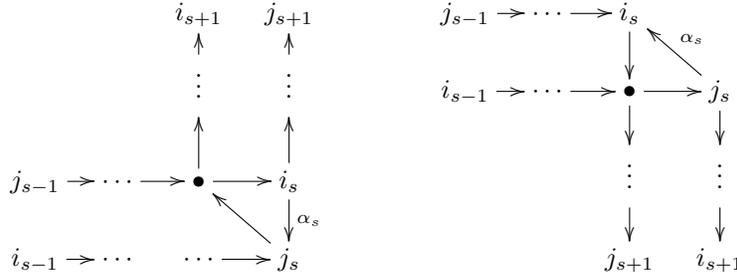
$$\begin{array}{ccc} \bigoplus_{h=1}^n P(j_h) & \xrightarrow{f_\gamma} & \bigoplus_{h=1}^m P(i_h) \\ \downarrow g_j & & \downarrow g_i \\ \bigoplus_{h=1}^n P(j_h) & \xrightarrow{f_\gamma} & \bigoplus_{h=1}^m P(i_h) \end{array}$$

for some matrices $g_j = (a_{h,h'})$ and $g_i = (b_{h,h'})$. Then $f_{\gamma_{s,t}} = 0$ or $b_{s,s} = a_{t,t}$ for all s, t .

Proof. This is proved in section A.3. □

Lemma 4.17. *Let γ be a representative of a 2-diagonal in \mathcal{S} . Suppose that the two steps from $s-1$ to $s+1$ are forward in the crossing sequence for γ . Then there is a valid path $i_{s-1} \rightsquigarrow j_{s+1}$ if and only if the arrow α_s is oriented $\alpha_s : j_s \rightarrow i_s$. In particular, at least one of $f_{\gamma_{s,s}}, f_{\gamma_{s-1,s+1}}$ is nonzero.*

Proof. This follows from the two pictures below, where we depict the two cases depending on the orientation of the arrow α_s .



Moreover, either there is an arrow $j_s \rightarrow i_s$ and $f_{\gamma_{s-1,s+1}}$ is a nonzero valid path $i_{s-1} \rightsquigarrow j_{s+1}$ or there is an arrow $i_s \rightarrow j_s$ which gives the nonzero entry $f_{\gamma_{s,s}}$. □

Proposition 4.18. *Let γ be a representative of a 2-diagonal in \mathcal{S} , then $\text{coker } f_\gamma$ is indecomposable.*

Proof. Let $f = f_\gamma : \bigoplus_{h=1}^n P(j_h) \rightarrow \bigoplus_{h=1}^m P(i_h)$. Suppose we have a commutative diagram

$$\begin{array}{ccc} \bigoplus_{h=1}^n P(j_h) & \xrightarrow{f_\gamma} & \bigoplus_{h=1}^m P(i_h) \\ \downarrow g_j & & \downarrow g_i \\ \bigoplus_{h=1}^n P(j_h) & \xrightarrow{f_\gamma} & \bigoplus_{h=1}^m P(i_h) \end{array}$$

for some matrices $g_j = (a_{h,h'})$ and $g_i = (b_{h,h'})$. By Lemma 4.16 we have $f_{s,t} = 0$ or $b_{s,s} = a_{t,t}$ for all s, t .

Suppose now without loss of generality that the step from s to $s+1$ is forward, then $f_{s,s+1} \neq 0$ and the above equation with $t = s+1$ yields

$$(4.2) \quad b_{s,s} = a_{s+1,s+1}.$$

If the step from $s-1$ to s is forward too, then $f_{s-1,s} \neq 0$ and hence

$$(4.3) \quad b_{s-1,s-1} = a_{s,s}.$$

By Lemma 4.17, we have $f_{s,s} \neq 0$ or $f_{s-1,s+1} \neq 0$. Suppose first $f_{s,s} \neq 0$. Then $b_{s,s} = a_{s,s}$ and thus

$$(4.4) \quad b_{s-1,s-1} = a_{s,s} = b_{s,s} = a_{s+1,s+1}$$

where the first equality follows from (4.3) and the last equality follows from (4.2).

Suppose now $f_{s-1,s+1} \neq 0$. Then $b_{s-1,s-1} = a_{s+1,s+1}$ and thus

$$a_{s,s} = b_{s-1,s-1} = a_{s+1,s+1} = b_{s,s}$$

where again the first equality follows from (4.3) and the last equality follows from (4.2). This is the same equation as (4.4).

If the step from $s-1$ to s is not forward but the step from s to $s+1$ is still forward, then $f_{s,s-1} \neq 0$, and thus

$$(4.5) \quad b_{s,s} = a_{s-1,s-1}.$$

Moreover, since the step from $s-1$ to s and the step from s to $s+1$ are in opposite direction, the arrow $\alpha_s : i_s \longrightarrow j_s$ is not a boundary arrow and therefore $f_{s,s} \neq 0$. Thus $b_{s,s} = a_{s,s}$ and thus equations (4.2) and (4.5) imply

$$b_{s,s} = a_{s-1,s-1} = a_{s,s} = a_{s+1,s+1}.$$

Similar, if both steps from $s-1$ to $s+1$ are not forward we get

$$a_{s-1,s-1} = b_{s,s} = a_{s,s} = b_{s+1,s+1}$$

and if the first step is forward but the second is not we get

$$a_{s,s} = b_{s-1,s-1} = b_{s,s} = b_{s+1,s+1}.$$

This implies that in all possible orientations of the steps we obtain $a_{s,s} = a_{t,t} = b_{s,s} = b_{t,t}$ for all s and t . In other words, all diagonal entries in the matrices g_j, g_i are equal.

Thus, $g_j = \lambda I + g'_j$ and $g_i = \lambda I + g'_i$ where g'_j and g'_i are nilpotent. Thus, every endomorphism of the two term complex f_γ is of the form $\lambda \cdot \text{Id} + N'$ with $\lambda \in \mathbb{k}$ and N' nilpotent. In particular, every endomorphism of $\text{coker } f_\gamma$ is of the form $\lambda \cdot \text{Id}_{\text{coker } f_\gamma} + N$, where N is nilpotent, and thus the endomorphism ring of $\text{coker } f_\gamma$ is local. Therefore, $\text{coker } f_\gamma$ is indecomposable. \square

4.5. Syzygy M_γ . In light of the results we obtained in the previous sections, we can finally make the following definition.

Definition 4.19. Given a 2-diagonal $\gamma \in \text{Diag}(\mathcal{S})$ define $M_\gamma := \text{coker } f_\gamma$.

Note that the definition of the map f_γ depends on the representative of the 2-diagonal, however by Proposition 4.13 its cokernel does not. This implies that M_γ is well-defined.

Next, we obtain a number of important results that summarize the key properties of M_γ .

Theorem 4.20. *If $\gamma \in \text{Diag}(\mathcal{S})$ then $\Omega M_\gamma \cong M_{R(\gamma)}$.*

Proof. By definition of M_γ and Proposition 4.12 we have an exact sequence in $\text{mod } B$ as follows.

$$P_1(R(\gamma)) \xrightarrow{\bar{f}_{R(\gamma)}} P_1(\gamma) \xrightarrow{f_\gamma} P_0(\gamma) \longrightarrow M_\gamma \longrightarrow 0$$

By Remark 4.8(b) and the definition of $M_{R(\gamma)}$ we have $\text{coker } \bar{f}_{R(\gamma)} \cong \text{coker } f_{R(\gamma)} \cong M_{R(\gamma)}$. Since modules $P_1(\gamma), P_0(\gamma)$ and $P_1(\gamma), P_1(R(\gamma))$ are nonzero modules that have no summands in common by Proposition 3.34(a), we conclude that the sequence above is the beginning of a minimal projective resolution of M_γ . This implies that $\Omega M_\gamma \cong M_{R(\gamma)}$. \square

Corollary 4.21. *Let \mathcal{S} be a checkerboard polygon of size $2N$, then M_γ is a nonzero indecomposable non-projective syzygy in $\text{mod } B$ and*

$$M_\gamma \cong \begin{cases} \Omega^N M_\gamma & \text{if } \gamma \text{ is a diameter} \\ \Omega^{2N} M_\gamma & \text{otherwise.} \end{cases}$$

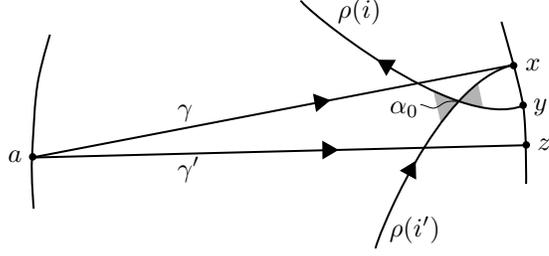


FIGURE 8. Proof of Lemma 5.1

Lemma 5.1. *Let γ, γ' be 2-diagonals in \mathcal{S} such that γ' is obtained from γ by a 2-pivot. Suppose there exist vertices i, i' in Q such that*

- (a) γ crosses the radical line $\rho(i)$ and γ' crosses the radical line $\rho(i')$ and these crossings have the same degree,
- (b) γ does not cross $\rho(i')$ and γ' does not cross $\rho(i)$.

Then there is an arrow $i' \rightarrow i$ in Q .

Proof. We use the notation in Figure 2. In particular the common endpoint of γ and γ' is the vertex a and the other endpoints are x and z . Without loss of generality, we may assume that γ is oriented from a towards x . Then γ' is oriented from a towards z .

Since γ crosses $\rho(i)$, but γ' doesn't, it follows that one endpoint of $\rho(i)$ is y or z and the other lies to the left of γ . Similarly, one endpoint of $\rho(i')$ is x or y and the other lies to the right of γ' .

Suppose first that $\rho(i)$ and $\rho(i')$ both have endpoint y . Then both $\rho(i), \rho(i')$ are oriented from y towards their other endpoint. Thus $\rho(i)$ crosses γ in degree 0, but $\rho(i')$ crosses γ' in degree 1. This implies that the crossings have different degrees, a contradiction to our assumption.

If $\rho(i)$ has endpoint z and $\rho(i')$ has endpoint x we also obtain this contradiction.

Now suppose $\rho(i)$ starts at vertex y and $\rho(i')$ ends at vertex x , see Figure 8. Then $\rho(i)$ and $\rho(i')$ cross at an interior point α_0 of the checkerboard pattern of \mathcal{S} . This interior vertex corresponds to an arrow α that connects i and i' in Q . Moreover, the 4 regions adjacent to α_0 are either shaded or not, and the shading is determined by the orientation of the radical lines $\rho(i)$ and $\rho(i')$ according to Lemma 3.25. Namely, the segments of the radical lines form an oriented path around the shaded regions and they are alternating around the white regions. Thus the shading at α_0 must be as in Figure 8. Now, the direction of the arrow α is determined by going around the shaded regions in counterclockwise direction. When we do this in either of the two shaded region, we encounter $\rho(i')$ first and then $\rho(i)$. Therefore the arrow α is oriented accordingly $\alpha: i' \rightarrow i$, and the proof is complete in this case.

The only remaining case has $\rho(i)$ ending at z and $\rho(i')$ ending at y , and its proof is similar. □

Next, we define maps g_0, g_1 from the summands of a projective resolution of a 2-diagonal to its 2-pivot.

Definition 5.2. Let γ, γ' be 2-diagonals in \mathcal{S} such that γ' is obtained from γ by a 2-pivot. Let $f_\gamma: P_1 \rightarrow P_0$ and $f_{\gamma'}: P'_1 \rightarrow P'_0$ be the morphisms associated to γ and γ' in Definition 4.5. We define morphisms $g_1: P_1 \rightarrow P'_1$ and $g_0: P_0 \rightarrow P'_0$ on the indecomposable summands of P_1 and P_0 as follows.

If $P(i)$ is an indecomposable summand in both P_0 and P'_0 we let the component $P(i) \rightarrow P(i)$ of g_0 be the identity map and the components $P(i) \rightarrow P(j)$ and $P(j) \rightarrow P(i)$ be the zero map, for all $j \neq i$. If $P(i)$ is an indecomposable summand of P_0 but not of P'_0 and $P(i')$ is an indecomposable summand of P'_0 but not of P_0 , we let the component $P(i) \rightarrow P(i')$ of g_0 be the map given by composing with the arrow $i' \rightarrow i$ given by Lemma 5.1. We define the components $P(j) \rightarrow P(j')$ to be the zero map, for any other j, j' .

The morphism $g_1: P_1 \rightarrow P'_1$ is defined in the same way.

Given a representative of a 2-diagonal γ and its 2-pivot γ' , recall Definition 4.9(a) of compatibility of γ, γ' . In particular, this means that the two crossing sequences for a compatible pair of arcs agree except at the very end.

The next lemma gives a precise description of the matrices for the maps g_0, g_1 .

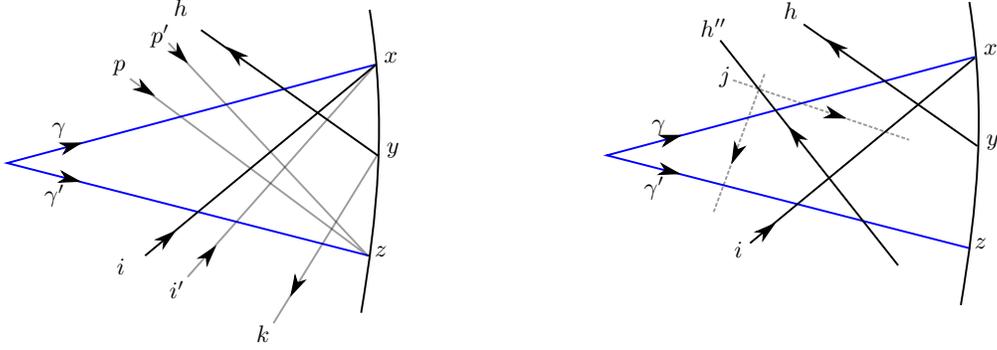


FIGURE 9. The proof of Lemma 5.3. Here the two dashed lines in the right figure illustrate the two possibilities for the arc j .

Lemma 5.3. *Let γ, γ' be compatible 2-diagonals in \mathcal{S} such that γ' is obtained from γ by a 2-pivot, and let g_0, g_1 be the maps given in Definition 5.2. Then, up to reversing the direction of the crossing sequences, the matrices for g_0, g_1 are of the form I_n or $[I_n \ 0]$ or $\begin{bmatrix} I_n \\ 0 \end{bmatrix}$ or $\begin{bmatrix} I_n & 0 \\ 0 & M \end{bmatrix}$, where I_n is the identity matrix with $n \geq 0$ and M equals one of the following $[i \rightarrow h]$, $[i \rightarrow h \ i \rightarrow h']$, $\begin{bmatrix} i \rightarrow h' \\ i' \rightarrow h \end{bmatrix}$. Moreover, the number of arrows in g_0 and g_1 combined is at most three.*

Proof. Construct the crossing sequences for γ, γ' by starting at the common endpoint of the arcs and then moving towards the other endpoints. Then the two crossing sequences for γ, γ' start with a common subsequence of degree zero crossings i_1, \dots, i_n in that order. Moreover, we assume that this subsequence is maximal, meaning that γ, γ' have different crossings in degree zero after i_n . By definition of g_0 we conclude that its matrix contains a block I_n in its top left corner. If there are no other arrows in g_0 , then its matrix is one of I_n , $[I_n \ 0]$, or $\begin{bmatrix} I_n \\ 0 \end{bmatrix}$ and the lemma follows.

For the remainder of this proof, we suppose that there is at least one arrow in g_0 . By definition of g_0 its matrix is of the form $\begin{bmatrix} I_n & 0 \\ 0 & M \end{bmatrix}$ for some matrix M , because any row and column with entry $1_{P(i_k)}$ for some k has all other entries being zero.

First, we claim that g_0 contains at most two arrows. By definition, an arrow $i \rightarrow h$ in g_0 comes from a pair of crossing radical lines $\rho(i), \rho(h)$ in \mathcal{S} such that $\rho(h)$ starts at y and crosses γ while $\rho(i)$ crosses γ' and ends in x , see Figure 9 on the left. There are at most two radical lines at every boundary vertex of \mathcal{S} , by Lemma 3.12(c). If there are also $\rho(h'), \rho(i')$ with endpoints y, x and crossing γ, γ' respectively, then we obtain at least four crossings, where $\rho(i), \rho(i')$ each cross both of $\rho(h), \rho(h')$. Then Q admits a subquiver that is given below on the left.



This yields a contradiction, because there should be a sequence of 3-cycles in Q between the arrows $i \rightarrow h$ and $i \rightarrow h'$, which means that Q contains interior vertices. This shows that either $\rho(i')$ or $\rho(h')$ or both are not in \mathcal{S} , so g_0 contains an arrow $i \rightarrow h$ and at most one of $i \rightarrow h', i' \rightarrow h$. This shows the claim that g_0 contains at most two arrows.

Now we show that the total number of arrows in g_0 and g_1 is at most three. Suppose on the contrary that both g_0, g_1 contain two arrows each. An arrow $k \rightarrow p$ in g_1 comes from a pair of crossing radical lines $\rho(k), \rho(p)$ in \mathcal{S} such that $\rho(k)$ starts at y and crosses γ' while $\rho(p)$ crosses γ and ends in z . An arrow $i \rightarrow h$ in g_0 yields another pair of radical lines with endpoints x, y . In particular, there are two radical lines $\rho(k), \rho(h)$ at vertex y , so there cannot be any additional radical lines at this vertex. This implies that if g_0, g_1 contain two arrows each then there are two radical lines i, i' ending in x and crossing γ' , and two radical lines p, p' crossing γ and ending in z , see Figure 9 on the left. In this case we obtain the subquiver of Q given above on the right, which contains interior vertices. This yields a contradiction, and shows that the total number of arrows in g_0 and g_1 is at most three.

Now we study the structure of M . In the case when M contains two arrows we label the radical lines so that γ crosses h before h' and similarly γ' crosses i before i' .

Now suppose that M contains a row of zeros, which correspond to entries in M that are paths starting at some fixed vertex k . Then there is a 2-diagonal k that crosses γ' in degree zero but it does not cross γ . Then k crosses γ' and ends in x , so in particular it crosses h , as we assume g_0 contains at least one arrow $i \rightarrow h$. But then, by definition of g_0 , the matrix M would contain an entry $k \rightarrow h$ and not a row of zeros as we assumed above. This gives a contradiction. We obtain a similar contradiction to M containing a column of zeros, which shows that M cannot contain a row or a column of zeros.

Now we show that the top left entry of M is an arrow $i \rightarrow h$. By definition every entry of M is either zero, $1_{P(h'')}$, or an arrow. If the top left entry of M is $1_{P(h'')}$, then we obtain a contradiction to i_1, \dots, i_n being the maximal initial subsequence of the degree zero crossings that γ, γ' have in common. Hence, suppose that the top left entry of M is zero. By the above, M does not contain any row or column of zeros, so suppose that the first row of M contains an arrow $i \rightarrow h$ and then the first column of M must contain $1_{P(h'')}$ for some vertex h'' . Then h'' is a common crossing of γ, γ' in degree zero such that γ' first crosses i and then h'' , while γ first crosses h'' and then h . Thus, we are in the situation of Figure 9 on the right. Since h'' is not the last crossing of γ in degree zero, it follows that (h'', j) is a crossing pair for γ for some vertex j . If in addition j crosses γ' , then by definition of compatibility between γ, γ' we conclude that (h'', j) is also a crossing pair for γ' . Then γ' crosses h'', j after crossing i . In particular, j and i must cross, and we obtain arrows $j \rightarrow i \rightarrow h''$ and $j \rightarrow h''$, which is not a possible subquiver of Q . Thus, we obtain a contradiction in the case when j also crosses γ' . If j does not cross γ' , then it would end at vertex z . But then we still obtain that j must cross i and we arrive at the same contradiction. This shows that it is not possible for the first row of M to contain an arrow $i \rightarrow h$ and for the first column to contain $1_{P(h'')}$. If M contains an arrow $i \rightarrow h$ in the first column and $1_{P(h'')}$ in the first row then we also obtain a contradiction in a similar way. This shows that the top left entry of M is an arrow $i \rightarrow h$.

Now suppose that M contains an entry $1_{P(h'')}$. Since $i \rightarrow h$ lies in the top left corner of M , it follows that $1_{P(h'')}$ cannot be in the first row or column of M . Then γ, γ' cross h'' after their crossings with h, i respectively. Then it follows that h'' crosses both h and i , and in particular we obtain arrows $i \rightarrow h$ and $i \rightarrow h'' \rightarrow h$ in the quiver. This is not a possible configuration, so we obtain a contradiction. This shows that the matrix M cannot contain any entry of the form $1_{P(h'')}$.

Thus, we conclude that M cannot contain a row or a column of zeros, or any entry of the form $1_{P(h'')}$, moreover we know that M contains at most two arrows. This implies that for the map g_0 the matrix M must be of the form given in the statement of the lemma.

The proof in the case of g_1 follows similarly. \square

The above lemma implies that in a given row or column of g_i with $i = 0, 1$, there are at most two nonzero entries. If g_i contains two arrows in the same row, then let g_i^r denote the map obtained from g_i by changing the sign of the arrow in the last column of the matrix, and otherwise let $g_i^r = g_i$. Similarly, if g_i contains two arrows in the same column, then let g_i^c denote the map obtained from g_i by changing the sign of the arrow in the last row, and otherwise let $g_i^c = g_i$.

Definition 5.4. The morphism (g_0^r, g_1^c) defined above is called the *pivot morphism* associated to the 2-pivot $\gamma \mapsto \gamma'$.

Theorem 5.5. Let γ, γ' be compatible 2-diagonals in \mathcal{S} such that γ' is obtained from γ by a 2-pivot and let (g_0^r, g_1^c) be the morphism defined above. Then we have the following commutative diagram with exact rows

$$(5.1) \quad \begin{array}{ccccccc} P_1 & \xrightarrow{f_\gamma} & P_0 & \xrightarrow{\pi_\gamma} & M_\gamma & \longrightarrow & 0 \\ g_1^c \downarrow & & \downarrow g_0^r & & \downarrow g & & \\ P'_1 & \xrightarrow{f_{\gamma'}} & P'_0 & \xrightarrow{\pi_{\gamma'}} & M_{\gamma'} & \longrightarrow & 0 \end{array}$$

where g is the induced morphism on the cokernels. In particular, g is a morphism of syzygies in CMP B .

Proof. This is proved in section B of the appendix. \square

Definition 5.6. Let γ, γ' be 2-diagonals in \mathcal{S} and let $M_\gamma, M_{\gamma'}$ be the corresponding syzygies in $\underline{\text{CMP}} B$. If γ' is obtained from γ by a 2-pivot then the corresponding morphism $g: M_\gamma \rightarrow M_{\gamma'}$ in $\underline{\text{CMP}} B$ is called *pivot morphism*.

6. AUSLANDER-REITEN TRIANGLES OF THE SYZYGY CATEGORY

In this section, we give a combinatorial description of the Auslander-Reiten triangles in $\underline{\text{CMP}} B$ in terms of the 2-pivots in the category of 2-diagonals $\text{Diag}(\mathcal{S})$. In subsection 6.1, we show that the indecomposable objects in $\underline{\text{CMP}} B$ do not admit any nonzero nilpotent endomorphisms. As a consequence we see that the dimension of Ext^1 between an indecomposable M and its AR-translate is equal to one. In subsection 6.2, we show that the Auslander-Reiten triangles of $\text{Diag}(\mathcal{S})$ give rise to commutative diagrams in $\underline{\text{CMP}} B$. We use these results in subsection 6.3, where we construct the short exact sequences in $\text{mod } B$ that induce the AR-triangles $\underline{\text{CMP}} B$.

6.1. Nilpotent endomorphisms of M_γ . In this subsection, we show that the syzygy M_γ admits no nonzero nilpotent endomorphisms in $\underline{\text{CMP}} B$.

Let γ be a 2-diagonal in \mathcal{S} and $f_\gamma: P_1(\gamma) \rightarrow P_0(\gamma)$ be the morphism defined in Definition 4.5. Let $g: M_\gamma \rightarrow M_\gamma$ be a nilpotent endomorphism, and let $g_0: P_0(\gamma) \rightarrow P_0(\gamma)$, $g_1: P_1(\gamma) \rightarrow P_1(\gamma)$ be the induced endomorphisms of the projectives. Thus we have the following commutative diagram with exact rows in $\text{mod } B$.

$$(6.1) \quad \begin{array}{ccccccc} P_1(\gamma) & \xrightarrow{f_\gamma} & P_0(\gamma) & \xrightarrow{\pi} & M_\gamma & \longrightarrow & 0 \\ g_1 \downarrow & & \downarrow g_0 & & \downarrow g & & \\ P_1(\gamma) & \xrightarrow{f_\gamma} & P_0(\gamma) & \xrightarrow{\pi} & M_\gamma & \longrightarrow & 0 \end{array}$$

The main result of this subsection is Theorem 6.1 which states that g is zero in the stable category $\underline{\text{CMP}} B$. The proof of this theorem requires a detailed analysis of the entries of the matrix of g_0 . We partition the matrix twice into column blocks and row blocks and then reduce g_0 successively by removing one block at a time.

As a direct consequence, we show in Corollary 6.2 that in $\underline{\text{CMP}} B$ the dimension of the first extension group between $\Omega^2 M_\gamma$ and M_γ is equal to 1. The following is proved in the appendix as Theorem C.25.

Theorem 6.1. *Let γ be a 2-diagonal and M_γ the associated indecomposable syzygy over B . Then M_γ does not admit any nonzero nilpotent endomorphisms in $\underline{\text{CMP}} B$.*

Recall that $\underline{\text{CMP}} B$ is a triangulated category with inverse shift given by Ω . It has almost-split triangles where τ^{-1} is given by Ω^2 , so in particular $\dim \text{Ext}_{\underline{\text{CMP}} B}^1(\Omega^2 M, M) \geq 1$. Now we show that the dimension of this space is actually equal to 1.

Corollary 6.2. *Let γ be a 2-diagonal in \mathcal{S} and $M = M_\gamma$ be the associated indecomposable syzygy in $\underline{\text{CMP}} B$. Then*

$$\dim \text{Ext}_{\underline{\text{CMP}} B}^1(\Omega^2 M, M) = 1.$$

Proof. Consider the following sequence of isomorphisms

$$\text{Hom}_{\underline{\text{CMP}} B}(M, M) \cong \text{Ext}_{\underline{\text{CMP}} B}^1(M, \Omega M) \cong \text{DExt}_{\underline{\text{CMP}} B}^2(\Omega M, M) \cong \text{DExt}_{\underline{\text{CMP}} B}^1(\Omega^2 M, M)$$

where the first and the last step follow because Ω is the inverse shift in $\underline{\text{CMP}} B$ and the second step follows because $\underline{\text{CMP}} B$ is a 3-Calabi-Yau. By Theorem 6.1 there are no nilpotent endomorphisms of M in $\underline{\text{CMP}} B$, and since M is indecomposable, by Proposition 4.18, [S2, Corollary 4.20] implies

$$\dim \text{Hom}_{\underline{\text{CMP}} B}(M, M) = 1.$$

Then the result follows. \square

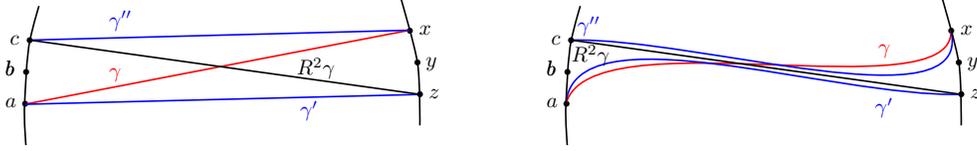


FIGURE 10. The left picture shows the 2-diagonals corresponding to the mesh $\gamma \rightarrow \gamma' \oplus \gamma'' \rightarrow R^2\gamma$ in the case where the 2-diagonal is not short. The right picture shows deformations of these 2-diagonals that are pairwise compatible, where we assume that none of $\gamma, R\gamma, R^2\gamma$ is a radical line.

6.2. Mesh relations. The main result of this subsection, Proposition 6.3, shows that the Auslander-Reiten triangles of the category of 2-diagonals $\text{Diag}(\mathcal{S})$ give rise to commutative diagrams in the stable category $\underline{\text{CMP}} B$. In subsection 6.3, we will show that these commutative diagrams actually are Auslander-Reiten triangles in $\underline{\text{CMP}} B$.

Let γ be a 2-diagonal in \mathcal{S} . We say that γ is *short* if one of the two polygons obtained by cutting \mathcal{S} along γ is a quadrilateral. In other words, the short 2-diagonals are those that are the start of a unique 2-pivot, and the non-short 2-diagonals are those that are the start of exactly two 2-pivots.

If γ is not short, let γ', γ'' denote the 2-diagonals that are obtained from γ by a 2-pivot and let $R^2\gamma$ be the 2-diagonal obtained from γ by applying the rotation R twice, see Figure 10. Thus $R^2\gamma$ can be obtained from γ in two ways by a sequence of two 2-pivots either passing through γ' or γ'' . In the category $\text{Diag}(\mathcal{S})$ this corresponds to the AR triangle

$$\gamma \rightarrow \gamma' \oplus \gamma'' \rightarrow R^2\gamma \rightarrow R\gamma$$

and a mesh in the AR quiver.

In the case where γ is short, only one of the two diagonals γ', γ'' , say γ' , exists. In this case the AR triangle is of the form

$$\gamma \rightarrow \gamma' \rightarrow R^2\gamma \rightarrow R\gamma.$$

We denote the corresponding pivot morphisms g', g'', h', h'' in $\underline{\text{CMP}} B$ as illustrated in the following diagrams

$$(6.2) \quad \begin{array}{ccc} & M_{\gamma'} & \\ g' \nearrow & & \searrow h' \\ M_{\gamma} & & M_{R^2\gamma} \\ g'' \searrow & & \nearrow h'' \\ & M_{\gamma''} & \end{array} \quad \begin{array}{ccc} & M_{\gamma'} & \\ g' \nearrow & & \searrow h' \\ M_{\gamma} & & M_{R^2\gamma} \end{array}$$

where the diagram on the left corresponds to the case where the 2-diagonal γ is not short and the diagram on the right to the case where γ is short.

Proposition 6.3. *Let γ be a 2-diagonal in \mathcal{S} such that $R\gamma$ is not a radical line. With the notation of diagram (6.2), we have the following identities in $\underline{\text{CMP}} B$.*

- (a) *If γ is not short then $h'g' = h''g''$.*
- (b) *if γ is short then $h'g' = 0$.*

Proof. This result is proved in section C.3 of the appendix. □

6.3. Auslander-Reiten triangles for M_{γ} . The main result of this subsection, Theorem 6.6 shows that the mesh relations given by 2-pivots and rotation R^2 in $\text{Diag}(\mathcal{S})$ correspond to AR-triangles in $\underline{\text{CMP}} B$. To prove this result, we first show that in each rotation orbit in $\text{Diag}(\mathcal{S})$ there is a mesh relation that induces a short exact sequence in $\text{CMP} B$ between M_{γ} and $\Omega^2 M_{\gamma}$. Then using that $\dim \text{Ext}_{\underline{\text{CMP}} B}^1(\Omega^2 M_{\gamma}, M_{\gamma}) = 1$, proved in Corollary 6.2, we see that this sequence induces an AR-triangle in $\underline{\text{CMP}} B$. Finally, the proof of the main result then follows from the correspondence $R \sim \Omega$.

Before stating the first result, recall the definition of γ', γ'' for a 2-diagonal γ which give rise to the corresponding sequence pivot morphisms $M_\gamma \rightarrow M_{\gamma'} \oplus M_{\gamma''} \rightarrow M_{R^2\gamma}$, see diagram (6.2).

Proposition 6.4. *Suppose γ is a 2-diagonal in \mathcal{S} such that $R\gamma$ is not a radical line.*

(a) *If γ is a radical line $\rho(i)$ is not short then there exists a short exact sequence in $\text{mod } B$*

$$0 \rightarrow M_\gamma \rightarrow M_{\gamma'} \oplus M_{\gamma''} \oplus P(i) \rightarrow M_{R^2\gamma} \rightarrow 0.$$

(b) *If γ is a radical line $\rho(i)$ is short then there exists a short exact sequence in $\text{mod } B$*

$$0 \rightarrow M_\gamma \rightarrow M_{\gamma'} \oplus P(i) \rightarrow M_{R^2\gamma} \rightarrow 0.$$

(c) *If $O(\gamma)$ does not contain any radical lines then γ is not short and there exists a short exact sequence in $\text{mod } B$*

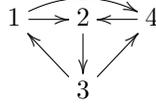
$$0 \rightarrow M_\gamma \rightarrow M_{\gamma'} \oplus M_{\gamma''} \rightarrow M_{R^2\gamma} \rightarrow 0.$$

Proof. For a proof see section C.4 of the appendix. \square

Let $\{R^i\gamma\}$ denote the orbit of a 2-diagonal γ under the action of R . The next lemma says that except for the case of the hexagon $\{R^i\gamma\}$ cannot consist entirely of radical lines.

Lemma 6.5. *Let \mathcal{S} be a checkerboard polygon of size at least 8, and let γ be a 2-diagonal in \mathcal{S} that is not short. For every orbit $\{R^i\gamma\}$ there exists some $\gamma' \in \{R^i\gamma\}$ such that $R(\gamma')$ is not a radical line in \mathcal{S} .*

Proof. Suppose on the contrary that every 2-diagonal in $\{R^i\gamma\}$ is a radical line in \mathcal{S} . Then in particular, $\gamma, R\gamma, R^2\gamma, R^3\gamma$ are four distinct radical lines in \mathcal{S} which we can label $\rho(1), \rho(2), \rho(3), \rho(4)$ respectively. Each pair of these arcs intersects, because γ is not short, and we obtain the following subquiver.



This contradicts the assumptions on the quiver that every vertex is boundary, see Lemma 3.24. \square

The main theorem of this subsection says that the exact sequences in $\text{mod } B$ discussed above give rise to Auslander-Reiten triangles in $\underline{\text{CMP}} B$.

Theorem 6.6. *Let γ be a 2-diagonal in \mathcal{S} .*

(a) *If γ is not short then*

$$M_\gamma \rightarrow M_{\gamma'} \oplus M_{\gamma''} \rightarrow M_{R^2\gamma} \rightarrow M_\gamma[1]$$

is an Auslander-Reiten triangle in $\underline{\text{CMP}} B$.

(b) *If γ is short then*

$$M_\gamma \rightarrow M_{\gamma''} \rightarrow M_{R^2\gamma} \rightarrow M_\gamma[1]$$

is an Auslander-Reiten triangle in $\underline{\text{CMP}} B$.

Proof. First suppose that γ is not short. Moreover, suppose that $\gamma = \rho(i)$ such that $R(\gamma)$ is not a radical line or $\{R^i\gamma\}$ does not contain any radical lines, then we obtain a short exact sequence in $\text{CMP } B$ as in Proposition 6.4 (a) or (c) respectively. This sequence reduces to a triangle

$$(6.3) \quad M_\gamma \rightarrow M_{\gamma'} \oplus M_{\gamma''} \rightarrow M_{R^2\gamma} \rightarrow M_\gamma[1]$$

by passing to $\underline{\text{CMP}} B$. If the connecting map $M_{R^2\gamma} \rightarrow M_\gamma[1]$ were to be zero then taking its cone and completing it to a triangle we obtain the following sequence.

$$M_\gamma \rightarrow M_{\gamma'} \oplus M_{\gamma''} \rightarrow M_{R^2\gamma} \xrightarrow{0} M_\gamma[1] \rightarrow M_{R^2\gamma}[1] \oplus M_\gamma[1]$$

Then $(M_{\gamma'} \oplus M_{\gamma''})[1] \cong M_{R^2\gamma}[1] \oplus M_\gamma[1]$ which is a contradiction, since all of these modules are distinct. Therefore the connecting map $M_{R^2\gamma} \rightarrow M_\gamma[1]$ is nonzero, which means that the triangle in (6.3) gives a nonzero element of $\text{Ext}_{\underline{\text{CMP}} B}^1(M_{R^2\gamma}, M_\gamma)$. By Theorem 4.20, we have $\Omega^2 M_\gamma \cong M_{R^2\gamma}$, so Corollary 6.2 implies that the triangle in (6.3) generates $\text{Ext}_{\underline{\text{CMP}} B}^1(M_{R^2\gamma}, M_\gamma)$. Since $\text{CMP } B$ is a triangulated category with inverse Auslander-Reiten translation given by Ω^2 it follows that this triangle is an Auslander-Reiten triangle in $\underline{\text{CMP}} B$. Hence part (a) holds whenever $\gamma = \rho(i)$ such that $R\gamma$ is not a radical line or whenever $\{R^i\gamma\}$ does not contain any radical lines.

Recall that the rotation R corresponds to Ω , which is the same as the inverse shift in $\underline{\text{CMP}} B$. Then an Auslander-Reiten triangle starting in $M_{R^i\gamma}$ is obtained from the Auslander-Reiten triangle starting with M_γ given in (6.3) by applying Ω^i . This completes the proof of part (a), since every 2-diagonal γ' in \mathcal{S} either lies in the same orbit under R as some radical line $\rho(i)$, and by Lemma 6.5 we can always choose $\rho(i)$ whose rotation is not a radical line, or its orbit $\{R^i\gamma'\}$ does not contain any radical lines.

The proof of part (b) follows from in the same way as above starting from a short exact sequence in Proposition 6.4(b). \square

7. MAIN RESULT

We are now ready for the main result of the paper. It proves Conjecture 1.1 in a special case.

Theorem 7.1. *Let B be a dimer tree algebra given by a quiver Q with potential that satisfies Definition 3.3 and such that every chordless cycle has length three. Let \mathcal{S} be the polygon constructed in subsection 3.1 and, for every 2-diagonal γ in \mathcal{S} , denote by f_γ the morphism defined in subsection 4.1. Then the mapping $\gamma \mapsto \text{coker } f_\gamma$ induces an equivalence of categories*

$$F: \text{Diag}(\mathcal{S}) \rightarrow \underline{\text{CMP}} B$$

such that

- (a) the radical line $\rho(i)$ corresponds to the radical of the indecomposable projective $P(i)$, for all $i \in Q_0$,

$$F(\rho(i)) = \text{rad } P(i).$$

- (b) The clockwise rotation R of \mathcal{S} corresponds to the inverse shift Ω and the counterclockwise rotation R^{-1} to the shift Ω^{-1} in $\underline{\text{CMP}} B$,

$$F(R(\gamma)) = \Omega F(\gamma) \quad \text{and} \quad F(R^{-1}(\gamma)) = \Omega^{-1} F(\gamma).$$

- (c) The square of the rotations corresponds to the Auslander-Reiten translations in $\underline{\text{CMP}} B$,

$$F(R^2(\gamma)) = \tau^{-1} F(\gamma) \quad \text{and} \quad F(R^{-2}(\gamma)) = \tau F(\gamma).$$

- (d) F induces a bijection between the 2-pivots in $\text{Diag}(\mathcal{S})$ and the irreducible morphisms between indecomposables in $\underline{\text{CMP}} B$.

- (e) F induces an isomorphism of Auslander-Reiten quivers

$$F: \Gamma_{\text{Diag}(\mathcal{S})} \rightarrow \Gamma_{\underline{\text{CMP}} B}.$$

Proof. On indecomposable objects, the functor F is defined by $F(\gamma) = \text{coker } f_\gamma$. It is well-defined on objects because the cokernel of f_γ is a syzygy by Proposition 4.12 and it is indecomposable by Proposition 4.18. The morphisms in $\text{Diag}(\mathcal{S})$ are given by compositions of 2-pivots modulo mesh relations. The functor F is defined on a 2-pivot by mapping it to the pivot morphism introduced in Definition 5.4. On compositions of 2-pivots, F is defined as the composition of the images. This is well-defined, because, as shown in subsection 6.2, F respects the mesh relations in $\text{Diag}(\mathcal{S})$. Moreover, the dimension of $\text{Hom}_{\text{Diag}(\mathcal{S})}(\gamma, \delta)$ is either 0 or 1 for any pair of 2-diagonals γ, δ , and this shows that F is faithful.

The case where the polygon \mathcal{S} has 6 sides is illustrated in Example 3.1. In this case, the theorem follows by inspection. Indeed, the category $\text{Diag}(\mathcal{S})$ has three indecomposable objects, the three 2-diagonals in \mathcal{S} , and the category $\underline{\text{CMP}} B$ also has three indecomposables, the three simple modules. The Auslander-Reiten quiver of each category consists of three vertices and no arrows.

Suppose now that \mathcal{S} has at least 8 sides.

Part (a) of the theorem is proved in Proposition 4.22 and part (b) in Theorem 4.20. Part (c) follows from (b) and the fact that $\Omega^2 = \tau^{-1}$ in $\underline{\text{CMP}} B$.

The Auslander-Reiten quiver of $\text{Diag}(\mathcal{S})$ consists of a single connected component which is finite. Theorem 6.6 shows that the functor F maps the Auslander-Reiten triangles in $\text{Diag}(\mathcal{S})$ to Auslander-Reiten triangles in $\underline{\text{CMP}} B$, and 2-pivots in $\text{Diag}(\mathcal{S})$ to irreducible morphisms in $\underline{\text{CMP}} B$. Therefore, the image of the Auslander-Reiten quiver of $\text{Diag}(\mathcal{S})$ is a connected component Γ of the Auslander-Reiten quiver of $\underline{\text{CMP}} B$. We will show that this is the only component of the Auslander-Reiten quiver of $\underline{\text{CMP}} B$. Suppose there is another component Γ' . Suppose first that there exists a nonzero morphism $f: M \rightarrow M'$ with $M \in \Gamma, M' \in \Gamma'$ indecomposable. Since f is not irreducible it must factor through one of the irreducible morphisms (hence an arrow in Γ) $g_1: M \rightarrow M_1$ starting at M , thus $f = f_1 g_1$ for some morphism $f_1: M_1 \rightarrow M'$.

Repeating the argument with f_1 we get a factorization $f = f_2 g_2 g_1$, where $g_2 g_1$ is the composition of two arrows in Γ . Continuing this way we obtain arbitrary long factorizations $f = f_t g_t \cdots g_2 g_1$, where $g_t \cdots g_2 g_1$ is a path in Γ . However, Γ is a finite component and the dimension of $\text{Hom}(M, N)$ between two indecomposable objects M, N in Γ is at most 1. Thus any composition of a large enough number of arrows in Γ is zero. This implies that $f = 0$.

Therefore there is no nonzero morphism from Γ to Γ' . Let M' be an indecomposable object in Γ' and let $f: P \rightarrow M'$ be its projective cover. Let $P(i)$ be an indecomposable summand of P . Then $\text{rad } P(i)$ is the image of the radical line $\rho(i)$ under our functor F . In particular, $\text{rad } P(i)$ lies in Γ . Now, since $P(i)$ is projective-injective in $\text{CMP } B$, the morphism f factors through $\tau^{-1} \text{rad } P(i)$. Thus there exists a nonzero morphism $\tau^{-1} \text{rad } P(i) \rightarrow M'$ from an object in Γ to an object in Γ' , a contradiction.

This shows that the Auslander-Reiten quiver of $\text{CMP } B$ consists of a single finite connected component that, under the functor F , is isomorphic to the Auslander-Reiten quiver of $\text{Diag}(\mathcal{S})$. This shows part (e) of the theorem and proves that the functor F is an equivalence. \square

8. CONSEQUENCES OF THE MAIN RESULT

In this section, we give some immediate corollaries of Theorem 7.1. Throughout the section, we let B be a dimer tree algebra for which all chordless cycles are of length three.

8.1. Rigidity and τ -rigidity of syzygies. Recall that a B -module M is said to be *rigid* if $\text{Ext}_B^1(M, M) = 0$.

Corollary 8.1. *The indecomposable syzygies over B are rigid B -modules.*

Proof. Let M be an indecomposable syzygy over B . Under the equivalence of categories in Theorem 7.1, M corresponds to a 2-diagonal in $\text{Diag}(\mathcal{S})$ which, because of Lemma 2.8, is a rigid object in $\text{Diag}(\mathcal{S})$. Thus M is rigid in $\text{CMP } B$. Now the result follows from Corollary 2.4. \square

Corollary 8.2. *Let M, N be indecomposable syzygies over B . Then the dimension of $\text{Ext}_B^1(M, N) \oplus \text{Ext}_B^1(N, M)$ is equal to the number of crossing points between the corresponding 2-diagonals. In particular, the dimension is either 1 or 0.*

Proof. This follows from the equivalence of categories of Baur and Marsh in Theorem 2.7 together with Theorem 7.1 and Proposition 2.3. \square

The question of τ -rigidity is much more subtle. This is because the Auslander-Reiten translation $\tau_{\text{CMP } B}$ in the syzygy category is different from the Auslander-Reiten translation τ_B in the module category $\text{mod } B$.

Recall that a B -module is said to be τ -rigid if $\text{Hom}_B(M, \tau_B M) = 0$, see [AIR]. Since B is a 2CY-tilted algebra, there is a bijection between indecomposable τ -rigid B -modules and indecomposable rigid objects in the cluster category of B , and hence with the cluster variables in the corresponding cluster algebra.

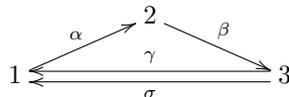
In general, the indecomposable syzygies over a 2CY-tilted algebra are not τ -rigid, see the example below. For the 2CY-tilted algebras B considered here however, we conjecture the following.

Conjecture 8.3. *Let B be a 2-Calabi-Yau tilted algebra whose quiver satisfies Definition 3.3. Then the indecomposable syzygies are τ -rigid in $\text{mod } B$.*

If the conjecture holds then the 2-triangulations of the checkerboard polygon \mathcal{S} correspond to (non-maximal) sets of compatible cluster variables in the cluster algebra of Q . We think it would be interesting to study the completions of these partial clusters in the cluster algebra as well as in $\text{mod } B$.

We give an example of a 2CY-tilted algebra that does not satisfy Definition 3.3 for which the conjecture fails.

Example 8.4. Let B be the 2-Calabi-Yau tilted algebra of type $\tilde{A}_{2,1}$ with quiver



bound by the relations $\alpha\beta = \beta\gamma = \gamma\alpha = 0$. Then $\text{rad } P(2) = \begin{smallmatrix} 3 \\ 2 \end{smallmatrix}$ is an indecomposable syzygy which has a nonzero morphism into $\tau \text{ rad } P(2) = \begin{smallmatrix} 2 \\ 1 \end{smallmatrix}$. Note that the algebra B does not satisfy the condition (Q1) because the quiver has no chordless cycles. On the other hand, the syzygy category of B is finite.

8.2. Syzygies and cosyzygies. Recall from Section 2.2 that the Auslander-Reiten translations induce equivalences of categories

$$(8.1) \quad \tau: \underline{\text{CMP}} B \rightarrow \underline{\text{CMI}} B \quad \tau^{-1}: \underline{\text{CMI}} B \rightarrow \underline{\text{CMP}} B.$$

Thus it follows immediately from Theorem 7.1 that $\underline{\text{CMI}} B$ is also equivalent to the category of 2-diagonals $\text{Diag}(\mathcal{S})$, Thus every 2-diagonal γ defines a morphism $f_\gamma^*: I_0(\gamma) \rightarrow I_1(\gamma)$ between injective modules such that $\ker f_\gamma^* \in \underline{\text{CMI}} B$ is a cosyzygy. In particular, the indecomposable projective $P(j)$ is a direct summand of $P_1(\gamma)$ if and only if the indecomposable injective $I(j)$ is a direct summand of $I_0(\gamma)$, while the indecomposable projective $P(i)$ is a direct summand of $P_0(\gamma)$ if and only if the indecomposable injective $I(i)$ is a direct summand of $I_1(\gamma)$. Therefore the Nakayama functor $\nu = D\text{Hom}_B(-, B)$ gives the following commutative diagram with exact rows

$$\begin{array}{ccccccc} P_1(\gamma) & \xrightarrow{f_\gamma} & P_0(\gamma) & \longrightarrow & M_\gamma & \longrightarrow & 0 \\ & & \downarrow \nu & & \downarrow \nu & & \\ 0 & \longrightarrow & \tau_B M_\gamma & \longrightarrow & I_0(\gamma) & \xrightarrow{f_\gamma^*} & I_1(\gamma) \end{array}$$

It follows that the syzygy M_γ in $\underline{\text{CMP}} B$ and its Auslander-Reiten translation $\tau_B M_\gamma$ in $\underline{\text{CMI}} B$ are both represented by the same 2-diagonal γ in $\text{Diag}(\mathcal{S})$. We have proved the following result.

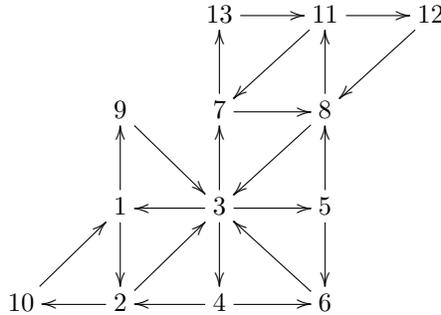
Corollary 8.5. *The following diagram commutes.*

$$\begin{array}{ccc} \underline{\text{CMP}} B & \xrightarrow{\tau} & \underline{\text{CMI}} B \\ \uparrow \text{cok } f_\gamma & & \uparrow \ker f_\gamma^* \\ \text{Diag}(\mathcal{S}) & \xrightarrow{1} & \text{Diag}(\mathcal{S}) \end{array}$$

9. EXAMPLES

In this section we give several examples.

Example 9.1. Let Q be the quiver



The corresponding checkerboard polygon has 24 vertices and is depicted in Figure 11. The 2-diagonal γ that runs from the boundary vertex 18 to the boundary vertex 9 is drawn in red. Its crossing sequence is $(2, 1), (3, 4), (6, 5), (8, 7)$. The corresponding syzygy M_γ is the cokernel of the associated morphism

$$f_\gamma: P(1) \oplus P(4) \oplus P(5) \oplus P(7) \longrightarrow P(2) \oplus P(3) \oplus P(6) \oplus P(8).$$

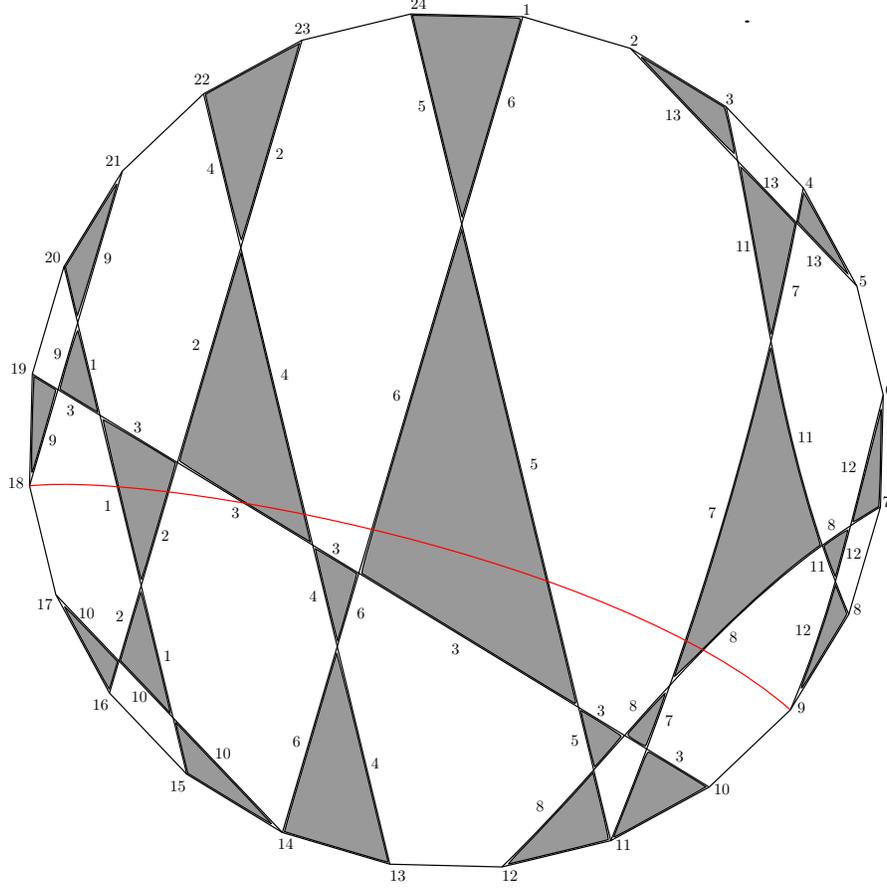


FIGURE 11. Checkerboard polygon for the quiver in Example 9.1.

The matrix of f_γ is

$$\begin{bmatrix} 2 \rightsquigarrow 1 & 0 & 0 & 0 \\ 3 \rightarrow 1 & 3 \rightarrow 4 & 3 \rightarrow 5 & 3 \rightarrow 7 \\ 0 & 0 & 6 \rightsquigarrow 5 & 6 \rightsquigarrow 7 \\ 0 & 0 & 0 & 8 \rightsquigarrow 7 \end{bmatrix}$$

The rotation $R(\gamma)$ is the 2-diagonal from boundary vertex 10 to boundary vertex 19. This is the radical line ρ_3 . The corresponding syzygy is the radical of $P(3)$ which is also given by the cokernel of the morphism

$$f_{R(\gamma)}: P(9) \oplus P(2) \oplus P(6) \oplus P(8) \longrightarrow P(1) \oplus P(4) \oplus P(5) \oplus P(7)$$

whose matrix is equal to

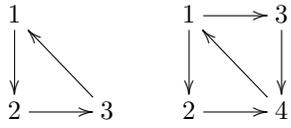
$$\begin{bmatrix} 1 \rightarrow 9 & 1 \rightarrow 2 & 0 & 0 \\ 0 & 4 \rightarrow 2 & 4 \rightarrow 6 & 0 \\ 0 & 0 & 5 \rightarrow 6 & 5 \rightarrow 8 \\ 0 & 0 & 0 & 7 \rightarrow 8 \end{bmatrix}$$

We have the following projective resolution.

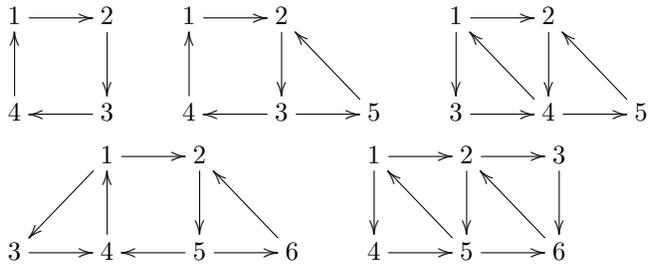
$$\dots \longrightarrow P_2(\gamma) \xrightarrow{\bar{f}_{R(\gamma)}} P_1(\gamma) \xrightarrow{f_\gamma} P_0(\gamma) \longrightarrow M_\gamma \longrightarrow 0$$

Example 9.2. In this example we classify the quivers whose checkerboard polygon has 6, 8 or 10 vertices. The quivers are listed below and the polygons are given in the same order in Figure 12.

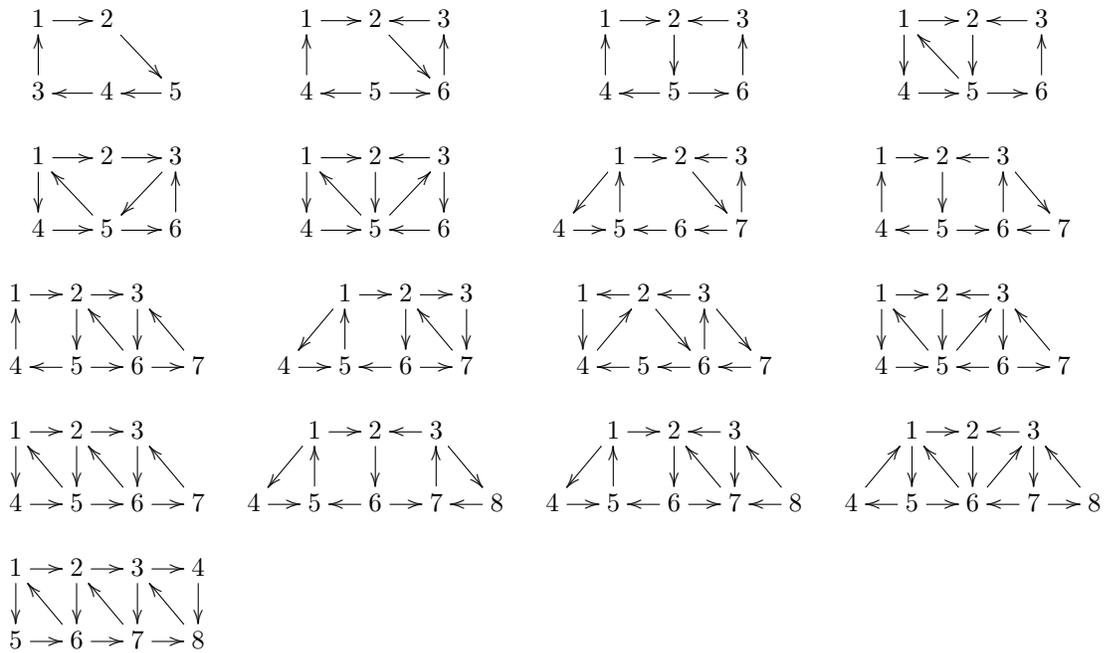
2 Hexagons



5 Octagons



17 Decagons



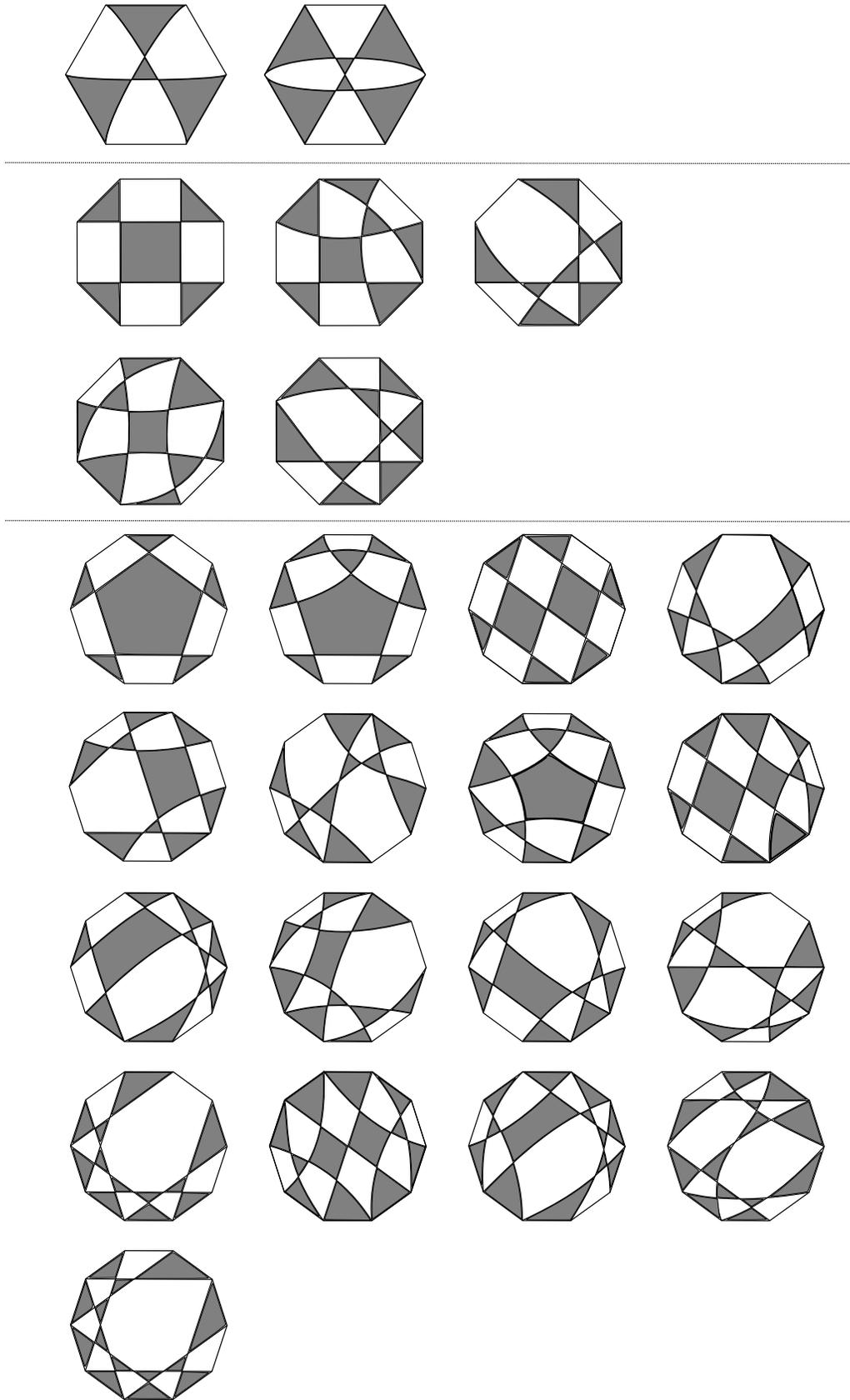


FIGURE 12. All checkerboard polygons up to 10 vertices.

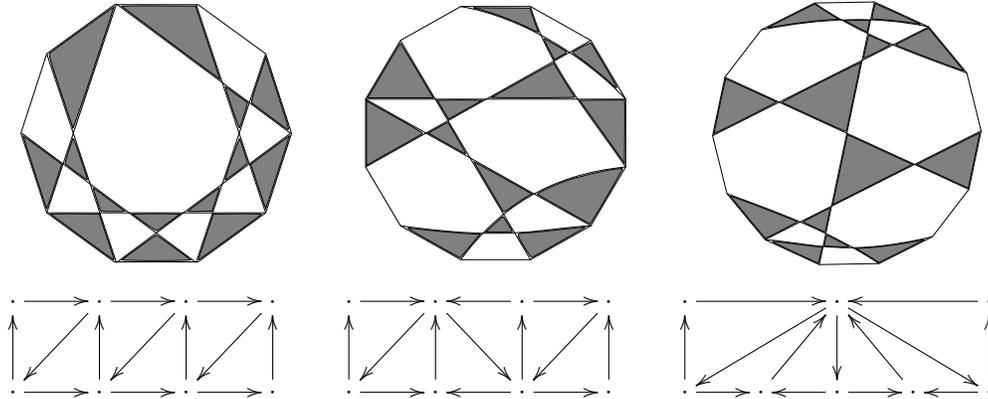


FIGURE 13. Three checkerboard polygons together with their quiver.

Example 9.3. Figure 13 shows three checkerboard polygons \mathcal{S} together with their respective quiver Q . The three quivers have 8 vertices, while the three polygons respectively have 10, 12 and 14 vertices. The two quivers on the left are of Dynkin type \mathbb{E}_8 or Grassmannian type $\text{Gr}(3, 8)$. The quiver on the right is of affine type $\tilde{\mathbb{E}}_7$.

REFERENCES

- [AIR] T. Adachi, O. Iyama and I. Reiten, τ -tilting theory, *Compos. Math.* **150** (2014), no. 3, 415–452.
- [Am] C. Amiot, Cluster categories for algebras of global dimension 2 and quivers with potential, *Ann. Inst. Fourier* **59** no 6, (2009), 2525–2590.
- [As] I. Assem, A course on cluster tilted algebras. Homological methods, representation theory, and cluster algebras, 127–176, CRM Short Courses, Springer, Cham, 2018.
- [ASS] I. Assem, D. Simson and A. Skowroński, *Elements of the Representation Theory of Associative Algebras, 1: Techniques of Representation Theory*, London Mathematical Society Student Texts 65, Cambridge University Press, 2006.
- [BHL] J. Bastian, T. Holm and S. Ladkani, Towards derived equivalence classification of the cluster-tilted algebras of Dynkin type D . *J. Algebra* **410** (2014), 277–332.
- [BHL2] J. Bastian, T. Holm and S. Ladkani, Derived equivalence classification of the cluster-tilted algebras of Dynkin type E . *Algebr. Represent. Theory* **16** (2013), no. 2, 527–551.
- [BKM] K. Baur, A. King and B. Marsh, Dimer models and cluster categories of Grassmannians. *Proc. Lond. Math. Soc.* (3) **113** (2016), no. 2, 213–260.
- [BM] K. Baur and B. Marsh, A geometric description of m -cluster categories, *Trans. Amer. Math. Soc* **360** (2008), no. 11, 5789–5803.
- [BM2] K. Baur and B. Marsh, A geometric description of the m -cluster categories of type D_n , *Int. Math. Res. Not. IMRN* (2007), no. 4, Art. ID rnm011, 19 pp.
- [Bo] R. Bocklandt, A dimer ABC. *Bull. Lond. Math. Soc.* **48** (2016), no. 3, 387–451.
- [BuMa] A. B. Buan, and B. Marsh, Cluster-tilting theory. Trends in representation theory of algebras and related topics, 1–30, *Contemp. Math.*, **406**, Amer. Math. Soc., Providence, RI, 2006.
- [BMRRT] A. B. Buan, B. Marsh, M. Reineke, I. Reiten and G. Todorov, *Tilting theory and cluster combinatorics*, Adv. Math. **204** (2006), no. 2, 572–618.
- [BMR] A. B. Buan, B. Marsh and I. Reiten, Cluster-tilted algebras, *Trans. Amer. Math. Soc.* **359** (2007), no. 1, 323–332 (electronic).
- [BRT] A. B. Buan, I. Reiten, and H. Thomas, From m -clusters to m -noncrossing partitions via exceptional sequences. *Math. Z.* **271** (2012), no. 3-4, 1117–1139.
- [BR] A. Beligiannis and I. Reiten, Homological and homotopical aspects of torsion theories, *Mem. Amer. Math. Soc.* **188** (2007).
- [Bu] R. Buchweitz, Maximal Cohen-Macaulay modules and Tate-cohomology over Gorenstein rings (unpublished).
- [CCS] P. Caldero, F. Chapoton and R. Schiffler, Quivers with relations arising from clusters (A_n case), *Trans. Amer. Math. Soc.* **358** (2006), no. 3, 1347–1364.
- [C] X. Chen, The singularity category of a quadratic monomial algebra. *Q. J. Math.* **69** (2018), no. 3, 1015–1033.
- [C2] X. Chen, The singularity category of an algebra with radical square zero. *Doc. Math.* **16** (2011), 921–936.
- [CL] X. Chen and M. Lu, Singularity categories of skewed-gentle algebras. *Colloq. Math.* **141** (2015), no. 2, 183–198.
- [CGL] X. Chen, S. Geng and M. Lu, The singularity categories of the cluster-tilted algebras of Dynkin type. *Algebr. Represent. Theory* **18** (2015), no. 2, 531–554.

- [CDZ] X. Chen, D. Shen and G. Zhou, The Gorenstein-projective modules over a monomial algebra. *Proc. Roy. Soc. Edinburgh Sect. A* **148** (2018), no. 6, 1115–1134.
- [DWZ] H. Derksen, J. Weyman and A. Zelevinsky, Quivers with potentials and their representations. I. Mutations. *Selecta Math. (N.S.)* **14** (2008), no. 1, 59–119.
- [GES] A. Garcia Elsener and R. Schiffler, On syzygies over 2-Calabi-Yau tilted algebras. *J. Algebra* **470** (2017), 91–121.
- [HK] A. Hanany and K. D. Kennaway, Dimer models and toric diagrams, (2005), arXiv:hep-th/0503149.
- [Ha] D. Happel. *Triangulated categories in the representation theory of finite-dimensional algebras*, volume 119 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1988.
- [IY] O. Iyama, and Y. Yoshino, Mutation in triangulated categories and rigid Cohen-Macaulay modules. *Invent. Math.* **172** (2008), no. 1, 117–168.
- [JKS] B. T. Jensen, A. D. King, and X. Su, A categorification of Grassmannian cluster algebras. *Proc. Lond. Math. Soc.* (3) **113** (2016), no. 2, 185–212.
- [Ka] M. Kalck, Singularity categories of gentle algebras. *Bull. Lond. Math. Soc.* **47** (2015), no. 1, 65–74.
- [K] B. Keller, On triangulated orbit categories, *Documenta Math.* **10** (2005), 551–581.
- [KR] B. Keller and I. Reiten, Cluster-tilted algebras are Gorenstein and stably Calabi-Yau, *Adv. Math.* **211** (2007), no. 1, 123–151.
- [L] S. Ladkani, 2CY tilted algebras that are not Jacobian. preprint arXiv:1403.6814.
- [LW] G. Leuschke and R. Wiegand, Cohen-Macaulay representations. *Mathematical Surveys and Monographs*, 181. American Mathematical Society, Providence, RI, 2012. xviii+367 pp.
- [Lu] M. Lu, Singularity Categories of some 2-CY-tilted Algebras, *Algebr. Represent. Theory* **19** (2016), no. 6, 1257–1295.
- [LZ] M. Lu and B. Zhu, Singularity categories of Gorenstein monomial algebras. *J. Pure Appl. Algebra* **225** (2021), no. 8, 106651, 39 pp.
- [M] M. Mastroeni, Matrix factorizations and singularity categories in codimension two. *Proc. Amer. Math. Soc.* **146** (2018), no. 11, 4605–4617.
- [O] D. Orlov, Derived categories of coherent sheaves and triangulated categories of singularities. *Algebra, arithmetic, and geometry: in honor of Yu. I. Manin. Vol. II*, 503–531, *Progr. Math.*, **270**, Birkhäuser Boston, Boston, MA, 2009.
- [PV] A. Polishchuk and A. Vaintrob, Matrix factorizations and singularity categories for stacks. *Ann. Inst. Fourier (Grenoble)* **61** (2011), no. 7, 2609–2642.
- [Po] A. Postnikov, Total positivity, Grassmannians, and networks, arXiv:math/0609764.
- [Pr] M. Pressland, Calabi-Yau properties of Postnikov diagrams, arXiv:1912.12475.
- [R] I. Reiten, 2-Calabi-Yau tilted algebras, *São Paulo J. Math. Sci.* **4**, 3 (2010), 529–545.
- [Ri] C. Riedtmann. Algebren, Darstellungsköcher, Überlagerungen und zurück. *Comment. Math. Helv.*, 55(2):199–224, 1980.
- [S1] R. Schiffler. A geometric model for cluster categories of type D_n . *J. Algebraic Combin.*, 27(1):1–21, 2008.
- [S2] R. Schiffler, *Quiver Representations*, CMS Books in Mathematics, Springer International Publishing, 2014.
- [SS] R. Schiffler and K. Serhiyenko, A geometric model for syzygies over 2-Calabi-Yau tilted algebras II.
- [Sh] D. Shen, The singularity category of a Nakayama algebra. *J. Algebra* **429** (2015), 1–18.
- [T] H. Thomas, Defining an m -cluster category. *J. Algebra* **318** (2007), no. 1, 37–46.
- [Tor] H. A. Torkildsen, A geometric realization of the m -cluster category of affine type A . *Comm. Algebra* **43** (2015), no. 6, 2541–2567.
- [Z] P. Zhang, Gorenstein-projective modules and symmetric recollement, *J. Algebra* **388** (2013) 65–80.

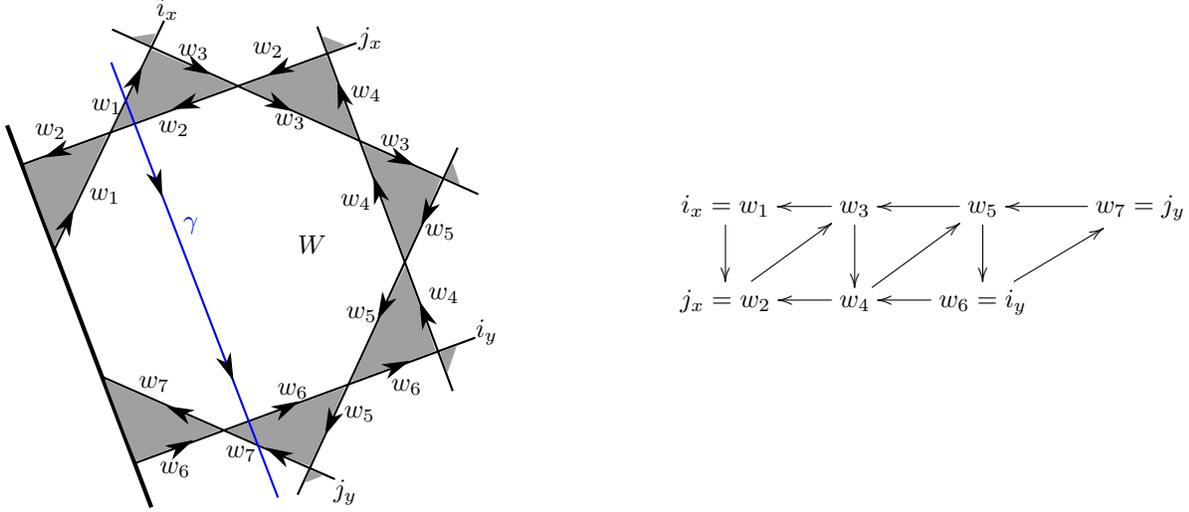


FIGURE 14. An arc γ traverses a white region W and contains two pairs $(i_x, j_x), (i_y, j_y)$ in its crossing sequence. In this case $Q(W) = Q(W, \gamma)$.

APPENDIX A. PROOFS OF SECTION 4

This is the first of three appendices which provide detailed proofs for sections 4, 5, and 6.

A.1. **Properties of f_γ .** In this section we establish certain properties of the map f_γ , and in particular of valid paths in the quiver Q . These results will be important for many proofs.

A.1.1. *Construction of $Q(\gamma)$.* Here we define a full subquiver $Q(\gamma)$ of Q determined by the crossing sequence for γ . This quiver will be instrumental during the remainder of the paper. We begin with considering the portion of the subquiver coming from two consecutive crossing pairs.

Let W be a white region in \mathcal{S} with bounding radical lines labeled w_1, w_2, \dots, w_m in the clockwise orientation such that w_1 and w_m are attached at the boundary of \mathcal{S} . Recall that by Lemma 3.12(a) a white region has either an edge with two vertices or a single vertex on the boundary of \mathcal{S} . Suppose that an arc γ enters W by crossing two of its boundary edges w_x, w_{x+1} either in the order w_x, w_{x+1} or w_{x+1}, w_x . Similarly, suppose γ exits W by crossing two of its boundary edges w_y, w_{y+1} either in this order or the opposite order. For an example see Figure 14. Let $Q(W)$ be the full subquiver of Q on vertices w_1, \dots, w_m . Note that three consecutive vertices w_l, w_{l+1}, w_{l+2} for $l = 1, \dots, m - 2$ form a 3-cycle in Q with arrows $w_l \rightarrow w_{l+1} \rightarrow w_{l+2} \rightarrow w_l$. Also, let $Q(W, \gamma)$ be the full subquiver of $Q(W)$ with vertices $w_x, w_{x+1}, \dots, w_y, w_{y+1}$ if $x < y$ or with vertices $w_{x+1}, w_x, \dots, w_{y+1}, w_y$ if $x > y$. Then there are four possibilities for $Q(W, \gamma)$ depending on the relationship between x and y , and whether there is an even or odd number of vertices in $Q(W, \gamma)$. See Figure 15.

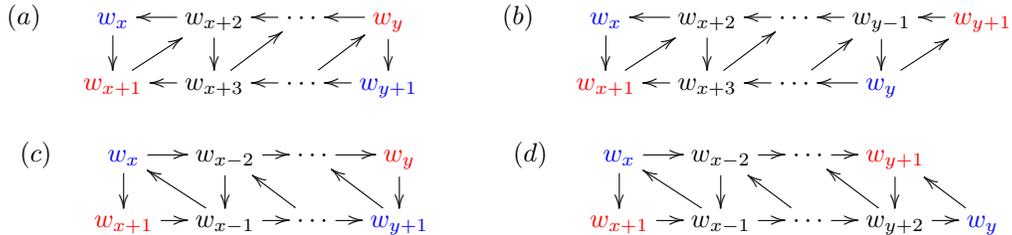


FIGURE 15. The four possibilities for $Q(W, \gamma)$. Cases (a) and (b) occur when $x < y$, and cases (c) and (d) occur when $x > y$. The step from x to y is rectangular in cases (a) and (c) and it is trapezoidal in cases (b) and (d). Vertices of the same color represent the same degree of crossings between the corresponding edges and γ .

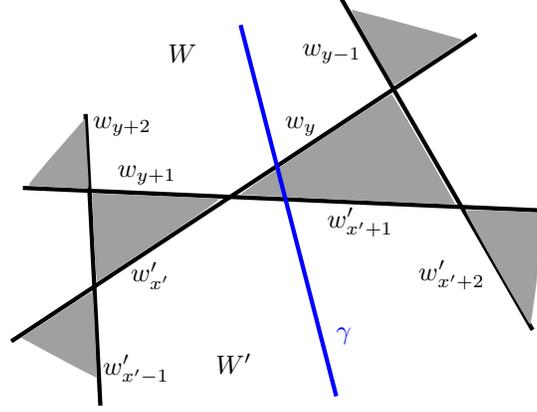


FIGURE 16. An arc γ traverses a pair of consecutive white regions W, W' .

We say that a step from x to y in the crossing sequence for γ is *rectangular* if $Q(W, \gamma)$ has an even number of vertices and the step from x to y is *trapezoidal* if $Q(W, \gamma)$ has an odd number of vertices.

Now, we analyze the subquiver of Q determined by three consecutive crossing pairs in the crossing sequence for γ . By the above we obtain a subquiver determined by two consecutive pairs and we want to analyze how to piece these subquivers together. For this construction it will not be important whether each of the steps is rectangular or trapezoidal, but the key distinction will be whether these steps are forward or not. Let W, W' be two white regions that are traversed consecutively by γ . Let w_1, \dots, w_m and w'_1, \dots, w'_m be the bounding radical lines of W and W' respectively, labeled clockwise around the white region such that w_1, w_m, w'_1, w'_m meet the boundary of \mathcal{S} in W, W' . Then γ crosses $w_x, w_{x+1}, w_y, w_{y+1}$ in W and γ crosses $w'_{x'}, w'_{x'+1}, w'_{y'}, w'_{y'+1}$ in W' , where $w'_{x'} = w_y$ and $w_{y+1} = w'_{x'+1}$ is the unique pair of edges in $W \cap W'$ that is crossed by γ . Moreover, there is an arrow between w_y and w_{y+1} which we call the *connecting arrow*, see Figure 16. The subquivers $Q(W, \gamma)$ and $Q(W', \gamma)$ share this connecting arrow. There are four cases coming from $(x < y$ or $x > y)$ and $(x' < y'$ or $x' > y')$, see Figure 17. The connecting arrow is labeled α . These quivers are fundamentally different in the following sense.

- $Q(W, \gamma)$ and $Q(W', \gamma)$ share no other vertex besides the two endpoints of the connecting arrow if $(x < y$ and $x' > y')$ or $(x > y$ and $x' < y')$. In this case the connecting arrow is an interior arrow of the quiver on vertices $Q(W, \gamma)_0 \cup Q(W', \gamma)_0$. The subsequences of the crossing sequence for γ from x to y' and from y' to x are not forward.
- $Q(W, \gamma)$ and $Q(W', \gamma)$ share a 3-cycle containing the connecting arrow if $(x < y$ and $x' < y')$ or $(x > y$ and $x' > y')$. In this case the connecting arrow is a boundary arrow of the quiver on vertices $Q(W, \gamma)_0 \cup Q(W', \gamma)_0$. The subsequence of the crossing sequence for γ from x to y' is forward if $(x > y$ and $x' > y')$ and from y' to x is forward if $(x < y$ and $x' < y')$.

Given any subsequence of the crossing sequence for γ made up of consecutive crossing pairs we can use the above construction recursively to construct a subquiver $Q(\gamma)$ of Q . In order to define the quiver $Q(\gamma)$ for the entire crossing sequence we first consider the case when γ crosses only one side of a shaded region S . This occurs only at the endpoints of γ when S lies on the boundary of \mathcal{S} . Then γ starts at the boundary of \mathcal{S} traverses S and crosses a radical line w_x as it exits the shaded region and enters a white region W . As γ traverses W and exits the white region it crosses another set of radical lines w_y, w_{y+1} or a single line w_y . Label the boundary of W by w_1, \dots, w_m in the clockwise direction as before. Since S lies on the boundary of \mathcal{S} it follows that w_x is either the first or the last edge in W , so $x = 1$ or $x = m$. If γ crosses only a single radical line w_y as it exits W then $y = m$ or $y = 1$ respectively. Moreover, the crossing sequence for γ is $(w_x), (w_y)$, and it does not contain any crossing pairs. In this case we define $Q(\gamma) = Q(W)$ to be the full subquiver of Q on the vertices w_1, \dots, w_m . If γ crosses two radical lines w_y, w_{y+1} as it exits W then the crossing sequence for γ starts with $(w_x), (w_y, w_{y+1})$ if w_y is of degree 0 or $(w_x), (w_{y+1}, w_y)$ if w_{y+1} is of degree 0. We define $Q(W, \gamma)$ to be the full subquiver of $Q(W)$ on vertices w_1, \dots, w_y, w_{y+1} if $x = 1$ or the full subquiver of $Q(W)$ on vertices w_y, w_{y+1}, \dots, w_m if $x = m$. We define $Q(W, \gamma)$ similarly if γ crosses a single edge of a shaded region as it exits W .

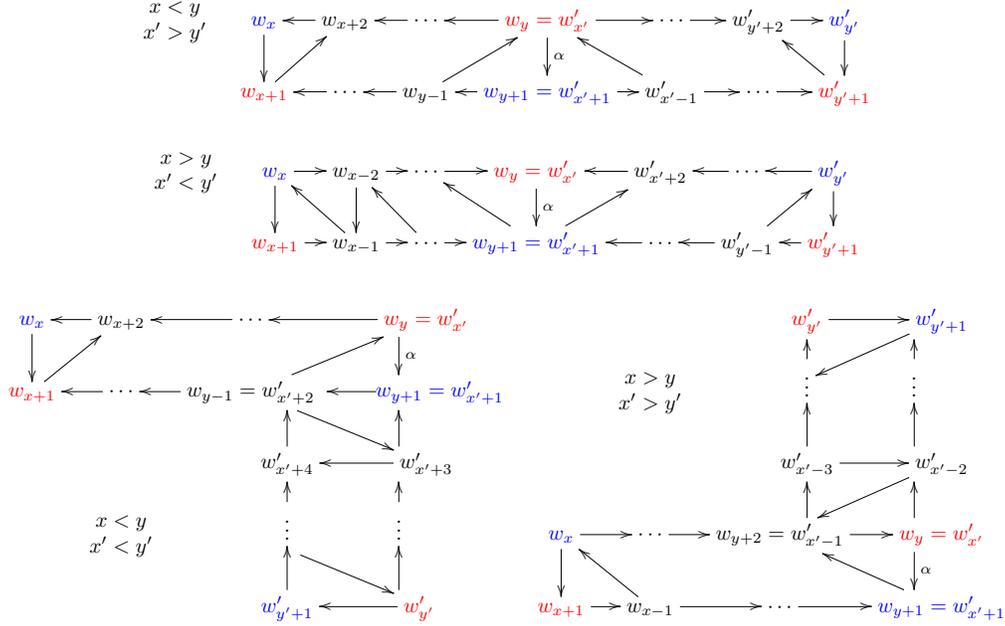


FIGURE 17. The subquivers of Q on vertices $Q(W, \gamma)_0 \cup Q(W', \gamma)_0$. Vertices of the same color represent the same degree of crossings between the corresponding edges and γ .

Definition A.1. Let γ be a representative of a 2-diagonal. The crossing sequence for γ determines a collection of subquivers $Q(W_1, \gamma), \dots, Q(W_t, \gamma)$ for some $t \geq 1$, and define $Q(\gamma)$ to be the full subquiver of Q on vertices $Q(W_1, \gamma)_0 \cup \dots \cup Q(W_t, \gamma)_0$.

In particular, $Q(\gamma)$ is obtained by piecing together $Q(W_i, \gamma)$ according to the construction in Figure 17.

A.1.2. *Valid paths.* Now we study properties of valid paths appearing in the definition of the map f_γ .

Suppose γ crosses pairs of radical lines w_x, w_{x+1} and then w_y, w_{y+1} . Then there are two consecutive crossing pairs $(i_x, j_x), (i_y, j_y)$ in the crossing sequence for γ where the following sets are equal $\{w_x, w_{x+1}\} = \{i_x, j_x\}$ and $\{w_y, w_{y+1}\} = \{i_y, j_y\}$. The next lemma is formulated in terms of valid paths, but it says that i_x and i_y (respectively j_x and j_y) appear diagonally opposite from each other in $Q(W, \gamma)$. First, we make the following observation.

Remark A.2. Consider the quiver determined by two consecutive crossing pairs w_x, w_{x+1} and w_y, w_{y+1} given in Figure 15. We observe that if $a > b$ and a and b have the same parity then there is a valid path from w_a to w_b in Q and there is no valid path from w_b to w_a .

Lemma A.3. *If $(i_x, j_x), (i_y, j_y)$ are two consecutive crossing pairs for a 2-diagonal γ then exactly one of the following is true:*

- (i) *there is a valid path from i_x to j_y and a valid path from j_x to i_y in Q ;*
- (ii) *there is a valid path from i_y to j_x and a valid path from j_y to i_x in Q .*

Proof. By Remark A.2, in order to prove the lemma it suffices to show that if $i_x = w_a$ for $a \in \{x, x+1\}$ then $j_y = w_b$ for $b \in \{y, y+1\}$ such that a and b have the same parity. Now, we consider the orientation of the radical lines in the surface \mathcal{S} and the orientation of the arc γ . By Lemma 3.25 the radical lines along a white region alternate in direction. Hence, if γ crosses w_a and w_b , such that a and b have the same parity, then γ crosses the two radical lines in opposite directions. That is, if w_a crosses γ from left to right then w_b crosses γ from right to left and vice versa. This completes the proof. \square

Remark A.4. The above lemma implies that in the crossing sequence for γ either the step from x to y is forward or the step from y to x is forward but not both, where x and y are consecutive. For example, in

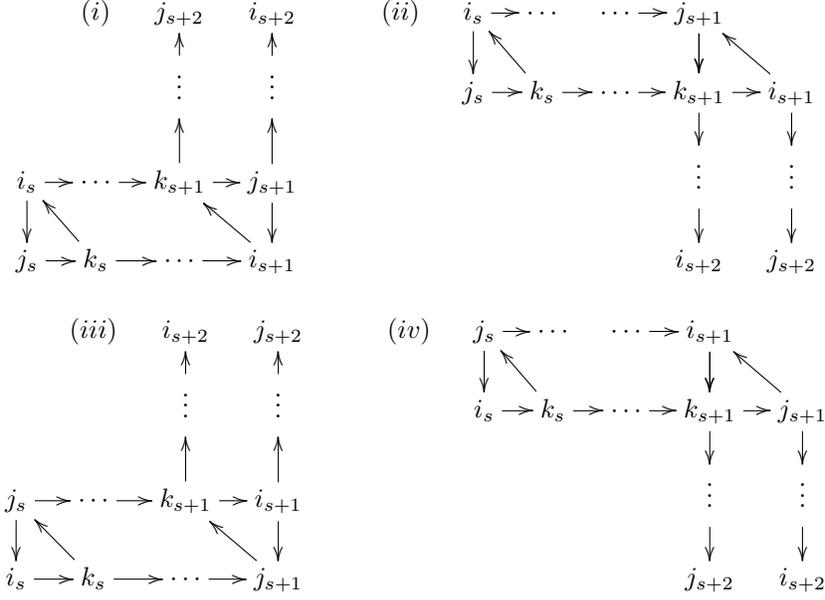
FIGURE 18. The quiver $Q(\gamma)$ in the proof of Lemma A.6.

Figure 15 the step from x to y is forward in cases (c) and (d), and the step y to x is forward in cases (a) and (b).

Remark A.5. Let $(i_x, j_x), (i_y, j_y)$ be two consecutive crossing pairs for γ contained in a white region W traversed by γ . Then valid paths from $i_x \rightsquigarrow j_y$ or $j_x \rightsquigarrow i_y$ are subpaths of maximal valid paths of W , see Definition 3.14. In particular, these paths correspond to moving counterclockwise around the boundary edges of W .

This completely describes valid paths coming from two consecutive crossing pairs. Next, we consider valid paths determined by three or more consecutive crossing pairs.

Lemma A.6. *Let $(i_s, j_s), (i_{s+1}, j_{s+1}), \dots, (i_t, j_t)$ be a forward subsequence of the crossing sequence for $\gamma \in \text{Diag}(\mathcal{S})$. If $t - s \geq 2$ then at most one of the paths $i_s \rightsquigarrow j_t, j_s \rightsquigarrow i_t$ is valid, and if $t - s = 2$ then exactly one of these paths is valid.*

Proof. We consider the portion of the quiver $Q(\gamma)$ coming from the first three crossing pairs $(i_s, j_s), (i_{s+1}, j_{s+1}), (i_{s+2}, j_{s+2})$. Since the steps from s to $s + 2$ are forward, $Q(\gamma)$ will look like the last quiver in Figure 17. There are four possibilities depending on the direction of the arrow between i_s and j_s and whether the step from s to $s + 1$ is rectangular or trapezoidal, see Figure 18. By Lemma A.3 there are valid paths $i_s \rightsquigarrow j_{s+1}$ and $j_{s+1} \rightsquigarrow i_{s+2}$, which says that in the figure j_{s+1} is directly horizontally from i_s and i_{s+2} is directly above or below j_{s+1} . This justifies the particular labeling of i, j vertices in the figure with subscripts $s + 1$ and $s + 2$. Moreover, we label the remaining vertex of the 3-cycle in $Q(\gamma)$ containing i_l, j_l by k_l for $l = s, s + 1, \dots, t$.

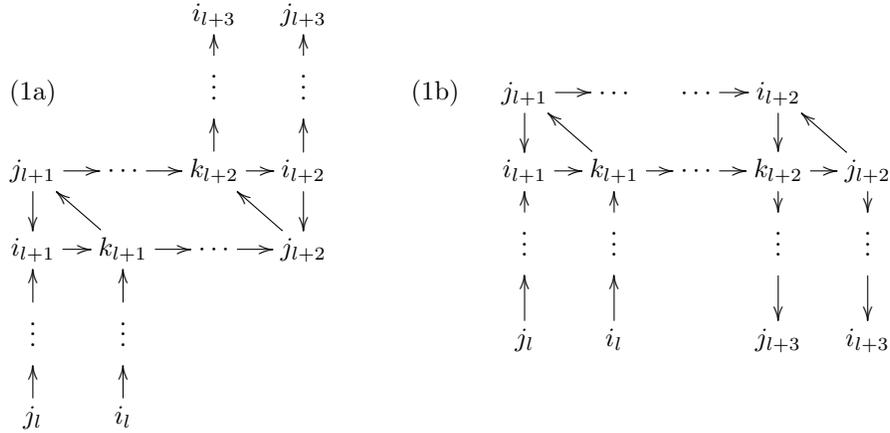
In cases (ii) and (iii) of Figure 17 any path starting in i_s and ending in j_t with $t - s \geq 2$ passes through vertex k_{s+1} . In particular, such a path is not valid because the path $i_s \rightsquigarrow k_{s+1}$ is not valid. Similarly, in cases (i) and (iv) any path starting in j_s and ending in i_t with $t - s \geq 2$ passes through k_{s+1} and it is not valid. Moreover, if $t - s = 2$ we see that in cases (ii) and (iii) there is a valid path $j_s \rightsquigarrow i_{s+2}$ and in cases (i) and (iv) there is a valid path $i_s \rightsquigarrow j_{s+2}$. This shows the lemma. \square

The following lemma describes the relationship between rectangular and trapezoidal steps and valid paths.

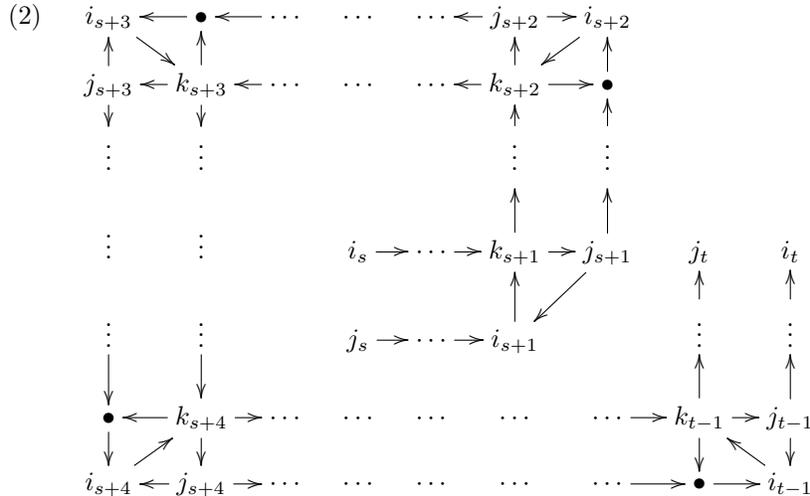
Lemma A.7. (*Rectangle-Trapezoid Lemma*) *Let $(i_s, j_s), (i_{s+1}, j_{s+1}), \dots, (i_t, j_t)$ be a forward subsequence of the crossing sequence for some γ . Then the following properties hold.*

- (1a) If the step from $l + 1$ to $l + 2$ is rectangular for $l = s, \dots, t - 3$ then either there exists a pair of valid paths $w_l : i_l \rightsquigarrow j_{l+2}, w_{l+1} : j_{l+1} \rightsquigarrow i_{l+3}$ or there exists a pair of valid paths $w_l : j_l \rightsquigarrow i_{l+2}, w_{l+1} : i_{l+1} \rightsquigarrow j_{l+3}$, and there is no valid path from i_l to j_{l+3} or from j_l to i_{l+3} .
- (1b) If the step from $l + 1$ to $l + 2$ is trapezoidal for $l = s, \dots, t - 3$ then there is exactly one valid path either $i_l \rightsquigarrow j_{l+3}$ or $j_l \rightsquigarrow i_{l+3}$.
- (2) There is a valid path $i_s \rightsquigarrow j_t$ or a valid path $j_s \rightsquigarrow i_t$ if and only if all the steps from $s + 1$ to $t - 1$ are trapezoidal.
- (3) If there is a valid path $i_s \rightsquigarrow j_t$ then there are valid paths $i_l \rightsquigarrow j_{l'}$ for all $l, l' = s, s + 1, \dots, t$ with $l < l'$.
- (4) If there exist valid paths $i_s \rightsquigarrow j_u$ and $i_t \rightsquigarrow j_w$ with $s < t < u < w$ then there exists a valid path $i_s \rightsquigarrow j_w$.

Proof. Parts (1a) and (1b) of the lemma follow from the structure of quivers below. Here we only depict the case when there is an arrow $j_{l+1} \rightarrow i_{l+1}$, and the quivers in the case when there is an arrow $i_{l+1} \rightarrow j_{l+1}$ are obtained from these by interchanging the labels of all i 's and j 's.



To show part (2) first suppose that there exists some l with $s \leq l \leq t - 3$ such that the step from $l + 1$ to $l + 2$ is rectangular. Then $Q(\gamma)$ contains as a full subquiver the quiver given (1a) above, where the labels of all i 's and j 's may be interchanged. Then a path w from i_s to j_t passes through some vertices in the rectangular portion of the quiver determined by $i_{l+1}, j_{l+1}, i_{l+2}, j_{l+2}$. In particular, w enters this rectangle at vertex k_{l+1} or i_{l+1} and exit the rectangle at vertex k_{l+2} or i_{l+2} . Then w contains two arrows in the same 3-cycle, which shows that it is not a valid path. Now, suppose that all steps from $s + 1$ to $t - 1$ are trapezoidal. Then we have the following subquiver of $Q(\gamma)$ if there is an arrow $j_{s+1} \rightarrow i_{s+1}$, and the labels of all i 's and j 's interchange if instead there is an arrow $i_{s+1} \rightarrow j_{s+1}$.



We remark that the number of vertices in the quiver between two consecutive crossing pairs does not necessarily increase as the figure might suggest, this is only a convenient way to draw the quiver, which looks like it is spiraling out from the center as we move from s to t . Here we see that there is a valid path from i_s to j_t that passes through vertices $k_{s+1}, k_{s+2}, \dots, k_{t-1}$ in this order. Moreover, there is no valid path from j_s to i_t in this case. This shows part (2) of the lemma.

To show part (3), suppose that there is a valid path $i_s \rightsquigarrow j_t$. If $t = s + 1$, then the statement holds trivially, and if $t = s + 2$ then the statement follows from Lemma A.3. Finally, if $t \geq s + 3$, then part (2) of this lemma implies that all steps from $s + 1$ to $t - 1$ are trapezoidal. Moreover, since $i_s \rightsquigarrow j_t$ is valid, rather than $j_s \rightsquigarrow i_t$ being valid, then there is an arrow $j_l \rightarrow i_l$ for all $l = s + 1, \dots, t - 1$ as in the quiver above. Because all the steps from $s + 1$ to $t - 1$ are trapezoidal, then again part (2) of this lemma implies that there are valid paths $i_l \rightsquigarrow j_{l'}$ for all $l, l' = s, \dots, t$ and $l < l'$. This shows part (3) of the lemma.

Now we show the last part of the lemma. If the sequence s, t, u, w is consecutive, then the step from t to u must be trapezoidal since we have a pair of valid paths $i_s \rightsquigarrow j_u, i_t \rightsquigarrow j_w$, see parts (1a) and (1b) of the lemma. Thus part (1b) of the lemma implies that there is a valid path $i_s \rightsquigarrow j_w$ and completes the proof in this case.

If the sequence s, t, u is consecutive and $w > u + 1$ then all the step from u to $w - 1$ are trapezoidal by part (2) of the lemma. Moreover, if the step from t to u is rectangular then (1a) together with the existence of a valid path $i_s \rightsquigarrow j_u$ implies that there is a valid path $j_t \rightsquigarrow i_{u+1}$. Then Lemma A.6 implies that there is no valid path $i_t \rightsquigarrow j_{u+1}$. Thus by part (3) of the lemma there is no valid path $i_t \rightsquigarrow j_w$, which is a contradiction. Thus the step from t to u is trapezoidal. In particular, all steps from t to $w - 1$ are trapezoidal and (2) yields a valid path $i_s \rightsquigarrow j_w$.

The case when t, u, w is consecutive and $s < t - 1$ is similar. It remains to consider the case when s, t, u is not consecutive and t, u, w is not consecutive. In this situation all steps from $s + 1$ to $w - 1$ are trapezoidal by (2), and again by (2) there is a valid path $i_s \rightsquigarrow j_w$. This completes the proof of (4). \square

Lemma A.8. *Let γ be a representative of a 2-diagonal in \mathcal{S} . If in the crossing sequence for γ all steps from s to t are forward then there are no valid paths in Q from i_s to i_t or from j_s to j_t .*

Proof. If $|s - t| = 1$ then the statement follows from the structure of the quiver $Q(\gamma)$ depicted in Figure 15. Indeed, i_s, i_t are vertices of a rectangle or a trapezoid that are diagonally opposite from each other by Lemma A.3, so any path $i_s \rightsquigarrow i_t$ uses two arrows in the same 3-cycle and hence it is not valid. Similar statement holds for j_s, j_t .

If $|s - t| = 2$, then lemma follows from the quivers in Figure 18. Now, suppose that $|s - t| > 2$ and without loss of generality assume $t > s$. If there is a rectangular step from $l + 1$ to $l + 2$ for some $l \in \{s, \dots, t - 3\}$, then the quiver $Q(\gamma)$, up to interchanging all i 's and j 's, is given in picture (1a) in the proof of Lemma A.7. We see that any path $i_s \rightsquigarrow i_t$ or $j_s \rightsquigarrow j_t$ passes through the rectangle determined by $i_{l+1}, i_{l+2}, j_{l+1}, j_{l+2}$. In particular, it cannot be valid. If all the steps from $s + 1$ to $t - 1$ are trapezoidal, then the quiver $Q(\gamma)$, up to interchanging all i 's and j 's, is given in picture (2) in the proof of the same lemma. Again we observe that there are no valid paths from i_s to i_t or from j_s to j_t . \square

A.1.3. Relations in B . In this section we prove two lemmas describing the structure of certain paths in Q that compose to zero or commute in the algebra B .

Recall from Section 3.1.1 that the trunk of the dual graph of Q consists of chordless cycles connected by interior arrows.

Lemma A.9. *Let η be a nonzero path in the algebra B such that the composition $\alpha\eta = 0$ for some arrow α . Then up to commutativity η starts with η' where*

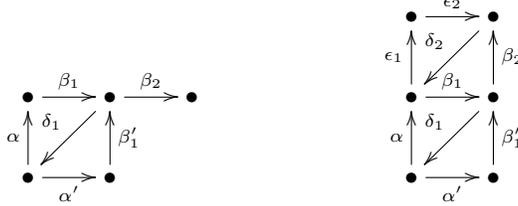
- (a) $\eta' = \beta_1\delta_1$ such that $\alpha, \beta_1, \delta_1$ are arrows in the same 3-cycle, or
- (b) $\eta' = \beta_1 \dots \beta_k$ such that α, β_1 are arrows in the same 3-cycle, β_1, \dots, β_k are arrows that lie in distinct 3-cycles C_1, \dots, C_k such that the path $C_1 - \bullet - C_2 - \bullet - \dots - \bullet - C_k$ is in trunk of the dual graph of Q where C_k is a leaf. In particular, there is the following subquiver of Q .



Proof. Let η' be a minimal nonzero subpath of η appearing at the start of η such that $\alpha\eta' = 0$. Since $\alpha\eta' = 0$ we may choose (up to commutativity) the first arrow in η' to be β_1 such that α, β_1 lie in a common 3-cycle C_1 . Let δ_1 be the third arrow in C_1 . Moreover, $\alpha\beta_1$ must lie in a relation that propagates further via commutativity and zero relations in B that results in $\alpha\eta'$ being zero.

If $\eta' = \beta_1$ then δ_1 is a boundary arrow, and we obtain case (a) of the lemma.

Suppose η' consists of at least two arrows $\beta_1\beta_2$. By minimality of η' the composition $\alpha\beta_1 \neq 0$. Then the arrow δ_1 is not boundary, and there exist arrows α', β'_1 such that $\alpha\beta_1 = \alpha'\beta'_1$, see the quiver below on the left.



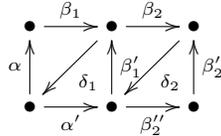
If $\beta_2 = \delta_1$, then $\alpha_1\beta_1\delta_1 = 0$ and $\eta' = \beta_1\delta_1$ by the minimality of η' . This gives case (a) of the lemma. If the second arrow in η' is $\beta_2 \neq \delta_1$, then $\alpha\eta'$ starts with $\alpha\beta_1\beta_2 = \alpha'\beta'_1\beta_2$. Because $\eta' \neq 0$ but $\alpha\eta' = 0$ then $\beta_1\beta_2$ or $\beta'_1\beta_2$ lie in a relation. Thus, one of the paths $\beta_1\beta_2$ or $\beta'_1\beta_2$ lies in a 3-cycle. We claim that the former case is not possible.

Suppose on the contrary that $\beta_1\beta_2$ lies in a 3-cycle, and let δ_2 be the third arrow in this 3-cycle. Because $\beta_1\beta_2 \neq 0$ it follows that δ_2 is not a boundary arrow. Then there exists another 3-cycle containing δ_2 and two other arrows ϵ_1, ϵ_2 , see the quiver above on the right. Note

$$\alpha\beta_1\beta_2 = \alpha\epsilon_1\epsilon_2 = \alpha'\beta'_1\beta_2 \neq 0.$$

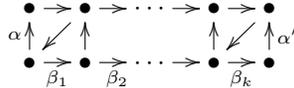
The arrows α, ϵ_1 cannot belong to a common 3-cycle, because $t(\alpha)$ would be an interior vertex of Q , which is a contradiction. Thus there is no relation on the path $\alpha\epsilon_1$. Then $0 = \alpha\eta' = \alpha\epsilon_1\epsilon_2\eta''$ implies $0 = \epsilon_1\epsilon_2\eta'' = \eta'$, a contradiction. This shows the claim that $\beta_1\beta_2$ cannot lie in a 3-cycle.

Then β_2, β'_1 lie a common 3-cycle C_2 . Let δ_2 be the third arrow in C_2 . If δ_2 is a boundary arrow, then $\alpha\beta_1\beta_2 = 0$ and $\eta' = \beta_1\beta_2$, and case (b) of the lemma is satisfied. If δ_2 is not a boundary arrow then $\alpha\eta'$ is equivalent to a path starting with $\alpha'\beta''_1\beta'_2$, where $\delta_2, \beta''_1, \beta'_2$ form another 3-cycle containing δ_2 . Note that α', β''_1 cannot lie in a common 3-cycle, because $t(\alpha')$ would be an interior vertex of Q . Thus, there are no more commutativity moves possible that involve only the first three arrows in $\alpha\eta'$.



Next we can continue in the same way considering the next arrow in η' . Eventually, we obtain $\eta' = \beta_1 \dots \beta_k$ for some k as in the quiver in the statement of the lemma where the arrow δ_k is boundary. \square

Lemma A.10. *Let α be an arrow and $\eta = \beta_1 \dots \beta_k$ be a path in the quiver Q that lie in the full subquiver of Q shown below. Suppose $\alpha\eta' = \eta\nu$ is a nonzero path in B for some paths η', ν . Then up to commutativity ν starts with the arrow α' .*



Proof. By the assumptions of the lemma, we have the full subquiver of Q as in Figure 19, where we label the vertices x_1, \dots, x_{k+1} and y_1, \dots, y_{k+1} . Moreover, since $\alpha\eta', \eta\nu$ are two equal nonzero path in B , then they end in some common vertex c . Since there are no cycles in B by Proposition 3.30, then the two paths do not self-intersect.

We claim that the path η' starts with the arrow α_1 . Suppose not. Then η' is a path starting in x_1 and ending in c that does not pass through x_2 . We show the two possibilities for η' in Figure 19. If η' is the

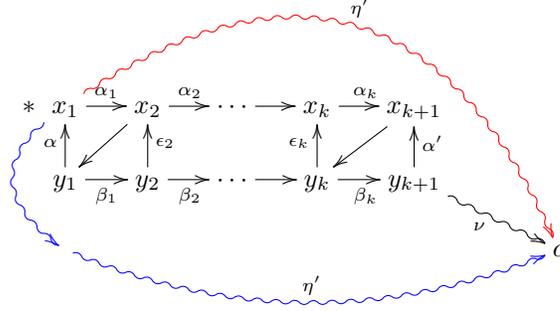


FIGURE 19. Proof of Lemma A.10.

upper path then x_2 is an internal vertex, and if η' is the lower path then y_1 is an internal vertex. This shows the claim that η' starts with the arrow α_1 .

Now we have $\eta' = \alpha_1 \eta''$ for some path η'' and we obtain the following equalities of nonzero paths.

$$\alpha \eta' = \alpha \alpha_1 \eta'' = \beta_1 \epsilon_2 \eta'' = \beta_1 \beta_2 \dots \beta_k \nu = \eta \nu$$

In particular, $\epsilon_2 \eta'' = \beta_2, \dots, \beta_k \nu$ are nonzero paths in B . By the same reasoning as before we can show that that first arrow in η'' is α_2 . Continuing in this way we obtain the equality

$$\epsilon_k \alpha_k \eta^{(k+1)} = \beta_k \alpha' \eta^{(k+1)} = \beta_k \nu$$

for some path $\eta^{(k+1)}$ starting in x_{k+1} and ending in c . Therefore, $\alpha' \eta^{(k+1)} = \nu$, and we obtain that up to commutativity ν starts with the arrow α' . This completes the proof of the lemma. \square

A.2. Exactness. Here we prove Proposition 4.12, see Proposition A.16 at the end of this subsection.

Lemma A.11. *Let $\gamma, R(\gamma)$ be a pair of compatible arcs in \mathcal{S} . Then $f_\gamma \circ \bar{f}_{R(\gamma)} = 0$.*

Proof. Let $A = f_\gamma, B = \bar{f}_{R(\gamma)}$ be two matrices with entries $A = (a_{h,l}), B = (b_{h,l})$. To prove the lemma we need to show that the product AB is zero. The (s, t) -entry of the matrix AB is the following sum of products of paths

$$\sum_l f_{\gamma^{i_s, j_l}} \bar{f}_{R(\gamma)_{j_l, i_t}}$$

from i_s in the crossing sequence for γ to i_t in the crossing sequence for $R(\gamma)$. This is trivially zero if for all l in the crossing sequence for γ the step from s to l is not forward or if in the crossing sequence for $R(\gamma)$ the step from l to t is not forward. Now, we fix t such that there exist l and s such that all steps from s to l are forward and all steps from l to t are forward. We may index the crossing pairs in the crossing sequences so that $s \leq l \leq t$. Let B_t be the t -th column of B and let $l \leq t - 1$ be least integer such that $b_{l,t}$ is a valid path. Then either $l = 1$ or $b_{l-1, l+1}$ is not a valid path in B , because by Lemma A.6(4) it would yield a valid path $b_{l-1, t}$ contradicting minimality of l .

Next, we study the structure of valid paths in the matrix A given below, where valid paths are denoted by \star . Let $\Delta(l, t)$ denote the triangular region in A or B whose vertices are at positions $(l, t), (l, l+2), (t-2, t)$. Since $b_{l,t}$ up to sign is a valid path $j_l \rightsquigarrow i_t$ then every path in $\Delta(l, t)$ in B is also valid by Lemma A.7(3). By Lemma A.6 each entry of A in $\Delta(l, t)$ is zero, and thus again by Lemma A.7(3) each entry of A in the rectangular region above $\Delta(l, t)$ is zero. The entry $a_{l-1, l+1}$ in A is a valid path, because otherwise the entry $b_{l-1, l+1}$ in B would be a valid path by Lemma A.6 and we have seen above that $b_{l-1, l+1}$ is not valid.



FIGURE 20. Proof of Lemma A.11.

	k		l	$l+1$	$l+2$		t
\dots	\dots	\dots	\dots	\dots	\dots	\dots	\dots
k	$\star \star 0 0 \dots \dots \dots 0 0 \dots \dots 0$						
$k+1$	$0 \star \star \star \star \dots \dots \star 0 \dots \dots 0$						
	$0 \star \star \star \dots \dots \star 0 \dots \dots 0$						
	$0 \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots$						
	$0 0 \star \star \star \star \star 0 \dots \dots 0$						
$l-1$							
l							
$l+1$							
$t-2$							
$t-1$							
t							

Let k be the least positive integer such that $a_{k+1,l+1}$ is valid, or, if no such k exists, we let $k = 1$. Then every entry in A in the triangle $\Delta(k+1, l+1)$ is a valid path by Lemma C.1(3), and $a_{k,l+1} = 0$ by minimality of l . Therefore,

$$a_{k,k+2} = a_{k,k+3} = \dots = a_{k,l+1} = 0$$

by Lemma A.7(4). Then Lemma A.7(3) implies that every entry $a_{h,k+2}, \dots, a_{h,l+1}$ with $h \leq k$ is zero. If $l > 1$, then multiplying AB_t yields a vector whose first k entries are zero. The s -th entry of AB_t for $k+1 < s < t-1$ is a difference of two paths $i_s \rightsquigarrow j_l \rightsquigarrow i_t, i_s \rightsquigarrow j_{l+1} \rightsquigarrow i_t$, which is zero, because all parallel paths in Q are equal in B by Proposition 3.31. The t -th entry of AB_t is a single path $i_t \rightsquigarrow j_t \rightsquigarrow i_t$, which is zero, because there are no cycles in B by Proposition 3.30.

If $l = 1$ then $k = 1$. In addition, if the first row of A contains two nonzero entries then the same argument as above implies that AB_t yields a vector whose first t entries are zero. Now suppose $l = 1$, but the first row of A contains only a single entry. Then the crossing sequence for γ starts with $(i_1), (i_2, j_2)$ and the first row of A contains a single path $i_1 \rightsquigarrow j_2$. In particular, there is no valid path from i_1 to j_3 . It suffices to show that the first entry in AB_t is zero. Since the crossing sequence for γ starts with $(i_1), (i_2, j_2)$, then we are in the situation of Figure 20 on the left, where the arrow $j_1 \rightarrow i_1$ is a boundary arrow. Moreover, since there is no valid path $i_1 \rightarrow j_3$, then we have the corresponding quiver as in Figure 20 on the right.

The first entry in AB_t is a composition $i_1 \rightsquigarrow j_2 \rightsquigarrow i_t$, for some $t \geq 2$. From the quiver we see that this path is zero, because it would start with the path $i_1 \rightsquigarrow k$, which is zero, because $j_1 \rightarrow i_1$ is a boundary arrow. This shows that in all cases AB_t is zero in the first t entries.

Finally, if the step from t to $t+1$ is forward, then remaining entries $t+1, t+2, \dots$ are zero since $b_{h,t} = 0$ for all $h \geq t+1$. If the step from t to $t+1$ is not forward, then we let $l' \geq t+1$ be the largest integer such that $b_{l',t} \neq 0$, and we let k' be the largest positive integer such that $a_{k'-1,l'-1}$ is valid. Then we consider the structure of A in rows $t, t+1, \dots, k'$, which is analogous to the one we have above but instead A is lower triangular in that region instead of being upper triangular. Then a similar argument as before shows that AB_t is zero in positions greater than t . Because t was arbitrary this shows that $AB = 0$. \square

The above lemma says that in particular $\text{Im } \bar{f}_{R(\gamma)} \subset \ker f_\gamma$. The rest of this section is dedicated to showing the reverse inclusion. We shall need three preparatory lemmas. In the first, we consider special cases which we treat separately.



FIGURE 21. Proof of Lemma A.12.

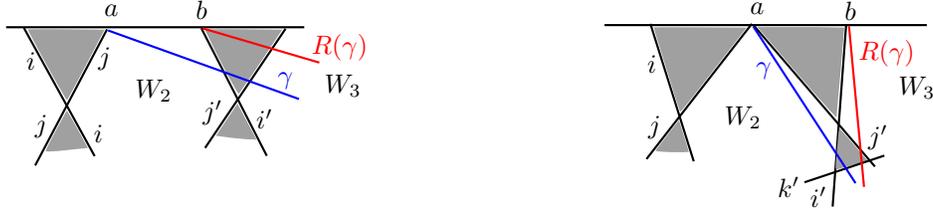


FIGURE 22. Cases (1.1) and (2.3) in the proof of Lemma A.12.

Lemma A.12. *Let γ be an arc in \mathcal{S} such that the crossing sequence for γ consists of $n - 1$ steps that go forward from 1 to n . Let $x = [x_1 \ 0 \ \dots \ 0]^T \in \ker f_\gamma$, where x_1 is a nonzero path in Q . Then $x \in \text{Im } \bar{f}_{R(\gamma)}$.*

Proof. We need to consider the following cases separately, see Figure 21. These cases represent all possible starting configurations for γ and $R(\gamma)$. The starting point a of γ is a boundary point of the checkerboard polygon \mathcal{S} , and the boundary edge incident to it in clockwise direction is either in a white region W_2 (case 1) or in a shaded region (case 2). The terminal point b of $R(\gamma)$ is the other endpoint of this edge.

Recall that an edge between i, j in a crossing pair represents an arrow $i \rightarrow j$ or $j \rightarrow i$. The crossing sequence for γ will start in one of the following ways.

- $(i_1 - j_1), (i_2 - j_2), \dots$ or
 $(i_0), (i_1 - j_1), (i_2 - j_2), \dots$ or
 $(j_0), (i_1 - j_1), (i_2 - j_2), \dots$

The path x_1 is an element of the first indecomposable summand $P(j')$ of $P_1(\gamma)$. Thus $s(x_1) = j' = j_1$ in the first two cases and $s(x_1) = j' = j_0$ in the third case.

Now we distinguish eight difference cases in the crossing sequences for γ and $R(\gamma)$, where the first three come from case 1 and the remaining five come from case 2 as in Figure 21. Here recall that, because the direction of γ and $R(\gamma)$ changes, it follows that if (i, j) is a crossing pair for γ , and $R(\gamma)$ also crosses i, j , then (j, i) is a crossing pair of $R(\gamma)$.

Cases The crossing sequences start with

- (1.1) $(i' \rightarrow j')$ for γ and (j') for $R(\gamma)$, where $i' \rightarrow j'$ is a boundary arrow.
(1.2) (i) for γ and $(i \rightarrow j)$ for $R(\gamma)$, where $i \rightarrow j$ is a boundary arrow.
(1.3) $(i_1 - j_1)$ for γ and $(j_1 - i_1)$ for $R(\gamma)$, where $i_1 - j_1$ is an interior arrow coming from W_2 .
(2.1) (i) for γ and $(j'), (i \rightarrow j)$ for $R(\gamma)$, where $i \rightarrow j$ is a boundary arrow.
(2.2) $(i_1 - j_1)$ for γ and $(j'), (j_1 - i_1)$ for $R(\gamma)$, where $i_1 - j_1$ is an interior arrow coming from W_2 .
(2.3) $(i' \rightarrow k')$ for γ and $(k' \rightarrow j')$ for $R(\gamma)$, where there is a 3-cycle $i' \rightarrow k' \rightarrow j'$ with $j' \rightarrow i'$ a boundary arrow.
(2.4) $(i'), (i_1 - j_1)$ for γ and $(j_1 - i_1)$ for $R(\gamma)$, where $i_1 - j_1$ is either an interior arrow coming from W_3 or the other boundary arrow coming from W_3 , provided W_3 has a boundary edge.
(2.5) $(i'), (i_1 - j_1)$ for γ and (j_1) for $R(\gamma)$, where $i_1 - j_1$ is the other boundary arrow from W_3 , provided W_3 has no boundary edge.

We only provide the details of the proof in the cases (1.1), (2.3), and (2.4), because the other cases are proved in a similar way.

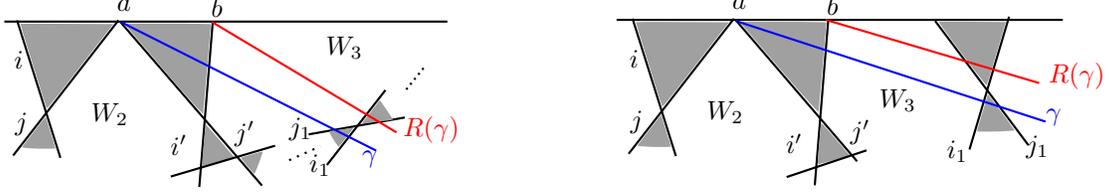


FIGURE 23. Case (2.4) in the proof of Lemma A.12.

Case (1.1) In this case the arc γ starts in vertex a , enters the white region W_2 , then crosses i', j' and enters the white region W_3 . Then the corresponding $R(\gamma)$ starts in vertex b , crosses only j' and enters region W_3 , see Figure 22.

By assumption $0 = f_\gamma(x)$, which means that the composition of the valid path $i' \rightarrow j'$ with the path x_1 is zero in B . Hence x_1 is a path $j' \rightsquigarrow c$ starting in j' and ending in some vertex c . Since $x_1 \neq 0$, Lemma A.9(b) shows that the path $j' \rightsquigarrow c$ starts with a path $j' \rightsquigarrow i_2 \rightsquigarrow z$, and there is a boundary arrow ∂_2 starting at z as in the quiver below. The black portion of the quiver lies in $Q(\gamma)$ and the red portion does not. Moreover, ∂_1 is a boundary arrow.

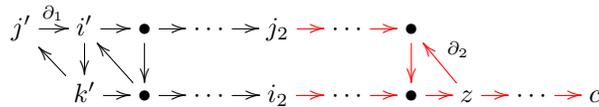


The arc $R(\gamma)$ does not cross i' , so the map $\bar{f}_{R(\gamma)} = \begin{bmatrix} j' \rightsquigarrow i_2 & \dots \\ -(j_2 \rightsquigarrow i_2) & \dots \\ 0 & \dots \\ \vdots & \vdots \end{bmatrix}$ has only two valid paths in the first column as all steps in γ are assumed to be forward. Thus

$$\bar{f}_{R(\gamma)} \begin{bmatrix} i_2 \rightsquigarrow z \rightsquigarrow c \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} j' \rightsquigarrow i_2 \rightsquigarrow z \rightsquigarrow c \\ -(j_2 \rightsquigarrow i_2 \rightsquigarrow z \rightsquigarrow c) \\ 0 \\ \vdots \\ 0 \end{bmatrix} = x,$$

because the path $j_2 \rightsquigarrow i_2 \rightsquigarrow z$ is zero as ∂_2 is a boundary arrow. This shows that $x \in \text{Im } \bar{f}_{R(\gamma)}$ and proves the lemma in this case.

Case (2.3) This case is special, since it is the only one that involves the diagonal k' , see Figure 22. By assumption $0 = f_\gamma(x)$, which means that the composition of the valid path $i' \rightarrow k'$ with x_1 is zero in B . Hence x_1 is a path $k' \rightsquigarrow c$ starting in k' and ending in some vertex c . Consider the following subquiver of Q , where ∂_1, ∂_2 are boundary arrows. The black portion lies in $Q(\gamma)$ and the red portion does not. Then by Lemma A.9, either $k' \rightsquigarrow c$ factors through i_2 and z , and then we proceed as in case (1.1), or $k' \rightsquigarrow c$ factors through j' .



The arc $R(\gamma)$ crosses j', k' but not i' , so the map $\bar{f}_{R(\gamma)} = \begin{bmatrix} k' \rightsquigarrow j' & \dots \\ 0 & \dots \\ \vdots & \vdots \end{bmatrix}$ has only one valid path in the first column. Then $\bar{f}_{R(\gamma)} \begin{bmatrix} j' \rightsquigarrow c \\ 0 \\ \vdots \\ 0 \end{bmatrix} = x$, which shows that $x \in \text{Im } \bar{f}_{R(\gamma)}$ and proves the lemma in this case.

Case (2.4) In this case, the arc γ starts in vertex a crosses i' and moves into region W_3 , where it crosses a pair i_1, j_1 . Then $R(\gamma)$ starts in vertex b and then crosses i_1, j_1 in W_3 , see Figure 23. On the left, we show the case when the arrow between i_1, j_1 is an interior arrow of W_3 , and on the right, we show the case when the arrow between i_1, j_1 is a boundary arrow of W_3 . In the latter case, we must have $j_1 \rightarrow i_1$, because white regions have an even number of sides, by Lemma 3.25, and i', j_1 must have the same parity along the edges

of W_3 , by Remark A.2. In the former case, the arrow between i_1 and j_1 may be in either direction. We have the following subquiver of Q , where ∂_1 is a boundary arrow.

$$\begin{array}{ccccccc} j' & \rightarrow & \bullet & \rightarrow & \cdots & \rightarrow & i_1 \\ \partial_1 \downarrow & \swarrow & \downarrow & & & & \\ i' & \rightarrow & \bullet & \rightarrow & \cdots & \rightarrow & j_1 \end{array}$$

Then x_1 is a path $j_1 \rightsquigarrow c$ and by assumption $0 = f_\gamma(x) = \begin{bmatrix} i' \rightsquigarrow j_1 \rightsquigarrow c \\ i_1 \rightsquigarrow j_1 \rightsquigarrow c \\ 0 \\ \vdots \\ 0 \end{bmatrix}$. We only prove the case when there

is an arrow $i_1 \xrightarrow{\alpha} j_1$, since the other case when there is an arrow $j_1 \rightarrow i_1$ follows similarly. Then $i_1 \rightarrow j_1$ is not a boundary arrow, and by Lemma A.9 the vanishing of $i_1 \rightarrow j_1 \rightsquigarrow c$ implies that $j_1 \rightsquigarrow c$ factors through i_1 or z , where z is either above or to the right of j_1 as in the quiver below. However, the latter case is not possible because $f_\gamma(x) = 0$ implies that the valid path $i' \rightsquigarrow j_1$ composed with x_1 is also zero.

$$\begin{array}{ccccccc} & & & & c & & \\ & & & & \uparrow & & \\ & & & & \vdots & & \\ & & & & \uparrow & & \\ & & & & z & \rightarrow & \bullet \\ & & & & \uparrow & \searrow & \uparrow \\ & & & & \vdots & & \vdots \\ & & & & \uparrow & & \uparrow \\ & & & & i_2 & & j_2 \\ & & & & \uparrow & & \uparrow \\ & & & & \vdots & & \vdots \\ & & & & \uparrow & & \uparrow \\ j' & \rightarrow & \bullet & \rightarrow & \cdots & \rightarrow & \bullet & \rightarrow & i_1 & \rightarrow & \cdots & \rightarrow & \bullet \\ \partial_1 \downarrow & \swarrow & \downarrow & & & & \swarrow & \downarrow & & & & & \downarrow & \searrow & \partial_2 \\ i' & \rightarrow & \bullet & \rightarrow & \cdots & \rightarrow & \bullet & \rightarrow & j_1 & \rightarrow & \cdots & \rightarrow & \bullet & \rightarrow & z & \rightarrow & \cdots & \rightarrow & c \end{array}$$

We have $\bar{f}_{R(\gamma)} = \begin{bmatrix} j_1 \rightsquigarrow i_1 & -(j_1 \rightsquigarrow i_2) & \cdots \\ 0 & j_2 \rightsquigarrow i_2 & \cdots \\ \vdots & 0 & \cdots \\ \vdots & \vdots & \ddots \\ \vdots & \vdots & \ddots \end{bmatrix}$. If $j_1 \rightsquigarrow c$ factors through i_1 then $\bar{f}_{R(\gamma)} \begin{bmatrix} i_1 \rightsquigarrow c \\ 0 \\ \vdots \\ 0 \end{bmatrix} = x$. If $j_1 \rightsquigarrow c$

factors through i_2 then $\bar{f}_{R(\gamma)} \begin{bmatrix} 0 \\ i_2 \rightsquigarrow c \\ 0 \\ \vdots \\ 0 \end{bmatrix} = x$, because $j_2 \rightsquigarrow i_2 \rightsquigarrow c$ is zero as ∂_2 is a boundary arrow. This completes the proof in case (2.4). \square

Lemma A.13. *Let γ be an arc in \mathcal{S} such that the crossing sequence for γ consists of $n - 1$ steps that go forward from 1 to n . Let $x = [x_1 \cdots x_k \ 0 \cdots 0]^T \in \ker f_\gamma$, where x_k is a nonzero path in Q starting in j_k , for $k \geq 2$. Then x_k factors through i_k or i_{k+1} , if $k \neq n$.*

Proof. Let f_γ be given by a matrix $A = (a_{h,l})$. Let A_l denote the l -th row of A . Because all the steps in the crossing sequence for γ are forward $a_{h,l} = 0$ for $h < l$. In particular, in A_k the first $k - 1$ entries $a_{k,1}, a_{k,2}, \dots, a_{k,k-1}$ are zero. Then $A_k x = a_{k,k} x_k$, where $a_{k,k}$ is a path $i_k \rightsquigarrow j_k$. We have $a_{k,k} x_k = 0$, because $x \in \ker f_\gamma$ by assumption.

The path $x_k : j_k \rightsquigarrow c$ is ending in some vertex c in Q and $a_{k,k} x_k$ is a path $i_k \rightsquigarrow j_k \rightsquigarrow c$ in Q that is zero in the algebra B . We consider two cases depending on whether $a_{k,k}$ is given by a single arrow or a composition of two arrows.

zero, the path x_k must go through a 3-cycle that contains a boundary arrow that is not on x_k . Thus, the quiver implies that x_k must factor through i_{k+1} , a contradiction.

Thus $a_{k-1,k}x_k \neq 0$ and $A_{k-1}x$ is given by equation (A.1). By Lemma A.10, the path x_k starts with the arrow α , which contradicts the assumption that x_k does not factor through i_k . This proves case (ii). \square

Lemma A.14. *Let W be a white region. Label its edges w_1, \dots, w_{2m} in clockwise order such that w_1 and w_2 meet in the interior of the polygon \mathcal{S} and w_{2m}, w_1 meet at the boundary vertex. In particular, if W has a boundary edge then it is the edge w_{2m} . Let γ be an arc with endpoint in W .*

- (1a) *If the endpoint of γ is incident to the edge w_1 , then any crossing of γ with an even-indexed edge is of degree 0, and any crossing with an odd-indexed edge is of degree 1.*
- (1b) *If the endpoint of γ is not incident to the edge w_1 , then W has a boundary edge $2m$, any crossing of γ with an even-indexed edge is of degree 1, and any crossing with an odd-indexed edge is of degree 0.*
- (2) *The full subquiver $Q(W)$ of Q is given below, where ∂_1, ∂_2 denote boundary arrows of Q and depending on whether W has a boundary edge (right) or not (left).*

$$\begin{array}{ccc}
 w_1 \leftarrow w_3 \leftarrow \cdots \leftarrow w_{2m-1} & & w_1 \leftarrow w_3 \leftarrow \cdots \leftarrow w_{2m-3} \leftarrow w_{2m-1} \\
 \partial_1 \downarrow \nearrow & \downarrow & \partial_1 \downarrow \nearrow & \downarrow & \nearrow & \downarrow & \nearrow \\
 w_2 \leftarrow w_4 \leftarrow \cdots \leftarrow w_{2m} & & w_2 \leftarrow w_4 \leftarrow \cdots \leftarrow w_{2m-2} & & & & \partial_2
 \end{array}$$

Proof. Parts (1a) and (1b) follow from the fact that the direction of the edges alternates around the white region, by Lemma 3.25. Part (2) follows from the construction, see for example Figure 14. \square

We are ready to prove the complementary statement to Lemma A.11.

Lemma A.15. *Let $\gamma, R(\gamma)$ be a pair of compatible arcs in \mathcal{S} . Then $\ker f_\gamma \subset \text{Im } \bar{f}_{R(\gamma)}$.*

Proof. Let $x \in \ker f_\gamma$. First we prove the lemma in the case when all steps in the crossing sequence for γ are forward. Let k be the largest integer such that the k -th position in x is nonzero. We proceed by induction on k , and we outline the main steps of the proof below.

Induction on k :

$k = 1$ Apply Lemma A.12 to conclude the base case.

$k > 1$ (a) Write $x = x' + x_1$ with

- $x', x_1 \in \ker f_\gamma$, and
- x_1 has one less term in position k than x and is zero in positions greater than k , and
- $x' \in \text{Im } \bar{f}_{R(\gamma)}$.

(b) Repeat with x_1 , where $x_1 = x'_1 + x_2$, and continue until we obtain $x_p \in \ker f_\gamma$ such that x_p is zero in positions greater or equal to k . By induction $x_p \in \text{Im } \bar{f}_{R(\gamma)}$, so

$$x = x_p + x'_1 + x'_2 + \cdots + x'_{p-1} \in \text{Im } \bar{f}_{R(\gamma)}.$$

Therefore, we need to show the base case $k = 1$ and part (a) of the inductive step. Then applying the argument in part (b) completes the proof.

If $k = 1$ then x is nonzero only in the first position. The first entry in x is $\sum_{c \in Q_0} \lambda_c(j \rightsquigarrow c)$, a linear combination of paths starting in vertex j , where j is the first vertex of degree 1 in the crossing sequence for γ . Moreover, we may assume that the sum runs over distinct vertices $c \in Q_0$, because parallel paths are equal in B , by Proposition 3.31. Since $x \in \ker f_\gamma$, we obtain that the vector $[\lambda_c(j \rightsquigarrow c) \ 0 \ \cdots \ 0]^T \in \ker f_\gamma$, for all c . By Lemma A.12, every such vector is contained in $\text{Im } \bar{f}_{R(\gamma)}$, so by linearity we obtain that $x \in \text{Im } \bar{f}_{R(\gamma)}$. This completes the proof in the case $k = 1$.

Part (a). Suppose $k > 1$. Similarly to the base case, the k -th entry of x is a linear combination of paths starting in j_k , because the k -th column of f_γ consists of paths ending in j_k . Moreover, we may assume again that the sum runs over paths ending in distinct vertices of Q . Let $w : j_k \rightsquigarrow c$ be one of the nonzero paths in the position k of x . By Lemma A.13, the path w factors through i_k or i_{k+1} , if $k \neq n$. We will prove the case when $k = n$ and w does not factor through i_n later (call this case (c)), and now we assume that w factors through i_k . The case when w factors through i_{k+1} is similar.

Then $w = w'u : j_k \rightsquigarrow^{w'} i_k \rightsquigarrow^u c$. Let $s < k$ be the largest integer such that there is a valid path from i_s to j_{s+2} and, if no such s exists, we let $s = 1$. This choice of s can be seen in the matrix of f_γ as the largest row index for which there exists a non-zero entry on the third diagonal. Let $r \leq s$ be the largest row index

in f_γ such that there exists a valid path $i_r \rightsquigarrow j_{s+2}$. Assuming γ crosses j_1 , the matrix of f_γ is as follows, where the entry in position (p, q) is labeled \star if there is a valid path from i_p to j_q in that position and $p \neq q$. For $p = q$ the symbol \star denotes a path that is either a single arrow or a composition of two arrows in the same 3-cycle. If γ does not cross j_1 then one needs to remove the first column of this matrix; we call this case (d) and deal with it later.

$$\begin{array}{cccccccccccc}
 & 1 & & & r & & s & & s+2 & & k-2 & & k \\
 1 & \star & \star & \cdots & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 \\
 & 0 & \ddots & \ddots & \cdots & \vdots & & & \vdots & \vdots & \vdots & \vdots & \vdots \\
 r-1 & & 0 & \star & \star & 0 & 0 & \cdots & 0 & 0 & & \vdots & \vdots \\
 r & & & & 0 & \star & \star & \star & \cdots & \star & 0 & \vdots & \vdots \\
 & & & & & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\
 s & & & & & & 0 & \star & \star & \star & 0 & \vdots & \vdots \\
 s+1 & & & & & & & 0 & \star & \star & 0 & \vdots & \vdots \\
 s+2 & & & & & & & & 0 & \star & \star & 0 & \vdots \\
 & & & & & & & & & \ddots & \ddots & \ddots & \vdots \\
 k-2 & & & & & & & & & & 0 & \star & \star & 0 \\
 k-1 & & & & & & & & & & & 0 & \star & \star \\
 k & & & & & & & & & & & & 0 & \star
 \end{array}$$

Moreover, for each $h \in \{s+1, \dots, k-3\}$ the step in the crossing sequence for γ from $h+1$ to $h+2$ cannot be rectangular, because in that case, Lemma A.7(1a) would imply that there is a valid path $i_h \rightsquigarrow j_{h+2}$ or $i_{h+1} \rightsquigarrow j_{h+3}$. Then all these steps from $s+2$ to $k-1$ must be trapezoidal, and Lemma A.7(2) implies that there is a valid path $j_{s+1} \rightsquigarrow i_k$, so part (3) of the same lemma yields valid paths $w'_h : j_h \rightsquigarrow i_k$ for all $h \in \{s+1, \dots, k-1\}$.

Define

$$x' = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ (-1)^{k-s-1} w_{s+1} \\ \vdots \\ w_{k-2} \\ -w_{k-1} \\ w_k \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

where w_k is our path $w : j_k \rightsquigarrow^{w'} i_k \rightsquigarrow^u c$ and $w_h := w'_h u = j_h \rightsquigarrow^{w'_h} i_k \rightsquigarrow^u c$ for $h \in \{s+1, \dots, k\}$. In particular, all these paths end in the same subpath u .

Let the entries in the matrix f_γ be denoted by $a_{i,j}$. Then by construction

$$f_\gamma(x') = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ (-1)^{k-s-1} (a_{r,s+1} w_{s+1} - a_{r,s+2} w_{s+2}) \\ \vdots \\ (-1)^{k-s-1} (a_{s+1,s+1} w_{s+1} - a_{s+1,s+2} w_{s+2}) \\ (-1)^{k-s-2} (a_{s+2,s+2} w_{s+2} - a_{s+2,s+3} w_{s+3}) \\ \vdots \\ -(a_{k-1,k-1} w_{k-1} - a_{k-1,k} w_k) \\ a_{k,k} w_k \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

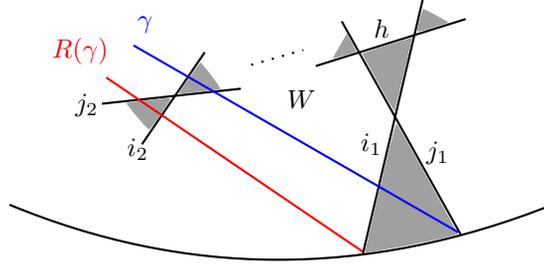


FIGURE 24. Case (d) in the proof of Lemma A.15.

where $a_{k,k}w_k : i_k \rightsquigarrow j_k \rightsquigarrow i_k \rightsquigarrow c$ is a path that includes an oriented cycle from i_k to i_k and hence it is zero by Proposition 3.30. All other entries are differences of parallel paths and hence zero by Proposition 3.31. Thus $x' \in \ker f_\gamma$.

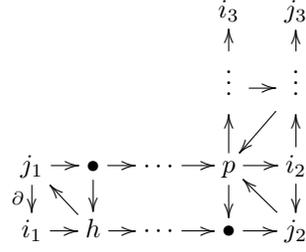
Let $x_1 = x - x'$, then $x_1 \in \ker f_\gamma$, since both $x, x' \in \ker f_\gamma$. Moreover, x_1 has one less term in position k than x and is zero in positions greater than k . We will show that $x' \in \text{Im } \bar{f}_{R(\gamma)}$. Let $y' = [0 \dots 0 u 0 \dots 0]^T$, where u is our path $u : i_k \rightsquigarrow c$ and it sits in the l -th row in y' where l is such that the l -th column of $\bar{f}_{R(\gamma)}$ consists of valid paths ending in i_k . Thus, the l -th column of $\bar{f}_{R(\gamma)}$ is

$$\begin{bmatrix} 0 \\ \vdots \\ 0 \\ \pm w'_{s+1} \\ \vdots \\ \pm w'_{k-1} \\ \mp w'_k \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

where the first s entries are zero, because from the definition of s we know there is a valid path $i_s \rightsquigarrow j_{s+2}$, thus there is no valid path $j_s \rightsquigarrow i_{s+2}$, by Lemma A.6, and thus there is no valid path $j_s \rightsquigarrow i_k$, by Lemma A.7(3). Moreover, the signs in the column alternate, by definition of $\bar{f}_{R(\gamma)}$. Therefore, $\bar{f}_{R(\gamma)}(y') = \pm x'$, so $x' \in \text{Im } \bar{f}_{R(\gamma)}$. This shows that $x = x' + x_1$ where x', x_1 satisfy the three conditions of part (a) in the outline of the proof. This completes the proof of part (a) excluding the case when $k = n$ and w does not factor through i_n (case (c)) and excluding the case when γ does not cross j_1 (case (d)).

Case (d). Suppose that γ does not cross j_1 . The map f_γ has the same structure as before, except we remove the first column. If $s > 1$, or $s = 1$ and there is a valid path $i_1 \rightsquigarrow j_3$ then the same argument as above goes through, so suppose $s = 1$ and there is no valid path from i_1 to j_3 . Then the crossing sequence for γ starts with $(i_1), (i_2, j_2), \dots$, so γ starts in a shaded boundary region given by the diagonals i_1, j_1 , then crosses the diagonal i_1 , thereby entering a white region W , and then γ crosses i_2, j_2 exiting W , see Figure 24. Note that if instead we drew γ starting at the endpoint of i_1 , then moving to the right and crossing j_1 , then j_1 would cross γ from left to right. In particular, $P(j_1)$ would be a summand of $P_1(\gamma)$, contrary to the assumption that the crossing sequence of γ starts with a vertex corresponding to a summand of $P_0(\gamma)$. Hence, Figure 24 illustrates the only possibility for γ under the given assumptions.

The quiver $Q(W, \gamma)$ is the full subquiver of Q whose vertices correspond to the edges of the white region W moving from i_1 to i_2, j_2 in the counterclockwise direction. Since i_1 is the first edge in W it has precisely two neighbors j_1 and h in $Q(W, \gamma)$, whereas j_1 is the second edge in W and therefore it has precisely three neighbors in $Q(W, \gamma)$. Since $s = 1$ and there is no valid path from i_1 to j_3 , we have the following subquiver of $Q(\gamma)$, where ∂ is a boundary arrow.

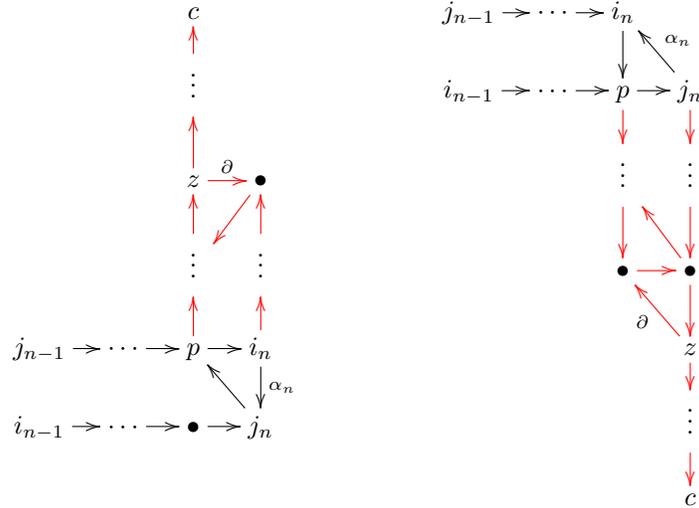


Define $x' = \begin{bmatrix} \pm w_2 \\ \vdots \\ -w_{k-1} \\ w_k \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, where w_k are as before. Then $f_\gamma(x') = \begin{bmatrix} \pm(i_1 \rightsquigarrow j_2)w_2 \\ \pm(i_2 \rightsquigarrow j_2)w_2 \mp (i_2 \rightsquigarrow j_3)w_3 \\ \vdots \\ -(i_{k-1} \rightsquigarrow j_{k-1})w_{k-1} + (i_{k-1} \rightsquigarrow j_k)w_k \\ (i_k \rightsquigarrow j_k)w_k \\ 0 \\ \vdots \\ 0 \end{bmatrix}$.

The first entry is zero, because the path $w_2 = j_2 \rightarrow p \rightarrow i_2 \rightsquigarrow c$, and the path from i_1 to p is zero, since ∂ is a boundary arrow. The proof that the other entries are zero is similar to the previous case. Moreover, $x' = f_{R(\gamma)}(y')$ with y' as before. This completes the proof in case (d).

Case (c). It remains to consider the case where $k = n$ and w does not factor through i_n . There are two possibilities depending on whether γ crosses i_n or not.

First, suppose that γ crosses i_n . By the same argument as in the proof of Lemma A.13, we are in one of the following situations, because $w : j_n \rightsquigarrow^{\bar{w}} z \rightsquigarrow^{\bar{u}} c$ does not factor through i_n . The two situations below depend on the direction of the arrow α_n between i_n and j_n . Here ∂ denotes a boundary arrow.



It suffices to show that $R(\gamma)$ crosses z , because then $\bar{f}_{R(\gamma)} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \bar{u} \end{bmatrix} = x'$. Suppose not. Then γ crosses i_n, j_n ,

enters a white region W and ends on the boundary of W , because (i_n, j_n) is the last pair in the crossing sequence for γ . Label the edges of W as in the statement of Lemma A.14. Because $R(\gamma)$ does not cross z , then w_{2m} is a boundary segment, the endpoint of γ is incident to the edges w_{2m} and w_{2m-1} , and $R(\gamma)$ has endpoint incident to the edges $2m$ and 1 . By Lemma A.14(1b), j_n is an edge of W with even index, say $j_n = w_{2h}$, and i_n is an edge with odd index. If $\alpha_n : i_n \rightarrow j_n$ then $i_n = w_{2h-1}$ and if $\alpha_n : j_n \rightarrow i_n$ then $i_n = w_{2h+1}$, by Lemma A.14(2). Moreover, since z is a source of a boundary arrow, we have $z = w_1$, and thus z lies on the same side as i_n in $Q(W)$, meaning there is a valid path $i_n = w_{2h+1} \rightarrow w_{2h-1} \rightarrow \dots \rightarrow w_3 \rightarrow w_1 = z$. This contradicts the two quivers above and completes the proof of part (c) when γ crosses i_n .

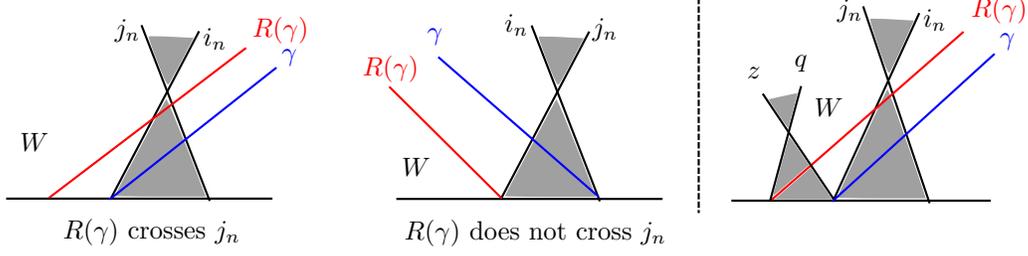


FIGURE 25. Case (c) in the proof of Lemma A.15.

To show the remaining part of (c), suppose γ does not cross i_n and we still have that $k = n$. Then γ ends in a shaded boundary region formed by the diagonals i_n, j_n . Since γ and $R(\gamma)$ both cross j_n , by Proposition 3.34(b), we have the left picture and not the middle one in Figure 25. In particular, $R(\gamma)$ crosses i_n .

Then the arrow α_n is oriented $\alpha_n : j_n \rightarrow i_n$, and the quiver is shown above on the right. Then

$$f_\gamma = \begin{bmatrix} \ddots & \vdots & \vdots & \vdots \\ 0 & i_{n-2} \rightsquigarrow j_{n-2} & i_{n-2} \rightsquigarrow j_{n-1} & \vdots \\ 0 & 0 & i_{n-1} \rightsquigarrow j_{n-1} & i_{n-1} \rightsquigarrow j_n \end{bmatrix}$$

where \star is either 0 or a valid path $i_{n-2} \rightsquigarrow j_n$ if such a path exists. The last row of x contains our nonzero path $w : j_n \rightsquigarrow c$. So looking at the last row of f_γ , either

$$(A.2) \quad (i_{n-1} \rightsquigarrow j_n)w = 0$$

or there exists a path $w_{n-1} : j_{n-1} \rightsquigarrow c$ in the $(n-1)$ -st row of x such that

$$(A.3) \quad (i_{n-1} \rightsquigarrow j_{n-1})w_{n-1} = (i_{n-1} \rightsquigarrow j_n)w.$$

If w factors through i_n , then the same argument as in the general case applies, since $R(\gamma)$ crosses i_n . If equation (A.3) is satisfied then, by Lemma A.10, the path w factors through i_n and we are done. Otherwise, equation (A.2) is satisfied, and w does not factor through i_n . Then w factors through z , $w = j_n \rightsquigarrow z \rightsquigarrow c$ as in the previous quiver on the right, where ∂ is a boundary arrow.

Let $x' = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ w \end{bmatrix}$, then $f_\gamma(x') = \begin{bmatrix} \vdots \\ (i_{n-2} \rightsquigarrow j_n)w \\ (i_{n-1} \rightsquigarrow j_n)w \end{bmatrix} = 0$, because all paths $i_h \rightsquigarrow j_n$ factor through p and $(p \rightarrow j_n)w = 0$. We will show that $R(\gamma)$ crosses z . Indeed, z is the starting point of a boundary arrow ∂ , and the number of 3-cycles between the two boundary arrows ∂ and α_n is even. Therefore, the corresponding white region W has no boundary edge, by Lemma A.14(2). Thus, we can complete our previous picture as in Figure 25 on the right.

In particular, $R(\gamma)$ crosses $z = i_{n+1}$. Then $\pm x' = \bar{f}_{R(\gamma)} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \bar{u} \end{bmatrix} \in \text{Im } \bar{f}_{R(\gamma)}$, as the last column of $\bar{f}_{R(\gamma)}$ contains only one nonzero entry $\pm(j_n \rightarrow i_{n+1})$ in the last position. This completes the proof in case (c).

This finishes the argument of the inductive step and completes the proof of the lemma in the case when the crossing sequence for γ consists of all forward steps. On the other hand, if none of the steps are forward then the proof is very similar using a dual argument working with the first nonzero entry of $x \in \ker f_\gamma$ instead of the last nonzero entry.

Finally, if some of the steps are forward and some are not, we can start at a position t where the step from $t-1$ to t is forward followed by a non-forward step from t to $t+1$. The matrix of f_γ at row t contains a single valid path $i_t \rightsquigarrow j_t$ and otherwise has the following shape, where all entries not marked by stars are zero.

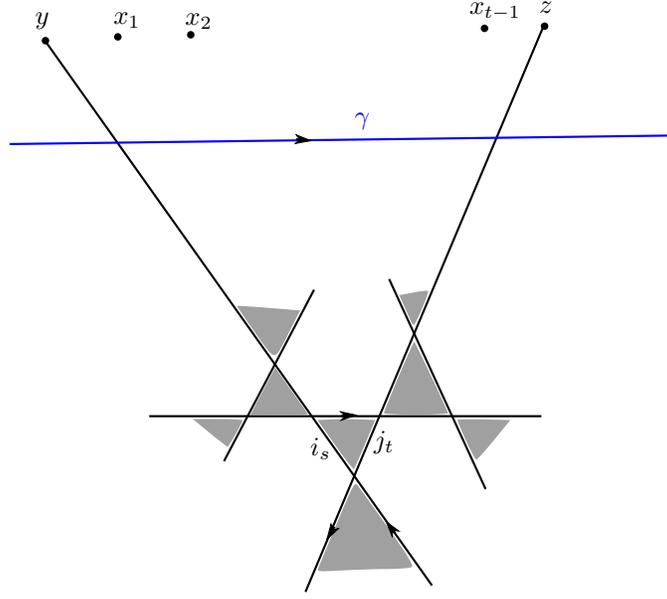
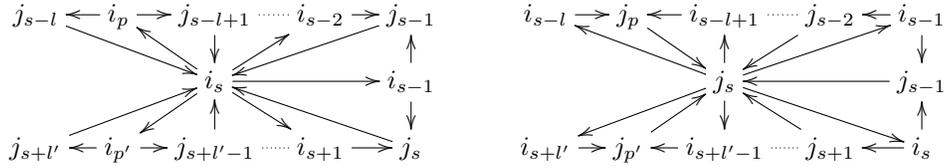


FIGURE 26. Proof of Lemma A.17.

Lemma A.18. *Suppose (i_s, j_s) is a crossing pair for γ such that there is an arrow $j_s \rightarrow i_s$. If the path $i_s \rightsquigarrow j_s$ factors through some other i , or some other j respectively, in the crossing sequence for γ , then there exist l, l' and p, p' such that their quivers $Q(\gamma, [i_s])$, or $Q(\gamma, [j_s])$ respectively, are as follows.*



In particular, the quivers $Q(\gamma, [i_s]), Q(\gamma, [j_s])$ have connected dual graphs and consist of an even number of 3-cycles with i 's and j 's alternating around the boundary of the quivers. Moreover, $p \leq s - l$ is the least positive integer such that there is an arrow $i_s \rightarrow i_p$ or $j_p \rightarrow j_s$ and $p' \geq s + l'$ is the largest integer such that there is an arrow $i_s \rightarrow i_{p'}$ or $j_{p'} \rightarrow j_s$ respectively.

Proof. Suppose (i_s, j_s) is a crossing pair for γ such that there is an arrow $j_s \rightarrow i_s$, and the path $i_s \rightsquigarrow j_s$ factors through some i_r . Then there is a 3-cycle $i_s \rightarrow i_r \rightarrow j_s$ in Q . Without loss of generality we may assume $r < s$. This situation is illustrated in the left picture in Figure 27, where by assumption γ crosses i_s, j_s, i_r .

As we move in the direction of γ , if the crossing of γ with i_r occurs after the crossing of γ with i_s, j_s then i_r crosses γ from left to right. This means that i_r is a degree 1 crossing for γ , which is a contradiction. Therefore, the crossing of γ and i_r occurs prior to the crossing of γ with i_s, j_s .

Now, we claim that the arrow $i_s \rightarrow i_r$ is not a boundary arrow. If $i_s \rightarrow i_r$ were a boundary arrow, then the segment with endpoints x_1, x_2 , as in Figure 27, would be a boundary segment of \mathcal{S} . Since γ crosses i_r before crossing i_s, j_s , it would follow that γ starts in x_1 or some other point counterclockwise from x_1 . If γ starts in x_1 then it has the same starting point as i_s , so i_s cannot be in the crossing sequence for γ . If γ starts in another point counterclockwise from x_1 then γ crosses i_s twice, which is also a contradiction. This shows that $i_s \rightarrow i_r$ is not a boundary arrow.

Then we have the following subquiver of Q .

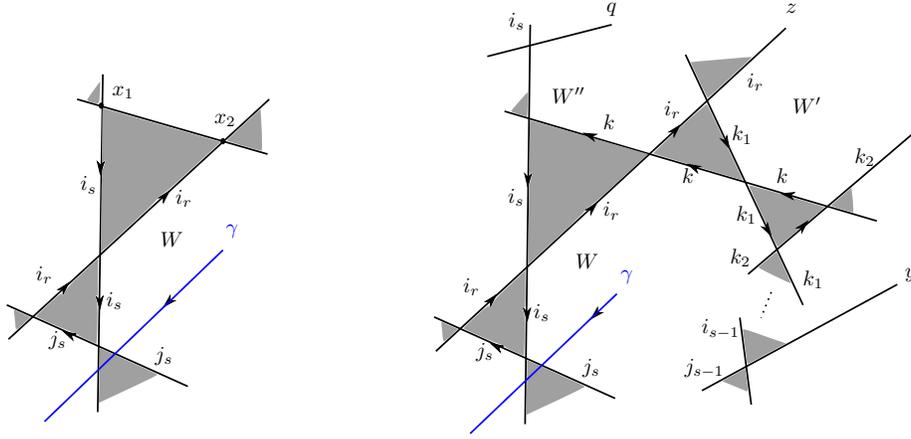
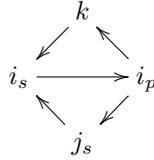


FIGURE 27. Proof of Lemma A.18.



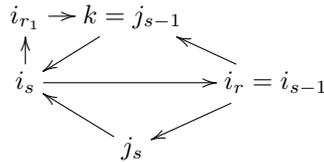
We claim that $k = j_{s-1}$. This case is illustrated in Figure 27 on the right.

Let W denote the white region containing j_s, i_s, i_r, k on the boundary. If γ crosses k as it enters W then we obtain $k = j_{s-1}$ as claimed. Otherwise, γ crosses some other pair i_{s-1}, j_{s-1} that lie clockwise from k along the boundary of W . For example, i_{s-1}, j_{s-1} may equal k_1, k_2 respectively. In the figure we label the arcs for convenience so that there is an arrow $i_{s-1} \rightarrow j_{s-1}$, but this will not be important for the proof if we interchange the labels of these arcs. Let W' denote the white region containing k_2, k, k_1, i_r . As we move along j_{s-1} in the direction of its orientation we note that if j_{s-1} crosses i_r then we obtain that W' is an interior white region, which is not possible. Therefore, j_{s-1} and i_r do not cross. Let z, y denote the endpoints of i_r, j_{s-1} respectively. Then y is to the right of i_r . Moreover, since γ crosses i_{s-1}, j_{s-1} , then γ is to the right of i_r so it cannot cross i_r , which is a contradiction. This shows the claim that $k = j_{s-1}$.

If there are no other arrows $i_s \rightarrow i_{s'}, i_s \rightarrow j_{s'}$ with $s' < s$ and $i_{s'}, j_{s'}$ in the crossing sequence of γ , then $l = 1, r = p$ and the lemma holds. Note that the steps from r to s are forward by Lemma A.17 as there is an arrow $i_r \rightarrow j_s$.

Now, suppose that there is some other arrow $i_s \rightarrow q$ with $q \in \{i_{s'}, j_{s'}\}$ for $s' < s$. Then γ crosses q prior to crossing i_r and j_{s-1} . We claim that in this case $i_r = i_{s-1}$. Suppose to the contrary $i_r \neq i_{s-1}$. We have $j_{s-1} = k$ and γ crosses (k_1, k) as it enters the white region W , see the picture above. Moreover, we observe that k_1 and q cannot cross, otherwise the white region W''' with edges k_1, i_r, k, i_s would be interior. But this means that γ would have to cross k_1 again before it can cross q , a contradiction. This shows that $i_r = i_{s-1}$.

Now, given that there is some other arrow $i_s \rightarrow q$ with $q \in \{i_{s'}, j_{s'}\}$ for $s' < s$, we claim that there exists i_{r_1} in the crossing sequence for γ with $r_1 < s - 1$ and such that i_{r_1}, i_s, j_{s-1} are in the same 3-cycle. That is, we want to show that we have the following quiver.



Consider Figure 28. We see that in order for γ to cross q , it would have to cross h as well, otherwise the white region W'''' would have to be interior. Moreover, h is of degree 0 in the crossing sequence for γ so

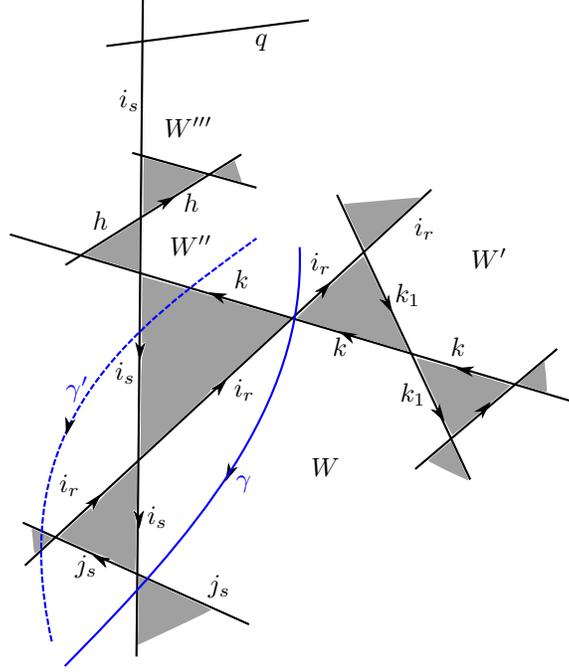


FIGURE 28. Proof of Lemma A.18.

$h = i_{r_1}$ for some $r_1 < s - 1$. This shows the claim that i_{r_1}, i_s, j_{s-1} are in the crossing sequence for γ and form a 3-cycle.

Finally, we can homotope γ to obtain a new arc γ' that has the same crossing sequence as γ except we replace the crossing pairs $(i_s, j_s), (i_{s-1}, j_{s-1})$ in γ by $(i_{s-1}, j_s), (i_s, j_{s-1})$ in γ' . Now we can continue the same argument as before replacing γ with γ' and i_s, j_s with i_s, j_{s-1} . In this way we obtain the desired part of the quiver $Q(\gamma, [i_s])$ coming from the crossings of γ before the pair (i_s, j_s) . Moreover, if $i_s \rightsquigarrow j_s$ factors through another $i_{r'}$ with $i_{r'} \neq i_r$ and $r' > s$, then we can proceed in the same way as above analyzing the crossings of γ after it crosses (i_s, j_s) . This shows the lemma. \square

Lemma A.19. *Let γ be a representative of a 2-diagonal in \mathcal{S} , and let $f_\gamma : \bigoplus_{h=1}^n P(j_h) \rightarrow \bigoplus_{h=1}^m P(i_h)$ be the associated morphism. Suppose we have a commutative diagram*

$$\begin{array}{ccc} \bigoplus_{h=1}^n P(j_h) & \xrightarrow{f_\gamma} & \bigoplus_{h=1}^m P(i_h) \\ \downarrow g_j & & \downarrow g_i \\ \bigoplus_{h=1}^n P(j_h) & \xrightarrow{f_\gamma} & \bigoplus_{h=1}^m P(i_h) \end{array}$$

for some matrices $g_j = (a_{h,h'})$ and $g_i = (b_{h,h'})$. Then $f_{\gamma_{s,t}} = 0$ or $b_{s,s} = a_{t,t}$ for all s, t .

Proof. By commutativity we have $g_i f_\gamma = f_\gamma g_j$ as matrices. The diagonal entries $a_{t,t}, b_{s,s}$ in g_j, g_i are elements in the field \mathbb{k} and off diagonal entries are non-constant paths $j_h \rightsquigarrow j_{h'}$ or $i_h \rightsquigarrow i_{h'}$. Note that the statement of the lemma is about the diagonal entries only. Denote f_γ simply by f , and consider position (s, t) in this composition

$$\sum_h b_{s,h} f_{h,t} = \sum_h f_{s,h} a_{h,t}$$

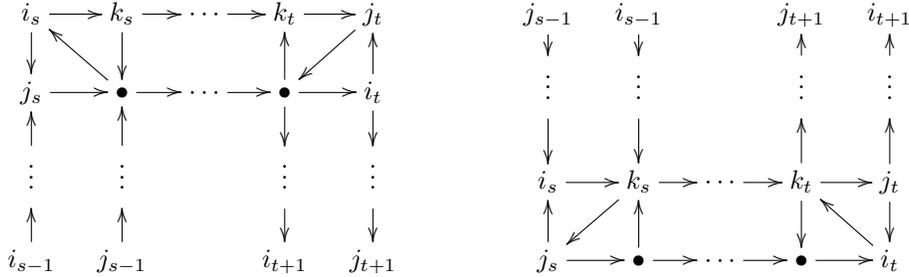
where we index the position (s, t) in f so that it corresponds to a path from i_s to j_t . Then we get

$$(A.4) \quad b_{s,s} f_{s,t} + \sum_{h \neq s} b_{s,h} f_{h,t} = f_{s,t} a_{t,t} + \sum_{h \neq t} f_{s,h} a_{h,t}.$$

If $f_{s,t} = 0$ then the conclusion holds, thus suppose $f_{s,t} \neq 0$. Then by definition of the map $f = f_\gamma$ all steps from s to t are forward and there is a valid path $i_s \rightsquigarrow j_t$, or $s = t$ and there is a nonzero path $i_s \rightsquigarrow j_s$. Without loss of generality we can assume $t \geq s$.

If the path $i_s \rightsquigarrow j_t$ does not factor through any other $i_h, j_{h'}$ then equation (A.4) becomes $b_{s,s}f_{s,t} = f_{s,t}a_{t,t}$ with $f_{s,t} \neq 0$ and the lemma holds. In particular, if the path $i_s \rightsquigarrow j_t$ is just a single arrow then it is valid, so $f_{s,t} \neq 0$, and it does not factor nontrivially through any other $i_h, j_{h'}$ and the lemma holds. Therefore, we can assume that the path $i_s \rightsquigarrow j_t$ is a composition of two or more arrows.

Now, suppose $f_{s,t} \neq 0$ and the path $i_s \rightsquigarrow j_t$ factors through some i_h or $j_{h'}$ or both with $h \neq s, h' \neq t$. Then we claim that $i_s \rightarrow i_h$ or $j_{h'} \rightarrow j_t$ or both of these paths consist of a single arrow. If $s = t$ then the claim clearly holds as $f_{s,s}$ consists of at most two arrows. If $s \neq t$, then $f_{s,t}$ is a valid path. If $h, h' \in \{s+1, \dots, t-1\}$ then $i_s \rightsquigarrow i_h \rightsquigarrow j_t$ or $i_s \rightsquigarrow j_{h'} \rightsquigarrow j_t$ is not valid because $i_s \rightsquigarrow i_h, j_{h'} \rightsquigarrow j_t$ is not valid by Lemma A.8. Therefore, $h, h' \notin \{s, \dots, t\}$. Let $Q(\gamma, [p, q])$ denote the subquiver of $Q(\gamma)$ determined by the subsequence $(i_p, j_p), \dots, (i_q, j_q)$ of the crossing sequence for γ . We know that step from s to $s+1$ is forward, and if the step from $s-1$ to s is not forward, then the two subquivers $Q(\gamma, [1, s]), Q(\gamma, [s, t])$ of $Q(\gamma)$ only have two vertices i_s, j_s connected by a single arrow in common. In particular, in this case the path $i_s \rightsquigarrow j_t$ cannot factor through any $i_h, j_{h'}$ for $h, h' < s$. This is a contradiction, so the step from $s-1$ to s must be forward. A similar argument implies that the step from t to $t+1$ must also be forward. Then we have the following possibilities for $Q(\gamma)$, where in the first picture there are arrows $i_s \rightarrow j_s, i_t \rightarrow j_t$ and in the second one the arrows go in the opposite direction.



In the first case, we see that $i_s \rightsquigarrow j_t$ does not pass through any vertex outside of $Q(\gamma, [s, t])$. Therefore, this path cannot factor through any other $i_h, j_{h'}$ with $h, h' < s$ or $h, h' > t$, a contradiction. Then we must be in the situation as in the quiver on the right. Here, the path $i_s \rightsquigarrow j_t$ factors through k_s, k_t which are the only vertices of $Q(\gamma, [s, t])$ that can also be part of $Q(\gamma, [1, s])$ or $Q(\gamma, [t, \max(m, n)])$ apart from i_s, j_s or i_t, j_t respectively, and where m, n are as in the statement of the lemma.

The mixed cases when there are arrows $i_s \rightarrow j_s, j_t \rightarrow i_t$ or $j_s \rightarrow i_s, i_t \rightarrow j_t$ follow in the same way, and here the path $i_s \rightsquigarrow j_t$ factors through one of k_t or k_s respectively. This proves the claim that if $i_s \rightsquigarrow j_t$ factors through i_h or $j_{h'}$ or both, then $i_s \rightarrow i_h$ or $j_{h'} \rightarrow j_t$ or both of these paths respectively consist of a single arrow.

We are now ready to prove the lemma. Suppose first $s \neq t$ and $i_s \rightsquigarrow j_t$ is a valid path that factors through both k_s, k_t such that γ crosses k_s, k_t . In addition, we can take $k_s \neq k_t$. Thus, here we consider the most general situation, and when γ crosses only one of k_s, k_t then it becomes a special case of the one we have. By the above we also know that all steps from $s-1$ to $t+1$ must be forward. Note that if k_s is a $j_{h'}$ for $h' < s$ in the crossing sequence for γ , then the arrow $i_s \rightarrow j_{h'}$ implies that the steps from s to h' are forward by Lemma A.17. Since $k_s \neq k_t$ this is a contradiction, so the crossing between k_s and γ is in degree zero. We consider the quiver $Q(\gamma, [i_s])$. By Lemma A.18, we have the following situation, where $l \geq 1$ is such that the path $i_s \rightsquigarrow j_{s-l}$ factors through only one other vertex i_p in the crossing sequence of γ , $p \leq s-l$, and all steps from p to s are forward. Moreover, $k_s = i_{s-1}$ if $l > 2$ and $k_s = i_p$ for some $p \leq s-1$ if $l = 1$, and $k_t = j_{h'}$ for some $h' > t$.

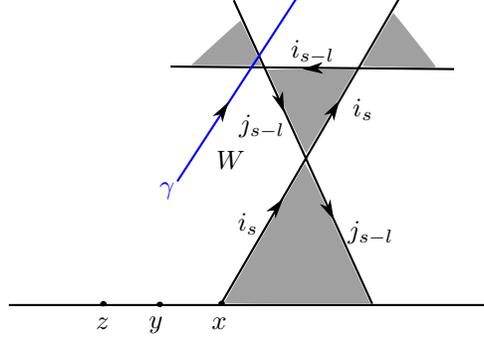
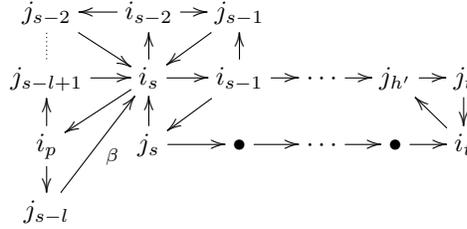


FIGURE 29. Proof of Lemma 4.16.



The position (s, t) in the matrix equation $g_i f_\gamma = f_\gamma g_j$ yields the following equation.

$$(s, t) \quad b_{s,s} f_{s,t} + b_{s,s-1} f_{s-1,t} = f_{s,t} a_{t,t} + f_{s,h'} a_{h',t}$$

In order to prove the lemma it suffices to show that $b_{s,s-1} = a_{h',t} = 0$. We will show that $b_{s,s-1} = 0$ and the remaining part $a_{h',t} = 0$ follows dually.

We also have the following sets of equations coming from the other positions in the matrix equation (A.4). They correspond to paths in the quiver $Q(\gamma, [i_s])$ starting at i_s and ending at $j_s, j_{s-1}, \dots, j_{s-l}$ respectively. Except for the first and the last path here all of them factor through exactly two other i 's in the crossing sequence for γ . For example, there are exactly two different paths in Q starting at i_s and ending at j_{s-1} , which are $i_s \rightarrow i_{s-1} \rightarrow j_{s-1}$ and $i_s \rightarrow i_{s-2} \rightarrow j_{s-1}$. This means that in the equation $(s, s-1)$ there are four terms because a path $i_s \rightsquigarrow j_{s-1}$ factors through four vertices that are in the crossing sequence for γ , namely $i_s, i_{s-1}, i_{s-2}, j_{s-1}$.

$$\begin{aligned} (s, s) & \quad b_{s,s} f_{s,s} + b_{s,s-1} f_{s-1,s} = f_{s,s} a_{s,s} \\ (s, s-1) & \quad b_{s,s} f_{s,s-1} + b_{s,s-1} f_{s-1,s-1} + b_{s,s-2} f_{s-2,s-1} = f_{s,s-1} a_{s-1,s-1} \\ & \quad \vdots \\ (s, s-l+1) & \quad b_{s,s} f_{s,s-l+1} + b_{s,s-l+1} f_{s-l+1,s-l+1} + b_{s,p} f_{p,s-l+1} = f_{s,s-l+1} a_{s-l+1,s-l+1} \\ (s, s-l) & \quad b_{s,s} f_{s,s-l} + b_{s,p} f_{p,s-l} = f_{s,s-l} a_{s-l,s-l} \end{aligned}$$

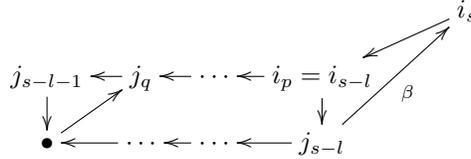
In equation $(s, s-l)$ we have $f_{s,s-l} = 0$ because $p \leq s-l < s$ and the steps from p to s are forward and $f_{p,s-l} \neq 0$ because there exists an arrow $i_p \rightarrow j_{s-l}$. If β is not a boundary arrow, then the path $i_s \rightarrow i_p \rightarrow j_{s-l}$ is nonzero in the algebra B , and we conclude that $b_{s,p} = 0$. Now, consider equation $(s, s-l+1)$. We have that $f_{s,s-l+1} = 0$ because the steps from $s-l+1$ to s are forward. Since $b_{s,p} = 0$ we conclude that $b_{s,s-l+1} = 0$. Continuing in this way we obtain that $b_{s,s-2} = 0$ from equation $(s, s-2)$. Then equation $(s, s-1)$ implies the desired claim that $b_{s,s-1} = 0$. This completes the proof of the lemma in the case when β is not a boundary arrow.

Now, suppose β is a boundary arrow. First, we want to show that $b_{s,s-l} = 0$. Since the step from $s-l$ to $s-l+1$ is forward and β is boundary we must have that $i_p = i_{s-l}$. We want to show that γ crosses j_{s-l-1} . Suppose not. Then i_{s-l}, j_{s-l} is the first crossing pair for γ and we are in the situation of Figure 29.

Let W be the white region that γ traverses before crossing i_{s-l}, j_{s-l} . Let x be a boundary point of the polygon \mathcal{S} in W where i_s starts, and let y, z be neighboring boundary points of \mathcal{S} clockwise from x as in the picture. Since γ also crosses i_s , it cannot start at x . Because j_{s-l} crosses γ from left to right, it follows

that γ cannot start at the point y . Therefore, γ starts in z or at some point clockwise from z . If γ starts at some other point clockwise from z then it crosses another pair of arcs in W before crossing i_{s-l}, j_{s-l} , which is a contradiction to (i_{s-l}, j_{s-l}) being the first crossing pair for γ . Then γ starts in z . If the white region W has no edges on the boundary of \mathcal{S} , then the boundary edge with endpoints y, z lies in a shaded region. In particular, γ crosses another pair of arcs before crossing i_{s-l}, j_{s-l} , which is again a contradiction. Then W has an edge on the boundary with endpoints x, y , and the edge with endpoints z, y bounds a triangular shaded region. Moreover, the arc attached to y crosses γ from left to right, which means that this arc is j_{s-l-1} in the crossing sequence for γ . This shows the claim.

Then we have the following quiver, which is an extension of the quiver $Q(\gamma, [i_s])$ given earlier. Note that it may happen that the path $i_{s-l} \rightarrow j_{s-l-1}$ factors through some other j_q with $q < s-l-1$, and here we depict the most general situation.



The equation coming from position $(s, s-l-1)$ corresponds to paths in the quiver starting in i_s and ending in j_{s-l-1} . Because there is a valid path from i_s to j_{s-l-1} then it does not pass through any relations, and so it is the unique path nonzero path in B between these vertices. The path $i_s \rightsquigarrow j_{s-l-1}$ factors through exactly four vertices $i_s, i_{s-l}, j_q, j_{s-l-1}$ that are in the crossing sequence for γ . This means that the equation $(s, s-l-1)$ contains four terms as shown below.

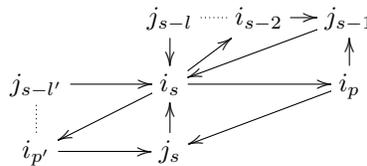
$$(s, s-l-1) \quad b_{s,s}f_{s,s-l-1} + b_{s,s-l}f_{s-l,s-l-1} = f_{s,s-l-1}a_{s-l-1,s-l-1} + f_{s,q}a_{q,s-l-1}$$

Here $f_{s,s-l-1} = f_{s,q} = 0$ because the steps from s to $s-l$ are not forward. Then we obtain $b_{s,s-l}f_{s-l,s-l-1} = 0$, where $f_{s-l,s-l-1} \neq 0$ because it is a valid path and the step from $s-l$ to $s-l-1$ is forward. Therefore, we still obtain that $b_{s,s-l} = 0$ when β is a boundary arrow. Then we can continue as before to conclude that $b_{s,s-1} = 0$ from equations $(s, s-l), \dots, (s, s-1)$. This proves the lemma in the case $s \neq t$.

Now, suppose $s = t$, $f_{s,s} \neq 0$ and the path $i_s \rightsquigarrow j_s$ factors through some other i or j . If the path $i_s \rightsquigarrow j_s$ factors through one or two i 's, say $i_p, i_{p'}$ then we obtain the following equation.

$$(s, s) \quad b_{s,s}f_{s,s} + b_{s,p}f_{p,s} + b_{s,p'}f_{p',s} = f_{s,s}a_{s,s}$$

We then conclude that $b_{s,p}, b_{s,p'}$ are zero in the same manner as in the case $s \neq t$ above. First, we construct the quiver $Q(\gamma, [i_s])$, see below. Then we use equations $(s, s-1), \dots, (s, s-l)$ and equations $(s, s+1), \dots, (s, s+l')$ to show that $b_{s,p}, b_{s,p'}$ are zero respectively.



If instead the path $i_s \rightsquigarrow j_s$ factors through one or two j 's then again we obtain the desired result by considering the quiver $Q(\gamma, [j_s])$ and applying the dual argument.

Finally, if $i_s \rightsquigarrow j_s$ factors through i_p and j_q then to show that $b_{s,p} = 0$ we consider the subquiver $Q(\gamma, [i_s])$. Then to show that $a_{q,s} = 0$ we consider the subquiver $Q(\gamma, [j_s])$. This completes the proof of the lemma in all cases. \square

The following statement is a direct consequence of the proof of Lemma 4.16 above, which we will need later in Section C.2.3. Note that here we assume that the diagonal entries of the matrices g_i, g_j are zero.

Corollary A.20. *Let γ be a representative of a 2-diagonal in \mathcal{S} , and let $f_\gamma : \bigoplus_{h=1}^n P(j_h) \rightarrow \bigoplus_{h=1}^m P(i_h)$ be the associated morphism. Suppose we have a commutative diagram*

$$\begin{array}{ccc} \bigoplus_{h=1}^n P(j_h) & \xrightarrow{f_\gamma} & \bigoplus_{h=1}^m P(i_h) \\ \downarrow g_j & & \downarrow g_i \\ \bigoplus_{h=1}^n P(j_h) & \xrightarrow{f_\gamma} & \bigoplus_{h=1}^m P(i_h) \end{array}$$

for some nilpotent morphisms g_j and g_i . Then $(g_i f_\gamma)_{s,t} = 0$ for all s, t such that $(f_\gamma)_{s,t} \neq 0$.

APPENDIX B. PROOFS OF SECTION 5

Here we prove Theorem 5.5 and extend the definition of pivot morphisms to complexes of projectives in Theorem B.7 which will be used in later sections.

Theorem B.1 (Theorem 5.5). *Let γ, γ' be compatible 2-diagonals in \mathcal{S} such that γ' is obtained from γ by a 2-pivot and let (g_0^r, g_1^c) be the morphism defined above. Then we have the following commutative diagram with exact rows*

$$(B.1) \quad \begin{array}{ccccccc} P_1 & \xrightarrow{f_\gamma} & P_0 & \xrightarrow{\pi_\gamma} & M_\gamma & \longrightarrow & 0 \\ g_1^c \downarrow & & \downarrow g_0^r & & \downarrow g & & \\ P'_1 & \xrightarrow{f_{\gamma'}} & P'_0 & \xrightarrow{\pi_{\gamma'}} & M_{\gamma'} & \longrightarrow & 0 \end{array}$$

where g is the induced morphism on the cokernels. In particular, g is a morphism of syzygies in CMP B.

Proof. It suffices to show the commutativity on a pair of indecomposable summands $P(j)$ of P_1 and $P(i)$ of P'_0 . Thus without loss of generality, we may assume that $P_1 = P(j)$ and $P'_0 = P(i)$. By definition, the pivot morphism g_1^c is a column vector $g_1^c: P(j) \rightarrow P_0$ whose components are either the identity morphism $1_{P(j)}$, multiplication by arrows $h \rightarrow j$ ending in j , or equal to zero. Moreover, if a component is $1_{P(j)}$ then all other components are zero, and if no component is $1_{P(j)}$ then $P(j)$ is not a direct summand of P'_1 , so by Lemma 5.3, there may be up to two arrows $h_1 \rightarrow j$ and $-h_2 \rightarrow j$ in g_1^c and all other components are zero.

Similarly, the map g_0^r is a row vector, $g_0^r: P_0 \rightarrow P(i)$ whose entries are $0, 1_{P(i)}$ or arrows $i \rightarrow \ell$. Moreover, if an entry is $1_{P(i)}$ then all other entries are zero, and if no entry is $1_{P(i)}$ then $P(i)$ is not a summand of P_0 , there may be up to two arrows $i \rightarrow \ell_1, -i \rightarrow \ell_2$ in g_0^r and all other entries are zero.

We use the notation and orientation of Figure 8. Thus a is the common endpoint of γ and γ' and the 2-pivot move is from vertex x to vertex z , skipping vertex y . Recall that the vertex y , as any boundary vertex, is the endpoint of exactly one or two radical lines, by Lemma 3.12(c).

Also note that $P(j)$ is not a summand of P'_0 , because even if the radical line $\rho(j)$ were to cross both γ and γ' then these crossings would have the same degree, and thus the projective $P(j)$ would have the same degree in either of the two-term complexes of f_γ and $f_{\gamma'}$, and, in that case, $P(j)$ would be a summand of P'_1 but not of P'_0 . A similar argument shows that $P(i)$ is not a summand of P_1 .

We consider several cases determined by the geometric configuration in the pivot area of the polygon \mathcal{S} .

(1) Suppose first that $P(j)$ is not a summand of P'_1 and $P(i)$ is not a summand of P_0 . Thus the radical line $\rho(j)$ crosses γ from left to right but does not cross γ' . This implies that $\rho(j)$ must end at vertex z . Similarly, the radical line $\rho(i)$ crosses γ' from right to left but does not cross γ , and therefore $\rho(i)$ must end at vertex x .

(1a) Suppose there are two radical lines $\rho(h_1)$ and $\rho(h_2)$ at y and both of them cross γ or both of them cross γ' . We may assume without loss of generality that both cross γ , see the left picture in Figure 30. In this case, the map g_1^c is zero and g_0^r is the row vector

$$g_0^r = [0 \quad \cdots \quad 0 \quad i \rightarrow h_1 \quad -i \rightarrow h_2].$$

We need to show $g_0^r f_\gamma = 0$. Consider the white region W incident to the crossing point of $\rho(i)$ and $\rho(h_1)$ and to the right of $\rho(i)$. Since every white region has an even number of sides, there must be some radical lines missing in the left picture in Figure 30. Moreover, an additional radical line cannot eliminate the vertex y from the region W , because every white region has at least one boundary vertex, by Lemma 3.12(a). We therefore must have a radical line $\rho(\ell)$ that eliminates the crossing point between $\rho(i)$ and $\rho(h_2)$ from W , see

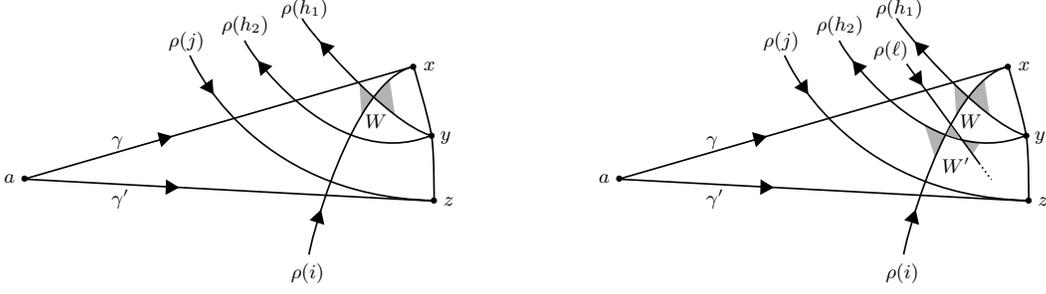
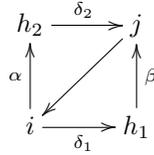


FIGURE 30. Proof of Theorem 5.5.

the right hand picture in Figure 30. Note that the orientation of $\rho(\ell)$ must be as indicated in the figure, since the orientation of the edges of W must alternate around the white region, by Lemma 3.25. Furthermore, $\rho(\ell)$ must end at vertex z , because otherwise, it would cross $\rho(j)$ and γ' and thereby create a white region W' without boundary vertex, which is impossible. Thus $\rho(\ell)$ ends at z .

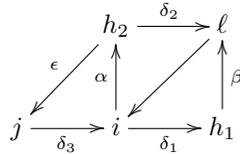
(i) Suppose first that $\rho(\ell)$ is equal to $\rho(j)$. Then the white region W has four sides, $\rho(h_2), \rho(j), \rho(i)$ and $\rho(h_1)$ and its cycle path is $\mathfrak{c}(W) = h_2 \rightarrow j \rightarrow i \rightarrow h_1$. Therefore the quiver Q contains the following subquiver

(B.2)



with δ_1, δ_2 boundary arrows. Now consider the map $f_\gamma: P(j) \rightarrow P_0$. The subquiver above implies that the arrows δ_2 and β are valid paths and hence nonzero components of f_γ by Lemma A.17. Recall that, on the other hand, the nonzero components of g_0^r are the arrows α and δ_1 . Therefore, we have $g_0^r f_\gamma = \delta_1 \beta - \alpha \delta_2 = 0$. Since g_1^c is zero, this completes the commutativity in this case.

(ii) Now suppose that $\rho(\ell) \neq \rho(j)$. Since the white region W' must have a boundary vertex, this vertex must be z and the sides of W' in clockwise order are $\rho(j), \rho(i), \rho(h_2)$ and $\rho(\ell)$. Therefore its cycle path in Q is $\mathfrak{c}(W') = j \xrightarrow{\delta_3} i \xrightarrow{\alpha} h_2 \xrightarrow{\delta_2} \ell$. In particular, the subpath $\delta_3 \alpha$ lies in an oriented cycle whose third arrow we denote by ϵ . Thus the regions W, W' give rise to the following subquiver of Q with δ_1, δ_2 and δ_3 boundary arrows.



It follows that the arrow ϵ is a valid path and that there is no valid path from h_1 to j . Therefore, we have $g_0^r f_\gamma = \alpha \epsilon$ which is zero since δ_3 is a boundary arrow. This completes the proof in case (1a).

(1b) Suppose there are two radical lines $\rho(h_1)$ and $\rho(h_2)$ at vertex y , one crossing γ and the other crossing γ' , see Figure 31. Then by the same argument as in case (1a), the white region W has four sides $\rho(h_2), \rho(j), \rho(i)$ and $\rho(h_1)$ in clockwise order. Thus again we have the subquiver (B.2) In this situation however, both g_0^r and g_1^c are nonzero. The only nonzero component of g_0^r is the arrow δ_1 and the only nonzero component of g_1^c is the arrow δ_2 . On the other hand, the arrow α is a valid path and hence a component of $f_{\gamma'}$, and the arrow β is a component of f_γ . Thus we have $f_{\gamma'} g_1^c = \alpha \delta_2$ and $g_0^r f_\gamma = \delta_1 \beta$. These paths are equal by Lemma 3.29(b), and this completes the proof in case (1b).

(1c) Suppose there is exactly one radical line $\rho(h)$ at y . Then $\rho(h)$ must cross either γ or γ' . These two cases are similar, and we only prove one of them. Suppose therefore that $\rho(h)$ crosses γ , see Figure 32. In

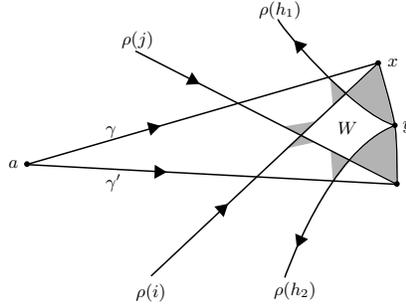


FIGURE 31. Proof of Theorem 5.5 case (1b).

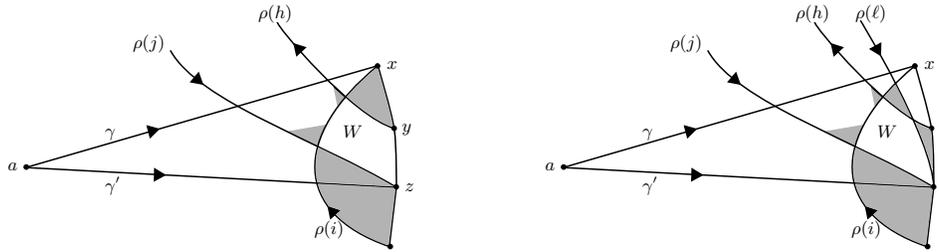
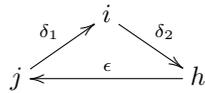


FIGURE 32. Proof of Theorem 5.5 case (1c).

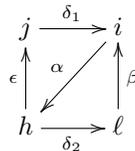
this situation the map $g_1: P(j) \rightarrow P'_1$ is zero, by definition. Thus we must show $g_0 f_\gamma = 0$. Let W be the white region incident to the crossing point of $\rho(i)$ and $\rho(h)$ and to the right of $\rho(i)$. Since there is only one radical line at vertex y , the region W must contain the vertex z , and the arrow $\delta_1: j \rightarrow i$ is a boundary arrow in Q . Thus the radical line $\rho(i)$ must start at the clockwise neighbor of z on the boundary of \mathcal{S} .

(i) Suppose first that the arrow $i \rightarrow h$ is also a boundary arrow and denote it by δ_2 , see the left picture in Figure 32. Then the white region W has 4 sides and hence its cycle path is $\mathbf{c}(W) = j \xrightarrow{\delta_1} i \xrightarrow{\delta_2} h$, with δ_1, δ_2 boundary arrows in Q . Moreover, since $\mathbf{c}(W)$ is a subpath of a chordless cycle in Q , there must be an arrow $\epsilon: h \rightarrow j$. Thus the radical lines $\rho(h)$ and $\rho(j)$ must cross; this crossing is not shown in Figure 32. Thus Q contains the following subquiver



In this situation, the map f_γ is given by the arrow ϵ and the map g_0^r by the arrow δ_2 . Therefore we have $g_0^r f_\gamma = \delta_2 \epsilon$, which is zero, because δ_1 is a boundary arrow.

(ii) Now suppose that the arrow $i \rightarrow h$ is not a boundary arrow, see the right picture in Figure 32. Then there is a radical line $\rho(\ell)$ that crosses $\rho(h)$ and $\rho(i)$ forming a triangular shaded region that corresponds to a chordless cycle in Q . The white region W has 4 sides and its cycle path is $\mathbf{c}(W) = j \xrightarrow{\delta_1} i \xrightarrow{\alpha} h \xrightarrow{\delta_3} \ell$, with boundary arrows δ_1 and δ_3 . Moreover, the paths $\delta_1 \alpha$ and $\alpha \delta_3$ lie in two different chordless cycles in Q . Hence Q has the following subquiver



In this situation, we still have $g_0^r f_\gamma = \alpha \epsilon$ which is zero because δ_1 is a boundary arrow. This completes the proof in case (1).

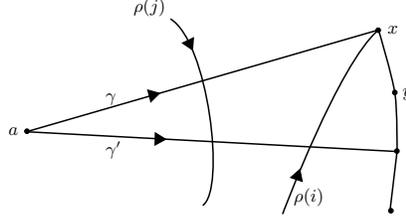


FIGURE 33. Proof of Theorem 5.5 case (2b).

(2) Suppose that $P(j)$ is a summand of P_1^r . Then $\rho(j)$ crosses both γ and γ' and the map g_1^c is the identity on the component $P(j) \rightarrow P(j)$ and zero on all components $P(j) \rightarrow P(\ell)$ with $\ell \neq j$.

(2a) Suppose $P(i)$ is a summand of P_0 . Then g_0^r is the identity on the component $P(i) \rightarrow P(i)$ and zero on all components $P(\ell) \rightarrow P(i)$. Therefore $g_0^r f_\gamma: P(j) \rightarrow P(i)$ is equal to f_γ and $f_{\gamma'} g_1^c: P(j) \rightarrow P(i)$ is equal to $f_{\gamma'}$. These two maps are equal since γ and γ' are compatible and each cross both $\rho(i)$ and $\rho(j)$.

(2b) Suppose $P(i)$ is not a summand of P_0 . Then the radical line $\rho(i)$ crosses γ' but not γ , see Figure 33.

(i) Suppose first that there is no radical line that starts at vertex y and crosses γ . Then $g_0^r: P_0 \rightarrow P(i)$ is zero and thus $g_0^r f_\gamma: P(j) \rightarrow P(i)$ is zero. On the other hand, $g_1^c: P(j) \rightarrow P_1^r$ is the identity on $P(j)$ and thus $f_{\gamma'} g_1^c$ is given by the component $f_{\gamma'}: P(j) \rightarrow P(i)$. So it suffices to show that there is no valid path from i to j in Q .

Using Lemma 3.12(b), we let W denote the unique white region containing the vertex x . If W lies to the left of $\rho(i)$, then, since there is no radical line that crosses γ and ends at y , γ' crosses $\rho(j)$ first and runs through W before crossing $\rho(i)$. Moreover, the cycle path $\mathbf{c}(W)$ starts at i and one of the maximal valid paths $\mathbf{v}(W)$ ends at i , since the cycle path goes clockwise around W and the maximal valid paths go counterclockwise. In particular, there is no valid path starting at i and going along the sides of W , because a valid path is the composition of valid paths along white regions. Hence there is no valid path from i to j and we are done.

On the other hand, if W lies to the right of $\rho(i)$, then there is a shaded region S containing x and lying to the left of $\rho(i)$. Thus S is a boundary region, since it contains x , and therefore it must also contain the counterclockwise neighbor of x on the boundary. Hence there exists a radical line $\rho(\ell)$ starting at the counterclockwise neighbor of x such that $\rho(i)$ crosses $\rho(\ell)$. Let W' be the white region incident to this crossing point and to the left of $\rho(i)$. Now the result follows from the argument above replacing W by W' .

(ii) Now suppose there is a radical line $\rho(h)$ that starts at vertex y and crosses γ . In this case, g_0^r is given by the arrow $\alpha: i \rightarrow h$, and $g_0^r f_\gamma$ is a the composition of α with a valid path from h to j . Recall that g_1^c is the identity on $P(j)$, and $f_{\gamma'} g_1^c$ is given by a valid path from i to j . So we must show that every valid path from i to j factors through the arrow α .

The radical lines $\rho(h)$ and $\rho(i)$ cross, and, from their orientations, we see that there exists a shaded region S adjacent to the crossing point that lies to the left of both $\rho(h)$ and $\rho(i)$. Let $\rho(\ell)$ be the third side of S . Let W be the white region adjacent to S via the edge $\rho(\ell)$. Then its cycle path is of the form $\mathbf{c}(W) = \dots \rightarrow h \rightarrow \ell \rightarrow i \rightarrow \dots$, and the arrow $\alpha: i \rightarrow h$ is a valid path.

Now, let u be a valid path from h to j and write $u = \beta u'$, where β is the initial arrow of u and u' is the remaining subpath. Then either $\alpha\beta$ is a subpath of a maximal valid path of the white region W , or γ lies in a different white region. In both cases, $\alpha\beta$ is not a subpath of a chordless cycle and hence $\alpha\beta u'$ is a valid path from i to j . By Proposition 3.31, every valid path from i to j is of this form. This completes the proof in case (2). \square

We will also need the following result, which provides the commutative diagram analogous to the one in Theorem 5.5 when replacing the maps $f_\gamma, f_{\gamma'}$ by their modifications $\bar{f}_\gamma, \bar{f}_{\gamma'}$ introduced in Section 4.2.

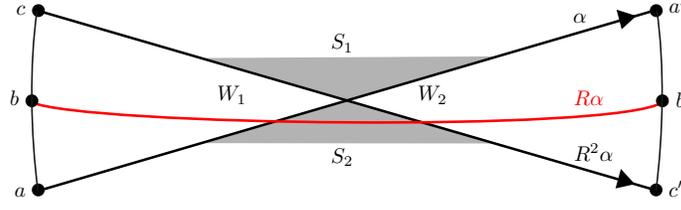


FIGURE 34. Proof of Lemma B.3.

Corollary B.2. *Let γ, γ' be compatible 2-diagonals in \mathcal{S} such that γ' is obtained from γ by a 2-pivot and let (g_0^c, g_1^r) be the morphism defined above. Then the following diagram commutes.*

$$\begin{array}{ccc} P_1 & \xrightarrow{\bar{f}_\gamma} & P_0 \\ g_1^r \downarrow & & \downarrow g_0^c \\ P'_1 & \xrightarrow{\bar{f}_{\gamma'}} & P'_0 \end{array}$$

Proof. Recall that, given a map $f : P_1 \rightarrow P_0$, the corresponding map \bar{f} is defined as $\bar{f} = J_{|P_0|} f J_{|P_1|}$, where J_k is an isomorphism that squares to the identity. Theorem 5.5 shows that $g_0^r f_\gamma = f_{\gamma'} g_1^c$ and we obtain,

$$\overline{g_0^r} \bar{f}_\gamma = J g_0^r J f_\gamma J = \overline{g_0^r} \bar{f}_\gamma = \overline{f_{\gamma'}} \overline{g_1^c} = J f_{\gamma'} J J g_1^c J = \bar{f}_{\gamma'} \overline{g_1^c}.$$

It therefore suffices to show that $\overline{g_1^c} = g_1^r$ and $\overline{g_0^c} = g_0^r$. The maps $\overline{g_1^c}, \overline{g_0^c}$ are obtained from g_1^c, g_0^r respectively, by introducing alternating signs in their matrices, see Remark 4.8(a). By Lemma 5.3 each of g_1^c, g_0^r has at most one nonzero off-diagonal entry, which implies the desired result. \square

B.1. The induced morphism of complexes on the projective resolutions. Our next goal is to give an explicit description of the complete projective resolution and the morphism of complexes induced by the pivot maps. Recall that if two representatives γ_1, γ_2 of the same 2-diagonal γ are not compatible then they give rise to two different morphisms $f_{\gamma_1}, f_{\gamma_2}$ in the projective presentation. Moreover, the sequence

$$P_2(\gamma) \xrightarrow{\bar{f}_{R\gamma}} P_1(\gamma) \xrightarrow{f_\gamma} P_0(\gamma)$$

is exact only if the arcs $R\gamma$ and γ are compatible, see Proposition 4.12. On the other hand, the commutativity of the pivot diagram in Corollary B.2 relies on the compatibility of the arcs γ and its pivot γ' . In this subsection, we will give an explicit construction that will control this double compatibility condition.

Since the syzygies in the projective resolution of M_γ correspond to the elements $R^i\gamma$ in the rotation orbit of γ , we shall need the following result about the distribution of radical lines in the rotation orbit.

Lemma B.3. *Let $\alpha, R\alpha, R^2\alpha$ be three consecutive 2-diagonals under rotation. If α and $R^2\alpha$ are radical lines then $R\alpha$ is a radical line as well.*

Proof. Say $\alpha = \rho(x)$ and $R^2\alpha = \rho(y)$ and let p denote their crossing point in \mathcal{S} . Recall that there are two shaded regions S_1, S_2 and two white regions W_1, W_2 at p . Also recall that the orientation of the radical lines is alternating along the white regions and directed along the shaded regions, see Lemma 3.25. Now, since α and $R^2\alpha$ have the same orientation, we see that one white region W_1 at p lies between the two incoming segments of α and $R^2\alpha$ and that the other white regions W_2 lies between the two outgoing segments of α and $R^2\alpha$, see Figure 34.

Because of our assumption that every chordless cycle in Q is of length 3, the shaded region S_1 is a triangle. Then either the third side of S_1 is a boundary edge of the polygon connecting the points c and a' , or there exists a radical line $\rho(z)$ that comes from W_1 , crosses $R^2\alpha$, then forms a boundary edge of S_1 and crosses α , and continues in W_2 . In particular, such a $\rho(z)$ must end at the points b and b' , because it cannot cross α or $R^2\alpha$ twice. Thus $\rho(z)$ is homotopic to $R\alpha$ and we are done. A similar analysis for the shaded region S_2 shows that the only remaining case is when both S_1 and S_2 have the third side on the boundary. In that case the polygon has exactly six vertices a, b, c, a', b', c' , and therefore it must be one of the two shown in the first row of Figure 12. In particular $R\alpha$ is a radical line. \square

Next, we need to characterize the compatibility of the arcs in the rotation orbit. This is done in the following two lemmata. The first lemma specifies how to choose compatible representatives of a 2-diagonal and its rotation if one of the two is a radical line. To illustrate the problem, consider the red arc γ in Figure 11. It is not compatible with the radical line $\rho(3)$, because the crossing sequence of γ is $(2, 1), (3, 4), (6, 5), (8, 7)$ and the one of $\rho(3)$ is either $(9), (1, 2)(4, 6), (5, 8), (7)$ or $(1, 9), (4, 2), (5, 6), (7, 8)$, depending on whether its representative runs on the left or the right of the radical line. Neither of the two crossing sequences of $\rho(3)$ is compatible with the crossing sequence of γ . In order to obtain a compatible representative, we need to deform γ homotopically so that the crossing sequence becomes either $(3, 1), (2, 4), (6, 5), (8, 7)$, where the crossing with 3 occurs at the very beginning, or $(2, 1), (6, 4), (8, 5), (3, 7)$, where the crossing with 3 occurs at the very end. We can describe the representative of the first crossing sequence by saying that γ starts at the boundary vertex labeled 18, then runs clockwise along the boundary of its first white region, crosses $\rho(3)$, then runs parallel to $\rho(3)$ until the white region that contains the other endpoint of γ and then ends by following the boundary of that white region until the boundary vertex labeled 9.

Lemma B.4. *Let γ be a representative of a 2-diagonal whose rotation $R\gamma$ is a radical line $\rho(x)$. Then γ and $R\gamma$ are compatible if γ starts by following the boundary of its first white region in clockwise direction, crosses $\rho(x)$ and then follows $R\gamma$ up to its last white region at the other end and then ends by following the boundary of that white region in clockwise direction.*

Proof. Suppose (y, z) is a crossing pair for γ and that $R\gamma$ crosses both $\rho(y)$ and $\rho(z)$. Since γ runs parallel to a subpath of $R\gamma$, the pair (y, z) also is in the crossing sequence of $R\gamma$. Thus γ and $R\gamma$ are compatible. \square

The next lemma will allow us to fix suitable representatives of the arcs $R^i\gamma$.

Lemma B.5. *We can choose representatives of the elements of the rotation orbit $\{R^i\gamma\}$ of γ such that for all i*

- (a) $R^{2i+1}\gamma$ is compatible with $R^{2i+2}\gamma$
- (b) There exists a homotopic deformation $(R^{2i+1}\gamma)'$ of $R^{2i+1}\gamma$ that is compatible with $R^{2i}\gamma$.

Proof. First note that if $R^j\gamma, R^{j+1}\gamma$ are two consecutive arcs in the rotation orbit and none of them is a radical line then we can always choose compatible representatives. Indeed, the only type of problem that we need to avoid can come from a radical line $\rho(x)$ that crosses one but not the other; say $\rho(x)$ crosses $R^{j+1}\gamma$ but not $R^j\gamma$. Then $\rho(x)$ shares an endpoint a with $R^j\gamma$ on the boundary of \mathcal{S} . To achieve compatibility, we only need to ensure that $R^{j+1}\gamma$ crosses $\rho(x)$ as close as possible to this point a and then follows $R^j\gamma$.

Suppose now that we have three consecutive arcs $R^j\gamma, R^{j+1}\gamma, R^{j+2}\gamma$, none of which is a radical line. By the argument above, it is possible to choose representatives such that $R^j\gamma$ and $R^{j+1}\gamma$ are compatible, and it is possible to choose representatives such that $R^{j+1}\gamma$ and $R^{j+2}\gamma$ are compatible. If j is even, this proves the statement. If j is odd however, we need to make sure that we can choose the two compatible pairs simultaneously. Therefore suppose that it is possible to apply an elementary homotopy to $R^{j+1}\gamma$. Thus $R^{j+1}\gamma$ crosses all four radical lines of an hourglass shape as in Figure 7. If $R^j\gamma$ and $R^{j+2}\gamma$ also cross all four sides of the hourglass then there is no problem. So the only obstruction would be if there is a radical line $\rho(x)$ that crosses $R^{j+1}\gamma$ but not $R^j\gamma$, so that we obtain a condition on the shape of $R^{j+1}\gamma$, and there is another radical line $\rho(y)$ that crosses $R^{j+1}\gamma$ but not $R^{j+2}\gamma$, so that we obtain a second condition on the shape of $R^{j+1}\gamma$, see Figure 35. In this situation, the arc $R^{j+1}\gamma$ would pass below the crossing point of $\rho(x)$ and $\rho(y)$ for compatibility with $R^j\gamma$, and it would pass above the crossing point for compatibility with $R^{j+2}\gamma$. Moreover, this difference would be a non-trivial homotopy if and only if there are two further radical lines $\rho(w), \rho(z)$ that cross all arcs involved and make an hourglass shape as shown in the figure. This, however, is impossible, because the radical lines $\rho(x)$ and $\rho(y)$ are oriented in the same direction, since the endpoint c is the second neighbor of the endpoint a on the boundary, and thus the orientation of the boundary segments of the shaded regions in the figure is not cyclic, contradicting Lemma 3.25. This completes the proof in the absence of radical lines in the rotation orbit.

Suppose now that a set of three consecutive arcs $\alpha, R\alpha, R^2\alpha$ contains a radical line. If there is exactly one radical line, there are three possibilities.

- (i) If α is the radical, we can deform $R\alpha$ to be compatible with α as explained in Lemma B.4. On the other hand, we can deform $R\alpha$ to be compatible with $R^2\alpha$, since none of the two is a radical line.
- (ii) If $R\alpha$ is the radical line, we can deform both α and $R^2\alpha$ to be compatible with $R\alpha$ as explained in Lemma B.4.

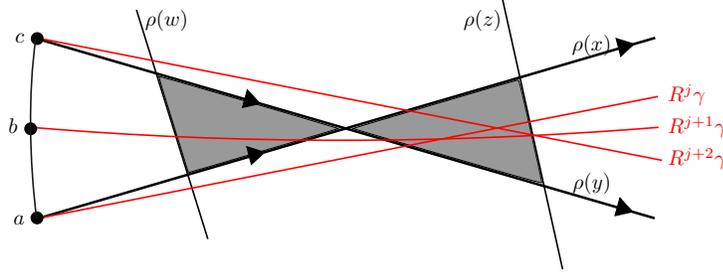


FIGURE 35. Proof of Lemma B.5

(iii) The case where $R^2\alpha$ is the radical line is symmetric to case (i).

This proves the statement if the set of three arcs $R^{2i}\gamma, R^{2i+1}\gamma, R^{2i+2}\gamma$ contains exactly one radical line. Now suppose that there are exactly two radical lines among $\alpha, R\alpha, R^2\alpha$. Again there are three cases.

- (iv) If α and $R\alpha$ are the radical lines, then they are automatically compatible, by Definition 4.9. Moreover $R^2\alpha$ can be deformed to be compatible with $R\alpha$, by Lemma B.4.
- (v) If α and $R^2\alpha$ are the radical lines, then Lemma B.4 implies that $R\alpha$ is too. In this case, all three are automatically compatible.
- (vi) The case where $R\alpha$ and $R^2\alpha$ are the radical lines is symmetric to case (iv).

This proves the statement if the set of three arcs $R^{2i}\gamma, R^{2i+1}\gamma, R^{2i+2}\gamma$ contains exactly two radical lines. If all three arcs are radical lines, then they are automatically compatible, so there is nothing to show. \square

We now construct a projective resolution of M_γ . We assume that the representatives $R^i\gamma$ satisfy the conditions in Lemma B.5. Because of condition (a), Proposition 4.12 implies

$$(B.3) \quad \text{im} f_{R^{2i+2}\gamma} = \ker \bar{f}_{R^{2i+1}\gamma}, \text{ for all } i.$$

We use condition (b) in order to obtain a similar identity between the degrees $2i$ and $2i+1$ as follows. Let $(R^{2i+1}\gamma)'$ be as in condition (b). From Corollary 4.14 we know that the homotopy gives rise to an automorphism

$$(\varphi_{2i+1,1}, \varphi_{2i+1,0}) \in \text{Aut} P_{2i+2}(\gamma) \oplus \text{Aut} P_{2i+1}(\gamma)$$

such that

$$(B.4) \quad \bar{f}_{(R^{2i+1}\gamma)'} = \varphi_{2i+1,0} \bar{f}_{R^{2i+1}\gamma} \varphi_{2i+1,1}^{-1}$$

is the conjugation of $f_{R^{2i+1}\gamma}$ by this automorphism. Because $(R^{2i+1}\gamma)'$ and $R^{2i}\gamma$ are compatible, Proposition 4.12 yields

$$(B.5) \quad \text{im} \bar{\varphi}_{2i+1,0} \bar{f}_{R^{2i+1}\gamma} \bar{\varphi}_{2i+1,1}^{-1} = \ker f_{R^{2i}\gamma}, \text{ for all } i.$$

We are now ready to write the projective resolution of M_γ .

Proposition B.6. *With the above notation, the following sequence is a projective resolution.*

$$\begin{aligned} \cdots P_3(\gamma) &\xrightarrow{\bar{\varphi}_{1,1} f_{R^2\gamma}} P_2(\gamma) \xrightarrow{\bar{\varphi}_{1,0} \bar{f}_{R\gamma} \bar{\varphi}_{1,1}^{-1}} P_1(\gamma) \xrightarrow{f_\gamma} P_0(\gamma) \longrightarrow M_\gamma \longrightarrow 0 \\ \cdots P_{2i+2}(\gamma) &\xrightarrow{\bar{\varphi}_{2i+1,0} \bar{f}_{R^{2i+1}\gamma} \bar{\varphi}_{2i+1,1}^{-1}} P_{2i+1}(\gamma) \xrightarrow{\bar{\varphi}_{2i-1,1} f_{R^{2i}\gamma}} P_{2i}(\gamma) \cdots \end{aligned}$$

Proof. The sequence is exact at $P_1(\gamma)$ by equation (B.5) with $i=0$. To show exactness at $P_{2i}(\gamma)$, we note that on the one hand, $\text{im}(\bar{\varphi}_{2i-1,1} f_{R^{2i}\gamma}) = \bar{\varphi}_{2i-1,1}(\text{im} f_{R^{2i}\gamma})$, because $\bar{\varphi}_{2i-1,1}$ is an isomorphism, and on the other hand, $\ker(\bar{\varphi}_{2i-1,0} \bar{f}_{R^{2i-1}\gamma} \bar{\varphi}_{2i-1,1}^{-1}) = \bar{\varphi}_{2i-1,1}(\ker \bar{f}_{R^{2i-1}\gamma})$, because $\bar{\varphi}_{2i-1,0}$ and $\bar{\varphi}_{2i-1,1}^{-1}$ are isomorphisms. Now exactness follows from equation (B.3).

It remains to show exactness at $P_{2i+1}(\gamma)$. We have $\ker(\bar{\varphi}_{2i-1,1} f_{R^{2i}\gamma}) = \ker f_{R^{2i}\gamma}$, since $\bar{\varphi}_{2i-1,1}$ is an isomorphism. Now the result follows from equation (B.5). \square

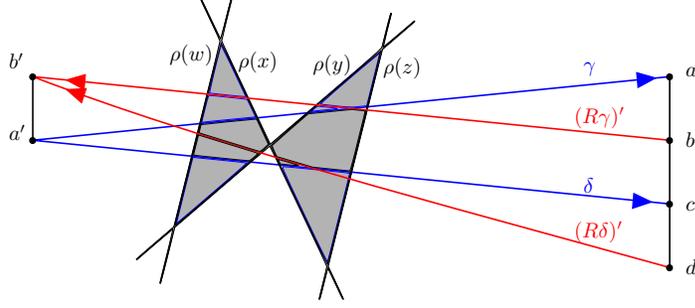


FIGURE 36. Proof of Proposition B.7. In this picture , for the sake of argument, the arcs γ and δ are not compatible.

We are now ready for the main result of this section. It describes the morphism of complexes induced by a 2-pivot on the projective resolutions.

Theorem B.7. *Let γ be a representative of a 2-diagonal and δ a representative of the 2-diagonal obtained from γ by a 2-pivot such that γ and δ are compatible. Choose representatives of the rotation orbits of γ and δ that satisfy conditions (a) and (b) in Lemma B.5. Then the pivot morphism $g: M_\gamma \rightarrow M_\delta$ induces a morphism of complexes on the projective resolutions so that the following diagram commutes.*

$$\begin{array}{ccccccccc}
\cdots & \rightarrow & P_3(\gamma) & \xrightarrow{\bar{\varphi}_{1,1} f_{R^2\gamma}} & P_2(\gamma) & \xrightarrow{\bar{\varphi}_{1,0} \bar{f}_{R\gamma} \bar{\varphi}_{1,1}^{-1}} & P_1(\gamma) & \xrightarrow{f_\gamma} & P_0(\gamma) & \longrightarrow & M_\gamma & \longrightarrow & 0 \\
& & \downarrow & & \downarrow g_2^r & & \downarrow g_1^c & & \downarrow g_0^d & & \downarrow g & & \\
\cdots & \rightarrow & P_3(\delta) & \xrightarrow{\bar{\chi}_{1,1} f_{R^2\delta}} & P_2(\delta) & \xrightarrow{\bar{\chi}_{1,0} \bar{f}_{R\delta} \bar{\chi}_{1,1}^{-1}} & P_1(\delta) & \xrightarrow{f_\delta} & P_0(\delta) & \longrightarrow & M_\delta & \longrightarrow & 0
\end{array}$$

Proof. To fix notation let a' be the common endpoint of γ and δ and let b' be its clockwise neighbor on the boundary of \mathcal{S} . Let a, b, c denote consecutive vertices on the boundary in clockwise order, such that a is the second endpoint of γ and c is the second endpoint of δ . Then $R\gamma$ has endpoints b and b' , see Figure 36.

First let us check that we can simultaneously choose the representatives of the arcs in the two rotation orbits such that they satisfy both Lemma B.5 as well as the condition that γ and δ are compatible. If $R\gamma$ is not radical line then the lemma does not impose any restriction on γ and there is nothing to show. If $R\gamma$ is a radical line $\rho(x)$, then we will choose γ and $R\gamma$ according to the compatibility condition stated in Lemma B.4. There are exactly two possibilities for $R\gamma$ since it can run along either side of the radical line $\rho(x)$. We choose $R\gamma$ to run from b to b' on the left of $\rho(x)$ so that it is closer to δ . Then we can choose γ to start at a' , then follow the boundary of its first white region clockwise, then follow $R\gamma$ until its last shaded region at b , then cross $R\gamma = \delta(x)$ and end by following the boundary of the white region in the counterclockwise direction until a . In the case where $R\delta$ is a radical we perform the ‘same’ deformation to δ as to γ so that both are crossing $R\gamma$ close to the endpoint a' . This choice assures that γ and δ are compatible.

Therefore Theorem 5.5 implies that $g_0^d f_\gamma = f_\delta g_1^c$ and thus the degree 0,1 square of our diagram commutes. We also know that the rows of our diagram are exact, thanks to Proposition B.6. Moreover, because of equation (B.4) with $i = 0$, we have

$$(B.6) \quad f_{(R\gamma)'} = \varphi_{1,0} f_{R\gamma} \varphi_{1,1}^{-1},$$

thus the map from degree 2 to degree 1 in the first row of our diagram is equal to $\bar{f}_{(R\gamma)'}$, where $(R\gamma)'$ is a representative of $R\gamma$ that is compatible with γ . Similarly, the corresponding map in the second row is given as $\bar{f}_{(R\delta)'}$, where $(R\delta)'$ is an arc that is compatible with δ . Thus we have the following chain of compatibility $(R\delta)' \sim \delta \sim \gamma \sim (R\gamma)'$.

We will now show that this implies that $(R\delta)'$ and $(R\gamma)'$ are compatible as well. The relative position of the arcs involved is illustrated in Figure 36, however the arcs γ and δ are not compatible in the picture. Suppose that (w, x) is a pair in the crossing sequence of $(R\gamma)'$ such that $(R\delta)'$ crosses $\rho(w)$ and $\rho(x)$. If

(w, x) also is a pair in the crossing sequence of $(R\delta)'$ we are done. Otherwise, there exist two additional radical lines $\rho(y), \rho(z)$ that together with $\rho(w)$ and $\rho(x)$ form two triangular shaded regions meeting at the crossing point of $\rho(x)$ and $\rho(y)$, such that none of the vertices of the shaded triangles lies on the boundary of \mathcal{S} and the arcs $(R\gamma)'$ and $(R\delta)'$ also cross $\rho(y)$ and $\rho(z)$, as shown in the figure. Then

$$R\gamma \sim R\delta \Leftrightarrow (x, w) \text{ is a crossing pair in both} \Leftrightarrow (y, z) \text{ is a crossing pair in both.}$$

This implies that both γ and δ cross at least three of the four radical lines and so both cross both arcs of one of the crossing pairs (x, w) or (y, z) . Without loss of generality, we assume that they cross the pair (x, w) . Now the compatibility of $(R\gamma)'$ and γ implies that (x, w) is a pair in the crossing sequence of γ . Therefore we also have the pair (x, w) in the crossing sequence of δ , because γ and δ are compatible. But this implies that (w, x) is a pair in the crossing sequence of $(R\delta)'$, because of the compatibility of $(R\delta)'$ and δ . This shows that $(R\gamma)'$ and $(R\delta)'$ are compatible.

Because of equation (B.6), this means that the two horizontal maps from degree 2 to degree 1 in the proposition are given by two compatible arcs $(R\gamma)'$ and $(R\delta)'$. Therefore Corollary B.2 implies that the degree 1,2 square of the diagram commutes as well, and the proof is complete. \square

Remark B.8. We did not specify the vertical map in degree 3 in Proposition B.7. If the horizontal maps from degree 3 to degree 2 are also induced from a representative of a 2-diagonal, then we can iterate the argument of the proof and conclude that the vertical map in degree 3 is g_3^c .

APPENDIX C. PROOFS OF SECTION 6

This section contains the proofs of several results from section 6. We start by studying the matrix shape of our maps f_γ in greater detail in subsection C.1.

C.1. The staircase shape of f_γ . Throughout this subsection, γ is a representative of a 2-diagonal in the checkerboard polygon \mathcal{S} . Recall that the morphism $f_\gamma: P_1(\gamma) \rightarrow P_0(\gamma)$ is defined in terms of the crossing sequence of γ in Definition 4.5. We assume without loss of generality that the crossing sequence is of the form $(i_1, j_1), \dots, (i_n, j_n)$, since the cases where there is an i_0 (or j_0) that is not part of a pair, follow by the same arguments. The new key ingredient in this section is to consider the steps in the crossing sequence. Also recall that the subquiver $Q(\gamma)$ of Q was introduced in Definition A.1.

Let γ be a 2-diagonal with crossing subsequence

$$(i_s, j_s) \xrightarrow{\Sigma_s} (i_{s+1}, j_{s+1}) \xrightarrow{\Sigma_{s+1}} \dots \xrightarrow{\Sigma_{t-1}} (i_t, j_t)$$

and suppose that every step Σ_l is forward. Recall that the crossings between γ and i_l are of degree zero and the crossings between γ and j_l are of degree one, for $l = s, s+1, \dots, t$.

Let α_l be the arrow that connects i_l and j_l in the quiver Q . Then, by condition (Q_3) of subsection 3.1.1, α_l lies in at most two 3-cycles in Q and exactly one of them lies in the subquiver $Q(\gamma)$. We denote the third vertex of that 3-cycle by k_l , so the cycle is

$$\begin{array}{ccc} i_l & \xrightarrow{\alpha} & j_l \\ & \searrow & \swarrow \\ & k_l & \end{array}$$

When we remove the vertex k_l and the arrow α_l from the quiver $Q(\gamma)$ we obtain two connected components. We say that the vertex i_l is the *entry* to the l -th pair (i_l, j_l) if it belongs to the same connected component as the previous pair (i_{l-1}, j_{l-1}) , and the *exit* from the l -th pair otherwise.

In Figure 9, the vertex i_{s+1} is the entry in cases (i) and (iv) and j_{s+1} is the entry in cases (ii) and (iii).

We say that the step Σ_l from l to $l+1$ in the crossing sequence has *degree zero* if the entry to the l -th pair is i_l , and we say it has *degree one* if the entry is j_l .

The following lemma is a reformulation of Lemma A.7.

Lemma C.1. (*Rectangle-Trapezoid Lemma*) *With the notation above, the following properties hold for all $l \geq s$.*

Indeed, the two consecutive T_0 in steps 2 and 3 imply the nonzero entries in positions 13, 14, 15, 24, 25 and 35. The first R_0 yields a nonzero entry in position 35 and a zero in position 36, which then gives zeros in positions 16 and 26. The fifth step T_1 yields zeros in positions 46, 47 and 57. The R_1 in step 6 gives a zero in position 57 and a nonzero entry in position 68, and the R_0 in step 7 gives a nonzero entry in position 68 and a zero in position 79, which also implies that position 69 is zero. The last step R_1 gives a zero in position 79.

The matrix f_γ thus has staircase shape with blocks limited by the columns 5,6,8 and 9.

We now describe in general the staircase shape of f_γ observed in the example. Suppose the crossing sequence of γ is $(i_1, j_1) \xrightarrow{\Sigma_1} (i_2, j_2) \xrightarrow{\Sigma_2} \cdots \xrightarrow{\Sigma_{n-1}} (i_n, j_n)$ and that every step is forward. We define the *source* $s(\Sigma_l)$ and the *target* $t(\Sigma_l)$ of the step Σ_l to be the integers $s(\Sigma_l) = l$ and $t(\Sigma_l) = l + 1$.

Definition C.5. Define a sequence of integers $1 = t_0 < t_1 < \cdots < t_p = n$ recursively by setting

$$t_1 = \min \left\{ \begin{array}{l} \text{target of the first rectangular step of degree 0 after step 1,} \\ \text{target of the first trapezoidal step of degree 1 after step 1,} \\ \text{source of the first rectangular step of degree 1 after step 1} \end{array} \right\}$$

and recursively,

$$t_{l+1} = \min \left\{ \begin{array}{l} \text{target of the first rectangular step of degree 0 after step } t_l - 1, \\ \text{target of the first trapezoidal step of degree 1 after step } t_l - 1, \\ \text{source of the first rectangular step of degree 1 after step } t_l \end{array} \right\}$$

if it exists, and $t_{l+1} = n$ and $p = l + 1$, otherwise. This sequence is called the *staircase sequence* of f_γ .

Note that the definition of the staircase sequence is independent of the first step in the crossing sequence.

In Example C.4, we have $t_1 = 5$, because step four is R_0 , $t_2 = 6$, because step five is T_1 , and $t_3 = 8$, because step seven is R_0 . Thus the staircase sequence in this example is $1 < 5 < 6 < 8 < 9$, as observed in the matrix of Example C.4.

Proposition C.6. *The matrix f_γ has an upper triangular staircase shape whose nonzero blocks are limited by the columns t_1, t_2, \dots, t_n as illustrated in the left picture in Figure 37.*

In other words, f_γ is nonzero on the main diagonal $(f_{ss})_{s=1, \dots, n}$ (and on the diagonal $(f_{s(s+1)})_{s=1, \dots, n-1}$) as well as on p triangular regions J_1, J_2, \dots, J_p , where the first row of the region J_1 is the row t_{l-1} , the last column of J_l is the column t_l and the hypotenuse of J_l is given by the second diagonal $(f_{s(s+1)})_{t_{l-1} \leq s \leq t_l - 1}$.

We now define two types of blocks A_l and B_l , both being subsets of column t_l of the matrix of f_g .

Definition C.7. (a) Let a_l be the least positive integer such that the entry in f_γ at position (a_l, t_l) is an arrow. We denote by A_l the following block in f_γ

$$A_l = [f_{a_l t_l}, f_{(a_l+1)t_l}, \dots, f_{t_l t_l}]^T$$

Then each of the positions in A_l is given by an arrow in the quiver. In the example in the left picture of Figure 37, the positions of A_l are marked by the symbol \bullet .

(b) We denote by B_l the last column of the triangular region J_l in the proposition. Thus

$$B_l = [f_{t_{l-1} t_l}, f_{(t_{l-1}+1)t_l}, \dots, f_{(t_l-1)t_l}]^T$$

In the left picture of Figure 37, these blocks are the framed in red color.

Let a_l be the least positive integer such that the entry in f_γ at position (a_l, t_l) is an arrow. Then $a_l \leq t_l$ and each of the positions $(a_l, t_l), (a_l + 1, t_l), \dots, (t_l, t_l)$ in f_γ is given by an arrow in the quiver. These positions are marked by a \bullet in the left picture of Figure 37.

Corollary C.8. *With the notation above, the sequence of steps $\Sigma_{t_{l-1}}, \Sigma_{t_{l-1}+1}, \dots, \Sigma_{t_l-1}$ is equal to*

$$\left\{ \begin{array}{ll} T_1 & \text{if } t_l - t_{l-1} = 1; \\ R_1 \underbrace{T_0 \cdots T_0}_{t_l - t_{l-1} - 2} R_0 & \text{if } t_l - t_{l-1} \geq 2. \end{array} \right.$$

In particular, $\Sigma_{t_{l-1}} = R_1$ unless $t_l - t_{l-1} = 1$.

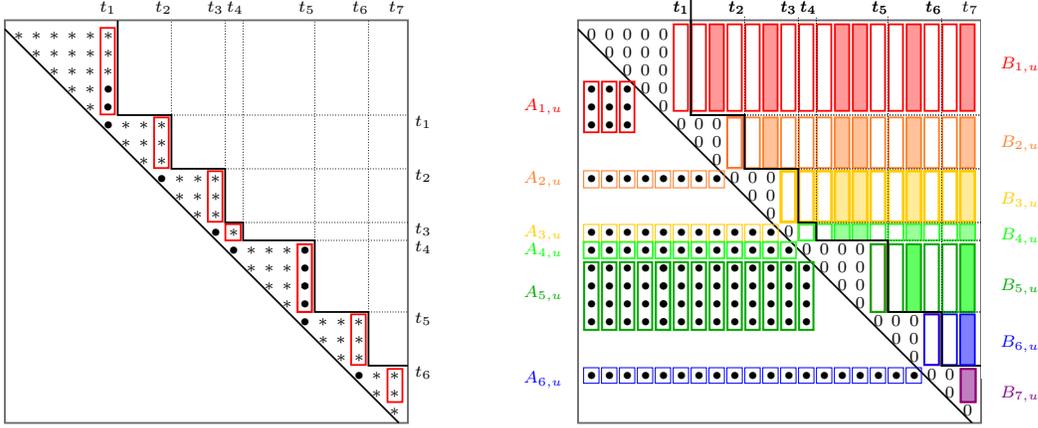


FIGURE 37. On the left, the staircase shape of the matrix of f_γ . Entries in blank positions are zero and entries in positions marked with a bullet point are given by arrows. The regions B_l are framed in red. The picture on the right indicates the blocks of the matrix g_0 . The blocks $B_{l,u}$ are the framed columns above the diagonal; the entries in the shaded blocks are zero. The blocks $A_{l,u}$ are the framed columns with bullet points. These blocks project horizontally onto the positions of the arrows marked by bullet points in the matrix of f_γ .

Proof. Let $m = t_l - t_{l-1}$. Since t_{l-1} is in the staircase sequence of Definition C.5, we must have $\Sigma_{t_{l-1}-1} = R_0$ or T_1 , or $\Sigma_{t_{l-1}} = R_1$.

Suppose first that $\Sigma_{t_{l-1}-1} = R_0$. Then $\Sigma_{t_{l-1}}$ is of degree 1, by Lemma C.3. If $\Sigma_{t_{l-1}} = T_1$ then $t_l = t(\Sigma_{t_{l-1}}) = t_{l-1} + 1$, which is the case $m = 1$ of the statement. If $\Sigma_{t_{l-1}} = R_1$ then its terminal is not in the staircase sequence, and thus $m > 1$. The following step $\Sigma_{t_{l-1}+1}$ is of degree 0. If it is R_0 then its terminal is t_l , and we have $m = 2$. Otherwise, $\Sigma_{t_{l-1}+1} = T_0$, and the following step $\Sigma_{t_{l-1}+2}$ is of degree 0. Continuing this way, we construct the sequence in the corollary.

If $\Sigma_{t_{l-1}-1} = T_1$, then again $\Sigma_{t_{l-1}}$ is of degree 1, by Lemma C.3, and the proof is the same as above.

Suppose now that $\Sigma_{t_{l-1}} = R_1$. Then the degree of $\Sigma_{t_{l-1}+1}$ is 0. If $\Sigma_{t_{l-1}+1} = R_0$ then $m = 2$ and the sequence is R_1R_0 . Otherwise $\Sigma_{t_{l-1}+1} = T_0$ and the following step $\Sigma_{t_{l-1}+2}$ is of degree 0. Again we obtain the sequence in the corollary by repeating this argument. \square

The next lemma discusses the position of arrows in f .

Lemma C.9. *Suppose f_{st} is an arrow such that $t - s \geq 2$. Then exactly one of the following statements holds.*

- (i) $f_{st}, f_{(s+1)t}, \dots, f_{tt}$ are arrows and $f_{s't'}$ are not arrows for all $s' \geq s, t' < t, s' \leq t'$ and $(s', t') \neq (s, s)$.
- (ii) $f_{ss}, f_{s(s+1)}, \dots, f_{st}$ are arrows and $f_{s't'}$ are not arrows for all $s' > s, t' \leq t, s' \leq t'$ and $(s', t') \neq (t, t)$.

Proof. Since f_{st} is an arrow, it follows that the entries $f_{s't'}$ in the triangular region with $s' \geq s, t' \leq t, s' \leq t'$ are given by valid paths. Then Lemma C.3(c) implies that $f_{s's'}$ with $s < s' < t$ are not arrows. In particular, since $t - s \geq 2$ we have that $f_{(s+1)(s+1)}$ is not an arrow.

First, suppose that $f_{s(s+1)}$ is not an arrow, then we want to show that statement (i) of the lemma holds. Then there is an arrow $j_t \rightarrow j_{s+1}$, because if this path were of length two or more then the path i_{s+1} to j_t would not be valid, hence $f_{(s+1)t} = 0$, a contradiction. Then since f_{st} is an arrow, there is a path of length two $i_s \rightarrow j_t \rightarrow j_{s+1}$, see Figure 38 on the left. Note that the path from j_s to i_{s+1} is either a single arrow or a composition of two arrows. If $t = s + 2$ then we identify the two vertices labeled j_t and j_{s+2} given in the figure. Then both $f_{(s+1)t}, f_{(s+2),t}$ are arrows and statement (i) holds. Otherwise, if $t > s + 2$ then the quiver is as in the figure. In particular, $f_{(s+1)(s+2)}$ is not an arrow, so there is a path of length two $i_{s+1} \rightarrow j_t \rightarrow j_{s+2}$. If $t = s + 3$ then condition (i) holds by the same reasoning as above, and otherwise if $t > s + 3$ we can continue the argument in the same way. This shows the lemma in the case when $f_{s(s+1)}$ is not an arrow.

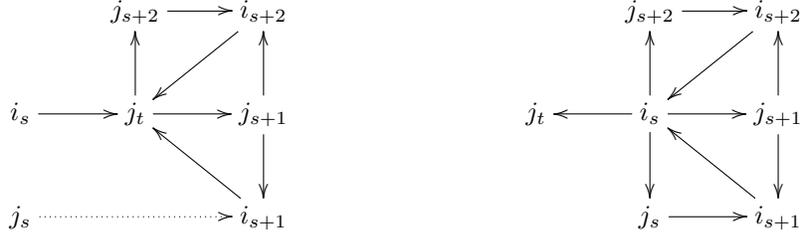


FIGURE 38. Proof of Lemma C.9.

Now, suppose that $f_{s(s+1)}$ is an arrow, then we want to show that statement (ii) of the lemma holds. We are in the situation of Figure 38 on the right. In particular, since $f_{s(s+1)}$ is an arrow then $f_{(s+1)(s+1)}$ is not an arrow, because otherwise there would be no valid path from i_s to j_t and thus $f_{st} = 0$. Then f_{ss} is an arrow. Moreover $f_{s(s+1)}$ is an arrow, because otherwise again the path from i_s to j_t would not be valid. If $t = s + 2$, then $f_{ss}, f_{s(s+1)}, f_{s(s+2)}$ are arrows and (ii) holds. If $t > s + 2$ then $f_{s(s+3)}$ is an arrow and we can continue the argument in the same way to conclude that condition (ii) holds. This shows the lemma in the case when $f_{s(s+1)}$ is an arrow and completes the proof. \square

In the example of Figure 37 we have $f_{t_4 t_5}$ is an arrow and $f_{t_4(t_5-1)}$ is not, so the positions $f_{(t_4+1)t_5}, \dots, f_{t_5 t_5}$ are arrows.

C.2. Nilpotent endomorphisms of M_γ . In this subsection, we provide the proofs for section 6.1.

C.2.1. A preparatory lemma. We need the following lemma from homological algebra.

Lemma C.10. *Let $g : M \rightarrow M'$ be a morphism in $\text{mod } B$ that induces maps g_0, g_1 on the projective presentations of M and M' such that the following diagram commutes.*

$$\begin{array}{ccccccc}
 P_1 & \xrightarrow{f} & P_0 & \xrightarrow{\pi} & M & \longrightarrow & 0 \\
 g_1 \downarrow & \nearrow h_1 & \downarrow g_0 & \nearrow h & \downarrow g & & \\
 P'_1 & \xrightarrow{f'} & P'_0 & \xrightarrow{\pi'} & M' & \longrightarrow & 0
 \end{array}$$

Then g is zero in $\underline{\text{mod}} B$ if and only if $g_0 = f'h_1 + g'$ for some maps h_1, g' such that $g'f = 0$.

Proof. The following sequence of arguments yields the desired result.

g is zero in $\underline{\text{mod}} B$	iff	g factors through the projective cover of M'
	iff	$g = \pi'h$ for some $h : M \rightarrow P'_0$
	iff	$g\pi = \pi'h\pi$ because π is surjective
	iff	$\pi'g_0 = \pi'h\pi$ because the diagram commutes
	iff	$\pi'(g_0 - h\pi) = 0$
	iff	$g_0 - h\pi$ factors through f'
	iff	$g_0 - h\pi = f'h_1$ for some $h_1 : P_0 \rightarrow P'_1$
	iff	$g_0 = f'h_1 + h\pi$
	iff	$g_0 = f'h_1 + g'$ such that $g'f = 0$. \square

C.2.2. The compositions $g_0 f_\gamma$ and $f_\gamma g_1$. The following lemmata describe the matrix entries in the composition $g_0 f_\gamma$ column by column. There are four cases to consider depending on the direction of the two steps in the crossing sequence immediately before and after the crossing pair at position t . The first lemma deals with the case where both steps are forward. Part (a) describes the entries above the diagonal and part (b) the entries below the diagonal.

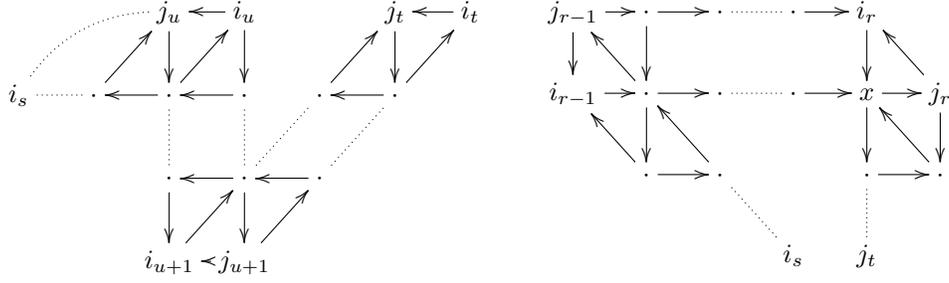


FIGURE 39. Proof of Lemma C.11(a). On the left, $s < u = t - 2, t = t_{(l+1)}$. On the right, $s < u < t_{(l+1)}$ and the step $(i_{r-1}, j_{r-1}) \rightarrow (i_r, j_r)$ is trapezoidal of degree 0.

Lemma C.11. (The case $\rightarrow (i_t, j_t) \rightarrow$). Let $[0 \ \dots \ 0 \ f_{ut} \ f_{(u+1)t} \ \dots \ f_{tt} \ 0 \ \dots \ 0]^T$ be the column t of f_γ with $u < t$.

(a) For all $s = 1, 2, \dots, u - 1$

$$(C.1) \quad (g_0 f_\gamma)_{st} = i_{su} f_{ut} + i_{s(u+1)} f_{(u+1)t}$$

Moreover, if $f_{uu}: i_u \rightarrow j_u$ is a single arrow and the steps before and after (i_u, j_u) are of the form

$$(i_{u-1}, j_{u-1}) \xleftarrow{\Sigma_u} (i_u, j_u) \xrightarrow{\Sigma_{u+1}} (i_{u+1}, j_{u+1}) \text{ then } i_{s(u+1)} f_{(u+1)t} = 0, \text{ so } (g_0 f_\gamma)_{st} = i_{su} f_{ut}.$$

(b) For all $s = t + 1, t + 2, \dots, n$

$$(C.2) \quad (g_0 f_\gamma)_{st} = i_{st} f_{tt} + i_{st'} f_{t't}$$

where $t' < t$ is the largest integer such that $f_{t't}$ is an arrow and

$$\begin{cases} i_{st} f_{tt} = 0 & \text{if } f_{tt} \text{ is not an arrow;} \\ i_{st'} f_{t't} = 0 & \text{if no such } t' \text{ exists.} \end{cases}$$

Proof. Since the column t of f_γ is nonzero above the diagonal and zero below the diagonal, we know that the crossing sequence at position t is of the form

$$(i_{t-1}, j_{t-1}) \xrightarrow{\Sigma_t} (i_t, j_t) \xrightarrow{\Sigma_{t+1}} (i_{t+1}, j_{t+1})$$

with both steps going forward. Therefore the staircase shape of f_γ implies that there exists a unique l such that $t_l + 1 < t \leq t_{(l+1)}$ and $u = t_l$.

(a) Let $s < u$. Since $f_{rt} = 0$ for $r < u$ and $r > t$, we have

$$(C.3) \quad (g_0 f_\gamma)_{st} = \sum_{r=u}^t i_{sr} f_{rt}.$$

So we must show that $i_{sr} f_{rt} = 0$, for $r = u + 2, u + 3, \dots, t$. For such an r , we would have $t_l = u < u + 1 < r \leq t \leq t_{(l+1)}$, and in particular $t_{(l+1)} - t_l \geq 2$. In this situation, Corollary C.8 implies that the sequence of steps $(\Sigma_{t_l}, \Sigma_{t_l+1}, \dots, \Sigma_{t_{(l+1)}-1})$ is equal to $(R_1, T_0, \dots, T_0, R_0)$ with $t_{(l+1)} - t_l - 2$ steps T_0 in the middle.

If $t_{(l+1)} - t_l = 2$, we have $r > u + 1 = t_l + 1$, hence $r \geq t_l + 2 = t_{(l+1)}$. Thus $r = t = t_{(l+1)}$, and we want to show $i_{st} f_{tt} = 0$, where $s < u$. The quiver is shown in the left picture of Figure 39. The path i_{st} runs from i_s to i_t and must factor through j_t . Therefore $i_{st} f_{tt} = 0$ since it contains a cyclic subpath at j_t .

If $t_{(l+1)} - t_l > 2$, then our sequence of steps contains $t_{(l+1)} - t_l - 2$ steps T_0 in the middle. The case where $r = t = t_{(l+1)}$ uses the same argument as above. In all other cases, the crossing pair (i_r, j_r) is the end of a trapezoidal step of degree 0 and at the start of some step of degree 0. The quiver is shown in the right picture of Figure 39.

In this case, i_{sr} is a path from i_s to i_r with $s < r - 1$ and it must therefore factor through the vertex labeled x that forms a 3-cycle with i_r and j_r . On the other hand, f_{rt} is a path from i_r to j_t with $t \geq r$ and this path also factors through x . Consequently $i_{sr} f_{rt} = 0$, because it contains a cyclic subpath at x . This proves equation (C.3).

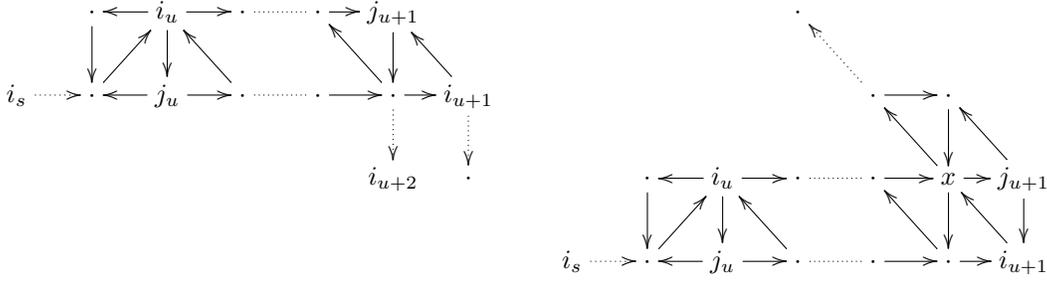


FIGURE 40. Proof of the second statement of Lemma C.11 (a). On the left, the step $(i_u, j_u) \rightarrow (i_{u+1}, j_{u+1})$ is trapezoidal, and on the right it is rectangular.

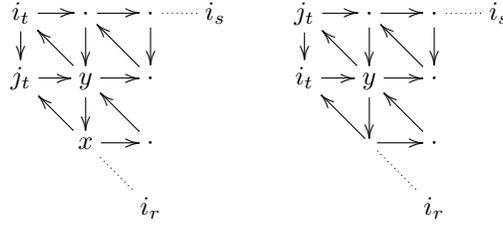


FIGURE 41. Proof of Lemma C.11 (b), $r \leq t < s$. On the left, $f_{tt}: i_t \rightarrow j_t$ is an arrow and on the right it is a path $i_t \rightarrow x \rightarrow j_t$.

To prove the moreover statement of the lemma, suppose now that $f_{uu}: i_u \rightarrow j_u$ is a single arrow and that the step Σ_u is backward and the step Σ_{u+1} . We want to show that $f_{s(u+1)}f_{(u+1)t} = 0$. The step Σ_{u+1} is either trapezoidal or rectangular and both cases are illustrated in Figure 40. If it is trapezoidal then $i_{s(u+1)}$ factors through j_{u+1} , as shown in the left picture of the figure. Thus if $t = u + 1$ then $i_{s(u+1)}f_{(u+1)t} = 0$, because it contains a cyclic subpath at j_{u+1} . If $t > u + 1$ then $f_{ut} = 0$, because every path from i_u to any vertex past j_{u+1} will contain two arrows of the same 3-cycle and thus cannot be valid. This gives a contradiction to the assumption of the lemma that $f_{ut} \neq 0$.

Suppose now that Σ_{n+1} is rectangular, see the right picture in Figure 40. Then the path $i_{s(u+1)}$ factors through the vertex x in the figure. The path $f_{(u+1)t}$ also factors through x , which implies that $i_{s(u+1)}f_{(u+1)t} = 0$, since it contains a cyclic subpath at x . This completes the proof of part (a).

(b) Let $s > t$. Again we have equation (C.3) and we may thus assume that $u \leq r \leq t$. Then i_{sr} is a path from i_s to i_r that runs against the direction of the steps in the crossing sequence, and f_{rt} is a path from i_r to j_t that follows the direction of the steps, see Figure 41.

First suppose that $f_{tt}: i_t \rightarrow j_t$ is an arrow. This situation is shown in the left picture of the figure.

If the vertex $x \neq i_r$ then i_{sr} factors through the vertex x . On the other hand, the path f_{rt} from i_r to j_t also factors through x . Therefore $i_{sr}f_{rt} = 0$, since it contains a cyclic subpath at x .

If $x = i_r$ then, because of the position of x in the quiver, $r = t'$ is the largest integer such that $f_{t't}$ is an arrow. In this situation, $i_{st'}f_{t't}$ is nonzero. Moreover, if $y = j_{t-1}$ then $t' = t - 1$; otherwise t' can be strictly smaller than $t - 1$.

If $r = t$ then we are considering $i_{st}f_{tt}$ which is nonzero since $f_{tt}: i_t \rightarrow j_t$ is a single arrow.

Now suppose that f_{tt} is not an arrow. Then there is an arrow $j_t \rightarrow i_t$. This situation is shown in the right picture of Figure 41. If the vertex y is equal to some $i_{t'}$ in the crossing sequence, the argument is similar to the one above. Otherwise, the path i_{sr} factors through the vertex y that forms a 3-cycle with i_t and j_t . On the other hand, f_{rt} is a path from i_r to j_t and it also must factor through y . Thus if the vertex y is different from i_r then $i_{sr}f_{rt} = 0$, since it contains a cyclic subpath at y . Suppose now that $y = i_r$. Then

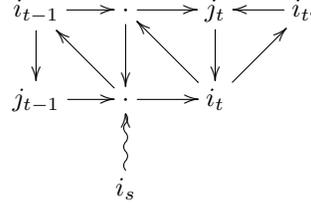


FIGURE 42. The exceptional case in Lemma C.13

$r = t'$, where t' is the largest integer such that $f_{t't}$ is an arrow. Thus, we have an arrow $i_{t'} \rightarrow j_t$ and the composition $i_{st'}f_{t't}$ is nonzero. \square

Lemma C.12. (The case $\leftarrow (i_t, j_t) \leftarrow$). Let $[0 \ \dots \ 0 \ f_{tt} \ \dots \ f_{vt} \ 0 \ \dots \ 0]^T$ be the column t of f_γ with $t < v$.

(a) For all $s = 1, 2, \dots, t-1$

$$(C.4) \quad (g_0 f_\gamma)_{st} = i_{st}f_{tt} + i_{st'}f_{t't}$$

where $t' > t$ is the smallest integer such that $f_{t't}$ is an arrow and

$$\begin{cases} i_{st}f_{tt} = 0 & \text{if } f_{tt} \text{ is not an arrow;} \\ i_{st'}f_{t't} & \text{if no such } t' \text{ exists.} \end{cases}$$

(b) For all $s = v+1, v+2, \dots, n$

$$(C.5) \quad (g_0 f_\gamma)_{st} = i_{sv}f_{vt} + i_{s(v-1)}f_{(v-1)t}$$

Moreover, if $f_{vv}: i_v \rightarrow j_v$ is a single arrow and the steps before and after (i_v, j_v) are of the form

$$(i_{v-1}, j_{v-1}) \xleftarrow{\Sigma_v} (i_v, j_v) \xrightarrow{\Sigma_{v+1}} (i_{v+1}, j_{v+1}) \text{ then } i_{s(v-1)}f_{(v-1)t} = 0, \text{ so } (g_0 f_\gamma)_{st} = i_{sv}f_{vt}.$$

Proof. This is the dual statement to Lemma C.11. \square

Lemma C.13. (The case $\rightarrow (i_t, j_t) \leftarrow$). Let $[0 \ \dots \ 0 \ f_{ut} \ \dots \ f_{tt} \ \dots \ f_{vt} \ 0 \ \dots \ 0]^T$ be the column t of f_γ with $u < t < v$. Then the entry $(g_0 f_\gamma)_{st}$ is computed according to formula (C.1), if $s < u$, and according to formula (C.4), if $s > v$, unless we are in one of the following two exceptional form

If $u = t-1$, f_{tt} is not an arrow, the steps before and after (i_{t-1}, j_{t-1}) are of the form

$$(i_{t-2}, j_{t-2}) \xrightarrow{\Sigma_{t-1}} (i_{t-1}, j_{t-1}) \xrightarrow{\Sigma_t} (i_t, j_t),$$

and there exists $t' > t$ such that $f_{t't}: i_{t'} \rightarrow j_t$ is an arrow, then $(g_0 f_\gamma)_{st}$ has an additional term $i_{st'}f_{t't}$, for all $s < u$.

If $v = t+1$, f_{tt} is not an arrow, the steps before and after (i_{t+1}, j_{t+1}) are of the form

$$(i_t, j_t) \xleftarrow{\Sigma_{t+1}} (i_{t+1}, j_{t+1}) \xleftarrow{\Sigma_{t+2}} (i_{t+2}, j_{t+2}),$$

and there exists $t' < t$ such that $f_{t't}: i_{t'} \rightarrow j_t$ is an arrow, then $(g_0 f_\gamma)_{st}$ has an additional term $i_{st'}f_{t't}$, for all $s > v$.

Proof. The proof of the general case is analogous to the relevant parts of the two previous lemmata. The exceptional case is illustrated in Figure 42. Clearly $(g_0 f_\gamma)_{st} = i_{s(t-1)}f_{(t-1)t} + i_{st}f_{tt} + i_{st'}f_{t't}$. Since $t-1 = u$ and $t = u+1$, the first two terms are precisely those given by equation (C.1), while the third term is the additional term given in the statement. \square

Lemma C.14. (The case $\leftarrow (i_t, j_t) \rightarrow$). Let $[0 \ \dots \ 0 \ f_{tt} \ 0 \ \dots \ 0]^T$ be the column t of f_γ . Then for all $s \neq t$

$$(C.6) \quad (g_0 f_\gamma)_{st} = \begin{cases} i_{st}f_{tt} & \text{if } f_{tt}: i_t \rightarrow j_t \text{ is an arrow;} \\ 0 & \text{if } f_{tt}: i_t \rightarrow \cdot \rightarrow j_t \text{ is not an arrow.} \end{cases}$$

Proof. We have $(g_0 f_\gamma)_{st} = \sum_u i_{su} f_{ut} = i_{st} f_{tt}$, where the last identity follows from the assumption that f_{tt} is the only nonzero entry in the t -th column of f_γ . It only remains to show that $i_{st} f_{tt} = 0$ when f_{tt} is not an arrow. In this situation, we have the following subquiver

$$\begin{array}{c} \cdot \leftarrow j_t \rightarrow \cdot \\ \nearrow \downarrow \nwarrow \\ k_1 \leftarrow i_t \rightarrow k_2 \end{array}$$

Thus there is an arrow $j_t \rightarrow i_t$ that lies in two 3-cycles whose third vertex is labeled k_1 or k_2 in the figure, and f_{tt} is equal to either one of the two paths $i_t \rightarrow k_1 \rightarrow j_t$ and $i_t \rightarrow k_2 \rightarrow j_t$. On the other hand, i_{st} , being a path ending at i_t , must end in one of the two subpaths $k_1 \rightarrow j_t \rightarrow i_t$ and $k_2 \rightarrow j_t \rightarrow i_t$. In both cases, the composition $i_{st} f_{tt}$ is zero, because it contains a cyclic subpath at k_1 or k_2 . \square

Lemmata C.11-C.14 also imply the following dual statements for the composition $f_\gamma g_1$, obtained by reformulation using the rows of f_γ instead of the columns. The proof follows directly from the property of the transpose $(f_\gamma g_1)^T = g_1^T f_\gamma^T$.

Lemma C.15. *The entries of the matrix of $f_\gamma g_1$ satisfy the following equations.*

- (1) (The case $\rightarrow (i_s, j_s) \rightarrow$). Let $[0 \ \dots \ 0 \ f_{ss} \ \dots \ f_{sr} \ 0 \ \dots \ 0]$ be the row s of f_γ , with $r > s$.
(a) For all $t = r + 1, \dots, n$

$$(C.7) \quad (f_\gamma g_1)_{st} = f_{sr} j_{rt} + f_{s(r-1)} j_{(r-1)t}$$

Moreover, if $f_{rr}: i_r \rightarrow j_r$ is a single arrow and the steps before and after (i_r, j_r) are of the form

$$(i_{r-1}, j_{r-1}) \xrightarrow{\Sigma_r} (i_r, j_r) \xleftarrow{\Sigma_{r+1}} (i_{r+1}, j_{r+1}) \text{ then } f_{s(r-1)} j_{(r-1)t} = 0, \text{ so } (f_\gamma g_1)_{st} = f_{sr} j_{rt}.$$

- (b) For all $t = 1, 2, \dots, s - 1$

$$(C.8) \quad (f_\gamma g_1)_{st} = f_{ss} j_{st} + f_{ss'} j_{s't}$$

where $s' > s$ is the smallest integer such that $f_{ss'}$ is an arrow,

$$\text{and } \begin{cases} f_{ss} j_{st} = 0 & \text{if } f_{ss} \text{ is not an arrow;} \\ f_{ss'} j_{s't} = 0 & \text{if no such } s' \text{ exists.} \end{cases}$$

- (2) (The case $\leftarrow (i_s, j_s) \leftarrow$). Let $[0 \ \dots \ 0 \ f_{sr} \ \dots \ f_{ss} \ 0 \ \dots \ 0]$ be the row s of f_γ , with $r < s$.
(a) For all $t = s + 1, \dots, n$

$$(C.9) \quad (f_\gamma g_1)_{st} = f_{ss} j_{st} + f_{ss'} j_{s't}$$

where $s' < s$ is the largest integer such that $f_{s's}$ is an arrow,

$$\text{and } \begin{cases} f_{ss} j_{st} = 0 & \text{if } f_{ss} \text{ is not an arrow;} \\ f_{ss'} j_{s't} = 0 & \text{if no such } s' \text{ exists.} \end{cases}$$

- (b) For all $t = 1, 2, \dots, r - 1$

$$(C.10) \quad (f_\gamma g_1)_{st} = f_{sr} j_{rt} + f_{s(r+1)} j_{(r+1)t}.$$

Moreover, if $f_{rr}: i_r \rightarrow j_r$ is a single arrow and the steps before and after (i_r, j_r) are of the form

$$(i_{r-1}, j_{r-1}) \xrightarrow{\Sigma_r} (i_r, j_r) \xleftarrow{\Sigma_{r+1}} (i_{r+1}, j_{r+1}) \text{ then } f_{s(r+1)} j_{(r+1)t} = 0, \text{ so } (f_\gamma g_1)_{st} = f_{sr} j_{rt}.$$

- (3) (The case $\leftarrow (i_s, j_s) \rightarrow$). Let $[0 \ \dots \ 0 \ f_{sr} \ \dots \ f_{ss} \ \dots \ f_{sr'} \ 0 \ \dots \ 0]$ be the row s of f_γ , with $r < s < r'$. Then the entry $(f_\gamma g_1)_{st}$ is computed according to formula (C.7) if $t > r'$ and according to formula (C.10) if $t < r$, unless we are in one of the following two exceptional cases.

If $r' = s + 1$, f_{ss} is not an arrow, the steps before and after (i_{s+1}, j_{s+1}) are of the form

$(i_s, j_s) \xrightarrow{\Sigma_{s+1}} (i_{s+1}, j_{s+1}) \xrightarrow{\Sigma_{s+2}} (i_{s+2}, j_{s+2})$, and there exists $s' < s$ such that $f_{ss'}: i_s \rightarrow j_{s'}$ is an arrow, then $(f_\gamma g_1)_{st}$ has an additional term $f_{ss'} j_{s't}$, for all $t > r'$.

If $r = s - 1$, f_{ss} is not an arrow, the steps before and after (i_{s-1}, j_{s-1}) are of the form

$(i_{s-2}, j_{s-2}) \xleftarrow{\Sigma_{s-1}} (i_{s-1}, j_{s-1}) \xleftarrow{\Sigma_s} (i_s, j_s)$, and there exists $r'' > s$ such that $f_{sr''}: i_s \rightarrow j_{r''}$ is an arrow, then $(f_\gamma g_1)_{st}$ has an additional term $f_{sr''} j_{r''t}$, for all $t < r$.

- (4) (The case $\rightarrow (i_s, j_s) \leftarrow$). Let $[0 \quad \dots \quad f_{ss} \quad 0 \quad \dots \quad 0]$ be the row s of f_γ .
Then for all $s \neq t$

$$(C.11) \quad (f_\gamma g_1)_{st} = \begin{cases} f_{ss} j_{st} & \text{if } f_{ss}: i_s \rightarrow j_t \text{ is an arrow;} \\ 0 & \text{if } f_{ss}: i_s \rightarrow \cdot \rightarrow j_s \text{ is not an arrow.} \end{cases}$$

C.2.3. *Blocks of g_0 .* We will think of the result in Section C.2.2 as relations in the matrix g_0 . To make this idea precise, we introduce the following block structure in the matrix g_0 . Recall that both maps f_γ and g_0 are given by square matrices of the same size, and that we have defined the blocks A_l and B_l in f_γ as the subcolumns of the column t_l . Each of these blocks A_l and B_l induces a family of column blocks $A_{l,u}$ and $B_{l,u}$ in g_0 as follows.

Definition C.16 (Column Blocks). Let g be an endomorphism of M_γ and let g_0 be the induced endomorphism on the projective cover $P_0(\gamma) = \bigoplus_{s=1}^n P(i_s)$. We treat the change of direction in parts (c) and (d). For parts (a) and (b) below, we assume that the crossing sequence is forward without change of direction, and we use the notation of Definition C.7, thus $1 = t_0 < t_1 < \dots < t_p = n$ is the staircase sequence of f_γ and a_l is the least positive integer such that the entry in f_γ at position (a_l, t_l) is an arrow. We define the following *column blocks* in the matrix of g_0 ,

- (a) (i) If $a_l \neq t_{l-1}$ or $f_{t_{l-1}t_{l-1}}$ is not an arrow, define

$$A_{l,u} = [i_{a_l u} \ i_{(a_l+1)u} \ \dots \ i_{t_l u}]^T, \text{ where } u \leq a_l - 1.$$

- (ii) If $a_l = t_{l-1}$ and $f_{t_{l-1}t_{l-1}}$ is an arrow, define

$$A_{l,u} = [i_{(a_l+1)u} \ i_{(a_l+2)u} \ \dots \ i_{t_l u}]^T, \text{ where } u \leq a_l.$$

In the right picture of Figure 37, the A -blocks are the colored framed regions with bullet points below the diagonal. In the example of Figure 37, the block $A_{5,u}$ is of type (a)(ii). For a crossing sequence that is non-forward the definition is symmetric.

- (b) For all $1 \leq l \leq p$ and $t_l \leq u$ let

$$B_{l,u} = [i_{t_{l-1}u} \ i_{(t_{l-1}+1)u} \ \dots \ i_{(t_l-1)u}]^T.$$

In the right picture of Figure 37, the B -blocks are the colored framed regions above the diagonal. The shading in some of these blocks will be explained in Example C.22. For a crossing sequence that is non-forward the definition is symmetric.

- (c) Assume the crossing sequence has a change of direction at t_l of the form $\rightarrow (i_{t_l}, j_{t_l}) \leftarrow$ and $f_{t_l t_l}$ is an arrow. In addition to the A and B blocks defined above we also have a gluing of two of these as follows.

- (i) If $a_l \neq t_{l-1}$ or $f_{t_{l-1}t_{l-1}}$ is not an arrow, define

$$AB_{l,u} = [i_{a_l u} \ i_{(a_l+1)u} \ \dots \ i_{t_l u} \ i_{(t_l+1)u} \ \dots \ i_{(t_{l+1})u}]^T, \text{ where } u \leq a_l - 1.$$

- (ii) If $a_l = t_{l-1}$ and $f_{t_{l-1}t_{l-1}}$ is an arrow, define

$$AB_{l,u} = [i_{(a_l+1)u} \ \dots \ i_{t_l u} \ i_{(t_l+1)u} \ \dots \ i_{(t_{l+1})u}]^T, \text{ where } u \leq a_l.$$

In a similar way we would also have blocks of type $BA_{l,u}$ with \bar{a}_l defined as the largest integer such that $f_{\bar{a}_l t_l}$ is an arrow and $u \geq \bar{a}_l + 1$ in type (i) or $u \geq \bar{a}_l$ in type (ii).

- (d) Assume the crossing sequence has a change of direction at t_l of the form $\leftarrow (i_{t_l}, j_{t_l}) \rightarrow$. Then the $A_{l,u}, B_{l,u}$ blocks are empty. The $A_{l+1,u}, A_{l-1,u}$ blocks are as in part (a)(ii) and the $B_{l+1,u}, B_{l-1,u}$ are as in part (b).

The following lemma examines the entries of the composition $f_\gamma g_1$ in positions corresponding to the column blocks of g_0 .

Lemma C.17. Let $[i_{s' u} \ i_{(s'+1)u} \ \dots \ i_{s'' u}]^T$ be a column block in g_0 , and let $s' \leq s \leq s''$. Assume that if the column block is $A_{l,u}$ of Definition C.16 and there exists $t' > s''$ such that $f_{s'' t'}$ is an arrow, then $s' \leq s < s''$. Then the coefficients of $(f_\gamma g_1)_{su}$ are equal for all s .

Proof. First, suppose that the column block is of type $A_{l,u}$ and it lies below the main diagonal as in Figure 37. Then $f_{s's''}, f_{(s'+1)s''}, \dots, f_{s''s''}$ are arrows. Moreover, there are no arrows in f_γ to the left of these entries by Definition C.16(a) of the column blocks. By Lemma C.15(1)(b) $(f_\gamma g_1)_{su} = f_{ss''} j_{s''u}$ for all $s' \leq s < s''$. Then the coefficient of $(f_\gamma g_1)_{su}$ is independent of s , so the lemma follows. If $s = s''$ and no t' as in the statement of the lemma exists then we still have the same equation for $(f_\gamma g_1)_{s''u}$, and the lemma holds.

If the column block is of type $AB_{l,u}$ or $BA_{l,u}$ then the argument is similar to the one above and also uses the moreover statement of part (1) and part (2) of Lemma C.15.

If the column block is of type $B_{l,u}$, and without loss of generality we may assume that $s'' < u$. Hence, the column block is to the right of the main diagonal in g_0 as in Figure 37, and using notation of Definition C.16 we have that $s' = t_{l-1}$ and $s'' = t_l - 1$ for some l . Also, note that $t' \geq t_l$. Then the row s of f_γ is

$$[0 \quad \dots \quad 0 \quad f_{ss} \quad \dots \quad f_{s(t_l-1)} \quad f_{s(t_l)} \quad 0 \quad \dots \quad 0]$$

and Lemma C.15 implies that

$$(f_\gamma g_1)_{su} = f_{s(t_l-1)} j_{(t_l-1)u} + f_{s(t_l)} j_{(t_l)u}$$

for all $s' \leq s < s''$ and $u > t_l$. In particular, the coefficient of $(f_\gamma g_1)_{su}$ is the sum of the coefficients of $j_{(t_l-1)u}$ and $j_{(t_l)u}$, which is independent of s . If $u = t_l$ then by Corollary A.20 we have that $(f_\gamma g_1)_{st_l} = 0$ because $f_{st_l} \neq 0$. This is again independent of s , so the lemma holds. \square

Next we will introduce row blocks in the matrix of g_0 . We assume that the steps in the crossing sequence are forward. The non-forward case is symmetric. The changes of direction will be discussed below as well.

Definition C.18. (a) We define *maximal one-step sequence* in the matrix of f_γ to be a maximal sequence of consecutive columns $m, m+1, \dots, m'$ where the diagonal entry i_{ll} and the entry right above the diagonal $i_{(l-1)l}$ are nonzero, and all other entries above the diagonal are zero.

(b) We define *maximal one-step sequence of arrows* in the matrix of f_γ to be a maximal sequence of consecutive columns $m, m+1, \dots, m'$ where the diagonal entry and the entry right above the diagonal are arrows.

Example C.19. In the example in Figure 37, there is one maximal one-step sequence of length two $m = t_4, m' = t_4 + 1$, and all other maximal one-step sequences are of length one $m = m' = t_l + 1$, with $l = 1, 2, 5, 6$.

The only two maximal one-step sequences of arrows are $m = m' = t_1$, or t_5 .

Remark C.20. Let $m, m+1, \dots, m'$ be a maximal one-step sequence. If there is no change of direction at $m-1$ and m' , then it follows from the definition of the staircase sequence C.5 of f_γ that there exists an l such that $m = t_l, m+1 = t_{l+1}, \dots, m'-1 = t_{l+m'-m-1}$, and m' is not one of the t_i of the staircase sequence. Moreover, we have $m-1 = t_{l-1}$. In this situation the columns $m-1$ and $m'+1$ has at least three nonzero entries on and above the diagonal.

If there is a change of direction at m' then $m' = t_{l+m'-m}$ is also in the staircase sequence. In this case, the column $m'+1$ is zero at every position above the diagonal.

If there is a change of direction at $m-1$ then the column $m-1$ is zero except at the diagonal position and $m-1$ is not in the staircase sequence.

Definition C.21 (Row Blocks). Let g be an endomorphism of M_γ and let g_0 be the induced endomorphism on the projective cover $P_0(\gamma) = \bigoplus_{s=1}^n P(i_s)$. We treat the change of direction in parts (a)(ii), (b)(ii) and (b)(iii). Otherwise, we assume that the crossing sequence is forward without change of direction. We define the following *row blocks* in the matrix of g_0 .

- (a) For each maximal one-step sequence $m, m+1, \dots, m'$, we define
 (i) Unless we are in the situation of case (ii) below, let

$$F_{v,m} = [i_{v,m-1} \ i_{vm} \ \dots \ i_{vm'}], \text{ where } v < m-1.$$

- (ii) If there is a change of direction at $m-1$ whose diagonal entry $f_{(m-1)(m-1)}$ is an arrow then let

$$F_{v,m} = [i_{vm} \ \dots \ i_{vm'}], \text{ where } v \leq m-1.$$

- (b) For each maximal one-step sequence of arrows $m, m+1, \dots, m'$ we have the following row blocks below the diagonal.

- (i) Unless we are in the situation of case (ii) or (iii) below, let

$$E_{v,m} = [i_{v(m-1)} \ i_{vm} \ \dots \ i_{vm'}], \text{ where } v \geq m' + 1.$$

- (ii) If there is a change of direction at $m' + 1$ such that $f_{m'(m'+1)}$ is an arrow, $f_{(m'+1)(m'+1)}$ is not an arrow, and $f_{(m'+3)(m'+1)} = 0$ then let

$$E_{v,m} = [i_{v(m-1)} \ i_{vm} \ \dots \ i_{vm'} \ i_{v(m'+1)}], \text{ where } v \geq m' + 3.$$

- (iii) If there is a change of direction at $m - 1$ whose diagonal entry is not an arrow then the block E_{vm} extends to the left into the adjacent F -block as follows,

$$FE_{v,m} = [i_{v(m'')} \ \dots \ i_{v(m-1)} \ i_{vm} \ \dots \ i_{vm'}], \text{ where } v \geq m' + 1$$

and m'' is the first column of the (backward) maximal one-step sequence ending at $m' - 1$.

In a similar way we would also have blocks of type $EF_{v,m}$ above the diagonal.

- (iv) In a case (i), (ii) or (iii) above we have $f_{(m-1)(m-1)}$ or $f_{m'(m'+1)}$ is an arrow then the row block will behave somewhat differently in what follows, and we refer to it as a row block of type (iv).

Example C.22. In the example of Figure 37 the row blocks $F_{v,m}$ are formed by those colored blocks $B_{l,u}$ that are non-shaded. For example row one of the matrix has the following five row blocks.

$$[i_{1t_1} \ i_{1(t_1+1)}], [i_{1t_2} \ i_{1(t_2+1)}], [i_{1t_3} \ i_{1t_4} \ i_{1(t_4+1)}], [i_{1t_5} \ i_{1(t_5+1)}], [i_{1t_6} \ i_{1(t_6+1)}]$$

C.2.4. *The cases where g_0 is constant on the column blocks or alternating on the row blocks.* Each entry of the matrix of g_0 is given by a product of a scalar coefficient and path. If C is any submatrix, we say that g_0 is *constant on C* if each entry of g_0 in C has the same scalar coefficient. If the submatrix C has only one row, we say that g_0 is *alternating on C* if the scalar coefficient is constant up to sign and the sign alternates along the row.

Lemma C.23. *Let g be a nilpotent endomorphism of M_γ . Then $g = 0$ in CMP B in each of the following cases.*

- (i) g_0 is constant on a block $B_{l,u}$ and zero elsewhere,
- (ii) g_0 is constant on a block $A_{l,u}$ and zero elsewhere,
- (iii) g_0 is constant on $A_{l,u} \cup \{i_{a_l u}\}$ and zero elsewhere, where $A_{l,u} = [i_{(a_l+1)u} \ i_{(a_l+2)u} \ \dots \ i_{t_l u}]^T$ is a column block as in part (a)(ii) of Definition C.16,
- (iv) g_0 is constant on a block $AB_{l,u}$ or a block $BA_{l,u}$ and zero elsewhere.

Proof. We are going to construct a morphism $h: P_0(\gamma) \rightarrow P_1(\gamma)$ such that $g_0 = f_\gamma h$. Then the result will follow from Lemma C.10.

(i) In order to construct h , we first must take a closer look at g_0 . Let (s, u) be a nonzero position in the matrix of g_0 that lies in the block $B_{l,u}$. Thus $t_{l-1} \leq s \leq t_l - 1$ and $t_l \leq u$. Then $(g_0)_{su}$ is a scalar multiple of a path from i_s to i_u . We shall show that this path factors through j_{t_l} .

Indeed, if $t_l - t_{l-1} = 1$ then $s = t_l - 1$ and the block $B_{l,u}$ consists of a single position $(g_0)_{(t_l-1)u}$, $u \geq t_l$, and the step Σ_{t_l-1} in the crossing sequence of γ is trapezoidal of degree 1, by Corollary C.8. It is not hard to see that in this case the path $(g_0)_{(t_l-1)u}$ from i_{t_l-1} to i_u factors through j_{t_l-1} .

On the other hand, if $t_l - t_{l-1} \geq 2$ then Σ_{t_l-1} is rectangular of degree zero, again by Corollary C.8. This case is illustrated in Figure 43. Thus the path in $(g_0)_{su}$ from i_s to i_u with $s \leq t_l - 1$, $u \geq t_l$ factors through j_{t_l-1} .

Therefore, up to a scalar coefficient, $(g_0)_{su}$ is a path $w: i_s \rightsquigarrow j_{t_l-1} \rightsquigarrow i_u$. Moreover, since $t_{l-1} \leq s \leq t_l - 1$, the staircase shape of f_γ implies that $f_{s(t_l-1)} \neq 0$, and thus there exists a valid path $f_{s(t_l-1)}: i_s \rightsquigarrow j_{t_l-1}$. Let $h_{(t_l-1)u}$ denote the subpath of w from j_{t_l-1} to i_u , multiplied by the coefficient of $(g_0)_{su}$. Here we use the fact that this coefficient does not depend on s , since g_0 is constant on the block $B_{l,u}$. Then $h_{(t_l-1)u}$ is a morphism from $P(i_u)$ to $P(j_{t_l-1})$ and we have

$$(g_0)_{su} = f_{s(t_l-1)} h_{(t_l-1)u}.$$

Note that the right hand side of this equation is the product of a single entry of f_γ with $h_{(t_l-1)u}$. We now shall consider the product of f_γ with $h_{(t_l-1)u}$.

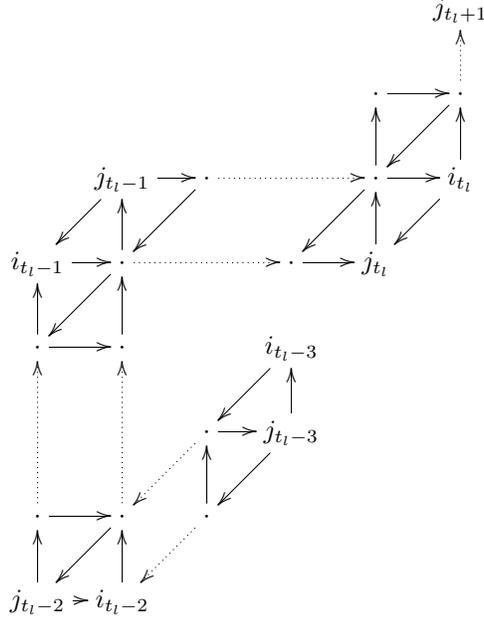


FIGURE 43. Proof of Lemma C.23. The steps $\Sigma_{t_l-3}, \Sigma_{t_l-2}$ are both T_0 and the step Σ_{t_l-1} is R_0 .

If the block $B_{l,u}$ is of size at least 2, that is $t_l - t_{l-1} \geq 2$ then

$$f_\gamma h_{(t_l-1)u} = (f_{s'(t_l-1)} h_{(t_l-1)u})_{t_{l-1} \leq s' \leq t_l-1} = g_0|_{B_{l,u}}.$$

In this case, we define $h = h_{(t_l-1)u}$ at position $(t_l-1)u$ and zero elsewhere.

If the block $B_{l,u}$ is of size 1, that is $t_l - t_{l-1} = 1$, then

$$f_\gamma h_{(t_l-1)u} = (f_{s'(t_l-1)} h_{(t_l-1)u})_{t_{l-2} \leq s' \leq t_l-1}$$

is bigger than $B_{l,u}$. In this case, we defined h to be $h_{(t_l-1)u}$ at position $(t_l-1)u$, and $-h_{(t_{l-1}-1)u}$ at position $(t_{l-1}-1)u$. Then if the previous block $B_{l-1,u}$ has size greater than 1 then $(f_\gamma h) = g_0$. And if $B_{l-1,u}$ has size 1 as well, then we add $h_{(t_{l-2}-1)u}$ to h in order to compensate again. Continuing this way will produce the desired map h .

(ii) and (iii) Recall that a_l is the least integer such that in column t_l of the matrix of f_γ the positions a_l, a_l+1, \dots, t_l are given by arrows $i_{a_l} \rightarrow j_{t_l}, i_{a_l+1} \rightarrow j_{t_l}, \dots, i_{t_l} \rightarrow j_{t_l}$. Suppose first that the arrow between i_{a_l} and j_{a_l} is in the direction $j_{a_l} \rightarrow i_{a_l}$. The quiver in this case is illustrated in the left picture of Figure 44. We call the configuration given by the full subquiver on the vertices $i_{a_l}, j_{a_l}, \dots, i_{t_l}, j_{t_l}$ a flower centered at j_{t_l} . In this situation the position (a_l, a_l) in the matrix f_γ is not an arrow. Thus the block $A_{l,u}$ is of type (a)(i) in Definition C.16. Hence, it consists of the positions a_l, a_l+1, \dots, t_l in column u of the matrix g_0 , where $u \leq a_l - 1$. The entries in this block are paths starting at one of the points i_{a_l}, \dots, i_{t_l} and ending at i_u . From the figure it is clear that each of these paths must factor through j_{t_l} . Define h_1 to be the path $j_{t_l} \rightsquigarrow i_u$ at position (t_l, u) and zero elsewhere. Then $f_\gamma h_1 = g_0|_{A_{l,u}}$, and we are done.

Now suppose that the arrow is $i_{a_l} \rightarrow j_{a_l}$. The quiver in this case is illustrated in the right picture of Figure 44. In this situation, the flower centered at j_{t_l} does not contain the vertex j_{a_l} , and the position (a_l, a_l) in the matrix f_γ is an arrow. Thus the block $A_{l,u}$ consists of the positions a_l+1, a_l+2, \dots, t_l in column u of the matrix g_0 , where again $u \leq a_l$. Again each of the entries in this block is a path that factor through j_{t_l} . In case (iii) this completes the proof. However, in case (ii), the map h_1 defined above will not work, because its composition with f_γ would contain the nonzero path $i_{a_l} \rightarrow j_{t_l} \rightsquigarrow i_u$ which is not part of the block $A_{l,u}$. In order to compensate, we note that this path factors through j_{a_l} as well. So we can define $h_2 = h_1 - (j_{a_l} \rightsquigarrow i_u)$, so h_2 has two nonzero positions $j_{t_l} \rightsquigarrow i_u$ and $-(j_{a_l} \rightsquigarrow i_u)$, and then $f_\gamma h_2$ is equal to g_0 on the block $A_{l,u}$ and is zero at position (a_l, u) . Thus if there is no other arrow from a vertex i_s to the vertex j_{a_l} with $s < a_l$ then this h_2 has the desired property.

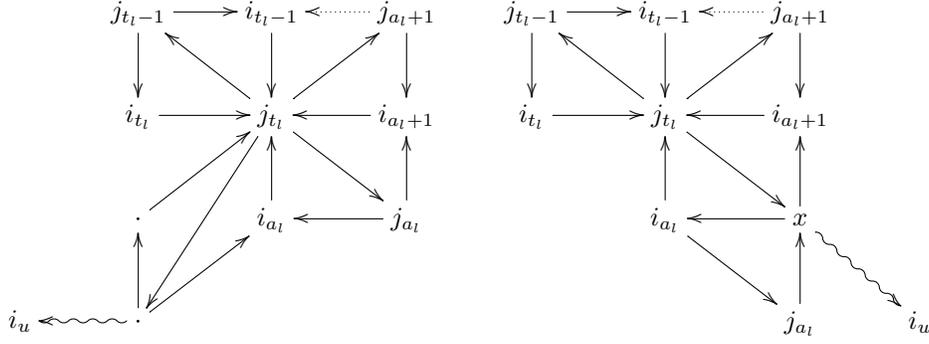


FIGURE 44. Local configuration in the situation of block $A_{l,u}$. On the left, the arrow of the a_l -th crossing pair is $j_{a_l} \rightarrow i_{a_l}$; on the right, the arrow is $i_{a_l} \rightarrow j_{a_l}$.

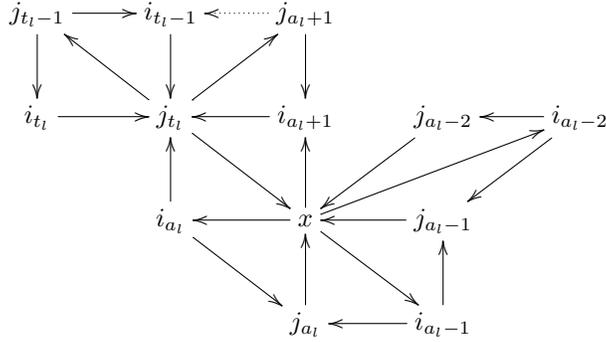


FIGURE 45. Local configuration in the situation of block $A_{l,u}$ in the case where $i_{a_l} \rightarrow j_{a_l}$, $i_{a_l-1} \rightarrow j_{a_l}$ and $i_{a_l-1} \rightarrow j_{a_l-1}$.

Now suppose there is an arrow $i_s \rightarrow j_{a_l}$ with $s < a_l$. Here we need to consider two subcases, depending on whether or not the vertex x in the right picture in Figure 44 is the vertex j_{a_l-1} . If it is, then the path $i_{a_l} \rightarrow j_{t_l} \rightsquigarrow i_u$ would be

$$i_{a_l} \rightarrow j_{t_l} \rightarrow j_{a_l-1} \rightarrow i_{a_l-1} \rightarrow j_{a_l} \rightsquigarrow i_u,$$

and, since the initial piece $i_{a_l} \rightarrow j_{t_l} \rightarrow j_{a_l-1}$ factors through j_{a_l} , this path is zero unless $u = a_l - 1$. Hence our first morphism h_1 has the desired property if $u < a_l - 1$, and our second morphism h_2 has the desired property if $u = a_l - 1$. If, on the other hand, the vertex x is not j_{a_l-1} then either there is no arrow $i_{a_l-1} \rightarrow j_{a_l}$ and in this case we can use the map h_2 , or there is an arrow $i_{a_l-1} \rightarrow j_{a_l}$, see Figure 45.

In that figure, we have an arrow from j_{a_l-1} to x which implies that the quiver contains a flower at x and thus there exists a path from x to some i_u with $u < a_l - 1$ (for example $u = a_l - 2$) that does not factor through i_{a_l-1} . In this situation, the composition $f_\gamma h_2$ contains the term $-(i_{a_l-1} \rightarrow j_{a_l} \rightsquigarrow i_u)$ which is not in the block $A_{l,u}$, and again we need to compensate. In this case, the map $h_3 = h_2 + (j_{a_l-1} \rightsquigarrow i_u)$ has the desired property.

This process will continue if there is another flower at a vertex y that is not one of the vertices i, j of the crossing sequence at that has an arrow from x . After a finite number of steps, it will produce a map h such that $f_\gamma h = g_0$ and we are done.

Let us also remark that it is possible that the crossing sequence has a change of direction at the pair (i_{a_l}, j_{a_l}) . In that situation, the map $h = \begin{cases} h_1 & \text{if } i_{a_l} \rightarrow j_{a_l} \text{ is an arrow;} \\ h_2 & \text{if } i_{a_l} \rightsquigarrow j_{a_l} \text{ is not an arrow} \end{cases}$ will have the desired property.

(iv) This case is similar to case (ii). □

Next we are going to study the case where g_0 is alternating on row blocks.

Lemma C.24. *Let g be a nilpotent endomorphism of M_γ such that g_0 is alternating on a row block and zero elsewhere else.*

- (a) *If the row block is not of type (iv) then $g_0 f_\gamma = 0$. In particular $g = 0$ in CMP B .*
- (b) *If the row block is of type (iv) then $g_0 f_\gamma$ is zero everywhere except possibly at two positions.*

Proof. We may assume without loss of generality that the crossing sequence is forward. Suppose first that the row block is $F_{v,m}$ as in part (a) of Definition C.21. Let (v, t) be a position in $F_{v,m}$. Suppose first that $t \geq m$. Since we have a maximal one-step sequence, the parameter u in part (a) of Lemma C.11 is equal to $t - 1$. Therefore equation (C.1) implies that $(g_0 f_\gamma)_{vt} = i_{v(t-1)} f_{(t-1)t} + i_{vt} f_{tt}$. Both terms in this expressions are equal to scalar multiples of the unique path from i_v to j_t and the scalar is equal to the coefficient of $i_{v(t-1)}$ and i_{vt} , respectively. Thus the expression is zero because g_0 is alternating on the block.

Now suppose $t = m - 1$. Then the column t is not part of the maximal one-step sequence. Suppose first that there is not change of direction at $m - 1$. Then Lemma C.11(a) implies $(g_0 f_\gamma)_{v(m-1)} = i_{vu} f_{u(m-1)} + i_{v(u+1)} f_{(u+1)(m-1)}$, with $u < m - 2$. Thus the positions $(v, u), (v, u + 1)$ are not part of the block $F_{v,m}$, hence $i_{vu} = i_{v(u+1)} = 0$, and we are done. Now suppose there is a change of direction at $m - 1$. Then the moreover statement of Lemma C.11(a) implies that $i_{vm} f_{mm} = 0$ and thus $(g_0 f_\gamma)_{vm} = i_{v(m-1)} f_{(m-1)m}$. But $i_{v(m-1)} = 0$, since $(v, m - 1)$ is not part of the block, because of Definition C.21(a)(ii). This shows that $(g_0 f_\gamma)_{vm} = 0$. Moreover, $(v, m - 1)$ not being part of the block also implies $(g_0 f_\gamma)_{v(m-1)} = 0$. This completes the proof in the case when the row block is of type F .

Now suppose that the row block is $E_{v,m}$ as in part (b)(i) of Definition C.21. Let (v, t) be a position in $E_{v,m}$. Suppose first that $m \leq t \leq m'$. In this case, we have a maximal one-step sequence of arrows, and thus the parameter t' in part (b) of Lemma C.11 is equal to $t - 1$. Therefore equation (C.2) implies that $(g_0 f_\gamma)_{vt} = i_{vt} f_{tt} + i_{v(t-1)} f_{(t-1)t}$ and both terms are nonzero. By the same argument as in the previous case, we see that the expression is zero because g_0 is alternating on the block.

Now suppose that $t = m - 1$ and assume first that $E_{v,m}$ is not of type (iv). In particular $f_{(m-1)(m-1)}$ is not an arrow. Then in the composition $g_0 f_\gamma$ in column $m - 1$, only the term $f_{(m-1)(m-1)}$ may contribute since g_0 is zero in columns t with $t < m - 1$. Now, since $f_{(m-1)(m-1)}$ is not an arrow then by Lemma C.11 (b), $g_0 f_\gamma$ is zero in column $m - 1$. Finally suppose that $t = m' + 1$. Similarly to the previous case, $f_{m'(m'+1)}$ is not an arrow. Note, that since $f_{m'm'}$ is an arrow then $f_{(m'-1)(m'+1)} = 0$, by Lemma C.3 (c). Thus in column $m' + 1$ of f_γ , we will have two terms $f_{(m'+1)(m'+1)}, f_{m'(m'+1)}$, and then $(g_0 f_\gamma)_{v(m'+1)} = i_{v(m'+1)} f_{(m'+1)(m'+1)} + i_{vm'} f_{m'(m'+1)}$. Now, $i_{vm'} f_{m'(m'+1)} = 0$, because of Lemma C.11(b) using the fact that $f_{m'(m'+1)}$ is not an arrow, and $i_{v(m'+1)} f_{(m'+1)(m'+1)} = 0$, because the position $(v, m' + 1)$ is not in the block.

If the row block $E_{v,m}$ is as in part (b)(ii) of Definition C.21 then Lemma C.13 with $t' = m', t = m' + 1$ implies that $(g_0 f_\gamma)_{v(m'+1)}$ has an additional term $i_{vm'} f_{m'(m'+1)}$. Thus

$$(g_0 f_\gamma)_{v(m'+1)} = i_{vm'} f_{m'(m'+1)} + i_{v(m'+1)} f_{(m'+1)(m'+1)} + i_{v(m'+2)} f_{(m'+2)(m'+1)}.$$

However, the position $(v, (m' + 2))$ is not in $E_{v,m}$ so $i_{v(m'+2)} = 0$ and thus the above expression has only two non-zero terms. Again since g_0 is alternating on the block, the sum is zero.

If the row block is $FE_{v,m}$ or $EF_{v,m}$ as in part (b)(iii) of Definition C.21 then it consists of an E -block and an F -block that are joined. This case follows by combining the previous ones. Moreover, Lemma C.14 guarantees that the $g_0 f_\gamma$ is also zero at position $(v, m - 1)$, because $f_{(m-1)(m-1)}$ is not an arrow.

Thus in all cases we have $g_0 f_\gamma = 0$. In particular, Lemma C.10 implies $g = 0$ in CMP B .

(b) Now assume that $E_{v,m}$ is of type (iv). In this case the entries $(g_0 f_\gamma)_{vt}$, with t the first or last entry of the block may be nonzero. \square

C.2.5. Proof of Theorem 6.1.

Theorem C.25. *Let γ be a 2-diagonal and M_γ the associated indecomposable syzygy over B . Then M_γ does not admit any nonzero nilpotent endomorphisms in CMP B .*

Proof. Let g be a nilpotent endomorphism of M_γ with the corresponding maps g_0, g_1 on the projective presentation of M_γ as in the diagram (6.1). Let $g_0 = (i_{st}), g_1 = (j_{st})$ denote the entries of these matrices. Recall that i_{st} is a path $i_s \rightsquigarrow j_t$ scaled by some coefficient from the field \mathbb{k} , and since g is nilpotent $i_{st} = 0$ if $s = t$. The strategy of the proof is as follows. Given a nonzero entry i_{st} of g_0 we construct a map $g'_0 : P_0(\gamma) \rightarrow P_0(\gamma)$ that induces a nilpotent endomorphism g' of M_γ such that $g' = 0$ in CMP B . Moreover,

$(g'_0)_{st} = i_{st}$ and $g_0 - g'_0$ has fewer nonzero entries than g_0 . Then replacing g_0 with $g_0 - g'_0$ and repeating the argument we conclude that $g_0 = 0$ and, hence, $g = 0$ in $\underline{\text{CMP}} B$.

Let i_{st} be a nonzero entry of g_0 . Next we consider several cases based on whether i_{st} belongs to a column block and/or a row block of g_0 .

(1) Suppose that i_{st} belongs to a column block that consists of a single term. Let g'_0 be the map that contains the entry i_{st} in position (s, t) and zero everywhere else. Then by Lemma C.23 g'_0 induces an endomorphism of M_γ that is zero in $\underline{\text{CMP}} B$. By construction, $g_0 - g'_0$ has fewer nonzero entries than g_0 . This shows the desired conclusion in this case.

(2) Suppose that i_{st} does not belong a column block and $f_{st} \neq 0$. Without loss of generality we may assume that $s < t$, then t is not one of the t_l in the staircase sequence, see Figure 37. In particular, $t_{l-1} \leq s < t < t_l$ for some l , and $f_{tu} \neq 0$ if and only if $u = t, t+1, \dots, t_l$. Moreover, note that $f_{su} \neq 0$ for $u = t, \dots, t_l$. Let g'_0 be the map that contains the entry i_{st} in position (s, t) and zero everywhere else. Then the composition $g'_0 f$ is zero everywhere except in positions (s, u) where $(g'_0 f_\gamma)_{su} = i_{st} f_{tu}$ and $u = t, t+1, \dots, t_l$. However, since $f_{su} \neq 0$, Corollary A.20 implies that $(g'_0 f_\gamma)_{su} = 0$. Hence, $g'_0 f_\gamma = 0$. In particular, g'_0 induces an endomorphism g' of M_γ which is zero in $\underline{\text{CMP}} B$. By construction, $g_0 - g'_0$ has fewer nonzero entries than g_0 , which yields the desired conclusion in this case.

(3) Suppose that i_{st} does not belong to a column block and $f_{st} = 0$. Without loss of generality we may assume that $f_{s(s-1)} = 0$, so in particular $f_{su} = 0$ for all $u < s$. Then since i_{st} does not belong to a column block and $f_{st} = 0$, it follows that $s > t$ and f_{su} is not an arrow for any $u \geq s$, see Figure 37. In this case, Lemma C.15 parts (1b) and (4) imply the following equation.

$$(C.12) \quad (fg_1)_{su} = 0 \text{ for } u \leq s$$

Now we consider two subcases depending on whether i_{st} belongs to a row block or not.

(3a) Suppose that i_{st} belongs to a row block

$$[i_{st'} \quad i_{s(t'+1)} \quad \dots \quad i_{st} \quad \dots \quad i_{st''}].$$

Then by definition of a row block there is a maximal one-step sequence in f_γ and thus column u of f_γ has exactly two nonzero entries. Then the composition

$$(C.13) \quad (g_0 f)_{su} = i_{sv} f_{vu} + i_{s(v+1)} f_{(v+1)u} \text{ for } t' < u, v < t''$$

where $v = u$ if the row block is an F block and $v+1 = u$ if the row block is an E block. The case when the row block is an EF or an FE block follows similarly, so we omit it from the discussion. If the row block is not of type (iv) then equation (C.13) also holds when $u = t''$ and $u = t'$. In this case (C.12) implies that two consecutive entries in g_0 in the row block have opposite signs. Let g'_0 be the map that agrees with g_0 on the entries of the row block and is zero everywhere else. By Lemma C.24(a) the map g'_0 induces an endomorphism of M_γ that is zero in $\underline{\text{CMP}} B$. By construction, $g_0 - g'_0$ has fewer nonzero entries than g_0 , which yields the desired conclusion in the case when the row block is not of type (iv).

Otherwise, suppose that the row block is of type (iv). Then $f_{t't'}$ is an arrow, and $i_{st'} f_{t't'}$ is a summand of $(g_0 f)_{st'}$ or $f_{t''(t'+1)}$ is an arrow and $i_{st''} f_{t''(t'+1)}$ is a summand of $(g_0 f)_{s(t'+1)}$ or both of these cases hold. If $(g_0 f)_{st'} = i_{st'} f_{t't'}$ then equations (C.12) and (C.13) imply that g_0 is constantly zero on the row block. In particular, $i_{st} = 0$, contrary to our assumption in the beginning of the proof, or $(g_0 f)_{st'} = 0$ and here we can proceed in the same way as if the row block is not of type (iv). If $f_{t''(t'+1)}$ is an arrow, then $f_{(t'+1)(t'+1)}$ is not an arrow because the maximal one-step sequence of arrows ends at t'' , so $(g_0 f)_{s(t'+1)} = i_{st''} f_{t''(t'+1)}$ by Lemma C.11(b). Then similarly to the above we conclude that $i_{st} = 0$. It remains to consider the case when $(g_0 f)_{st'} = i_{st'} f_{t't'} + i_{st''} f_{t''t'}$ where $f_{t''t'}$ is an arrow and $t''' < t'$ is maximal. We claim that $i_{st'''} = 0$, so again we can proceed in the same way as before. Note that $t''' < t' - 1$ because the maximal one-step sequence of arrows starts at $t'+1$. Hence, $f_{(t'-1)t'}$ is not an arrow and then Lemma C.9 implies that $f_{t''(t'-1)}$ is an arrow and t''' is maximal such that $f_{t''(t'-1)}$ is an arrow. In particular, Lemma C.11(b) implies that $(g_0 f)_{s(t'-1)} = i_{st'''} f_{t''(t'-1)}$, which equals zero by (C.12). This shows the claim that $i_{st'''} = 0$, and completes the proof in case (3a).

(3b) Suppose that i_{st} does not belong to a row block and $f_{st} = 0$. Then let g'_0 be the map that contains the entry i_{st} in position (s, t) and zero everywhere else. First, assume that $f_{(t+1)t} \neq 0$. Then $f_{(t+2)t} \neq 0$ since t is not part of a maximal one-step sequence. In particular, if $f_{tu} \neq 0$ then $f_{(t+1)u}, f_{(t+2)u}$ are also

nonzero. Then the composition $g'_0 f_\gamma$ is zero everywhere except possibly in position (s, u) , where it equals $i_{st} f_{tu}$ for $u \leq s$. However, the parameter v in Lemma C.12(b) is at least $t + 2$ and equation (C.5) implies that f_{tu} does not contribute to $(g'_0 f_\gamma)_{su}$. Therefore, $(g'_0 f_\gamma)_{su} = 0$. This completes the proof in case (3b) whenever $f_{(t+1)t} \neq 0$. Note that in this argument we did not need the stronger assumption of (3) that i_{st} does not belong to a column block.

Now suppose that $f_{(t+1)t} = 0$. Then $(g'_0 f)_{st'}$ equal zero or it equals $i_{st} f_{tt'}$ if $f_{tt'}$ is an arrow. However, equation (C.12) implies that $i_{st} f_{tt'} = 0$ and so $g'_0 f = 0$. Hence, $g'_0 f = 0$. In particular, g'_0 induces an endomorphism g' of M_γ which is zero in $\underline{\text{CMP}} B$. By construction, $g_0 - g'_0$ has fewer nonzero entries than g_0 , which yields the desired conclusion. This completes the proof of case (3).

(4) Suppose that i_{st} belongs to a column block

$$[i_{s't} \quad i_{(s'+1)t} \quad \dots \quad i_{st} \quad \dots \quad i_{s''t}]^T$$

of size greater than one. Assume without loss of generality that $s > t$.

(4a) Suppose that i_{st} also belongs to a row block

$$[i_{st'} \quad i_{s(t'+1)} \quad \dots \quad i_{st} \quad \dots \quad i_{st''}].$$

Let C be the submatrix of g_0 with entries i_{uv} where $s' \leq u \leq s''$ and $t' \leq v \leq t''$. Hence, every column of C is a column block of g_0 and every row of C is a row block of g_0 . Let g'_0 be map that agrees with g_0 on the entries in C and zero everywhere else. We will show that g'_0 induces a nilpotent endomorphism g' of M_γ . Recall that every i_{uv} of C is a path $i_u \rightsquigarrow i_v$ multiplied by a coefficient $a_{uv} \in \mathbb{k}$. Denote by $A = (a_{uv})$ the matrix of the coefficients in C .

First, suppose that the row blocks in C are not of type (iv). Assume further that if the column block is $A_{l,u}$ of Definition C.16 then there does not exist a $v > s''$ such that $f_{s''v}$ is an arrow. In this situation, we may use Lemma C.17 to conclude that for all $s' \leq u \leq s''$ and $t' \leq v \leq t''$ we have relations $a_{uv} + a_{u(v+1)} = a_{(u+1)v} + a_{(u+1)(v+1)}$ which imply

$$(C.14) \quad a_{(u+1)v} - a_{uv} = -(a_{(u+1)(v+1)} - a_{u(v+1)})$$

Define a matrix A_{col} of the same size as A such that every column of A_{col} is constant and equal to the first entry in the corresponding column of A . Then

$$A - A_{col} = \begin{bmatrix} 0 & 0 & 0 & \dots \\ a_{(s'+1)s'} - a_{s's'} & a_{(s'+1)(s'+1)} - a_{s'(s'+1)} & a_{(s'+1)(s'+2)} - a_{s'(s'+2)} & \dots \\ a_{(s'+2)s'} - a_{s's'} & a_{(s'+2)(s'+1)} - a_{s'(s'+1)} & a_{(s'+2)(s'+2)} - a_{s'(s'+2)} & \dots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

We will show that this matrix is alternating on the rows. Indeed, for the first row this is trivial, and for the second row it follows directly from the relation (C.14) with $u = s'$. For the row indexed by $u > s' + 1$ it follows from the following telescoping argument

$$a_{uv} - a_{1v} = \sum_{l=1}^{u-1} a_{(l+1)v} - a_{lv} \stackrel{(C.14)}{=} \sum_{l=1}^{u-1} -(a_{(l+1)(v+1)} - a_{l(v+1)}) = -(a_{u(v+1)} - a_{1(v+1)}).$$

Thus g'_0 can be written as a sum $g'_0 = g_{col} + g_{row}$, where g_{col} is constant on each block and g_{row} is alternating on the rows. Now Lemmata C.23(i) and C.24(a) imply that $g' = 0$ in $\underline{\text{CMP}} B$.

Assume now that the column block is $A_{l,u}$ of Definition C.16 and there exists $v > s''$ such that $f_{s''v}$ is an arrow. Since $s > t$ this implies that the crossing sequence is forward. Since Lemma C.17 also applies in this case except for the last row s'' of the column blocks, we can use the above argument for all rows except for s'' . Hence we may assume that g_0 is zero on all positions (s, t) in these blocks with $s < s''$. We want to find a g'_0 that is equal to g_0 on row s'' and zero on all rows above such that there is a morphism h such that $g'_0 = f_\gamma h$.

In the very special case where $v = s'' + 1$ and $f_{(s''+1)(s''+1)}$ is not an arrow, we have $f_{(s''-1)(s''-1)}$ is not an arrow and $f_{(s''-1)s''}$ is an arrow (because the column block is not of size one), $f_{(s''-1)(s''+1)}$ is zero (by Lemma C.3(c)) and therefore it is easy to see that the map h that is equal to the path $j_{(s''+1)} \rightsquigarrow i_t$ and zero elsewhere realizes a factorization $g'_0 = f_\gamma h$ whenever g'_0 is zero everywhere except for a path $i_{s''} \rightsquigarrow i_t$. So we are done. Otherwise, $f_{s''v}, f_{(s''+1)v}, \dots, f_{vv}$ are arrows and we get a second set of column blocks $A_{l+1,u}$ with the property that the top entry of $A_{l+1,u}$ lies directly below the bottom entry of $A_{l,u}$. Then the block

$A_{l+1,u}$ is of type (a)(ii) in Definition C.16 and therefore satisfies the conditions of Lemma C.23(iii). Define g'_0 to be equal to $(g_0)_{s''u}$ on each entry of $A_{l+1,u} \cup \{i_{s''u}\}$ and zero elsewhere. Then Lemma C.23(iii) implies g' is zero in $\underline{\text{CMP}} B$. Subtracting g'_0 from g_0 , we obtain a zero in each position on row s'' and we are done. This completes the case (4a) when the row blocks are not of type (iv).

Now, suppose that the row blocks are of type (iv). Then at least one of $f_{t't'}$, $f_{t''(t''+1)}$ is an arrow. We consider the case when $f_{t't'}$ is an arrow and the remaining cases follow similarly. Then we still obtain the same relations in (C.14), however the map g_{row} which is alternating on the row blocks may no longer induce an endomorphism of M_γ , see Lemma C.24(b). Since $f_{t't'}$ is an arrow and $f_{(t'-1)t'}$ is not an arrow, because the maximal one-step sequence of arrows starts with $t' + 1$. Then by Lemma C.15(2a).

$$(C.15) \quad (g_0 f)_{ut'} = i_{ut''} f_{t't'} + i_{ut'''} f_{t''t'} \text{ for } s' \leq u \leq s''$$

where $t''' < t' + 1$ is maximal such that $f_{t''t'}$ is an arrow, and if no such t''' exists then this term does not appear in the equation. Since $f_{t''t'}$ is an arrow and $f_{(t'-1)t'}$ is not an arrow then Lemma C.9 implies that $f_{t''(t'-1)}$ is also an arrow and no entry in f below this one is an arrow. Lemma C.15(2a) implies that

$$(g_0 f)_{u(t'-1)} = i_{ut'''} f_{t''(t'-1)} \text{ for } s' \leq u \leq s''$$

Then Lemma C.17 implies that the coefficients $a_{ut''}$ for $s' \leq u \leq s''$ are all the same (except maybe for s'' , and this case is similar to the previous case). The same lemma together with (C.15) imply that $a_{ut'}$ for all $s' \leq u \leq s''$ are also the same. Similarly, (C.14) then shows that for a fixed v the coefficients a_{uv} are the same for all $s' \leq u \leq s''$. In particular, this means that the matrix $A = A_{\text{col}}$ is constant on the columns. Hence, we conclude that $g' = 0$ in $\underline{\text{CMP}} B$. This completes the proof in the case (4a).

(4b) Suppose that i_{st} does not belong to a row block. If $f_{(t+1)t} \neq 0$, then we proceed in the same way as in the first part of (3b). Note that here $f_{st} = 0$, since i_{st} lies in a column block but not in row block. Therefore, we can assume that $f_{(t+1)t} = 0$. Since i_{st} is not in a row block then it is not the case that f_{tt} , $f_{(t-1)t}$ are both arrows. Let g'_0 be the map that agrees with g_0 on the column block and is zero everywhere else. We will show that g'_0 gives a nilpotent endomorphism g' of M_γ , so $g_0 - g'_0$ has fewer nonzero entries than g_0 , which yields the desired conclusion.

From Lemma C.11(b) it follows that if the row t of f_γ does not contain any arrows then the composition $g_0 f_\gamma$ does not contain any term that involves entries of g_0 coming from the column block. Then $g'_0 f_\gamma = 0$ and thus g'_0 gives a nilpotent endomorphism g' of M_γ and the conclusion holds.

If f_{tt} is an arrow then $f_{(t-1)t}$ is not an arrow and

$$(C.16) \quad (g_0 f)_{vt} = i_{vt} f_{tt} + i_{vt'} f_{t't} \text{ for } v = s', \dots, s''$$

where $t' < t - 1$ is maximal such that $f_{t't}$ is an arrow. If no such t' exists then the second summand does not appear in equation (C.16). We suppose that t' exists, and the other case is a special situation of this one. Since $f_{(t-1)t}$ is not an arrow, by Lemma C.9 it follows that $f_{t'(t-1)}$ is an arrow and $f_{u(t-1)}$ with $u > t'$ is not an arrow. Then

$$(C.17) \quad (g_0 f)_{v(t-1)} = i_{vt'} f_{t'(t-1)} \text{ for } v = s', \dots, s''$$

The entries of $(f g_1)_{v(t-1)}$ and $(f g_1)_{vt}$ do not depend on v by Lemma C.17 (except maybe for the last row s'' which is similar as in case (4a)), so equations (C.16) and (C.17) imply that either $i_{vt} f_{tt} = 0$ for all $v = s', \dots, s''$ and in that case $g'_0 f_\gamma = 0$, or the coefficients of $i_{vt'}$ are nonzero and do not depend on v , and hence the coefficients of i_{vt} are nonzero and do not depend on v , thus g_0 is constant on the column block. Either way we conclude that g'_0 gives a nilpotent endomorphism g' of M_γ and the conclusion holds in the case when f_{tt} is an arrow.

Finally suppose that f_{tt} is not an arrow and $f_{t't}$ is an arrow for some $t' > t$ that is minimal. First suppose that $t' = t + 1$. Since i_{st} does not belong to a row block, then it is not the case that $f_{t(t+1)}$, $f_{(t+1)(t+1)}$ are both arrows. Hence $f_{(t+1)(t+1)}$ is not an arrow. Then

$$(g_0 f_\gamma)_{v(t+1)} = i_{vt} f_{t(t+1)} \text{ for } v = s', \dots, s''$$

so by the same argument as above we reach the desired conclusion. Otherwise, if $t' > t + 1$ then $f_{t(t+1)}$ is not an arrow and Lemma C.9 implies that $f_{(t+1)t'}$ is an arrow. Then the entries in the column block of g_0 do not appear in the composition $g_0 f_\gamma$ by Lemma C.11(b), and thus $g'_0 f_\gamma = 0$. This completes the proof in case (4), and finishes the proof of the theorem. \square

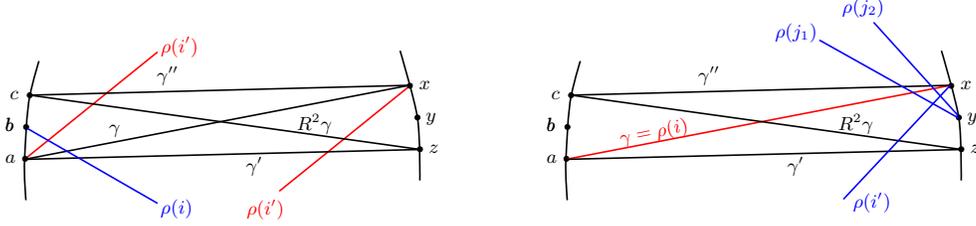


FIGURE 46. Proof of Proposition 6.3. The radical line $\rho(i')$ is different from γ in the picture on the left and $\gamma = \rho(i)$ in the picture on the right.

C.3. Proof of Proposition 6.3.

Proposition C.26. *Let γ be a 2-diagonal in \mathcal{S} such that $R\gamma$ is not a radical line. With the notation of diagram (6.2), we have the following identities in $\underline{\text{CMP}} B$.*

- (a) *If γ is not short then $h'g' = h''g''$.*
- (b) *if γ is short then $h'g' = 0$.*

Proof. (a) We consider two cases based on whether or not γ is a radical line.

(1) First, suppose that neither γ nor $R^2\gamma$ is a radical line. Since none of $\gamma, R\gamma, R^2\gamma$ is a radical line, we can choose representatives $\gamma, \gamma', \gamma''$ and $R^2\gamma$, as shown in Figure 10 on the right, such that $\gamma, R^2\gamma$ are compatible with both γ', γ'' .

Each of the modules M_α in the diagrams (6.2) is defined via its projective presentation

$$P_1(\alpha) \xrightarrow{f_\alpha} P_0(\alpha) \longrightarrow M_\alpha \longrightarrow 0$$

associated to the 2-diagonal $\alpha \in \text{Diag}(\mathcal{S})$. Each of the pivot morphisms $b \in \{g', g'', h', h''\}$ is defined on these presentations as in diagram (B.1) as a triple $b = (b_1, b_0, b)$. Let $a = h'g' - h''g''$. We want to show that $a = 0$ in $\underline{\text{CMP}} B$. Consider the following commutative diagram.

$$(C.18) \quad \begin{array}{ccccccc} P_1(\gamma) & \xrightarrow{f_\gamma} & P_0(\gamma) & \longrightarrow & M_\gamma & \longrightarrow & 0 \\ a_1 \downarrow & & \downarrow a_0 & & \downarrow a & & \\ P_1(R^2\gamma) & \xrightarrow{f_{R^2\gamma}} & P_0(R^2\gamma) & \longrightarrow & M_{R^2\gamma} & \longrightarrow & 0 \end{array}$$

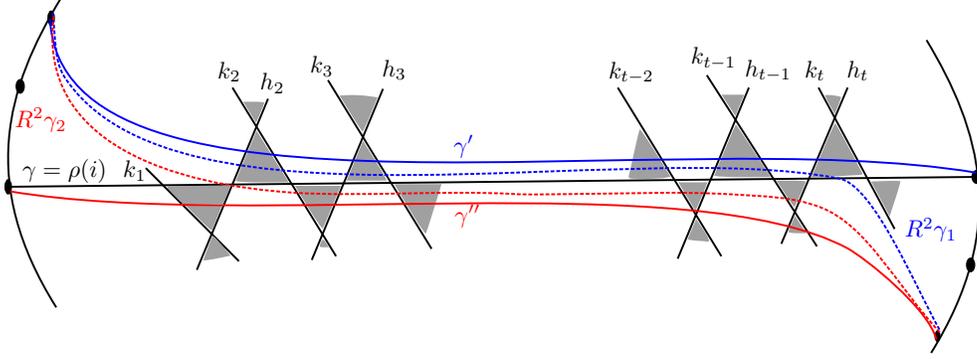
It suffices to show that $a_0 = 0$, and for that it suffices to show that, for each indecomposable summand $P(i)$ of $P_0(\gamma)$ and each indecomposable summand $P(i')$ of $P_0(R^2\gamma)$, the component $a_0(i, i')$ of a_0 that maps $P(i)$ to $P(i')$ is zero.

If $P(i)$ is also a summand of $P_0(R^2\gamma)$ then $a_0(i, i') = 0$, by Definition 5.2 of the pivot morphisms g', g'', h', h'' . For the same reason, we have $a_0(i, i') = 0$ if $P(i')$ is also a summand of $P_0(\gamma)$.

Suppose therefore that $P(i)$ is not a summand of $P_0(R^2\gamma)$ and $P(i')$ is not a summand of $P_0(\gamma)$. Thus the radical line $\rho(i)$ crosses γ in degree 0 but it does not cross $R^2\gamma$, and the radical line $\rho(i')$ crosses $R^2\gamma$ in degree 0 but it does not cross γ .

By assumption $\gamma \neq \rho(i')$. This case is illustrated in the left picture in Figure 46. Then the radical line $\rho(i)$ is incident to one of the points b or y and it crosses γ' or γ'' respectively. Without loss of generality, we may assume that $\rho(i)$ is incident to b . Thus γ' crosses $\rho(i)$ and γ'' doesn't. Consequently, the map g' acts as the identity on $P(i)$. On the other hand, the radical line $\rho(i')$ crosses $R^2\gamma$ and is incident to either a or x . Both cases are illustrated in Figure 46 in red. If $\rho(i')$ is incident to a then $\rho(i)$ and $\rho(i')$ cross, and thus there exists an arrow $\alpha: i' \rightarrow i$. Then the maps $g'', h': P(i) \rightarrow P(i')$ are given by the arrow α , and the map $h'': P(i') \rightarrow P(i')$ is the identity. It follows that $a_0(i, i') = \alpha - \alpha = 0$.

If $\rho(i')$ is incident to x then it does not cross γ'' and hence the map $h''g'': P(i) \rightarrow P(i')$ is zero. Indeed, if $h''g''$ factors through some $P(k)$, a summand of $P_0(\gamma'')$ then $k = i, i'$, because no radical line $\rho(k)$ can cross γ'' without also crossing γ or $R^2\gamma$. On the other hand, the map g' is the identity on $P(i)$ and hence zero on $P(i) \rightarrow P(i')$, whereas the map h' is the identity on $P(i')$ and hence also zero on $P(i) \rightarrow P(i')$. It follows

FIGURE 47. The proof of Proposition 6.3 when $\gamma = \rho(i)$ is not short.

that $a_0(i, i') = 0 - 0 = 0$. This proves the proposition in the case when γ is not short and neither γ nor $R^2\gamma$ is a radical line.

(2) Now suppose that $\gamma = \rho(i)$ is a radical line. See the right picture in Figure 46. If $R^2\gamma$ is also a radical line, then Lemma B.3 implies that $R\gamma$ is a radical line. This contradicts the assumption of the proposition, hence we are in the situation $\gamma = \rho(i)$ and neither $R\gamma$ nor $R^2\gamma$ is a radical line. In this case, it is not possible to find all four compatible representatives $\gamma, \gamma', \gamma'', R^2\gamma$. Therefore, we choose particular representatives and apply automorphisms of the homotopy as follows. In the case of $\rho(i)$ there are exactly two possible representatives γ_1, γ_2 of $\rho(i)$ following $\rho(i)$ infinitesimally close but always staying to the right or the left of $\rho(i)$ respectively, see Figure 47. Moreover, these two representatives give rise to the same map $f_{\gamma_1} = f_{\gamma_2} = f_{\rho(i)}$, see Proposition 4.13. Then the representatives for γ', γ'' are obtained from γ_1, γ_2 respectively by a pivot move so that the pairs γ', γ_1 and γ'', γ_2 are compatible.

Now, we define $f_{R^2\gamma}$ to be the map coming from a representative of $R^2\gamma$ after applying a pivot move to this γ' . In this way, we see that the two chosen representatives $R^2\gamma$ and γ' are compatible, and we conclude that $f_{R^2\gamma}h'_1 = h'_0f_{\gamma'}$. However, applying the pivot move to our chosen representative of γ'' we generally obtain a different representative of the 2-diagonal $R^2\gamma$, so that we may no longer guarantee that the equation $f_{R^2\gamma}h''_1 = h''_0f_{\gamma''}$ is satisfied for the same map $f_{R^2\gamma}$. However, according to Corollary 4.14 there exists an isomorphism $(\varphi_1, \varphi_0) \in \text{Aut}(P_1(R^2\gamma)) \oplus \text{Aut}(P_0(R^2\gamma))$ such that

$$(C.19) \quad f_{R^2\gamma}\varphi_1h''_1 = \varphi_0h''_0f_{\gamma''}.$$

Next, we find an explicit description of φ_1, φ_0 . Label the two representatives of $R^2\gamma$ coming from γ_1, γ_2 by $R^2\gamma_1, R^2\gamma_2$ respectively. Then $R^2\gamma_1, R^2\gamma_2$ have potentially different crossing sequences. In Figure 47 if we move from the left to the right along $R^2\gamma_1, R^2\gamma_2$, then the two representatives agree until they reach a crossing pair $(k_2, h_2), (i, h_2)$ respectively, afterwards they have different crossing sequences, and then the two arcs agree again after the crossing pair $(i, h_t), (k_{t-1}, h_t)$ respectively. Thus, we have the following crossing sequences in these cases.

$$\begin{aligned} R^2\gamma_1 : & \quad \dots, (k_2, h_2), (k_3, h_3), \dots, (k_{t-1}, h_{t-1}), (i, h_t), \dots \\ R^2\gamma_2 : & \quad \dots, (i, h_2), (k_2, h_3), \dots, (k_{t-2}, h_{t-1}), (k_{t-1}, h_t), \dots \end{aligned}$$

Moreover, we note that the difference in the crossing sequences comes from 2-diagonals k_2, \dots, k_{t-1} and h_2, \dots, h_t that are common crossings for all of the following $\gamma, \gamma', \gamma'', R^2\gamma$.

Then $R^2\gamma_2$ can be homotopically deformed to $R^2\gamma_1$ by a sequence of moves that first replaces $(i, h_2), (k_2, h_3)$ by $(k_2, h_2), (i, h_3)$ in its crossing sequence, and then replaces $(i, h_3), (k_3, h_4)$ by $(k_3, h_3), (i, h_4)$, and continues in this way until reaching the crossing sequence for $R^2\gamma_1$. Each step $j = 1, \dots, t-2$ in this process yields an isomorphism $\varphi^{(j)}$ of $P_0(R^2\gamma)$ where $\varphi^{(j)}$ is obtained from $1_{P_0(R^2\gamma)}$ by replacing a 2×2 block $\begin{bmatrix} 0 & 1_{P(i)} \\ 1_{P(k_{j+1})} & 0 \end{bmatrix}$ in the matrix for $1_{P_0(R^2\gamma)}$ with $\begin{bmatrix} i \rightarrow k_{j+1} & -1_{P(i)} \\ 1_{P(k_{j+1})} & 0 \end{bmatrix}$ and leaving the remaining entries unchanged, see the proof of Proposition 4.13. Then $\varphi_0 = \varphi^{(t-2)} \dots \varphi^{(2)}\varphi^{(1)}$ and $\varphi_1 = 1$, so we conclude $\varphi_0f_{R^2\gamma_2} = f_{R^2\gamma_1}$.

Thus, equation (C.19) becomes $f_{R^2\gamma}h''_1 = \varphi_0h''_0f_{\gamma''}$, and we obtain a commutative diagram as in (C.18), where now we let $a_0 = h'_0g'_0 - \varphi_0h''_0g''_0$ and $a_1 = h'_0g'_0 - h''_0g''_0$ as before. We want to compute $a_0(k, i')$:

$P(k) \rightarrow P(i')$ for all k and i' , but first, we consider what happens to the entries of h''_0 when we compose it with φ_0 .

Since $P(k_2)$ is a summand of both $P_0(R^2\gamma)$ and $P_0(\gamma'')$, it follows from the definition of the pivot morphism that in the matrix of h''_0 the entry $1_{P(k_2)}$ is the only nonzero entry in its row and column. Therefore, $\varphi^{(1)}h''_0$ replaces column $[0 \dots 0 \ 1_{P(k_2)} \ 0 \dots 0]^T$ with $[0 \dots 0 \ 1_{P(k_2)} \ 0 \dots 0 \ i \rightarrow k_2]^T$ and changes the sign of all entries in the (last) row of h''_0 that correspond to paths starting in i . Similarly, $\varphi^{(j+1)}$ affects $\varphi^{(j)}h''_0$ by replacing a zero in the first row with an arrow $i \rightarrow k_{j+2}$ and changing the signs of all entries in this row. This yields the desired description of $\varphi_0h''_0$.

To compute $a_0(k, i') : P(k) \rightarrow P(i')$, $a_0 = h'_0g'_0 - \varphi_0h''_0g''_0$, we consider several cases.

(2.1) Suppose $i' \neq i$. Then we claim that $a_0(k, i') = 0$. Since i' crosses $R^2\gamma$ in degree zero, then it crosses γ' or γ'' or both in the same degree. If i' crosses both of these then it also crosses γ and $\varphi_0h''_0g''_0, h'_0g'_0 : P(k) \rightarrow P(i')$ is the identity map if $k = i'$ and otherwise is the zero map. In both of these cases, we conclude that $a_0(k, i') = 0$.

Now, suppose that $i' \neq i$ crosses $R^2\gamma, \gamma'$ but not γ'' . Then i' ends in x and does not cross γ , see Figure 46 on the right. In particular $k \neq i'$. Then the restriction of $(h'_0, -\varphi_0h''_0)$ to $P_0(\gamma') \oplus P_0(\gamma'') \rightarrow P(i')$ equals $P(i') \oplus P(j_1) \oplus P(j_2) \rightarrow P(i')$ with the map $[1, -(i' \rightarrow j_1), i' \rightarrow j_2]$, where $P(j_1) \oplus P(j_2)$ is a summand of $P(\gamma'')$ such that the arcs j_1, j_2 cross γ'' but not $R^2\gamma$, as in the figure. Note that it may happen that either j_2 or both j_1, j_2 are not present, but here we depict the most general situation. Moreover, the signs appearing here follow by Lemma 5.3. Then the restriction of $(g'_0, g''_0)^T$ to $P(k) \rightarrow P(i') \oplus P(j_1) \oplus P(j_2)$ is given by the zero map if $k \neq j_1, j_2$ and otherwise it is given by $[i' \rightarrow j_1, 1, 0]^T, [-i' \rightarrow j_2, 0, 1]^T$ if $k = j_1, j_2$ respectively. Again we conclude that $a_0(k, i') = 0$ for all k . The case when $i' \neq i$ crosses $R^2\gamma, \gamma''$ but not γ' follows similarly. This shows the claim that $a_0(k, i') = 0$ if $i \neq i'$.

(2.2) Now, suppose that $i = i'$. Observe, that for every summand $P(k)$ of $P_0(\gamma)$ there is an arrow $\alpha : i \rightarrow k$, because $P_0(\gamma)$ is the projective cover of $\text{rad } P(i)$. We claim that $a_0(k, i) = \pm\alpha$ where the sign is the same as the sign of the arrow α appearing in the map $(h'_0, -\varphi_0h''_0)$. Observe that the maps $h'_0g'_0, \varphi_0h''_0g''_0 : P(k) \rightarrow P(i)$ are either zero maps or they are given by multiplication with the arrow α . Therefore, we can restrict our attention to summands $P(k)$ in $P_0(\gamma'), P_0(\gamma'')$. If $P(k)$ is a summand of both $P_0(\gamma'), P_0(\gamma'')$ then by the description of $\varphi_0h''_0$ obtained earlier, the restriction of $h'_0, -\varphi_0h''_0$ to $P(k) \rightarrow P(i)$ is given by zero, $\pm\alpha$ respectively. On the other hand, $g'_0, g''_0 : P(k) \rightarrow P(k)$ are given by the identity map, and we conclude that $a_0(k, i) = \pm\alpha$. If $P(k)$ is a summand of $P_0(\gamma')$ but not $P_0(\gamma'')$ then it is not a summand of $P_0(R^2\gamma)$. Thus, $\varphi_0h''_0g''_0 : P(k) \rightarrow P(i)$ is given by the zero map while $h'_0g'_0 : P(k) \rightarrow P(i)$ is given by multiplication with the arrow $\pm\alpha$. The case when $P(k)$ is a summand of $P_0(\gamma'')$ but not $P_0(\gamma')$ follows similarly. This shows the desired claim that $a_0(k, i) = \pm\alpha$.

We will show that the induced morphism $a : M_\gamma \rightarrow M_{R^2\gamma}$ on the cokernels factors through the projective $P(i)$. This will then imply that a is the zero morphism in $\underline{\text{CMP}} B$. To show the factorization, let $u : M_\gamma = \text{rad } P(i) \rightarrow P(i)$ be the inclusion morphism. As a map of projective presentations, u is given by the zero map in degree 1 and by the arrows in degree 0 as follows. For every indecomposable summand $P(k)$ of $P_0(\gamma)$, in degree 0 we define $u_0 : P_0(\gamma) \rightarrow P(i)$ on each component $P(k) \rightarrow P(i)$ to be the multiplication with the arrow $\alpha : i \rightarrow k$ or $-\alpha$. We can easily choose the signs in u_0 so that $u_0f_\gamma = 0$, because every row of the matrix of f_γ consists of at most two nonzero entries in adjacent columns, since γ is a radical line. An explicit description of the map f_γ is given in the last paragraph of the proof of Proposition 4.13. Then we obtain the following diagram.

$$(C.20) \quad \begin{array}{ccccccc} P_1(\gamma) & \xrightarrow{f_\gamma} & P(k) \oplus \overline{P_0}(\gamma) & \longrightarrow & M_\gamma = \text{rad } P(i) & \longrightarrow & 0 \\ 0 \downarrow & & \downarrow u_0 = [\pm\alpha \ *] & & \downarrow u & & \\ 0 & \xrightarrow{0} & P(i) & \longrightarrow & P(i) & \longrightarrow & 0 \end{array}$$

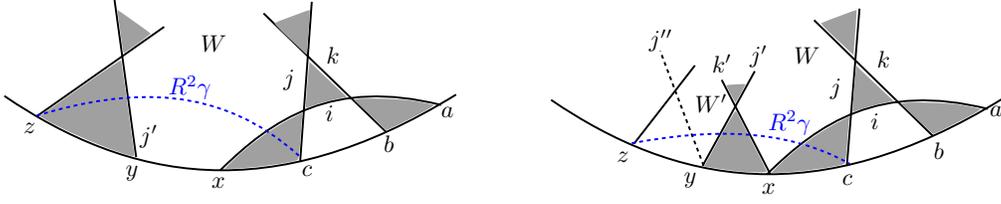


FIGURE 48. The proof of Proposition 6.3 when γ is short, and the boundary segment between c and b belongs to a white region.

Define a map $v: P(i) \rightarrow M_{R^2\gamma}$ as the cokernel map of the following diagram.

$$(C.21) \quad \begin{array}{ccccccc} 0 & \xrightarrow{0} & P(i) & \longrightarrow & P(i) & \longrightarrow & 0 \\ 0 \downarrow & & \downarrow v_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} & & \downarrow v & & \\ P_1(R^2\gamma) & \xrightarrow{f_{R^2\gamma}} & P(i) \oplus \overline{P_0}(R^2\gamma) & \longrightarrow & M_{R^2\gamma} & \longrightarrow & 0 \end{array}$$

Thus in degree 0, the component $v_0 u_0 : P(k) \rightarrow P(i')$ of the composition vu is given by $\pm\alpha$ if $i = i'$ and otherwise it equals zero. Furthermore, we can arrange the signs, by possibly replacing u with $-u$ so that the sign of every arrow α appearing in u_0 is opposite to the sign of α appearing in $(h'_0, -\varphi_0 h''_0)^T$. Using our computation of $a_0(k, i')$ above, we see that the difference $a(k, i') - vu$ is zero in degree 0. This shows that $a = vu$, and hence a factors through the projective $P(i)$ and hence it is zero in the stable category $\underline{\text{CMP}} B$. This completes the proof of (a).

(b) Now, suppose that γ is short. First assume that $\gamma = \rho(i)$ is a radical line. Again using Lemma B.3 and our hypothesis that $R\gamma$ is not a radical line, we conclude that $R^2\gamma$ is also not a radical line. Next, we consider two cases depending on whether the boundary segment between vertices c and b belongs to a white or a shaded region.

Figure 48 illustrates the situation when the boundary segment between vertices c and b belongs to a white region. Here the picture on the left shows the case when the white region W contains a boundary edge and the picture on the right shows the case when W contains only a single vertex on the boundary of \mathcal{S} . In the right picture, the 2-diagonal j'' may or may not be present depending on whether the white region W' has only a vertex or an entire edge on the boundary of \mathcal{S} . Thus, here we depict the most general situation.

Then we obtain a commutative diagram with exact rows as follows coming from the the situation of Figure 48 on the right. The diagram for the other case can be obtained from the given one by removing the extra summands in the projective presentations that do not cross $R\gamma', R^2\gamma$ according to Figure 48 on the left.

$$(C.22) \quad \begin{array}{ccccccc} P(j) & \xrightarrow{[k \rightarrow j]} & P(k) & \longrightarrow & M_\gamma & \longrightarrow & 0 \\ \downarrow g_1 & & \downarrow g_0 & & \downarrow g & & \\ P(j'') \oplus P(j') \oplus P(j) & \xrightarrow{\begin{bmatrix} k' \rightsquigarrow j'' & k' \rightarrow j' & 0 \\ 0 & k \rightsquigarrow j' & k \rightarrow j \\ 0 & 0 & 0 \end{bmatrix}} & P(k') \oplus P(k) \oplus P(i) & \longrightarrow & M_{\gamma'} \oplus P(i) & \longrightarrow & 0 \\ \downarrow h_1 & & \downarrow h_0 & & \downarrow h & & \\ P(j'') \oplus P(j') & \xrightarrow{\begin{bmatrix} k' \rightsquigarrow j'' & k' \rightarrow j' \\ 0 & i \rightarrow j' \end{bmatrix}} & P(k') \oplus P(i) & \longrightarrow & M_{R^2\gamma} & \longrightarrow & 0 \end{array}$$

The vertical maps are given as follows.

$$g_1 = \begin{bmatrix} 0 \\ 0 \\ 1_{P(j)} \end{bmatrix} \quad g_0 = \begin{bmatrix} 0 \\ 1_{P(k)} \\ i \rightarrow k \end{bmatrix} \quad h_1 = \begin{bmatrix} 1_{P(j'')} & 0 & 0 \\ 0 & 1_{P(j')} & 0 \end{bmatrix} \quad h_0 = \begin{bmatrix} 1_{P(k')} & 0 & 0 \\ 0 & -i \rightarrow k & 1_{P(i)} \end{bmatrix}$$

Note that here the pivot map $g'_0 = [0, 1_{P(k)}]^T$ appears in the matrix g_0 , and the pivot map $h'_0 = \begin{bmatrix} 1_{P(k')} & 0 \\ 0 & -i \rightarrow k \end{bmatrix}$ appears in the matrix h_0 .

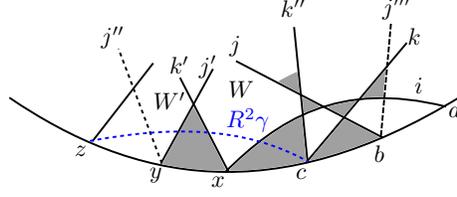


FIGURE 49. The proof of Proposition 6.3 when γ is short, and the boundary segment between c and b belongs to a shaded region.

It is easy to see that in degree zero the composition $h_0 g_0$ equals zero. By the same reasoning as in case 2.2 of part (a), we conclude that the pivot map $M_\gamma \rightarrow M_{\gamma'} \rightarrow M_{R^2\gamma}$ factors through $P(i)$, so it is zero in $\underline{\text{CMP}} B$. This completes the proof in the case when the boundary segment between vertices c and b belongs to a white region.

Now, suppose that the boundary segment between c and b belongs to shaded triangular region. This situation is illustrated in Figure 49. Note that the shaded triangle with edges k'' and j must not be boundary, because otherwise $j = R\gamma$, contrary to our assumption. Furthermore, here we depict the most general situation when the white region W has only a single vertex on the boundary. As in the previous case, the dashed diagonals j'', j''' may or may not be present. Then we obtain the following commutative diagram with exact rows.

$$(C.23) \quad \begin{array}{ccccccc} P(k) \oplus P(k'') & \xrightarrow{\begin{bmatrix} j \rightarrow k & j \rightarrow k'' \\ j'' \rightarrow k & 0 \end{bmatrix}} & P(j) \oplus P(j''') & \longrightarrow & M_\gamma & \longrightarrow & 0 \\ \downarrow g_1 & & \downarrow g_0 & & \downarrow g & & \\ P(j'') \oplus P(j') & \xrightarrow{\begin{bmatrix} 0 & j \rightarrow j' & j \rightarrow k'' & j \rightarrow k \\ k' \rightsquigarrow j'' & k' \rightarrow j' & 0 & 0 \\ 0 & 0 & 0 & j''' \rightarrow k \\ 0 & 0 & 0 & 0 \end{bmatrix}} & P(j) \oplus P(k') & \longrightarrow & M_{\gamma'} \oplus P(i) & \longrightarrow & 0 \\ \oplus P(k'') \oplus P(k) & & \oplus P(j''') \oplus P(i) & & \downarrow h & & \\ \downarrow h_1 & & \downarrow h_0 & & \downarrow h & & \\ P(j') \oplus P(j'') & \xrightarrow{\begin{bmatrix} k' \rightsquigarrow j' & k' \rightsquigarrow j'' \\ i \rightsquigarrow j' & 0 \end{bmatrix}} & P(k') \oplus P(i) & \longrightarrow & M_{R^2\gamma} & \longrightarrow & 0 \end{array}$$

The vertical maps are given as follows.

$$g_1 = \begin{bmatrix} 0 & 0 \\ 0 & 1_{P(k'')} \\ 1_{P(k)} & 0 \end{bmatrix} \quad g_0 = \begin{bmatrix} 1_{P(j)} & 0 \\ 0 & 0 \\ 0 & 1_{P(j''')} \\ -i \rightarrow j & i \rightarrow j''' \end{bmatrix}$$

$$h_1 = \begin{bmatrix} 0 & 1_{P(j')} & 0 & 0 \\ 1_{P(j'')} & 0 & 0 & 0 \end{bmatrix} \quad h_0 = \begin{bmatrix} 0 & 1_{P(k')} & 0 & 0 \\ i \rightarrow j & 0 & -i \rightarrow j''' & 1_{P(i)} \end{bmatrix}$$

By the same reasoning as before, we conclude that the pivot map $M_\gamma \rightarrow M_{\gamma'} \rightarrow M_{R^2\gamma}$ factors through $P(i)$, so it is zero in $\underline{\text{CMP}} B$. This completes the proof in the case when $\gamma = \rho(i)$ is short.

The case when γ is not a radical line but $R^2\gamma$ is a radical line is similar to the case when γ is a radical line discussed above, and we omit the detailed discussion. \square

C.4. Proof of Proposition 6.4. We need the following result from homological algebra.

Lemma C.27. *Consider a commutative diagram with exact rows.*

$$\begin{array}{ccccccc}
 P_1(A) & \xrightarrow{f_A} & P_0(A) & \xrightarrow{\pi_A} & A & \longrightarrow & 0 \\
 \downarrow g_1 & & \downarrow g_0 & & \downarrow g & & \\
 P_1(B) & \xrightarrow{f_B} & P_0(B) & \xrightarrow{\pi_B} & B & \longrightarrow & 0 \\
 \downarrow h_1 & & \downarrow h_0 & & \downarrow h & & \\
 P_1(C) & \xrightarrow{f_C} & P_0(C) & \xrightarrow{\pi_C} & C & \longrightarrow & 0
 \end{array}$$

If h_1 is surjective and $\ker h_0 \subset \text{im } g_0$ then $\ker h \subset \text{im } g$.

Proof. Let $b \in B$ such that $h(b) = 0$. Since the rows are exact, it follows that π_B is surjective and there exists some $b_0 \in P_0(B)$ such that $\pi_B(b_0) = b$. By commutativity $\pi_C h_0(b_0) = h \pi_B(b_0) = 0$. In particular, $h_0(b_0) \in \ker \pi_C$. Because the rows in the diagram are exact, it follows that there exists some $c_1 \in P_1(C)$ such that $f_C(c_1) = h_0(b_0)$. By assumption h_1 is surjective, so there exists some $b_1 \in P_1(B)$ such that $h_1(b_1) = c_1$. Then we have

$$h_0 f_B(b_1) = f_C h_1(b_1) = f_C(c_1) = h_0(b_0)$$

and therefore $b_0 - f_B(b_1) \in \ker h_0$. By assumption $\ker h_0 \subset \text{im } g_0$, so there exists some $a_0 \in P_0(A)$ such that $g_0(a_0) = b_0 - f_B(b_1)$. Finally, we obtain

$$g \pi_A(a_0) = \pi_B g_0(a_0) = \pi_B(b_0 - f_B(b_1)) = \pi_B(b_0) = b$$

where the last step follows because the rows in the diagram are exact so $\pi_B f_B = 0$. This shows that $b \in \text{im } g$, thus $\ker h \subset \text{im } g$ as desired. \square

Proposition C.28. *Suppose γ is a 2-diagonal in \mathcal{S} such that $R\gamma$ is not a radical line.*

(a) *If γ is a radical line $\rho(i)$ is not short then there exists a short exact sequence in $\text{mod } B$*

$$0 \rightarrow M_\gamma \rightarrow M_{\gamma'} \oplus M_{\gamma''} \oplus P(i) \rightarrow M_{R^2\gamma} \rightarrow 0.$$

(b) *If γ is a radical line $\rho(i)$ is short then there exists a short exact sequence in $\text{mod } B$*

$$0 \rightarrow M_\gamma \rightarrow M_{\gamma'} \oplus P(i) \rightarrow M_{R^2\gamma} \rightarrow 0.$$

(c) *If $O(\gamma)$ does not contain any radical lines then γ is not short and there exists a short exact sequence in $\text{mod } B$*

$$0 \rightarrow M_\gamma \rightarrow M_{\gamma'} \oplus M_{\gamma''} \rightarrow M_{R^2\gamma} \rightarrow 0.$$

Proof. We construct these short exact sequences from the projective presentations of these modules. To prove part (a) consider the following diagram with exact rows coming from the projective presentations of the modules in the right column. Moreover, we choose representatives for $\gamma, \gamma', \gamma'', R^2\gamma = R^2\gamma_1$ as in Figure 47.

$$\begin{array}{ccccccc}
 P_1(\gamma) & \xrightarrow{f_\gamma} & P_0(\gamma) & \xrightarrow{\pi_\gamma} & M_\gamma \cong \text{rad } P(i) & \longrightarrow & 0 \\
 \downarrow g_1 & & \downarrow g_0 & & \downarrow g & & \\
 P_1(\gamma') \oplus P_1(\gamma'') & \xrightarrow{\begin{bmatrix} f_{\gamma'} & 0 \\ 0 & f_{\gamma''} \\ 0 & 0 \end{bmatrix}} & P_0(\gamma') \oplus P_0(\gamma'') \oplus P(i) & \xrightarrow{\begin{bmatrix} \pi_{\gamma'} & 0 & 0 \\ 0 & \pi_{\gamma''} & 0 \\ 0 & 0 & 1_{P(i)} \end{bmatrix}} & M_{\gamma'} \oplus M_{\gamma''} \oplus P(i) & \longrightarrow & 0 \\
 \downarrow h_1 & & \downarrow h_0 & & \downarrow h & & \\
 P_1(R^2\gamma) & \xrightarrow{f_{R^2\gamma}} & P_0(R^2\gamma) & \xrightarrow{\pi_{R^2\gamma}} & M_{R^2\gamma} & \longrightarrow & 0
 \end{array}$$

The vertical maps we define as follows

$$g_1 = \begin{bmatrix} g'_1 \\ g''_1 \end{bmatrix} \quad g_0 = \begin{bmatrix} g'_0 \\ g''_0 \\ u \end{bmatrix} \quad h_0 = [h'_0 \quad -\varphi_0 h''_0 \quad v] \quad h_1 = [h'_1 \quad -h''_1].$$

The maps g'_0, g'_1 and g''_0, g''_1 are the pivot morphisms coming from $\gamma \mapsto \gamma'$ and $\gamma \mapsto \gamma''$ respectively. The maps h'_0, h'_1 and h''_0, h''_1 are the pivot morphisms coming from $\gamma' \mapsto R^2\gamma$ and $\gamma'' \mapsto R^2\gamma$ respectively. Here we omit superscripts r and c for the pivot morphisms between the projective modules to simplify the notation.

The map φ_0 is an automorphism of $P_0(R^2\gamma)$ needed to make the lower left square commute. The same construction appears in the proof of Proposition 6.3, which we refer to for further details regarding the commutativity of the diagram and the precise construction of φ_0 .

Observe that $R^2\gamma$ crosses $\gamma = \rho(i)$ so $P(i) \in \text{add } P_0(R^2\gamma)$, and we define the map $v : P(i) \rightarrow P_0(R^2\gamma)$ to be the inclusion that maps $P(i) \rightarrow P(i)$ via identity. Finally, for every indecomposable summand $P(k)$ of $P_0(\gamma)$ there exists an arrow $\alpha : i \rightarrow k$ in the quiver, because $P_0(\gamma)$ is the projective cover of $\text{rad } P(i)$, so we define $u : P_0(\gamma) \rightarrow P(i)$ on each component $P(k) \rightarrow P(i)$ to be the multiplication with the arrow α or $-\alpha$. We can easily choose the signs in u so that $uf_\gamma = 0$ because every row of the matrix of f_γ consists of at most two nonzero entries in adjacent columns, since γ is a radical line. An explicit description of the map f_γ is given in the last paragraph of the proof of Proposition 4.13. This defines the vertical maps between the projective presentations.

The maps between the projective presentations induce the corresponding maps g, h between their cokernels as in the diagram, making the two squares on the right also commute. To prove part (a) it suffices to show that g is injective, h is surjective, and $\text{im } g = \ker h$.

Observe that g is injective, because its component $\text{rad } P(i) \rightarrow P(i)$ induced from the map u is the inclusion map. The module $P_0(R^2\gamma)$ is a direct summand of $P_0(\gamma') \oplus P_0(\gamma'') \oplus P(i)$ so by the definition of the pivot maps and the map v it follows that h_0 is surjective. Therefore, h is surjective, because both $\pi_{R^2\gamma}$ and h_0 are surjective. Furthermore, $\text{im } g \subset \ker h$ because of the mesh relations established in Proposition 6.3. Therefore, it remains to show that $\ker h \subset \text{im } g$.

Since $R\gamma$ is not a radical line, it follows that h_1 is surjective because $P_1(R^2\gamma)$ is a direct summand of $P_1(\gamma') \oplus P_1(\gamma'')$. Then Lemma C.27 implies that $\ker h \subset \text{im } g$ provided that $\ker h_0 \subset \text{im } g_0$. Therefore, it suffices to show that $\ker h_0 \subset \text{im } g_0$, so we discuss these maps in more detail below.

We already have a good understanding of the structure of the pivot maps, and we have an explicit description of u and v . Now, we recall an explicit description of $\varphi_0 h_0''$ obtained in the proof of Proposition 6.3. In particular, the entries of the matrix $\varphi_0 h_0''$ are obtained from those in h_0'' by replacing a zero with an arrow $\pm\alpha$, where $\alpha : i \rightarrow k$, if $P(k)$ is a summand of both $P_0(\gamma')$ and $P_0(\gamma'')$. Moreover, the sign of the arrows in the matrix of h_0 satisfy the following important observation:

$$(C.24) \quad \begin{array}{l} \text{by definition of the maps there is an arrow } x \rightarrow y \text{ in the matrix of } h_0 \\ \text{if and only if the same arrow appears in the matrix of } g_0. \end{array}$$

Furthermore, by possibly replacing u with $-u$ we can choose the signs so that $\pm(i \rightarrow k)$ is in u if and only if $\mp(i \rightarrow k)$ is in h_0 .

Now, we claim that $\ker h_0 \subset \text{im } g_0$. Let $m \in \ker h_0$. We can write $m = \sum_{c \in Q_0} m e_c$ where e_c is the idempotent at vertex c of the quiver Q , and it suffices to show that $m e_c \in \text{im } g_0$ for every c . Therefore, without loss of generality we may assume that every entry of m consists of paths ending in some fixed vertex c of Q . We think of m as being a column vector, and we denote its entries corresponding to paths starting at a vertex k by m_k . In particular, because there is at most one path in B between any two vertices, then every entry m_k of m is of the form $\lambda_k(k \rightsquigarrow c)$, a scalar multiple of a single path $k \rightsquigarrow c$ for some element λ_k of the underlying field.

Our strategy to show that $\ker h_0 \subset \text{im } g_0$ is as follows. We consider several cases, where we write $m = (m - m') + m'$ for some $m' \in \ker h_0 \cap \text{im } g_0$. Then $m - m' \in \ker h_0$ and it suffices to show that this element is also in $\text{im } g_0$.

(a1) Suppose that $m_k \neq 0$ and $P(k)$ is a summand of both $P_0(\gamma')$ and $P_0(\gamma'')$. Then the map

$$h_0 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ * & \vdots & * & \vdots & * & \vdots \\ 0 & \cdots & 0 & 1_{P(k)} & 0 & \cdots & 0 & -1_{P(k)} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ * & \vdots & * \\ 0 & \cdots & 0 & \pm\alpha & 0 & \cdots & 0 & \pm\alpha & 0 & \cdots & 1_{P(i)} \end{bmatrix}$$

Since m is in the kernel of h_0 it follows that $m = [* \cdots * m_k * \cdots * m_k * \cdots *]^T$, where the positions of the m_k align with the columns of h_0 that contain $1_{P(k)}$. Define $m' = [0 \cdots 0 m_k 0 \cdots 0 m_k 0 \cdots \mp\alpha m_k]^T$. Then it is easy to see that $h_0(m') = 0$. On the other hand, $m' \in \text{im } g_0$, because g_0 contains a column $[0 \cdots 0 1_{P(k)} 0 \cdots 0 1_{P(k)} 0 \cdots \mp\alpha]^T$. Thus $m' \in \ker h_0 \cap \text{im } g_0$ and $m - m' \in \ker h_0$. Hence, in order to show that $m = (m - m') + m'$ is in the image of g_0 it suffices to show that $m - m'$ is in the image of g_0 .

(a2) Suppose that $m_k \neq 0$ and $P(k)$ is a summand of $P_0(\gamma')$ but not $P_0(\gamma'')$. Then $1_{P(k)}$ is an entry in h'_0 . Since $m \in \ker h_0$ then the row of h_0 containing $1_{P(k)}$ also contains at least one other nonzero entry, an arrow $-(k \rightarrow k'')$, and such that the path m_k starts with the arrow $k \rightarrow k''$. In particular, we can write $m_k = (k \rightarrow k'') \circ m''_k$ where m''_k is a path $\lambda_k(k'' \rightsquigarrow c)$. Because of the observation (C.24), we conclude that $P(k'')$ is a summand of $P_0(\gamma)$. Therefore, there is an arrow $\pm(i \rightarrow k'')$ in u . But then by the same observation there should also be an arrow $\mp(i \rightarrow k'')$ in h_0 . In particular, $h'_0, -\varphi h''_0$ contain columns $[0 \dots 0 \ 1_{P(k)} \ 0 \dots 0]^T$, $[0 \dots 0 \ -k \rightarrow k'' \ 0 \dots \mp i \rightarrow k'']^T$ respectively, and v is as in case (a1). Then define

$$m' = [0 \dots 0 \ m_k \ 0 \dots \ m''_k \ 0 \dots 0 \ \pm(i \rightarrow k'') \circ m''_k]^T.$$

Then $m' \in \ker h_0$, because the matrix of g_0 contains a column

$$[0 \dots 0 \ k \rightarrow k'' \ 0 \dots 0 \ 1_{P(k'')} \ 0 \dots 0 \ \pm(i \rightarrow k'')]^T$$

and again we can see that m' is the image under g_0 of the vector that contains a single nonzero entry m''_k . Therefore, applying the process in (a2) recursively we can construct m' such that $m - m'$ is a vector whose every entry $(m - m')_k$ is zero whenever $P(k)$ is a summand of $P_0(\gamma')$ but not $P_0(\gamma'')$.

(a3) The case when $m_k \neq 0$ and $P(k)$ is a summand of $P_0(\gamma'')$ but not $P_0(\gamma')$ follows in the same way as (a2) so we omit the detailed discussion.

Therefore, by applying cases (a1)-(a3) and replacing m with $m - m'$, it suffices to show that an element m whose every entry m_k is zero whenever $P(k)$ is a summand of $P_0(\gamma') \oplus P_0(\gamma'')$ belongs to $\text{im } g_0$.

(a4) Now suppose that h_0 contains arrows $k \rightarrow k', k \rightarrow k''$ in the same row of its matrix with $i \neq k$. Then these arrows would also appear in g_0 by our earlier observation, so in u we would have arrows $i \rightarrow k', i \rightarrow k''$ up to sign. Since $i \neq k$, then there is a subquiver of Q of the form

$$\begin{array}{ccc} k & \rightarrow & k' \\ \downarrow & & \uparrow \\ k'' & \leftarrow & i \end{array}$$

which is not possible because there must be a sequence of 3-cycles at i connecting k' and k'' and another sequence of 3-cycles at k connecting k' and k'' . In particular, Q would have interior vertices, which is a contradiction. This shows that no row of h_0 contains two arrows except possibly for the last one corresponding to paths starting in i .

(a5) Now suppose that $m_i \neq 0$, recall that $M_\gamma = \text{rad } P(i)$. Since $m \in \ker h_0$ there exists $m_{k''} \neq 0$ such that the path m_i starts with the arrow $i \rightarrow k''$. We can write $m_i = (i \rightarrow k'') \circ m'_{k''}$ for some path $m'_{k''}$. By steps (a1)-(a3) we have that $1_{P(k'')}$ is not in h_0 , because otherwise $m_{k''}$ would be zero. Then either $\pm i \rightarrow k''$ is the only nonzero entry in its column of h_0 or there exists another arrow $-k \rightarrow k''$ in this column. If $-(k \rightarrow k'') \circ m'_{k''} \neq 0$ then, since $m \in \ker h_0$, it follows that there is another arrow $k \rightarrow k'''$ or $1_{P(k)}$ in h_0 in the same row as $-(k \rightarrow k'')$ such that $m_{k''}$ or m_k are nonzero. This contradicts (a4) and (a3) respectively. This shows that $\pm i \rightarrow k''$ is either the only nonzero entry in its column or there exists another nonzero entry $-(k \rightarrow k'')$ such that $-(k \rightarrow k'') \circ m'_{k''} = 0$. Then define

$$m' = [0 \dots 0 \ \mp m'_{k''} \ 0 \dots 0 \ m_i]^T$$

which belongs to the kernel of h_0 . The map g_0 contains a column

$$[0 \dots 0 \ k \rightarrow k'' \ 0 \dots 0 \ 1_{P(k'')} \ 0 \dots 0 \ \mp i \rightarrow k'']^T$$

where the first arrow is possibly zero. We see that g_0 applied to the vector that has entry $\mp m'_{k''}$ and all other entries being zero yields m' . Therefore, by the same reasoning as before we can reduce m to $m - m'$, the element whose every entry $(m - m')_k$ is zero whenever $P(k)$ is a summand of $P_0(\gamma') \oplus P_0(\gamma'') \oplus P(i)$.

(a6) Suppose that $m_k \neq 0$ and there is an arrow $\pm i \rightarrow k$ in h_0 . By above we know that $1_{P(k)}$ does not appear in h_0 . If $(i \rightarrow k) \circ m_k = 0$ then by our assumptions on m if there exists another arrow $k' \rightarrow k$ in h_0 then $(k' \rightarrow k) \circ m_k = 0$. Thus, whether or not there exists this second arrow ending in k , define $m' = [0 \dots 0 \ m_k \ 0 \dots 0]^T$ belongs to the kernel of h_0 . There is a column

$$[0 \dots 0 \ k' \rightarrow k \ 0 \dots 0 \ 1_{P(k)} \ 0 \dots 0 \ \mp i \rightarrow k]^T$$

in g_0 where $k' \rightarrow k$ may or may not be there. Again we can conclude that this $m' \in \text{im } g_0$ if $(i \rightarrow k) \circ m_k = 0$.

Now, suppose that $(i \rightarrow k) \circ m_k \neq 0$. By our assumptions on m we have $m_i = 0$, so since $m \in \ker h_0$ we conclude that there exists some other $m_{k'} \neq 0$ such that $(i \rightarrow k) \circ m_k - (i \rightarrow k') \circ m'_{k'}$ for some path $m'_{k'}$. It follows that the two arrows $i \rightarrow k, i \rightarrow k'$ must form a square as follows.

$$\begin{array}{ccc} & i & \rightarrow & k \\ & \downarrow & \searrow & \downarrow \\ k' & \rightarrow & & \bullet \end{array}$$

By construction of h_0 the two arrows $i \rightarrow k, i \rightarrow k'$ would come with opposite signs. By our assumptions on m entries $1_{P(k)}, 1_{P(k')}$ do not appear in h_0 . Hence, the two relevant columns of h_0 are as follows, where $k'' \rightarrow k, k''' \rightarrow k'$ may or may not be there but we depict the most general situation. Moreover, these arrows may also have some signs, but it will not be important for what follows.

$$[0 \dots 0 \ k'' \rightarrow k \ 0 \ \dots 0 \ \pm i \rightarrow k]^T \quad [0 \dots 0 \ k''' \rightarrow k' \ 0 \ \dots 0 \ \mp i \rightarrow k']^T$$

Note that $k'' \neq k'''$ by (a4), and again by our assumptions on m we must have that $(k'' \rightarrow k) \circ m_k = 0$ and $(k''' \rightarrow k') \circ m_{k'} = 0$. The latter also implies that $(k''' \rightarrow k') \circ m'_{k'} = 0$. In particular, we conclude that m' which is zero except for entries $m_k, m'_{k'}$ is in the kernel of h_0 . Furthermore, by similar computations as before m' is then in the image of g_0 , where g_0 contains two relevant columns given below.

$$[0 \dots 0 \ k'' \rightarrow k \ 0 \ \dots 0 \ 1_{P(k)} \ 0 \ \dots 0 \ \mp i \rightarrow k]^T \quad [0 \dots 0 \ k''' \rightarrow k' \ 0 \ \dots 0 \ 1_{P(k')} \ 0 \ \dots 0 \ \pm i \rightarrow k']^T$$

Finally $m - m'$ satisfying all the earlier assumptions in cases (a1)-(a6) reduces m to the zero element. This completes the proof that $\ker h_0 \subset \text{im } g_0$ and shows part (a) of the proposition.

To show part (b) we have $\gamma = \rho(i)$ is short and $R\gamma$ is not a radical line. In this case $\rho(i)$ is given as in Figure 48 or Figure 49. In Figure 48, the picture on the left shows the case when the white region W contains a boundary edge and the picture on the right shows the case when W contains only a single vertex on the boundary of \mathcal{S} . In the right picture, the 2-diagonal j'' may or may not be present depending on whether the white region W' has only a vertex or an entire edge on the boundary of \mathcal{S} . Thus, here we depict the most general situation. Analogous cases may occur in the situation of Figure 49, but here we illustrate the most general scenario.

Then we obtain a commutative diagram (C.22) with exact rows as follows coming from the the situation of Figure 48 on the right. The precise description of the vertical maps are given below the diagram. The diagram for the situation of Figure 48 on the left can be obtained from the given one by removing the extra summands in the projective presentations that do not cross $R\gamma'', R^2\gamma$.

By the same reasoning as in part (a) we conclude that h is surjective, g is injective, and $\text{im } g \subset \ker h$. Finally, we can easily see that $\ker h_0 \subset \text{im } g_0$, so Lemma C.27 again implies that $\text{im } g = \ker h$. This shows part (b) of the proposition if $\rho(i)$ is as in Figure 48.

If $\rho(i)$ is as in Figure 49, then we obtain a commutative diagram (C.23) with exact rows. Similar reasoning shows that part (b) of the proposition holds in this case as well. This completes the proof of part (b).

To prove part (c) of the proposition we first observe that if γ is short then $\{R^i\gamma\}$ contains a radical line $\rho(i)$ where vertex i is a leaf in the dual graph of the quiver Q . Since by assumption $\{R^i\gamma\}$ does not contain any radical lines we conclude that γ is not short. Then we construct a commutative diagram with exact rows, which is analogous to the one in case (a) except that we remove the direct summand $P(i)$ from the exact sequence in the middle row together with the appropriate maps. Then we obtain maps h, g as before. Because γ is not a radical line it follows that h_0 is surjective, hence h is surjective. Again we obtain that $\text{im } g \subset \ker h$ by Proposition 6.3. We can also show that $\ker g_0 \subset \text{im } h_0$ by a similar argument as in case (a), so we omit the detailed discussion. Note that the argument in this case would be simpler as the maps g_0, h_0 would not contain u and v . Since $R(\gamma)$ is not a radical line then h_1 is surjective, so together with $\ker g_0 \subset \text{im } h_0$ we conclude that $\ker h \subset \text{im } g$ by Lemma C.27. It remains to show that g is injective.

Consider the following commutative diagram, where we now show the superscripts c and r on the vertical map as they will be important. We know that the square on the left commutes. In the square on the right we use the notation \tilde{f} to denote a map $\varphi_0 f \varphi_1$ for some automorphisms φ_1, φ_0 , and it is constructed as follows. Recall that Proposition B.6 gives a projective resolution of M_γ starting with the map f_γ and choosing the appropriate representatives for $f_{R^i\gamma}$ to produce the maps $\tilde{f}_{R^i\gamma}$ for $i \geq 0$ as in the statement. Then Proposition B.7 describes the first three vertical maps g_0^r, g_1^c, g_2^c induced by the pivot morphism from γ to δ . A dual construction allows us to produce a projective co-resolution by starting with f_γ and then moving to the right with $f_{R^{-1}\gamma}$ so that we obtain a commuting square on the right as in the diagram below.

Note that the vertical maps are now g_1^c, g_0^r, g_{-1}^c moving from left to right. Then $\tilde{f}_{R^{-1}\gamma}$ factors through M_γ , and $\tilde{f}_{R^{-1}\gamma'} \oplus \tilde{f}_{R^{-1}\gamma''}$ factors through $M_{\gamma'} \oplus M_{\gamma''}$ as shown below.

$$\begin{array}{ccccc}
 P_1(\gamma) & \xrightarrow{f_\gamma} & P_0(\gamma) & \xrightarrow{\tilde{f}_{R^{-1}\gamma}} & P_0(R^{-1}\gamma) \\
 \downarrow g_1^c & & \downarrow g_0^r & \searrow & \downarrow g_{-1}^c \\
 & & & M_\gamma & \\
 & & & \downarrow g & \\
 & & & M_{\gamma'} \oplus M_{\gamma''} & \\
 & & & \swarrow & \searrow \\
 P_1(\gamma') \oplus P_1(\gamma'') & \xrightarrow{f_{\gamma'} \oplus f_{\gamma''}} & P_0(\gamma') \oplus P_0(\gamma'') & \xrightarrow{\tilde{f}_{R^{-1}\gamma'} \oplus \tilde{f}_{R^{-1}\gamma''}} & P_0(R^{-1}\gamma') \oplus P_0(R^{-1}\gamma'')
 \end{array}$$

Because $R^{-1}\gamma$ is not a radical line then $P_0(R^{-1}\gamma)$ is a direct summand of $P_0(R^{-1}\gamma') \oplus P_0(R^{-1}\gamma'')$, and we conclude that g_{-1}^c is injective. Since the square on the right commutes and g_{-1}^c is injective it follows that g is also injective. This completes the proof of part (c). \square

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CONNECTICUT, STORRS, CT 06269-1009, USA
 Email address: schiffler@math.uconn.edu

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF KENTUCKY, LEXINGTON, KY 40506-0027, USA
 Email address: khrystyna.serhiyenko@uky.edu