THE ROLE OF GENTLE ALGEBRAS IN HIGHER HOMOLOGICAL ALGEBRA

JOHANNE HAUGLAND, KARIN M. JACOBSEN, AND SIBYLLE SCHROLL

ABSTRACT. We investigate the role of gentle algebras in higher homological algebra. In the first part of the paper, we show that if the module category of a gentle algebra Λ contains a d-cluster tilting subcategory for some $d \geq 2$, then Λ is a radical square zero Nakayama algebra. This gives a complete classification of weakly d-representation finite gentle algebras. In the second part, we use a geometric model of the derived category to prove a similar result in the triangulated setup. More precisely, we show that if $\mathcal{D}^b(\Lambda)$ contains a d-cluster tilting subcategory that is closed under $[d]$, then Λ is derived equivalent to an algebra of Dynkin type A. Furthermore, our approach gives a geometric characterization of all d-cluster tilting subcategories of $\mathcal{D}^b(\Lambda)$ that are closed under $[d]$.

1. INTRODUCTION

The research field of higher homological algebra was initiated by Iyama [28, 29]. It concerns the study of d-abelian and $(d + 2)$-angulated categories, as well as further generalizations [16, 27, 38]. Distinguished sequences consisting of $d + 2$ objects, for a fixed positive integer d, play a fundamental role in these structures. In the case $d = 1$, one recovers the short exact sequences and distinguished triangles of abelian and triangulated categories, and the theory corresponds to classical homological algebra.

Iyama's work and the axiomatizations of associated categorical structures inspired extensive research activity, and many ideas from classical homological algebra have been shown to have an analogue in the higher setting [20, 26, 36, 37, 39, 41, 46, 51]. As connections between higher homological algebra and other branches of mathematics have been developed, the importance of the research field has become increasingly evident. Higher homological algebra is intimately related to higher Auslander–Reiten (AR) theory and representation theory of finite dimensional algebras [29, 30, 40]. It has connections to commutative algebra, commutative and non-commutative algebraic geometry, combinatorics and conformal field theory [1, 15, 24, 34, 42, 49, 59]. The research field has recently seen interesting applications in homological mirror symmetry, through which it relates to symplectic geometry and Fukaya categories [13].

The notion of d-cluster tilting subcategories plays a crucial role in higher homological algebra. A d-cluster tilting subcategory of an abelian category is d-abelian [38, Theorem 3.16], and every d-abelian category has been shown to arise in this way [14, 44]. Similarly, a d-cluster tilting subcategory of a triangulated category carries a $(d + 2)$-angulated structure given that it is closed under d-suspension [16, Theorem 1]. We investigate the role of gentle algebras in higher homological algebra by studying the d-cluster tilting subcategories both of their module and their derived categories. While the questions we answer in this paper are of a higher homological nature, geometric models play a crucial role in our proofs. In particular, it seems difficult to prove our main result without applying the geometric insights offered in [47]. Thus, in addition to the new understanding our results provide on the role of

gentle algebras in higher homological algebra, this also demonstrates the power of geometric models.

Recall that a finite dimensional algebra is called weakly d-representation finite if it has a module that generates a d-cluster tilting subcategory. The study of such algebras has played an important role in the development of higher homological algebra as we know it today. From the viewpoint of higher AR-theory, the class of weakly d-representation finite algebras can be thought of as a higher analogue of algebras of finite representation type. In particular, the definition coincides with the classical notion of a representation finite algebra in the case $d = 1$.

Just as the classification of (hereditary) algebras of finite representation type has been one of the fundamental questions in classical representation theory, the classification of weakly d-representation finite algebras is an important question in higher representation theory. In general, this is a difficult problem, but significant progress has been made for particularly nice classes of algebras. Darpö and Iyama characterize weakly d-representation finite cyclic Nakayama algebras with homogeneous relations in [11, Theorem 5.1]. The acyclic case was first studied by Jasso [38, Proposition 6.2], and Vaso gives a complete classification in [57, Theorem 2]. In the same paper, Vaso also characterizes all d-representation finite d-hereditary Nakayama algebras [57, Theorem 3]. A classification of iterated tilted d-representation finite d-hereditary algebras in the case $d = 2$ is given by Iyama and Oppermann [33, Theorem 3.12]. Very recently, similar classification results have been obtained in the context of radical square zero algebras [58], monomial algebras [53], and symmetric algebras [12].

A natural question to ask is whether the classification results mentioned above can be extended to more general classes of algebras. Gentle algebras constitute a large class of algebras which naturally extends many of the known examples where a classification has been obtained. In this paper we give a complete classification of weakly d-representation finite gentle algebras, as well as d-representation finite d-hereditary gentle algebras, see Corollary 3.3 and Corollary 3.4. The main step towards these results is the theorem below, where we show that only very few gentle algebras are weakly d-representation finite. More precisely, we prove the following.

Theorem 1 (see Theorem 3.1). *Let Λ be a gentle algebra. If $\mathrm{mod}\Lambda$ contains a d-cluster tilting subcategory for some $d \geq 2$, then Λ is a radical square zero Nakayama algebra.*

While the existence of d-cluster tilting subcategories of module categories is well-studied for certain classes of algebras, less is known in the triangulated setup. The main aim of this paper is to increase this understanding in the case of derived categories associated to gentle algebras. The class of gentle algebras is special in that not only are these algebras of tame representation type, but they are also derived tame. The indecomposable objects in the bounded derived category of a gentle algebra are classified in [7], and a basis of the morphism space between indecomposable objects is described in [2]. In [47] a geometric model for the derived category of a gentle algebra is given, see also [18, 45].

Using the geometric model, we characterize d-cluster tilting subcategories of the derived category of a gentle algebra that are closed under d-suspension. Recall that these subcategories give examples of $(d + 2)$-angulated categories. The most important step towards the classification is the following theorem.

Theorem 2 (see Theorem 4.10). *Let Λ be a gentle algebra. If $\mathcal{D}^b(\Lambda)$ contains a d-cluster tilting subcategory that is closed under $[d]$ for some $d \geq 2$, then Λ is derived equivalent to an algebra of Dynkin type A.*

A crucial tool in the proof of this result is Proposition 4.1, where we observe that, for any finite dimensional algebra, an indecomposable perfect object contained in a d-cluster

tilting subcategory that is closed under d-suspension has no non-zero morphisms from its AR-translate to itself. In particular, the middle term in the AR-triangle ending in such an object is indecomposable. This excludes a large class of objects.

Knowing that the only possible examples arise in the type A case, we classify the d-cluster tilting subcategories that are closed under d-suspension when our algebra is derived equivalent to an algebra of Dynkin type A. This classification is given in Theorem 4.11. In particular, this yields a geometric interpretation of the examples coming from d-representation finite d-hereditary gentle algebras. Combining Theorem 4.10 and Theorem 4.11, we see that all d-cluster tilting subcategories of the derived category of a gentle algebra that are closed under d-suspension arise in this way.

Altogether, our work reveals a lack of d-cluster tilting subcategories arising from gentle algebras, both in the module category and the derived category. Through our characterization results, we see that the examples amount to those one can already obtain by studying Nakayama algebras. This suggests that the role of gentle algebras in higher homological algebra is limited, which is surprising due to the otherwise rich theory of this class of algebras. Our results also show that the situation in the derived setup is even more restrictive than in the module category. In particular, the derived category of a cyclic Nakayama algebra never contains d-cluster tilting subcategories closed under d-suspension, even though the associated module category may contain d-cluster tilting subcategories as described in [11].

The paper is structured as follows. In Section 2 we give an overview of some necessary background and preliminaries. This includes the definition of d-cluster tilting subcategories and notions related to d-representation finiteness, as well as an introduction to the geometric model for the derived category of a gentle algebra. In Section 3 we present our results related to the module category of a gentle algebra, before we discuss the derived case in Section 4.

1.1. **Conventions and notation.** Throughout this paper, let d denote a positive integer. We will typically assume $d \geq 2$. All algebras considered are connected and finite dimensional over an algebraically closed field K. The field is assumed to be algebraically closed to be consistent with [2], but as noted in that paper, this condition could be omitted.

Given a quiver Q, we denote its set of vertices by Q_0 and its set of arrows by Q_1. For an arrow α in Q_1, we write $s(\alpha)$ for the start vertex of α and $t(\alpha)$ for the end vertex of α. Given an arrow β with $t(\alpha) = s(\beta)$, we write $\alpha\beta$ for the non-zero product in the path algebra KQ.

We denote the category of finitely generated right modules over an algebra Λ by $\mathrm{mod}\Lambda$. The subcategory of projectives in $\mathrm{mod}\Lambda$ is denoted $\mathrm{proj}\,\Lambda$. We use the notation $\mathcal{D}^b(\Lambda)$ for the bounded derived category of $\mathrm{mod}\Lambda$. The AR-translation (where it exists) is denoted by τ and the suspension functor in $\mathcal{D}^b(\Lambda)$ by $[1]$.

All subcategories are assumed to be full. Given a set of objects $\mathcal{S}$ in an additive category $\mathcal{C}$, we use the notation $\mathrm{add}\,\mathcal{S}$ for the subcategory of $\mathcal{C}$ consisting of direct summands of finite direct sums of objects in $\mathcal{S}$.

2. Background and preliminaries

2.1. d**-cluster tilting subcategories.** The notions of d-abelian and $(d+2)$-angulated categories were introduced in [16, 38] to axiomatize properties of d-cluster tilting subcategories of abelian and triangulated categories. The definition of such subcategories plays a crucial role in this paper.

Before giving the definition, let us recall what it means for a subcategory $\mathcal{U}$ of some category $\mathcal{C}$ to be functorially finite. Given an object X in $\mathcal{C}$, a morphism $f: U \to X$ with U in $\mathcal{U}$ is a *right $\mathcal{U}$-approximation* of X if any morphism $U' \to X$ with U' in $\mathcal{U}$ factors through

f. The subcategory $\mathcal{U}$ is called *contravariantly finite* if every object in $\mathcal{C}$ admits a right $\mathcal{U}$-approximation. The notions of *left $\mathcal{U}$-approximations* and *covariantly finite* subcategories are defined dually. A subcategory is *functorially finite* if it is both covariantly and contravariantly finite.

Given a subcategory $\mathcal{U}$ of some abelian or triangulated category $\mathcal{C}$, we associate the subcategories

$$\mathcal{U}^{\perp_d} = \{X \in \mathcal{C} \mid \operatorname{Ext}^i_{\mathcal{C}}(\mathcal{U}, X) = 0 \ \text{ for } \ 1 \le i \le d-1\}$$
$$^{\perp_d}\mathcal{U} = \{X \in \mathcal{C} \mid \operatorname{Ext}^i_{\mathcal{C}}(X, \mathcal{U}) = 0 \ \text{ for } \ 1 \le i \le d-1\}.$$

Note that we write $\operatorname{Ext}^i_{\mathcal{C}}(X, Y) = \operatorname{Hom}_{\mathcal{C}}(X, Y[i])$ in the triangulated case. Using this notation, we give the definition of a d-cluster tilting subcategory. Recall that in the case where $\mathcal{C}$ is abelian, our subcategory $\mathcal{U}$ is called *generating* (resp. *cogenerating*) if for each object X in $\mathcal{C}$ there exists an epimorphism $U \twoheadrightarrow X$ (resp. monomorphism $X \rightarrowtail U$) with U in $\mathcal{U}$.

Definition 2.1 (see [31, 41, 43]). A functorially finite subcategory $\mathcal{U}$ of an abelian or triangulated category $\mathcal{C}$ is d-*cluster tilting* if it is generating-cogenerating (in the abelian case) and

$$\mathcal{U} = \mathcal{U}^{\perp_d} = {}^{\perp_d}\mathcal{U}.$$

It follows immediately from the definition that any d-cluster tilting subcategory necessarily contains all projective and all injective objects. In particular, a d-cluster tilting subcategory is automatically generating-cogenerating when the ambient category is a module category.

Following [39], a finite dimensional algebra Λ is called *weakly d-representation finite* if it has a d-cluster tilting Λ-module. This means that there is a Λ-module M such that $\operatorname{add}(M)$ is a d-cluster tilting subcategory of $\operatorname{mod}\Lambda$. A weakly d-representation finite algebra is called d-*representation finite d-hereditary* if the global dimension of Λ is at most d. The reader should note that terminology related to higher representation finiteness varies in the literature. For instance, a d-representation finite d-hereditary algebra is in many papers known simply as d-*representation finite*, see for example [21–23, 25, 32, 33].

The module category of a d-representation finite d-hereditary algebra Λ contains a unique d-cluster tilting subcategory $\mathcal{U} \subseteq \operatorname{mod}\Lambda$ [31, Theorem 1.6]. In this situation, the subcategory

$$\mathcal{U}[d\mathbb{Z}] = \operatorname{add}\{X[di] \mid X \in \mathcal{U} \ \text{ and } \ i \in \mathbb{Z}\} \subseteq \mathcal{D}^b(\Lambda)$$

is a d-cluster tilting subcategory of the bounded derived category [31, Theorem 1.23]. This subcategory is closed under $[d]$, and thus yields an example of a $(d+2)$-angulated category [16].

Example 2.2. Consider $\Lambda = KA_3/J^2$, where A_3 is the quiver $1 \to 2 \to 3$ and J denotes the arrow ideal. The algebra Λ is 2-representation finite 2-hereditary. Figure 1 shows the AR-quiver of $\operatorname{mod}\Lambda$ with the unique 2-cluster tilting subcategory $\mathcal{U} = \operatorname{add} M$, where M is given by the direct sum of all indecomposable projectives and injectives. The lift $\mathcal{U}[2\mathbb{Z}]$ to $\mathcal{D}^b(\Lambda)$ is shown in Figure 2.
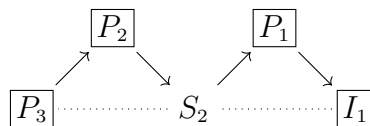

FIGURE 1. The AR-quiver of $\operatorname{mod}\Lambda$ with rectangles around the indecomposable objects in the 2-cluster tilting subcategory $\mathcal{U}$.
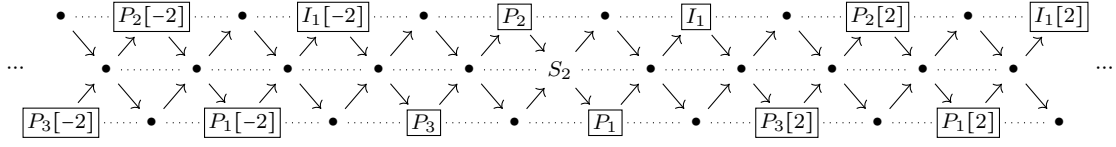
FIGURE 2. The AR-quiver of $\mathcal{D}^b(\Lambda)$ with rectangles around the indecomposable objects in the 2-cluster tilting subcategory $\mathcal{U}[2\mathbb{Z}]$. Note that we identify a Λ-module with its associated stalk complex concentrated in degree 0.

2.2. Gentle algebras.

Gentle algebras constitute a large and well-studied class of algebras. They first appeared as iterated tilted algebras of Dynkin type A [3] and $\widetilde{A}$ [5], and can be seen as generalizations of algebras of Dynkin type A.

Definition 2.3. An algebra of the form KQ/I is called *gentle* if the following conditions hold:

(1) The quiver Q is finite;
(2) For all $\alpha \in Q_1$, there exists at most one arrow β such that $\alpha\beta \notin I$ and at most one arrow γ such that $\gamma\alpha \notin I$;
(3) For all $\alpha \in Q_1$, there exists at most one arrow β with $t(\alpha) = s(\beta)$ such that $\alpha\beta \in I$ and at most one arrow γ with $t(\gamma) = s(\alpha)$ such that $\gamma\alpha \in I$;
(4) The ideal I is admissible and generated by the relations in (3).

Gentle algebras are special biserial and their indecomposable modules have been classified in terms of string and band combinatorics [9]. In this paper we only need to work with string modules, so we briefly recall their definition here.

For every arrow α in Q_1 with $s(\alpha) = x$ and $t(\alpha) = y$, we define its formal inverse α^{-1} by setting $s(\alpha^{-1}) = y$ and $t(\alpha^{-1}) = x$. A *walk* is a sequence $\alpha_1 \ldots \alpha_r$ of arrows and inverse arrows such that $t(\alpha_i) = s(\alpha_{i+1})$ and $\alpha_{i+1} \neq \alpha_i^{-1}$ for all $1 \leq i \leq r-1$. A *string* is a walk w in Q such that no subword of w or of w^{-1} is in I. If w is a string, the associated *string module* $M(w)$ is given by the quiver representation obtained by replacing every vertex in w by a copy of K and every arrow by the identity map.

2.3. The geometric model.

In [47], building on [54] and [55], a geometric model for the derived category of a gentle algebra is given in terms of surface dissections. It is closely related to the partially wrapped Fukaya category of surfaces with stops described in [18] and further studied in [45]. We give a brief introduction to the geometric model, emphasizing aspects that are needed in Section 4. The reader is referred to [47] for more detailed explanations.

The geometric model is based on a bijection between gentle algebras and certain surface dissections as described in [47], see also [6, 50]. The construction is as follows. Consider a pair (S, M), where S is a compact oriented surface with boundary and M is a set of marked points on the boundary. Let Γ be a *dissection* of (S, M) into polygons. That is, the vertices of Γ are exactly the marked points in M and the complement of Γ in S is a disjoint union of polygons. We call the dissection Γ *admissible* if each polygon either has exactly one boundary segment or, if it has no boundary segment, encloses a boundary component with no marked points on it.

Given an admissible dissection Γ of (S, M), we describe how to obtain a quiver Q. The vertices of Q are in bijection with the edges of Γ. If two edges γ and γ' of Γ are incident with the same vertex of Γ such that γ' directly follows γ in the orientation of the surface, there is an arrow from γ to γ' in Q. We define an ideal of relations I of KQ as follows. Suppose α and β are two composable arrows in Q. If the edges of Γ corresponding to the vertices $s(\alpha), s(\beta)$ and $t(\beta)$ in Q are incident with the same vertex of Γ and directly follow each other in the

orientation of S, then $\alpha\beta \notin I$. Otherwise, we set $\alpha\beta \in I$. The resulting algebra $\Lambda = KQ/I$ is gentle, and every gentle algebra arises in this way.

Example 2.4. Consider the gentle algebra $\Lambda = KA_n/J^2$, where A_n is the linearly oriented quiver of Dynkin type A with n vertices and J is the arrow ideal. This algebra arises from a dissection of the disk with $n + 1$ marked points on the boundary, as illustrated in Figure 3.
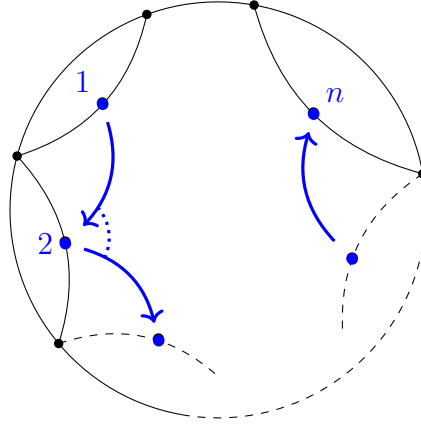

FIGURE 3. The geometric model associated to the gentle algebra KA_n/J^2. The dissection Γ is given by the thinner outer curves, while the quiver A_n is drawn with thicker blue arrows. The relations are indicated by the dotted line.

Given a surface (S, M) with admissible dissection Γ, we define the dual (embedded) graph L of Γ as follows. The vertices of L lie on the boundary of S in such a way that on each boundary component, the vertices of L and Γ alternate. If a boundary component does not contain any marked points of M, then we replace it by a vertex of L and refer to it as a *puncture*. The edges of the dual graph L are in bijection with the edges of Γ, and for each edge γ in Γ there exists a unique edge ℓ in L crossing γ and crossing no other edge of Γ. The dual graph L gives an admissible dissection of (S, M).

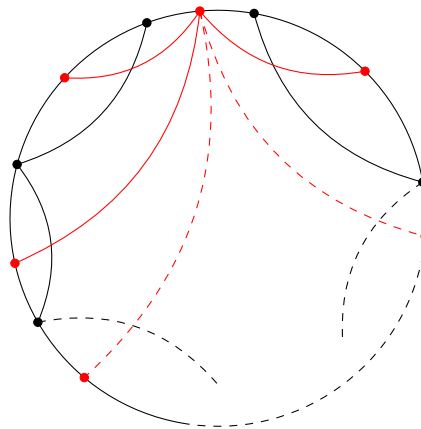**Example 2.5.** The dual graph L of the dissection Γ given in Example 2.4 is shown in Figure 4.

FIGURE 4. The dual graph (in red) of an admissible dissection of the disk.

We let the notion *closed curve* refer to a homotopy class of closed curves in (S, M). Similarly, an *arc* is a homotopy class of curves between marked points or wrapping around

punctures on one or both ends (following the orientation of S). By abuse of notation, we typically let a representative γ of an arc also denote the arc itself. If the underlying curve of an arc connects two marked points, we call the arc *finite*. A finite arc is called *minimal* if it is homotopy equivalent to a segment of the boundary with no marked points. In particular, any minimal arc has both endpoints on the same boundary component.

Let γ be an arc or a closed curve in (S, M). A *grading* of γ is a function $f: L \cap \gamma \to \mathbb{Z}$ subject to the condition described below. Let p_1 and p_2 be two consecutive (in the orientation of γ) intersection points of γ with L. The points p_1 and p_2 correspond to edges l_1 and l_2 of a unique polygon P of the dissection given by L in such a way that γ enters P via l_1 and leaves via l_2. The polygon P has exactly one boundary segment. The grading f satisfies $f(p_2) = f(p_1) + 1$ if this boundary segment lies to the left of γ with respect to the orientation of γ and $f(p_2) = f(p_1) - 1$ otherwise. A grading f of γ is hence determined by its value $f(p)$ at one point $p \in L \cap \gamma$. Note that in the case where γ is a closed curve, there need not exist a grading of γ. In the case where a grading f of γ does exist, we refer to the pair (γ, f) as a *graded arc* or a *graded closed curve*.

By [7] the indecomposable objects in $\mathcal{D}^b(\Lambda)$ are in bijection with so-called graded homotopy strings and bands. This bijection allows us to divide the indecomposable objects into *string objects* and *band objects*. The graded homotopy strings are in bijection with graded arcs. Consequently, graded arcs are in bijection with string objects in $\mathcal{D}^b(\Lambda)$. Band objects occur in one parameter families, which are in bijection with graded closed curves. Altogether, this yields a correspondence between indecomposable objects in $\mathcal{D}^b(\Lambda)$ and graded arcs and graded closed curves in (S, M). Given a string object X in $\mathcal{D}^b(\Lambda)$, we use the notation (γ_X, f_X) for the corresponding graded arc. When we do not need to describe the grading explicitly, we refer to the graded arc simply by γ_X.
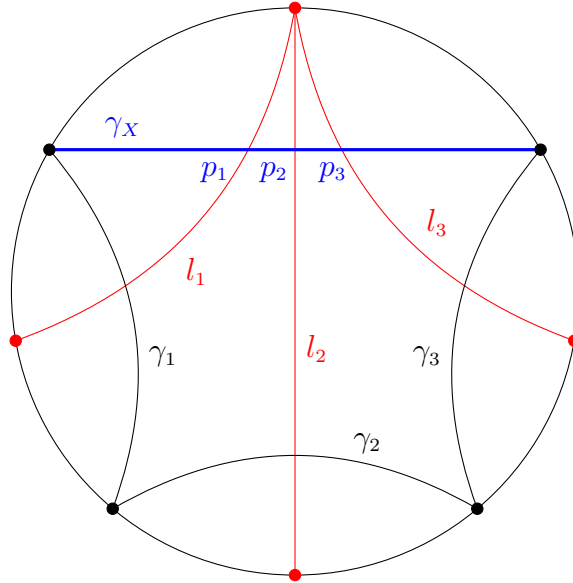

FIGURE 5. The geometric model of KA_3/J^2, with the dual graph L given by the curves l_1, l_2 and l_3. Objects corresponding to minimal graded arcs are described in Example 2.6.

Example 2.6. We illustrate how graded arcs in the surface of our running example correspond to indecomposable objects in the derived category. In Figure 5 we display the geometric model of $\Lambda = KA_3/J^2$, where the dual graph L is drawn in red. Recall that we label the vertices in A_3 by $1 \to 2 \to 3$. Edges of the dual graph are denoted l_i for $i \in \{1, 2, 3\}$. The arcs corresponding

to indecomposable projectives up to shift are drawn in black, so that the arc γ_i corresponding to the stalk complex of the projective in i crosses l_i.

Consider the blue arc γ_X which intersects all three edges of L. The intersection of γ_X and l_i is labeled p_i. As the grading of an arc is determined by its grading in one intersection point, we choose $f_X(p_1) = 0$, which gives $f_X(p_2) = -1$ and $f_X(p_3) = -2$. Thus, by the description in [47], the indecomposable object X in $\mathcal{D}^b(\Lambda)$ corresponding to the graded arc (γ_X, f_X) is given by

$$\cdots \to 0 \to \overset{-2}{P_3} \to \overset{-1}{P_2} \to \overset{0}{P_1} \to 0 \to \cdots,$$

where the grading is written above the complex. We recognize this as the projective resolution of the simple in 1, so X is isomorphic in $\mathcal{D}^b(\Lambda)$ to the stalk complex with the simple in 1 in degree 0.

In Section 4 we also consider the perfect derived category of a gentle algebra Λ. This is the full isomorphism closed subcategory $\mathcal{D}^b(\operatorname{proj}\Lambda) \subseteq \mathcal{D}^b(\Lambda)$ consisting of bounded complexes of finitely generated projective Λ-modules. The indecomposable perfect objects correspond to finite graded arcs and graded closed curves. If Λ has finite global dimension, the associated surface has no punctures and the perfect derived category is equivalent to the bounded derived category.

By [19], the perfect derived category of a finite dimensional algebra Λ admits a Serre functor. This is equivalent to existence of AR-triangles in $\mathcal{D}^b(\operatorname{proj}\Lambda)$. In the case where Λ is gentle, the AR-translate of an indecomposable object X corresponding to a finite graded arc (γ_X, f_X) can be computed in terms of the geometric model. More precisely, the AR-translate τX corresponds to the arc $\gamma_{\tau X}$ obtained by moving the endpoints of γ_X to the next marked points on the boundary (following the orientation of S) equipped with a suitable grading.

Given two homotopy classes of graded curves (γ_X, f_X) and (γ_Y, f_Y) corresponding to indecomposable objects X and Y in $\mathcal{D}^b(\Lambda)$, there is an explicit bijection between the oriented graded intersections of γ_X with γ_Y and $\operatorname{Hom}_{\mathcal{D}^b(\Lambda)}(X, Y)$. Note that when considering intersections of homotopy classes of curves, we always choose representatives such that the number of intersections is minimal. We give a summary of the definition of an oriented graded intersection using Figure 6 and Figure 7, and refer the reader to [47, Definition 3.7] for more details.
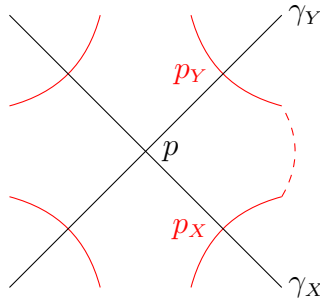

FIGURE 6. Illustration of oriented graded intersection from (γ_X, f_X) to (γ_Y, f_Y). Intersection points of γ_X and γ_Y with edges of the polygon given by L as a dissection of the surface are labeled by p_X and p_Y, as indicated.

In the first case, let γ_X and γ_Y intersect such that the intersection point p is not at a puncture. The intersection lies in one of the polygons given by the dual graph as a dissection of the surface, and this polygon has exactly one boundary segment. In Figure 6 we assume that this boundary segment does not lie between p_X and p_Y, as indicated by the red dotted line.

Note that the intersection point p can be on the boundary of S and that the edges of the dual graph are not necessarily distinct. Given a situation as in Figure 6, there is an oriented graded intersection from (γ_X, f_X) to (γ_Y, f_Y), and hence a non-zero morphism from X to Y in the derived category, whenever $f_X(p_X) = f_Y(p_Y)$.
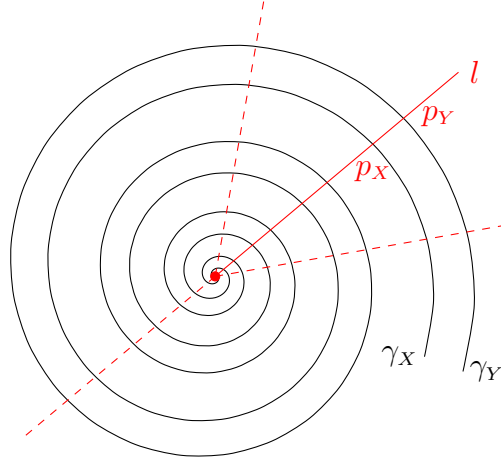

FIGURE 7. Illustration of oriented graded intersection from (γ_X, f_X) to (γ_Y, f_Y) in the case where γ_X and γ_Y are infinite arcs wrapping around the same puncture at one end.

In the second case, assume that γ_X and γ_Y are infinite arcs, wrapping around the same puncture p at one end. We then say that the arcs intersect at p (even though they technically only approach the point of intersection asymptotically). Consider an edge l in the dual graph L ending at p. Let p_X and p_Y be intersection points of l with γ_X and γ_Y, respectively. The situation is shown in Figure 7. In this setup, there is an oriented graded intersection from γ_X to γ_Y if $f_X(p_X) = f_Y(p_Y)$.

The possibilities for an intersection point p of graded curves γ_X and γ_Y to give rise to a non-zero morphism in the derived category can be summarized as follows:

- If p corresponds to a point in the interior of the surface and f_X is any grading of γ_X, then there exists a unique grading f_Y of γ_Y such that p corresponds to an oriented graded intersection from (γ_X, f_X) to (γ_Y, f_Y) and to an oriented graded intersection from (γ_Y, f_Y) to $(\gamma_X, f_X[1])$.
- If p is on the boundary of S and f_X is any grading of γ_X, then there exists a unique grading f_Y of γ_Y such that p corresponds to either an oriented graded intersection from (γ_X, f_X) to (γ_Y, f_Y) or an oriented graded intersection from (γ_Y, f_Y) to (γ_X, f_X).
- It remains to consider the case where p is a puncture. For this, let w denote the number of endpoints of edges of L that are incident with p. If f_X is any grading of γ_X, then there exists a grading f_Y of γ_Y such that p corresponds to a family of oriented graded intersections from (γ_X, f_X) to $(\gamma_Y, f_Y[mw])$ and a family of oriented graded intersections from (γ_Y, f_Y) to $(\gamma_X, f_X[(m+1)w])$ for $m \geq 0$.

Example 2.7. We illustrate the correspondence between morphisms and oriented graded intersections in our running example. Consider the arcs γ_X and γ_3 from Example 2.6, see Figure 5. Let f_X be the grading of γ_X that is described in Example 2.6 and define a grading f_3 on γ_3 by $f_3(p) = -2$ for the sole intersection point $p \in L \cap \gamma_3$. The graded arc (γ_3, f_3) corresponds to the object $P_3[2]$, by which we mean the stalk complex with the projective P_3 in degree -2. As $f_X(p_3) = f_3(p)$, there is an oriented graded intersection from (γ_X, f_X) to (γ_3, f_3). This corresponds to a non-zero morphism from X to $P_3[2]$ in the derived category.

3. Weakly d-representation finite gentle algebras

In this section we give a complete classification of the weakly d-representation finite gentle algebras, as well as d-representation finite d-hereditary gentle algebras.

We denote the quivers

$$1 \rightarrow 2 \rightarrow \cdots \rightarrow n \qquad \text{and} \qquad \text{(quiver diagram with vertices } 0, 1, \ldots, n \text{ arranged in a cycle)}$$

by A_n and $\widetilde{A}_n$, respectively. A *Nakayama algebra* is a path algebra of one of these quivers modulo an admissible ideal, see for instance [4, Chapter V] for more details. Recall that such an algebra is called *radical square zero* if the admissible ideal is given by J^2, where J denotes the arrow ideal. In our next result, we show that if Λ is a gentle algebra and $\mathrm{mod}\Lambda$ admits a d-cluster tilting subcategory for some $d \geq 2$, then Λ is a radical square zero Nakayama algebra. In particular, this yields that the only examples of d-cluster tilting subcategories of module categories that arise from gentle algebras are the ones known from [11, 57].

Theorem 3.1. *Let Λ be a gentle algebra. If $\mathrm{mod}\Lambda$ contains a d-cluster tilting subcategory for some $d \geq 2$, then Λ is a radical square zero Nakayama algebra.*

Proof. Any vertex x in the quiver of the gentle algebra Λ is part of a subquiver

with relations as indicated by the dotted lines. Note that we allow arrows to be non-existent. We write $\alpha = \varnothing$ in the case where the arrow α in the above figure does not exist.

The projective module P_x associated to the vertex x is represented by the string $u_\gamma^{-1}\gamma^{-1}\delta u_\delta$, where u_δ and u_γ are the (possibly trivial) maximal strings such that δu_δ and γu_γ are non-zero strings. Similarly, the injective I_x is represented by $v_\alpha \alpha \beta^{-1} v_\beta^{-1}$, where v_α and v_β are maximal strings such that $v_\alpha \alpha$ and $v_\beta \beta$ are non-zero strings. By [56], there is a short exact sequence

$$0 \rightarrow M(u_\gamma^{-1}\gamma^{-1}\delta u_\delta) \rightarrow M(v_\alpha \alpha \delta u_\delta) \oplus M(v_\beta \beta \gamma u_\gamma) \rightarrow M(v_\alpha \alpha \beta^{-1} v_\beta^{-1}) \rightarrow 0$$

starting in P_x and ending in I_x. This sequence does not split as long as the following conditions are satisfied:

(i) If $\alpha = \varnothing$, then $\gamma \neq \varnothing$;
(ii) If $\beta = \varnothing$, then $\delta \neq \varnothing$.

In this case the non-split short exact sequence is known as an *overlap extension* [10, Definition 3.1], see also [8], and we have $\mathrm{Ext}^1_\Lambda(I_x, P_x) \neq 0$.

Assume that $\mathrm{mod}\Lambda$ contains a d-cluster tilting subcategory for some $d \geq 2$. As a d-cluster tilting subcategory necessarily contains all projective and injective modules, this implies that $\mathrm{Ext}^1_\Lambda(I_x, P_x) = 0$ for every vertex x. By our argument above, this means that for every vertex in the quiver of Λ, at least one of (i) or (ii) does not hold. Considering all possible configurations of the subquiver associated to a vertex x, we see that this only happens if there is exactly one arrow adjacent to x or if we have a situation of the type

$$\longrightarrow x \longrightarrow$$
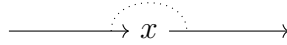
with exactly one incoming and one outgoing arrow. This yields the result. □

Remark 3.2. Note that the result above does not hold in the more general case where Λ is assumed to be a string algebra. For examples of this, see for instance [58].

Combining Theorem 3.1 with previously known classification results for Nakayama algebras [11, 57], we obtain the following characterization of weakly d-representation finite gentle algebras. Recall that we denote the arrow ideal of a path algebra KQ by J.

Corollary 3.3. *Let Λ be a gentle algebra and assume $d \geq 2$. Then Λ is weakly d-representation finite if and only if one of the following statements holds:*
 (1) $\Lambda = KA_n/J^2$ with $n = dk + 1$ for some $k \geq 0$;
 (2) $\Lambda = K\widetilde{A}_n/J^2$ with either $n = dk - 1$ for some $k \geq 1$ or d is even and $n = 2dk - 1$ for some $k \geq 1$.

Proof. If one of the statements *(1)* or *(2)* holds, it follows from [57, Theorem 2] and [11, Theorem 5.1] that Λ is weakly d-representation finite. For this, notice that existence of a d-cluster tilting subcategory implies existence of a d-cluster tilting module as our algebra is representation finite.

Assume next that Λ is weakly d-representation finite. In particular, this means that $\mathrm{mod}\Lambda$ contains a d-cluster tilting subcategory. By Theorem 3.1, this yields $\Lambda = KA_n/J^2$ or $\Lambda = K\widetilde{A}_n/J^2$ for some n. Our conclusion now follows by applying [57, Theorem 2] and [11, Theorem 5.1] and using that Λ has Loewy length 2. □

Our approach also yields a classification of d-representation finite d-hereditary gentle algebras. Recall that a weakly d-representation finite algebra is called d-representation finite d-hereditary in the case where the global dimension is at most d.

Corollary 3.4. *Let Λ be a gentle algebra of global dimension d with $d \geq 2$. Then Λ is d-representation finite d-hereditary if and only if $\Lambda = KA_n/J^2$ with $n = d + 1$.*

Proof. This is an immediate consequence of Theorem 3.1 combined with Vaso's classification of d-representation finite d-hereditary Nakayama algebras [57, Theorem 3]. □

In the case where $d \geq 3$ or the preprojective algebra of Λ is a planar quiver with potential, Corollary 3.4 can be recovered by work of Sandøy and Thibault [53, Theorem B]. However, their result does not cover all gentle algebras, since there exist gentle algebras with a non-planar quiver and global dimension 2. One algebra of this type is shown in Example 3.5, and it gives rise to an infinite family of such algebras.
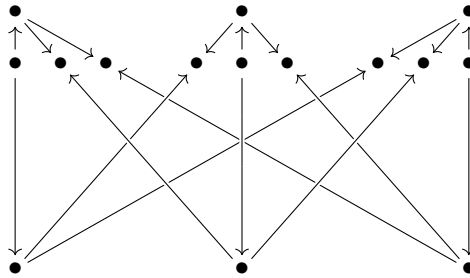

FIGURE 8. The non-planar quiver used in Example 3.5.

Example 3.5. Consider the path algebra $\Lambda = KQ/I$, where Q is the quiver in Figure 8 and I is any set of relations making Λ gentle, see Definition 2.3. Then Λ has global dimension 2.

4. d-CLUSTER TILTING SUBCATEGORIES OF THE DERIVED CATEGORY

In this section we study the derived category of a gentle algebra. We discuss under what circumstances this category contains d-cluster tilting subcategories that are closed under $[d]$. Our main tool in this investigation is the geometric model from [47], as described in Section 2.3. All Hom sets in this section are considered in the derived category.

Our proofs are based on the observation below, which gives a useful condition satisfied by perfect objects contained in a d-cluster tilting subcategory closed under $[d]$. This result can be deduced from [35, Proposition 3.4] in the case where the algebra has finite global dimension. Note that Proposition 4.1 holds for any finite dimensional algebra.

Proposition 4.1. *Let Λ be a finite dimensional algebra and consider a d-cluster tilting subcategory $\mathcal{U} \subseteq \mathcal{D}^b(\Lambda)$ that is closed under $[d]$ for some $d \geq 2$. If an indecomposable perfect object X is contained in $\mathcal{U}$, then $\mathrm{Hom}(\tau X, X) = 0$.*

Proof. Recall that as the subcategory $\mathcal{U}$ is d-cluster tilting in $\mathcal{D}^b(\Lambda)$, we have

$$\mathcal{U} = \{Y \in \mathcal{D}^b(\Lambda) \mid \mathrm{Hom}(\mathcal{U}, Y[i]) = 0 \ \text{ for } \ 1 \leq i \leq d-1\}$$
$$= \{Y \in \mathcal{D}^b(\Lambda) \mid \mathrm{Hom}(Y, \mathcal{U}[i]) = 0 \ \text{ for } \ 1 \leq i \leq d-1\}.$$

As in Section 2.1, we denote the first of these sets by $\mathcal{U}^{\perp_d}$ and the second one by $^{\perp_d}\mathcal{U}$.

Suppose towards a contradiction that there exists an indecomposable perfect object X in $\mathcal{U}$ with $\mathrm{Hom}(\tau X, X) \neq 0$. This yields

$$\mathrm{Hom}(\tau X[1], X[1]) \cong \mathrm{Hom}(\tau X, X) \neq 0.$$

Since $d \geq 2$ and $\mathcal{U}$ is d-cluster tilting, this means that $\tau X[1] \notin {}^{\perp_d}\mathcal{U} = \mathcal{U}^{\perp_d}$. Thus, there exists an object $Y \in \mathcal{U}$ such that $\mathrm{Hom}(Y, \tau X[1][i]) \neq 0$ for some $1 \leq i \leq d-1$.

Recall that the derived category of perfect complexes over Λ has a Serre functor, which coincides with $\tau[1]$, see [52, Proposition I.2.3]. Moreover, it follows from the fact that gentle algebras are Gorenstein [17] combined with [48, Example 3.7] that $\tau[1] \colon \mathcal{D}^b(\mathrm{proj}\, \Lambda) \to \mathcal{D}^b(\Lambda)$ is a partial Serre functor in the sense of [48]. Using this, we obtain

$$\mathrm{Hom}(Y, \tau X[1][i]) \cong \mathrm{Hom}(Y[-i], \tau X[1])$$
$$\cong D\,\mathrm{Hom}(X, Y[-i])$$
$$\cong D\,\mathrm{Hom}(X[d], Y[d-i]) \neq 0.$$

Since $\mathcal{U}$ is closed under $[d]$, we have $X[d] \in \mathcal{U}$. Noting that $1 \leq d-i \leq d-1$, this yields $Y \notin \mathcal{U}^{\perp_d} = \mathcal{U}$, which is a contradiction. $\qquad\square$

Our next aim is to show that for gentle algebras there are limitations on arcs corresponding to objects in a d-cluster tilting subcategory. Recall that we use the notation (γ_X, f_X) for the graded arc that corresponds to a string object X in the derived category and that a finite arc is called minimal if it is homotopy equivalent to a segment of the boundary with no marked points.

Lemma 4.2. *Let Λ be a gentle algebra. If $X \in \mathcal{D}^b(\Lambda)$ corresponds to a finite graded arc that is not minimal, then $\mathrm{Hom}(\tau X, X) \neq 0$.*

Proof. As the graded arc γ_X corresponding to X is finite, we can use the algorithm described in [47, Section 5] to compute τX. We obtain the graded arc $\gamma_{\tau X}$ corresponding to τX by moving the endpoints of γ_X to the next marked points following the orientation of the boundary. This is illustrated in Figure 9, where the marked points in the figure are not necessarily distinct.
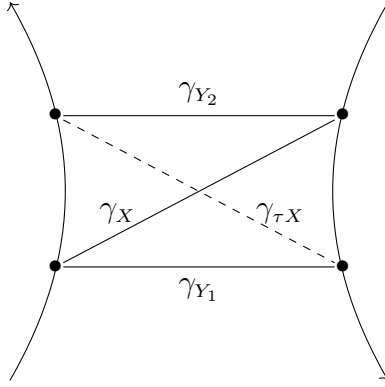
FIGURE 9. Geometric computation of τX.

Note that the arc γ_X is minimal if and only if either γ_{Y_1} or γ_{Y_2} is contractible to a point. When γ_X is not minimal, the objects Y_1 and Y_2 are hence non-zero. This yields an almost split sequence

$$\tau X \to Y_1 \oplus Y_2 \to X$$

with two indecomposable summands in the middle term. The morphism $\tau X \to Y_1 \to X$ is non-zero, and the conclusion follows. $\square$

As a consequence of our two previous results, we find that if an indecomposable perfect object is contained in a d-cluster tilting subcategory that is closed under $[d]$, then it corresponds to a minimal graded arc.

Lemma 4.3. *Let Λ be a gentle algebra and consider a d-cluster tilting subcategory $\mathcal{U} \subseteq \mathcal{D}^b(\Lambda)$ that is closed under $[d]$ for some $d \geq 2$. If an indecomposable perfect object is contained in $\mathcal{U}$, then it corresponds to a minimal graded arc.*

Proof. Let X be a perfect object that is contained in $\mathcal{U}$. If X is a band object, then $\tau X \cong X$, which contradicts Proposition 4.1. As X is perfect, this implies that γ_X is a finite graded arc. Our statement now follows by combining Lemma 4.2 and Proposition 4.1. $\square$

We next investigate how the gradings of minimal arcs giving rise to objects in a d-cluster tilting subcategory are related. Let Λ be a gentle algebra and recall that L denotes the associated dual graph as explained in Section 2.3.

Consider two graded arcs (γ_X, f_X) and (γ_Y, f_Y) with a common endpoint m, as indicated in Figure 10. Let p_X (resp. p_Y) denote the intersection point of γ_X (resp. γ_Y) and L that is closest to m. We say that γ_X and γ_Y have *compatible grading in m* if $f_X(p_X) = f_Y(p_Y)$. Note that by the description of morphisms in the bounded derived category in terms of graded intersections, this implies that $\mathrm{Hom}(Y, X) \neq 0$. Similarly, we say that the two arcs have *d-compatible grading in m* if $f_X(p_X) \equiv f_Y(p_Y) \pmod{d}$.
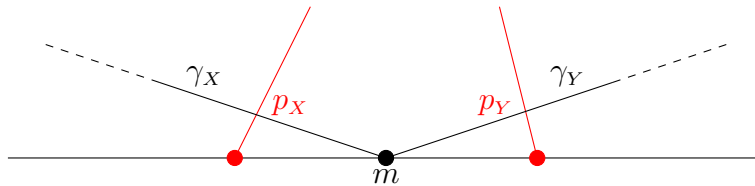
FIGURE 10. Illustration of compatible grading. The graded arcs γ_X and γ_Y have d-compatible grading in m if $f_X(p_X) \equiv f_Y(p_Y) \pmod{d}$.

Consider an object X in $\mathcal{D}^b(\Lambda)$ corresponding to a minimal graded arc. Let $\mathcal{B}_X$ be the boundary component containing the endpoints of γ_X. Denote the marked points on $\mathcal{B}_X$ by $m_1, m_2, \ldots, m_t$, where m_1 and m_t are the endpoints of γ_X. The marked point m_{j+1} follows m_j, as illustrated in Figure 11. Use the notation $\gamma_1 = \gamma_X$ for the graded arc corresponding to $X_1 = X$. For $j = 2, \ldots, t$, let γ_j be the minimal graded arc with endpoints m_{j-1} and m_j for which γ_{j-1} and γ_j have compatible grading in m_{j-1}. Denote the object corresponding to γ_j by X_j. We now define $\mathcal{V}_X^d$ to be the subcategory

$$\mathcal{V}_X^d = \mathrm{add}\{X_j[di] \mid i \in \mathbb{Z} \text{ and } 1 \le j \le t\} \subseteq \mathcal{D}^b(\Lambda).$$

Note that as every indecomposable object in $\mathcal{V}_X^d$ corresponds to a minimal graded arc, we have $\mathcal{V}_X^d \subseteq \mathcal{D}^b(\mathrm{proj}\,\Lambda)$.
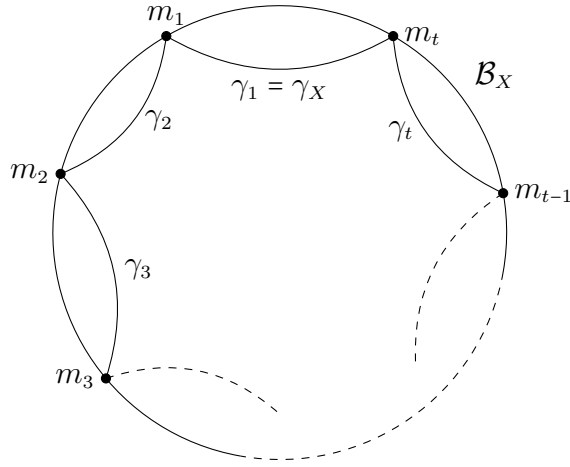

FIGURE 11. Minimal arcs sharing a boundary with γ_X.

Example 4.4. In Example 2.2 we considered the algebra $\Lambda = KA_3/J^2$, which is 2-representation finite 2-hereditary. Note that the 2-cluster tilting subcategory $\mathcal{U}[2\mathbb{Z}] \subseteq \mathcal{D}^b(\Lambda)$ as illustrated in Figure 2 can also be obtained as $\mathcal{V}_X^2$ for any indecomposable object $X \in \mathcal{U}[2\mathbb{Z}]$.

Our next two lemmas demonstrate that the notion of d-compatibility is useful to describe when indecomposable objects are contained in a d-cluster tilting subcategory. In Lemma 4.5 we consider the general case, before restricting to perfect objects in Lemma 4.6.

Lemma 4.5. *Let Λ be a gentle algebra and consider a d-cluster tilting subcategory $\mathcal{U} \subseteq \mathcal{D}^b(\Lambda)$ that is closed under $[d]$ for some $d \ge 2$. The following statements hold for an indecomposable object X in $\mathcal{U}$:*

(1) $X[j] \in \mathcal{U}$ if and only if d divides j.

(2) Let Y correspond to a graded arc γ_Y and assume that γ_X and γ_Y have a common endpoint m. If $Y \in \mathcal{U}$, then γ_X and γ_Y have d-compatible grading in m.

Proof. Notice that as the subcategory $\mathcal{U}$ is closed under $[d]$, it is also closed under $[id]$ for every integer i. It follows that if d divides j, then $X[j] \in \mathcal{U}$. Suppose that d does not divide j. We may assume without loss of generality that $1 \le j \le d - 1$. As we have $\mathrm{Hom}(X[j], X[j]) \ne 0$, this implies that $X[j]$ is not contained in ${}^{\perp_d}\mathcal{U} = \mathcal{U}$, which proves the first statement.

For the second statement, denote by p_X (resp. p_Y) the intersection point of γ_X (resp. γ_Y) and L that is closest to m. Assume that γ_X and γ_Y do not have d-compatible grading in m, i.e. that $f_X(p_X) \not\equiv f_Y(p_Y) \pmod{d}$. By *(1)*, it is enough to consider the case where $f_X(p_X) = 0$ and

$f_Y(p_Y) = i$ for some $1 \leq i \leq d-1$. It follows from the description of morphisms in terms of graded intersections that this yields $\mathrm{Hom}(X, Y[i]) \neq 0$ or $\mathrm{Hom}(Y[i], X) \neq 0$. In either case, this implies that Y is not contained in $\mathcal{U}$ by the definition of a d-cluster tilting subcategory. $\square$

For perfect objects, there is a converse to part *(2)* of the lemma above, as described in Lemma 4.6 part *(1)*. This yields a complete description of the perfect objects that are contained in a d-cluster tilting subcategory $\mathcal{U}$ that is closed under $[d]$. In particular, we see that the inclusion of one indecomposable perfect object X in $\mathcal{U}$ fully determines what other perfect objects associated to the boundary $\mathcal{B}_X$ are contained in $\mathcal{U}$. Note that the object X in the lemma below must correspond to a minimal graded arc by Lemma 4.3.

Lemma 4.6. *Let Λ be a gentle algebra and consider a d-cluster tilting subcategory $\mathcal{U} \subseteq \mathcal{D}^b(\Lambda)$ that is closed under $[d]$ for some $d \geq 2$. The following statements hold for an indecomposable perfect object X in $\mathcal{U}$:*

> *(1) Let Y correspond to a minimal graded arc γ_Y that shares an endpoint m with γ_X. Then $Y \in \mathcal{U}$ if and only if γ_X and γ_Y have d-compatible grading in m.*
>
> *(2) Let Y correspond to a minimal graded arc with endpoints on the same boundary component as γ_X. Then $Y \in \mathcal{U}$ if and only if $Y \in \mathcal{V}_X^d$. In particular, we have $\mathcal{V}_X^d \subseteq \mathcal{U}$.*

Proof. If $Y \in \mathcal{U}$, the arcs γ_X and γ_Y have d-compatible grading in m by part *(2)* of Lemma 4.5. For the reverse direction, assume that $Y \notin \mathcal{U}$. As $\mathcal{U} = {}^{\perp_d}\mathcal{U}$, this means that there is an indecomposable object $Z \in \mathcal{U}$ such that $\mathrm{Hom}(Y, Z[i]) \neq 0$ for some $1 \leq i \leq d-1$. Without loss of generality we can assume the intersection of γ_X and γ_Y in m to be of the type illustrated in Figure 10, as the proof in the case where the position of the two arcs is interchanged is dual. Consequently, using that γ_Y is minimal, a non-zero morphism from Y to $Z[i]$ must correspond to a graded intersection of γ_Y and $\gamma_{Z[i]}$ in the endpoint m. In particular, this implies that the arcs γ_Y and γ_Z do not have d-compatible grading in m. On the other hand, the gradings of γ_X and γ_Z are d-compatible in m by Lemma 4.5 part *(2)*, as X and Z are in $\mathcal{U}$. Combining this, we see that γ_X and γ_Y do not have d-compatible grading in m.

The second statement follows directly. $\square$

Using the description in Lemma 4.6, we are now able to prove that the only examples of d-cluster tilting subcategories of the perfect derived category that are closed under $[d]$ arise in the type A case. This result provides an important step in the proof of Theorem 4.10, where we show that the analogue statement holds for $\mathcal{D}^b(\Lambda)$.

Remark 4.7. Note that Proposition 4.1, Lemma 4.3, Lemma 4.5 and Lemma 4.6 hold also when considering a d-cluster tilting subcategory $\mathcal{U} \subseteq \mathcal{D}^b(\mathrm{proj}\,\Lambda)$. These results can hence be applied in the setup of Proposition 4.8.

Proposition 4.8. *Let Λ be a gentle algebra. If $\mathcal{D}^b(\mathrm{proj}\,\Lambda)$ contains a d-cluster tilting subcategory that is closed under $[d]$ for some $d \geq 2$, then Λ is derived equivalent to an algebra of Dynkin type A.*

Proof. Assume that Λ is not derived equivalent to an algebra of Dynkin type A. By [47, Corollary 1.23], this means that the surface in the geometric model of $\mathcal{D}^b(\Lambda)$ is not a disk.

Assume towards a contradiction that there exists a d-cluster tilting subcategory $\mathcal{U}$ of $\mathcal{D}^b(\mathrm{proj}\,\Lambda)$ that is closed under $[d]$ for some $d \geq 2$. Let X be an indecomposable object in $\mathcal{U}$, which by Lemma 4.3 corresponds to a minimal graded arc (γ_X, f_X) as X is perfect. The boundary component containing the endpoints of γ_X is denoted by $\mathcal{B}_X$. Lemma 4.6 part *(2)* yields that an object corresponding to a minimal graded arc with endpoints on $\mathcal{B}_X$ is contained in $\mathcal{U}$ if and only if it is in $\mathcal{V}_X^d$.

Denote one of the endpoints of γ_X by m. As the surface in the geometric model associated to Λ is not a disk, there exists an arc γ starting and ending in m that is not contractible to a point. For simplicity, we assume that γ has no self-intersections except in the endpoint. Let γ_i be the arc obtained by concatenating i copies of γ and γ_X in such a way that γ_i has i self-intersections. Note that when choosing a representative of γ_i, it is helpful to choose one such that all the self-intersections are separated and occur before the concatenation with γ_X. See Figure 12 for an illustration of γ_2.
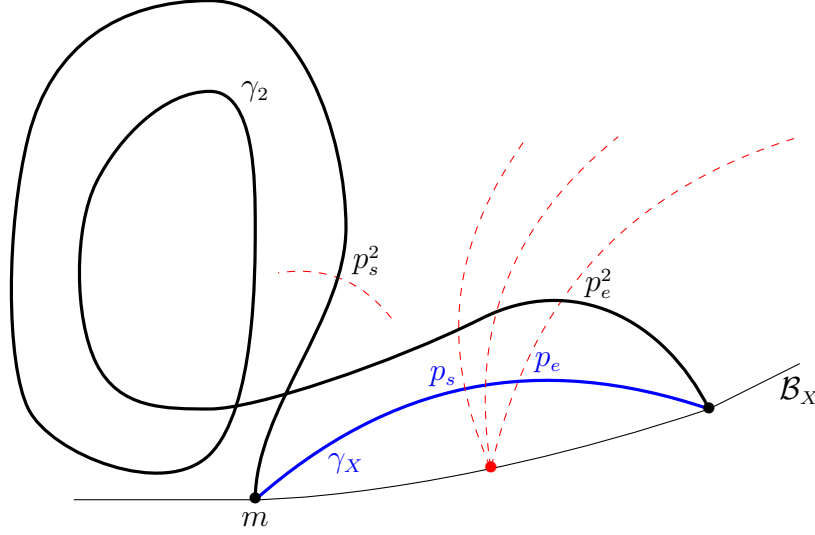

FIGURE 12. The arc γ_i for $i = 2$, as used in the proof of Proposition 4.8.

Denote by p_s (resp. p_s^i) the intersection point of γ_X (resp. γ_i) and L that is closest to m, as indicated in Figure 12. Similarly, we use the notation p_e (resp. p_e^i) for the intersection point of γ_X (resp. γ_i) and L that is closest to the second endpoint of γ_X. Note that p_s and p_e might coincide. Let f_i be the grading of γ_i for which $f_i(p_s^i) = f_X(p_s)$. Denote by Y_i the indecomposable object in $\mathcal{D}^b(\operatorname{proj}\Lambda)$ corresponding to the finite graded arc (γ_i, f_i).

Notice that for each loop of γ_i, the grading increases by an integer w. As the last segment in the construction of γ_i follows the trajectory of γ_X, this yields $f_i(p_e^i) = f_X(p_e) + iw$. Choosing $i = d$, we hence obtain $f_d(p_e^d) \equiv f_X(p_e) \pmod{d}$. By construction, the arc γ_d does not intersect any minimal arcs except in the endpoints. This implies that

$$\operatorname{Hom}(\mathcal{U}, Y_d[j]) = \operatorname{Hom}(\mathcal{V}_X^d, Y_d[j]) = 0$$

for $1 \leq j \leq d - 1$. Consequently, the object Y_d is contained in $\mathcal{U}^{\perp_d} = \mathcal{U}$. This contradicts Lemma 4.3 as γ_d is not a minimal arc. $\qquad\qquad\square$

Our next aim is to show that the same conclusion as in Proposition 4.8 holds when working with the entire bounded derived category. We hence need to consider objects corresponding to graded arcs wrapping around punctures on one or both ends. To determine whether such objects can be contained in a d-cluster tilting subcategory, the following lemma is useful.

Lemma 4.9. *Let Λ be a gentle algebra and consider a d-cluster tilting subcategory $\mathcal{U} \subseteq \mathcal{D}^b(\Lambda)$ that is closed under $[d]$ for some $d \geq 2$. If X and Y are two indecomposable objects in $\mathcal{U}$, then the corresponding graded arcs γ_X and γ_Y do not intersect in the interior of the surface in the geometric model.*

Proof. Let X and Y be in $\mathcal{U}$ and assume that γ_X and γ_Y do intersect in the interior of the surface in the geometric model. This intersection lies in a polygon P of the dissection given

by the dual graph L as illustrated in Figure 13. Denote intersection points of γ_X and γ_Y with the edges of P as indicated in the figure. Using that $\mathrm{Hom}(Y, X[i]) = 0$ for $1 \leq i \leq d-1$ and applying Lemma 4.5 part *(1)*, we can assume $f_X(p_X) = f_Y(p_Y)$. As $f_Y(p'_Y) = f_Y(p_Y) + 1$, this implies that $\mathrm{Hom}(X, Y[1]) \neq 0$, which yields a contradiction. $\qquad\square$
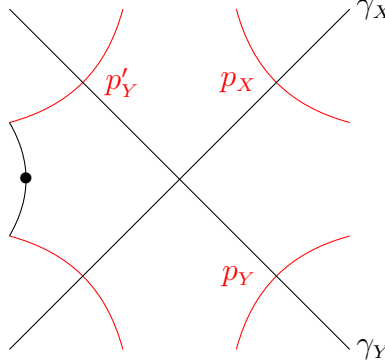

FIGURE 13. The oriented graded intersection discussed in the proof of Lemma 4.9.

We are now ready to prove Theorem 4.10.

Theorem 4.10. *Let Λ be a gentle algebra. If $\mathcal{D}^b(\Lambda)$ contains a d-cluster tilting subcategory that is closed under $[d]$ for some $d \geq 2$, then Λ is derived equivalent to an algebra of Dynkin type A.*

Proof. If Λ has finite global dimension, then the bounded derived category coincides with the perfect derived category and the result follows from Proposition 4.8. We hence assume that Λ has infinite global dimension, which is equivalent to the existence of at least one puncture in the surface of the associated geometric model. The strategy from here is to show that if there exists a d-cluster tilting subcategory $\mathcal{U} \subseteq \mathcal{D}^b(\Lambda)$, then $\mathcal{U}$ contains a certain non-perfect object X. This enables us to construct a perfect object Z that must be contained in $\mathcal{U}$ but does not correspond to a minimal graded arc. Similarly as in the proof of Proposition 4.8, this leads to a contradiction.

So suppose there exists a d-cluster tilting subcategory $\mathcal{U} \subseteq \mathcal{D}^b(\Lambda)$. We claim that $\mathcal{U}$ must contain an object X corresponding to a graded arc that starts in a marked point and ends wrapping around a puncture. To see this, assume to the contrary that each indecomposable object in $\mathcal{U}$ is either perfect or corresponds to a graded arc wrapping around punctures on both ends. Let $\mathcal{B}$ denote a boundary component with at least one marked point, and consider the arc γ that starts and ends in this marked point and follows the boundary $\mathcal{B}$ up to homotopy. Note that by our assumptions, this arc is not contractible to a point. Applying the iterative construction from the proof of Proposition 4.8 to γ, we obtain a finite arc γ_d. This arc can be equipped with a grading that is d-compatible with the grading of any minimal graded arc that corresponds to an object in $\mathcal{U}$ and starts or ends in the endpoints of γ_d. By Lemma 4.3 combined with our assumption on the non-perfect objects in $\mathcal{U}$, this yields that the object Y_d corresponding to the graded arc γ_d is contained in $\mathcal{U}$. Similarly as in the proof of Proposition 4.8, this is a contradiction as γ_d is not minimal.

Thus, we can assume that $\mathcal{U}$ contains an indecomposable object X such that the corresponding graded arc (γ_X, f_X) starts in a marked point m and ends wrapping around a puncture r. Before defining the object Z, we label some useful points in the model and make some observations. Denote by l an edge in the dual graph L that is adjacent to r. Let p be an intersection of γ_X and l such that after this intersection, the arc γ_X wraps infinitely many times around the

single puncture r. The next intersection of γ_X and l is denoted by p', as indicated in Figure 14. Note that by our assumption on p, any arc that intersects l between p and p' and does not intersect γ_X, has an end that wraps infinitely many times around r. Let w be the integer defined by the equation $f_X(p') = f_X(p) + w$, and observe that the grading at the intersections of γ_X and l increases by w each time γ_X loops around r. By the description of morphisms arising from punctures, we deduce that d divides w as X is in $\mathcal{U}$.

We now construct the arc γ_Z by concatenating the following four segments:

(i) The first segment of γ_Z starts in m and follows the trajectory of γ_X until the point p';

(ii) The second segment of γ_Z follows l from p' to p;

(iii) The third segment of γ_Z follows the trajectory of γ_X from p and back to m;

(iv) The last segment of γ_Z follows the minimal arc γ_m starting in m and ending in a marked point m' in such a way that γ_Z has exactly one self-intersection.

We equip γ_Z (resp. γ_m) with the grading f_Z (resp. f_m) that is compatible with f_X in m and denote the indecomposable object in $\mathcal{D}^b(\Lambda)$ corresponding to (γ_Z, f_Z) by Z.
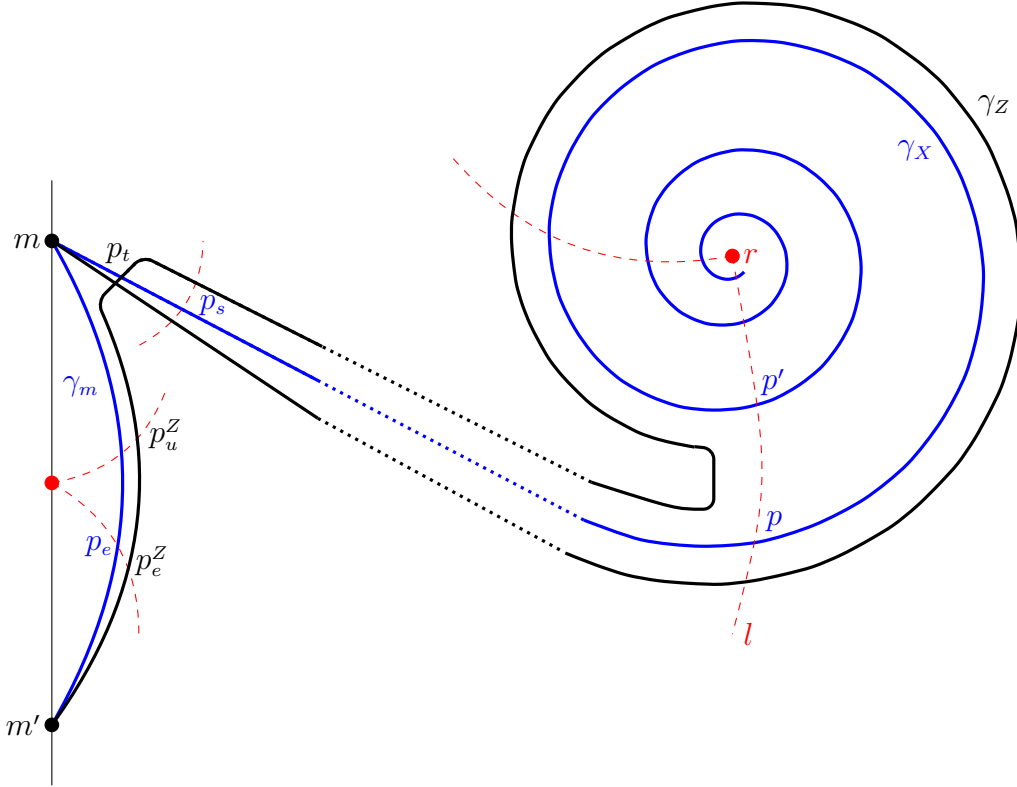

FIGURE 14. The arc γ_Z as used in the proof of Theorem 4.10

The last step in our proof is to show that $Z \in {}^{\perp_d}\mathcal{U} = \mathcal{U}$. Let p_e (resp. p_e^Z) denote the intersection point of γ_m (resp. γ_Z) that is closest to the marked point m'. By similar arguments as in the proof of Proposition 4.8, we see that $f_Z(p_e^Z) = f_m(p_e) + w$. As d divides w, this implies that the gradings of γ_Z and γ_m are d-compatible in m'. Lemma 4.5 part (2) hence allows us to conclude that intersections in the endpoints of γ_Z give rise to no non-zero morphisms from Z to $\mathcal{U}[i]$ for $1 \le i \le d-1$.

We claim that the same holds for any intersection between γ_Z and a graded arc corresponding to an object in $\mathcal{U}$ in the interior of the surface. Notice first that this is immediate for the segments of γ_Z described in (i) and (iii), as an intersection here would also yield an intersection with γ_X and hence contradict Lemma 4.9. If a graded arc corresponding to an object in $\mathcal{U}$ intersects

γ_Z in the interior along the segment described in (iv), this yields a graded intersection of the same type as the one denoted by p_t in Figure 14.

To study this graded intersection, let p_s (resp. p_u^Z) denote the intersection of γ_X (resp. the fourth segment of γ_Z) and L that is closest to m, as indicated in Figure 14. By similar arguments as before, we have $f_Z(p_u^Z) = f_X(p_s) + w$. Again using that d divides w, this implies that the intersection in p_t does not give rise to non-zero morphisms from Z to $\mathcal{U}[i]$ for $1 \le i \le d-1$.

It remains to consider intersections of graded arcs corresponding to objects in $\mathcal{U}$ along the segment of γ_Z described in (ii). As such a graded arc does not intersect γ_X, it wraps around r on one end by the assumption on p. Close to r, the grading of such an arc hence agrees with that of γ_X up to shifts by d. Using this, we look at the polygon of the dissection given by L that the intersection lies in. Considering different possibilities for the marked point of this polygon, we see that an intersection of this type also does not give rise to non-zero morphisms from Z to $\mathcal{U}[i]$ for $1 \le i \le d-1$. This allows us to conclude that $Z \in {}^{\perp_d}\mathcal{U} = \mathcal{U}$, which contradicts Lemma 4.3 as γ_Z is finite but not minimal. $\qquad\square$

The fact that a gentle algebra is derived equivalent to an algebra of Dynkin type A if and only if the surface in the associated geometric model is a disk [47, Corollary 1.23], played an important role in the proofs of Proposition 4.8 and Theorem 4.10. We now move on to characterizing d-cluster tilting subcategories of the derived category in this case. Our classification gives a geometric interpretation of the d-cluster tilting subcategories of the derived category arising from d-representation finite d-hereditary gentle algebras, as described in Section 2.1. By our classification in Corollary 3.4, these algebras are of the form KA_n/J^2 for $n = d+1$.

Theorem 4.11. *Assume $n \ge 3$ and let Λ be a gentle algebra that is derived equivalent to an algebra of Dynkin type A_n. A subcategory $\mathcal{U} \subseteq \mathcal{D}^b(\Lambda)$ is d-cluster tilting and closed under $[d]$ for some $d \ge 2$ if and only if $d = n-1$ and $\mathcal{U} = \mathcal{V}_X^d$ for some object X corresponding to a minimal graded arc.*

Proof. Without loss of generality, we can assume $\Lambda \cong KA_n/J^2$. As demonstrated in Example 2.4, the geometric model associated to Λ is a disk with $n+1$ marked points on the boundary. The dual graph L is shown in Example 2.5.
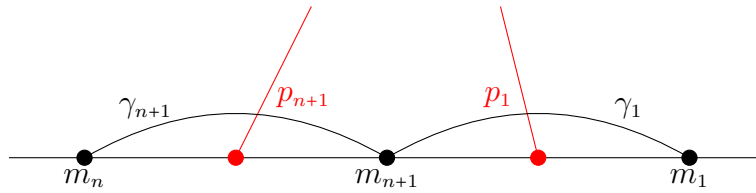

FIGURE 15. The notation used in the proof of Theorem 4.11

Given an object X in $\mathcal{D}^b(\Lambda)$ corresponding to a minimal graded arc γ_X, we follow the notation introduced in the definition of $\mathcal{V}_X^d$ with $t = n+1$, see Figure 11. We use the notation f_i for the grading of γ_i. Consider the intersection p_1 (resp. p_{n+1}) of $\gamma_1 = \gamma_X$ (resp. γ_{n+1}) with the dual graph L that is closest to the endpoint m_{n+1}, as indicated in Figure 15. By the description of L and the compatibility of grading from the construction of $\mathcal{V}_X^d$, we see that

$$(*) \qquad\qquad f_{n+1}(p_{n+1}) = f_1(p_1) + n - 1.$$

Assume that $\mathcal{U} \subseteq \mathcal{D}^b(\Lambda)$ is d-cluster tilting and closed under $[d]$ for some $d \ge 2$. Let X be an indecomposable object in $\mathcal{U}$, and note that X is perfect as Λ has finite global dimension.

By Lemma 4.3, the corresponding graded arc γ_X is hence minimal. As the surface in the geometric model associated to Λ only has one boundary component, Lemma 4.6 part *(2)* implies that $\mathcal{U} = \mathcal{V}_X^d$.

Since X and X_{n+1} are contained in $\mathcal{V}_X^d = \mathcal{U}$, Lemma 4.6 part *(1)* combined with $(*)$ yields that d divides $n-1$. If $d < n-1$, there exists a non-minimal arc that crosses precisely $d+1$ edges of L. For this arc, we can choose a grading such that the corresponding object is contained in $\mathcal{U}^{\perp_d} = \mathcal{U}$. This contradicts Lemma 4.3, and we can thus conclude that $d = n - 1$.

For the reverse direction, let $d = n - 1$. As $n \geq 3$, this yields $d \geq 2$. Consider $\mathcal{U} = \mathcal{V}_X^d$ for some indecomposable object X in $\mathcal{D}^b(\Lambda)$ corresponding to a minimal graded arc. Note that $\mathcal{V}_X^d$ is closed under $[d]$ by definition. It remains to show that $\mathcal{V}_X^d$ is a d-cluster tilting subcategory of $\mathcal{D}^b(\Lambda)$.

By the compatibility of grading in the definition of $\mathcal{V}_X^d$ combined with $(*)$ and the assumption $d = n-1$, graded arcs corresponding to indecomposable objects in $\mathcal{U} = \mathcal{V}_X^d$ have d-compatible grading in common endpoints. This implies that

$$\mathrm{Hom}(\mathcal{V}_X^d, \mathcal{V}_X^d[j]) = 0$$

whenever $d \nmid j$. In particular, we have $\mathcal{V}_X^d \subseteq \mathcal{U}^{\perp_d}$ and $\mathcal{V}_X^d \subseteq {}^{\perp_d}\mathcal{U}$.

Consider an indecomposable object Y that is not contained in $\mathcal{V}_X^d$. If the corresponding graded arc γ_Y is minimal, this means that the grading of γ_Y in its endpoints is not d-compatible with the graded arcs corresponding to indecomposable objects in $\mathcal{V}_X^d$. This gives $Y \notin \mathcal{U}^{\perp_d}$ and $Y \notin {}^{\perp_d}\mathcal{U}$. If γ_Y is not minimal, we see that γ_Y crosses precisely l edges of L for some $2 \leq l \leq d$. This implies that for any possible grading of γ_Y, one has $Y \notin \mathcal{U}^{\perp_d}$ and $Y \notin {}^{\perp_d}\mathcal{U}$, which yields $\mathcal{U}^{\perp_d} \subseteq \mathcal{V}_X^d$ and ${}^{\perp_d}\mathcal{U} \subseteq \mathcal{V}_X^d$. We can hence conclude that $\mathcal{V}_X^d = \mathcal{U}^{\perp_d} = {}^{\perp_d}\mathcal{U}$.

It remains to observe that $\mathcal{V}_X^d$ is functorially finite in $\mathcal{D}^b(\Lambda)$. For this, consider an indecomposable object Y in the derived category. If the grading of γ_Y in its endpoints is not compatible with the grading of any minimal arc corresponding to an object in $\mathcal{V}_X^d$, left and right approximations of Y are given by the zero morphism. If the grading is compatible at one of the endpoints, we get left and right approximations by the morphisms corresponding to the graded intersection. Our subcategory $\mathcal{U} = \mathcal{V}_X^d$ is hence d-cluster tilting, which finishes the proof. □

Note that it is possible to show a version of Theorem 4.11 by working directly with the AR-quiver of $\mathcal{D}^b(\Lambda)$ instead of using the geometric model.

Remark 4.12. As $\Lambda = KA_n/J^2$ is d-representation finite d-hereditary for $n = d+1$, the module category contains a unique d-cluster tilting subcategory $\mathcal{U} \subseteq \mathrm{mod}\,\Lambda$. Notice that the d-cluster tilting subcategories described in the theorem above are equivalent to the subcategory $\mathcal{U}[d\mathbb{Z}]$ of $\mathcal{D}^b(\Lambda)$, see Example 2.2 and Example 4.4. Combining Theorem 4.10 and Theorem 4.11, we see that all d-cluster tilting subcategories of the derived category of a gentle algebra that are closed under $[d]$ arise in this way.

Combining our results in this section with [47, Corollary 1.23], we obtain the following corollary.

Corollary 4.13. *Let $\Lambda = KQ/I$ be a gentle algebra which is not a field. The following statements are equivalent:*

   (1) There exists a d-cluster tilting subcategory $\mathcal{U} \subseteq \mathcal{D}^b(\Lambda)$ that is closed under $[d]$ for some $d \geq 2$.
   (2) The algebra Λ is derived equivalent to an algebra of Dynkin type A_n with $n \geq 3$.
   (3) The quiver Q is a tree with $|Q_0| \geq 3$.

(4) *The surface in the geometric model associated to Λ is a disk with at least four marked points on the boundary.*

Remark 4.14. In the case where $\Lambda = K$ is a field, there exists a d-cluster tilting subcategory $\mathcal{U}_d \subseteq \mathcal{D}^b(\Lambda)$ that is closed under $[d]$ for any $d \geq 1$. This subcategory is given by

$$\mathcal{U}_d = \mathrm{add}\{K[di] \mid i \in \mathbb{Z}\} \subseteq \mathcal{D}^b(\Lambda),$$

where the notation K is used for the stalk complex with K in degree 0. Any d-cluster tilting subcategory of $\mathcal{D}^b(\Lambda)$ is equivalent to $\mathcal{U}_d$.

References

[1] C. Amiot, O. Iyama, and I. Reiten, *Stable categories of Cohen-Macaulay modules and cluster categories*, Amer. J. Math. **137** (2015), no. 3, 813–857.

[2] K. K. Arnesen, R. Laking, and D. Pauksztello, *Morphisms between indecomposable objects in the bounded derived category of a gentle algebra*, J. Algebra **467** (2016), 1–46.

[3] I. Assem and D. Happel, *Generalized tilted algebras of type A_n*, Comm. Algebra **9** (1981), no. 20, 2101–2125, 1981.

[4] I. Assem, D. Simson, and A. Skowroński, *Elements of the representation theory of associative algebras. Vol. 1*, London Mathematical Society Student Texts, vol. 65, Cambridge University Press, Cambridge, 2006. Techniques of representation theory.

[5] I. Assem and A. Skowroński, *Iteraterd tilted algebras of type $\tilde{A}_n$*, Math. Z. **195** (1987), no. 2, 269-290.

[6] K. Baur and R. Coelho Simões, *A Geometric Model for the Module Category of a Gentle Algebra*, International Mathematics Research Notices (2019).

[7] V. Bekkert and H. A. Merklen, *Indecomposables in Derived Categories of Gentle Algebras*, Algebras and Representation Theory **6** (2003), 285–302.

[8] T. Brüstle, G. Douville, K. Mousavand, H. Thomas, and E. Yıldırım, *On the combinatorics of gentle algebras*, Canad. J. Math. **72** (2020), no. 6, 1551–1580.

[9] M. C. R. Butler and C. M. Ringel, *Auslander–Reiten sequences with few middle terms and applications to string algebras*, Comm. Algebra. **15** (1987), no. 1-2, 145-179.

[10] İ Çanakçı, D. Pauksztello, and S. Schroll, *On extensions for gentle algebras*, Canad. J. Math. **73** (2021), no. 1, 249–292.

[11] E. Darpö and O. Iyama, *d-representation-finite self-injective algebras*, Adv. Math. **362** (2020), 106932, 50.

[12] E. Darpö and T. Kringeland, *d-Representation-finite symmetric Nakayama algebras and trivial extensions of quiver algebras*, arXiv:2103.15380 (2021).

[13] T. Dyckerhoff, G. Jasso, and Y. Lekili, *The symplectic geometry of higher Auslander algebras: symmetric products of disks*, Forum Math. Sigma **9** (2021), Paper No. e10, 49.

[14] R. Ebrahimi and A. Nasr-Isfahani, *Higher Auslander's formula*, Int. Math. Res. Not. IMRN **0** (2021), 1–18.

[15] D. E. Evans and M. Pugh, *The Nakayama automorphism of the almost Calabi-Yau algebras associated to $SU(3)$ modular invariants*, Comm. Math. Phys. **312** (2012), no. 1, 179–222.

[16] C. Geiss, B. Keller, and S. Oppermann, *n-angulated categories*, J. Reine Angew. Math. **675** (2013), 101–120.

[17] C. Geiss and I. Reiten, *Gentle algebras are Gorenstein*, Representations of algebras and related topics, Fields Inst. Commun., vol. 45, Amer. Math. Soc., Providence, RI, 2005, pp. 129–133.

[18] F. Haiden, L. Katzarkov, and M. Kontsevich, *Flat surfaces and stability structures*, Publ.math.IHES **126** (2017), 247–318.

[19] D. Happel, *Auslander-Reiten triangles in derived categories of finite-dimensional algebras*, Proc. Amer. Math. Soc. **112** (1991), no. 3, 641–648.

[20] J. Haugland, *The Grothendieck group of an n-exangulated category*, Appl. Categ. Structures **29** (2021), no. 3, 431–446.

[21] J. Haugland and M. H. Sandøy, *Higher Koszul duality and connections with n-hereditary algebras*, arXiv:2101.12743 (2021).

[22] M. Herschend and O. Iyama, *n-representation-finite algebras and twisted fractionally Calabi–Yau algebras*, Bull. Lond. Math. Soc. **43** (2011), no. 3, 449–466.

[23] ———, *Selfinjective quivers with potential and 2-representation-finite algebras*, Compos. Math. **147** (2011), no. 6, 1885–1920.

[24] M. Herschend, O. Iyama, H. Minamoto, and S. Oppermann, *Representation theory of Geigle-Lenzing complete intersections*, to appear in Mem. Amer. Math. Soc.

[25] M. Herschend, O. Iyama, and S. Oppermann, *n-representation infinite algebras*, Adv. Math. **252** (2014), 292–342.

[26] M. Herschend, P. Jørgensen, and L. Vaso, *Wide subcategories of d-cluster tilting subcategories*, Trans. Amer. Math. Soc. **373** (2020), no. 4, 2281–2309.

[27] M. Herschend, Y. Liu, and H. Nakaoka, *n-exangulated categories (I): Definitions and fundamental properties*, J. Algebra **570** (2021), 531–586.

[28] O. Iyama, *Auslander correspondence*, Adv. Math. **210** (2007), no. 1, 51–82.

[29] ———, *Higher-dimensional Auslander–Reiten theory on maximal orthogonal subcategories*, Adv. Math. **210** (2007), no. 1, 22–50.

[30] ———, *Auslander–Reiten theory revisited*, Trends in representation theory of algebras and related topics, EMS Ser. Congr. Rep., Eur. Math. Soc., Zürich, 2008, pp. 349–397.

[31] ———, *Cluster tilting for higher Auslander algebras*, Adv. Math. **226** (2011), no. 1, 1–61.

[32] O. Iyama and S. Oppermann, *n-representation-finite algebras and n-APR tilting*, Trans. Amer. Math. Soc. **363** (2011), no. 12, 6575–6614.

[33] ———, *Stable categories of higher preprojective algebras*, Adv. Math. **244** (2013), 23–68.

[34] O. Iyama and M. Wemyss, *Maximal modifications and Auslander-Reiten duality for non-isolated singularities*, Invent. Math. **197** (2014), no. 3, 521–586.

[35] O. Iyama and Y. Yoshino, *Mutation in triangulated categories and rigid Cohen-Macaulay modules*, Invent. Math. **172** (2008), no. 1, 117–168.

[36] K. M. Jacobsen and P. Jørgensen, *d-abelian quotients of $(d + 2)$-angulated categories*, J. Algebra **521** (2019), 114–136.

[37] ———, *Maximal τ_d-rigid pairs*, J. Algebra **546** (2020), 119–134.

[38] G. Jasso, *n-abelian and n-exact categories*, Math. Z. **283** (2016), no. 3-4, 703–759.

[39] G. Jasso and J. Külshammer, *Higher Nakayama algebras I: Construction*, Adv. Math. **351** (2019), 1139–1200. With an appendix by J. Külshammer and C. Psaroudakis and an appendix by S. Kvamme.

[40] G. Jasso and S. Kvamme, *An introduction to higher Auslander–Reiten theory*, Bull. Lond. Math. Soc. **51** (2019), no. 1, 1–24.

[41] P. Jørgensen, *Torsion classes and t-structures in higher homological algebra*, Int. Math. Res. Not. IMRN **13** (2016), 3880–3905.

[42] ———, *Tropical friezes and the index in higher homological algebra*, Math. Proc. Cambridge Philos. Soc. **171** (2021), no. 1, 23–49.

[43] B. Keller and I. Reiten, *representation tilted algebras are Gorenstein and stably Calabi–Yau*, Adv. Math. **211** (2007), no. 1, 123–151.

[44] S. Kvamme, *Axiomatizing Subcategories of Abelian Categories*, to appear in J. Pure Appl. Algebra.

[45] Y. Lekili and A. Polishchuk, *Derived equivalences of gentle algebras via Fukaya categories*, Math. Ann. 376 (2020), 187–225.

[46] Y. Mizuno, *A Gabriel-type theorem for cluster tilting*, Proc. Lond. Math. Soc. (3) **108** (2014), no. 4, 836–868.

[47] S. Opper, P.-G. Plamondon, and S. Schroll, *A geometric model for the derived category of gentle algebras*, arXiv:1801.09659 (2018).

[48] S. Oppermann, C. Psaroudakis, and T. Stai, *Partial Serre duality and cocompact objects*, arXiv:2104.12498 (2021).

[49] S. Oppermann and H. Thomas, *Higher-dimensional cluster combinatorics and representation theory*, J. Eur. Math. Soc. (JEMS) **14** (2012), no. 6, 1679–1737.

[50] Y. Palu, V. Pilaud, and P.-G. Plamondon, *Non-kissing and non-crossing complexes for locally gentle algebras*, J. Comb. Alg. **3** (2019), no. 4, 401-438.

[51] J. Reid, *Indecomposable objects determined by their index in higher homological algebra*, Proc. Amer. Math. Soc. **148** (2020), no. 6, 2331–2343.

[52] I. Reiten and M. Van den Bergh, *Noetherian hereditary abelian categories satisfying Serre duality*, J. Amer. Math. Soc. **15** (2002), no. 2, 295–366.

[53] M. H. Sandøy and L.-P. Thibault, *Classification results for n-hereditary monomial algebras*, arXiv:2101.12746 (2021).

[54] S. Schroll, *Trivial extensions of gentle algebras and Brauer graph algebras*, J. Algebra **444** (2015), 183–200.

[55] ———, *Brauer graph algebras: a survey on Brauer graph algebras, associated gentle algebras and their connections to cluster theory*, Homological methods, representation theory, and cluster algebras, 2018, pp. 177–223.

[56] J. Schröer, *Modules without self-extensions over gentle algebras*, J. Algebra **216** (1999), no. 1, 178–189.

[57] L. Vaso, *n-cluster tilting subcategories of representation-directed algebras*, J. Pure Appl. Algebra **223** (2019), no. 5, 2101–2122.

[58] ———, *n-cluster tilting subcategories for radical square zero algebras*, arXiv:2105.05830 (2021).

[59] N. J. Williams, *New interpretations of the higher Stasheff–Tamari orders*, arXiv:2007.12664 (2020).

DEPARTMENT OF MATHEMATICAL SCIENCES, NTNU, NO-7491 TRONDHEIM, NORWAY
Email address: `johanne.haugland@ntnu.no`

DEPARTMENT OF MATHEMATICS, AARHUS UNIVERSITET, NY MUNKEGADE 118, DK-8000 AARHUS C, DENMARK
Email address: `karin.jacobsen@ntnu.no`

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF COLOGNE, 50931 COLOGNE, GERMANY, AND DEPARTMENT OF MATHEMATICAL SCIENCES, NTNU, NO-7491 TRONDHEIM, NORWAY
Email address: `schroll@math.uni-koeln.de`