A Varadhan estimate for big order differential generators

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1 Introduction

Let us consider a compact Riemannian manifold M of dimension d endowed with its normalized Riemannian measure dx $(x \in M)$. Associated to it, we consider the Laplace-Beltrami operator Δ and the heat semi-group associated to it P_t

$$\frac{\partial}{\partial t} P_t f = -\Delta/2 P_t f \tag{1}$$

if f is a smooth function on M and $P_0(f) = f$.

The heat semi-group is represented by an heat-kernel if $t > 0$:

$$P_t f(x) = \int_M p_t(x,y) f(y) dy \tag{2}$$

where $(x,y) \to p_t(x,y)$ is smooth positive.

Associated to the Riemannian structure, we consider the Riemannian distance $(x,y) \to d_R(x,y)$ which is continuous positive. Varadhan's type estimate state that

$$lim_{t\to 0} 2t \log p_t(x,y) = -d_R^2(x,y) \tag{3}$$

For a subelliptic operator, we can consider the essociated semi-group. By Hoermander's theorem, there is still an heat-kernel. There is the generalization in this case of the Riemannian distance called the Sub-Riemannian distance $d^2_{S.R}(x,y)$ which is still continuous positive finite. Under some technical condition by using a mixture of the Malliavin Calculus and large deviation estimates we have shown in [3]

$$\overline{\lim}_{t\to 0} 2t\log p_t(x,y) \leq -d^2_{S.R}(x,y) \tag{4}$$

Our goal is to repeat the strategy of [3] for non-markovian semi-groups. We consider some vector fields X_i smooth without divergence on the manifold and we consider the operator

$$L = (-1)^k \sum_{i=1}^m X_i^{2k} \tag{5}$$

for some strictly positive integer k. We suppose that at each point x of M, the vectors fields spann the tangent space $T_x(M)$. In such case the operator is elliptic positive symmetric. By abstract theory [2], it admits a self-adjoint extension and is essentially self-adjoint.

We can consider the heat semi-group associated to it

$$\frac{\partial}{\partial t} P_t^L f = -\Delta^L P_t f \tag{6}$$

if f is a smooth function on M and $P_0^L(f) = f$. The main difference with the case of the Laplacian is that the semi-group does-not preserve positivity. Classically in analysis, the semi group P_t^L has an heat kernel $p_t^L(x,y)$ which changes of sign [2]. We have shown this result by using this result by using the tools of the Malliavin Calculus for non-markovian semi-groups (See [6] for a review).

To L is associated an Hamiltonian H. It is an application of $T^*(M)$, the cotangent bundle of M given by if $\xi \in T_x^*(M)$

$$(x,\xi) \to \sum_{i=1}^m < X_i(x), \xi >^{2k} \tag{7}$$

Due to the hypothesis of ellipticity, we have

$$|H(x,\xi)| \geq |\xi|^{2k} \tag{8}$$

According to the theory of large deviation, we introduce the Lagrangian associated to it. It is a function from $T(M)$, the tangent bundle of M into $\mathbb{R}$

$$L(x,p) = \sup_\xi (< \xi, p > -H(x,\xi)) \tag{9}$$

$(x \in T_x(M); \xi \in T_x^*(M))$. L is continuous positive. If $t \to \gamma(t)$ is a finite energy curve on M, we define its action

$$S(\gamma) = \int_0^1 L(\gamma(t), d/dt\gamma(t))dt \tag{10}$$

and we put

$$l(x,y) = \inf_{\gamma(0)=x;\gamma(1)=y} S(\gamma) \tag{11}$$

By standard method, due to the estimate (8), $(x, y) \to l(x, y)$ is continuous.

The goal of this note is to show a Varadhan type estimate for $p_t^L(x, y)$:

Theorem 1 *When $t \to 0$, we have uniformly*

$$\overline{\lim} t^{\frac{1}{2k-1}} \log |p_t^L(x, y)| \leq -l(x, y) \tag{12}$$

This estimate has to be compare with the standard estimates of harmonic-analysis (See for instance [1]). We adapt the method of [3] in this non-markovian context. Let us remark that we have already got similar estimates in [4], [5], [6] for rightinvariant elleptic differential operators on a compact Lie group, by mixing tools of the Malliavin Calculus for non-markovian semi-groups and Wentzel-Freidlin estimates for non-markovian semi groups.

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2 Proof of the theorem

Since L is symmetric,

$$\int_M f(x) P_t^l g(x) dx = \int_M g(x) P_t^l f(x) dx \tag{13}$$

and therefore $p_t^L(x, y) = p_t^L(y, x)$.

Let us recall some results of [6]. In part 4 of [6], we shown by using the intrinsic Malliavin Calculus on the semi-group generated by L that:

$$|p_t^L(x, y)| \leq \frac{C}{t^l} \tag{14}$$

for $t \leq 1$ uniformly in (x, y).

Moreover in part 5 of [6] we have shown if O is an open ball uniformy in x

$$\overline{\lim} t^{\frac{1}{2k-1}} \log |P_t^L|[1_0](x) \leq - \inf_{y \in O} l(x, y) \tag{15}$$

This means in other words that

$$\int_O |p_t^L(x, y)| dy \leq \exp[\frac{-\inf_{y \in O} l(x, y) + \eta}{t^{\frac{1}{2k-1}}}] \tag{16}$$

We have shown in [6] lemma 9 the following result. For all δ, all C, there exists s_δ such that if $s \leq s_\delta$

$$|P_{st}^L|[1_{B(x,\delta)^c}](x) \leq \exp[-\frac{C}{t^{\frac{1}{2k-1}}}] \tag{17}$$

This means

$$\int_{B(x,\delta)^c} |p_{st}^L|(x, y) dy \leq \exp[-\frac{C}{t^{\frac{1}{2k-1}}}] \tag{18}$$

The two previous estimates are uniform. (14), (15) and (18) will allow to conclude. By the semi-group property

$$p_t^L(x,y) = \int_M p_{(1-s)t}^L(x,z)p_{st}^L(z,y)dz \tag{19}$$

We deduce that

$$|p_t^L(x,y)| \leq A + B \tag{20}$$

with

$$A = \int_{B(y,\delta)} |p_{(1-s)t}^L(x,z)||p_{st}^L(z,y)|dz \tag{21}$$

and

$$B = \int_{B(y,\delta)^c} |p_{(1-s)t}^L(x,z)||p_{st}^L(z,y)|dz \tag{22}$$

If s is small enough,

$$B \leq \frac{C}{((1-s)t)^l} \exp[-\frac{C}{t^{\frac{1}{2k-1}}}] \tag{23}$$

If t is small enough,

$$A \leq \frac{C}{(st)^l} \exp[-\frac{\inf_{z\in B(y,\delta)} l(x,z) + \eta}{t^{\frac{1}{2k-1}}}] \tag{24}$$

The conclusion holds if we choose δ very small such that $\inf_{z\in B(y,\delta)} l(x,z)$ is close from $l(x,y)$ because $(x,z) \to l(x,z)$ is continuous.

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