RSK TABLEAUX AND BOX-BALL SYSTEMS

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Abstract. A box-ball system is a discrete dynamical system whose dynamics come from the balls jumping according to certain rules. A permutation on n objects gives a box-ball system state by assigning its one-line notation to n consecutive boxes. After a finite number of steps, a box-ball system will reach a steady state. From any steady state, we can construct a tableau called the soliton decomposition of the box-ball system. We prove that if the soliton decomposition of a permutation w is a standard tableau or if its shape coincides with the Robinson–Schensted (RS) partition of w, then the soliton decomposition of w and the RS insertion tableau of w are equal. We also use row reading words, Knuth moves, RS recording tableaux, and a localized version of Greene's theorem (proven recently by Lewis, Lyu, Pylyavskyy, and Sen) to study various properties of a box-ball system.

Contents

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1. Introduction

A *box-ball system (BBS)* is a collection of discrete time states. At each state, we have an injective map from n balls (labeled by the integers from 1 to n) to boxes (labeled by the natural numbers); each box can fit at most one ball. The dynamics come from the balls jumping according to certain rules. Let S_n denote the set of permutations on $[n] = \{1, 2, \ldots, n\}$. A permutation w in S_n gives a box-ball system state by assigning the one-line notation of the permutation to n consecutive boxes. Given a BBS state at time t, we compute the BBS state at time $t+1$ by applying one BBS move, which is the process of moving each integer to the nearest empty box to its right, beginning with the smallest. See Figure 1. This version of the box-ball system was introduced in [Tak93] and is an extension of the box-ball system first invented by Takahashi and Satsuma in [TS90].

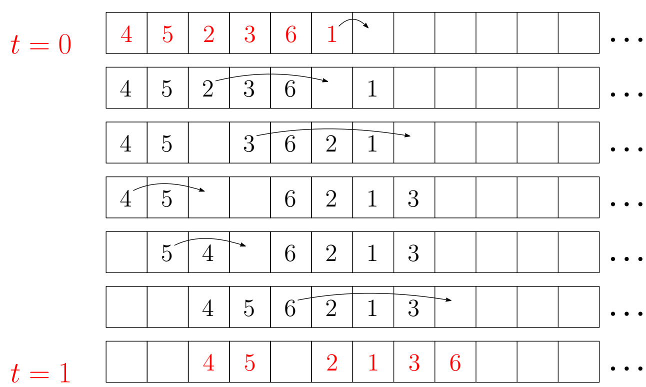

FIGURE 1. Performing a BBS move on $w = 452361$

A *soliton* is a maximal consecutive increasing sequence of balls which is preserved by all subsequent BBS moves. After a finite number of BBS moves, a box-ball system containing a configuration w will reach a *steady state*, decomposing into solitons whose sizes are weakly decreasing from right to left, that is, forming an integer partition shape. From such a state, we can construct the *soliton decomposition of the box-ball system*, denoted SD, by stacking solitons so that the rightmost soliton is placed on the first row, the soliton to its left is placed on the second row, and so on. We obtain a tableau where each row is increasing but which may or may not be standard. The *soliton decomposition of a permutation* w is the soliton decomposition of the box-ball system containing w.

Figure 2 shows the state of the box-ball system containing $w = 452361$ from $t = 0$ to $t = 4$. Note that steady state is first reached at $t = 3$. The soliton decomposition of $w = 452361$ is the tableau

$$\mathrm{SD}(w) = \begin{array}{|c|c|c|}\hline 1 & 3 & 6 \\ \hline 2 & 5 \\ \cline{1-2} 4 \\ \cline{1-1}\end{array}\,.$$

In this example, the soliton decomposition is a standard tableau, but most permutations have soliton decompositions which are not standard. The tableau $\mathrm{SD}(w)$ has shape $(3, 2, 1)$. We will refer to the shape of the soliton decomposition as the *BBS soliton partition*.

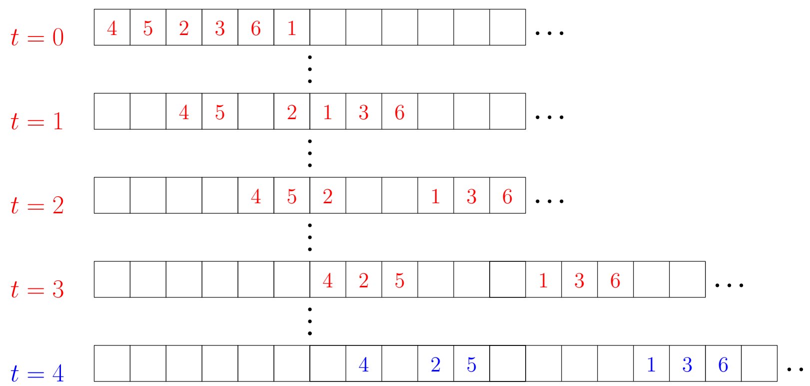

FIGURE 2. BBS moves starting at $w = 452361$

The well-known Robinson–Schensted (RS) insertion algorithm is a bijection

$$w \mapsto (\mathrm{P}(w), \mathrm{Q}(w))$$

from S_n onto pairs of standard size-n tableaux of the same shape [Sch61]. The tableau $\mathrm{P}(w)$ is called the *insertion tableau* of w, and the tableau $\mathrm{Q}(w)$ is called the *recording tableau* of w. The shape of these tableaux is called the *RS partition of* w.

The *row reading word* of a tableau is the permutation formed by concatenating the rows of the tableau from bottom to top, left to right.

$$\text{If } r \text{ is the row reading word of a standard tableau } T, \text{ then } \mathrm{P}(r) = T. \tag{1.1}$$

For example, if $w = 452361$, then

$$\mathrm{P}(w) = \begin{array}{|c|c|c|} \hline 1 & 3 & 6 \\ \hline 2 & 5 \\ \cline{1-2} 4 \\ \cline{1-1} \end{array}\,, \quad \mathrm{Q}(w) = \begin{array}{|c|c|c|} \hline 1 & 2 & 5 \\ \hline 3 & 4 \\ \cline{1-2} 6 \\ \cline{1-1} \end{array}\,.$$

The tableau $\mathrm{P}(w)$ has row reading word $r = 425136$. The insertion tableau of r is the tableau $\mathrm{P}(w)$. For more information, see for example the textbook [Sag20, Section 7.5].

Our goal is to study the connection between the soliton decompositions and RS tableaux of permutations. We now describe our main results.

1.1. **Insertion tableaux and soliton decompositions.** For the permutation $w = 452361$ used in the above example, we have $\mathrm{SD}(w) = \mathrm{P}(w)$. However, in general the soliton decomposition and the RS insertion tableau of a permutation do *not* coincide. Surprisingly, having a standard soliton decomposition tableau or having a BBS soliton partition which equals the RS partition is enough to guarantee that the soliton decomposition and the RS insertion tableau coincide.

Theorem A (Theorem 4.2)**.** Suppose w is a permutation. Then the following are equivalent:

(1) $\mathrm{SD}(w) = \mathrm{P}(w)$.
(2) $\mathrm{SD}(w)$ is a standard tableau.

(3) The shape of $\mathrm{SD}(w)$ equals the shape of $\mathrm{P}(w)$.

Our proof uses Greene's theorem (Theorem 2.2) and a result of Fukuda which says that the RS insertion tableau is an invariant of a box-ball system (Theorem 3.3). The proof that part (3) implies part (2) was suggested to us by Darij Grinberg.

1.2. **Tableau reading words.** We study the connection between steady-state configurations and row reading words. It turns out that a permutation is in steady state if and only if it is the row reading word of a standard tableau.

Proposition B (Proposition 5.1)**.** A permutation r reaches its soliton decomposition at time $t = 0$ if and only if r is the row reading word of a standard tableau.

Next, we represent a box-ball system state as an array containing integers from 1 to n called the *configuration array*. This array has increasing rows but not necessarily increasing columns; it also may not have a valid skew shape and it may be disconnected. Proposition B turns out to be a special case of the following.

Proposition C (Proposition 5.2)**.** A BBS configuration w is in steady state if and only if the configuration array of w is a standard skew tableau whose rows are weakly decreasing in length.

As we will explain in Section 5, Proposition C is a corollary of a characterization for steady state given by Lewis, Lyu, Pylyavskyy, and Sen in [LLPS19, Proof of Lemma 2.1 and 2.3].

1.3. **Recording tableaux and time to steady state.** We also study the relationship between the RS recording tableau of a permutation and the behavior of its box-ball system. The number of BBS moves required for a permutation w to reach a steady state is called the *steady-state time* of w. For example, as illustrated in Figure 2, the steady-state time of the permutation 452361 is 3.

Theorem D (Theorem 6.7)**.** If $n \geq 5$, let

$$\widehat{Q} := \begin{array}{ccccc} \boxed{1} & \boxed{2} & \cdots & \boxed{n\text{-}2} & \boxed{n\text{-}1} \\ \boxed{3} & \boxed{4} & & & \\ \boxed{n} & & & & \end{array}.$$

If $\mathrm{Q}(w) = \widehat{Q}$, then w first reaches steady state at time $n - 3$.

This particular recording tableau is special; we conjecture that all other permutations in S_n have steady-state time smaller than $n - 3$.

Conjecture 1.1. A permutation in S_n whose recording tableau is not equal to $\widehat{Q}$ has steady-state time smaller than $n - 3$.

Furthermore, we conjecture that Theorem D is a special case of the following general phenomenon.

Conjecture 1.2. If two permutations π and w are such that $\mathrm{Q}(\pi) = \mathrm{Q}(w)$, then π and w have the same steady-state time.

Conjecture 1.2 is proven in a forthcoming paper [CFG+].

1.4. **Types of Knuth moves.** The RS insertion tableau is preserved under any Knuth move [Knu70]. In contrast, the soliton decomposition is only preserved under certain types of Knuth moves.

Definition 1.3 (Knuth Moves)**.** Suppose π, $w \in S_n$ and $x < y < z$.

(1) We say that π and w differ by a Knuth relation of the *first kind* (K_1) if

$$\pi = \pi_1 \dots yxz \dots \pi_n \text{ and } w = \pi_1 \dots yzx \dots \pi_n \text{ or vice versa}$$

(2) We say that π and w differ by a Knuth relation of the *second kind* (K_2) if

$$\pi = \pi_1 \dots xzy \dots \pi_n \text{ and } w = \pi_1 \dots zxy \dots \pi_n \text{ or vice versa}$$

In addition, We say that π and w differ by a Knuth relation of *both kinds* (K_B) if they differ by a Knuth relation of the first kind (K_1) and of the second kind (K_2), that is,

$$\pi = \pi_1 \dots y_1xzy_2 \dots \pi_n \text{ and } w = \pi_1 \dots y_1zxy_2 \dots \pi_n \text{ or vice versa}$$

where $x < y_1 < z$ and $x < y_2 < z$.

Note that, when we apply a K_1 move (respectively, a K_2 move), the move may or may not be a K_B move. If we apply a K_B move, then it is both a K_1 move and a K_2 move.

When performing a Knuth move, if we replace an "xz" pattern with a "zx" pattern, we denote this with a superscript "+." Otherwise, if we replace a "zx" pattern with an "xz" pattern, we denote this with a superscript "−." For example, if $x < y_1 < z$ and $x < y_2 < z$, the move $y_1xzy_2 \mapsto y_1zxy_2$ is denoted K_B^+.

We say that π and w are *Knuth equivalent* if they differ by a finite sequence of Knuth relations.

Using the localized version of Greene's Theorem given in Section 2.2, we prove a partial characterization of the BBS soliton partition in terms of types of Knuth moves.

Theorem E (Theorem 7.1)**.** *If π and w are related by a sequence of K_1 or K_2 moves (but not K_B), then $\operatorname{sh}\operatorname{SD}(\pi) = \operatorname{sh}\operatorname{SD}(w)$. If π and w are related by a sequence of Knuth moves containing an odd number of K_B moves, then $\operatorname{sh}\operatorname{SD}(\pi) \neq \operatorname{sh}\operatorname{SD}(w)$.*

We also use a non-K_B Knuth move to give a family of permutations which have steady-state time 1.

Theorem F (Theorem 7.4)**.** *Suppose r is the row reading word of a standard tableau. If w is a permutation one K_1 or K_2 (but not K_B) move away from r, then the steady-state time of w is 1.*

1.5. **An algorithm with multiple carriers.** The *single-carrier algorithm* (which we review in Section 3) is a way to transform a box-ball configuration at time t into the configuration at time $t + 1$. At each step in the algorithm, we insert and bump numbers in and out of a *carrier* filled with a weakly increasing sequence, following a rule which should remind the reader of the Robinson–Schensted–Knuth (RSK) insertion algorithm. In fact, the carrier algorithm can be viewed as a sequence of Knuth transformations (see Remark 3.2).

In Section 8, we define the *M-carrier algorithm* (Algorithm 2) which is equivalent to performing the carrier algorithm M times (Proposition 8.2). Given a large enough M, the M-carrier algorithm gives us an RSK-like insertion algorithm which maps a permutation to its soliton decomposition.

The paper is organized as follows. In the next two sections, we review materials in the literature that we will use to prove our results. First, we review Greene's theorem in Section 2.1 and Lewis, Lyu, Pylyavskyy, and Sen's localized Greene's theorem in Section 2.2. Next, we review Fukuda's carrier algorithm and its connection to the RS insertion tableaux in Section 3. In Section 4, we prove Theorem A. In Section 5, we define the configuration array and use the carrier algorithm to prove Proposition C. Section 6 is devoted to the proof of Theorem D. We prove the two results involving types of Knuth moves (Theorem E and Theorem F) in Section 7. Finally, we define the M-carrier algorithm in Section 8.

2. Greene's theorem and a localized version of Greene's theorem

In the 1970s, Greene showed that the RS partition of a permutation and its conjugate record the numbers of disjoint unions of increasing and decreasing sequences of the permutation, which we explain in Section 2.1. Lewis, Lyu, Pylyavskyy, and Sen recently showed that the BBS soliton partition of a permutation and its conjugate record a localized version of Greene's theorem statistics. They studied an alternate version of the box-ball system, so in Section 2.2 we reframe their result to match our box-ball convention.

2.1. Greene's theorem and RS partition.

In this section, we review Greene's theorem [Gre74, Theorem 3.1], which states that the RS partition of a permutation and its conjugate record the numbers of disjoint unions of increasing and decreasing sequences of the permutation. For more details, see for example Chapter 3 of the textbook [Sag01].

Definition 2.1 (longest k-increasing and k-decreasing subsequences)**.** A subsequence σ of w is called *k-increasing* if, as a set, it can be written as a disjoint union

$$\sigma = \sigma_1 \sqcup \sigma_2 \sqcup \cdots \sqcup \sigma_k$$

where each σ_i is an increasing subsequence of w. If each σ_i is a decreasing subsequence of w, we say that σ is *k-decreasing*. Let

$$\mathrm{i}_k(w) \text{ denote the length of a longest } k\text{-increasing subsequence of } w$$

and

$$\mathrm{d}_k(w) \text{ denote the length of a longest } k\text{-decreasing subsequence of } w.$$

Theorem 2.2 ([Gre74, Theorem 3.1])**.** Suppose $w \in S_n$. Let $\lambda = (\lambda_1, \lambda_2, \lambda_3, \dots)$ denote the RS partition of w, that is, let $\lambda = \operatorname{sh} \mathrm{P}(w)$. Let $\mu = (\mu_1, \mu_2, \mu_3, \dots)$ denote the conjugate of λ. Then, for any k,

$$\mathrm{i}_k(w) = \lambda_1 + \lambda_2 + \ldots + \lambda_k,$$
$$\mathrm{d}_k(w) = \mu_1 + \mu_2 + \ldots + \mu_k.$$

Example 2.3. Let $w = 5623714$. For short, we write $\mathrm{i}_k \coloneqq \mathrm{i}_k(w)$ and $\mathrm{d}_k \coloneqq \mathrm{d}_k(w)$. The longest 1-increasing subsequences are

$$567, \quad 237, \quad \text{and} \quad 234.$$

The longest 2-increasing subsequence is given by

$$562374 = 567 \sqcup 234.$$

A longest 3-increasing subsequence (among others) is given by

$$5623714 = 56 \sqcup 237 \sqcup 14.$$

Thus,

$$\mathrm{i}_1 = 3, \qquad \mathrm{i}_2 = 6, \qquad \text{and} \qquad \mathrm{i}_k = 7 \ \text{ if } \ k \geq 3.$$

Similarly, the longest 1-decreasing subsequences are

$$521, \quad 621, \quad 531, \quad \text{and} \quad 631.$$

A longest 2-decreasing subsequence (among others) is given by

$$52714 = 521 \sqcup 74.$$

A longest 3-decreasing subsequence (among others) is given by

$$5623714 = 52 \sqcup 631 \sqcup 74.$$

Thus,

$$\mathrm{d}_1 = 3, \qquad \mathrm{d}_2 = 5, \qquad \text{and} \qquad \mathrm{d}_k = 7 \ \text{ if } \ k \geq 3.$$

By Theorem 2.2, the RS partition is equal to $\lambda = (\mathrm{i}_1, \mathrm{i}_2 - \mathrm{i}_1, \mathrm{i}_3 - \mathrm{i}_2) = (3, 3, 1)$ and the conjugate of the RS partition is $\mu = (\mathrm{d}_1, \mathrm{d}_2 - \mathrm{d}_1, \mathrm{d}_3 - \mathrm{d}_2) = (3, 2, 2)$. We can verify this by computing the RS tableaux

$$\mathrm{P}(w) = \begin{array}{|c|c|c|} \hline 1 & 3 & 4 \\ \hline 2 & 6 & 7 \\ \hline 5 \\ \cline{1-1} \end{array}, \qquad \mathrm{Q}(w) = \begin{array}{|c|c|c|} \hline 1 & 2 & 5 \\ \hline 3 & 4 & 7 \\ \hline 6 \\ \cline{1-1} \end{array}.$$

2.2. **Localized Greene's theorem and BBS soliton partition.** In [LLPS19, Lemma 2.1] and the blog post [Lew], Lewis, Lyu, Pylyavskyy, and Sen presented a localized version of Greene's theorem. They studied an alternate version of the box-ball system, and in this section we reframe their result to match our box-ball convention.

Definition 2.4 (A localized version of longest k-increasing subsequences)**.** If u is a sequence, let $\mathrm{i}(u)$ denote the length of a longest increasing subsequence of u.

For $w \in S_n$ and $k \geq 1$, we define

$$\mathrm{I}_k(w) = \max_{w = u_1 | \cdots | u_k} \sum_{j=1}^{k} \mathrm{i}(u_j),$$

where the maximum is taken over ways of writing w as a concatenation $u_1 \mid \cdots \mid u_k$ of consecutive subsequences. That is, we consider all ways to break w into k consecutive subsequences, sum the $\mathrm{i}(u_j)$ values for each way, and let $\mathrm{I}_k(w)$ be the maximum sum.

Definition 2.5 (A localized version of longest k-decreasing subsequences)**.** If u is a sequence of ℓ elements, an integer $m \in [\ell - 1]$ is called a *descent* of u if $u_m > u_{m+1}$. Let $\mathrm{D}(u) \coloneqq 1 + |\{\text{descents of } u\}|$.

For $w \in S_n$ and $k \geq 1$, we define

$$\mathrm{D}_k(w) = \max_{w = u_1 \sqcup \cdots \sqcup u_k} \sum_{j=1}^{k} \mathrm{D}(u_j),$$

where the maximum is taken over ways to write w as the union of disjoint subsequences u_j of w. Notice that we only require $u_1, \ldots, u_k$ to be disjoint, *not* consecutive, in contrast to the procedure for calculating $\mathrm{I}_k(w)$.

The following lemma is a corollary of [LLPS19, Lemma 2.1].

Lemma 2.6 (A localized version of Greene's theorem)**.** Suppose $w \in S_n$. Let $\Lambda = (\Lambda_1, \Lambda_2, \Lambda_3, \dots)$ denote the BBS soliton partition of w, that is, let $\Lambda = \operatorname{sh}\operatorname{SD}(w)$. Let $M = (M_1, M_2, M_3, \dots)$ denote the conjugate of Λ. Then, for any k,

$$\mathrm{I}_k(w) = \Lambda_1 + \Lambda_2 + \ldots + \Lambda_k,$$
$$\mathrm{D}_k(w) = M_1 + M_2 + \ldots + M_k.$$

Example 2.7. Let $w = 5623714$, the permutation used in Example 2.3. For short, we write $\mathrm{I}_k \coloneqq \mathrm{I}_k(w)$ and $\mathrm{D}_k \coloneqq \mathrm{D}_k(w)$. Then

$\mathrm{I}_1 = \mathrm{i}(w) = 3$ (since the longest increasing subsequences are 567, 237, and 234),

$\mathrm{I}_2 = 5$ (witnessed by 56|23714 or 56237|14),

$\mathrm{I}_3 = 7$ (witnessed uniquely by 56|237|14), and

$\mathrm{I}_k = 7$ for all $k \geq 3$.

We have

$\mathrm{D}_1 = \mathrm{D}(w) = 1 + |\text{descents of } 5623714| = 1 + |\{2, 5\}| = 3$,

$\mathrm{D}_2 = 6$ (one can take subsequences 531 and 6274, among other partitions),

$\mathrm{D}_3 = 7$ (one can take subsequences 52, 631, and 74, among other partitions), and

$\mathrm{D}_k = 7$ for all $k \geq 3$.

By Lemma 2.6, $\operatorname{sh}\operatorname{SD}(w) = (\mathrm{I}_1, \mathrm{I}_2 - \mathrm{I}_1, \mathrm{I}_3 - \mathrm{I}_2) = (3, 2, 2)$ and its conjugate is $(\mathrm{D}_1,\ \mathrm{D}_2 - \mathrm{D}_1,\ \mathrm{D}_3 - \mathrm{D}_2) = (3, 3, 1)$. We can verify this by computing the soliton decomposition $\operatorname{SD}(w)$, which turns out to be the nonstandard tableau

$$\begin{array}{|c|c|c|} \hline 1 & 3 & 4 \\ \hline 2 & 7 & \\ \cline{1-2} 5 & 6 & \\ \cline{1-2} \end{array}\ .$$

Note that, in this example, $\operatorname{SD}(w) \neq \mathrm{P}(w)$, demonstrating Theorem A. Also, in this example, $\operatorname{sh}\operatorname{SD}(w) = (3, 2, 2)$ is smaller than $\operatorname{sh}\mathrm{P}(w) = (3, 3, 1)$ in the dominance partial order.

Corollary 2.8. If $w \in S_n$, then the BBS soliton partition of w is smaller or equal to the RS partition of w in the dominance partial order.

Proof. Let $\Lambda = (\Lambda_1, \Lambda_2, \Lambda_3, \dots)$ denote $\operatorname{sh}\operatorname{SD}(w)$ and let $\lambda = (\lambda_1, \lambda_2, \lambda_3, \dots)$ denote $\operatorname{sh}\mathrm{P}(w)$. Then, for all $k = 1, 2, \dots$, we have

$$\begin{aligned} \Lambda_1 + \Lambda_2 + \cdots + \Lambda_k &= \mathrm{I}_k(w) && \text{by localized Greene's theorem (Lemma 2.6)} \\ &\leq \mathrm{i}_k(w) && \text{since } \mathrm{I}_k(w) \text{ gives the length of a } k\text{-increasing subsequence of } w \\ &= \lambda_1 + \lambda_2 + \cdots + \lambda_k && \text{by Greene's theorem (Theorem 2.2).} \end{aligned}$$

□

3. Fukuda's carrier algorithm

In this section, we review the carrier algorithm and the fact that the RS insertion tableau is an invariant of a box-ball system (BBS).

3.1. **Carrier algorithm.** The carrier algorithm is a way to describe a BBS move as a sequence of local operations of inserting and bumping numbers in and out of a carrier filled with a weakly increasing string. A version of the carrier algorithm was first introduced in [TM97], and the version of the carrier algorithm we use in this paper comes from [Fuk04, Section 3.3]. Given a BBS state at time t, the carrier algorithm is used to calculate the state at time $t+1$. We describe the process in Algorithm 1. Note that, after each insertion and ejection step, the sequence in the carrier is weakly increasing.

Algorithm 1 The 1-carrier algorithm [Fuk04]

1: **begin** carrier algorithm
2: Set $e \coloneqq n+1$, so that e is considered to be larger than any ball
3: Set $B \coloneqq$ the configuration of the BBS at time t, where each empty box is replaced with an e and the first (leftmost) element of B is the integer in the first (leftmost) nonempty box in the configuration and the last (rightmost) element of B is the integer in the last (rightmost) nonempty box of the configuration
4: Let ℓ denote the number of elements (including the e's) of B
5: Fill a "carrier" $\mathcal{C}$ —depicted —with n copies of e
6: Write B to the right of $\mathcal{C}$
7: **begin** Process 1: insertion process
8: **for all** i in $\{1, 2, \ldots, \ell\}$ **do**
9: Set p to be the i^{th} leftmost element of B
10: **begin** element ejection process
11: **if** an element in $\mathcal{C}$ is larger than p **then**
12: Set $s \coloneqq$ the smallest element in $\mathcal{C}$ larger than p. If $s = e$, pick the leftmost e
13: Eject s from $\mathcal{C}$ and put it immediately to the left of $\mathcal{C}$
14: insert p in the place of s
15: **else**
16: Set $s \coloneqq$ the smallest element in $\mathcal{C}$
17: Eject s from $\mathcal{C}$ and put it immediately to the left of $\mathcal{C}$
18: ▶ Note: There are now $n-1$ elements in $\mathcal{C}$
19: Place p in the rightmost location in $\mathcal{C}$
20: ▶ Note: There are now n elements in $\mathcal{C}$
21: **end if**
22: **end** element ejection process
23: **end for**
24: **end** Process 1: insertion process
25: **begin** Process 2: flushing process
26: **while** there are non-e elements in $\mathcal{C}$ **do**
27: Set $p \coloneqq e$
28: Perform the element ejection process (see line 10)
29: **end while**
30: **end** Process 2: flushing process
31: ▶ Note: The current elements to the left of $\mathcal{C}$ correspond to the $t+1$ state of the BBS
32: **end** carrier algorithm

Example 3.1. We compute the configuration at time $t = 3$ of the box-ball system from Figure 2 by applying the carrier algorithm to the configuration at time $t = 2$. Following Algorithm 1, we set $B \coloneqq 452ee136$. The carrier algorithm then proceeds as follows.

begin Process 1: insertion process

$$\underbracket{eeeeee}\,452ee136$$
$$e\,\underbracket{4eeeee}\,52ee136$$
$$ee\,\underbracket{45eeee}\,2ee136$$
$$ee4\,\underbracket{25eeee}\,ee136$$
$$ee42\,\underbracket{5eeeee}\,e136$$
$$ee425\,\underbracket{eeeeee}\,136$$
$$ee425e\,\underbracket{1eeeee}\,36$$
$$ee425ee\,\underbracket{13eeee}\,6$$
$$ee425eee\,\underbracket{136eee}$$

end insertion process

begin Process 2: flushing process

$$ee425eee\,\underbracket{136eee} \leftarrow e$$
$$ee425eee1\,\underbracket{36eeee} \leftarrow e$$
$$ee425eee13\,\underbracket{6eeeee} \leftarrow e$$
$$ee425eee136\,\underbracket{eeeeee}$$

end flushing process

The elements $ee425eee136$ to the left of $\mathcal{C}$ correspond to the configuration at time $t = 3$ given in Figure 2.

3.2. **The RS insertion tableau is an invariant of a box-ball system.**

Remark 3.2 ([Fuk04, Remark 4])**.** The carrier algorithm can be viewed as a sequence of Knuth moves. Consider the insertion of $\mathbf{p}$ into the carrier. Note that, since our carrier can carry n elements, if $\mathbf{p} \neq e$, then the carrier must contain a number (possibly e) greater than $\mathbf{p}$. If $\mathbf{p} = e$, then no number in the carrier is greater than $\mathbf{p}$.

First, suppose $\mathbf{p} \neq e$, and let C_p denote the smallest element in the carrier which is greater than $\mathbf{p}$.

(i) If C_p is the smallest element in the carrier, then the insertion process is equivalent to applying a sequence of K_1^- moves

$$\begin{gathered}
\underbracket{C_p z_1 z_2 \cdots z_{\ell-1} z_\ell}\ \mathbf{p} \\
C_p z_1 z_2 \cdots z_{\ell-1} \mathbf{p} z_\ell \\
\vdots \\
C_p z_1 \mathbf{p} z_2 \cdots z_{\ell-1} z_\ell \\
C_p\,\underbracket{\mathbf{p} z_1 z_2 \ldots\ldots z_\ell}\,.
\end{gathered}$$

(ii) If C_p is the largest element in the carrier, then the insertion process is equivalent to applying a sequence of K_2^+ moves

$$\begin{gathered}
\underbracket{x_1 x_2 \cdots x_{m-1} x_m C_p}\ \mathbf{p} \\
x_1 x_2 \cdots x_{m-1} C_p x_m \mathbf{p} \\
\vdots \\
x_1 C_p x_2 \cdots x_{m-1} x_m \mathbf{p} \\
C_p\,\underbracket{x_1 x_2 \cdots x_{m-1} x_m \mathbf{p}}\,.
\end{gathered}$$

(iii) If C_p is neither the smallest nor the largest element in the carrier, then the insertion process is equivalent to applying a sequence of K_1^- moves

$$\begin{gathered}
\underbracket{x_1x_2\cdots x_{m-1}x_mC_pz_1z_2\cdots z_{\ell-1}z_\ell}\ \mathbf{p} \\
x_1x_2\cdots x_{m-1}x_mC_pz_1z_2\cdots z_{\ell-1}\mathbf{p}z_\ell \\
\vdots \\
x_1x_2\cdots x_{m-1}x_mC_pz_1\mathbf{p}z_2\cdots z_{\ell-1}z_\ell \\
x_1x_2\cdots x_{m-1}x_mC_p\mathbf{p}z_1z_2\cdots z_{\ell-1}z_\ell
\end{gathered}$$

followed by a sequence of K_2^+ moves

$$\begin{gathered}
x_1x_2\cdots x_{m-1}x_mC_p\mathbf{p}z_1z_2\cdots z_{\ell-1}z_\ell \\
x_1x_2\cdots x_{m-1}C_px_m\mathbf{p}z_1z_2\cdots z_{\ell-1}z_\ell \\
\vdots \\
x_1C_px_2\cdots x_{m-1}x_m\mathbf{p}z_1z_2\cdots z_{\ell-1}z_\ell \\
C_p\,\underbracket{x_1x_2\cdots x_{m-1}x_m\mathbf{p}z_1z_2\cdots z_{\ell-1}z_\ell}\,.
\end{gathered}$$

Next, suppose $\mathbf{p} = e$. Then $\mathbf{p}$ is greater than or equal to every element in the carrier, and the insertion process is equivalent to applying the trivial transformation

$$\begin{gathered}
\underbracket{x_1x_2\cdots x_n}\ \mathbf{p} \\
x_1\ \underbracket{x_2\cdots x_n\ \mathbf{p}}\,.
\end{gathered}$$

Theorem 3.3 ([Fuk04, Theorem 3.1])**.** The RS insertion tableau is a conserved quantity under the time evolution of the BBS, i.e., the RS insertion tableau is preserved under each BBS move. More precisely, let B_t be the state of a box-ball system at time t. Let B_t' be the permutation created from B_t by removing all e's. Then $\mathrm{P}(B_t')$ is identical for all t.

Example 3.4. As shown in Figure 2, the configurations 452361, $ee45e2136$, $eeee452ee136$, and $eeeeee425eee136$ are in the same box-ball system. As Theorem 3.3 tells us, the permutations 452361, 452136, and 425136 have the same RS insertion tableau

$$\mathrm{P}(452361) = \mathrm{P}(452136) = \mathrm{P}(425136) = \begin{array}{|c|c|c|}\hline 1 & 3 & 6 \\ \hline 2 & 5 \\ \cline{1-2} 4 \\ \cline{1-1}\end{array}\,.$$

Corollary 3.5. Let w be a permutation. If r is the row reading word of $\mathrm{SD}(w)$, then $\mathrm{P}(w) = \mathrm{P}(r)$.

Proof. Let r be the row reading word of $\mathrm{SD}(w)$. By definition of the soliton decomposition tableau, we know that r is the order in which the balls of w are configured once we reach a steady state. Therefore, r is a state in the box-ball system containing w. Theorem 3.3 tells us that the RS insertion tableau is preserved under a sequence of box-ball moves, so $\mathrm{P}(w) = \mathrm{P}(r)$. □

Example 3.6. Let $w = 5623714$, the permutation from Section 2, and let r be the row reading word of $\mathrm{SD}(w)$. We have

$$\mathrm{SD}(w) = \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 7 \\ \cline{1-2} 5 & 6 \\ \cline{1-2}\end{array}\,, \quad r = 5627134, \quad \text{and} \quad \mathrm{P}(w) = \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 6 & 7 \\ \hline 5 \\ \cline{1-1}\end{array} = \mathrm{P}(r).$$

In Example 3.4, the soliton decomposition coincides with the RS insertion tableau of the box-ball system, but in Example 3.6 these two tableaux do not coincide. In the next section we discuss when $\mathrm{SD}(w) = \mathrm{P}(w)$.

4. When the soliton decomposition and the RS insertion tableau coincide

In this section, we will prove Theorem 4.2. One direction of our proof uses the following lemma, which was communicated to us by Darij Grinberg.

Lemma 4.1. Suppose S is a row-strict tableau, that is, every row is increasing (with no restrictions on the columns). Let r be the row reading word of S. If $\operatorname{sh} S = \operatorname{sh} \mathrm{P}(r)$, then S is standard, that is, every column of S is increasing.

Proof. Suppose S is not standard. Then S has two adjacent entries in a column which are out of order. Indexing our rows from top to bottom and our columns from left to right, this means there is a column (say, column c) for which the entry in some row k is bigger than the entry immediately below it. Let y be the entry in the k-th row, c-th column of S, and let x be the entry immediately below it (in the $k+1$-th row, c-th column of S).

Since r is the row reading word of S and since each row of S is increasing, we can construct a list of k disjoint increasing subsequences of r: The first $k-1$ increasing subsequences of r are the first $k-1$ rows of S. The k-th increasing subsequence starts in row $k+1$, column 1 of S, moving along the same row until we get to column c (with entry x), then going up to row k above (which has entry y), then continuing to the end of row k.

The length of the k-th increasing subsequence is larger (by 1) than the length of the k-th row of S. So the total number of letters in our list of k disjoint increasing subsequences of r is larger by 1 than the total length of the first k rows of S. Thus, Greene's theorem (Theorem 2.2) says that the total length of the first k rows of the RS insertion tableau $\mathrm{P}(r)$ of r is larger (at least by 1) than the total length of the first k rows of S. Therefore, the shape of S is not equal to the shape of $\mathrm{P}(r)$. ☐

The following theorem gives a characterization of permutations whose soliton decompositions are equal to their RS insertion tableaux.

Theorem 4.2. Let w be a permutation. Then the following are equivalent:

(1) $\mathrm{SD}(w) = \mathrm{P}(w)$.
(2) $\mathrm{SD}(w)$ is a standard tableau.
(3) The shape of $\mathrm{SD}(w)$ equals the shape of $\mathrm{P}(w)$.

Proof. Certainly (1) implies (2) and (3). We will show that (2) implies (1) and (3) implies (2).

Let r be the row reading word of $\mathrm{SD}(w)$. By Corollary 3.5, we have

$$\mathrm{P}(w) = \mathrm{P}(r). \tag{4.1}$$

First, we show that (2) implies (1). Suppose that $\mathrm{SD}(w)$ is a standard tableau T. Since r is the row reading word of T, we have $\mathrm{P}(r) = T$ by (1.1). Combining this equality with (4.1), we get $\mathrm{P}(w) = \mathrm{P}(r) = T = \mathrm{SD}(w)$.

Next, we show that (3) implies (2). Let S denote $\mathrm{SD}(w)$, and note that $\mathrm{SD}(w)$ is a row-strict tableau by construction. Suppose $\operatorname{sh} S = \operatorname{sh} \mathrm{P}(w)$. Since $\mathrm{P}(w) = \mathrm{P}(r)$ by (4.1), we have

$\operatorname{sh} \mathrm{P}(w) = \operatorname{sh} \mathrm{P}(r)$, so $\operatorname{sh} S = \operatorname{sh} \mathrm{P}(w) = \operatorname{sh} \mathrm{P}(r)$. Because S is a row-strict tableau and the permutation r is the row reading word of S and $\operatorname{sh} S = \operatorname{sh} \mathrm{P}(r)$, Lemma 4.1 tells us that S is standard. □

Corollary 4.3. Let w be a permutation. Then the following five statements are equivalent:

(1) $\mathrm{SD}(w) = \mathrm{P}(w)$.
(2) $\mathrm{SD}(w)$ is a standard tableau.
(3) The shape of $\mathrm{SD}(w)$ equals the shape of $\mathrm{P}(w)$.
(4) For all $k \geq 1$, we have
$$\mathrm{I}_k(w) = \mathrm{i}_k(w).$$
(5) For all $k \geq 1$, we have
$$\mathrm{D}_k(w) = \mathrm{d}_k(w).$$

The symbols I_k and D_k are the statistics from localized Greene's theorem (Section 2.2) and i_k and d_k are the statistics from Greene's theorem (Section 2.1).

Proof. For short, we write $\mathrm{i}_k \coloneqq \mathrm{i}_k(w)$, $\mathrm{I}_k \coloneqq \mathrm{I}_k(w)$, $\mathrm{d}_k \coloneqq \mathrm{d}_k(w)$, and $\mathrm{D}_k \coloneqq \mathrm{D}_k(w)$. By localized Greene's theorem (Lemma 2.6),

$$\text{the shape of } \mathrm{SD}(w) \text{ is } (\mathrm{I}_1, \mathrm{I}_2 - \mathrm{I}_1, \mathrm{I}_3 - \mathrm{I}_2, \dots) \text{ and}$$
$$\text{the shape of the conjugate of } \mathrm{SD}(w) \text{ is } (\mathrm{D}_1, \mathrm{D}_2 - \mathrm{D}_1, \mathrm{D}_3 - \mathrm{D}_2, \dots).$$

By Greene's theorem (Theorem 2.2),

$$\text{the shape of } \mathrm{P}(w) \text{ is } (\mathrm{i}_1, \mathrm{i}_2 - \mathrm{i}_1, \mathrm{i}_3 - \mathrm{i}_2, \dots) \text{ and}$$
$$\text{the shape of the conjugate of } \mathrm{P}(w) \text{ is } (\mathrm{d}_1, \mathrm{d}_2 - \mathrm{d}_1, \mathrm{d}_3 - \mathrm{d}_2, \dots).$$

Combining these facts, we conclude that $\operatorname{sh} \mathrm{SD}(w) = \operatorname{sh} \mathrm{P}(w)$ if and only if $\mathrm{I}_k = \mathrm{i}_k$ for all $k \geq 1$ if and only if $\mathrm{D}_k = \mathrm{d}_k$ for all $k \geq 1$. □

Example 4.4. Let $w = 5623714$. From Examples 2.3 and 2.7, we know that $\mathrm{I}_2(w) = 5 < 6 = \mathrm{i}_2(w)$. So all the other items of Corollary 4.3 must also be false.

5. Reading words and steady states

We study the steady-state configurations of a box-ball system. The main result of this section (Proposition 5.2) is a corollary of [LLPS19, Proof of Lemma 2.1 and 2.3].

5.1. **Reading words of standard tableaux.**

The permutations which reach their steady state at time 0 are precisely the row reading words of standard tableaux.

Proposition 5.1. A permutation r is the row reading word of a standard tableau if and only if r reaches its soliton decomposition at time $t = 0$.

In particular, if r is the row reading word of a standard tableau T, then $T = \mathrm{SD}(r)$. In the next section, the standard tableau in Proposition 5.1 is generalized to standard skew tableaux whose rows are weakly decreasing in length.

5.2. **Reading words of standard skew tableaux.** A BBS state can be represented as a *configuration array* containing the integers from 1 to n as follows: scanning the boxes from right to left, each *increasing run* (maximal consecutive increasing string of balls) becomes a row in the array. A string of g empty boxes indicates that the next row below should be shifted g spaces to the left. Note that this array has increasing rows but not necessarily increasing columns; it may be disconnected and it may not have a valid skew shape.

Proposition 5.2. A BBS configuration C is in steady state if and only if its configuration array is a standard (possibly disconnected) skew tableau whose rows are weakly decreasing in length.

We will give a proof in Section 5.3.

Example 5.3. Let $w = 5623714$, the example we use in Section 2. The following are the box-ball system states from time $t = 0$ to $t = 4$ and their configuration arrays.

$$t = 0 \qquad 5\,6\,2\,3\,7\,1\,4\,e\ldots \qquad \begin{array}{|c|c|c|} \hline 1 & 4 & \\ \hline 2 & 3 & 7 \\ \hline 5 & 6 & \\ \hline \end{array}$$

$$t = 1 \qquad e\,e\,5\,6\,e\,2\,7\,1\,3\,4\,e\ldots \qquad \begin{array}{cccc} & 1 & 3 & 4 \\ & 2 & 7 & \\ 5 & 6 & & \end{array}$$

$$t = 2 \qquad e\,e\,e\,e\,5\,6\,e\,2\,7\,e\,1\,3\,4\,e\ldots \qquad \begin{array}{ccccc} & & 1 & 3 & 4 \\ & 2 & 7 & & \\ 5 & 6 & & & \end{array}$$

$$t = 3 \qquad e\,e\,e\,e\,e\,e\,5\,6\,e\,2\,7\,e\,e\,1\,3\,4\,e\ldots \qquad \begin{array}{cccccc} & & & 1 & 3 & 4 \\ & 2 & 7 & & & \\ 5 & 6 & & & & \end{array}$$

$$t = 4 \qquad e\,e\,e\,e\,e\,e\,e\,e\,5\,6\,e\,2\,7\,e\,e\,e\,1\,3\,4\,e\ldots \qquad \begin{array}{ccccccc} & & & & 1 & 3 & 4 \\ & 2 & 7 & & & & \\ 5 & 6 & & & & & \end{array}$$

In this box-ball system, all configurations at time $t \geq 1$ are in steady state.

Example 5.4. The following is an example of a non-steady-state BBS configuration and its configuration array. Note that the configuration array is a standard skew tableau but its rows are not weakly decreasing in length.

$$\ldots e\,1\,3\,7\,e\,2\,4\,6\,9\,e\,e\,5\,8\,e\ldots \qquad \begin{array}{ccccc} & & & 5 & 8 \\ & 2 & 4 & 6 & 9 \\ 1 & 3 & 7 & & \end{array}$$

5.3. **Separation condition.** A 'separation condition' for steady state is given in statement (43) in [LLPS19]. In Lemmas 5.5 and 5.6, we reframe this characterization for steady state in terms of our version of the box-ball system. Proposition 5.2 follows directly from these two lemmas.

Lemma 5.5 (Separation condition)**.** Let a BBS configuration be in steady state. Suppose two adjacent solitons L (the left soliton with length ℓ) and R (the right soliton) are separated by g empty boxes, where $g < \ell$. Then, for $i = 1, 2, \ldots, \ell - g$,

the i-th smallest ball of the right soliton R is smaller than
the $(i+g)$-th smallest ball of the left soliton L.

Proof. We apply one BBS move to the configuration via the carrier algorithm. Suppose $L = L_1L_2\dots L_\ell$ and $R = R_1R_2\dots R_r$ are the two leftmost solitons.

Our initial setup with n copies of e in the carrier is

$$\underbrace{ee\cdots e}_{\text{carrier}}\ L_1\ \dots\ L_\ell\ \overbrace{e\dots e}^{g\text{ copies}}\ R_1\ \dots\ R_r\ \dots$$

First, we simply insert $L_1,\dots,L_\ell$ into the carrier. Since L is increasing, each time we insert a ball of L, we eject a copy of e. We get

$$\overbrace{e\dots e}^{\ell\text{ copies}}\ \underline{L_1\ \cdots\ L_\ell\ ee\cdots e}\ \overbrace{e\dots e}^{g\text{ copies}}\ R_1\ \dots\ R_r\ \dots \tag{5.1}$$

Next, we insert the g copies of e into the carrier and eject $L_1,\dots,L_g$:

$$e\dots e\ \underbrace{L_1\ \dots\ L_g}_{\text{first } g \text{ balls}}\ \underline{\overbrace{L_{g+1}\ \cdots L_\ell}^{\ell-g\text{ balls}}\ ee\cdots e}\ R_1\ \dots\ R_r\ \dots$$

Since we started with a steady-state configuration, the left soliton L must stay intact at the end of the carrier algorithm. So, for each $i = 1,\dots,\ell-g$, as we insert R_i, we must eject L_{g+i}, and get

$$e\dots e\, L_1\ \dots\ L_g\ \underbrace{L_{g+1}\ \dots\ L_\ell}_{\ell-g\text{ balls}}\,\underline{R_1\ \cdots\ R_{\ell-g}\ ee\cdots e}\,R_{\ell-g+1}\dots R_r\dots$$

So we must have $R_i < L_{g+i}$ for $i = 1,2,\dots,\ell-g$, as needed.

After we insert the rest of the elements of R into the carrier, we have

$$e\dots e\, L_1\ \dots\ L_\ell\ \overbrace{ee\dots e}^{\substack{r-\ell+g\\ \text{copies}}}\ \underline{R_1\ \cdots\ R_r\ ee\cdots e}\dots$$

If we have a third soliton located to the right of R, we would be in the same situation as (5.1). We then repeat the same process for the rest of the solitons and arrive at the same conclusion. ☐

Lemma 5.6 (Sufficient condition for steady state)**.** Suppose a BBS configuration w satisfies the following.

(1) The configuration array of w has rows of weakly decreasing length.

(2) The configuration array of w is standard; that is, if two adjacent maximal consecutive increasing blocks L (the left block with length ℓ) and R (the right block) of w are separated by g empty boxes such that $g < \ell$, then, for $i = 1,2,\dots,\ell-g$,

the i-th ball of the right block R is smaller than
the $(i+g)$-th ball of the left block L.

Then w is in steady state.

Proof. Suppose w is the configuration at time t. We apply the carrier algorithm to get the configuration at time $t+1$. Suppose $L = L_1L_2\dots L_\ell$ and $R = R_1R_2\dots R_r$ are the two leftmost increasing runs (maximal consecutive increasing blocks of balls).

Prior to applying the carrier algorithm, we have

$$\underbracket{ee\cdots e}_{\text{carrier}}\,\underbrace{L_1\dots L_\ell}_{\text{first run}}\overbrace{e\dots e}^{g\text{ copies}}\underbrace{R_1\dots R_r}_{\text{second run}}\dots$$

First, we insert each of $L_1,\dots,L_\ell$ into the carrier and eject an e each time. We get

$$\overbrace{e\dots e}^{\ell\text{ copies}}\,\underbracket{L_1\cdots L_\ell\ ee\cdots e}\ \overbrace{e\dots e}^{g\text{ copies}}R_1\dots R_r \tag{5.2}$$

Next, we insert the g copies of e into the carrier and eject $L_1,\dots,L_g$. There are two cases: either (a) $g\geq\ell$ or (b) $g<\ell$.

(a) First, suppose that $g\geq\ell$. Then all of $L_1,\dots,L_\ell$ are ejected and the carrier is now empty:

$$e\dots e\ \underbrace{L_1\dots L_\ell}_{\text{first run}}\overbrace{e\dots e}^{g-\ell}\,\underbracket{ee\cdots e}\,\underbrace{R_1\dots R_r}_{\text{second run}}\dots$$

We proceed by inserting $R_1,\dots,R_r$ into the carrier. Since R is increasing, we eject r copies of e's:

$$e\dots e\,L_1\dots L_\ell\overbrace{e\dots e}^{g-\ell}\overbrace{e\dots e}^{r}\,\underbracket{R_1\cdots R_r\ ee\cdots e}\dots$$

(b) Second, suppose $g<\ell$. After $L_1,\dots,L_g$ are ejected, we have

$$e\dots e\,\underbrace{L_1\dots L_g}_{\text{first } g\text{ balls}}\,\underbracket{\overbrace{L_{g+1}\cdots L_\ell}^{\ell-g\text{ balls}}ee\cdots e}\,\underbrace{R_1\dots R_r}_{\text{second run}}\dots$$

We proceed by inserting $R_1,\cdots,R_r$ into the carrier. We have $\ell\leq r$ by assumption part (1) and $R_i<L_{g+i}$ for $i=1,2,\dots,l-g$ by assumption part (2). Therefore, as we insert $R_1,\dots,R_{\ell-g}$, we must eject $L_{g+1},\dots,L_\ell$, and we get

$$e\dots e\,L_1\dots L_g\,\underbrace{L_{g+1}\dots L_\ell}_{\ell-g\text{ balls}}\,\underbracket{R_1\cdots R_{\ell-g}\ ee\cdots e}\,R_{\ell-g+1}\dots R_r\dots$$

After we insert the rest of the elements of R into the carrier, we have

$$e\dots e\,L_1\dots L_\ell\ \overbrace{ee\dots e}^{r-\ell+g}\ \underbracket{R_1\cdots R_r\ ee\cdots e}\dots$$

In both cases, at time $t+1$ there are at least $r-\ell+g$ empty boxes to the right of L. Since $\ell\leq r$, we have $g\leq r-\ell+g$, so there are at least as many empty boxes to the right of L as at time t. Furthermore, the increasing run L stays together.

If we have a third increasing run $S=S_1\dots S_s$ to the right of R (with a gap of g' empty boxes), we would be in the same situation as (5.2). After inserting the elements of S into the carrier, we would have

$$e\dots e\,L_1\dots L_\ell\ \overbrace{ee\dots e}^{r-\ell+g}\ R_1\dots R_r\ \overbrace{ee\dots e}^{s-r+g'}\ \underbracket{S_1\cdots S_s\ ee\cdots e}\dots$$

Again, there are at least as many empty boxes to the right of R at time $t+1$ than at time t, and R stays together.

At the end of the carrier algorithm, the increasing runs stay together, their order stays the same, and the gap of empty boxes between each pair of adjacent sequences is at least as large as at time t.

The new configuration satisfies both part (1) and (2) of the assumption. By induction, subsequent carrier algorithm applications leave the order of the increasing runs unchanged, so these increasing runs are in fact solitons. □

By the two lemmas above, we have Proposition 5.2: a box-ball configuration is in steady state if and only if its configuration array (1) has rows of weakly decreasing length and (2) each of its column is increasing.

6. A recording tableau giving n–3 steady-state time

In this section, we prove Theorem 6.7, which states that all permutations in S_n with a certain recording tableau have box-ball steady-state time $n-3$. We conjecture that all other permutations in S_n have steady-state time less than $n-3$ (Conjecture 1.1).

Theorem 6.7 turns out to be a special case of a general phenomenon, which is proven in [CFG+]: if two permutations have the same recording tableau, then they have the same BBS steady-state time (Conjecture 1.2).

6.1. A recording tableau giving n–3 steady-state time.

Definition 6.1. If $n \geq 5$, let $\widehat{Q}$ denote the tableau

$$\begin{array}{ccccc} \boxed{1} & \boxed{2} & \ldots & \boxed{n-2} & \boxed{n-1} \\ \boxed{3} & \boxed{4} & & & \\ \boxed{n} & & & & \end{array}.$$

Let $S_n(\widehat{Q})$ be the set of permutations $w \in S_n$ such that its recording tableau $\mathrm{Q}(w)$ is equal to $\widehat{Q}$.

Example 6.2. For $n = 5$, the five permutations of $S_n(\widehat{Q})$ are the following.

45132   25143   35142   45231   35241

For $n = 6$, the sixteen permutations of $S_n(\widehat{Q})$ are as follows.

451362	251463	351462	452361	352461	561243	261354	361254
461253	561342	261453	361452	461352	562341	362451	462351

Note that one of our running examples, 452361, is in $S_6(\widehat{Q})$. As illustrated in Figure 2, its steady-state time is $3 = 6 - 3$.

Remark 6.3. It follows from Definition 6.1 that the RS algorithm induces a bijection from $S_n(\widehat{Q})$ onto the set of standard tableaux of shape $(n-3, 2, 1)$, so $S_n(\widehat{Q})$ is counted by the sequence [Lan02].

The rest of this section is devoted to proving Theorem 6.7, which states that every permutation in $S_n(\widehat{Q})$ has steady state time $n-3$.

6.2. Lemmas for Theorem 6.7.

Lemma 6.4. Let $n \geq 5$, and suppose $w \in S_n(\widehat{Q})$. Then w is *not* the union of two increasing subsequences.

Proof. The recording tableau of w is equal to $\widehat{Q}$, which has height 3. Therefore, the RS partition of w has three parts. By Greene's theorem (Theorem 2.2), w is not the union of two increasing subsequences. $\square$

Lemma 6.5. Let $n \geq 5$, and suppose $w = w_1 w_2 \dots w_n \in S_n(\widehat{Q})$. Then w satisfies the following.

(1) $w_3 < w_4 < \cdots < w_{n-1}$
(2) $w_n < w_2$
(3) $w_1 < w_2$
(4) $w_3 < w_1$
(5) $w_3 < w_2$
(6) $w_4 < w_2$

Proof. Since $w \in S_n(\widehat{Q})$, the recording tableau of w is equal to $\widehat{Q}$. We will use the inverse RS algorithm[1] to construct w. Let $P = \mathrm{P}(w)$ and $Q = \mathrm{Q}(w)$. Denote the entries in the top row of P by $a_1, a_2, \dots, a_q$ (where $q = n-3$), the second row of P by b_1 and b_2, and the entry in the third row of P by c_1. Hence, the starting pair P and Q is

$$P = \begin{array}{|c|c|c|c|c|c|c|}\hline a_1 & a_2 & a_3 & a_4 & \dots & a_{q-1} & a_q \\ \hline b_1 & b_2 \\ \cline{1-2} c_1 \\ \cline{1-1}\end{array} \qquad Q = \begin{array}{|c|c|c|c|c|c|c|}\hline 1 & 2 & 5 & 6 & \dots & n-2 & n-1 \\ \hline 3 & 4 \\ \cline{1-2} n \\ \cline{1-1}\end{array}$$

Since P is standard, we know that $b_1 < c_1$. The other entry b_2 in the second row is larger than b_1. If $b_2 < c_1$, let b_y equal b_2. Otherwise, let b_y be b_1. In other words, we let b_y denote the largest element in the second row which is smaller than c_1. Similarly, let a_x denote the largest element in the first row which is smaller than b_y. The first step of the inverse RS algorithm tells us that $w_n = a_x$.

After the first step in the inverse RS algorithm, we get the pair of tableaux

$$P_{n-1} = \begin{array}{|c|c|c|c|c|c|c|}\hline \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \dots & \alpha_{q-1} & \alpha_q \\ \hline \beta_1 & \beta_2 \\ \cline{1-2}\end{array} \qquad Q_{n-1} = \begin{array}{|c|c|c|c|c|c|c|}\hline 1 & 2 & 5 & 6 & \dots & n-2 & n-1 \\ \hline 3 & 4 \\ \cline{1-2}\end{array}.$$

We now pause to observe two facts that will be referenced at the end of this proof. First, note that P_{n-1} is standard by definition of the inverse RS algorithm. Thus,

$$\alpha_1, \alpha_2, \dots, \alpha_q \text{ is increasing.} \tag{6.1}$$

Second, we note that

$$a_x < \beta_2, \tag{6.2}$$

as we now explain. Recall that $w_n = a_x$, so, using the original RS algorithm, we insert a_x into P_{n-1} to get P. Since row 1 of P_{n-1} and row 1 of P have the same size, we know that a_x bumps a number in row 1 of P_{n-1} to row 2. Let

a_i denote the smallest entry in row 1 of P_{n-1} which is greater than a_x.

The RS algorithm replaces a_i with a_x and bumps a_i to row 2. Since row 2 of P_{n-1} and row 2 of P have the same size, we know that a_i bumps a number in row 2 of P_{n-1}. So a_i must be smaller than β_2. Since $a_x < a_i$, we have $a_x < \beta_2$. This concludes our explanation for (6.2).

[1]For definition of the inverse RS algorithm, see, for example, the textbook [Sag01, Section 3.1].

We also note that

$$\beta_1 < \beta_2, \tag{6.3}$$
$$\alpha_1 < \beta_1, \quad \text{and} \tag{6.4}$$
$$\alpha_2 < \beta_2, \tag{6.5}$$

since P_{n-1} is standard. We will reference these inequalities at the end of this proof.

If $n > 5$, the numbers $n-1, n-2, \ldots, 6, 5$ are in the first row of Q, so the next steps in the inverse RS algorithm are to remove elements $\alpha_q, \alpha_{q-1}, \ldots, \alpha_4, \alpha_3$ from P_{n-1}, in that order. Hence, the last $n-4$ letters of w are $\alpha_3, \alpha_4, \ldots, \alpha_{q-1}, \alpha_q, a_x$.

The new pair of tableaux is

$$P_4 = \begin{array}{|c|c|}\hline \alpha_1 & \alpha_2 \\ \hline \beta_1 & \beta_2 \\ \hline \end{array} \qquad Q_4 = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array}.$$

Note that 4 is the bottom right corner of Q_4. Since $\alpha_2 < \beta_2$ by (6.5), we know that α_2 is the largest element in row 1 of P which is smaller than β_2. So $w_4 = \alpha_2$, and the last $n-3$ letters of w are $\alpha_2, \alpha_3, \alpha_4, \ldots, \alpha_q, \alpha_{q-1}, a_x$.

The new pair of tableaux is

$$P_3 = \begin{array}{|c|c|}\hline \alpha_1 & \beta_2 \\ \hline \beta_1 \\ \cline{1-1} \end{array} \qquad Q_3 = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1} \end{array}.$$

Note that 3 is in the second row of Q_3. We know from (6.3) that β_2 is larger than β_1, so α_1 is the largest element in row 1 smaller than β_1. Thus, $w_3 = \alpha_1$. So the last $n-2$ letters of w are $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \ldots, \alpha_q, \alpha_{q-1}, a_x$. The new pair of tableaux is

$$P_2 = \begin{array}{|c|c|}\hline \beta_1 & \beta_2 \\ \hline \end{array} \qquad Q_2 = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline \end{array}.$$

We then remove β_2 and β_1 from P_2, in that order.

Therefore,

$$w = \underbrace{\beta_1\beta_2}_{\text{increasing}} \underbrace{\alpha_1\alpha_2\alpha_3\alpha_4 \ldots \alpha_{q-1}\alpha_q}_{\text{increasing}} a_x.$$

We now have all the necessary information to prove all parts of the lemma.

(1) The subsequence w_3, w_4, $\ldots$, w_{n-1} is increasing because it is equal to the sequence α_1, α_2, $\ldots$, α_q, which is increasing due to (6.1). This proves part (1).
(2) We have $w_n < w_2$ from (6.2), since $w_n = a_x$ and $w_2 = \beta_2$. This proves part (2).
(3) We have $w_1 < w_2$ from (6.3), since $w_1 = \beta_1$ and $w_2 = \beta_2$. This proves part (3).
(4) We have $w_3 < w_1$ from (6.4), since $w_3 = \alpha_1$ and $w_1 = \beta_1$. This proves part (4).
(5) We have $w_3 < w_2$ since $w_1 < w_2$ and $w_3 < w_1$. This proves part (5).
(6) We have $w_4 < w_2$ from (6.5), since $w_4 = \alpha_2$ and $w_2 = \beta_2$. This proves part (6).

□

Lemma 6.6. Suppose $w = w_1 \ldots w_n \in S_n(\widehat{Q})$.

(1) Either $w_n = 1$ or $w_3 = 1$.
(2) If $w_3 = 1$, then $w_1 = 2$, $w_4 = 2$, or $w_n = 2$.
(3) If $w_3 = 1$ and $w_1 = 2$, then $w_4 = 3$ or $w_n = 3$.

Proof. (1) Suppose $w_n \neq 1$. Since both w_1, w_2 and $w_3, \ldots, w_{n-1}$ are increasing subsequences by Lemma 6.5(3),(1), either $w_1 = 1$ or $w_3 = 1$. Since $w_3 < w_1$ by Lemma 6.5(4), we must have $w_3 = 1$.

(2) Assume $w_3 = 1$. We will show that $w_2 \neq 2$ and that none of $w_5, \ldots, w_{n-1}$ is equal to 2 (hence $w_1 = 2$, $w_4 = 2$, or $w_n = 2$). Since $w_3 < w_2$ and $w_4 < w_2$ by Lemma 6.5(5),(6) and since w is a permutation, we must have $2 < w_2$. Similarly, since $w_3 < w_4 < w_5 < \cdots < w_{n-1}$ by Lemma 6.5(1) and since w is a permutation, each of $w_5, \ldots, w_{n-1}$ must be larger than 2.

(3) Suppose $w_3 = 1$ and $w_1 = 2$. We will prove that $w_2 \neq 3$ and none of $w_5, \ldots, w_{n-1}$ is equal to 3 (hence $w_4 = 3$ or $w_n = 3$). Since $w_n \notin \{1, 2\}$, we have $2 < w_n$. By Lemma 6.5(2), we have $w_n < w_2$. So $2 < w_n < w_2$, which implies that w_2 is larger than 3 (since w is a permutation). Similarly, since $w_3 < w_4$ by Lemma 6.5(1) and $w_1 = 2$, we must have $2 < w_4 < w_5 < \cdots < w_{n-1}$ by Lemma 6.5(1). So each of $w_5, \ldots, w_{n-1}$ is larger than 3 (since w is a permutation).

□

6.3. **Proof of Theorem 6.7.**

Theorem 6.7. If $n \geq 5$, every permutation in $S_n(\widehat{Q})$ has steady-state time $n - 3$.

Proof. Suppose $w = w_1 \ldots w_n \in S_n(\widehat{Q})$ is the box-ball configuration at time 0. We will show that w first reaches steady state at time $t = n - 3$.

Let j be the smallest number in $\{3, 4, \ldots, n-1\}$ such that $w_n < w_j$. We claim that the box-ball configuration at time $t = 1$ is

$$e\,e\ \ \underbrace{w_1\, w_2}_{\substack{\text{increasing}\\ \text{block}}}\ \overbrace{e\,e\,e\,\ldots\,e}^{\substack{n-5\\ \text{copies}}}\ x\ \underbrace{1\, y_1\, y_2\, \ldots\, y_{n-4}}_{\text{increasing block}}, \tag{6.6}$$

where $x = w_j$, there are $(n-5)$ copies of e between w_2 and x, and $y_1 < y_2 < \cdots < y_{n-4}$.

To prove this claim, consider the following cases. Due to Lemma 6.6, these five cases cover all possibilities.

(1) $w_n = 1$
(2) $w_3 = 1$ and $w_n = 2$
(3) $w_3 = 1$, $w_1 = 2$, and $w_n = 3$
(4) $w_3 = 1$, $w_1 = 2$, and $w_4 = 3$
(5) $w_3 = 1$ and $w_4 = 2$

First, suppose $w_n = 1$. Lemma 6.5 tells us that w_3 is smaller than each w_i except for $w_n = 1$, so we must have $w_3 = 2$ and $j = 3$:

$$w_1\, w_2\, \underbrace{w_3}_{2}\, w_4\, w_5\, \ldots\, w_{n-1}\, \underbrace{w_n}_{1}.$$

Since $w_1 < w_2$ and $w_4 < w_5 < \cdots < w_{n-1}$ and since $w_4 < w_2$, applying one box-ball move to w results in the configuration

$$e\,e\,w_1\,w_2\,\overbrace{e\,e\,e\,\ldots\,e}^{n-5}\,\underbrace{w_3}_{x}\ 1\,w_4\,w_5\,\ldots\,w_{n-1}$$

where there are $(n-5)$ copies of e between w_2 and $x = w_3 = 2$.

Second, suppose $w_3 = 1$ and $w_n = 2$:

$$w_1\, w_2\, \underbrace{w_3}_{1}\, w_4\, w_5\, \ldots\, w_{n-1}\, \underbrace{w_n}_{2}\,.$$

Since $w_1 < w_2$ and $w_4 < w_5 < \cdots < w_{n-1}$ and since $w_4 < w_2$, applying one box-ball move to w results in the configuration

$$e\, e\, w_1\, w_2\, \overbrace{e\, e\, e\, \ldots\, e}^{n-5 \text{ copies}}\, \underbrace{w_4}_{x}\, 1\, 2\, w_5\, w_6\, \ldots\, w_{n-1}$$

where there are $(n-5)$ copies of e between w_2 and $x = w_4$. In this case, $w_3 = 1$ is not bigger than $w_n = 2$, but w_4 must be bigger than $w_n = 2$ since $w_4 \notin \{1, 2\}$, so $j = 4$.

Third, suppose $w_3 = 1$ and $w_1 = 2$ and $w_n = 3$. Lemma 6.5 tells us that w_4 is smaller than each of the w_i (except for $w_3 = 1$, $w_1 = 2$, and $w_n = 3$), so w_4 must be 4:

$$\underbrace{w_1}_{2}\, w_2\, \underbrace{w_3}_{1}\, \underbrace{w_4}_{4}\, w_5\, \ldots\, w_{n-1}\, \underbrace{w_n}_{3}\,.$$

Using the same reasoning as in the previous two cases, applying one box-ball move to w results in the configuration

$$e\, e\, \underbrace{w_1}_{2}\, w_2\, \overbrace{e\, e\, e\, \ldots\, e}^{n-5 \text{ copies}}\, \underbrace{w_4}_{x}\, 1\, w_n\, w_5\, w_6\, \ldots\, w_{n-1}$$

where there are $(n-5)$ copies of e between w_2 and $x = w_4$. In this case, $j = 4$ since $w_3 = 1$ is not larger than $w_n = 3$ but $w_4 = 4$ is.

Finally, suppose we have one of the last two cases, so $w_3 = 1$ and $w_4 < w_n$:

$$w_1\, w_2\, \underbrace{w_3}_{1}\, w_4\, w_5\, \ldots\, w_{n-1}\, \underbrace{w_n}_{\substack{\text{larger} \\ \text{than } w_4}}$$

Since $w_1 < w_2$ and $w_4 < w_5 < \cdots < w_{j-1} < w_n < w_j < \cdots < w_{n-1}$ and since $w_4 < w_2$, applying one box-ball move to w results in the configuration

$$e\, e\, w_1\, w_2\, \overbrace{e\, e\, e\, \ldots\, e}^{n-5 \text{ copies}}\, \underbrace{w_j}_{x}\, 1\, w_4\, w_5\, \ldots\, w_{j-1}\, w_n\, w_{j+1}\, \ldots\, w_{n-1}$$

where there are $(n-5)$ e's between w_2 and $x = w_j$. In this case, $j \geq 5$ since w_4 is smaller than w_n. This concludes the proof of our claim that the box-ball configuration at time $t = 1$ is as given in (6.6).

Now we perform another box-ball move to reach the configuration at $t = 2$. If $n > 5$, in the configuration at $t = 2$, there are $(n-6)$ e's between w_2 and x:

$$e\,e\,e\,e\,w_1\,w_2\,\overbrace{e\,e\,\ldots\,e}^{\substack{n-6\\\text{copies}}}\;x\;\overbrace{e\,e\,\ldots\,e}^{\substack{n-4\\\text{copies}}}\,\underbrace{1\,y_1\,y_2\,\ldots\,y_{n-4}}_{\text{increasing block}}.$$

In fact, at each BBS move, the increasing sequence w_1, w_2 moves together two spaces to the right, the singleton x moves one space to the right, and the increasing sequence $1, y_1, y_2 \ldots, y_{n-4}$ moves $n-3$ spaces to the right. So the number of e's between w_2 and x decreases by 1 after each BBS move. The configuration at $t = n-4$ is

$$\ldots\,e\,e\,e\,w_1\,w_2\,x\,e\;e\,e\,\ldots\,e\,e\,e\;\underbrace{1\,y_1\,y_2\,\ldots\,y_{n-4}}_{\text{increasing block}}.$$

We claim that

$$x < w_2,$$

which we now prove. Recall that $x = w_j$, where j is the smallest number in $\{3, 4, \ldots, n-1\}$ such that $w_n < w_j$. If $w_2 < w_j$, then $w_1 < w_2 < w_j < w_{j+1} < \cdots < w_{n-1}$ and the remaining w_i's form two increasing subsequences of w whose union is w. This contradicts Lemma 6.4, so indeed $x < w_2$.

Since $x < w_2$, we have either $x < w_1 < w_2$ or $w_1 < x < w_2$. If $x < w_1 < w_2$, then the configuration at $t = n-3$ is

$$\ldots\,e\,e\,w_1\quad\underbrace{x\,w_2}_{\substack{\text{increasing}\\\text{block}}}\quad e\,e\,\ldots\,e\,e\;\underbrace{1\,y_1\,y_2\,\ldots\,y_{n-4}}_{\text{increasing block}}.$$

If $w_1 < x < w_2$, then the configuration at $t = n-3$ is

$$\ldots\,e\,e\,w_2\quad\underbrace{w_1\,x}_{\substack{\text{increasing}\\\text{block}}}\quad e\,e\,\ldots\,e\,e\;\underbrace{1\,y_1\,y_2\,\ldots\,y_{n-4}}_{\text{increasing block}}.$$

Either way, the configuration array at $t = n-3$ is a standard skew tableau whose rows have length $n-3$, 2, and 1. By Proposition 5.2, the configuration at $t = n-3$ is in steady state.

The configuration at $t = n-4$ is not yet in steady-state, as the relative positions of w_1, w_2, and x in the configuration at $t = n-4$ differ from the configuration at $t = n-3$. Therefore, $t = n-3$ is the minimum steady-state time of w. □

7. Knuth moves

We study how types of Knuth moves (Definition 1.3) play a role in a box-ball system. In Section 7.1, we prove that a non-K_B Knuth move preserves the shape of a soliton decomposition and that a K_B move changes it (Theorem 7.1). In Section 7.2, we prove that every permutation which is one non-K_B Knuth move away from a row reading word has steady state time 1 (Theorem 7.4).

7.1. Soliton decompositions are preserved by certain Knuth moves.

Using the localized version of Greene's Theorem given in Section 2.2, we prove a partial characterization of the shape of SD in terms of types of Knuth moves.

Theorem 7.1. Suppose π and w are two permutations in the same Knuth equivalence class.

(1) If π and w are related by a sequence of Knuth moves containing an odd number of K_B moves, then $\mathrm{SD}(\pi) \neq \mathrm{SD}(w)$.

(2) If π and w are related by a sequence of non-K_B Knuth moves, then $\operatorname{sh}\mathrm{SD}(\pi) = \operatorname{sh}\mathrm{SD}(w)$.

Proof. To prove part (1), we observe that a K_B^+ move decreases the number of descents by 1, and a K_B^- move increases the number of descents by 1. Since the height the partition $\operatorname{sh}\mathrm{SD}(w)$ is equal to

$$\mathrm{D}_1(w) = 1 + |\{\text{descents of } w\}|$$

by Lemma 2.6, it follows that applying an odd number of K_B moves to w changes $\operatorname{sh}\mathrm{SD}(w)$.

To prove part (2), suppose $x, y \in S_n$ are related by a K_1 or K_2 move which is not K_B. Due to Lemma 2.6, it suffices to prove that $\mathrm{D}_k(x) = \mathrm{D}_k(y)$ for all k. This breaks down into two main cases: case (i), where $y = K_1^+(x)$, and case (ii), where $y = K_2^+(x)$. These further divide into the following subcases, where $a < b < c$ in all cases:

i.	(a)	$y = \cdots bca$	or	$y = \cdots bcad \cdots$ with $c < d$
		$x = \cdots bac$	or	$x = \cdots bacd \cdots$
	(b)	$y = \cdots bca$	or	$y = \cdots bcaa'$ with $a' < a$
		$x = \cdots bac$	or	$x = \cdots baca'$
ii.	(a)	$y = cab \cdots$	or	$y = \cdots dcab \cdots$ with $c < d$
		$x = acb \cdots$	or	$x = \cdots dacb \cdots$
	(b)	$y = cab \cdots$	or	$y = \cdots a'cab \cdots$ with $a' < a$
		$x = acb \cdots$	or	$x = \cdots a'acb \cdots$

The proofs are similar for each case. We include a partial proof of case (ia). Suppose

$$y = \cdots bca$$
$$x = \cdots bac$$

or

$$y = \cdots bcad \cdots$$
$$x = \cdots bacd \cdots$$

where $a < b < c < d$. The idea is to show that $\mathrm{D}_k(y) \leq \mathrm{D}_k(x)$ and $\mathrm{D}_k(x) \leq \mathrm{D}_k(y)$ for all k, from which the result follows.

Let $k \geq 1$. To show $\mathrm{D}_k(y) \leq \mathrm{D}_k(x)$, suppose that $u_1, \ldots, u_k$ are disjoint subsequences of y such that

$$\mathrm{D}_k(y) = \mathrm{D}(u_1) + \cdots + \mathrm{D}(u_k).$$

We will produce disjoint subsequences $u_1', \ldots, u_k'$ of x where

$$\mathrm{D}(u_1) + \cdots + \mathrm{D}(u_k) \leq \mathrm{D}(u_1') + \cdots + \mathrm{D}(u_k').$$

First, suppose that c and a are in different subsequences. Then set $u_i' := u_i$ for each $1 \leq i \leq k$. Since $\mathrm{D}(u_1) + \cdots + \mathrm{D}(u_k) = \mathrm{D}(u_1') + \cdots + \mathrm{D}(u_k')$, we have $\mathrm{D}_k(y) \leq \mathrm{D}_k(x)$.

Next, suppose that $b, c,$ and a are in the same subsequence u_j of y. Define u_j' to be the subsequence of x which is obtained from u_j by swapping c, a with a, c. Define $u_i' := u_i$ for all $i \neq j$.

Then, since $a < b < c$, we have

$$\mathrm{D}(u_j) = \mathrm{D}(\dots, b, c, a, \dots) \le \mathrm{D}(\dots, b, a, c, \dots) = \mathrm{D}(u'_j),$$

so $\mathrm{D}_k(y) \le \mathrm{D}_k(x)$.

Lastly, suppose that c and a are in the same subsequence, say u_1, and b is in a different subsequence, say u_2. Write u_1 as a concatenation

$$u_1 = \underbrace{(\dots, c)}_{u_1^1} \sqcup \underbrace{(a, \dots)}_{u_1^2}$$

of two subsequences u_1^1 and u_1^2, respectively. Write u_2 as a concatenation

$$u_2 = \underbrace{(\dots, b)}_{u_2^1} \sqcup \underbrace{(\dots)}_{u_2^2}$$

of two subsequences u_2^1 and u_2^2, respectively. Define

$$u'_1 := u_2^1 \sqcup u_1^2 = (\dots, b) \sqcup (a, \dots),$$
$$u'_2 := u_1^1 \sqcup u_2^2 = (\dots, c) \sqcup (\dots),$$

and $u'_i := u_i$ for all $i \notin \{1, 2\}$. Then, since $a < b < c$,

$$\mathrm{D}(u_1) + \mathrm{D}(u_2) \le \mathrm{D}(u'_1) + \mathrm{D}(u'_2),$$

so $\mathrm{D}_k(y) \le \mathrm{D}_k(x)$. The proof of the reverse inequality $\mathrm{D}_k(x) \le \mathrm{D}_k(y)$ is similar. $\square$

Theorem 7.1 allow us to use Knuth moves to find a subset of permutations whose soliton decomposition and RS insertion tableau coincide.

Corollary 7.2 (Corollary of Theorem 4.2 and Theorem 7.1)**.** Let $w \in S_n$ and let $T = \mathrm{P}(w)$.

(1) If w is related to the row reading word of T by a sequence of Knuth moves such that an odd number of the moves are K_B moves, then $\mathrm{SD}(w) \neq \mathrm{P}(w) = T$.

(2) If w is a sequence of K_1 or K_2 moves (but not K_B) away from the row reading word of T, then $\mathrm{SD}(w) = \mathrm{P}(w) = T$.

Example 7.3. The permutation $r = 362514$ is the reading word of the tableau

1	4
2	5
3	6

Figure 3 shows all permutations in the Knuth equivalence class of r. The corresponding soliton decomposition is drawn next to each permutation. The edge with label K_1 (respectively, K_2) indicates that the move is only K_1 and not K_2 (respectively, K_2 and not K_1). An edge with label K_B indicates that the move is both K_1 and K_2. The permutations are arranged in such that they form the Hasse diagram of a subposet of the right weak order[2] on the symmetric group S_6.

7.2. **Permutations one Knuth move away from a reading word with steady-state time 1.**

[2]For definition of the right weak order, see, for example, the textbook [BB05, Section 3.1].

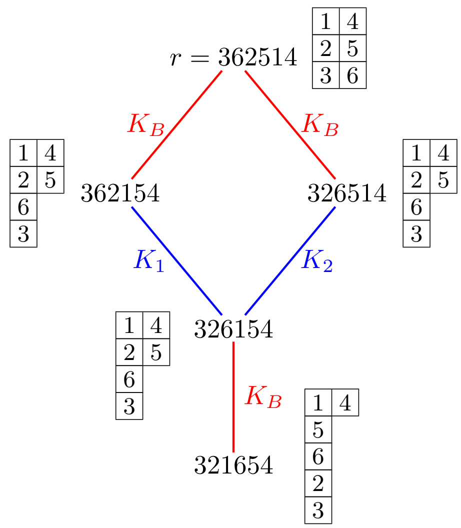

FIGURE 3. Soliton decompositions of the Knuth equivalence class of $r = 362514$

Theorem 7.4. Suppose r is the row reading word of a standard tableau. Let w be a permutation one K_1 or K_2 (but not K_B) move away from r. Then w first reaches its steady state after one BBS move.

If w is one K_B move away from the row reading word r of a standard tableau, then w may first reach its steady state after more than one BBS move. See Example 7.5.

Example 7.5. Figure 4 shows all permutations in the Knuth equivalence class of $r = 362514$ from Example 7.3 and their corresponding steady-state times. The edge with label K_1 (respectively, K_2) indicates that the move is only K_1 and not K_2 (respectively, K_2 and not K_1). An edge with label K_B indicates that the move is both K_1 and K_2.

The permutation 362154 is one K_B^- move from r, and it first reaches steady state at $t = 2$. Another permutation, 326514, is also one K_B^- move from r, and it first reaches steady state at $t = 1$.

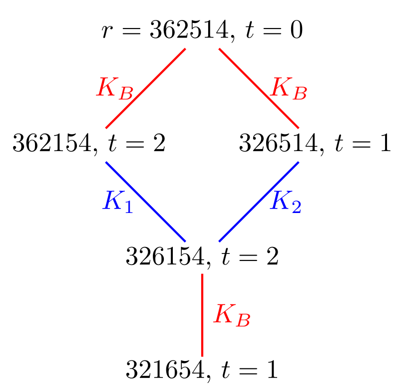

FIGURE 4. Steady-state times of the Knuth equivalence class of $r = 362514$

7.2.1. *Proof of Theorem 7.4.* Theorem 7.4 follows from the following four lemmas.

Lemma 7.6. Let $r = r_1 r_2 \dots r_n$ be the row reading word of a standard tableau P.

(1) If one performs a K_1^- move on r, the move is K_B.
(2) Suppose we are able to perform a K_1^+ move $yxz \mapsto yzx$ (where $x < y < z$) on r. If $r_1 \neq y$, we must have

$$r = \underbrace{r_1 \, \dots \, r_\ell \, y \, x}_{\text{decreasing}} \, z \, \dots \, r_{n-1} \, r_n \tag{7.1}$$

where $r_1 > r_2 > \cdots > r_\ell > y > x$. The tableau P must be of the form given in Figure 5, where the entry y is in its own row, and the row immediately above y starts with entries x, z.

$$\begin{array}{lllll}
\vdots & \cdots & & & \\
\cline{1-3}
\multicolumn{1}{|c|}{a_1} & \multicolumn{1}{c|}{a_2} & \multicolumn{1}{c|}{a_3} & \cdots & \\
\cline{1-4}
\multicolumn{1}{|c|}{x} & \multicolumn{1}{c|}{z} & \multicolumn{1}{c|}{b_1} & \multicolumn{1}{c|}{b_2} & \cdots \\
\cline{1-4}
\multicolumn{1}{|c|}{y} & \multicolumn{4}{l}{\text{(possibly with no } b_i\text{'s)}} \\
\cline{1-1}
\multicolumn{1}{|c|}{r_\ell} & & & & \\
\cline{1-1}
\vdots & & & & \\
\cline{1-1}
\multicolumn{1}{|c|}{r_1} & & & & \\
\cline{1-1}
\end{array}$$

FIGURE 5. General form of a standard tableau P whose row reading word can undergo a K_1^+ move

(3) If one performs a K_1^+ move on r, the move is *not* K_B.

Proof. First, we prove part (1) of the lemma. Suppose we perform a K_1^- move $yzx \mapsto yxz$ (where $x < y < z$) on r. Since r is the row reading word of P, the tableau P must contain a subtableau

$$\begin{array}{|c|c|} \hline x & b \\ \hline y & z \\ \hline \end{array} \qquad \text{or} \qquad \begin{array}{cccc} \cline{1-1} \multicolumn{1}{|c|}{x} & & \cdots & \multicolumn{1}{|c|}{b} \\ \cline{1-1} \cline{3-4} & \cdots & \multicolumn{1}{|c|}{y} & \multicolumn{1}{c|}{z} \\ \cline{3-4} \end{array}.$$

Since the rows and columns of P are increasing, we must have $x < b < z$. Thus, r must contain a consecutive subsequence $yzxb'$ where $x < b' \leq b < z$, so the K_1^- move $yzx \mapsto yxz$ is K_B^-.

Now suppose we perform a K_1^+ move $yxz \mapsto yzx$ on r. First, we prove part (2). Since $x < y < z$ and P is standard, the entry y must be the only element in its row in P, that is, the rows of P containing x, y, z are of the form

$$\begin{array}{|c|c|l} \hline x & z & \cdots \\ \hline y & \multicolumn{2}{l}{} \\ \cline{1-1} \end{array}$$

If $r_1 = y$, then we are done. Suppose $r_1 \neq y$, and write $r = r_1 r_2 \dots r_\ell yxz \dots r_n$. Since the rows of P are weakly decreasing in length, the rows of P below y are of size 1. Since P is standard, we have $r_1 > r_2 > \cdots > r_\ell > y$. So r is of the form given in (7.1) and P is of the form given in Figure 5.

Finally, to prove part (3) of the lemma, we prove that this K_1^+ move is not a K_B move. If $r_n = z$, then we know this K_1^+ move is not K_B. Suppose $r_n \neq z$, so $r = r_1 \dots yxzb \dots r_n$ for some b. Since r is the row reading word of P, either the entry b is immediately above x in P or the entry b is immediately to the right of z in P:

$$
\begin{array}{|c|c}
\hline
b & \cdots \\
\hline
x & \multicolumn{1}{|c|}{z} \\
\hline
y & \multicolumn{1}{|c}{} \\
\cline{1-1}
\end{array}
\qquad \text{or} \qquad
\begin{array}{|c|c|c|c}
\hline
x & z & b & \cdots \\
\hline
y & \multicolumn{1}{|c}{} & & \\
\cline{1-1}
\end{array}
$$

Since P is standard, either $b < x$ or $z < b$. Either way, this K_1^+ move is not K_B. $\square$

Lemma 7.7. Let $r = r_1 r_2 \ldots r_n$ be the row reading word of a standard tableau P.

(1) It is impossible to perform a K_2^+ move on r.

(2) Suppose we are able to perform a K_2^- move $zxy \mapsto xzy$ (where $x < y < z$) which is not a K_B move on r. If $r_1 \neq z$, we have

$$r = \underbrace{r_1 \ldots r_\ell \, z \, x}_{\text{decreasing}} \, y \ldots r_{n-1} \, r_n \tag{7.2}$$

where $r_1 > r_2 > \cdots > r_\ell > z$. The tableau P must be of the form given in Figure 6, where the entry z is in its own row, and the row immediately above z starts with entries x, y.

$$
\begin{array}{|c|c|c|c|c|c|c|}
\multicolumn{1}{c}{\vdots} & \multicolumn{6}{l}{\cdots} \\
\hline
a_1 & a_2 & a_3 & \multicolumn{4}{l}{\cdots} \\
\hline
x & y & b_1 & b_2 & b_3 & \multicolumn{1}{c}{\cdots} & b_m \\
\hline
z & \multicolumn{6}{l}{\quad \text{(possibly with no } b_i\text{'s)}} \\
\cline{1-1}
r_\ell & \multicolumn{6}{l}{} \\
\cline{1-1}
\multicolumn{1}{c}{\vdots} & \multicolumn{6}{l}{} \\
\cline{1-1}
r_2 & \multicolumn{6}{l}{} \\
\cline{1-1}
r_1 & \multicolumn{6}{l}{} \\
\cline{1-1}
\end{array}
$$

FIGURE 6. *General form of a standard tableau P whose row reading word can undergo a K_2^- move which is not K_B*

Proof. First, we prove part (1) of the lemma. Assume (for the sake of contradiction) that one could perform a K_2^+ move on r. Then r must contain a xzy pattern. Hence, since r is the row reading word of P, the tableau P must contain the following subtableau:

$$
\begin{array}{|c|cc}
\cline{1-1}
y & \multicolumn{2}{l}{\cdots} \\
\hline
\multicolumn{1}{c}{} & \multicolumn{1}{|c|}{x} & \multicolumn{1}{c|}{z} \\
\cline{2-3}
\end{array}
$$

Notice that y is north or northwest of x but $x < y$. This is a contradiction to the fact that P is a standard tableau. Therefore, we cannot perform a K_2^+ move on r.

Next, we prove part (2) of the lemma. Suppose we perform a K_2^- move $zxy \mapsto xzy$ on r which is not a K_B move. If $r_1 = z$, then the last two rows of P are of the form

$$
\begin{array}{|c|c|c}
\hline
x & y & \cdots \\
\hline
z & \multicolumn{1}{|c}{} & \\
\cline{1-1}
\end{array}\ ,
$$

so P is of the form given in Figure 6.

Suppose $r_1 \neq z$, and write $r = r_1 \ldots r_\ell\, z\, x\, y \ldots r_{n-1}\, r_n$. Since our K_2^- move is not K_B, we must have either $r_\ell < x$ or $z < r_\ell$. Since P is standard and x is in the first column, we cannot have $r_\ell < x$. So $z < r_\ell$. Therefore z is in its own row in P. Since the rows of P are weakly decreasing in length, the rows of P below z are of size 1. Since P is standard, we have $r_1 > r_2 > \cdots > r_\ell$. So r is of the form given in (7.2) and P is of the form given in Figure 6. ☐

Remark 7.8. In general, a K_2^- move on the row reading word of a standard tableau may (or may not) be K_B.

The proofs of the next two lemmas, Lemmas 7.9 and 7.10, are similar.

Lemma 7.9. Suppose $r = r_1 r_2 \ldots r_n \in S_n$ is the row reading word of a standard tableau P. Let w be a permutation which differs from r by one K_1 move which is not K_B. Then w first reaches its steady state at $t = 1$.

Proof. By Lemma 7.6, applying a K_1 move that is not K_B to r must be a K_1^+ move $yxz \mapsto yzx$ such that

$$r = \underbrace{r_1\, r_2\, \ldots\, r_\ell\, y}_{\text{decreasing}}\, x\, z\, \ldots\, r_{n-1}\, r_n$$

$$w = K_1^+(r) = \underbrace{r_1\, r_2\, \ldots\, r_\ell\, y}_{\text{decreasing}}\, z\, x\, \ldots\, r_{n-1}\, r_n$$

where $r_1 > r_2 > \cdots > r_\ell > y$ (if $r_1 \neq y$) and $x < y < z$.

We apply the carrier algorithm to w. First, we insert $r_1, r_2, \ldots, r_\ell, y$ into the carrier. Since these are decreasing, we eject $e, r_1, r_2, \ldots, r_\ell$ from the carrier in consecutive order:

$$\underbracket{e\, e\, \cdots\, e}_{\text{carrier}}\; \underbrace{r_1\, r_2\, r_3\, \ldots\, r_{\ell-1}\, r_\ell\, y}_{\text{decreasing}}\, z\, x\, \ldots\, r_n$$

$$e\; \underbracket{r_1\, e\, e\, \cdots\, e}\; \underbrace{r_2\, r_3\, \ldots\, r_{\ell-1}\, r_\ell\, y}_{\text{decreasing}}\, z\, x\, \ldots\, r_n$$

$$e\, r_1\; \underbracket{r_2\, e\, e\, \cdots\, e}\; \underbrace{r_3 \ldots\, r_{\ell-1}\, r_\ell\, y}_{\text{decreasing}}\, z\, x\, \ldots\, r_n$$

$$\vdots$$

$$e\, r_1\, r_2\, \ldots\, r_\ell\; \underbracket{y\, e\, e\, \cdots\, e}\; z\, x\, \ldots\, r_n$$

Next, we insert z into the carrier. Since the only non-e entry in the carrier, y, is smaller than z, we eject an e:

$$e\, e\, r_1\, r_2 \ldots r_\ell\, e\; \underbracket{y\, z\, e\, e\, \cdots\, e}\; x\, r_{\ell+4}\, \ldots\, r_n$$

Next, we insert x into the carrier. Since $x < y < z$, we eject y and get

$$e\, e\, r_1\, r_2 \ldots r_\ell\, e\, y\; \underbracket{x\, z\, e\, e\, \cdots\, e}\; r_{\ell+4}\, \ldots\, r_n$$

Note that the string

$$x\, z\, r_{\ell+4}\, \ldots\, r_{n-1}\, r_n$$

is equal to the consecutive subsequence $r_{\ell+2}\ldots r_{n-1}\,r_n$ of r. This string is the row reading word of the subtableau (possibly with no b_i's)

$$\begin{array}{ccccc} \vdots & \cdots & & & \\ \cline{1-3} \multicolumn{1}{|c|}{a_1} & \multicolumn{1}{c|}{a_2} & \multicolumn{1}{c|}{a_3} & \cdots & \\ \hline \multicolumn{1}{|c|}{x} & \multicolumn{1}{c|}{z} & \multicolumn{1}{c|}{b_1} & \multicolumn{1}{c|}{b_2} & \cdots \\ \cline{1-4} \end{array}$$

of P, where P is given in Figure 5. Since this subtableau has the shape of a partition and has increasing rows and columns, completing the carrier algorithm yields the configuration at time $t=1$:

$$e\,e\,r_1\,r_2\ldots r_\ell\,e\,y\,\overbrace{e\,e\,\ldots\,e}^{\substack{\text{0 or more}\\ \text{copies}}}\,x\,z\,b_1\,b_2\,b_3\ \ldots\ a_1\,a_2\ \ldots\ldots r_{n-1}\,r_n\ \underline{e\,e\cdots e}.$$

The configuration array at $t=1$ is the skew tableau created by taking P and shifting some of the rows to the right. Since P is standard tableau with partition shape to begin with, the configuration array is a standard skew tableau with weakly increasing rows. By Proposition 5.2, the configuration at $t=1$ is in steady state. $\square$

Lemma 7.10. Suppose $r=r_1r_2\ldots r_n\in S_n$ is the row reading word of a standard tableau P. Let w be a permutation which differs from r by one K_2 move which is not K_B. Then w first reaches its steady state at $t=1$.

Proof. By Lemma 7.7, applying a K_2 move that is not K_B to r must be a K_2^- move $zxy\mapsto xzy$ to r such that

$$r=r_1\ldots r_\ell\,z\,x\,y\ \ldots\ r_{n-1}\,r_n$$
$$w=K_2^-(r)=r_1\ldots r_\ell\,x\,z\,y\ \ldots\ r_{n-1}\,r_n$$

where $r_1>r_2>\cdots>r_\ell>z$ (if $r_1\neq z$) and $x<y<z$.

As in the proof of Lemma 7.9, we apply the carrier algorithm to w. We insert the decreasing sequence $r_1,r_2,\ldots,r_\ell,x$ into the carrier and eject $e,r_1,r_2,\ldots,r_\ell$, in that order. As we insert z and y, we eject e and z, in that order:

$$\underbrace{e\cdots e}_{\text{carrier}}\ \underbrace{r_1\,r_2\,r_3\,\ldots r_\ell\,x}_{\text{decreasing}}\ z\,y\ \ldots\ r_{n-1}\,r_n$$

$$e\ \underline{r_1\,e\,e\cdots e}\ \underbrace{r_2\,r_3\,\ldots r_\ell\,x}_{\text{decreasing}}\ z\,y\ \ldots\ r_{n-1}\,r_n$$

$$e\,r_1\ \underline{r_2\,e\,e\cdots e}\ \underbrace{r_3\ldots r_\ell\,x}_{\text{decreasing}}\ z\,y\ \ldots\ r_{n-1}\,r_n$$

$$\vdots$$

$$e\,r_1\,r_2\ \ldots\ r_\ell\ \underline{x\,e\ \ldots\ e}\,z\,y\ \ldots\ r_{n-1}\,r_n$$

$$e\,r_1\,r_2\ \ldots\ r_\ell\,e\ \underline{x\,z\,e\ \ldots\ e}\,y\,r_{\ell+4}\ \ldots\ r_n$$

$$e\,r_1\,r_2\ \ldots\ r_\ell\,e\,z\ \underline{x\,y\,e\ \ldots\ e}\,r_{\ell+4}\ \ldots r_n$$

Note that the string

$$x\,y\,r_{\ell+4}\ \ldots\ r_{n-1}\,r_n$$

is equal to the consecutive subsequence $r_{\ell+2}\ldots r_{n-1}\,r_n$ of r. This string is the row reading word of the subtableau (possibly with no b_i's)

$$
\begin{array}{l}
\vdots \quad \cdots \\
\begin{array}{|c|c|c|l}
\hline
a_1 & a_2 & a_3 & \cdots \\
\hline
x & y & b_1 & \multicolumn{1}{|c|}{b_2} \; \cdots \\
\hline
\end{array}
\end{array}
$$

of P, where P is given in Figure 6. Since this subtableau has the shape of a partition and has increasing rows and columns, completing the carrier algorithm yields the configuration at time $t = 1$:

$$
e\,e\,r_1\,r_2\ldots r_\ell\,e\,z\;\overbrace{e\,\ldots\,e}^{\substack{\text{0 or more}\\ \text{copies}}}\;x\,y\;b_1\,b_2\,b_3\;\cdots\;a_1\,a_2\;\ldots\;r_{n-1}\,r_n\;\underbrace{e\,e\,\cdots\,e}\,.
$$

The configuration array at $t = 1$ is the skew tableau created by taking P and shifting some of the rows to the right. Since P is standard tableau with partition shape to begin with, the configuration array is a standard skew tableau with weakly increasing rows. By Proposition 5.2, the configuration at $t = 1$ is in steady state. $\square$

8. M-carrier algorithm

In Algorithm 2, we define the *M-carrier algorithm* which is equivalent to performing the carrier algorithm M times (Proposition 8.2). In addition to improving the efficiency of the box-ball system calculations, the M-carrier algorithm enables us to compare the RS insertion algorithm and the box-ball system more directly. Given a large enough M, the M-carrier algorithm gives us an RS-like insertion algorithm which sends a permutation to its soliton decomposition.

Example 8.1. We apply the M-carrier algorithm with $M = 3$ to one of our running examples $w = 452361$, the permutation whose box-ball system is illustrated in Figure 2.

begin Process 1: insertion process

$$
\begin{array}{r}
\underbrace{eeeeee}\,\underbrace{eeeeee}\,\underbrace{eeeeee}\,452361 \\
e\,\underbrace{eeeeee}\,\underbrace{eeeeee}\,\underbrace{4eeeee}\,452361 \\
ee\,\underbrace{eeeeee}\,\underbrace{eeeeee}\,\underbrace{45eeee}\,2361 \\
eee\,\underbrace{eeeeee}\,\underbrace{4eeeee}\,\underbrace{25eeee}\,361 \\
eeee\,\underbrace{eeeeee}\,\underbrace{45eeee}\,\underbrace{23eeee}\,61 \\
eeeee\,\underbrace{4eeeee}\,\underbrace{5eeeee}\,\underbrace{236eee}\,1 \\
eeeeee\,\underbrace{45eeee}_{\substack{\text{carrier}\\ M=3}}\,\underbrace{2eeeee}_{\text{carrier 2}}\,\underbrace{136eee}_{\text{carrier 1}}
\end{array}
$$

end insertion process

begin Process 2: flushing process

$$
\begin{array}{r}
eeeeee\,\underbrace{45eeee}\,\underbrace{2eeeee}\,\underbrace{136eee} \leftarrow e \\
eeeeee4\,\underbrace{25eeee}\,\underbrace{1eeeee}\,\underbrace{36eeee} \leftarrow e \\
eeeeee42\,\underbrace{5eeeee}\,\underbrace{13eeee}\,\underbrace{6eeeee} \leftarrow e \\
eeeeee425\,\underbrace{eeeeee}\,\underbrace{136eee}\,\underbrace{eeeeee} \leftarrow e \\
eeeeee425e\,\underbrace{1eeeee}\,\underbrace{36eeee}\,\underbrace{eeeeee} \leftarrow e \\
eeeeee425ee\,\underbrace{13eeee}\,\underbrace{6eeeee}\,\underbrace{eeeeee} \leftarrow e \\
eeeeee425eee\,\underbrace{136eee}\,\underbrace{eeeeee}\,\underbrace{eeeeee} \leftarrow e \\
eeeeee425eee1\,\underbrace{36eeee}\,\underbrace{eeeeee}\,\underbrace{eeeeee} \leftarrow e \\
eeeeee425eee13\,\underbrace{6eeeee}\,\underbrace{eeeeee}\,\underbrace{eeeeee} \leftarrow e \\
eeeeee425eee136\,\underbrace{eeeeee}\,\underbrace{eeeeee}\,\underbrace{eeeeee}
\end{array}
$$

end flushing process

Algorithm 2 The M-carrier algorithm

1: **begin** M-carrier algorithm
2: Set $e \coloneqq n+1$
3: Set $B \coloneqq$ the configuration of the BBS at time t, where each empty box is denoted by the letter e and the first (leftmost) element of B is the integer in the first (leftmost) nonempty box in the configuration and the last (rightmost) element of B is the integer in the last (rightmost) nonempty box of the configuration
4: Let ℓ be the number of elements (including the e's) of B
5: Fill each of the M adjacent "carriers"—depicted $\sqcup \; \sqcup \cdots \sqcup$ — with n copies of e
6: Denote this string of carriers $\mathcal{C}$
7: Denote the rightmost carrier c_1, and in general, the j^{th} rightmost carrier c_j.
8: Write B to the right of $\mathcal{C}$
9: **begin** Process 1: insertion process
10: **for all** i in $\{1, 2, \ldots, \ell\}$ **do**
11: Set p to be the i^{th} leftmost element of B
12: **begin** element ejection process
13: **for all** j in $\{1, 2, \ldots, M\}$ **do**
14: **if** an element in c_j is larger than p **then**
15: Set $s \coloneqq$ the smallest element in c_j larger than p. If $s = e$, pick the first e
16: Eject s by replacing it with p and setting $p \coloneqq s$
17: **else**
18: Set $s \coloneqq$ the smallest element in c_j
19: Remove s from c_j
20: ▶ Note: There are now $n-1$ elements in c_j
21: Place p in the rightmost location in c_j
22: ▶ Note: There are now n elements in c_j
23: Set $p \coloneqq s$
24: **end if**
25: **if** $j = M$ **then**
26: Put p immediately to the left of $\mathcal{C}$
27: **end if**
28: **end for**
29: **end** element ejection process
30: **end for**
31: **end** Process 1: insertion process
32: **begin** Process 2: flushing process
33: **while** there are non-e elements in $\mathcal{C}$ **do**
34: Set $p \coloneqq e$
35: Perform the element ejection process (see line 12)
36: **end while**
37: **end** Process 2: flushing process
38: ▶ Note: The elements to the left of $\mathcal{C}$ correspond to the state of the BBS at time $t + M$
39: **end** M-carrier algorithm

Proposition 8.2. Performing the M-carrier algorithm is equivalent to performing the 1-carrier algorithm (Algorithm 1) M times. In particular, applying Algorithm 2 to a box-ball configuration at time t yields the box-ball configuration of at $t + M$.

Proof. Ejecting an element from a carrier c_i and then immediately inserting it into the next carrier c_{i+1} is equivalent to ejecting all the elements from c_i, forming a sequence and then inserting that sequence into c_{i+1}. $\square$

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