1 Introduction

Let us consider a compact Riemannian manifold $M$ of dimension $d$ endowed with its normalized Riemannian measure $dx$ ($x \in M$). Associated to it, we consider the Laplace-Beltrami operator $\Delta$ and the heat semi-group associated to it $P_t$

$$\frac{\partial}{\partial t} P_t f = -\Delta/2P_t f$$

if $f$ is a smooth function on $M$ and $P_0(f) = f$.

The heat semi-group is represented by an heat-kernel if $t > 0$:

$$P_t f(x) = \int_M p_t(x, y)f(y)dy$$

where $(x, y) \rightarrow p_t(x, y)$ is smooth positive.

Associated to the Riemannian structure, we consider the Riemannian distance $(x, y) \rightarrow d_R(x, y)$ which is continuous positive. Varadhan’s type estimate state that

$$\lim_{t \rightarrow 0} 2t \log p_t(x, y) = -d_R^2(x, y)$$

For a subelliptic operator, we can consider the associated semi-group. By Hörmander’s theorem, there is still an heat-kernel. There is the generalization in this case of the Riemannian distance called the Sub-Riemannian distance $d^2_{S,R}(x, y)$ which is still continuous positive finite. Under some technical condition by using a mixture of the Malliavin Calculus and large deviation estimates
we have shown in [3]
\[ \lim_{t \to 0} 2t \log p_t(x, y) \leq -d^2_{S.R}(x, y) \] (4)

Our goal is to repeat the strategy of [3] for non-markovian semi-groups. We consider some vector fields \( X_i \) smooth without divergence on the manifold and we consider the operator
\[ L = (-1)^k \sum_{i=1}^{m} X_i^{2k} \] (5)

for some strictly positive integer \( k \). We suppose that at each point \( x \) of \( M \), the vectors fields span the tangent space \( T_x(M) \). In such case the operator is elliptic positive symmetric. By abstract theory [2], it admits a self-adjoint extension and is essentially self-adjoint.

We can consider the heat semi-group associated to it
\[ \frac{\partial}{\partial t} P^L_t f = -\Delta^L P_t f \] (6)

if \( f \) is a smooth function on \( M \) and \( P^L_0(f) = f \). The main difference with the case of the Laplacian is that the semi-group does-not preserve positivity. Classically in analysis, the semi group \( P^L_t \) has an heat kernel \( p^L_t(x, y) \) which changes of sign [2]. We have shown this result by using this result by using the tools of the Malliavin Calculus for non-markovian semi-groups (See [6] for a review).

To \( L \) is associated an Hamiltonian \( H \). It is an application of \( T^*(M) \), the cotangent bundle of \( M \) given by if \( \xi \in T^*_x(M) \)
\[ (x, \xi) \to \sum_{i=1}^{m} < X_i(x), \xi >^{2k} \] (7)

Due to the hypothesis of ellipticity, we have
\[ |H(x, \xi)| \geq |\xi|^{2k} \] (8)

According to the theory of large deviation, we introduce the Lagrangian associated to it. It is a function from \( TM \), the tangent bundle of \( M \) into \( \mathbb{R} \)
\[ L(x, p) = \sup_{\xi} < \xi, p > -H(x, \xi) \] (9)

\((x \in T_x(M); \xi \in T^*_x(M))\). \( L \) is continuous positive. If \( t \to \gamma(t) \) is a finite energy curve on \( M \), we define its action
\[ S(\gamma) = \int_0^1 L(\gamma(t), d/dt\gamma(t))dt \] (10)

and we put
\[ l(x, y) = \inf_{\gamma(0)=x; \gamma(1)=y} S(\gamma) \] (11)
By standard method, due to the estimate (8), \((x, y) \to l(x, y)\) is continuous.

The goal of this note is to show a Varadhan type estimate for \(p^L_t(x, y)\):

**Theorem 1** When \(t \to 0\), we have uniformly

\[
\lim_{t \to 0} \frac{1}{t^{1/2}} \log |p^L_t(x, y)| \leq -l(x, y) \tag{12}
\]

This estimate has to be compared with the standard estimates of harmonic-analysis (See for instance [1]). We adapt the method of [3] in this non-markovian context. Let us remark that we have already obtained similar estimates in [4], [5], [6] for right-invariant elliptic differential operators on a compact Lie group, by mixing tools of the Malliavin Calculus for non-markovian semi-groups and Wentzel-Freidlin estimates for non-markovian semi groups.

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## 2 Proof of the theorem

Since \(L\) is symmetric,

\[
\int_M f(x)P^l_t g(x)dx = \int_M g(x)P^l_t f(x)dx \tag{13}
\]

and therefore \(p^L_t(x, y) = p^L_t(y, x)\).

Let us recall some results of [6]. In part 4 of [6], we shown by using the intrinsic Malliavin Calculus on the semi-group generated by \(L\) that:

\[
|p^L_t(x, y)| \leq \frac{C}{t^{1/2}} \tag{14}
\]

for \(t \leq 1\) uniformly in \((x, y)\).

Moreover in part 5 of [6] we have shown if \(O\) is an open ball uniformy in \(x\)

\[
\lim_{t \to 0} \frac{1}{t^{1/2}} \log |P^L_t[1_{O}](x)| \leq - \inf_{y \in O} l(x, y) \tag{15}
\]

This means in other words that

\[
\int_O |p^L_t(x, y)|dy \leq \exp\left[-\inf_{y \in O} l(x, y) + \frac{C}{t^{1/2}}\right] \tag{16}
\]

We have shown in [6] lemma 9 the following result. For all \(\delta\), all \(C\), there exists \(s_\delta\) such that if \(s \leq s_\delta\)

\[
|P^L_s[1_{B(x, \delta)}'](x)| \leq \exp\left[-\frac{C}{t^{1/2}}\right] \tag{17}
\]

This means

\[
\int_{B(x, \delta)} |P^L_s(x, y)|dy \leq \exp\left[-\frac{C}{t^{1/2}}\right] \tag{18}
\]
The two previous estimates are uniform. (14), (15) and (18) will allow to conclude. By the semi-group property

\[ p^L_t(x, y) = \int_M p^L_{(1-s)t}(x, z)p^L_{st}(z, y)dz \]  

(19)

We deduce that

\[ |p^L_t(x, y)| \leq A + B \]

(20)

with

\[ A = \int_{B(y, \delta)} |p^L_{(1-s)t}(x, z)||p^L_{st}(z, y)|dz \]  

(21)

and

\[ B = \int_{B(y, \delta)} |p^L_{(1-s)t}(x, z)||p^L_{st}(z, y)|dz \]  

(22)

If \( s \) is small enough,

\[ B \leq \frac{C}{((1-s)t)^t} \exp\left[-\frac{C}{t^{\frac{1}{m-1}}} \right] \]

(23)

If \( t \) is small enough,

\[ A \leq \frac{C}{(st)^t} \exp\left[-\frac{\inf_{z \in B(y, \delta)} l(x, z) + \eta}{t^{\frac{1}{m-1}}} \right] \]

(24)

The conclusion holds if we choose \( \delta \) very small such that \( \inf_{z \in B(y, \delta)} l(x, z) \) is close from \( l(x, y) \) because \( (x, z) \rightarrow l(x, z) \) is continuous.

References


