CORRIGENDUM TO "ON TWO CONJECTURES CONCERNING CONVEX CURVES", BY V. SEDYKH AND B. SHAPIRO

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Abstract. As was pointed out by S. Karp, Theorem B of paper [8] is wrong. Its claim is based on an erroneous example obtained by multiplication of three concrete totally positive 4×4 upper-triangular matrices, but the order of multiplication of matrices used to produce this example was not the correct one. Below we present a right statement which claims the opposite to that of Theorem B. Its proof can be essentially found in a recent paper [1].

1. Introduction

Recall that a classical result due to H. Schubert, [7] claims that for a generic $(k+1)(n-k)$-tuple of k-dimensional complex subspaces in $\mathbb{C}P^n$ there exist $\sharp_{k,n} = \frac{1!2!\dots(n-k-1)!((k+1)(n-k))!}{(k+1)!(k+2)!\dots(n)!}$ complex projective subspaces of dimension $(n-k-1)$ in $\mathbb{C}P^n$ intersecting each of the above k-dimensional susbspaces. (The number $\sharp_{k,n}$ is the degree of the Grassmannian of projective k-dimensional subspaces in $\mathbb{C}P^n$ considered as a projective variety embedded using Plücker coordinates.) The following conjecture has been formulated in early 1990's by the authors (unpublished); it has been proven in two fascinating papers [3, 2] some years ago. (Recently two novel and very different proofs of these results have been in [4] and [6]).

Conjecture on total reality. For the real rational normal curve $\rho_n : S^1 \to \mathbb{R}P^n$ and any $(k+1)(n-k)$-tuple of pairwise distinct real projective k-dimensional osculating subspaces to ρ_n, there exist $\sharp_{k,n}$ real projective subspaces of dimension $(n-k-1)$ in $\mathbb{R}P^n$ intersecting each of the above osculating subspaces.

Many discussions and further results related to the latter conjecture can be found in [9].

Originally, the authors suspected that the latter conjecture were also valid for convex curves and not just for the rational normal curve where a curve $\gamma : S^1 \to \mathbb{R}P^n$ (resp. $\gamma : [0,1] \to \mathbb{R}P^n$) is called *convex* if any hyperplane $H \subset \mathbb{R}P^n$ intersects γ at most n times counting multiplicities. (Discussions of various properties of convex curves can be found in a number of earlier papers by the authors as well as in other publications). In particular, at each point of a convex curve γ there exists a well-defined Frenet frame and therefore a well-defined osculating k-dimensional subspace for any $k = 1, \dots, n-1$.

Theorem B of [8] erroneously claims that there exists a convex curve in $\mathbb{R}P^3$ and a 4-tuple of its tangent lines such that there are no real lines intersecting all of them. (In this case $k = 1, n = 3$ and $\sharp_{1,3} = 2$). The correct statement is as follows.

Theorem 1. *For any convex curve $\gamma : S^1 \to \mathbb{R}P^3$ (resp. $\gamma : [0,1] \to \mathbb{R}P^3$) and any 4-tuple of its tangent lines $\mathcal{L} = (\ell_1, \ell_2, \ell_3, \ell_4)$, there exist two real distinct lines L_1 and L_2 intersecting each line in $\mathcal{L}$.*

In other words, Theorem 1 claims that total reality conjecture is valid in the special case $k = 1, n = 3$ for convex curves as well. Its proof follows straight forwardly from the next result of [1]. (We want to thank S. Karp for providing us the formulation and the proof of this statement.)

Theorem 2. *Let W_i, $i = 1,2,3,4$ be 4×2 real matrices, such that the 4×8 matrix formed by concatenating W_1, W_2, W_3, and W_4 has all its 4×4 minors positive. Then regarding each W_i as an element of the real Grassmannian $Gr_{2,4}(\mathbb{R})$, there exist two distinct $U \in Gr_{2,4}(\mathbb{R})$ such that $U \cap W_i \neq \emptyset$ for $i = 1,2,3,4$.*

Proof. Let $A := [W_1\ W_2\ W_3\ W_4]$ be the 4×8 matrix formed by concatenating W_1, W_2, W_3, and W_4. After acting on $\mathbb{R}^4$ by an element of a $GL_4(\mathbb{R})$ with positive determinant, we may assume that $A = [X\ Y]$, where X is a 4×4 totally positive matrix and

$$Y = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

Then $X = [W_1\ W_2]$ and $Y = [W_3\ W_4]$. Set

$$U := \begin{pmatrix} 1 & 0 \\ -x & 0 \\ 0 & -1 \\ 0 & y \end{pmatrix} \quad (x, y \in \mathbb{R}).$$

Then inside $Gr_{2,4}(\mathbb{R})$, we have $U \cap W_3 \neq \emptyset$ and $U \cap W_4 \neq \emptyset$. Also, we have $U \cap W_1 \neq \emptyset$ and $U \cap W_2 \neq \emptyset$ if and only if

$$\det[W_1\ U] = 0 \text{ and } \det[W_2\ U] = 0.$$

These conditions give the following two equations:

$$\Delta_{13,12}xy+\Delta_{14,12}x+\Delta_{23,12}y+\Delta_{24,12} = 0 \text{ and } \Delta_{13,34}xy+\Delta_{14,34}x+\Delta_{23,34}y+\Delta_{24,34} = 0,$$

where $\Delta_{I,J}$ denotes the determinant of the submatrix of X in rows I and columns J. Using the second equation to solve for y in terms of x and substituting into the first equation, we obtain a quadratic equation in x whose discriminant equals

$$D = (\Delta_{13,12}\Delta_{24,34} - \Delta_{24,12}\Delta_{13,34} - \Delta_{14,12}\Delta_{23,34} + \Delta_{23,12}\Delta_{14,34})^2$$
$$-4(\Delta_{13,12}\Delta_{14,34} - \Delta_{14,12}\Delta_{13,34})(\Delta_{23,12}\Delta_{24,34} - \Delta_{24,12}\Delta_{23,34}).$$

To settle Theorem 2 it suffices to show that under our assumptions $D > 0$.

Since X is totally positive, by the LoewnerWhitney theorem [5, 10] we can write

$$X = \begin{pmatrix} 1 & 0 & 0 & 0 \\ g+j+l & 1 & 0 & 0 \\ hj+hl+kl & h+k & 1 & 0 \\ ikl & ik & k & 1 \end{pmatrix} \cdot \begin{pmatrix} m & 0 & 0 & 0 \\ 0 & n & 0 & 0 \\ 0 & 0 & o & 0 \\ 0 & 0 & 0 & p \end{pmatrix} \cdot\!\!\cdot \begin{pmatrix} 1 & f+d+a & ab+ae+de & abc \\ 0 & 1 & b+e & bc \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where $a, \dots, p > 0$. Then we calculate

$$D = m^2n^2(FG + H^2),$$

where

$F = acehijmo + acehilmo + 2cdehijmo + cdehilmo + abhjmp + abhlmp + abklmp +$

$aehjmp+aehlmp+aeklmp+cehino+dehjmp+dehlmp+deklmp+bhnp+2bknp+ehnp+eknp,$

$$G = acehijmo+acehilmo+cdehilmo+abhjmp+abhlmp+abklmp+aehjmp+aehlmp+ aeklmp + cehino + dehjmp + dehlmp + deklmp + bhnp + ehnp + eknp,$$

$$H = bknp - cdehijmo.$$

Since F and G are positive in case when $a, \dots, p > 0$ we get that $D > 0$. □

In order to deduce Theorem 1 from Theorem 2 we need the following Lemma.

Lemma 3. *For any convex curve $\gamma : S^1 \to \mathbb{R}P^3$ (resp. $\gamma : [0, 1] \to \mathbb{R}P^3$) and any 4-tuple of its tangent lines $\mathcal{L} = (\ell_1, \ell_2, \ell_3, \ell_4)$, there exists a basic e_1, e_2, e_3, e_4 in $\mathbb{R}^4$ where $\mathbb{R}P^3 = (\mathbb{R}^4 \setminus 0)/\mathbb{R}^*$ and bases in the 2-dimensional subspace $\tilde{\ell}_1, \tilde{\ell}_2, \tilde{\ell}_3, \tilde{\ell}_4$ of $\mathbb{R}^4$ covering $\ell_1, \ell_2, \ell_3, \ell_4$ resp. such that the 4×2 matrices W_1, W_2, W_3, W_4 expressing the chosen bases of $\tilde{\ell}_1, \tilde{\ell}_2, \tilde{\ell}_3, \tilde{\ell}_4$ w.r.t e_1, e_2, e_3, e_4 satisfy the assumptions of Theorem 2.*

Proof. Notice that given a convex curve $\gamma : S^1 \to \mathbb{R}P^3$ (resp. $\gamma : [0, 1] \to \mathbb{R}P^3$) as above, one can always find its lift $\tilde{\gamma} : S^1 \to \mathbb{R}^4 \setminus 0$ (resp. $\tilde{\gamma} : [0, 1] \to \mathbb{R}^4 \setminus 0$) such that the projectivization map $\mathbb{R}P^3 = (\mathbb{R}^4 \setminus 0)/\mathbb{R}^*$ sends $\tilde{\gamma}$ to γ. Since γ is convex, the lift $\tilde{\gamma}$ satisfies the property that any linear hyperplane $H \subset \mathbb{R}^4$ intersects $\tilde{\gamma}$ at most 4 times counting multiplicities.

Now set $e_j = \tilde{\gamma}^{(j-1)}(0),\ j = 1, 2, 3, 4$ where $\tilde{\gamma}^{(s)}$ stands for the derivative of $\tilde{\gamma}$ of order s considered as a vector function with values in $\mathbb{R}^4$. By convexity, the vectors e_1, e_2, e_3, e_4 are linearly independent and therefore form a basis in $\mathbb{R}^4$. In what follows we consider coordinates in $\mathbb{R}^4$ with respect to the basis $\{e_j\}$.

The Wronski matrix of $\tilde{\gamma}$ at $t = 0$ written in these coordinates coincides with the identity matrix and therefore has determinant 1. In particular, this implies that the determinant of the 4×4 matrix whose rows are given by the coordinates of a 4-tuple of vectors $\tilde{\gamma}(\delta_i)$ in the latter basis where $0 \le \delta_1 < \delta_2 < \delta_3 < \delta_4 < \delta$ with sufficiently small δ is positive. Furthermore, by definition of convexity, the determinant of the 4×4 matrix with rows $\tilde{\gamma}(\theta_i),\ i = 1, 2, 3, 4$ does not vanish for any 4-tuple $0 \le \theta_1 < \theta_2 < \theta_3 < \theta_4 \le 1$. Thus, this determinant is positive since its value is close to 1 for sufficiently small θ_i's.

Thus all 4×4 minors of the matrix $U = (U_{i,j})_{\substack{1 \le i \le 8 \\ 1 \le j \le 4}}$, where $U_{i,j} = \tilde{\gamma}_j(t_i)$ are positive for any choice $0 \le t_1 < t_2 < t_3 < t_4 < t_5 < t_6 < t_7 < t_8 \le 1$. Choosing $0 < t_1 < t_3 < t_5 < t_7 < 1$ arbitrarily, set $t_{2i} = t_{2i-1} + \varepsilon$ for $i = 1, 2, 3, 4$ where ε is sufficiently small. Notice that $\tilde{\gamma}(t_{2i}) = \tilde{\gamma}(t_{2i-1}) + \varepsilon\tilde{\gamma}'(t_{2i-1}) + o(\varepsilon)$.

Now introduce the 8-tuple of vectors $\mathbf{w}_i$, where $\mathbf{w}_{2k-1} = \tilde{\gamma}(t_{2k-1}), k = 1, 2, 3, 4,$ and $\mathbf{w}_{2k} = \tilde{\gamma}(t_{2k-1}) + \varepsilon\gamma'(t_{2k-1})$. Define the 8×4 matrix $W = (W_{i,j})$, where $W_{ij} = (\mathbf{w}_i)_j$.

Then for any ordered index set $I = \{1 \le i_1 < i_2 < i_3 < i_4 \le 8\}$, let $U_I$and W_I denote the determinants of submatrices of U and W respectively formed by rows indexed by I.

Define $\varkappa_k = \begin{cases} 1, & \text{if } \{2k-1, 2k\} \subset I \\ 0, & \text{otherwise.} \end{cases}$, and $\varkappa_I := \sum_{k=1}^4 \varkappa_k$.

Obviously, $W_I = O(\varepsilon^{\varkappa_I})$ and $U_I = W_I + o(\varepsilon^{\varkappa_I})$. As we have noticed above, U_I's are positive for all index sets I which yields that all W_I's are positive as well if ε is sufficiently small. It remains to notice that matrix W satisfies the conditions of Theorem 2 and it consists of the 4-tuple of pairs of vectors spanning the 2-dimensional subspaces $\tilde{\ell}_1, \tilde{\ell}_2, \tilde{\ell}_3, \tilde{\ell}_4$ respectively. □

Problem 1. *Prove or disprove the total reality conjecture for convex curves for other values of parameters k and n.*

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