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GLOBAL WELL-POSEDNESS FOR THE THERMODYNAMICALLY REFINED PASSIVELY TRANSPORTED NONLINEAR MOISTURE DYNAMICS WITH PHASE CHANGES

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Abstract. In this work we study the global solvability of moisture dynamics with phase changes for warm clouds. We thereby in comparison to previous studies [15] take into account the different gas constants for dry air and water vapor as well as the different heat capacities for dry air, water vapor and liquid water, which leads to a much stronger coupling of the moisture balances and the thermodynamic equation. This refined thermodynamic setting has been demonstrated to be essential e.g. in the case of deep convective cloud columns in [14]. The more complicated structure requires careful derivations of sufficient a priori estimates for proving global existence and uniqueness of solutions.

1. Introduction

Precipitation causes one of the major uncertainties in weather forecast and climate modelling and thus also the incorporation of moisture and phase changes into atmospheric flow models is still actively debated, see e.g. [1]. In the framework of systematically derived reduced mathematical models for atmospheric dynamics, it has been shown that not only the inclusion of moist processes alone, but also the detailed structure of the moist-air submodels can decisively affect the overall flow dynamics, see, e.g., [25, 19, 26]. Often the difference of the gas constants for water vapor and dry air is neglected and further the simple form of the dry ideal gas law is assumed to hold. Even more typically also the dependence of the internal energy on the moisture components is neglected, which results in a much simpler form of the thermodynamic equation. So far global well-posedness of solutions to moisture models has only been proven based upon these assumptions, see also [2, 8, 10, 9, 15, 16]. As demonstrated e.g. in the asymptotical analysis in [14] for deep convective cloud columns, exactly these refined thermodynamics lead to a much stronger coupling of the thermodynamic equation (see below) to the moisture components and thereby even change the force balances to leading order. The aim here is to also incorporate them into the analysis, where the refined thermodynamical setting in comparison to [15] requires a different approach for proving a priori nonnegativity and uniform boundedness of the solution components, since the antidissipative term in the equation for temperature does not vanish anymore when rewriting it in terms of the potential temperature. We thus

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employ an iterative method similar to the one used by Coti Zelati et al. [9] to derive an upper bound on the temperature.

Coti Zelati et al. in [2, 8, 9, 10] analysed a basic moisture model consisting of one moisture quantity coupled to temperature and containing only the process of condensation during upward motion, see e.g. [17]. Since the source term there is modeled via a Heavy side function as a switching term between saturated and undersaturated regions, the analysis requires elaborate techniques. The approach based on differential inclusions and variational techniques has then further been applied to the moisture model coupled to the primitive equations in [9].

In preceding works [15, 16] we studied a moisture model consisting of three moisture quantities for water vapor, cloud water, and rain water, which contains besides the phase changes condensation and evaporation also the autoconversion of cloud water to rain water after a certain threshold is reached, as well as the collection of cloud water by the falling rain droplets. It corresponds to a basic form of a bulk microphysics model in the spirit of Kessler [18] and Grabowski and Smolarkiewicz [13]. In [15] we assumed the velocity field to be given and studied the moisture balances coupled to the thermodynamic equation through the latent heat. In [16] this moisture model has been successfully coupled to the primitive equations by taking over the ideas of Cao and Titi [7] for their breakthrough on the global solvability of the latter system.

In this work we extend this moisture model for warm clouds consisting of three moisture balances and the thermodynamic equation by the refined thermodynamic setting as explained above, which leads in particular to a much stronger coupling of the model equations.

In the remainder of this section we introduce the moisture model. In Section 2 we then formulate the full problem with boundary and side conditions and state the main result on the global existence and uniqueness of bounded solutions. In Section 3 we carry out the proof for the existence and uniqueness of strong solutions.

1.1. **Governing equations.** When modelling atmospheric flows in general, the full compressible governing equations need to be considered. However, under the assumption of hydrostatic balance, which in particular guarantees the pressure to decrease monotonically in height, the pressure p can be used as the vertical coordinate, which have the main advantage that the continuity equation takes the form of the incompressibility condition (see also (15) below). We therefore work in the following with the governing equations in the pressure coordinates (x, y, p) and as in [15] assume the velocity field to be given

$$\overline{\mathbf{v}} = (\overline{\mathbf{v}}_h, \overline{\omega}) = (\overline{u}, \overline{v}, \overline{\omega}),$$

where we note that the vertical velocity $\omega = \frac{dp}{dt}$ in pressure coordinates takes the inverse sign in comparison to cartesian coordinates for upward and downward motion. Also the horizontal and the vertical derivatives and accordingly the velocity components in pressure coordinates have different units. The total derivative in pressure coordinates reads

$$\frac{d}{dt} = \partial_t + \overline{\mathbf{v}}_h \cdot \nabla_h + \overline{\omega}\partial_p\,, \qquad \text{with} \quad \nabla_h = (\partial_x, \partial_y)\,. \tag{1}$$

For the closure of the turbulent and molecular transport we use

$$\mathcal{D}^* = \mu_* \Delta_h + \nu_* \partial_p \left(\left(\frac{gp}{R_d \bar{T}} \right)^2 \partial_p \right)\,, \qquad \text{with} \quad \Delta_h = \partial_x^2 + \partial_y^2\,, \tag{2}$$

where $\bar{T} = \bar{T}(p)$ corresponds to some background distribution being uniformly bounded from above and below and R_d is the individual gas constant for dry air. The operator $\mathcal{D}^*$ thereby provides a close approximation to the full Laplacian in cartesian coordinates, see also [21, 23]. The thermodynamic quanitities are related via the ideal gas law

$$p = p_d + p_v = R_d \rho_d T + R_v \rho_v T\,, \tag{3}$$

where R_v is the individual gas constant for water vapor and p_d, p_v, ρ_d, ρ_v denote the partial pressures and densities of dry air and water vapor, where we note that liquid water does not exert any pressure on the volume of air, see e.g. also [1, 11].

Before going more into details with the ideal gas law (see (7) below), we need to introduce the moisture quantities. In the case of moisture being present typically the water vapor mixing ratio, defined as the ratio of the density of ρ_v over the density of dry air ρ_d,

$$q_v = \frac{\rho_v}{\rho_d}\,,$$

is used for a measure of quantification. If saturation effects occur, then water is also present in liquid form as cloud water and rain water represented by the additional moisture quantities

$$q_c = \frac{\rho_c}{\rho_d}\,, \qquad q_r = \frac{\rho_r}{\rho_d}\,.$$

We focus here on warm clouds, where water is present only in gaseous and liquid form, i.e. no ice and snow phases occur. The total water content is therefore given by

$$q_T = q_v + q_c + q_r\,.$$

For these mixing ratios for water vapor, cloud water and rain water we have the following moisture balances

$$\frac{dq_v}{dt} = S_{ev} - S_{cd} + \mathcal{D}^{q_v} q_v\,, \tag{4}$$

$$\frac{dq_c}{dt} = S_{cd} - S_{ac} - S_{cr} + \mathcal{D}^{q_c} q_c\,, \tag{5}$$

$$\frac{dq_r}{dt} + V\partial_p \left(\frac{p}{R_d \bar{T}} q_r \right) = S_{ac} + S_{cr} - S_{ev} + \mathcal{D}^{q_r} q_r \tag{6}$$

with $\frac{d}{dt}$ as in (1) and $\mathcal{D}^{q_j}$ as in (2). Here $S_{ev}, S_{cd}, S_{ac}, S_{cr}$ are the rates of evaporation of rain water, the condensation of water vapor to cloud water and the inverse evaporation process, the auto-conversion of cloud water into rainwater by accumulation of microscopic droplets, and the collection of cloud water by falling rain. Moreover V denotes the terminal velocity of falling rain and is assumed to be constant.

Having introduced the mixing ratios, we can now reformulate the ideal gas law (3) as

$$p = \rho \widetilde{R} T \,, \tag{7}$$

where $\rho = \rho_d + \rho_v + \rho_c + \rho_r$ is the total density and $\widetilde{R}$ depends on the moisture content

$$\widetilde{R} = R_d \frac{1 + \frac{q_v}{E}}{1 + q_T} \,, \qquad \text{where} \qquad E = \frac{R_d}{R_v} \,,$$

see e.g. also [1, 11, 14]. The thermodynamic equation accounts for the diabatic source and sink terms, such as latent heating, radiation effects, etc., but we will in the following only focus on the effect of latent heat in association with phase changes (see e.g. also [20, 8, 9, 15]). The temperature equation in pressure coordinates then reads, see e.g. [14, 11],

$$\frac{dT}{dt} - \widetilde{\kappa} \frac{T}{p} \overline{\omega} + \frac{c_l}{\widetilde{C}} q_r V \partial_p T = \widetilde{L}(S_{cd} - S_{ev}) + \mathcal{D}^T T \,, \tag{8}$$

where

$$\widetilde{\kappa} = \frac{\widetilde{R}}{\widetilde{C}} \,, \qquad \text{and} \qquad \widetilde{C} = c_{pd} + c_{pv} q_v + c_l (q_c + q_r)$$

with the heat capacities c_{pd}, c_{pv} at constant pressure for dry air and water vapor and the heat capacity for liquid water c_l, respectively. For the latent heat term, we denote

$$\widetilde{L} = \frac{L}{\widetilde{C}} \,, \qquad \text{where} \qquad L(T) = L_0 - (c_l - c_{pv})(T - T_0) \qquad \text{and} \qquad L_0 = L(T_0), \tag{9}$$

where T_0 is the reference temperature which is typically chosen as $T_0 = 273.15K$. We emphasize here once more that so far only models with $L, \widetilde{R}, \widetilde{\kappa}, \widetilde{C}$ constant and $c_l = 0$ have been considered in mathematical analysis studies. Thus this physically more refined sestting has several extensions in comparison to existing studies.

Remark 1. *To describe the state of the atmosphere a common thermodynamic quantity used instead of the temperature is the potential temperature*

$$\theta = T \left(\frac{p_0}{p} \right)^{\kappa} \,, \qquad \text{where} \qquad \kappa = \frac{R_d}{c_{pd}} \,. \tag{10}$$

In case of the typical simplification $\widetilde{\kappa} = \kappa$ and $c_l = 0$, the left hand side of (8) *simply reduces to $\frac{T}{\theta} \frac{d}{dt} \theta$. This property was essential in the preceding works* [15] *and also* [10] *to derive a priori nonnegativity of the moisture quantities and temperature.*

1.2. **Explicit expressions for the source terms.** The saturation mixing ratio

$$q_{vs} = \frac{\rho_{vs}}{\rho_d} \,,$$

gives the threshold for saturation, i.e. $q_v < q_{vs}$ for undersaturation, $q_v = q_{vs}$ corresponds to saturation, and $q_v > q_{vs}$ accordingly holds in oversaturated regions. The saturation vapor mixing ratio satisfies

$$q_{vs}(p, T) = \frac{E e_s(T)}{p - e_s(T)},$$

with the saturation vapor pressure e_s as a function of T being defined by the Clausius-Clapeyron equation:

$$\frac{d \ln e_s}{dT} = \frac{L(T)}{R_v T^2} \,.$$

From this formula it is obvious that e_s increases in T (as long as $L(T)$ is positive). Since the temperature T is given in Kelvin K, only positive values are physical. It should be noted that the Clausius-Clapeyron equation is only meaningful for temperature ranges appearing in the troposphere, thus in particular we shall pose in the following the natural assumption

$$e_s(T) = 0\,Pa \quad \text{and} \quad q_{vs}(p,T) = 0 \quad \text{for } T \leq \underline{T}\,,$$

for some $\underline{T} \geq 0\,K$, which will also be helpful for proving nonnegativity of the moisture quantities and the temperature, see also [15].

Recalling the fact that $c_l - c_{pv} > 0$, we see from (9) that $L(T)$ decreases in T. In particular, there exists the critical temperature

$$T_{crit} = \frac{L_0}{(c_l - c_{pv})} - T_0 \tag{11}$$

at which the latent heat of evaporation vanishes, i.e. $L(T_{crit}) = 0$. At such high temperatures of about $700K$, the gaseous and liquid state become indistinguishable. Such temperatures however clearly exceed by far the ones present in the relevant atmospheric layers. Therefore, we in the following pose the natural assumption that

$$e_s(T) = 0\,Pa \quad \text{and} \quad q_{vs}(p,T) = 0 \quad \text{for } T \geq T_{crit}\,. \tag{12}$$

For deriving the uniqueness of the solutions we need additionally the uniform Lipschitz continuity of $q_{vs} \geq 0$ in T, i.e. we assume

$$|q_{vs}(p,T_1) - q_{vs}(p,T_2)| \leq C|T_1 - T_2|,$$

for a positive constant C independent of p.

For the source terms of the mixing ratios, we take over the setting of Klein and Majda [20] corresponding to a basic form of the bulk microphysics closure in the spirit of Kessler [18] and Grabowski and Smolarkiewicz [13], which has also been used in the preceding work [15]:

$$\begin{aligned} S_{ev} &= C_{ev}\widetilde{R}T(q_r^+)^\beta (q_{vs} - q_v)^+\,, \\ S_{cr} &= C_{cr}q_c q_r, \\ S_{ac} &= C_{ac}(q_c - q_{ac}^*)^+ \end{aligned}$$

where C_{ev}, C_{cr}, C_{ac} are dimensionless rate constants. Moreover, $(g)^+ = \max\{0, g\}$ and $q_{ac}^* \geq 0$ denotes the threshold for cloud water mixing ratio beyond which autoconversion of cloud water into precipitation become active. The cutoff of the negative part in q_r is only technical since clearly only nonnegative values for T and q_j for $j \in \{v, c, r\}$ are meaningful.

The exponent β in the evaporation term S_{ev} in the literature typically appears to be chosen as $\beta \approx 0.5$, see e.g. [13, 20] and the references therein. Exponent $\beta \in (0,1)$ causes difficulties in the analysis for the uniqueness of the solutions. In the case that both $\widetilde{C}$ and $\widetilde{R}$ are constants, this problem, however, was overcome in [15] by introducing new unknowns, which allow for certain cancellation properties of the source terms and reveal advantageous monotonicity properties. Here, however, we need to generalise the setting to incorporate the more complicated structure of the thermodynamic equation and in particular the nonconstant $\widetilde{C}, \widetilde{L}$.

We shall use the closure of the condensation term in a similar fashion to [20]

$$S_{cd} = C_{cd}(q_v - q_{vs})q_c + C_{cn}(q_v - q_{vs})^+ \,,$$

which is in the literature often defined implicitly via the equation of water vapor at saturation, see e.g. [13].

2. Formulation of the problem and main result

We analyse the moisture model consisting of the moisture equations (4)–(6) coupled to the thermodynamic equation (8). As in [15], we assume the velocity field $\bar{\mathbf{v}} = (\overline{\mathbf{v}}_h, \overline{\omega})$ to be given and to satisfy

$$\overline{\mathbf{v}}_h \in (L^2_{\text{loc}}([0,\infty); H^1(\mathcal{M})))^2 \cap (L^\infty_{\text{loc}}([0,\infty); L^2(\mathcal{M})))^2 \cap (L^r_{\text{loc}}([0,\infty); L^q(\mathcal{M})))^2, \quad (13)$$

$$\overline{\omega} \in L^\infty_{\text{loc}}([0,\infty); L^2(\mathcal{M})) \cap L^r_{\text{loc}}([0,\infty); L^q(\mathcal{M})), \quad (14)$$

for some $2 \le r \le \infty$ and $3 \le q \le \infty$ satisfying $\frac{2}{r} + \frac{3}{q} < 1$. Moreover, we assume mass conservation, taking in pressure coordinates the form of the incompressibility condition

$$\nabla_h \cdot \overline{\mathbf{v}}_h + \partial_p \overline{\omega} = 0 \qquad \text{in } \ \mathcal{M} \,, \quad (15)$$

and the no-penetration boundary condition

$$\overline{\mathbf{v}}_h \cdot \mathbf{n}_h + \overline{\omega}\, n_p = 0 \qquad \text{on } \ \partial\mathcal{M} \,. \quad (16)$$

This is motivated from the solution of the viscous primitive equations (without moisture) satisfying these required regularity properties in (13), see [3, 4, 5, 6, 7].

Similar as in [8, 15], we let $\mathcal{M}$ be a cylinder of the form

$$\mathcal{M} = \{(x,y,p) : (x,y) \in \mathcal{M}', p \in (p_1, p_0)\} \,,$$

where $\mathcal{M}'$ is a smooth bounded domain in $\mathbb{R}^2$ and $p_0 > p_1 > 0$. The boundary is given by

$$\Gamma_0 = \{(x,y,p) \in \overline{\mathcal{M}} : p = p_0\} \,,$$
$$\Gamma_1 = \{(x,y,p) \in \overline{\mathcal{M}} : p = p_1\} \,,$$
$$\Gamma_\ell = \{(x,y,p) \in \overline{\mathcal{M}} : (x,y) \in \partial\mathcal{M}', p_0 \ge p \ge p_1\} \,.$$

The boundary conditions read as

$$\Gamma_0 : \quad \partial_p T = \alpha_{0T}(T_{b0}(x,y,t) - T) \,, \quad \partial_p q_j = \alpha_{0j}(q_{b0j}(x,y,t) - q_j) \,, \quad j \in \{v,c,r\} \,, \ (17)$$

$$\Gamma_1 : \quad \partial_p T = 0 \,, \qquad \partial_p q_j = 0 \,, \quad j \in \{v,c,r\} \,, \quad (18)$$

$$\Gamma_\ell : \quad \partial_n T = \alpha_{\ell T}(T_{b\ell}(p,t) - T) \,, \quad \partial_n q_j = \alpha_{\ell j}(q_{b\ell j}(p,t) - q_j) \,, \quad j \in \{v,c,r\}, \quad (19)$$

where all given functions $\alpha_{0j}, \alpha_{\ell j}, \alpha_{0T}, \alpha_{\ell T}$ and $T_{b0}, T_{b\ell}, q_{b0j}, q_{b\ell j}$ are assumed to be non-negative, sufficiently smooth and uniformly bounded.

Throughout this paper, we use the abbreviation

$$\|f\| = \|f\|_{L^2(\mathcal{M})} \,, \qquad \|f\|_{L^p} = \|f\|_{L^p(\mathcal{M})} \,.$$

According to the weight in the vertical diffusion terms, we also introduce the weighted norms

$$\|f\|_w = \left\| \left(\frac{gp}{R_d\bar{T}}\right) f \right\|, \qquad \|f\|^2_{H^1_w} = \|f\|^2 + \|\nabla_h f\|^2 + \|\partial_p f\|^2_w \,,$$

where we emphasize that, since the weight $\frac{gp}{R_d\bar{T}}$ is uniformly bounded from above and below by positive constants, the $H^1_w(\mathcal{M})$-norm is equivalent to the $H^1(\mathcal{M})$-norm. Moreover, we shall often use for convenience the notation

$$\|(f_1, \ldots, f_n)\|^2 = \sum_{j=1}^{n} \|f_j\|^2.$$

For the initial data space of functions for the moisture components and the temperature, we introduce

$$\mathcal{X} = L^\infty(\mathcal{M}) \cap H^1(\mathcal{M})$$

and accordingly for strong solutions

$$\mathcal{Y}_{\mathcal{T}} = \{f | f \in L^\infty(\mathcal{M} \times (0, \mathcal{T})) \cap L^2(0, \mathcal{T}; H^2(\mathcal{M})), \partial_t f \in L^2(\mathcal{M} \times (0, \mathcal{T}))\}.$$

The main result of this paper is the following theorem.

Theorem 1. *Let $\beta = 1$ and assume that the given velocity field $(\overline{\mathbf{v}}_h, \overline{\omega})$ satisfies (13)–(16) and the initial data $(T_0, q_{v0}, q_{c0}, q_{r0}) \in \mathcal{X}^4$ is nonnegative. Then, for any $\mathcal{T} > 0$ there exists a unique global strong solution $(T, q_v, q_c, q_r) \in \mathcal{Y}^4_{\mathcal{T}}$ to system* (4)–(8)*, subject to* (17)–(19)*, on $\mathcal{M} \times (0, \mathcal{T})$, and the solution components (T, q_v, q_c, q_r) remain nonnegative and uniformly bounded from above with bounds growing with $\mathcal{T}$.*

The proof of this theorem will be presented in the next section.

Remark 2. *In* [15]*, we treated the more complicated case of an evaporation source with a general exponent $\beta \in (0, 1]$ of q_r that causes in particular difficulties in the uniqueness. To overcome this problem, we introduced the new unknowns $Q = q_v + q_r$ and $H = T - \widetilde{L}(q_c + q_r)$ in* [15]*, where we recall that $\widetilde{L}$ was assumed to be constant there. Due to the challenge here of treating the additional terms arising from the refined thermodynamics, we stick to the case corresponding $\beta = 1$ here and leave the more general case $\beta \in (0, 1]$ for future work.*

Throughout this paper, we use C to denote a general positive constant which may be different at different places. For the aim of the future studies on the coupled system of the moisture dynamics investigated in the present paper to the primitive equations with either isotropic or anisotropic dissipations, see [3, 4, 5, 6, 7], the dependence of the constant C on the a priori bounds of the given velocity field will be explicitly pointed out at the relevant places. However, the dependence of C on the initial data or the parameters in the system will not be paid attention to. We will also use $C_k, k \in \mathbb{N}$, to denote constants having relevant units.

3. Proof of Theorem 1

As a start point, we consider the following modified system, which however is equivalent to the original system (4)–(8) for nonnegative solutions:

$$\partial_t T + (\overline{\mathbf{v}}_h \cdot \nabla_h) T + \overline{\omega}\partial_p T - \widetilde{\kappa}^+ \frac{T}{p}\overline{\omega} + \frac{c_l q_r}{\widetilde{C}^+} V \partial_p T = \widetilde{L}^+ (S_{cd}^+ - S_{ev}^+) + \mathcal{D}^T T, \quad (20)$$

$$\partial_t q_v + \overline{\mathbf{v}}_h \cdot \nabla_h q_v + \overline{\omega}\partial_p q_v = S_{ev}^+ - S_{cd}^+ + \mathcal{D}^{q_v} q_v \,, \quad (21)$$

$$\partial_t q_c + \overline{\mathbf{v}}_h \cdot \nabla_h q_c + \overline{\omega}\partial_p q_c = S_{cd}^+ - S_{ac}^+ - S_{cr}^+ + \mathcal{D}^{q_c} q_c \,, \quad (22)$$

$$\partial_t q_r + \overline{\mathbf{v}}_h \cdot \nabla_h q_r + \overline{\omega}\partial_p q_r + V\partial_p \left(\frac{p}{R_d \overline{T}} q_r \right) = S_{ac}^+ + S_{cr}^+ - S_{ev}^+ + \mathcal{D}^{q_r} q_r \,, \quad (23)$$

where

$$\widetilde{\kappa}^+ = \frac{\widetilde{R}^+}{\widetilde{C}^+}, \quad \widetilde{L}^+ = \frac{L(T)}{\widetilde{C}^+}, \quad \widetilde{R}^+ = \frac{R_d + R_v q_v^+}{1 + q_v^+ + q_c^+ + q_r^+}, \quad \widetilde{C}^+ = c_{pd} + c_{pv} q_c^+ + c_l (q_c^+ + q_r^+),$$

$$S_{ev}^+ = C_{ev} \widetilde{R}^+ T^+ q_r^+ (q_{vs}(p,T) - q_v)^+, \quad S_{cr}^+ = C_{cr} q_c^+ q_r^+, \quad S_{ac}^+ = S_{ac} = C_{ac}(q_c - q_{ac}^*)^+,$$

$$S_{cd}^+ = C_{cd}(q_v^+ - q_{vs}(p,T)) q_c^+ + C_{cn}(q_v - q_{vs}(p,T))^+,$$

with $L(T)$ given by (9).

Since all the nonlinear terms $S_{ev}^+, S_{cr}^+, S_{ac}^+, S_{cd}^+$ and all the coefficients $\widetilde{\kappa}^+, \frac{1}{\widetilde{C}^+}, \widetilde{L}^+$ are Lipschitz with respect to q_v, q_c, q_r, and T, the local existence of strong solutions to the initial boundary value problem of system (20)–(23) follows by the standard fixed point arguments. In fact, by following the proof in [15], we can prove the following proposition on the local existence and uniqueness.

Proposition 1. *Assume that the given velocity field $(\overline{\mathbf{v}}_h, \overline{\omega})$ satisfies (13)–(16) and the initial data $(T_0, q_{v0}, q_{c0}, q_{r0}) \in \mathcal{X}^4$ is nonnegative. Then, there exists a positive time $\mathcal{T}_0$ depending only on the upper bound of $\|(T_0, q_{v0}, q_{c0}, q_{r0})\|_{H^1(\mathcal{M})}$, such that system (20)–(23), subject to (17)–(19), on $\mathcal{M} \times (0, \mathcal{T}_0)$, has a unique strong solution $(T, q_v, q_c, q_r) \in \mathcal{Y}^4_{\mathcal{T}_0}$.*

By applying Proposition 1 inductively, one can extend uniquely the solution (T, q_v, q_c, q_r) obtained there to the maximal time interval $(0, \mathcal{T}_{\max})$, where $T_{\max}$ is characterized as

$$\limsup_{\mathcal{T} \to \mathcal{T}_{\max}^-} \|(T, q_v, q_c, q_r)\|_{H^1(\mathcal{M})} = \infty, \quad \text{if } \mathcal{T}_{\max} < \infty. \quad (24)$$

Observe that if $T_{\max} = \infty$, then Proposition 1 implies Theorem 1. Therefore, our aim is to show that $T_{\max} = \infty$. To this end, we assume by contradiction that $T_{\max} < \infty$. Due to this fact, the following assumption will be made in the subsequent propositions throughout this section.

Assumption 1. *Let all the assumptions in Proposition 1 hold, and let the solution (T, q_v, q_c, q_r) obtained in Proposition 1 be extended uniquely to the maximal interval of existence $(0, \mathcal{T}_{max})$, where $\mathcal{T}_{max} < \infty$.*

The main part of this section is to carry out a series of a priori estimates on (T, q_v, q_c, q_r).

First, the following proposition about the nonnegative and uniform boundedness of the moisture components q_v, q_c, and q_r can be proved by slightly modifying the corresponding proof of Proposition 3.2 in [15].

Proposition 2. *Let Assumption 1 hold, then the solution* (T, q_v, q_c, q_r) *satisfies*

$$0 \le q_v \le q_v^*\,, \quad 0 \le q_c \le q_c^*\,, \quad 0 \le q_r \le q_r^*,$$

for any $\mathcal{T} \in (0, \mathcal{T}_{max})$*, where*

$$\begin{aligned} q_v^* &= \max\left\{\|q_{v0}\|_{L^\infty(\mathcal{M})}, \|q_{b0v}\|_{L^\infty((0,\mathcal{T})\times\mathcal{M}')}, \|q_{b\ell v}\|_{L^\infty((0,\mathcal{T})\times\Gamma_\ell)}, q_{vs}^*\right\}, \\ q_c^* &= q_c^*\big(\mathcal{T}, \|q_{c0}\|_{L^\infty(\mathcal{M})}, \|q_{b0c}\|_{L^\infty((0,T)\times\mathcal{M}')}, \|q_{b\ell c}\|_{L^\infty((0,T)\times\Gamma_\ell)}, q_v^*, q_{vs}^*\big)\,, \\ q_r^* &= q_r^*\big(\mathcal{T}, \|q_{r0}\|_{L^\infty(\mathcal{M})}, \|q_{b0r}\|_{L^\infty((0,T)\times\mathcal{M}')}, \|q_{b\ell r}\|_{L^\infty((0,T)\times\Gamma_\ell)}, q_c^*\big), \end{aligned}$$

with $q_{vs}^* = \max q_{vs}$ *and moreover* q_c^* *and* q_r^* *are continuous in* $\mathcal{T} \in (0, \infty)$.

Due to the nonnegativity of q_v, q_c, q_r, it is clear that $\widetilde{R}^+ = \widetilde{R}, \widetilde{C}^+ = \widetilde{C}, \widetilde{\kappa}^+ = \widetilde{\kappa}, \widetilde{L}^+ = \widetilde{L}, S_{cr}^+ = S_{cr}, S_{cd}^+ = S_{cd}$, $S_{ac}^+ = S_{ac}$, and

$$0 < \widetilde{\kappa} = \frac{\widetilde{R}}{\widetilde{C}} \le \kappa_1, \quad 0 \le \frac{c_l q_r}{\widetilde{C}} \le 1, \quad 0 < \frac{1}{\widetilde{C}} \le \frac{1}{c_{pd}}, \tag{25}$$

for some positive constant κ_1. Besides, by the uniform boundedness of q_v, q_c, q_r, one has

$$|S_{cd}| + |S_{cr}| + |S_{ac}| \le C, \tag{26}$$

for some positive constant C depending on q_v^*, q_c^*, q_r^*.

Proposition 3. *Let Assumption 1 hold, then for any* $\mathcal{T} \in (0, T_{max})$,

$$\sup_{0\le t\le\mathcal{T}} \|(q_v, q_c, q_r)\|^2 + \int_0^{\mathcal{T}} \|\nabla(q_v, q_c, q_r)\|^2 dt \le K_0(\mathcal{T}),$$

for a continuous bounded function $K_0(\mathcal{T})$ *determined by* q_v^*, q_c^**, and* q_r^*.

Proof. Testing (21), (22), and (23), respectively, with q_v, q_c, and q_r, summing the resultants up, using the uniform boundedness of $q_v, q_c, q_r, S_{ac}, S_{cr}, S_{cd}$ (due to Proposition 2 and (26)), and noticing that $S_{ev}^+ q_r \ge 0$, one deduces

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|(q_v, q_c, q_r)\|^2 - \sum_{j\in\{v,c,r\}} \int_{\mathcal{M}} q_j \mathcal{D}^{q_j} q_j d\mathcal{M} \\ = &- \sum_{j\in\{v,c,r\}} \int_{\mathcal{M}} (\overline{\mathbf{v}}_h \cdot \nabla_h q_j + \overline{\omega}\partial_p q_j) q_j d - V \int_{\mathcal{M}} q_r \partial_p \left(\frac{p q_r}{R_d \bar{T}}\right) d\mathcal{M} \\ &+ \int_{\mathcal{M}} [(S_{ev}^+ - S_{cd}) q_v + (S_{cd} - S_{ac} - S_{cr}) q_c + (S_{ac} + S_{cr} - S_{ev}^+) q_r] d\mathcal{M} \\ \le &- \sum_{j\in\{v,c,r\}} \int_{\mathcal{M}} (\overline{\mathbf{v}}_h \cdot \nabla_h q_j + \overline{\omega}\partial_p q_j) q_j d - V \int_{\mathcal{M}} q_r \partial_p \left(\frac{p q_r}{R_d \bar{T}}\right) d\mathcal{M} \\ &+ \int_{\mathcal{M}} S_{ev}^+ q_v d\mathcal{M} + C. \end{aligned} \tag{27}$$

The integrals in (27) are estimated as follows. For the diffusion terms, integration by parts and using the boundary conditions (17)–(19), one deduces

$$- \int_{\mathcal{M}} q_j \mathcal{D}^{q_j} q_j d\mathcal{M}$$

$$
\begin{aligned}
&= -\int_{\mathcal{M}} \left[\mu_{q_j}\Delta_h q_j + \nu_{q_j}\partial_p\left(\left(\frac{gp}{R\bar{T}}\right)^2 \partial_p q_j\right)\right] q_j d\mathcal{M} \\
&= -\mu_{q_j}\int_{\Gamma_\ell} (\partial_n q_j) q_j d\Gamma_\ell - \nu_{q_j}\int_{\mathcal{M}'} \left(\frac{gp}{R\bar{T}}\right)^2 (\partial_p q_j) q_j d\mathcal{M}'\Big|_{p_1}^{p_0} \\
&\quad + \int_{\mathcal{M}} \left[\mu_{q_j}\nabla_h q_j\cdot\nabla_h q_j + \nu_{q_j}\left(\frac{gp}{R\bar{T}}\right)^2 \partial_p q_j \partial_p q_j\right] d\mathcal{M} \\
&= \mu_{q_j}\|\nabla_h q_j\|^2 + \nu_{q_j}\|\partial_p q_j\|_w^2 - \mu_{q_j}\int_{\Gamma_\ell} \alpha_{\ell j}(q_{\ell j} - q_j) q_j d\Gamma_\ell \\
&\quad -\nu_{q_j}\int_{\mathcal{M}'} \left(\frac{gp_0}{R\bar{T}}\right)^2 \alpha_{b0j}(q_{b0j} - q_j) q_j d\mathcal{M}' \\
&= \mu_{q_j}\|\nabla_h q_j\|^2 + \nu_{q_j}\|\partial_p q_j\|_w^2 + \mu_{q_j}\int_{\Gamma_\ell} \alpha_{\ell j}\left[\left(q_j - \frac{q_{\ell j}}{2}\right)^2 - \frac{q_{\ell j}^2}{4}\right] d\Gamma_\ell \\
&\quad +\nu_{q_j}\int_{\mathcal{M}'} \left(\frac{gp_0}{R\bar{T}}\right)^2 \alpha_{b0j}\left[\left(q_j - \frac{q_{b0j}}{2}\right)^2 - \frac{q_{b0j}^2}{4}\right] d\mathcal{M}',
\end{aligned}
$$

for $j\in\{v,c,r\}$. This implies for $j\in\{v,c,r\}$ that

$$
-\int_{\mathcal{M}} q_j \mathcal{D}^{q_j} q_j d\mathcal{M} \geq \mu_{q_j}\|\nabla_h q_j^-\|^2 + \nu_{q_j}\|\partial_p q_j^-\|_w^2 - C, \tag{28}
$$

for a constant C depending only on the given inhomogeneous boundary functions $\alpha_{0j}, \alpha_{\ell j}$ and $q_{b0j}, q_{b\ell j}$. The integral containing the advection term vanishes due to (15) and (16), since

$$
\begin{aligned}
&\int_{\mathcal{M}} (\overline{\mathbf{v}}_h\cdot\nabla_h q_j + \overline{\omega}\partial_p q_j) q_j d\mathcal{M} = -\frac{1}{2}\int_{\mathcal{M}} (\overline{\mathbf{v}}_h\cdot\nabla_h + \overline{\omega}\partial_p)(q_j)^2 d\mathcal{M} \\
= \; & -\frac{1}{2}\int_{\partial\mathcal{M}} (\overline{\mathbf{v}}_h\cdot\mathbf{n}_h + \overline{\omega}n_p)(q_j)^2 d(\partial\mathcal{M}) + \frac{1}{2}\int_{\mathcal{M}} (q_j)^2(\nabla_h\cdot\overline{\mathbf{v}}_h + \partial_p\overline{\omega}) d\mathcal{M} = 0 .
\end{aligned} \tag{29}
$$

The Young inequality leads to

$$
-V\int_{\mathcal{M}} q_r\partial_p\left(\frac{pq_r}{R_d\bar{T}}\right) d\mathcal{M} \leq \frac{\nu_{q_r}}{8}\|\partial_p q_r\|_w^2 + C\|q_r\|^2. \tag{30}
$$

To estimate the term $\int_{\mathcal{M}} S_{ev}^+ q_v d\mathcal{M}$, we decompose the domain $\mathcal{M}$ as $\mathcal{M} = \mathcal{M}_+(t)\cup\mathcal{M}_-(t)$, where $\mathcal{M}_+(t) = \{(x,y,p)\in\mathcal{M} | T(x,y,p,t)\geq T_{\text{crit}}\}$ and $\mathcal{M}_-(t) = \mathcal{M}\setminus\mathcal{M}_+(t)$. Due to (12), one can check that $S_{ev}^+ = 0$ on $\mathcal{M}_+(t)$, while on $\mathcal{M}_-(t)$, due to the nonnegativity and uniform boundedness of q_v, q_c, q_r guaranteed by Proposition 2, one has $S_{ev}^+\leq C$. Therefore, we always have

$$
0\leq S_{ev}^+\leq C \quad \text{on } \mathcal{M}
$$

and, as result, it holds that $\int_{\mathcal{M}} S_{ev}^+ q_v d\mathcal{M}\leq C$. Thanks to this and combining (28)–(30), the conclusion follows from (27) by the Grönwall inequality. □

We would like to point out that the a priori estimates obtained in Proposition 2 and Proposition 3 do not depend on the a priori bounds of the given velocity $(\overline{\mathbf{v}}_h, \overline{\omega})$.

The following lemma will be used later.

Lemma 1. *Let $f \in L^\infty(0,\mathcal{T};L^2(\mathcal{M}))$ and $g,h \in L^2(0,\mathcal{T};H^1(\mathcal{M})) \cap L^\infty(0,\mathcal{T};L^2(\mathcal{M}))$. Then for some arbitrary $\delta_g, \delta_h > 0$ at a.e. $t \in (0,\mathcal{T})$ the following estimate holds*

$$\left|\int_{\mathcal{M}} f\, g\, h\, d\mathcal{M}\right| \leq \delta_g \|\nabla g\|^2_{L^2(\mathcal{M})} + \delta_h \|\nabla h\|^2_{L^2(\mathcal{M})} + C(\|g\|^2_{L^2(\mathcal{M})} + \|h\|^2_{L^2(\mathcal{M})}),$$

where $C = C(\delta_1, \delta_2, \|f\|_{L^\infty(0,\mathcal{T};L^2(\mathcal{M}))})$.

Proof. We first bound the integral by

$$\left|\int_{\mathcal{M}} f\, g\, h\, d\mathcal{M}\right| \leq \|f\|_{L^2(\mathcal{M})}\|g\, h\|_{L^2(\mathcal{M})} \leq C(\|g^2\|_{L^2(\mathcal{M})} + \|h^2\|_{L^2(\mathcal{M})})$$

where C depends on $\|f\|_{L^\infty(0,\mathcal{T};L^2(\mathcal{M}))}$. We next employ the Gagliardo Nirenberg inequality, see e.g.[12, Theorem 10.1] and [27, Theorem 1.1.4], to estimate

$$\begin{aligned}\|g^2\|_{L^2(\mathcal{M})} &= \|g\|^2_{L^4(\mathcal{M})} \leq C\|g\|^{2\vartheta}_{L^2(\mathcal{M})}\|\nabla g\|^{2(1-\vartheta)}_{L^2(\mathcal{M})} + \|g\|^2_{L^2(\mathcal{M})} \\ &\leq \delta_g\|\nabla g\|^2_{L^2(\mathcal{M})} + C(\delta_g)\|g\|^2_{L^2(\mathcal{M})}\end{aligned}$$

which holds for $\vartheta = \frac{1}{4}$. In the last estimate we used Young's inequality. The same estimate holds for h, which concludes the proof. □

Nonnegativity of the temperature is proved in the following proposition.

Proposition 4. *Let Assumption 1 hold, then the temperature T is nonnegative.*

Proof. As already mentioned above, for the refined thermodynamic modelling used in this work, the anti-dissipative term $\widetilde{\kappa}^+ \frac{T}{p}\overline{\omega}$ in the thermodynamic equation (20) does not vanish anymore, when switching to the potential temperature θ. We therefore perform the estimate directly from the equation for T.

Testing the equation for T with $-T^-$ yields

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_{\mathcal{M}}(T^-)^2 d\mathcal{M} + \int_{\mathcal{M}} T^- \mathcal{D}^T T d\mathcal{M} - \int_{\mathcal{M}}(\overline{\mathbf{v}}_h \cdot \nabla T + \overline{\omega}\partial_p T)T^- d\mathcal{M} \\ &= \int_{\mathcal{M}} \frac{\widetilde{\kappa}}{p}(T^-)^2\overline{\omega} d\mathcal{M} - \int_{\mathcal{M}} \frac{c_l q_r}{\widetilde{C}} V T^- \partial_p T^- d\mathcal{M} - \int_{\mathcal{M}} \widetilde{L} T^-(S_{cd} - S^+_{ev}) d\mathcal{M}. \qquad (31)\end{aligned}$$

The various terms in the above equality are estimated as follows. For the diffusion term, integration by parts and using the boundary conditions (17)–(19), one deduces

$$\begin{aligned}\int_{\mathcal{M}} T^- \mathcal{D}^T T d\mathcal{M} &= \int_{\mathcal{M}} \left[\mu_T \Delta_h T + \nu_T \partial_p \left(\left(\frac{gp}{R\bar{T}}\right)^2 \partial_p T\right)\right] T^- d\mathcal{M} \\ &= \mu_T \int_{\Gamma_\ell} (\partial_n T) T^- d\Gamma_\ell + \nu_T \int_{\mathcal{M}'} \left(\frac{gp}{R\bar{T}}\right)^2 (\partial_p T) T^- d\mathcal{M}'\Big|_{p_1}^{p_0} \\ &\quad - \int_{\mathcal{M}} \left[\mu_T \nabla_h T \cdot \nabla_h T^- + \nu_T \left(\frac{gp}{R\bar{T}}\right)^2 \partial_p T \partial_p T^-\right] d\mathcal{M} \\ &= \mu_T \|\nabla_h T^-\|^2 + \nu_T \|\partial_p T^-\|^2_w + \mu_T \int_{\Gamma_\ell} \alpha_{\ell T}(T_{bl} - T) T^- d\Gamma_\ell \\ &\quad + \nu_T \int_{\mathcal{M}'} \left(\frac{gp_0}{R\bar{T}}\right)^2 \alpha_{0T}(T_{b0} - T) T^- d\mathcal{M}'.\end{aligned}$$

Since the functions $\alpha_{\ell T}, T_{bl}, \alpha_{0T}, T_{b0}$ are nonnegative and $TT^- = -(T^-)^2$, the last two boundary integrals are nonnegative, and we obtain

$$\int_{\mathcal{M}} T^- \mathcal{D}^T T d\mathcal{M} \geq \mu_T \|\nabla_h T^-\|^2 + \nu_T \|\partial_p T^-\|_w^2. \tag{32}$$

Same as (29), the integral containing the advection terms vanishes due to (15) and (16). To bound the first term on the right-hand side containing the vertical velocity component, we apply Lemma 1 and use (25) as follows

$$\left|\int_{\mathcal{M}} \frac{\widetilde{\kappa}}{p}(T^-)^2 \overline{\omega} d\mathcal{M}\right| \leq C \int_{\mathcal{M}} |\overline{\omega}|(T^-)^2 d\mathcal{M} \leq \frac{\mu_T}{4}\|\nabla_h T^-\|^2 + \frac{\nu_T}{4}\|\partial_p T^-\|_w^2 + C\|T^-\|^2,$$

where C depends on $\|\overline{\omega}\|_{L^\infty(0,\mathcal{T};L^2(\mathcal{M}))}$. Using (25) again and by the Young inequality, we can estimate the second term as

$$\left|\int_{\mathcal{M}} \frac{c_l q_r}{\widetilde{C}} V T^- \partial_p T^- d\mathcal{M}\right| \leq \frac{\nu_T}{4}\|\partial_p T^-\|_w^2 + C\|T^-\|^2 .$$

It remains to bound the integral with the latent heating terms.

$$\begin{aligned} & -\int_{\mathcal{M}} \widetilde{L} T^- S_{cd} d\mathcal{M} \\ = & -\int \frac{L_0 + (c_l - c_{pv})T_0}{\widetilde{C}}[C_{cd}(q_v - q_{vs})q_c + C_{cn}(q_v - q_{vs})^+]T^- d\mathcal{M} \\ & -\int_{\mathcal{M}} \frac{c_l - c_{pv}}{\widetilde{C}}(T^-)^2[C_{cd}(q_v - q_{vs})q_c + C_{cn}(q_v - q_{vs})^+]d\mathcal{M} \leq 0 , \end{aligned}$$

where we used the fact $c_{pv} < c_l$, $q_{vs} = 0$ for $T \leq 0$, and the nonnegativity of all moisture quantities. The integral term with the evaporation term in (31) vanishes since $S_{ev}^+ T^- = C_{ev}\widetilde{R}T^+ q_r(q_{rs}(p,T) - q_v)^+ T^- = 0$.

Combining all bounds above, we thus obtain from (31) that

$$\frac{1}{2}\frac{d}{dt}\|T^-\|^2 + \frac{\mu_T}{2}\|\nabla_h T^-\|^2 + \frac{\nu_T}{2}\|\partial_p T^-\|_w^2 \leq C\|T^-\|^2,$$

and we can conclude by the Grönwall inequality the nonnegativity of T since $T_0^- = 0$. ☐

From now on, the a priori estimates to be carried out depend on the a priori bounds of the given velocity $(\overline{\mathbf{v}}_h, \overline{\omega})$. The relevant bounds of the velocity on which the solutions depend will be explicitly pointed out in the statements of the propositions.

Proposition 5. *Let Assumption 1 hold, then for any $\mathcal{T} \in (0, \mathcal{T}_{max})$*

$$\sup_{0\leq t\leq \mathcal{T}} \|T\|^2(t) + \int_0^{\mathcal{T}} \|\nabla T\|^2(t)dt \leq K_1(\mathcal{T}),$$

for a continuous bounded function $K_1(\mathcal{T})$ determined by the quantities $\|\overline{\omega}\|_{L^\infty(0,\mathcal{T};L^2(\mathcal{M}))}$, q_v^*, q_c^*, q_r^*.

Proof. By testing the temperature equation with T and by the same calculations as (28) and (29) to the diffusion terms and the convection terms, we have the following estimate

$$\frac{1}{2}\frac{d}{dt}\|T\|^2 + \mu_T\|\nabla_h T\|^2 + \nu_T\|\partial_p T\|_w^2$$

$$\leq \int_{\mathcal{M}} \frac{\widetilde{\kappa}}{p}\varpi T^2 d\mathcal{M} - \int_{\mathcal{M}} \frac{c_l q_r}{\widetilde{C}} V\partial_p T\, T d\mathcal{M} + \int_{\mathcal{M}} \widetilde{L}(S_{cd} - S_{ev}) T d\mathcal{M} + C.$$

To bound the first term on the right-hand side, we use Lemma 1 and (25) to get

$$\int_{\mathcal{M}} \frac{\widetilde{\kappa}}{p}\varpi T^2 d\mathcal{M} \leq C \int_{\mathcal{M}} |\varpi| T^2 d\mathcal{M} \leq \frac{\mu_T}{4}\|\nabla_h T\|^2 + \frac{\nu_T}{4}\|\partial_p T\|_w^2 + C\|T\|^2,$$

where C depends on $\|\varpi\|_{L^\infty(0,\mathcal{T};L^2(\mathcal{M}))}$. By Young's inequality and using (25) again, we obtain

$$\left|\int_{\mathcal{M}} \frac{c_l q_r}{\widetilde{C}} V\partial_p T\, T d\mathcal{M}\right| \leq \frac{\nu_T}{4}\|\partial_p T\|_w^2 + C\|T\|^2.$$

Recalling the expression of $L(T)$ given by (9), we obtain due to (25) and (26) that

$$\left|\int_{\mathcal{M}} \widetilde{L} S_{cd} T d\mathcal{M}\right| = \left|\int_{\mathcal{M}} \frac{L(T)}{\widetilde{C}} S_{cd} T d\mathcal{M}\right| \leq C(C_1 + \|T\|^2).$$

Since (12) implies $S_{ev} = C_{ev}\widetilde{R}Tq_r(q_{vs}(p,T) - q_v)^+ = C_{ev}\widetilde{R}Tq_r(-q_v)^+ = 0$, for $T > T_{\text{crit}}$, while (9) and (11) lead to $L(T) \geq 0$, for $0 \leq T \leq T_{\text{crit}}$, we therefore have

$$\widetilde{L} S_{ev} = \frac{L(T)}{\widetilde{C}} S_{ev} \geq 0 \tag{33}$$

and thus

$$-\int_{\mathcal{M}} \widetilde{L} S_{ev} T d\mathcal{M} \leq 0\,.$$

Combining all above estimates, we obtain

$$\frac{1}{2}\frac{d}{dt}\|T\|^2 + \mu_T\|\nabla_h T\|^2 + \nu_T\|\partial_p T\|_w^2 \leq C(C_2 + \|T\|^2),$$

leading to the conclusion by the Grönwall inequality. □

Uniform boundedness of T is stated and proved in the following proposition.

Proposition 6. *Let Assumption 1 hold, then for any* $\mathcal{T} \in (0, \mathcal{T}_{max})$

$$\sup_{0\leq t\leq \mathcal{T}} \|T\|_{L^\infty(\mathcal{M})}(t) \leq K_2(\mathcal{T}),$$

for a continuous bounded function $K_2(\mathcal{T})$ *determined by the quantities* $\|\varpi\|_{L^\infty(0,\mathcal{T};L^2(\mathcal{M}))}$, $\|T_0\|_{L^\infty(\mathcal{M})}$, $\|T_{b0}\|_{L^\infty((0,\mathcal{T})\times\mathcal{M}')}$, $\|T_{b\ell}\|_{L^\infty((0,\mathcal{T})\times\Gamma_\ell)}$, q_v^*, q_c^*, *and* q_r^*.

Proof. In [15], the upper bound for the temperature was derived by employing the potential temperature equation. Again here this does not alleviate the computations due to the stronger coupling of the thermodynamic equation (8) to the moisture quantities. We thus instead apply here the proof of Coti Zelati et al. [9] based on the De Giorgi technique.

Let $\lambda_k \geq \max\{\|T_0\|_{L^\infty(\mathcal{M})}, T_{\text{crit}}, \|T_{b\ell}\|_{L^\infty((0,\mathcal{T})\times\Gamma_\ell)}, \|T_{b\ell}\|_{L^\infty((0,\mathcal{T})\times\Gamma_\ell)}\}$, and denote $T_{\lambda_k} = (T - \lambda_k)^+$. We claim that

$$\widetilde{L} S_{cd} T_{\lambda_k} \leq 0. \tag{34}$$

In fact, if $T < \lambda_k$, then $T_{\lambda_k} = (T - \lambda_k)^+ = 0$ and, as a result, $\widetilde{L} S_{cd} T_{\lambda_k} = 0$, while if $T \geq \lambda_k$, it is clear by the definition of λ_k that $T \geq T_{\text{crit}}$, and as a result, noticing that in this case $L(T) \leq 0$ (due to (9) and (11)) and $q_{vs}(p,T) = 0$ (due to (12)), it holds that

$$\widetilde{L} S_{cd} T_{\lambda_k} = \frac{L(T)}{\widetilde{C}}[C_{cd}(q_v - q_{vs})q_c + C_{cn}(q_v - q_{vs})^+] T_{\lambda_k} \leq 0. \tag{35}$$

Testing equation (8) with T_{λ_k} and by similar calculations as (32) and (29) to the diffusion terms and the convection terms, we have the following estimate

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|T_{\lambda_k}\|^2+\mu_T\|\nabla_h T_{\lambda_k}\|^2+\nu_T\|\partial_p T_{\lambda_k}\|_w^2\\
\leq\ &\int_{\mathcal{M}}\frac{\widetilde{\kappa}}{p}\overline{\omega}TT_{\lambda_k}d\mathcal{M}-\int_{\mathcal{M}}\frac{c_lq_r}{\widetilde{C}}V\partial_pT_{\lambda_k}\,T_{\lambda_k}d\mathcal{M}+\int_{\mathcal{M}}\widetilde{L}(S_{cd}-S_{ev})T_{\lambda_k}d\mathcal{M}\\
\leq\ &C\int_{\mathcal{M}}|\overline{\omega}|(T_{\lambda_k}^2+\lambda_kT_{\lambda_k})d\mathcal{M}+C\int_{\mathcal{M}}|\partial_pT_{\lambda_k}|T_{\lambda_k}d\mathcal{M},
\end{aligned}\tag{36}$$

where (25), (33), (34), and (35) were used in the last step. By Lemma 1 and the Young inequality, it follows that

$$C\int_{\mathcal{M}}(|\bar{\omega}|T_{\lambda_k}^2+|\partial_pT_{\lambda_k}|T_{\lambda_k})d\mathcal{M}\leq\frac{1}{2}\left(\mu_T\|\nabla_hT_{\lambda_k}\|^2+\nu_T\|\partial_pT_{\lambda_k}\|_w^2\right)+C\|T_{\lambda_k}\|^2,$$

where C depends on $\|\bar{\omega}\|_{L^\infty(0,\mathcal{T};L^2(\mathcal{M}))}$, which substituted into (36) yields

$$\frac{d}{dt}\|T_{\lambda_k}\|^2+\mu_T\|\nabla_hT_{\lambda_k}\|^2+\nu_T\|\partial_pT_{\lambda_k}\|_w^2\leq C\|T_{\lambda_k}\|^2+C\lambda_k\int_{\mathcal{M}}|\bar{\omega}|T_{\lambda_k}d\mathcal{M}.$$

Due to $T_{\lambda_k}|_{t=0}=0$, we obtain by applying the Grönwall inequality to the above

$$\begin{aligned}
J_k\ :=\ &\sup_{t\in(0,\mathcal{T})}\|T_{\lambda_k}\|^2(t)+\int_0^{\mathcal{T}}(\mu_T\|\nabla_hT_{\lambda_k}\|^2+\nu_T\|\partial_pT_{\lambda_k}\|_w^2)dt\\
\leq\ &C\lambda_k\int_0^{\mathcal{T}}\int_{\mathcal{M}}|\bar{\omega}|T_{\lambda_k}d\mathcal{M}dt.
\end{aligned}\tag{37}$$

Let $M\geq 2\max\{\|T_0\|_{L^\infty(\mathcal{M})},T_{\mathrm{crit}},\|T_{b\ell}\|_{L^\infty((0,\mathcal{T})\times\Gamma_\ell)},\|T_{b\ell}\|_{L^\infty((0,\mathcal{T})\times\Gamma_\ell)}\}$ be a positive constant to be determined later and choose

$$\lambda_k:=M(1-2^{-k}),\quad Q_k:=\{(x,t)\in\mathcal{M}\times(0,\mathcal{T})|T(x,t)>\lambda_k\},\quad\text{for}\quad k\geq1.$$

For any $(x,t)\in Q_k$, noticing that $T(x,t)>\lambda_k>\lambda_{k-1}$, one deduces

$$T_{\lambda_{k-1}}(x,t)=(T-\lambda_{k-1})^+(x,t)=T(x,t)-\lambda_{k-1}>\lambda_k-\lambda_{k-1}=2^{-k}M$$

and, thus,

$$\chi_{Q_k}(x,t)\leq\frac{2^k}{M}T_{\lambda_{k-1}}(x,t).$$

Thanks to this and noticing that $T_{\lambda_k}\leq T_{\lambda_{k-1}}$ and $\lambda_k\leq M$, one deduces from (37) and the Gagliardo-Nirenberg inequality that

$$\begin{aligned}
J_k\ \leq\ &CM\int_0^{\mathcal{T}}\int_{\mathcal{M}}|\bar{\omega}|T_{\lambda_k}d\mathcal{M}dt=CM\int_0^{\mathcal{T}}\int_{\mathcal{M}}|\bar{\omega}|T_{\lambda_k}\chi_{Q_k}^{\frac{4}{3}}d\mathcal{M}dt\\
\leq\ &\frac{C}{M^{\frac{1}{3}}}16^{\frac{k}{3}}\int_0^{\mathcal{T}}\int_{\mathcal{M}}|\bar{\omega}|T_{\lambda_{k-1}}^{\frac{7}{3}}d\mathcal{M}dt\leq\frac{C}{M^{\frac{1}{3}}}16^{\frac{k}{3}}\int_0^{\mathcal{T}}\|\bar{\omega}\|\|\mathcal{T}_{\lambda_{k-1}}\|_{L^{\frac{14}{3}}(\mathcal{M})}^{\frac{7}{3}}dt\\
\leq\ &\frac{C}{M^{\frac{1}{3}}}16^{\frac{k}{3}}\int_0^{\mathcal{T}}\|\mathcal{T}_{\lambda_{k-1}}\|^{\frac{1}{3}}\left(\|\mathcal{T}_{\lambda_{k-1}}\|^2+\|\nabla T_{\lambda_{k-1}}\|^2\right)dt\\
\leq\ &\frac{C_*}{M^{\frac{1}{3}}}16^{\frac{k}{3}}J_{k-1}^{\frac{7}{6}},\qquad k=2,3,4,\cdots,
\end{aligned}$$

that is

$$J_k \le \frac{C_*}{M^{\frac{1}{3}}} 16^{\frac{k}{3}} J_{k-1}^{\frac{7}{6}}, \qquad k = 2, 3, 4, \cdots, \tag{38}$$

where C depends on $\|\bar{\omega}\|_{L^\infty(0,\mathcal{T};L^2(\mathcal{M}))}$. Setting

$$a = 64^7 C_*^6 M^{-2}, \quad b = 64, \quad S_k = ab^k J_k,$$

and by simple calculations, one can check from (38) that

$$S_{k+1} = ab^{k+1} J_{k+1} \le (ab^k J_k)^{\frac{7}{6}} = S_k^{\frac{7}{6}}$$

and, thus,

$$64^{k+8} C_*^6 M^{-2} J_{k+1} = S_{k+1} \le S_1^{\left(\frac{7}{6}\right)^k} = \left[64^8 C_*^6 M^{-2} J_1\right]^{\left(\frac{7}{6}\right)^k}, \quad k = 1, 2, \cdots. \tag{39}$$

Recalling the definition of J_k and applying Proposition 5 lead to

$$\begin{aligned} J_1 &= \sup_{t\in(0,\mathcal{T})} \|T_{\lambda_1}\|^2(t) + \int_0^{\mathcal{T}} (\mu_T \|\nabla_h T_{\lambda_1}\|^2 + \nu_T \|\partial_p T_{\lambda_1}\|_w^2) dt \\ &\le \sup_{t\in(0,\mathcal{T})} \|T\|^2(t) + \int_0^{\mathcal{T}} (\mu_T \|\nabla_h T\|^2 + \nu_T \|\partial_p T\|_w^2) dt \le C_{**}. \end{aligned} \tag{40}$$

Choose M large enough such that $M \ge 2\max\{\|T_0\|_{L^\infty(\mathcal{M})}, T_{\text{crit}}, \|T_{b\ell}\|_{L^\infty((0,\mathcal{T})\times\Gamma_\ell)}, \|T_{b\ell}\|_{L^\infty((0,\mathcal{T})\times\Gamma_\ell)}\}$ and $64^8 C_*^6 M^{-2} C_{**} \le \frac{1}{2}$. Then, it follows from (39) and (40) that

$$64^{k+8} C_*^6 M^{-2} J_{k+1} = S_{k+1} \le \left(\frac{1}{2}\right)^{\left(\frac{7}{6}\right)^k}$$

and, thus, $\lim_{k\to\infty} J_k = 0$. This leads to the desired bound $T \le M$ on $\mathcal{M} \times (0, \mathcal{T})$. □

Proposition 7. *If Assumption 1 holds, we have the estimates*

$$\sup_{0\le t\le\mathcal{T}} \|\nabla(q_v, q_c, q_r, T)\|^2(t) + \int_0^{\mathcal{T}} \|\nabla^2(q_v, q_c, q_r, T)\|^2(t) dt \le K_3(\mathcal{T}),$$

for a continuous bounded function $K_3(\mathcal{T})$ determined by the quantities q_v^*, q_c^*, q_r^*, $K_2(\mathcal{T})$, $\|T_0\|_{L^\infty(\mathcal{M})}$, $\|(T_0, q_{v0}, q_{r0}, q_{c0})\|_{H^1(\mathcal{M})}$, and $\|(\overline{\mathbf{v}}_h, \overline{\omega})\|_{L^r(0,\mathcal{T};L^q(\mathcal{M}))}$.

Proof. We only give the details about the proof for the estimate of T, those for the moisture components are similar (actually simpler).

We first estimate the vertical derivative $\partial_p T$. Multiplying the thermodynamic equation by $-\partial_p^2 T$ and integrating the resultant over $\mathcal{M}$ yields

$$\begin{aligned} &\int_{\mathcal{M}} (-\partial_t T + \mathcal{D}^T T) \partial_p^2 T d\mathcal{M} \\ = &\int_{\mathcal{M}} \left[\overline{\mathbf{v}}_h \cdot \nabla_h T + \overline{\omega} \partial_p T - \widetilde{\kappa} \frac{T}{p} \overline{\omega} + \frac{c_l q_r}{\widetilde{C}} V \partial_p T + \widetilde{L}(S_{cd} - S_{ev})\right] \partial_p^2 T d\mathcal{M}. \end{aligned} \tag{41}$$

Following the derivations in [16] (see (87) and (88) there) we have

$$-\int_{\mathcal{M}} \partial_t T \partial_p^2 T d\mathcal{M} \ge \frac{d}{dt}\left(\frac{\|\partial_p T\|^2}{2} + \alpha_{0T} \int_{\mathcal{M}'} \left(\frac{T^2}{2} - T T_{b0}\right) d\mathcal{M}'\Big|_{p_0}\right) - C(\|\partial_p T\| + C_3), \tag{42}$$

$$\int_{\mathcal{M}} \mathcal{D}^T T \partial_p^2 T d\mathcal{M} \geq \frac{3}{4}\left(\mu_T \|\nabla_h \partial_p T\|^2 + \nu_T \|\partial_p^2 T\|_w^2\right) - C(\|\partial_p T\|^2 + C_4). \tag{43}$$

By the Hölder, Sobolev, and Young inequalities, one deduces

$$\begin{aligned}
&\int_{\mathcal{M}} (\overline{\mathbf{v}}_h \cdot \nabla_h T + \overline{\omega}\partial_p T)\partial_p^2 T d\mathcal{M} \\
\leq\ & C\|(\overline{\mathbf{v}}_h, \overline{\omega})\|_{L^q(\mathcal{M})} \|\nabla T\|_{L^{\frac{2q}{q-2}}(\mathcal{M})} \|\partial_p^2 T\| \leq C\|(\overline{\mathbf{v}}_h, \overline{\omega})\|_{L^q(\mathcal{M})} \|\nabla T\|^{1-\frac{3}{q}} \|\nabla T\|_{H^1(\mathcal{M})}^{1+\frac{3}{q}} \\
\leq\ & \eta\|\nabla^2 T\|^2 + C_\eta \left(\|(\overline{\mathbf{v}}_h, \overline{\omega})\|_{L^q(\mathcal{M})}^{\frac{2q}{q-3}} + C_5 \right) \|\nabla T\|^2,
\end{aligned} \tag{44}$$

for any positive number η. Moreover, by (25) and Proposition 6, we obtain by the Young inequality that

$$-\int_{\mathcal{M}} \widetilde{\kappa} \frac{T}{p} \overline{\omega} \partial_p^2 T d\mathcal{M} + \int_{\mathcal{M}} \frac{c_l q_r}{\widetilde{C}} V \partial_p T \partial_p^2 T d\mathcal{M} \leq \eta \|\partial_p^2 T\|_w^2 + C_\eta(\|\overline{\omega}\|^2 + C_6 \|\partial_p T\|^2), \tag{45}$$

for any positive number η. Due to the nonnegativity and uniform boundedness of T, q_v, q_c, q_r, it follows from the Young inequality that

$$\int_{\mathcal{M}} \widetilde{L}(S_{cd} - S_{ev}) \partial_p^2 T d\mathcal{M} \leq C \int_{\mathcal{M}} |\partial_p^2 T| d\mathcal{M} \leq \eta \|\partial_p^2 T\|^2 + C_\eta, \tag{46}$$

for any positive number η. Substituting (42)–(46) into (41), one obtains

$$\begin{aligned}
&\frac{d}{dt}\left(\frac{\|\partial_p T\|^2}{2} + \alpha_{0T} \int_{\mathcal{M}'} \left(\frac{T^2}{2} - T T_{b0} \right) d\mathcal{M}' \Big|_{p_0} \right) + \frac{3}{4}(\mu_T \|\nabla_h \partial_p T\|^2 + \nu_T \|\partial_p^2 T\|_w^2) \\
&\qquad \leq 3\eta \|\nabla^2 T\|^2 + C_\eta \left(\|(\overline{\mathbf{v}}_h, \overline{\omega})\|_{L^q(\mathcal{M})}^{\frac{2q}{q-3}} + \|\overline{\omega}\|^2 + C_7 \right)(\|\nabla T\|^2 + C_8),
\end{aligned} \tag{47}$$

for any positive number η.

Next, we estimate the horizontal gradient $\nabla_h T$. Multiplying equation (20) by $-\Delta_h T$ and integrating the resultant over $\mathcal{M}$ yields

$$\begin{aligned}
&\int_{\mathcal{M}} (-\partial_t T + \mathcal{D}^T T) \Delta_h T d\mathcal{M} \\
=\ & \int_{\mathcal{M}} \left[\overline{\mathbf{v}}_h \cdot \nabla_h T + \overline{\omega} \partial_p T - \widetilde{\kappa} \frac{T}{p} \overline{\omega} + \frac{c_l q_r}{\widetilde{C}} V \partial_p T + \widetilde{L}(S_{cd} - S_{ev}) \right] \Delta_h T d\mathcal{M}.
\end{aligned} \tag{48}$$

Following the derivations in [16] (see (94) and (95) there) we have

$$\begin{aligned}
-\int_{\mathcal{M}} \partial_t T \Delta_h T d\mathcal{M} \ &\geq\ \frac{d}{dt}\left(\frac{\|\nabla_h T\|^2}{2} + \alpha_{\ell T} \int_{\Gamma_\ell} \left(\frac{T^2}{2} - T T_{b\ell} \right) d\Gamma_\ell \right) \\
&\quad -C(C_9 + \|\nabla_h T\|),
\end{aligned} \tag{49}$$

$$\int_{\mathcal{M}} \mathcal{D}^T T \Delta_h T d\mathcal{M} \ \geq\ \mu_T \|\Delta_h T\|^2 + \nu_T \|\nabla_h \partial_p T\|_w^2 - C. \tag{50}$$

Similar to (44)–(46), we have

$$\int_{\mathcal{M}} (\overline{\mathbf{v}}_h \cdot \nabla_h T + \overline{\omega} \partial_p T) \Delta_h T d\mathcal{M} \leq \eta \|\nabla^2 T\|^2 + C_\eta \left(\|(\overline{\mathbf{v}}_h, \overline{\omega})\|_{L^q(\mathcal{M})}^{\frac{2q}{q-3}} + C_{10} \right) \|\nabla T\|^2, \tag{51}$$

$$-\int_{\mathcal{M}} \widetilde{\kappa} \frac{T}{p} \overline{\omega} \Delta_h T d\mathcal{M} + \int_{\mathcal{M}} \frac{c_l q_r}{\widetilde{C}} V \partial_p T \Delta_h T d\mathcal{M} \leq \eta \|\Delta_h T\|^2 + C_\eta(\|\overline{\omega}\|^2 + C_{11} \|\partial_p T\|_w^2), \tag{52}$$

$$\int_{\mathcal{M}} \widetilde{L}(S_{cd} - S_{ev})\Delta_h T d\mathcal{M} \leq \eta\|\Delta_h T\|^2 + C_\eta, \tag{53}$$

for any positive number η. Substituting (49)–(53) into (48) gives

$$\begin{aligned} &\frac{d}{dt}\left(\frac{\|\nabla_h T\|^2}{2} + \alpha_{\ell T}\int_{\Gamma_\ell}\left(\frac{T^2}{2} - TT_{b\ell}\right)d\Gamma_\ell\right) + \frac{3}{4}(\mu_T\|\Delta_h T\|^2 + \nu_T\|\nabla_h\partial_p T\|_w^2) \\ &\qquad \leq 3\eta\|\nabla^2 T\|^2 + C_\eta\left(\|(\overline{\mathbf{v}}_h, \overline{\omega})\|_{L^q(\mathcal{M})}^{\frac{2q}{q-3}} + \|\overline{\omega}\|^2 + C_{12}\right)(\|\nabla T\|^2 + C_{13}), \end{aligned} \tag{54}$$

for any positive number η.

Summing (47) with (54) yields

$$\begin{aligned} &\frac{3}{4}\Big(\mu_T\|\nabla_h\partial_p T\|^2 + \nu_T\|\partial_p^2 T\|_w^2 + \mu_T\|\Delta_h T\|^2 + \nu_T\|\nabla_h\partial_p T\|_w^2\Big) \\ &+\frac{d}{dt}\left(\frac{\|\nabla T\|^2}{2} + \alpha_{0T}\int_{\mathcal{M}'}\left(\frac{T^2}{2} - TT_{b0}\right)d\mathcal{M}'\Big|_{p_0} + \alpha_{\ell T}\int_{\Gamma_\ell}\left(\frac{T^2}{2} - TT_{b\ell}\right)d\Gamma_\ell\right) \\ \leq\ & 6\eta\|\nabla^2 T\|^2 + C_\eta\left(\|(\overline{\mathbf{v}}_h, \overline{\omega})\|_{L^q(\mathcal{M})}^{\frac{2q}{q-3}} + \|\overline{\omega}\|^2 + C_{14}\right)(\|\nabla T\|^2 + C_{15}), \end{aligned} \tag{55}$$

for any positive number η. Applying the elliptic estimate (see Proposition A.2 in [15]) to the elliptic equation $\mathcal{D}^T T = f$ subject to the boundary condition (17)–(19) leads to

$$\|\nabla^2 T\| \leq C(\|\mathcal{D}^T T\| + C_{16}\|\nabla T\| + C_{17}) \leq C(\|\Delta_h T\| + \|\partial_p^2 T\|_w^2 + C_{18}\|\nabla T\| + C_{19}).$$

Thanks to this, and noticing that

$$\int_{\mathcal{M}'}\left(\frac{T^2}{2} - TT_{b0}\right)d\mathcal{M}' = \frac{1}{2}\int_{\mathcal{M}'}\left[(T - T_{b0})^2 - T_{b0}^2\right]d\mathcal{M}' \geq -\int_{\mathcal{M}'} T_{b0}^2 d\mathcal{M}' \geq -C,$$

and

$$\int_{\Gamma_\ell}\left(\frac{T^2}{2} - TT_{b\ell}\right)d\Gamma_\ell \ = \ \frac{1}{2}\int_{\Gamma_\ell}\left[(T - T_{b\ell})^2 - T_{b\ell}^2\right]d\Gamma_\ell \geq -\frac{1}{2}\int_{\Gamma_\ell} T_{b\ell}^2 d\Gamma_\ell \geq -C,$$

the conclusion follows by applying the Grönwall inequality to (55). □

We are now ready to give the proof of our main result, Theorem 1.

Proof of Theorem 1. By Proposition 1, there is a unique local solution (T, q_v, q_c, q_r) to system (20)–(23), subject to (17)–(19). Due to Proposition 2 and Proposition 4, q_c, q_c, q_r, and T are all nonnegative and, thus, (T, q_v, q_c, q_r) is a local solution to the original system, subject to the corresponding initial and boundary conditions. By applying Proposition 1 inductively, one can extend the local solution to the maximal time of existence $\mathcal{T}_{\max}$ characterized by (24). We need to prove $\mathcal{T}_{\max} = \infty$. Assume, by contradiction, that $\mathcal{T}_{\max} < \infty$. Then, by Propositions 2–7, we have the estimate

$$\sup_{0\leq t\leq\mathcal{T}}\|(T, q_v, q_c, q_r)\|_{H^1(\mathcal{M})} \leq C_0,$$

for any $\mathcal{T} \in (0, \mathcal{T}_{\max})$, and C_0 is a positive constant independent of $\mathcal{T} \in (0, \mathcal{T}_{\max})$. This contradicts (24) and, thus, $\mathcal{T}_{\max} = \infty$, proving the conclusion. □

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