Classifying spaces of finite groups of tame representation type

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Abstract. Thanks to the work of Karin Erdmann, we know a great deal about the representation theory of blocks of finite groups with tame representation type. Our purpose here is to examine the p-completed classifying spaces of these blocks and their loop spaces. We pay special attention to the A_∞ algebra structures, and singularity and cosingularity categories.

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Preface

The purpose of this document is to describe the A_∞ algebra structures on the cochains on the classifying space C^*BG, and on the chains on the loop space of the classifying space $C_*\Omega BG^\wedge_2$, when G is a finite group with dihedral, semidihedral, or generalised quaternion Sylow 2-subgroups. These are the groups whose group algebras have tame representation type. In some cases, we are able to completely describe the singularity and cosingularity categories of these, while in others this seems to be more difficult.

Part of the point of this work is to describe a variety of computational techniques for approaching questions such as these. The main tool in the dihedral and semidihedral case, as it was in the cyclic case [**22**, **23**], is to put a grading on the basic algebra of the principal block. This gives rise to a double grading on group cohomology, and a triple grading on the Hochschild cohomology of the group cohomology. This technique gives us no information in the generalised quaternion case, but an explicit computation involving minimal resolutions comes to our rescue in this case.

One curious outcome of this work is that if G has semidihedral or generalised quaternion Sylow 2-subgroups, and no normal subgroup of index two, then C^*BG is formal, meaning that it is quasi-isomorphic to its cohomology H^*BG with zero differential, see Theorems 3.7.16 and 4.7.4. The same happens in one of the other cases with semidihedral Sylow 2-subgroups, see Theorem 3.13.13. This also happens for finite groups with elementary abelian Sylow 2-subgroups in characteristic two, but necessary and sufficient conditions for this to occur are not known.

Acknowledgements. The investigations described in this work grew out of work with John Greenlees [**22**, **23**] in which we described the situation for finite groups with cyclic Sylow p-subgroups. I would like to take the opportunity to thank him for his influence on this work, which would not have been carried out without his input and encouragement. My thanks go to the University of Warwick (visits supported by EPSRC grant EP/P031080/1) and the Isaac Newton Institute in Cambridge (programme *'Groups, representations and applications: new perspectives'*, supported by EPSRC grant EP/R014604/1), both of whose hospitality allowed me extensive discussions with Greenlees in 2019, early 2020, and 2022, leading to the work from which this grew. I thank Srikanth Iyengar for patiently explaining various pieces of commutative algebra to me, and David Craven for correspondence about the structure of tame blocks and various errors in the literature. I have flagged these under the index entry "errors."

Finally, this work would probably never have happened without the confinement imposed by the Covid-19 pandemic, but there's frankly no way I'm going to thank this cursèd virus.

CHAPTER 1

Introduction and background

1.1. Introduction

This paper is a sequel to Benson and Greenlees [**22**, **23**]. In those papers, we examined the structure of H^*BG and $H_*\Omega BG$ as A_∞ algebras, for finite groups G with cyclic Sylow p-subgroups, over a field k of characteristic p, and elucidated the structure of their singularity and cosingularity categories. Here, we examine the case of blocks with dihedral, semidihedral, or (generalised) quaternion defect groups in characteristic two. The reason for the interest in this class of blocks is that these are the ones where the group algebra has tame representation type. The results are given in the introductory sections of the three chapters, dealing with the three types of defect groups. The ring structure on the homology of the loop space $H_*\Omega BG^\wedge_2$ was first computed by Levi [**167**] in these cases. The following table shows where the discussions of the various cases can be found. Note that for tame type, the homotopy type of BG is determined by the number of conjugacy classes of elements of order two and four; see Sections 2.7, 3.5, and 4.6.

Sylow p-subgroup	ccls of elts of order 2	4	HH^* of H^*BG	$H_*\Omega BG^\wedge_2$	$HH^*C^*BG \cong HH^*C_*\Omega BG^\wedge_2$	A_∞ structure of H^*BG	$H_*\Omega BG^\wedge_2$
Cyclic	—	—	[**22**]	[**22**]	[**22**]	[**23**]	[**23**]
Dihedral	1	1	2.8.6	2.8.9	2.8.12	2.8.7	2.8.10
	2	1	2.13.4	2.13.7	2.13.9	2.13.5	2.13.8
	3	1	2.3.2	[**200**], 2.6.3	[**200**], 2.6.3	2.4.2	2.5
Semidihedral	1	1	3.7.7	3.7.4	3.7.18	3.7.16	3.8.2
	2	1	3.13.3	3.14.1	3.13.15	3.13.13	3.14.2
	1	2					
	2	2		[**114**]	[**114**]		3.4
Generalised quaternion	1	1	4.7.5	4.7.5	4.7.5	4.7.4	4.7.5
	1	2	4.8.4			4.9.2	
	1	3	4.3.3	[**135**]	[**135**]		4.4

For ease of reference, we list here the cohomology algebras in the various cases of Sylow subgroups, and the loop space homology in those cases where the group is not p-nilpotent (because if G is p-nilpotent then $H_*\Omega BG^\wedge_p \cong \mathsf{k}G/O^p(G)$). The degrees are written homologically, followed by the degrees coming from the internal grading in the cases where they exist.

Cohomology algebras $H^*BG \cong \mathsf{Ext}^*_{\mathsf{k}G}(\mathsf{k},\mathsf{k})$

Cyclic, order p^n, inertial index $q|(p-1)$:

$$\mathsf{k}[x]\otimes\Lambda(t), \qquad |x| = -(2q, p^n), \qquad |t| = -(2q-1, p^n-(p^n-1)/q).$$

Klein four group, one class of involutions:

$$\mathsf{k}[\xi,\eta,t]/(\xi\eta+t^3), \qquad |\xi|=|\eta|=-(3,3), \qquad |t|=-(2,2).$$

Klein four group, three classes of involutions:

$$\mathsf{k}[x,y], \qquad |x|=|y|=-(1,1).$$

Dihedral, order $4q$ ($q\geqslant 2$), one class of involutions:

$$\mathsf{k}[\xi,\eta,t]/(\xi\eta), \quad |\xi|=-(3,q+1,q), \quad |\eta|=-(3,q,q+1), \quad |t|=-(2,q,q).$$

Dihedral, order $4q$ ($q\geqslant 2$), two classes of involutions:

$$\mathsf{k}[\xi,y,t]/(\xi y), \quad |\xi|=-(3,q+1,q), \quad |y|=-(1,0,1), \quad |t|=-(2,q,q).$$

Dihedral, order $4q$ ($q\geqslant 2$), three classes of involutions:

$$\mathsf{k}[x,y,t]/(xy), \quad |x|=-(1,1,0), \quad |y|=-(1,0,1), \quad |t|=-(2,q,q).$$

Semidihedral, order $8q$ ($q\geqslant 2$), one class of involutions, one of elements of order four:

$$\mathsf{k}[x,y,z]/(x^2y+z^2), \qquad |x|=-(3,q+1), \qquad |y|=-(4,4q), \qquad |z|=-(5,3q+1).$$

Semidihedral, order $8q$ ($q\geqslant 2$), two classes of involutions, one of elements of order four:

$$\mathsf{k}[x,y,z]/(x^2y+z^2), \qquad |x|=-(1,1-q), \qquad |y|=-(4,4q), \qquad |z|=-(3,q+1).$$

Semidihedral, order $8q$ ($q\geqslant 2$), one class of involutions, two of elements of order four:

$$\mathsf{k}[y,z,w,v]/(y^3,vy,yz,v^2+z^2w), \qquad |y|=-1, \quad |z|=-3, \quad |w|=-4, \quad |v|=-5.$$

Semidihedral, order $8q$ ($q\geqslant 2$), two classes of involutions, two of elements of order four:

$$\mathsf{k}[x,y,z,w]/(xy,y^3,yz,z^2+x^2w), \qquad |x|=|y|=-1, \qquad |z|=-3, \qquad |w|=-4.$$

Quaternion or generalised quaternion of order $8q$, one class of elements of order four:

$$\mathsf{k}[z]\otimes\Lambda(y), \qquad |z|=-4, \qquad |y|=-3.$$

Generalised quaternion of order $8q$, two classes of elements of order four:

$$\mathsf{k}[y,z]/(y^4), \qquad |y|=-1, \qquad |z|=-4.$$

Quaternion of order 8, three classes of elements of order four:

$$\mathsf{k}[u,v,z]/(u^2+uv+v^2, u^2v+uv^2), \qquad |u|=|v|=-1, \qquad |z|=-4.$$

Generalised quaternion of order $8q$ ($q\geqslant 2$), three classes of elements of order four:

$$\mathsf{k}[x,y,z]/(xy,x^3+y^3), \qquad |x|=|y|=-1, \qquad |z|=-4.$$

Loop space homology $H_*\Omega BG^\wedge_p$

Cyclic, order p^n, inertial index $q|(p-1)$ $(q \geqslant 2)$:

$$\mathsf{k}[\tau]\otimes\Lambda(\xi), \qquad |\tau| = (2q-2, p^n - (p^n-1)/q), \qquad |\xi| = (2q-1, p^n).$$

Dihedral, order $4q$ $(q \geqslant 1)$, one class of involutions:

$$\Lambda(\tau)\otimes\mathsf{k}\langle\alpha,\beta \mid \alpha^2=0, \beta^2=0\rangle, \quad |\tau| = (1,q,q), \quad |\alpha| = (2,q+1,q), \quad |\beta| = (2,q,q+1).$$

Dihedral, order $4q$ $(q \geqslant 2)$, two classes of involutions:

$$\Lambda(\tau)\otimes\mathsf{k}\langle\alpha,Y \mid \alpha^2=0, Y^2=0\rangle, \quad |\tau| = (1,q,q), \quad |\alpha| = (2,q+1,q), \quad |Y| = (0,0,1).$$

Semidihedral, order $8q$ $(q \geqslant 2)$, one class of involutions, one of elements of order four:

$$\Lambda(\hat{x},\hat{y})\otimes\mathsf{k}[\hat{z}], \qquad |\hat{x}| = (2,q+1), \qquad |\hat{y}| = (3,4q), \qquad |\hat{z}| = (4,3q+1).$$

Semidihedral, order $8q$ $(q \geqslant 2)$, two classes of involutions, one of elements of order four:

$$\Lambda(\hat{x},\hat{y})\otimes\mathsf{k}[\hat{z}], \qquad |\hat{x}| = (0,1-q), \qquad |\hat{y}| = (3,4q), \qquad |\hat{z}| = (2,q+1).$$

Semidihedral, order $8q$ $(q \geqslant 2)$, one class of involutions, two of elements of order four:

$$\Lambda(\eta)\otimes\mathsf{k}\langle\hat{y},\hat{z} \mid \hat{y}^2=\hat{z}^2=0\rangle, \qquad |\eta| = 1, \qquad |\hat{y}| = 0, \qquad |\hat{z}| = 2.$$

Quaternion or generalised quaternion, one class of elements of order four:

$$\Lambda(\hat{z})\otimes\mathsf{k}[\hat{y}], \qquad |\hat{z}| = 3, \qquad |\hat{y}| = 2.$$

Quaternion or generalised quaternion, two classes of elements of order four:

$$\Lambda(\hat{y},\hat{z})\otimes\mathsf{k}[\eta], \qquad |\hat{y}| = 0, \qquad |\hat{z}| = 3, \qquad |\eta| = 2.$$

1.2. Notation and conventions

In this chapter, we give some of the background, and set the stage for this project. We begin with the notations and conventions used throughout.

We use the following standard group theoretic notations. For a finite group G, we write $O_p(G)$ for the largest normal p-subgroup of G and $O_{p'}(G)$ for the largest normal p'-subgroup, i.e., the largest normal subgroup of order not divisible by p. When $p=2$, we write $O(G)$ for $O_{2'}(G)$, the largest normal odd order subgroup of G.

We write $O^p(G)$ for the smallest normal subgroup of G for which the quotient is a p-group, and $O^{p'}(G)$ for the smallest normal subgroup for which the quotient is a p'-group.

We write $\Gamma L(n,p^m)$, $GL(n,p^m)$, and $SL(n,p^m)$ for the groups of semi-linear automorphisms, linear automorphisms, and special (i.e., determinant one) linear automorphisms of a vector space of dimension n over $\mathbb{F}_{p^m}$. We write $P\Gamma L(n,p^m)$, $PGL(n,p^m)$, and $PSL(n,p^m)$ for the corresponding groups of projective transformations, namely the quotients of these groups by the subgroups of scalar transformations. Similarly, we write $\Gamma U(n,p^m)$, $GU(n,p^m)$, and $SU(n,p^m)$ for the corresponding groups of unitary transformations of a vector space of dimension n over $\mathbb{F}_{p^{2m}}$ with respect to the conjugation given by the Frobenius automorphism of order two of the field. We write $P\Gamma U(n,p^m)$, $PGU(n,p^m)$, and $PSU(n,p^m)$ for the quotient by the subgroup of scalar transformations. Closely related groups $SL^\pm(2,p^m)$, $SU^\pm(2,p^m)$

will be defined in Section 3.5, and their isoclinic groups $SL^\circ(2,p^m)$ and $SU^\circ(2,p^m)$ will be defined in Section 4.5.

All chains, cochains, homology and cohomology will have coefficients in a field k. So when we write C^*X, H^*X, C_*X, H_*X, we mean $C^*(X;\mathsf{k})$, $H^*(X;\mathsf{k})$, $C_*(X;\mathsf{k})$, $H_*(X;\mathsf{k})$ respectively. Since we are interested in both homology and cohomology, we shall write all degrees homologically, so that for example cohomology elements are given negative degrees.

We shall frequently use the fact that C^*BG, regarded as a differential graded algebra, is quasi-isomorphic to the differential graded algebra of endomorphisms of a projective resolution of the trivial module, $\mathsf{End}_{\mathsf{k}G}(P_*)$. Such a quasi-isomorphism induces an isomorphism in cohomology $H^*BG \cong \mathsf{Ext}^*_{\mathsf{k}G}(\mathsf{k},\mathsf{k})$. The link between the two is given by the Rothenberg–Steenrod construction, as explained for example in [**196**], or Section 4 of [**25**].

1.3. A_∞ algebras and quasi-isomorphisms

The concept of A_∞ algebra was introduced by Stasheff [**205**,**206**], and further information can be found in Kadeishvili [**151**], Keller [**154**, **155**], Boardman and Vogt [**27**].

Recall that an A_∞ algebra over a field k is a $\mathbb{Z}$-graded vector space $\mathfrak{a}$ with graded maps $m_n\colon \mathfrak{a}^{\otimes n} \to \mathfrak{a}$ of degree $n-2$ for $n \geqslant 1$ satisfying

$$\sum_{r+s+t=n} (-1)^{r+st} m_{r+1+t}(\mathsf{id}^{\otimes r} \otimes m_s \otimes \mathsf{id}^{\otimes t}) = 0 \tag{1.3.1}$$

for $n \geqslant 1$.

Remark 1.3.2. The first few cases of (1.3.1) are as follows.

$$\begin{aligned} m_1m_1 &= 0,\\ m_1m_2 &= m_2(m_1\otimes \mathsf{id} + \mathsf{id}\otimes m_1),\\ m_2(\mathsf{id}\otimes m_2 - m_2\otimes \mathsf{id}) &= m_1m_3 + m_3(m_1\otimes\mathsf{id}\otimes\mathsf{id} + \mathsf{id}\otimes m_1\otimes \mathsf{id} + \mathsf{id}\otimes\mathsf{id}\otimes m_1).\end{aligned}$$

The map m_1 is therefore a differential. The map m_2 is not necessarily associative, but it is a derivation with respect to m_1, and induces an associative product on $H_*\mathfrak{a}$. The map m_3 induces the Massey triple product on $H_*\mathfrak{a}$.

Example 1.3.3. A DG algebra (differential graded algebra) $\mathfrak{a}$ can be regarded as an A_∞ algebra with m_1 the differential, m_2 the product, and $m_i = 0$ for $i > 2$.

A morphism of A_∞ algebras $f\colon \mathfrak{a} \to \mathfrak{a}'$ consists of graded maps $f_n\colon \mathfrak{a}^{\otimes n} \to \mathfrak{a}'$ of degree $n-1$ satisfying

$$\sum_{r+s+t=n} (-1)^{r+st} f_{r+1+t}(\mathsf{id}^{\otimes r}\otimes m_s\otimes \mathsf{id}^{\otimes t}) = \sum_{i_1+\cdots+i_r=n} (-1)^\sigma m'_r(f_{i_1}\otimes\cdots\otimes f_{i_r}) \tag{1.3.4}$$

where in the sum on right hand side, $\sigma = \sum_{j=1}^{r-1}(r-j)(i_j-1)$ and m'_r are the operations in B.

Remark 1.3.5. The first two cases of (1.3.4) are as follows.

$$\begin{aligned} f_1m_1 &= m'_1f_1,\\ f_1m_2 &= m'_2(f_1\otimes f_1) + m'_1f_2 + f_2(m_1\otimes\mathsf{id} + \mathsf{id}\otimes m_1).\end{aligned}$$

Thus f_1 is a map of complexes with respect to the differential m_1, and commutes with the product m_2 up to a homotopy given by f_2.

DEFINITION 1.3.6. The morphism f is said to be a *quasi-isomorphism* of A_∞ algebras if f_1 induces an isomorphism in homology with respect to the differentials m_1, m'_1.

We say that an A_∞ algebra $\mathfrak{a}$ is *formal* if it is quasi-isomorphic to an A_∞ algebra with $m_i = 0$ for $i \neq 2$.

EXAMPLE 1.3.7. If $\mathfrak{a}$ is a DG algebra then it is formal as an A_∞ algebra if and only if it is formal as a DG algebra; namely if and only if there are quasi-isomorphisms of DG algebras $\mathfrak{a} \leftarrow \mathfrak{a}' \to H_*\mathfrak{a}$, where $H_*\mathfrak{a}$ is regarded as a DG algebra with zero differential.

A theorem of Kadeishvili [**151**] (see also Keller [**154**,**155**], Merkulov [**181**], Petersen [**190**]) may be stated as follows.

THEOREM 1.3.8. *Suppose that we are given an A_∞ algebra $\mathfrak{a}$ over a field k. Let $Z_*(\mathfrak{a})$ be the cocycles, $B^*(\mathfrak{a})$ be the coboundaries, and $H_*(\mathfrak{a}) = Z_*(\mathfrak{a})/B_*(\mathfrak{a})$, with respect to the differential m_1. Choose a vector space splitting $f_1\colon H_*(\mathfrak{a}) \to Z_*(\mathfrak{a}) \subseteq \mathfrak{a}$ of the quotient. Then the homology $H_*\mathfrak{a}$ has an A_∞ structure with $m_1 = 0$ and m_2 the multiplication on $H_*\mathfrak{a}$ induced by the multiplication on $\mathfrak{a}$, and f_1 extends to a quasi-isomorphism of A_∞ algebras $f\colon H_*\mathfrak{a} \to \mathfrak{a}$.*

If $\mathfrak{a}$ happens to carry auxiliary gradings respected by the maps m_i then $H_*\mathfrak{a}$ inherits the grading, and the A_∞ algebra structure maps m_i on $H_*\mathfrak{a}$ and the quasi-isomorphism f can be chosen to respect the gradings.

PROOF. The idea of the proof is an inductive procedure which goes as follows. The morphisms

$$m_2(f_1 \otimes f_1),\ f_1 m_2\colon H_*\mathfrak{a} \otimes H_*\mathfrak{a} \to \mathfrak{a}$$

are homotopic. In the presence of a grading on $\mathfrak{a}$, these maps preserve the grading, so a homogeneous homotopy can be chosen. This is a morphism $f_2\colon H_*\mathfrak{a} \to H_*\mathfrak{a} \to \mathfrak{a}$ of degree $(1,0)$ such that

$$f_1 m_2 = m_1 f_2 + m_2(f_1 \otimes f_1).$$

Next, consider the map

$$m_2(f_1 \otimes f_2 - f_2 \otimes f_1) + f_2(1 \otimes m_2 - m_2 \otimes 1)\colon (H_*\mathfrak{a})^{\otimes 3} \to \mathfrak{a}$$

of degree $(1,0)$. Composing with $m_1\colon \mathfrak{a} \to \mathfrak{a}$, a short calculation shows that we get zero. So we can add something in the image of f_1 to get a coboundary. Thus there exist maps $m_3\colon (H_*\mathfrak{a})^{\otimes 3} \to H_*\mathfrak{a}$ of degree $(1,0)$ and $f_3\colon (H_*\mathfrak{a})^{\otimes 3} \to \mathfrak{a}$ of degree $(2,0)$ such that

$$f_1 m_3 - m_1 f_3 = m_2(f_1 \otimes f_2 - f_2 \otimes f_1) + f_2(1 \otimes m_2 - m_2 \otimes 1).$$

Continuing this way, we obtain maps m_i of degree $(i-2,0)$ giving an A_∞ structure on $H_*\mathfrak{a}$, and f_i of degree $(i-1,0)$ giving a quasi-isomorphism $H_*\mathfrak{a} \to \mathfrak{a}$. Note that equation (1.3.4) simplifies slightly because $m_1 = 0$ on $H_*\mathfrak{a}$. If $\mathfrak{a}$ is a DG algebra then it simplifies further, because $m_i = 0$ for $i > 2$ on $\mathfrak{a}$. □

DEFINITION 1.3.9. We say that an A_∞ algebra $\mathfrak{a}$ is *minimal* if $m_1 = 0$. Given any A_∞ algebra $\mathfrak{a}$, a minimal A_∞ algebra quasi-isomorphic to $\mathfrak{a}$ is called a *minimal model* for $\mathfrak{a}$. Kadeishvili's Theorem 1.3.8 says that every A_∞ algebra $\mathfrak{a}$ has a minimal model, and it is $H_*\mathfrak{a}$ endowed with a suitable A_∞ structure. If $\mathfrak{a}$ carries an internal grading preserved by the structure maps then the structure maps on a minimal model may be taken to preserve the internal grading.

REMARK 1.3.10. In the inductive procedure described in the proof of Theorem 1.3.8, if it happens that

$$m_2(f_1 \otimes f_2 - f_2 \otimes f_1) + f_2(1 \otimes m_2 - m_2 \otimes 1) = 0$$

then it follows that we may take $m_i\colon (H_*\mathfrak{a})^{\otimes i} \to H_*\mathfrak{a}$ and $f_i\colon (H_*\mathfrak{a})^{\otimes i} \to \mathfrak{a}$ to be zero for all $i \geqslant 3$. In this case, we deduce that $\mathfrak{a}$ is formal. We shall apply this in Section 4.7 the case of finite groups with a generalised quaternion Sylow 2-subgroup and no normal subgroup of index two.

1.4. Hochschild cohomology

If $\mathfrak{a}$ is an A_∞ algebra, there is a spectral sequence

$$HH^*H_*\mathfrak{a} \Rightarrow HH^*\mathfrak{a}, \tag{1.4.1}$$

where the left hand side is the Hochschild cohomology of the homology algebra $H_*\mathfrak{a}$, not taking into account any higher structure. The right hand side is the Hochschild cohomology of $\mathfrak{a}$, which is described as follows (see §3 of Getzler and Jones [**123**], and Definition 12.6 of Stasheff [**207**]). The bar resolution $\mathbb{B}(\mathfrak{a}) = \bigoplus_{n\geqslant 0} \mathfrak{a}^{\otimes(n+2)}$ has a differential defined by

$$\begin{aligned}\partial(x\otimes[a_1|\dots|a_n]\otimes y) &= \sum_{j=0}^{n} \pm m_{j+1}(x,a_1,\dots,a_j)\otimes[a_{j+1}|\dots|a_n]\otimes y\\ &+ \sum_{0\leqslant i+j\leqslant n} \pm x\otimes[a_1|\dots|a_i|m_j(a_{i+1},\dots,a_{i+j})|a_{i+j+1}|\dots|a_n]\otimes y\\ &+ \sum_{j=0}^{n} \pm x\otimes[a_1|\dots|a_{n-j}]\otimes m_{j+1}(a_{n-j+1},\dots,a_n,y).\end{aligned}$$

The signs are given by the usual sign rules. If the only non-vanishing m_i is m_2 then this agrees with the classical notion of Hochschild cohomology of an algebra. If $\mathfrak{a}$ is a DG algebra, so the only non-vanishing m_i are m_1 and m_2, we can think of this as the total complex of the usual double complex defining Hochschild cohomology of a DG algebra.

If M is an $\mathfrak{a}$-$\mathfrak{a}$-bimodule, we have Hochschild cochains

$$\mathsf{Hom}_{\mathfrak{a},\mathfrak{a}}(\mathfrak{a}^{\otimes(n+2)}, M) \cong \mathsf{Hom}_k(\mathfrak{a}^{\otimes n}, M)$$

with differential

$$(\delta f)[a_1|\dots|a_n] = m_1 f[a_1|\dots|a_n] + \sum_{0\leqslant i+j\leqslant n} \pm f[a_1|\dots|a_i|m_j(a_{i+1},\dots,a_{i+j})|a_{i+j+1}|\dots|a_n].$$

The homology of this complex is $HH^*(\mathfrak{a},M)$. We write $HH^*\mathfrak{a}$ for $HH^*(\mathfrak{a},\mathfrak{a})$. The filtration of $\mathbb{B}(\mathfrak{a})$ by number of bars gives a filtration of Hochschild cochains, giving rise to the conditionally convergent spectral sequence (1.4.1). See Section 5 of [**22**] for details. The differentials in this spectral sequence are determined by the maps m_i.

Kadeishvili [**150**] discusses the relationship between A_∞ structure and Hochschild cohomology, and obtains the following.

Proposition 1.4.2. *Suppose that the action of m_i on $H_*\mathfrak{a}$ is zero for $i = 1$ and $2 < i < n$. Then $m_n \colon H_*\mathfrak{a}^{\otimes n} \to H_*\mathfrak{a}$ is a Hochschild n-cocycle on $H_*\mathfrak{a}$, of internal (homological) degree $n-2$.*

If $f \colon H_*\mathfrak{a} \to H_*\mathfrak{a}$ is a quasi-isomorphism to another A_∞ structure m'_i on $H_*\mathfrak{a}$ satisfying the same assumptions, with f_1 equal to the identity, then $m_n - m'_n$ is a Hochschild coboundary, and all such occur this way. Thus valid choices for m_n form a well defined class in degree $(-n, n-2)$ in $HH^*H_*\mathfrak{a}$.

If $\mathfrak{a}$ is graded and the A_∞ structure preserves internal degree then m_n represents a Hochschild class of degree $(-n, n-2, 0)$ on $H_*\mathfrak{a}$. Thus if, regarding $\mathfrak{a}$ as a doubly graded algebra and ignoring the m_n with $n > 2$, the Hochschild cohomology ring $HH^*H_*\mathfrak{a}$ has no non-zero elements of degree $(-n, n-2, 0)$ with $n > 2$, then the A_∞ structure on $\mathfrak{a}$ is formal.

Proof. Equation (1.3.1) implies that

$$-m_2(\mathsf{id} \otimes m_n) + \sum_{r=0}^{n-1} (-1)^r m_n(\mathsf{id}^{\otimes r} \otimes m_2 \otimes \mathsf{id}^{\otimes(n-r-1)}) + (-1)^n m_2(m_n \otimes \mathsf{id}) = 0$$

and so m_n is a Hochschild cocycle on the cohomology ring. If $f \colon H_*\mathfrak{a} \to H_*\mathfrak{a}$ is a quasi-isomomorphism with f_1 equal to the identity then equation 1.3.4 implies that

$$f_1 m_n + \sum_{r=0}^{n-2} (-1)^r f_{n-1}(\mathsf{id}^{\otimes r} \otimes m_2 \otimes \mathsf{id}^{\otimes(n-r-2)}) \\ = m'_2(f_1 \otimes f_{n-1}) + (-1)^n m'_2(f_{n-1} \otimes f_1) + m'_n(f_1 \otimes \cdots \otimes f_1).$$

Now f_1 is the identity and $m'_2 = m_2$, so this becomes

$$m'_n - m_n = -m_2(\mathsf{id} \otimes f_{n-1}) + \sum_{r=0}^{n-2} (-1)^r f_{n-1}(\mathsf{id}^{\otimes r} \otimes m_2 \otimes \mathsf{id}^{\otimes(n-r-2)}) + (-1)^{n-1} m_2(f_{n-1} \otimes \mathsf{id}).$$

The right hand side is the formula for the Hochschild coboundary of f_{n-1}, which may be taken to be any $(n-1)$-cochain.

The last part follows by an easy inductive argument on n, beginning with $n = 3$. □

Definition 1.4.3. We say that an A_∞ algebra $\mathfrak{a}$ is *intrinsically formal* if given another A_∞ algebra $\mathfrak{a}'$ and an isomorphism of associative algebras $H_*\mathfrak{a} \cong H_*\mathfrak{a}'$ there is a quasi-isomorphism $\mathfrak{a} \to \mathfrak{a}'$ inducing it. Clearly an intrinsically formal A_∞ algebra is formal.

If $\mathfrak{a}$ carries an internal grading then the isomorphism $H_*\mathfrak{a} \cong H_*\mathfrak{a}'$ is required to preserve the induced internal grading. So a graded A_∞ algebra may be intrinsically formal while the corresponding ungraded algebra is not.

Remark 1.4.4. Proposition 1.4.2 implies that if there are no non-zero classes of degree $(-n, n-2)$ in $HH^*H_*\mathfrak{a}$ with $n > 2$ then $\mathfrak{a}$ is intrinsically formal. If $\mathfrak{a}$ carries an internal grading then we only require that there are no non-zero classes of degree $(-n, n-2, 0)$, which is a weaker condition.

Theorem 1.4.5. *Let $x_1, \ldots, x_n$ ($n \geqslant 3$) be elements of the homology of a DG algebra $\mathfrak{a}$, and suppose that the Massey product $\langle x_1, \ldots, x_n \rangle$ is non-empty. Consider an A_∞ structure on $H_*\mathfrak{a}$ given by Kadeishvili's theorem, and suppose that $m_i = 0$ for $2 < i \leqslant n-2$. Then*

$$\varepsilon m_n(x_1, \ldots, x_n) \in \langle x_1, \ldots, x_n \rangle,$$

where $\varepsilon = (-1)^{\sum_{j=1}^{n-1}(n-j)|x_j|}$.

PROOF. This is described in Theorem 3.1 of Lu, Palmieri, Wu and Zhang [**177**], and corrected in Theorem 3.2 of Buijs, Moreno-Fernández and Murillo [**45**]. $\square$

1.5. The Gerstenhaber circle product

Let A be an associative k-algebra and M an A-A-bimodule. Gerstenhaber [**122**], introduced a circle product on Hochschild cocycles of A, that are related to A_∞ structure, as we now explain. If $f\colon \mathbb{B}(A) \to M$ is an m-cochain and $g\colon \mathbb{B}(A) \to M$ is an n-cochain, we define $f \circ_i g\colon \mathbb{B}(A) \to M$ to be the $(m+n-1)$-cochain given on the basis by

$$f \circ_i g\,[a_1|\dots|a_i|b_1|\dots|b_n|a_{i+1}|\dots|a_m] = f[a_1|\dots|a_i|\,g[b_1|\dots|b_n]\,|a_{i+1}|\dots|a_m].$$

We then define the *circle product*

$$f \circ g = \sum_{i=0}^{m} (-1)^{(n+1)i} f \circ_i g.$$

EXAMPLE 1.5.1. The statement $m_2 \circ m_2 = 0$ is equivalent to the associativity of multiplication, because

$$(m_2 \circ m_2)[a_1|a_2|a_3] = m_2(m_2(a_1,a_2),a_3) - m_2(a_1,m_2(a_2,a_3)) = (a_1a_2)a_3 - a_1(a_2a_3).$$

THEOREM 1.5.2. *The circle product is related to the differential, the cup product, and Gerstenhaber bracket on cochains by the formulas*

$$\begin{aligned}
\delta f &= (-1)^{|f|+1} m_2 \circ f - f \circ m_2 \\
f \cup g &= (m_2 \circ_0 f) \circ_{m-1} g \\
f \circ \delta g - \delta(f \circ g) + (-1)^{n-1}\delta f \circ g &= (-1)^{n-1}(g \cup f - (-1)^{mn} f \cup g) \\
[f,g] &= f \circ g - (-1)^{|f||g|} g \circ f.
\end{aligned}$$

where m_2 is the multiplication in A.

PROOF. This is proved in Sections 5–7 of [**122**]. $\square$

In terms of the circle product, equation 1.3.1 can be rewritten as

$$\sum_{i+j=n+1} (-1)^i m_i \circ m_j = 0. \tag{1.5.3}$$

So suppose, for example, that $\mathfrak{a}$ is an A_∞ algebra with $m_1 = 0$, and $m_i = 0$ for $2 < i < n$. We saw in Proposition 1.4.2 that m_n is a Hochschild n-cocycle on $A = H_*\mathfrak{a}$. We can see this easily using this formulation, since the condition reduces to $m_2 \circ m_n + (-1)^n m_n \circ m_2 = 0$, or equivalently $\delta m_n = 0$. It also follows from this formulation that under these circumstances, for $n < i < 2n-2$ the condition is again that m_i should be a Hochschild n-cocycle, and if these are all coboundaries then they can be rechosen to be zero. Then the condition for m_{2n-2} is

$$m_2 \circ m_{2n-2} + (-1)^n m_n \circ m_n + (-1)^{2n-2} m_{2n-2} \circ m_2 = 0,$$

which can be rewritten as

$$\delta m_{2n-2} = (-1)^n m_n \circ m_n.$$

Continuing this way, we obtain the following, which will help understand what is going on in Section 2.4.

Proposition 1.5.4. *Let $n, t \geqslant 2$, and let $\mathfrak{a}$ be an A_∞ algebra, such that that for $i < t$ we have $m_i = 0$ unless i is congruent to 2 modulo $n-2$. Then*

(1) *If t is not congruent to 2 modulo $n-2$ then m_t is a Hochschild cocycle.*
(2) *If $t = s(n-2)+2$ then m_t satisfies the coboundary condition*

$$\delta m_{s(n-2)+2} = \sum_{i=1}^{s-1} (-1)^{in} m_{i(n-2)+2} \circ m_{(s-i)(n-2)+2}.$$

(3) *Suppose that $HH^i H^* \mathfrak{a} = 0$ for i not congruent to 2 modulo $n-2$. Then an A_∞ structure on $H^*\mathfrak{a}$ quasi-isomorphic to that on $\mathfrak{a}$ may be chosen with $m_i = 0$ unless i is congruent to 2 modulo $n-2$.* □

1.6. Bousfield–Kan p-completion

We shall use the p-completion of Bousfield and Kan [**34**], namely the completion with respect to the field $\mathbb{F}_p$ of p elements. We write $X^\wedge_p$ for the p-completion of a space X. This comes with a natural map $X \to X^\wedge_p$, and has the following properties.

Theorem 1.6.1. *The Bousfield–Kan p-completion has the following properties.*

(i) *A map of spaces $X \to Y$ induces a mod p reduced homology equivalence $\tilde{H}_* X \to \tilde{H}_* Y$ if and only if it induces a weak homotopy equivalence between the completions $X^\wedge_p \to Y^\wedge_p$.*
(ii) *A space X is said to be* $\mathbb{F}_p$-good, *or* p-good, *if the map $\tilde{H}_* X \to \tilde{H}_* X^\wedge_p$ is an isomorphism, otherwise X is* $\mathbb{F}_p$-bad, *or* p-bad. *X is said to be* $\mathbb{F}_p$-complete, *or* p-complete, *if $X \to X^\wedge_p$ is a weak homotopy equivalence. The following are equivalent: (a) X is p-good, (b) $X^\wedge_p$ is p-complete, (c) $X^\wedge_p$ is p-good. Thus if X is p-bad, then however many times we complete it, it remains p-bad.*
(iii) *If $\pi_1 X$ is finite then X is p-good for all primes p. In this case, we have $\pi_1 X^\wedge_p \cong \pi_1 X / O^p(\pi_1 X)$.*

Proof. Parts (i) and (ii) are proved in Section I.I.5 of [**34**], while part (iii) is proved in Proposition I.VII.5.1 of [**34**]. □

Corollary 1.6.2. *If G is a finite group then the classifying space BG is p-good, its completion $BG^\wedge_p$ is a p-complete, nilpotent space and $\pi_1(BG^\wedge_p) \cong G/O^p(G)$. The space BG is already p-complete if and only if G is a finite p-group.*

The following are equivalent.

(1) *$BG^\wedge_p$ is simply connected.*
(2) *$\Omega BG^\wedge_p$ is connected.*
(3) *G has no normal subgroup of index p.*

Proof. This follows from Theorem 1.6.1. □

REMARK 1.6.3. In general, for a nilpotent space X with finite dimensional homology groups, the Eilenberg–Moore spectral sequence is a spectral sequence of Hopf algebras converging to the homology of the loop space:

$$\mathsf{Ext}^{*,*}_{H^*X}(\mathsf{k},\mathsf{k}) \cong \mathsf{Cotor}^{*,*}_{H_*X}(\mathsf{k},\mathsf{k}) \Rightarrow H_*\Omega X$$

(Eilenberg–Moore [**63**], Smith [**202**]). If the finite group G is not p-nilpotent, then BG is not a nilpotent space. However, $BG^\wedge_p$ is a nilpotent space, and $H^*BG^\wedge_p \cong H^*BG$. So we get a spectral sequence

$$\mathsf{Ext}^{*,*}_{H^*BG}(\mathsf{k},\mathsf{k}) \Rightarrow H_*\Omega BG^\wedge_p.$$

This expresses the cochain level statement for the DG algebra of endomorphisms

$$\mathcal{E}nd_{C^*BG}(\mathsf{k}) \simeq C_*\Omega BG^\wedge_p.$$

It follows that we have a functor $\mathcal{H}om_{C^*BG}(\mathsf{k},-)$ from C^*BG-modules to $C_*\Omega BG^\wedge_p$-modules.

In the other direction, for any path connected space X the Rothenberg–Steenrod construction [**196**] gives

$$\mathcal{E}nd_{C_*\Omega X}(\mathsf{k}) \simeq C^*X.$$

We can apply this either to $X = BG$ to obtain $\mathcal{E}nd_{\mathsf{k}G}(\mathsf{k}) \simeq C^*BG$ or to $X = BG^\wedge_p$ to obtain

$$\mathcal{E}nd_{C_*\Omega BG^\wedge_p}(\mathsf{k}) \simeq C^*BG^\wedge_p \simeq C^*BG.$$

In the latter case, we get a functor $\mathcal{H}om_{C_*\Omega BG^\wedge_p}(\mathsf{k},-)$ from $C_*\Omega BG^\wedge_p$-modules to C^*BG-modules.

LEMMA 1.6.4 (The fibre lemma). *Suppose that $F \to E \to B$ is a fibration sequence, and that the action of $\pi_1(B)$ on H_iF is nilpotent for all $i \geqslant 0$. Then $E^\wedge_p \to B^\wedge_p$ is a fibration, with fibre homotopy equivalent to $F^\wedge_p$.*

PROOF. Recalling that our convention is that H_*F denotes mod p homology, this is the case $R = \mathbb{F}_p$ of the Mod-R Fibre Lemma II.5.1 of Bousfield and Kan [**34**]. □

PROPOSITION 1.6.5. *Let G be a finite group, and embed G in a finite unitary group $G \to U(n)$. Then there are fibration sequences*

(i) *$U(n)/G \to BG \to BU(n)$, and*
(ii) *$(U(n)/G)^\wedge_p \to BG^\wedge_p \to BU(n)^\wedge_p$.*

PROOF. (i) Let $EU(n)$ be the complex Stiefel variety (or any contractible space on which $U(n)$ acts freely). Then we can use $EU(n)$ for EG, and the required fibration is

$$U(n)/G \to EU(n)/G \to EU(n)/U(n).$$

(ii) Since $\pi_1(BU(n))$ is trivial, this follows from Lemma 1.6.4 and part (i). □

1.7. Classifying spaces and fusion systems

Let G be a finite group and k be a field of characteristic p. Let EG be a contractible space with a free G-action, and let BG be the quotient EG/G. Since C_*EG is a free resolution of k as a $\mathsf{k}G$-module, the cohomology ring H^*BG is isomorphic to the group cohomology $H^*(G,\mathsf{k}) = \mathsf{Ext}^*_{\mathsf{k}G}(\mathsf{k},\mathsf{k})$. Furthermore, if P_* is any projective resolution of k as a $\mathsf{k}G$-module, then the DG algebra $\mathsf{Hom}_{\mathsf{k}G}(P_*,P_*)$ is quasi-isomorphic to C^*BG.

By a theorem of Cartan and Eilenberg (Theorem XII.10.1 of [**48**], the cohomology H^*BG only depends on the Sylow p-subgroup D of G and the fusion system on it defined by conjugation in G. This is defined as follows.

Definition 1.7.1. Let D be a Sylow p-subgroup of a finite group G. For subgroups H and K of D, we define $\mathsf{Hom}_G(H,K)$ to be the set of group homomorphisms from H to K that are induced by conjugation by some element of G, $\{\phi\colon H \to K \mid \exists g \in G\, \forall h \in H\ \phi(h) = ghg^{-1}\}$. The *fusion category* of G over D is the category $\mathcal{F}_D(G)$ whose objects are the subgroups of D, and whose morphisms are given by $\mathsf{Hom}_G(H,K)$. The *fusion system* of G over D consists of D together with the fusion category.

Abstract fusions systems are studied in the books of Aschbacher, Kessar and Oliver [**7**] and Craven [**52**]. There is a set of axioms, devised by Puig, and not every fusion system comes from a finite group in the above way. We shall assume that the saturation axiom is part of the definition.

Definition 1.7.2. Let D be a Sylow p-subgroup of a finite group G. A subgroup H of D is *p-centric* in G if $Z(H)$ is a Sylow p-subgroup of $C_G(H)$. This is equivalent to saying that $C_G(H) = Z(H) \times O_{p'}C_G(H)$. The *centric linking system* of G over D is the category $\mathcal{L}_D(G)$ whose objects are the subgroups of D that are p-centric in G, and whose morphisms are the quotient of $\{g \in G \mid gHg^{-1} \leqslant K\}$ by the action of $O_{p'}C_G(H)$. There is an obvious functor $\mathcal{L}_D(G) \to \mathcal{F}_D(G)$.

Again, there is a set of axioms for an abstract centric linking system $\mathcal{L}$ over a fusion system on a p-group D. A *p-local finite group* consists of a finite p-group D together with a fusion system $\mathcal{F}$ over D and a centric linking system $\mathcal{L} \to \mathcal{F}$.

Given a p-local finite group $(D,\mathcal{F},\mathcal{L})$, its *classifying space* $|\mathcal{L}|$ is defined to be the nerve of the category $\mathcal{L}$. It is a p-good space (Proposition 1.12 of Broto, Levi and Oliver [**41**]). The p-local finite group $(D,\mathcal{F},\mathcal{L})$ can be recovered from the homotopy type of $|\mathcal{L}|^\wedge_p$ (Theorem 7.4 of [**41**]).

Theorem 1.7.3. *The natural map $|\mathcal{L}_D(G)| \to BG$ is a mod p cohomology equivalence, and so induces a homotopy equivalence $|\mathcal{L}_D(G)|^\wedge_p \to BG^\wedge_p$.*

Proof. This is the main theorem of Broto, Levi and Oliver [**40**]. □

Theorem 1.7.4. *For a p-local finite group $(D,\mathcal{F},\mathcal{L})$, the cohomology $H^*|\mathcal{L}|$ is isomorphic to the ring of stable elements in H^*BG in the sense of Cartan and Eilenberg, Theorem XII.10.1 of [**48**].*

Proof. This is Theorem B of [**41**]. □

Theorem 1.7.5. *Suppose that G and G' are finite groups, and there is a fusion preserving isomorphism from the Sylow p-subgroup D of G to that of G'. Then there is a homotopy equivalence $BG^\wedge_p \to BG'^\wedge_p$.*

Proof. For $p = 2$ this is Theorem B of Oliver [**188**], while for odd primes it is Theorem B of Oliver [**187**]. □

The following stronger theorem was proved later, and Oliver's Theorem 1.7.5 is a consequence.

THEOREM 1.7.6. *Given a fusion system $\mathcal{F}$ on a finite p-group D, there exists a unique centric linking system $\mathcal{L}$ over $\mathcal{F}$.*

PROOF. This was first proved by Chermak [**49**] using his theory of localities. The proof used the classification of finite simple groups. Later, Oliver recast the proof in terms of obstruction theory. His proof depends on the Meierfrankenfeld–Stellmacher classification of quadratic best offenders [**180**], which again relies on the classification of finite simple groups. Finally, a classification free proof was given by Glauberman and Lynd [**125**]. $\square$

Using Theorem 1.7.6, given a fusion system $\mathcal{F}$ on a p-group D, it determines first a linking system $\mathcal{L}$ over $\mathcal{F}$, then the classifying space $|\mathcal{L}|$, then its p-completion $|\mathcal{L}|^\wedge_p$, and finally the cochains $C^*|\mathcal{L}|^\wedge_p \simeq C^*|\mathcal{L}|$.

REMARK 1.7.7. The corresponding theorem for discrete p-toral groups is proved in Levi and Libman [**170**]. This is relevant when trying to understand p-completed classifying spaces of compact Lie groups.

REMARK 1.7.8. Let B be a block of the group algebra $\mathsf{k}G$ of a finite group G over k. It follows from the work of Alperin and Broué [**2**] that one can associate to B a fusion system $\mathcal{F}_B$ describing the fusion of subpairs associated to the block. This is spelled out in Section 3 of Linckelmann [**175**]. Thus using Theorem 1.7.6, we can associate to B a linking system $\mathcal{L}_B$, and a classifying space $|\mathcal{L}_B|$, whose cohomology $H^*|\mathcal{L}_B|$ is the cohomology of B in the sense of Linckelmann [**174**]. In the case of a principal block, the defect groups are the Sylow p-subgroups of G, and the fusion system of the block is the same as the fusion system $\mathcal{F}_G(D)$ of the group.

Craven and Glesser [**54**] studied fusion systems on metacyclic groups. Theorem 1.1 of that paper shows that for dihedral, semidihedral and generalised quaternion 2-groups, all possible fusions systems are realised as fusion systems $\mathcal{F}_D(G)$ of some finite group G. It follows that when we discuss 2-completed classifying spaces of finite groups with these as Sylow 2-subgroups, we are really discussing the 2-completed classifying spaces associated to any fusion system on such a 2-group, including classifying spaces of blocks with these as defect groups.

1.8. Abelian Sylow subgroups

Let G be a finite group with abelian Sylow p-subgroup D, and let k be a field of characteristic p. Then by a classical theorem of Burnside, the normaliser $N_G(D)$ controls G-fusion in D. See for example Theorem 7.1.1 of Gorenstein [**126**]. This implies that the inclusion $N_G(D) \hookrightarrow G$ induces an isomorphism of fusion systems $\mathcal{F}_{N_G(D)}(D) \to \mathcal{F}_G(D)$. It follows that the ring of stable elements in H^*BD (see Theorem 1.7.4) is just the invariants of the normaliser. So we have the classical theorem of Swan.

THEOREM 1.8.1. *Suppose that G has an abelian Sylow p-subgroup D, and let k be a field of characteristic p. Then the inclusion $N_G(D) \to G$ and the quotient map $N_G(D) \to N_G(D)/O_{p'}N_G(D)$ induce isomorphisms*

$$H^*BG \cong H^*BN_G(D) \cong H^*B(N_G(D)/O_{p'}N_G(D)) \cong H^*BD^{N_G(D)/C_G(D)}.$$

PROOF. See Swan [**209**]. $\square$

The consequence for the p-completed classifying space is the following.

THEOREM 1.8.2. *Suppose that G has an abelian Sylow p-subgroup D, and let k be a field of characteristic p. Then we have homotopy equivalences*

$$B(N_G(D)/O_{p'}N_G(D))^\wedge_p \leftarrow BN_G(D)^\wedge_p \to BG^\wedge_p.$$

PROOF. By Theorem 1.8.1, the inclusion of a Sylow p-normaliser $N_G(D) \to G$ and the quotient map $N_G(D) \to N_G(D)/O_{p'}N_G(D)$ induce mod p cohomology equivalences

$$B(N_G(D)/O_{p'}N_G(D)) \leftarrow BN_G(D) \to BG.$$

Hence by Theorem 1.6.1 (i), after p-completion these give homotopy equivalences. $\square$

REMARK 1.8.3. The group $N_G(D)/O_{p'}N_G(D)$ is isomorphic to a semidirect product of D by a p'-subgroup H of $\mathsf{Aut}(D)$. This reduces the study of $BG^\wedge_p$ to the study of $B(D \rtimes H)^\wedge_p$.

We shall discuss the case where D is cyclic in Section 1.13 and the case where D is an elementary abelian 2-group in Theorem 5.2.2.

1.9. Singularity and cosingularity categories

Let $\mathfrak{a}$ be an A_∞ algebra over k. The *derived category* $\mathsf{D}(\mathfrak{a})$ has as its objects the A_∞ modules over $\mathfrak{a}$ and as arrows the homotopy classes of A_∞ morphisms. In this category, quasi-isomorphisms automatically have inverses. For details, see Keller [**154**, **155**], Lefèvre-Hasegawa [**165**].

If $H_*\mathfrak{a}$ is commutative Noetherian, we define the *bounded derived category* $\mathsf{D}^{\mathsf{b}}(\mathfrak{a})$ to be the thick subcategory of $\mathsf{D}(\mathfrak{a})$ whose objects are the modules with finitely generated homology.

We also need a suitable notion when $H_*\mathfrak{a}$ is not Noetherian, in order to deal with the case of $\mathfrak{a} = C_*\Omega BG^\wedge_p$. The appropriate condition there involves a suitable notion of Noether normalisation (Definition 3.7 of Greenlees and Stevenson [**133**]):

DEFINITION 1.9.1. We say that $\mathfrak{b} \to \mathfrak{a}$ is a *normalisation* of $\mathfrak{a} \to \mathsf{k}$ if both $\mathfrak{a}$ and k are in the thick subcategory $\mathsf{Thick}(\mathfrak{b}) \subseteq \mathsf{D}(\mathfrak{b})$ generated by $\mathfrak{b}$. For example, if $H_*\mathfrak{a}$ is finitely presented then the set of generators in a finite presentation leads to a normalisation (Theorem 3.13 of [**133**]).

If $\mathfrak{b} \to \mathfrak{a}$ is a normalisation, we define the *bounded derived category* $\mathsf{D}^{\mathsf{b}}(\mathfrak{a})$ to be full subcategory of $\mathsf{D}(\mathfrak{a})$ consisting of those objects whose restriction to $\mathfrak{b}$ are in $\mathsf{Thick}(\mathfrak{b}) \subseteq \mathsf{D}(\mathfrak{b})$. Under suitable hypotheses this is independent of the normalisation (Propositions 4.3 and 7.2 of [**133**]).

We define the *singularity category* $\mathsf{D}_{\mathsf{sg}}(\mathfrak{a})$ to be the Verdier quotient of $\mathsf{D}^{\mathsf{b}}(\mathfrak{a})$ by the thick subcategory $\mathsf{Thick}(\mathfrak{a})$ generated by $\mathfrak{a}$. We define the *cosingularity category* $\mathsf{D}_{\mathsf{csg}}(\mathfrak{a})$ to be the Verdier quotient of $\mathsf{D}^{\mathsf{b}}(\mathfrak{a})$ by the thick subcatgory $\mathsf{Thick}(\mathsf{k})$ generated by the field k.

We are interested in the cases of $C^*BG^\wedge_p$ and $C_*\Omega BG^\wedge_p$. Recall from Proposition 1.6.5 that we have a fibration sequence

$$(U(n)/G)^\wedge_p \to BG^\wedge_p \to BU(n)^\wedge_p.$$

This gives rise to maps

$$C^*BU(n)^\wedge_p \to C^*BG^\wedge_p \to C^*(U(n)/G)^\wedge_p$$

and
$$C_*\Omega(U(n)/G)^\wedge_p \to C_*\Omega BG^\wedge_p \to C_*U(n)^\wedge_p.$$
These have the property that $C^*BU(n)^\wedge_p \to C^*BG^\wedge_p$ and $C_*\Omega(U(n)/G)^\wedge_p \to C_*\Omega BG^\wedge_p$ are normalisations.

The following theorem expresses a version of Koszul duality between C^*BG and $C_*\Omega BG^\wedge_p$ (cf. Remark 1.6.3).

THEOREM 1.9.2. *For a finite group G, the functor $\mathcal{H}om_{C^*BG}(\mathsf{k},-)$ induces a triangulated equivalence of bounded derived categories $\mathsf{D}^{\mathsf{b}}(C^*BG) \xrightarrow{\sim} \mathsf{D}^{\mathsf{b}}(C_*\Omega BG^\wedge_p)$ that sends C^*BG to k and sends k to $C_*\Omega BG^\wedge_p$. It induces triangulated equivalences*
$$\mathsf{D}_{\mathsf{sg}}(C^*BG) \xrightarrow{\sim} \mathsf{D}_{\mathsf{csg}}(C_*\Omega BG^\wedge_p), \qquad \mathsf{D}_{\mathsf{csg}}(C^*BG) \xrightarrow{\sim} \mathsf{D}_{\mathsf{sg}}(C_*\Omega BG^\wedge_p).$$

PROOF. This follows from Theorem 9.1 and Example (10.6) of [**133**]. □

1.10. Tame blocks

The trichotomy theorem of Drozd [**62**] (see also Crawley-Boevey [**55**]) for finite dimensional algebras states that every finite dimensional algebra is is of finite, tame or wild representation type, and these types are mutually exclusive. Roughly speaking, finite representation type means that there are only finitely many isomorphism classes of finitely generated indecomposable modules. Tame representation type means that the finitely generated indecomposables of any particular dimension (over an infinite field) come in one parameter families with finitely many exceptions, and wild representation type means that the module theory for a free algebra on two generators can be encoded in the category of finite dimensional modules for the given algebra. For details, see for example Section 4.4 of [**15**].

In the case of blocks of finite groups, the representation type only depends on the defect group.

THEOREM 1.10.1. *Let B be a block of $\mathsf{k}G$ with defect group D. Then*

(i) *B has finite representation type if and only if D is cyclic.*
(ii) *B has tame representation type if and only if $p = 2$, and D is dihedral, generalised quaternion, or semidihedral.*
(iii) *In all other cases, B has wild representation type.*

PROOF. It follows from the work of Higman [**139**] that the representation type only depends on the defect group. For finite p-groups, the representation type was determined by Bondarenko and Drozd [**33**], see also Ringel [**194**]. □

Blocks with cyclic defect group were completely described in the work of Brauer [**35**] and Dade [**58**]. The case of tame representation type was the subject of extensive work of Erdmann [**66**–**79**], giving an almost complete description of the Morita types of these blocks. Our work leans heavily on these papers. To make life easy, it follows from a case by case analysis that for each isomorphism type of defect group of tame representation type and each fusion system on it, there is a principal block of some finite group G with the same fusion system, and all such have equivalent classifying spaces by Oliver's Theorem 1.7.5. Judicious choice of G minimises the work involved in understanding C^*BG and $C_*\Omega BG^\wedge_2$.

Remark 1.10.2. Finite dimensional local symmetric algebras of tame representation type are listed in Theorem III.1 of Erdmann [**74**]. The group algebras of finite 2-groups among these are as given as follows. The dihedral groups have type III.1 (c) with k a power of two (or type III.1 (b) for the Klein four group), the semidihedral groups have type III.1 (d′), and the generalised quaternion groups have type III.1 (e′). This is a slightly more precise statement than given in III.13 of [**74**]; see also Sections 3.2 and 4.2 for further comments.

1.11. Cohomology of complete intersections

We shall need to compute $\mathsf{Ext}^*_R(\mathsf{k},\mathsf{k})$ and $HH^*(R)$ in the case where $R = H^*BG$ is a complete intersection. For this reason, we give a brief review of cohomology of complete intersections, following Avramov [**8**], Sjödin [**201**] and Buchweitz and Roberts [**44**].

Let R be a complete intersection of the form Q/I, where $Q = \mathsf{k}[x_1,\dots,x_n]$ is a positively graded polynomial ring and I is generated by a homogeneous regular sequence $f_1,\dots,f_c$ in $\mathfrak{m}^2$, where $\mathfrak{m}$ is the ideal $(x_1,\dots,x_n)$.[1] We can take partial derivatives in the usual way to give polynomials

$$b_{i,k} = \frac{\partial f_k}{\partial x_i} \in Q.$$

We can then take their images in R, which we denote $\bar{b}_{i,k}$. But then there's a problem when it comes to second partial derivatives, because in characteristic two the second partial derivative of x^2 with respect to x vanishes. To remedy this, the second *divided partial derivative* is defined to be the second term in the Taylor expansion of the polynomial, so that for example $\dfrac{\partial^{(2)}(x^2)}{\partial x^2} = 1$. Thus we have $\dfrac{\partial^2 f_k}{\partial x_i^2} = 2\dfrac{\partial^{(2)} f_k}{\partial x_i^2}$. So now set

$$a_{i,j,k} = \begin{cases} \dfrac{\partial^2 f_k}{\partial x_i \partial x_j} & i \neq j \\[2ex] \dfrac{\partial^{(2)} f_k}{\partial x_i^2} & i = j, \end{cases}$$

as an element of Q, and write $\bar{a}_{i,j,k}$ for the image of $a_{i,j,k}$ in R. These are the coefficients of the *Hessian quadratic form* q associated to the relations defining R, see Section 2 of [**44**].

Definition 1.11.1. Following [**44**], we define $\mathsf{Cliff}(\mathsf{q})$ to be the differential bigraded algebra over R with generators $\hat{x}_i$ dual to the x_i, in degree $(-1,-|x_i|)$ $(1 \leqslant i \leqslant n)$ and s_k dual to the f_k, in degree $(-2,-|f_k|)$ $(1 \leqslant j \leqslant c)$. The multiplicative structure is given by making s_j central, and

$$\hat{x}_i\hat{x}_j + \hat{x}_j\hat{x}_i = \sum_{k=1}^{c} \bar{a}_{i,j,k}s_k \qquad (i \neq j), \qquad\qquad \hat{x}_i^2 = \sum_{k=1}^{c} \bar{a}_{i,i,k}s_k.$$

The differential $d\colon \mathsf{Cliff}(\mathsf{q}) \to \mathsf{Cliff}(\mathsf{q})$ vanishes on A and on the s_k, and on the $\hat{x}_i$ it is given by

$$d(\hat{x}_i) = \sum_{k=1}^{c} \bar{b}_{i,k}s_k.$$

[1] By negating degrees, results here apply equally well to negatively graded rings.

THEOREM 1.11.2. *We have* $\mathsf{Ext}^*_R(\mathsf{k},\mathsf{k}) = \mathsf{k} \otimes_R \mathsf{Cliff}(\mathsf{q})$.

PROOF. This is proved in Sjödin [**201**] when the characteristic is not two. The general statement can be obtained from Avramov [**8**] by combining Theorem 10.2.1 (5) and Example 10.2.2 there. □

REMARK 1.11.3. According to Theorem 1.11.2, $\mathsf{Ext}^*_R(\mathsf{k},\mathsf{k})$ is generated over k by elements $\hat{x}_i$ in degree $(-1, -|x_i|)$ $(1 \leqslant i \leqslant n)$ and s_k in degree $(-2, -|f_k|)$ $(1 \leqslant k \leqslant c)$. The elements s_k generate a central polynomial subring over which $\mathsf{Ext}^*_R(\mathsf{k},\mathsf{k})$ is a free module of rank 2^n. The relations express $\hat{x}_i\hat{x}_j + \hat{x}_j\hat{x}_i$ and $\hat{x}_i^2$ as linear combinations of the s_k with coefficients in k given by the constant terms $a_{i,j,k}(0)$ of the polynomials $a_{i,j,k}$. These only depend on the quadratic parts of the polynomials f_k, so we have $f_k = \sum_{i,j=1}^n a_{i,j,k}(0)x_ix_j +$ terms of degree at least three.

REMARK 1.11.4. The algebra $\mathsf{Ext}^*_R(\mathsf{k},\mathsf{k})$ carries a Hopf algebra structure for which the elements $\hat{x}_i$ and s_k are primitive. This gives the multiplication on the graded dual Hopf algebra $\mathsf{Tor}^R_*(\mathsf{k},\mathsf{k})$.

THEOREM 1.11.5. *We have* $HH^*(R) = H^*(\mathsf{Cliff}(\mathsf{q}), d)$*, the cohomology of* $\mathsf{Cliff}(\mathsf{q})$ *with respect to the differential* d*.*

PROOF. This is Theorem 2.11 of Buchweitz and Roberts [**44**]. □

REMARK 1.11.6. Since the relations f_k are required to be in $\mathfrak{m}^2$, we have $b_{i,k} \in \mathfrak{m}$, so $b_{i,k}(0) = 0$, and the differential d disappears on $\mathsf{k} \otimes_R \mathsf{Cliff}(\mathsf{q})$. So there is a natural map from $H^*(\mathsf{Cliff}(\mathsf{q}), d)$ to $\mathsf{k} \otimes_R \mathsf{Cliff}(\mathsf{q})$. This is the usual map $HH^*(R) \to \mathsf{Ext}^*_R(\mathsf{k},\mathsf{k})$ obtained by applying $- \otimes_R \mathsf{k}$ to a bimodule resolution of R to obtain a module resolution of k.

1.12. Koszul duality for graded algebras

For graded commutative rings whose relations are quadratic, Koszul duality provides a computation of both Ext and Hochschild cohomology.

DEFINITION 1.12.1. A *Koszul algebra* is a graded k-algebra R with the property that the minimal resolution of k as an R-module is *linear*. In other words, the maps in the minimal resolution are given by multiplication by linear combinations of the generators.

The relations in a Koszul algebra are quadratic, but not every graded algebra with quadratic relations is Koszul. However, a *graded commutative* algebra with quadratic relations is automatically Koszul.

If $R = \mathsf{k}\langle x_1, \dots, x_n\rangle/(S)$ is a Koszul algebra, with S a set of quadratic relations, then the *Koszul dual* is

$$R^! = \mathsf{Ext}^*_R(\mathsf{k},\mathsf{k}) \cong \mathsf{k}\langle \hat{x}_1, \dots, \hat{x}_n\rangle/(S^\perp).$$

If V is the vector space with basis $x_1, \dots, x_n$ then $\hat{x}_1, \dots, \hat{x}_n$ is the dual basis for V^*. The relations S form a linear subspace of $V \otimes V$, and $S^\perp$ is its annihilator in $V^* \otimes V^* \cong (V \otimes V)^*$.

THEOREM 1.12.2. *Let* $R = \mathsf{k}\langle x_1, \dots, x_n\rangle/(S)$ *be a graded Koszul algebra, with* S *a set of quadratic relations, and let* $R^! = \mathsf{k}\langle \hat{x}_1, \dots, \hat{x}_n\rangle/(S^\perp)$ *be the Koszul dual. Then as a* k*-algebra, the Hochschild cohomology* HH^*R *can be computed as* $H^*(R \otimes R^!, d)$*, where the differential* d *given by* $[e, -]$ *where* $e = x_1 \otimes \hat{x}_1 + \dots + x_n \otimes \hat{x}_n$*. Here, the variables* $\hat{x}_1, \dots, \hat{x}_n$ *are put in homological degree* -1 *in the complex, while* $x_1, \dots, x_n$ *are in homological degree zero.*

PROOF. In this form, this is proved in Theorem 1.2 of Negron [**186**], but see also the paper of Buchweitz, Green, Snashall and Solberg [**42**], where this is described in a more basis dependent way. □

REMARK 1.12.3. In the case of a complete intersection R with quadratic relations, of course S includes the commutativity relations, whereas the Koszul dual $R^!$ is usually non-commutative. In this case, the complex given in Theorem 1.12.2 is isomorphic to $(\mathsf{Cliff}(\mathsf{q}), d)$ appearing in Theorem 1.11.5. The advantage of this approach is that the same computation also computes $HH^*R^!$, but watching out for the change of degrees.

REMARK 1.12.4. If R and $R^!$ are Koszul dual algebras, there is a relation between the generating functions for the dimensions. For this purpose, it is necessary to give both R and $R^!$ an extra grading with the generators in degree one, so that a generator in degree n now has degree $(1, n)$. The Koszul dual generator is then in degree $(1, -n)$. This extra grading makes sense because the relations are quadratic, and therefore homogeneous in the new grading. Let

$$p_R(s,t) = \sum_{i,j} s^i t^j \dim_{\mathsf{k}} R_{i,j}.$$

Then we have

$$p_{R^!}(s,t) = 1/p_R(-st^{-1}, t^{-1}). \tag{1.12.5}$$

Without the internal grading, the formula reduces to the more well known formula

$$p_{R^!}(s) = 1/p_R(-s).$$

For example, if $R = \mathsf{k}[x, y]$ with x in degree -2 and y in degree -4 then $R^! = \Lambda(\hat{x}, \hat{y})$ with $\hat{x}$ in degree 1 and $\hat{y}$ in degree 3. Then $p_R(s,t) = 1/(1 - st^{-2})(1 - st^{-4})$ and

$$p_{R^!}(s,t) = (1 - (-st^{-1})t^2)(1 - (-st^{-1})t^4) = (1 + st)(1 + st^3).$$

1.13. The cyclic case

In this section, we summarise the results on groups with cyclic Sylow p-subgroups, from the papers [**22, 23**].

Let G be a finite group with cyclic Sylow p-subgroups, and let k be a field of characteristic p. Then by Theorem 1.8.2, the inclusion of a Sylow p-normaliser $N_G(D) \to G$ and the quotient map $N_G(D) \to N_G(D)/O_{p'}N_G(D)$ induce homotopy equivalences

$$B(N_G(D)/O_{p'}N_G(D))^\wedge_p \leftarrow BN_G(D)^\wedge_p \to BG^\wedge_p.$$

So it suffices to discuss the case $\mathbb{Z}/p^n \rtimes \mathbb{Z}/q$, where $q \geqslant 2$ is a divisor of $p-1$ and $\mathbb{Z}/q$ acts faithfully on $\mathbb{Z}/p^n$. Indeed, even in the case of a block with cyclic defect group of order p^n and inertial index q, the p-completed classifying space (see Remark 1.7.8) has the same homotopy type.

So set

$$G = \langle g, s \mid g^{p^n} = 1, s^q = 1, sgs^{-1} = g^\gamma \rangle \cong \mathbb{Z}/p^n \rtimes \mathbb{Z}/q,$$

where γ is a primitive qth root of unity modulo p^n. Setting

$$U = \sum_{\substack{1 \leqslant j \leqslant p^n - 1, \\ j^p \equiv j \pmod{p^n}}} g^j/j,$$

the group algebra is given by

$$\mathsf{k}G = \mathsf{k}\langle s, U \mid U^{p^n} = 0,\ s^q = 1,\ sU = \gamma U s\rangle$$

where γ is a primitive qth root of unity. This has a unique grading up to scalar multiplication. It is convenient to use a $\mathbb{Z}[\frac{1}{q}]$-grading and set $|s| = 0$, $|U| = 1/q$. With this grading, the cohomology is the doubly graded ring given by $H^*BG = \mathsf{k}[x] \otimes \Lambda(t)$ with $|x| = (-2q, -p^n)$ and $|t| = (-2q+1, -h)$ with $h = p^n - (p^n-1)/q$. The A_∞ structure is completely determined by

$$m_i(t, \dots, t) = \begin{cases} (-1)^{p^n(p^n-1)/2} x^h & i = p^n \\ 0 & \text{otherwise.} \end{cases}$$

The homology of the loop space on the p-completion $H_*\Omega BG^\wedge_p$ looks very similar. We have

$$H_*\Omega BG^\wedge_p = k[\tau] \otimes \Lambda(\xi)$$

where $|\tau| = (2q-2, h)$ and $|\xi| = (2q-1, p^n)$. The A_∞ structure is completely determined by

$$m_i(\xi, \dots, \xi) = \begin{cases} (-1)^{h(h-1)/2} & i = h \\ 0 & \text{otherwise.} \end{cases}$$

Thus the roles of h and p^n have been reversed. There is one exceptional case. If $h = 2$ then $q = 2$ and $p^n = 3$. In this case, the formula above gives $m_2(\xi, \xi) = -\tau^3$. Thus ξ is no longer an exterior variable, but rather we have the formal A_∞ algebra $H_*\Omega BG^\wedge_p = k[\tau, \xi]/(\xi^2 + \tau^3)$ in this case.

The category $\mathsf{D}_{\mathsf{sg}}(C^*BG) \simeq \mathsf{D}_{\mathsf{csg}}(\Omega BG^\wedge_p)$ is equivalent to $\mathsf{D}^{\mathsf{b}}(C_*\Omega BG^\wedge_p[\tau^{-1}])$. This is a finite Krull–Schmidt triangulated category with $(q-1)(h-1)$ isomorphism classes of indecomposable objects. The Auslander–Reiten quiver of this category is isomorphic to $\mathbb{Z}A_{h-1}/T^{q-1}$, a cylinder of height $h-1$ and circumference $q-1$. Here, T is the translation functor $\Sigma^{-2q} = \Sigma^{-2}$. The triangulated shift Σ reverses the ends of the cylinder, so that there are $[h/2]$ orbits of Σ on indecomposables.

The category $\mathsf{D}_{\mathsf{csg}}(C^*BG) \simeq \mathsf{D}_{\mathsf{sg}}(\Omega BG^\wedge_p)$ is equivalent to $\mathsf{D}^{\mathsf{b}}(C^*BG[x^{-1}])$. This is a finite Krull–Schmidt triangulated category with $q(p^n-1)$ isomorphism classes of indecomposable objects. The Auslander–Reiten quiver is isomorphic to $\mathbb{Z}A_{p^n-1}/T^q$, a cylinder of height p^n-1 and circumference q. The translation functor this time is $T = \Sigma^{2(q-1)}$, and the triangulated shift Σ again reverses the ends of the cylinder, so that there are $(p^n-1)/2$ orbits of Σ on indecomposables.

CHAPTER 2

The dihedral case

2.1. Introduction

In this chapter, we discuss the case of finite groups with dihedral Sylow 2-subgroups. The A_∞ structure on the cohomology of dihedral groups is given in the following theorem.

THEOREM 2.1.1. *Let* D *be a dihedral group of order* $4q$*, where* $q \geqslant 2$ *is a power of two, and let* k *be a field of characteristic two. Then as a ring, we have* $H^*B\mathrm{D} = \mathsf{k}[x,y,t]/(xy)$ *with* $|x| = |y| = -1$ *and* $|t| = -2$*. Up to quasi-isomorphism, the* A_∞ *structure on* $H^*B\mathrm{D}$ *is determined by*

$$m_{2q}(x,y,\dots,x,y) = m_{2q}(y,x,\dots,y,x) = t.$$

We give an explicit A_∞ structure within this quasi-isomorphism class in Theorem 2.4.2. It has $m_i \neq 0$ if and only if i is congruent to 2 modulo $2q-2$. For completeness, we also describe the case $q = 1$, which behaves differently.

The idea of the proof is to put a double grading on the group algebra $\mathsf{k}\mathrm{D}$. This gives a triple grading on $H^*B\mathrm{D}$, which then restricts the possibilities for the higher m_i. It is then easy to check that $m_i = 0$ unless i is congruent to 2 modulo $2q-2$. Then m_{2q} is interpreted as a Hochschild cocycle on $H^*B\mathrm{D}$, and quasi-isomorphism amounts to changing it by a Hochschild coboundary. We write down explicit formulas for all the m_i, using some Hochschild cohomology computations involving the circle product of Gerstenhaber.

Similar computations give the A_∞ structure on the cohomology of a finite group G with dihedral Sylow 2-subgroups of order $4q$. These groups were classified by Gorenstein and Walter. There are three possible 2-local structures, which are distinguished by the number of conjugacy classes of involutions (one, two or three). We examine the three possibilities in detail, and determine the A_∞ structures on H^*BG and $H_*\Omega BG^\wedge_2$ in each case. The case with three conjugacy classes is trivial, since G has a normal 2-complement in this case, so we concentrate on the remaining two cases.

Perhaps the most interesting case is the one where all involutions are conjugate, as this happens if and only if G has no subgroup of index two. In this case, if $q \geqslant 2$ we have

$$H^*BG = \mathsf{k}[t,\xi,\eta]/(\xi\eta)$$

with $|t| = -2$ and $|\xi| = |\eta| = -3$ (homological grading). If $q = 1$ then $H^*BG = \mathsf{k}[t,\xi,\eta]/(\xi\eta + t^2)$. This time, the A_∞ structure is determined up to quasi-isomorphism by

$$m_{2q}(\xi,\eta,\dots,\xi,\eta) = m_{2q}(\eta,\xi\dots,\eta,\xi) = t^{2q+1},$$

where the ξ and η alternate. Again the m_i are non-zero for i congruent to 2 modulo $2q-2$, and zero otherwise, and we give an explicit description of the non-zero ones.

The A_∞ structure on $H_*\Omega BG^\wedge_2$ in this case is easier to describe than that of H^*BG. This is because there are only two non-zero m_i.

THEOREM 2.1.2. *Let G be a finite group with dihedral Sylow 2-subgroups of order $4q$ with $q \geqslant 1$, and with no normal subgroup of index two, and let k be a field of characteristic two. Then the ring structure on the homology of $\Omega BG^\wedge_2$ is given by*

$$H_*\Omega BG^\wedge_2 = \Lambda(\tau) \otimes \mathsf{k}\langle \alpha, \beta \mid \alpha^2 = \beta^2 = 0\rangle$$

where $|\tau| = 1$, $|\alpha| = |\beta| = 2$. The A_∞ structure is determined by

$$m_{2q+1}(\tau, \dots, \tau) = s^q,$$

where $s = \alpha\beta + \beta\alpha$.

See Theorems 2.8.3 and 2.8.10 for details.

We describe a DG algebra Q which is quasi-isomorphic to $C_*\Omega BG^\wedge_2$, and use it to show that the degree 4 element $s = \alpha\beta + \beta\alpha$ is central. It may then be inverted, to obtain equivalences of categories

$$\mathsf{D}^{\mathsf{b}}(Q[s^{-1}]) \simeq \mathsf{D}^{\mathsf{b}}(C_*\Omega BG^\wedge_2[s^{-1}]) \simeq \mathsf{D}_{\mathsf{csg}}(C_*\Omega BG^\wedge_2) \simeq \mathsf{D}_{\mathsf{sg}}(C^*BG).$$

Finally, we observe that there is a Morita equivalence between $Q[s^{-1}]$ and one of the algebras discussed in [**22**]. This allows us to carry over the classification theorem there, to classify the indecomposable objects in $\mathsf{D}^{\mathsf{b}}(C_*\Omega BG^\wedge_2[s^{-1}])$, and hence also of $\mathsf{D}_{\mathsf{sg}}(C^*BG)$.

THEOREM 2.1.3. *Let G be a finite group with dihedral Sylow 2-subgroups of order $4q$ with $q \geqslant 1$, and with no subgroup of index two, and let k be a field of characteristic two. Then $\mathsf{D}_{\mathsf{sg}}(C^*BG) \simeq \mathsf{D}_{\mathsf{csg}}(C_*\Omega BG^\wedge_2)$ is a finite Krull–Schmidt category with $4q$ isomorphism classes of indecomposable objects, which come in q orbits of the suspension Σ, all of length four. The Auslander–Reiten quiver is isomorphic to $\mathbb{Z}A_{2q}/T^2$, where T is the translation functor Σ^2.*

This theorem is proved in Section 2.12 (Theorem 2.12.1). The corresponding theorem in the case where G has two conjugacy classes of involutions, so that G has a normal subgroup of index two but no normal subgroup of index four, is given in Theorem 2.13.10.

In contrast with Theorems 2.12.1 and 2.13.10, the category $\mathsf{D}_{\mathsf{csg}}(C^*BG) \simeq \mathsf{D}_{\mathsf{sg}}(C_*\Omega BG^\wedge_2)$ has infinite representation type.

2.2. Dihedral 2-groups

Let $\mathrm{D} = \langle g, h \mid g^2 = h^2 = (gh)^{2q} = 1\rangle$, a dihedral group of order $4q$, with $q \geqslant 1$ a power of two, and let k be a field of characteristic two. As elements of $\mathsf{k}\mathrm{D}$, let $X = g - 1$ and $Y = h - 1$. Then the group algebra can be rewritten as

$$\mathsf{k}\mathrm{D} = \mathsf{k}\langle X, Y \mid X^2 = Y^2 = 0, (XY)^q = (YX)^q\rangle.$$

This algebra has tame representation type, and the finitely generated $\mathsf{k}G$-modules were classified by Ringel [**195**].

We shall regard $\mathsf{k}\mathrm{D}$ as a $\mathbb{Z} \times \mathbb{Z}$-graded algebra, with $|X| = (1, 0)$ and $|Y| = (0, 1)$. With this bigrading, the relations are homogeneous. It is easy to compute the minimal resolution of k as a $\mathsf{k}G$-module, and hence the cohomology ring. Recall that we are using homological degrees throughout, so that cohomological degrees come out negative. We list first the homological degree, and then the two internal degrees.

The case $q = 1$ behaves differently from $q \geqslant 2$, so we discuss this case first. If $q = 1$ then $H^*B\mathrm{D} \cong \mathsf{Ext}^*_{\mathsf{k}G}(\mathsf{k}, \mathsf{k})$ is a formal A_∞ algebra $\mathsf{k}[x, y]$ with $|x| = -(1, 1, 0)$ and $|y| = -(1, 0, 1)$.

One way to see that it has to be formal is that the values m_i for $i > 2$ on non-zero elements land in zero groups for degree reasons.

We now assume, for the rest of this section, that $q \geqslant 2$. We have

$$H^*B\mathrm{D} \cong \mathsf{Ext}^*_{\mathsf{k}\mathrm{D}}(\mathsf{k},\mathsf{k}) \cong \mathsf{k}[x,y,t]/(xy)$$

where $|x| = -(1,1,0)$, $|y| = -(1,0,1)$ and $|t| = -(2,q,q)$. The elements x and y in $H^1B\mathrm{D}$ are dual to X and Y in $J(\mathsf{k}\mathrm{D})/J^2(\mathsf{k}\mathrm{D})$. The element

$$t \in H^2B\mathrm{D} \cong \mathsf{Ext}^1_{\mathsf{k}\mathrm{D}}(\Omega\mathsf{k},\mathsf{k})$$

is represented by the short exact sequence of bigraded modules

$$0 \to \mathsf{k} \to M \oplus N \to \Omega\mathsf{k} \to 0,$$

where M and N are uniserial modules of length $2q$. Examination of these uniserial $\mathsf{k}\mathrm{D}$-modules shows that we have Massey products

$$\langle x,y,\dots,x,y\rangle = \langle y,x,\dots,y,x\rangle = t$$

in $H^*B\mathrm{D}$. Here, in both expressions the arguments x and y alternate, and there are q of each, for a total of $2q$ terms. Note that these Massey products are only well defined up to adding multiples of x^2 and y^2. However, if we take the internal grading into account, the Massey product is well defined, with no ambiguity.

2.3. $HH^*H^*B\mathrm{D}$

Wishing to understand further the A_∞ structure of the cohomology of dihedral groups, it follows from Proposition 1.4.2 and Lemma 2.4.1 that we should next compute the Hochschild cohomology $HH^*H^*B\mathrm{D}$. Since $H^*B\mathrm{D}$ is a hypersurface (i.e., a codimension one complete intersection), we can compute Hochschild cohomology using Theorem 1.11.5. So first we compute the algebra $\mathsf{Cliff}(\mathsf{q})$ for $H^*B\mathrm{D}$, where D is a dihedral group of order $4q$ with $q \geqslant 2$. This will also be useful for computing $\mathsf{Ext}^{*,*}_{H^*B\mathrm{D}}(\mathsf{k},\mathsf{k})$ using Theorem 1.11.2. Recall that $H^*B\mathrm{D} = \mathsf{k}[x,y,t]/(xy)$ with $|x| = -(1,1,0)$, $|y| = -(1,0,1)$ and $|t| = -(2,q,q)$.

Proposition 2.3.1. *The DG algebra $\mathsf{Cliff}(\mathsf{q})$ is equal to $H^*B\mathrm{D}\langle\hat{x},\hat{y},\tau;s\rangle$, where s is central, and $\hat{x}^2 = 0$, $\hat{y}^2 = 0$, $\hat{x}\hat{y} + \hat{y}\hat{x} = s$, $\tau^2 = 0$, $\hat{x}\tau = \tau\hat{x}$, $\hat{y}\tau = \tau\hat{y}$. The degrees are given by $|x| = (0,-1,-1,0)$, $|y| = (0,-1,0,-1)$, $|t| = (0,-2,-q,-q)$, $|\hat{x}| = (-1,1,1,0)$, $|\hat{y}| = (-1,1,0,1)$, $|\tau| = (-1,2,q,q)$, $|s| = (-2,2,1,1)$. The differential is given by $d(\hat{x}) = ys$, $d(\hat{y}) = xs$, $d(\tau) = 0$, $d(s) = 0$.*

Proof. Let $f(x,y,t) = xy$. Then we have

$$\frac{\partial f}{\partial x} = y, \qquad \frac{\partial f}{\partial y} = x, \qquad \frac{\partial f}{\partial z} = 0,$$
$$\frac{\partial^{(2)} f}{\partial x^2} = 0, \qquad \frac{\partial^{(2)} f}{\partial y^2} = 0, \qquad \frac{\partial^{(2)} f}{\partial z^2} = 0,$$
$$\frac{\partial^2 f}{\partial x \partial y} = 1, \qquad \frac{\partial^2 f}{\partial x \partial z} = 0, \qquad \frac{\partial^2 f}{\partial y \partial z} = 0.$$

Plugging these into Definition 1.11.1, we get the given relations and differential for $\mathsf{Cliff}(\mathsf{q})$. □

Theorem 2.3.2. *Let G be a dihedral group of order $4q$ with $q \geqslant 2$ a power of two, and let k be a field of characteristic two. The Hochschild cohomology $HH^*H^*B\mathrm{D}$ has generators s, t, τ, x, y, u, v with*

$$\begin{aligned}
&|s| = (-2,2,1,1) && \\
&|t| = (0,-2,-q,-q) && |\tau| = (-1,2,q,q)\\
&|x| = (0,-1,-1,0) && |y| = (0,-1,0,-1)\\
&|u| = (-1,0,0,0) && |v| = (-1,0,0,0).
\end{aligned}$$

The relations are given by $u^2 = v^2 = uv = \tau^2 = 0$, $xy = 0$, $xv = yu = 0$, $xs = ys = 0$, and $us = vs$. The non-zero monomials and their degrees are as follows, with $i_1, i_2 \geqslant 0$, $\varepsilon_1, \varepsilon_2 \in \{0,1\}$. The first two cases overlap for $i_1 > 0$, the first and third, and the second and fourth overlap for $i_1 = 0$.

$$\begin{aligned}
|s^{i_1}t^{i_2}\tau^{\varepsilon_1}u^{\varepsilon_2}| &= (-2i_1-\varepsilon_1-\varepsilon_2, 2i_1-2i_2+2\varepsilon_1, i_1+q(-i_2+\varepsilon_1), i_1+q(-i_2+\varepsilon_1)),\\
|s^{i_1}t^{i_2}\tau^{\varepsilon_1}v^{\varepsilon_2}| &= (-2i_1-\varepsilon_1-\varepsilon_2, 2i_1-2i_2+2\varepsilon_1, i_1+q(-i_2+\varepsilon_1), i_1+q(-i_2+\varepsilon_1)),\\
|x^{i_1}t^{i_2}\tau^{\varepsilon_1}u^{\varepsilon_2}| &= (-\varepsilon_1-\varepsilon_2, -i_1-2i_2+2\varepsilon_1, -i_1+q(-i_2+\varepsilon_1), q(-i_2+\varepsilon_1))\\
|y^{i_1}t^{i_2}\tau^{\varepsilon_1}v^{\varepsilon_2}| &= (-\varepsilon_1-\varepsilon_2, -i_1-2i_2+2\varepsilon_1, q(-i_2+\varepsilon_1), -i_1+q(-i_2+\varepsilon_1))
\end{aligned}$$

(the top two coincide with the lower two when $i_1 = 0$, and are otherwise disjoint). There is only one monomial in degree $(-i, i-2, 0, 0)$ with $i > 2$, namely $s^q t$, with

$$|s^q t| = (-2q, 2q-2, 0, 0).$$

Proof. By Theorem 1.11.5, $HH^*H^*B\mathrm{D}$ is the cohomology of the DG algebra $\mathsf{Cliff}(\mathsf{q})$. By Proposition 2.3.1, this is therefore as described in the theorem, with $u = x\hat{x}$ and $v = y\hat{y}$. Since $\partial(\hat{x}\hat{y}) = (x\hat{x} + y\hat{y})s$, we have $us = vs$ in $HH^*H^*B\mathrm{D}$.

We also mention another approach to this computation, as this will become relevant in the proof of Proposition 2.8.9. Namely, we can use Theorem 1.12.2. This gives rise to the same complex as above. Here, the x_i are x, y and t and the $\hat{x}_i$ are $\hat{x}$, $\hat{y}$ and τ. The advantage of this approach is that it makes it easy to compute $HH^*A^!$ using the same computation, but watching out for the changes of degrees. This approach also makes it clear that $\hat{x}$ and $\hat{y}$ are really just avatars for the elements X and Y of $\mathsf{k}\mathrm{D}$.

For the last statement, we note that for the last two coordinates to be zero, the monomial must be of one of the first two types. Then we have

$$\begin{gathered}
i_1 + q(-i_2+\varepsilon_1) = 0,\\
(-2i_1-\varepsilon_1-\varepsilon_2) + (2i_1-2i_2+2\varepsilon_1) = -2.
\end{gathered}$$

Twice the first equation minus q times the second gives $(\varepsilon_1+\varepsilon_2)q + 2i_1 = 2q$, and so i_1 is either zero or q. If $i_1 = 0$ then $\varepsilon_1 = \varepsilon_2 = 1$, which then implies $i_2 = 1$, and the resulting monomials have $i = 2$. On the other hand, if $i_1 = q$ then $\varepsilon_1 = \varepsilon_2 = 0$, and again we have $i_2 = 1$, and the resulting monomial is $s^q t$. $\square$

2.4. A_∞ structure of $H^*B\mathrm{D}$

In this section, we completely describe the A_∞ structure of $H^*B\mathrm{D}$. This makes extensive use of Section 1.5, describing Gerstenhaber's circle product on Hochschild cochains and its relation to the structure maps of an A_∞ algebra.

Let D be a dihedral group of order $4q$ with $q \geqslant 2$ a power of two, and let k be a field of characteristic two.

Lemma 2.4.1. *For any A_∞ structure on $H^*B\mathrm{D}$ that preserves internal degrees, we have $m_n = 0$ unless $n-2$ is divisible by $2q-2$. In particular for $2 < n < 2q$ we have $m_n = 0$.*

Proof. Looking at the degrees of the generators x, y and z, for any monomial ζ in $H^*B\mathrm{D}$ of degree (a,b,c) we have $a \equiv b+c \pmod{2q-2}$. So for an n-tuple $(\zeta_1,\dots,\zeta_n)$, the degree of $m_n(\zeta_1,\dots,\zeta_i)$ satisfies $a \equiv b+c+n-2 \pmod{2q-2}$. It follows that for $m_n(\zeta_1,\dots,\zeta_n)$ to be non-zero we must have $n-2 \equiv 0 \pmod{2q-2}$. $\square$

Theorem 2.4.2. *The A_∞ structure on $H^*B\mathrm{D}$ is given as follows. The m_n are $\mathsf{k}[t]$-multilinear maps with $m_n = 0$ for n not congruent to 2 modulo $2q-2$. For $i, j \geqslant 1$,*

$$m_{2q}(x^i, y, x, y, \dots, x, y^j) = m_{2q}(y^j, x, y, x, \dots, y, x^i) = x^{i-1}y^{j-1}t$$

where the arguments alternate between x and y, and the right hand side is zero unless either $i=1$ or $j=1$; m_{2q} is zero on all other tuples of monomials not involving t. The maps $m_{\ell(2q-2)+2}$ with $\ell > 1$ similarly vanish on all tuples of monomials not involving t, except the ones which look as above, but for some choice of indices:

$$\begin{aligned} 1 \leqslant e_1 \leqslant e_2 \leqslant \cdots \leqslant e_{\ell-1} < e_{\ell-1} + (2q-2) + 1 \leqslant e_{\ell-2} + 2(2q-2) + 1 \\ \leqslant \cdots \leqslant e_1 + (\ell-1)(2q-2) + 1 \leqslant \ell(2q-2) + 2. \end{aligned}$$

the exponents on the terms are increased by one (or correspondingly more if an index is repeated). The value on these tuples is $x^{i-1}y^{j-1}t^\ell$. Thus

$$m_{\ell(2q-2)+2}(x^{i+\alpha_1}, y^{\alpha_2}, x^{\alpha_3}, \dots, x^{\alpha_{\ell(2q-2)+1}}, y^{j+\alpha_{\ell(2q-2)+2}}) = x^{i-1}y^{j-1}t^\ell$$

where each α_σ is one plus the number of indices in the list above that are equal to σ.

Remark 2.4.3. To illustrate this rather complicated looking condition, suppose that $q=4$. Then m_8 is given by

$$m_8(x^i, y, x, y, x, y, x, y^j) = m_8(y^j, x, y, x, y, x, y, x^i) = x^{i-1}y^{j-1}t,$$

and then m_{14} is the next non-zero m_n. The value of each of the following seven expressions is $x^{i-1}y^{j-1}t^2$:

$$\begin{gathered} m_{14}(x^{i+1}, y, x, y, x, y, x, y^2, x, y, x, y, x, y^j) \\ m_{14}(x^i, y^2, x, y, x, y, x, y, x^2, y, x, y, x, y^j) \\ m_{14}(x^i, y, x^2, y, x, y, x, y, x, y^2, x, y, x, y^j) \\ m_{14}(x^i, y, x, y^2, x, y, x, y, x, y, x^2, y, x, y^j) \\ m_{14}(x^i, y, x, y, x^2, y, x, y, x, y, x, y^2, x, y^j) \\ m_{14}(x^i, y, x, y, x, y^2, x, y, x, y, x, y, x^2, y^j) \\ m_{14}(x^i, y, x, y, x, y, x^2, y, x, y, x, y, x, y^{j+1}) \end{gathered}$$

There are seven more such expressions with non-zero values of m_{14}, where x and y have been interchanged. A typical non-zero value of m_{20}, which is the next non-zero m_n, corresponding to $\ell = 3$, is given by

$$m_{20}(x^i, y, x, y^2, x, y^2, x, y, x, y, x, y, x^2, y, x, y, x^2, y, x, y^j) = x^{i-1}y^{j-1}t^3,$$

where the indices $4 \leqslant 6 < 13 \leqslant 17$ come from $e_1 = 4$, $e_2 = 6$. An example with a repeated index is

$$m_{20}(x^i, y, x, y, x, y^3, x, y, x, y, x, y, x^2, y, x, y, x, y, x^2, y^j) = x^{i-1}y^{j-1}t^3,$$

with indices $6 \leqslant 6 < 13 \leqslant 19$ coming from $e_1 = e_2 = 6$.

Proof of Theorem 2.4.2. By Lemma 2.4.1, for every A_∞ structure preserving degrees, $m_n = 0$ for $2 < n < 2q$. So in order to determine m_{2q}, we invoke Proposition 1.4.2. We have Massey products

$$\langle x, y, \dots, x, y\rangle = \langle y, x, \dots, y, x\rangle = t,$$

well defined modulo the ideal generated by x and y. It follows from Theorem 1.4.5 that $m_{2q}(x, y, \dots, x, y)$ and $m_{2q}(y, x, \dots, y, x)$ are non-zero. So m_{2q} represents a non-zero Hochschild cohomology class in degree $(-2q, 2q-2, 0, 0)$ in $HH^*H^*B\mathrm{D}$. By Theorem 2.3.2, up to scalar multiplication, there is only one non-zero possibility for m_{2q} up to quasi-isomorphism. It is easy to check that the given formula for m_{2q} is indeed a Hochschild cocycle. Replacing t by a non-zero multiple of t if necessary (or by working over $\mathbb{F}_2$) makes this the correct Hochschild cohomology class.

Again using Lemma 2.4.1, we see that the next possible m_n after m_{2q} is m_{4q-2}. Using (1.3.1), this has to satisfy

$$m_2(\mathsf{id} \otimes m_{4q-2}) + \sum_{r=0}^{4q-3} m_{4q-2}(\mathsf{id}^{\otimes r} \otimes m_2 \otimes \mathsf{id}^{\otimes(4q-r-3)}) + m_2(m_{4q-2} \otimes \mathsf{id})$$
$$+ \sum_{r=0}^{2q-1} m_{2q}(\mathsf{id}^{\otimes r} \otimes m_{2q} \otimes \mathsf{id}^{\otimes 2q-r-1}) = 0.$$

Now the first three terms are the Hochschild coboundary of m_{4q-2}, while the last sum is the Gerstenhaber circle product $m_{2q} \circ m_{2q}$, see Section 1.5. So as in Proposition 1.5.4, we rewrite the above equation as

$$\delta m_{4q-2} = m_{2q} \circ m_{2q}, \tag{2.4.4}$$

where δ is the Hochschild coboundary. Subject to this condition, m_{4q-2} is well defined modulo Hochschild coboundaries. But by Theorem 2.3.2, the Hochschild cohomology $HH^*H^*B\mathrm{D}$ is zero in degree $(-4q+2, 4q-4, 0, 0)$, so any choice of m_{4q-2} satisfying (2.4.4) is valid. The one we have constructed satisfies this.

We continue by induction on ℓ. If we have constructed $m_{2q}, m_{4q-2}, \dots, m_{(\ell-1)(2q-2)+2}$, then the equation satisfied by $m_{\ell(2q-2)+2}$ is

$$\delta m_{\ell(2q-2)+2} = \sum_{i+j=\ell} m_{i(2q-2)+2} \circ m_{j(2q-2)+2}.$$

Again $HH^*H^*B\mathrm{D}$ is zero in degree $(-\ell(2q-2)-2, \ell(2q-2), 0, 0)$, and so any choice of $m_{\ell(2q-2)+2}$ satisfying this equation is valid. The one we have constructed satisfies this. $\square$

Remark 2.4.5. Let us illustrate the way equation (2.4.4) works, with the example of Remark 2.4.3. We have

$$\begin{aligned}
&(m_8 \circ m_8)(x^i, y, x, y^2, x, y, x, y, x, y, x, x, y, x, y^j) \\
&\qquad = m_8(x^i, y, x, m_8(y^2, x, y, x, y, x, y, x), x, y, x, y^j) \\
&\qquad = m_8(x^i, y, x, yt, x, y, x, y^j) \\
&\qquad = x^{i-1} y^{j-1} t^2,
\end{aligned}$$

and correspondingly,

$$\begin{aligned}
&\delta m_{14}(x^i, y, x, y^2, x, y, x, y, x, y, x, x, y, x, y^j) \\
&\qquad = m_{14}(x^i, y, x, y^2, x, y, x, y, x, y, x^2, y, x, y^j) \\
&\qquad = x^{i-1} y^{j-1} t^2.
\end{aligned}$$

2.5. Loops on $B\mathrm{D}_2^\wedge$

Since D is a finite 2-group, completing $B\mathrm{D}$ makes no difference to its homotopy type. So $\Omega B\mathrm{D}_2^\wedge$ has contractible connected components, and is homotopy equivalent to D with the group multiplication. So we should expect to see the Eilenberg–Moore spectral sequence converging to $\mathsf{k}\mathrm{D}$.

Proposition 2.5.1. *We have*

$$\mathsf{Ext}^{*,*}_{H^*B\mathrm{D}}(\mathsf{k}, \mathsf{k}) = \Lambda(\tau) \otimes \mathsf{k}\langle \hat{x}, \hat{y} \mid \hat{x}^2 = 0,\ \hat{y}^2 = 0 \rangle$$

with

$$|\tau| = (-1, 2, q, q), \qquad |X| = (-1, 1, 1, 0), \qquad |Y| = (-1, 1, 0, 1).$$

Proof. This follows by applying Theorem 1.11.2 to Proposition 2.3.1. The element s there is redundant, as it is equal to $\hat{x}\hat{y} + \hat{y}\hat{x}$. $\square$

The E^2 page of the spectral sequence

$$\mathsf{Ext}^{*,*}_{H^*B\mathrm{D}}(\mathsf{k}, \mathsf{k}) \Rightarrow \mathsf{k}\mathrm{D}$$

is given by the Proposition. There is a non-zero differential given by

$$d^{2q-1}(\tau) = (\hat{x}\hat{y} + \hat{y}\hat{x})^q.$$

Then

$$E^{2q} = E^\infty \cong \mathsf{k}\mathrm{D} = \mathsf{k}\langle \hat{x}, \hat{y} \mid \hat{x}^2 = 0,\ \hat{y}^2 = 0, (\hat{x}\hat{y} + \hat{y}\hat{x})^q = 0 \rangle,$$

concentrated in homological degree zero. Ungrading, X represents $\hat{x}$ and Y represents $\hat{y}$ to give an isomorphism with $\mathsf{k}\mathrm{D}$. Note that $\mathsf{k}\mathrm{D}$ is isomorphic to its associated graded with respect to the radical filtration, which is reflected in the fact that there is no ungrading to be done in this case. In the generalised quaternion and semidihedral situations, this will be more of an issue.

2.6. $HH^*\mathsf{k}\mathrm{D}$

In this section we use the spectral sequence

$$HH^*H^*B\mathrm{D} \Rightarrow HH^*C^*B\mathrm{D} \cong HH^*C_*\Omega BD \cong HH^*\mathsf{k}\mathrm{D} \tag{2.6.1}$$

to compute $HH^*\mathsf{k}\mathrm{D}$. This is not needed in the rest of the paper, but is an illustration of the power of the internal $\mathbb{Z}\times\mathbb{Z}$-grading on $\mathsf{k}\mathrm{D}$. The only differential comes from the analysis of the map m_{2q} in the A_∞ structure on $H^*B\mathrm{D}$, and there is just one ungrading problem, which turns out to be the only difficult part of the computation.

Theorem 2.6.2. *In the spectral sequence* (2.6.1) *we have* $d^{2q-1}(\tau) = s^q$.

Proof. We use the standard description of the Hochschild complex, see for example Section 5 of [**22**]. The element τ on the E^2 page corresponds to the Hochschild cochain $\tilde\tau\colon [t^i] \mapsto it^{i-1}$, all other monomials going to zero. Applying the formula for the differential, we have

$$(\delta\tilde\tau)[\underbrace{x, y, \dots, x, y}_{2q}] = \tilde\tau(m_{2q}(x, y, \dots, x, y)) = \tilde\tau(t) = 1,$$

and similarly

$$(\delta\tilde\tau)[\underbrace{y, x, \dots, y, x}_{2q}] = \tilde\tau(m_{2q}(y, x, \dots, y, x)) = \tilde\tau(t) = 1,$$

Since $s[x, y] = s[y, x] = 1$, $\delta\tilde\tau$ takes the same values as s^q, and hence $\delta\tilde\tau = s^q$. Examining the locations of these terms in the filtration of the bar complex giving rise to the spectral sequence, we deduce that this corresponds to the differential d^{2q-1} taking τ to s^q. $\square$

Theorem 2.6.3. *The algebra* $HH^*C^*B\mathrm{D} \cong HH^*C_*\Omega B\mathrm{D}^\wedge_2 \cong HH^*\mathsf{k}\mathrm{D}$ *has generators* s, t, x, y, u, v, w_1, w_2, w_3 *with* $|s| = (0, 1, 1)$, $|t| = (-2, -q, -q)$, $|x| = (-1, -1, 0)$, $|y| = (-1, 0, -1)$, $|u| = |v| = (-1, 0, 0)$, $|w_1| = (0, q-1, q)$, $|w_2| = (0, q, q-1)$, $|w_3| = (0, q, q)$. *These satisfy the degree zero relations*

$$w_1^2 = w_2^2 = w_3^2 = w_1w_2 = w_1w_3 = w_2w_3 = sw_1 = sw_2 = sw_3 = s^q = 0,$$

the degree -1 relations

$$vw_1 = uw_2 = uw_3 = vw_3 = xs = ys = 0,$$

$$us = vs, \quad xw_2 = yw_1 = us^{q-1}, \quad xw_3 = uw_1, \quad yw_3 = vw_2,$$

and the degree -2 relations

$$u^2 = v^2 = uv = xy = xv = yu = 0.$$

Proof. By the centraliser decomposition, we have $\dim_{\mathsf{k}} HH^n\mathsf{k}\mathrm{D} = 4n + q + 3$. In the spectral sequence $HH^*H^*B\mathrm{D} \Rightarrow HH^*\mathsf{k}\mathrm{D}$, we have $d^{2q-1}(\tau) = s^q$. Let w_1, w_2 and w_3 be representative of $x\tau$, $y\tau$ and $(u+v)\tau$. If this is the only differential then the dimensions at the E^∞ page already match those for $HH^n\mathsf{k}\mathrm{D}$. This is because HH^0 is spanned by s^i $(1 \leqslant i \leqslant q)$, w_1, w_2 and w_3, HH^1 is spanned by u, v, $us^i = vs^i$ $(1 \leqslant i \leqslant q)$, x, xw_1, xw_3, y, yw_2, yw_3, and for $n \geqslant 2$, HH^n is spanned by $t.HH^{n-2}$ together with the eight elements x^n, x^nw_1, x^nw_3, y^n, y^nw_2, y^nw_3, $x^{n-1}u$ and $y^{n-1}v$. So d^{2q-1} is the only differential, it's zero on all generators except τ, and we have $E^{2q} = E^\infty$. The E^∞ page is as above, but with $xw_2 = yw_1 = 0$ It remains to ungrade the relations.

We begin with degree zero. The dimension of the algebra $HH^0\mathsf{k}\mathrm{D} = Z(\mathsf{k}\mathrm{D})$ is $q+3$, and it is spanned by $s^i = (XY)^i + (YX)^i$ with $0 \leqslant i \leqslant q-1$, together with the elements $w_1 = (YX)^{q-1}Y$, $w_2 = (XY)^{q-1}X$, and $w_3 = (XY)^q = (YX)^q$. These have the required internal degrees, and satisfy the degree zero relations listed above. In particular, note that s^q is equal to zero and not to w_3, even though this has the right degree.

For the degree -1 and -2 relations, most have nothing lower in the filtration, in the right internal degree so they ungrade to the same relations. The exception is the relations $xw_2 = yw_1 = 0$, which ungrade to give some multiples of us^{q-1}. Since we can work over $\mathbb{F}_2$, the multiple is either zero or one, and by symmetry both are equal to the same multiple. The exact multiple is harder to determine, and a long computation in the centraliser decomposition shows that they are both equal to us^{q-1}. □

Remark 2.6.4. The algebra $HH^*\mathsf{k}\mathrm{D}$ was computed in Section 9 of Siegel and Witherspoon [**200**]. They chose a different basis, whose elements are not homogeneous with respect to our grading, and which complicates their relations. The relation $xw_2 = yw_1 = us^{q-1}$ can be read off from their computation. See also Generalov [**98**], where the Hochschild cohomology is computed for algebras in Erdmann's class III.1 (c) [**74**] for any parameter q. The degree -2 relations depend on the parity of q, but are determined already on the E^2 page of the spectral sequence. The same algebras in odd characteristic are discussed in Generalov [**97**], where generators of degree -3 and -4 also occur in the Hochschild cohomology.

2.7. Groups with dihedral Sylow 2-subgroups

The computation for groups with a dihedral Sylow 2-subgroup is analogous to the dihedral group case described above. These groups were classified by Gorenstein and Walter [**128**, **129**], see also Bender and Glauberman [**11**, **12**]. The representation theory was investigated by Brauer [**37**, **38**], Cabanes and Picaronny [**46**], Donovan and Freislich [**61**], Donovan [**60**], Erdmann [**66**, **68**, **74**, **75**], Erdmann and Michler [**78**], Holm [**141**], Holm and Zimmermann [**145**], Kauer [**152**], Koshitani [**159**], Koshitani and Lassueur [**160**], Landrock [**163**], Linckelmann [**173**]. The cohomology rings were investigated by Martino and Priddy [**179**], Asai [**5**], Asai and Sasaki [**6**], Generalov et al. [**10**, **86**, **88**, **107**, **109**–**111**, **115**], and the Hochschild cohomology in Generalov et al. [**83**, **90**, **97**, **98**, **112**, **113**, **116**, **117**], Holm [**143**], Taillefer [**210**]. The homology of $\Omega BG_2^\wedge$ was computed by Levi [**167**].

Let G be a finite group with a dihedral Sylow 2-subgroup D of order $4q$ with $q \geqslant 1$, and let k be a field of characteristic two. Then by the main theorem of Gorenstein and Walter [**129**], there are three mutually exclusive cases, according to the fusion on the dihedral groups. By Theorem 1.1 of Craven and Glesser [**54**], these also represent the only possible fusion systems on dihedral 2-groups.

Case 2.7.1. If G has one class of involutions then $G/O(G)$ is isomorphic to either the alternating group A_7 or a subgroup of $P\Gamma L(2,p^m)$ with p^m a power of an odd prime, containing $PSL(2,p^m)$ with odd index. The principal block of $\mathsf{k}G$ has three isomorphism classes of simple modules.

Case 2.7.2. If G has two classes of involutions then G has a normal subgroup of index two, but no normal subgroup of index four. In this case, $G/O(G)$ is a subgroup of $P\Gamma L(2,p^m)$ with p^m a power of an odd prime, containing $PGL(2,p^m)$ with odd index. The principal block of $\mathsf{k}G$ has two isomorphism classes of simple modules. In this case we have $q \geqslant 2$.

CASE 2.7.3. If G has three classes of involutions then $O(G)$ is a normal complement in G to a Sylow 2-subgroup D, so that $G/O(G) \cong \mathrm{D}$ and $H^*BG \cong H^*B\mathrm{D}$. The principal block of $\mathsf{k}G$ is isomorphic to $\mathsf{k}\mathrm{D}$, and has one isomorphism class of simple modules, namely the trivial module.

REMARK 2.7.1. For p is odd, we have

$$\begin{aligned}|P\Gamma L(2,p^m)| &= m(p^m-1)p^m(p^m+1),\\ |PGL(2,p^m)| &= (p^m-1)p^m(p^m+1),\\ |PSL(2,p^m)| &= (p^m-1)p^m(p^m+1)/2,\end{aligned}$$

and $PSL(2,p^m)$ is simple for $p \geqslant 5$.

PROPOSITION 2.7.2. *Suppose that G has a dihedral Sylow 2-subgroup* D. *Then the homotopy type of $BG^\wedge_2$ is determined by* $|\mathrm{D}|$ *and the number of conjugacy classes of involutions.*

PROOF. This follows from Theorem 1.7.5 and the main theorem of [**129**] described above. $\square$

We shall deal with Cases 2.7.1 and 2.7.2 in turn. Case 2.7.1 is the most interesting, because this is the case where G has no subgroup of index two, so $\Omega BG^\wedge_2$ is connected. Case 2.7.2 is computationally quite similar, but $\Omega BG^\wedge_2$ has two connected components, and so we give the details anyway for completeness. Case 2.7.3 is easy because $\Omega BG^\wedge_2$ is homotopy equivalent to D. Nonetheless, the Eilenberg–Moore spectral sequence has an interesting differential in this case, as we saw in Section 2.5.

We end this section with a table of the various cases of algebras of dihedral type in characteristic two, in Erdmann's classification.

Erdmann [**74**]	Case	Group	H^*	HH^*
III.I(a)	—	—	[**107**]	
III.I(b)	2.7.3	fours group		[**51**, **140**]
III.I(b′)	—	—		
III.I(c)	2.7.3	dihedral	[**185**]	[**98**, **200**]
III.I(c′)	—	—		
$\mathrm{D}(2\mathcal{A})$	2.7.2	$PGL(2,q),\ q \equiv 1 \pmod 4$	[**6**, **115**, **179**]	
$\mathrm{D}(2\mathcal{B})$	2.7.2	$PGL(2,q),\ q \equiv 3 \pmod 4$	[**6**, **88**, **179**]	[**112**, **113**, **116**]
$\mathrm{D}(3\mathcal{A})_1$	2.7.1	$PSL(2,q),\ q \equiv 1 \pmod 4$	[**6**, **9**, **179**]	[**90**]
$\mathrm{D}(3\mathcal{A})_2$	—	—		
$\mathrm{D}(3\mathcal{B})_1$	2.7.1	Alternating group A_7	[**6**, **86**, **179**]	[**90**]
$\mathrm{D}(3\mathcal{B})_2$	—	—		
$\mathrm{D}(3\mathcal{D})_1$	—	—		[**90**]
$\mathrm{D}(3\mathcal{D})_2$	—	—		
$\mathrm{D}(3\mathcal{K})$	2.7.1	$PSL(2,q),\ q \equiv 3 \pmod 4$	[**6**, **10**, **179**]	[**90**]
$\mathrm{D}(3\mathcal{L})$	—	—	[**109**]	
$\mathrm{D}(3\mathcal{Q})$	—	—	[**111**]	
$\mathrm{D}(3\mathcal{R})$	—	—	[**110**]	[**83**]

2.8. Loops on $BG^{\wedge}_2$: one class of involutions

In this section we begin the examination of Case 2.7.1. This is the case where G has a dihedral Sylow 2-subgroup D, and one conjugacy class of involution. In this case, G has no subgroup of index two, and it has three isomorphism classes of simple modules in the principal block.

REMARK 2.8.1. As we have already mentioned, Theorem 1.7.5 shows that up to quasi-isomorphism, C^*BG only depends on the fusion, and according to Proposition 2.7.2, for dihedral Sylow 2-subgroups this only depends on $|\mathrm{D}|$ and the number of simple modules. In fact more is true. Linckelmann [**173**] has proved that all blocks of finite groups with dihedral defect groups of a given order, and three isomorphism classes of simple modules, are derived equivalent. Explicit derived equivalences are described in that paper, and in the case of principal blocks, it can be checked that the derived equivalence may be chosen to take the trivial module to the trivial module. The endomorphism DGA of the trivial module is a derived invariant up to quasi-isomorphism, and is also quasi-isomorphic to C^*BG.

Let G be a group with dihedral Sylow 2-subgroup D of order $4q$, $q \geqslant 1$, and one conjugacy class of involutions. By Proposition 2.7.2, for the purpose of studying $BG^{\wedge}_2$, we may assume that $G = PSL(2,p)$ for a suitable prime $p \equiv 1 \pmod 4$. In this case, the principal block B_0 of $\mathsf{k}G$ has three simple modules, k, M and N, whose Ext^1 quiver is as follows:

$$\begin{tikzcd} \mathsf{M} \arrow[r, bend left, "e_2"] & \mathsf{k} \arrow[l, bend left, "e_1"] \arrow[r, bend left, "e_3"] & \mathsf{N} \arrow[l, bend left, "e_4"] \end{tikzcd}.$$

The relations are

$$e_1e_2 = 0, \qquad e_3e_4 = 0, \qquad (e_4e_3e_2e_1)^q = (e_2e_1e_4e_3)^q.$$

We put an internal grading on the basic algebra in this case by assigning degree $(\frac{1}{2}, 0)$ to e_1 and e_2 and degree $(0, \frac{1}{2})$ to e_3 and e_4. Thus we assign degree $\frac{1}{2}(n_1, n_2)$ to a path involving n_1 arrows of type e_1 or e_2, and n_2 arrows of type e_3 or e_4. This choice is appropriate, because the internal grading it induces in cohomology is compatible with restriction to the Sylow 2-subgroup.

REMARK 2.8.2. It is not clear *a priori* that there exists a grading on the principal block compatible with the restriction map in cohomology. This explains the need for the computation above. For a further discussion of gradings in this context, see Bogdanic [**28**].

Let $\bar{e}_1$ be the element of $\mathsf{Hom}_B(P_\mathsf{M}, P_\mathsf{k})$ opposite to e_1, and so on. Then the minimal resolution of k as a $\mathsf{k}G$-module takes the form

$$\cdots \longrightarrow P_\mathsf{M} \oplus P_\mathsf{k} \oplus P_\mathsf{k} \oplus P_\mathsf{N} \xrightarrow{\begin{pmatrix} \bar{e}_1 & v & 0 & 0 \\ 0 & \bar{e}_1\bar{e}_2 & \bar{e}_3\bar{e}_4 & 0 \\ 0 & 0 & u & \bar{e}_3 \end{pmatrix}} P_\mathsf{k} \oplus P_\mathsf{k} \oplus P_\mathsf{k} \xrightarrow{\begin{pmatrix} \bar{e}_1\bar{e}_2 & v & 0 \\ 0 & u & \bar{e}_3\bar{e}_4 \end{pmatrix}} P_\mathsf{k} \oplus P_\mathsf{k}$$

$$\xrightarrow{\begin{pmatrix} \bar{e}_2 v & 0 \\ \bar{e}_1\bar{e}_2 & \bar{e}_3\bar{e}_4 \\ 0 & \bar{e}_4 u \end{pmatrix}} P_\mathsf{M} \oplus P_\mathsf{k} \oplus P_\mathsf{N} \xrightarrow{\begin{pmatrix} \bar{e}_1 & v & 0 \\ 0 & u & \bar{e}_3 \end{pmatrix}} P_\mathsf{k} \oplus P_\mathsf{k} \xrightarrow{(\bar{e}_1\bar{e}_2 \ \ \bar{e}_3\bar{e}_4)} P_\mathsf{k} \xrightarrow{\begin{pmatrix} \bar{e}_2 v \\ \bar{e}_4 u \end{pmatrix}} P_\mathsf{M} \oplus P_\mathsf{N} \xrightarrow{(\bar{e}_1, \bar{e}_3)} P_\mathsf{k}$$

where $u = \bar{e}_1\bar{e}_2(\bar{e}_3\bar{e}_4\bar{e}_1\bar{e}_2)^{q-1}$ and $v = \bar{e}_3\bar{e}_4(\bar{e}_1\bar{e}_2\bar{e}_3\bar{e}_4)^{q-1}$. This is the total complex of the following double complex.

$$
\begin{tikzcd}
 & & & P_{\mathsf{k}} \arrow[d, "\bar{e}_3\bar{e}_4"] & \vdots & & \\
 & & P_{\mathsf{N}} \arrow[d, "\bar{e}_3"] & P_{\mathsf{k}} \arrow[l, "\bar{e}_4 u"'] \arrow[d, "\bar{e}_3\bar{e}_4"] & P_{\mathsf{k}} \arrow[l, "u"'] \arrow[d, "\bar{e}_3\bar{e}_4"] & & \\
 & & P_{\mathsf{k}} \arrow[d, "\bar{e}_3\bar{e}_4"] & P_{\mathsf{k}} \arrow[l, "u"'] \arrow[d, "\bar{e}_3\bar{e}_4"] & P_{\mathsf{k}} \arrow[l, "u"'] \arrow[d, "v"] & P_{\mathsf{k}} \arrow[l, "\bar{e}_1\bar{e}_2"'] \arrow[d, "v"] & \cdots \\
 & P_{\mathsf{N}} \arrow[d, "\bar{e}_3"] & P_{\mathsf{k}} \arrow[l, "\bar{e}_4 u"'] \arrow[d, "\bar{e}_3\bar{e}_4"] & P_{\mathsf{k}} \arrow[l, "u"'] \arrow[d, "v"] & P_{\mathsf{k}} \arrow[l, "\bar{e}_1\bar{e}_2"'] \arrow[d, "v"] & P_{\mathsf{k}} \arrow[l, "\bar{e}_1\bar{e}_2"'] \arrow[d, "\bar{e}_2 v"] & P_{\mathsf{k}} \arrow[l, "\bar{e}_1\bar{e}_2"'] \\
 & P_{\mathsf{k}} \arrow[d, "\bar{e}_3\bar{e}_4"] & P_{\mathsf{k}} \arrow[l, "u"'] \arrow[d, "v"] & P_{\mathsf{k}} \arrow[l, "\bar{e}_1\bar{e}_2"'] \arrow[d, "\bar{e}_2 v"] & P_{\mathsf{k}} \arrow[l, "\bar{e}_1\bar{e}_2"'] & P_{\mathsf{M}} \arrow[l, "\bar{e}_1"'] & \\
P_{\mathsf{N}} \arrow[d, "\bar{e}_3"] & P_{\mathsf{k}} \arrow[l, "\bar{e}_4 u"'] \arrow[d, "\bar{e}_2 v"] & P_{\mathsf{k}} \arrow[l, "\bar{e}_1\bar{e}_2"'] & P_{\mathsf{M}} \arrow[l, "\bar{e}_1"'] & & & \\
P_{\mathsf{k}} & P_{\mathsf{M}} \arrow[l, "\bar{e}_1"'] & & & & &
\end{tikzcd}
$$

So with this grading, if $q \geqslant 2$, the cohomology ring is given by $H^*(BG, \mathsf{k}) = \mathsf{k}[\xi, \eta, t]/(\xi\eta)$ where

$$|\xi| = -(3, q+1, q), \qquad |\eta| = -(3, q, q+1), \qquad |t| = -(2, q, q).$$

If $q = 1$, we assume that k contains $\mathbb{F}_4 = \{0, 1, \omega, \bar{\omega}\}$, and then the generators and degrees are the same, but the relation is $\xi\eta + t^3$ instead of $\xi\eta$. The restrictions to D are given by

$$\operatorname{res}_{\mathrm{D}}(\xi) = \begin{cases} (x+\omega y)^3 \\ xt \end{cases} \quad \operatorname{res}_{\mathrm{D}}(\eta) = \begin{cases} (x+\bar{\omega} y)^3 \\ yt \end{cases} \quad \operatorname{res}_{\mathrm{D}}(t) = \begin{cases} x^2+xy+y^2 & q = 1 \\ t & q \geqslant 2. \end{cases}$$

For $q = 1$ there are no non-zero Massey products, and the A_∞ structure is formal. For $q \geqslant 2$ we have Massey products

$$\langle \xi, \eta, \ldots, \xi, \eta \rangle = \langle \eta, \xi, \ldots, \eta, \xi \rangle = t^{2q+1}.$$

In both expressions the arguments ξ and η alternate, and there are $2q$ of them. These Massey products are only well defined up to adding elements of the ideal generated by ξ and η, but taking the grading into account, they are well defined with no ambiguity.

THEOREM 2.8.3. *Let G be a finite group with dihedral Sylow 2-subgroups of order $4q$ with $q \geqslant 1$ a power of two, and one class of involutions, and let k be a field of characteristic two. Then we have*

$$H_*\Omega BG^\wedge_2 = \Lambda(\tau) \otimes \mathsf{k}\langle \alpha, \beta \mid \alpha^2 = 0,\ \beta^2 = 0 \rangle.$$

with

$$|\tau| = (1, q, q), \qquad |\alpha| = (2, q+1, q), \qquad |\beta| = (2, q, q+1).$$

In homological degree $4n$ we have monomials $(\alpha\beta)^n$ and $(\beta\alpha)^n$, in degree $4n+2$ we have monomials $(\alpha\beta)^n\alpha$ and $(\beta\alpha)^n\beta$, and in odd degrees we have τ times all of these.

PROOF. For $q \geqslant 1$, the Eilenberg–Moore spectral sequence converging to $H_*\Omega BG^\wedge_2$ has as its E_2 page

$$\mathsf{Ext}^{*,*}_{H^*BG}(\mathsf{k}, \mathsf{k}) = \Lambda(\tau) \otimes \mathsf{k}\langle \alpha, \beta \mid \alpha^2 = 0,\ \beta^2 = 0 \rangle$$

where the generators have degrees

$$|\tau| = (-1, 2, q, q), \qquad |\alpha| = (-1, 3, q+1, q), \qquad |\beta| = (-1, 3, q, q+1).$$

The four degrees are first homological, then internal to H^*BG, and finally the two gradings internal to $\mathsf{k}G$. The elements τ, α and β come from the generators t, η and ξ, while the element $s = \alpha\beta + \beta\alpha$ in degree $(-2, 6, 2q+1, 2q+1)$ is the Eisenbud operator for the relation $\xi\eta = 0$ in H^*BG. There is no room for non-zero differentials, and there are no ungrading problems, so $E_2 = E_\infty = H_*\Omega BG^\wedge_2$. □

REMARK 2.8.4. Proposition II.4.1.5 of Levi [**167**] gets the correct additive structure for $H_*\Omega BG^\wedge_2$ but it is incorrectly claimed there that the ring structure is a polynomial tensor exterior algebra.

Note that the algebras H^*BG and $H_*\Omega BG^\wedge_2$ are Koszul dual to each other. This will play a role in the computation of Hochschild cohomology.

LEMMA 2.8.5. *For any A_∞ structure on H^*BG that preserves internal degrees, we have $m_i = 0$ unless $i-2$ is divisible by $2q-2$. In particular, for $2 < i < 2q$ we have $m_i = 0$.*

PROOF. The proof is the same as the proof of Lemma 2.4.1. □

PROPOSITION 2.8.6. *Let G be a group with a dihedral Sylow 2-subgroup D of order $4q$ with $q \geqslant 2$ a power of two, and one conjugacy class of involutions, and let k be a field of characteristic two. The Hochschild cohomology HH^*H^*BG has generators s, t, τ, ξ, η, u, v with*

$$\begin{aligned} &|s| = (-2, 6, 2q+1, 2q+1) && \\ &|t| = -(0, 2, q, q) && |\tau| = (-1, 2, q, q) \\ &|\xi| = -(0, 3, q+1, q) && |\eta| = -(0, 3, q, q+1) \\ &|u| = -(1, 0, 0, 0) && |v| = -(1, 0, 0, 0). \end{aligned}$$

The relations are given by $u^2 = v^2 = uv = \tau^2 = 0$, $\eta u = \xi v = 0$, $\xi s = \eta s = 0$, and $us = vs$. The non-zero monomials and their degrees are as follows, with $i_1, i_2 \geqslant 0$, $\varepsilon_1, \varepsilon_2 \in \{0, 1\}$.

$$\begin{aligned} |s^{i_1}t^{i_2}\tau^{\varepsilon_1}u^{\varepsilon_2}| &= (-2i_1 - \varepsilon_1 - \varepsilon_2, 6i_1 - 2i_2 + 2\varepsilon_1, i_1 + q(2i_1 - i_2 + \varepsilon_1), i_1 + q(2i_1 - i_2 + \varepsilon_1)), \\ |s^{i_1}t^{i_2}\tau^{\varepsilon_1}v^{\varepsilon_2}| &= (-2i_1 - \varepsilon_1 - \varepsilon_2, 6i_1 - 2i_2 + 2\varepsilon_1, i_1 + q(2i_1 - i_2 + \varepsilon_1), i_1 + q(2i_1 - i_2 + \varepsilon_1)), \\ |\xi^{i_1}t^{i_2}\tau^{\varepsilon_1}u^{\varepsilon_2}| &= (-\varepsilon_1 - \varepsilon_2, -3i_1 - 2i_2 + 2\varepsilon_1, -i_1 + q(-i_1 - i_2 + \varepsilon_1), q(-i_1 - i_2 + \varepsilon_1)) \\ |\eta^{i_1}t^{i_2}\tau^{\varepsilon_1}v^{\varepsilon_2}| &= (-\varepsilon_1 - \varepsilon_2, -3i_1 - 2i_2 + 2\varepsilon_1, q(-i_1 - i_2 + \varepsilon_1), -i_1 + q(-i_1 - i_2 + \varepsilon_1)) \end{aligned}$$

There is only one monomial in degree $(-i, i-2, 0, 0)$ with $i > 2$, namely $s^q t^{2q+1}$, with

$$|s^q t^{2q+1}| = (-2q, 2q-2, 0, 0).$$

PROOF. As in Theorem 2.3.2, we use the approach of Theorems 1.11.5 and 1.12.2. Thus HH^*H^*BG is the homology of the complex

$$(H^*BG \otimes H_*\Omega BG^\wedge_2, \partial),$$

where the generators t, ξ and η are in homological degree zero, the generators τ, α and β are in homological degree -1, and the differential is given by $\partial = [e, -]$ where $e = t\otimes\tau + \xi\otimes\alpha + \eta\otimes\beta$. Thus setting $s = \alpha\beta + \beta\alpha$, we have $\partial(t) = 0$, $\partial(\xi) = 0$, $\partial(\eta) = 0$, $\partial(\alpha) = \eta s$, $\partial(\tau) = 0$,

$\partial(\beta) = \xi s$. The generators and relations for the homology of this complex are therefore as given, with $u = \xi\alpha$ and $v = \eta\beta$.

For the last statement, the computation is similar to the corresponding part of the proof of Theorem 2.3.2. $\square$

THEOREM 2.8.7. *The A_∞ structure on H^*BG is given as follows. The m_n are $\mathsf{k}[t]$-multilinear maps with $m_n = 0$ for n not congruent to 2 modulo $2q-2$, and for $i, j \geqslant 1$*

$$m_{2q}(\xi^i, \eta, \xi, \eta, \ldots, \xi, \eta^j) = m_{2q}(\eta^j, \xi, \eta, \xi, \ldots, \eta, \xi^i) = \xi^{i-1}\eta^{j-1}t^{2q+1}$$

where the arguments alternate between ξ and η, and the right hand side is zero unless either $i = 1$ or $j = 1$; m_{2q} is zero on all other tuples of monomials not involving t. The maps $m_{\ell(2q-2)+2}$ with $\ell > 1$ similarly vanish on all tuples of monomials not involving t, except the ones which look as above, but for some choice of indices in the tuple:

$$\begin{aligned} 1 \leqslant e_1 \leqslant e_2 \leqslant \cdots \leqslant e_{\ell-1} < e_{\ell-1} + (2q-2) + 1 \leqslant e_{\ell-2} + 2(2q-2) + 1& \\ \leqslant \cdots \leqslant e_1 + (\ell-1)(2q-2) + 1 \leqslant \ell(2q-2) + 2.& \end{aligned}$$

the exponents on the terms are increased by one (or correspondingly more if an index is repeated). The value on these tuples is $\xi^{i-1}\eta^{j-1}t^{\ell(2q+1)}$. Thus

$$m_{\ell(2q-2)+2}(x^{i+\alpha_1}, y^{\alpha_2}, x^{\alpha_3}, \ldots, x^{\alpha_{\ell(2q-2)+1}}, y^{j+\alpha_{\ell(2q-2)+2}}) = x^{i-1}y^{j-1}t^{\ell(2q+1)}$$

where each α_σ is one plus the number of indices in the list above that are equal to σ.

PROOF. The proof is the same as the proof of Theorem 2.4.2, but using Lemma 2.8.5 and Proposition 2.8.6 in place of Lemma 2.4.1 and Theorem 2.3.2. $\square$

We now turn to the computation of the A_∞ structure on $H_*\Omega BG_2^\wedge$. This is easier to describe than the A_∞ structure on H^*BG.

LEMMA 2.8.8. *For any A_∞ structure on $H_*\Omega BG_2^\wedge$ that preserves internal degrees, we have $m_i = 0$ unless $i-2$ is divisible by $2q-1$. In particular, for $2 < i < 2q+1$ we have $m_i = 0$.*

PROOF. The proof is similar to the proof of Lemma 2.4.1. Looking at the degrees of the generators τ, α and β, for any monomial ζ in $H_*\Omega BG_2^\wedge$ we have $a \equiv b + c \pmod{2q-1}$. So for any i-tuple $(\zeta_1, \ldots, \zeta_i)$, the degree of $m_i(\zeta_1, \ldots, \zeta_i)$ satisfies $a \equiv b + c + i - 2 \equiv 0 \pmod{2q-1}$. So for $m_i(\zeta_1, \ldots, \zeta_i)$ to be non-zero we must have $i - 2 \equiv 0 \pmod{2q-1}$. $\square$

PROPOSITION 2.8.9. *The Hochschild cohomology $HH^*H_*\Omega BG_2^\wedge$ has generators s, t, τ, ξ, η, u, and v in degrees*

$$\begin{aligned} |s| &= (0, 4, 2q+1, 2q+1), & & \\ |t| &= -(1, 1, q, q), & |\tau| &= (0, 1, q, q), \\ |\xi| &= -(1, 2, q+1, q), & |\eta| &= -(1, 2, q, q+1), \\ |u| &= -(1, 0, 0, 0), & |v| &= -(1, 0, 0, 0). \end{aligned}$$

The relations are given by $\xi\eta = 0$, $u^2 = v^2 = uv = \tau^2 = 0$, $\eta u = \xi v = 0$, $\xi s = \eta s = 0$, and $us = vs$. The non-zero monomials and their degrees are given as follows, with $i_1, i_2 \geqslant 0$, $\varepsilon_1, \varepsilon_2 \in \{0, 1\}$.

$$|s^{i_1}t^{i_2}\tau^{\varepsilon_1}u^{\varepsilon_2}| = (-i_2 - \varepsilon_2, 4i_1 - i_2 + \varepsilon_1, (2i_1 - i_2 + \varepsilon_1)q + i_1, (2i_1 - i_2 + \varepsilon_1)q + i_1),$$

$$|s^{i_1}t^{i_2}\tau^{\varepsilon_1}v^{\varepsilon_2}| = (-i_2-\varepsilon_2, 4i_1-i_2+\varepsilon_1, (2i_1-i_2+\varepsilon_1)q+i_1, (2i_1-i_2+\varepsilon_1)q+i_1),$$
$$|\xi^{i_1}t^{i_2}\tau^{\varepsilon_1}u^{\varepsilon_2}| = (-i_1-i_2-\varepsilon_2, -2i_1-i_2+\varepsilon_1, -i_1+(-i_1-i_2+\varepsilon_1)q, (-i_1-i_2+\varepsilon_1)q),$$
$$|\eta^{i_1}t^{i_2}\tau^{\varepsilon_1}v^{\varepsilon_2}| = (-i_1-i_2-\varepsilon_2, -2i_1-i_2+\varepsilon_1, (-i_1-i_2+\varepsilon_1)q, -i_1+(-i_1-i_2+\varepsilon_1)q).$$

Thus there is only one monomial with degree $(-i, i-2, 0, 0)$ with $i > 2$, namely

$$|s^q t^{2q+1}| = (-2q-1, 2q-1, 0, 0).$$

Proof. Again, as in Theorem 2.3.2, we use the approach of Theorem 1.12.2. This time, $HH^*H_*\Omega BG_2^\wedge$ is the homology of the complex

$$(H_*\Omega BG_2^\wedge \otimes H^*BG, \partial),$$

where the generators τ, α and β of $H_*\Omega BG_2^\wedge$ are in homological degree zero, the generators t, ξ, η of H^*BG are in homological degree -1, and the differential is given by $\partial = [e, -]$ where $e = \tau\otimes t + \alpha\otimes\xi + \beta\otimes\eta$. So the answer is the same as in Proposition 2.8.6 but with the degrees changed.

For the last statement, the computation is again similar to the corresponding part of the proof of Theorem 2.3.2. □

Theorem 2.8.10. *In Case 2.7.1, the A_∞ structure on $H_*\Omega BG_2^\wedge$ is determined by*

$$m_{2q+1}(\tau, \tau, \ldots, \tau) = s^q,$$

where $s = \alpha\beta + \beta\alpha$. This implies that

$$m_{2q+1}(f_1(\alpha,\beta)\tau, f_2(\alpha,\beta)\tau, \ldots, f_{2q+1}(\alpha,\beta)\tau) = f_1(\alpha,\beta)\ldots f_{2q+1}(\alpha,\beta)s^q, \tag{2.8.11}$$

and all m_n for $n > 2$ on all other n-tuples of monomials give zero.

Proof. By Lemma 2.8.8, we have $m_n = 0$ for $2 < n < 2q+1$. So in order to determine m_{2q+1}, we invoke Proposition 1.4.2. This shows that m_{2q+1} has to be a Hochschild cocycle, well defined up to adding Hochschild coboundaries. By Proposition 2.8.9, the dimension of $HH^*H_*\Omega BG_2^\wedge$ is one dimensional in degree $(-2q-1, 2q-1, 0, 0)$. A representative for a non-zero cohomology class is given by (2.8.11). It is easy to check that this is a cocycle but not a coboundary. So by rescaling τ if necessary (or by working over $\mathbb{F}_2$) we may assume that either m_{2q+1} is either zero or as given in the theorem. In both cases we can check that the Gerstenhaber circle product $m_{2q+1}\circ m_{2q+1}$ is equal to the zero cocycle in degree $-4q$.

As in the proof of Theorem 2.4.2, we can rewrite Equation 1.3.1 in degree $-4q$ as

$$\delta m_{4q} = m_{2q+1}\circ m_{2q+1},$$

which as we just saw, is zero. Now by Proposition 2.8.9 again, $HH^*H_*\Omega BG_2^\wedge$ is zero in degree $(-4q, 4q-2, 0, 0)$. So m_{4q} is a Hochschild coboundary, and we can therefore take $m_{4q} = 0$, as it is only well defined modulo Hochschild coboundaries. At this point, for $\ell > 2$, the equation we obtain for $m_{\ell(2q-1)+2}$ is $\delta m_{\ell(2q-1)+2} = 0$. Again, $HH^*H_*\Omega BG_2^\wedge$ is zero in degree $(-\ell(2q-1)-2, \ell(2q-1), 0, 0)$, and so we may take $m_{\ell(2q-1)+2} = 0$.

This argument shows that there are two possibilities for the A_∞ structure up to isomorphism, namely the one given and the formal one with $m_n = 0$ for all $n > 2$. The latter is impossible, since it would imply that the A_∞ structure on H^*BG is also formal, which it is not. □

REMARK 2.8.12. In the spectral sequence $HH^*H_*\Omega BG_2^\wedge \Rightarrow HH^*C_*\Omega BG_2^\wedge$, we have $d^{2q}(\tau) = s^q t^{2q}$. This implies that after inverting s (we discuss this later), we have

$$HH^*C_*\Omega BG_2^\wedge[s^{-1}] = \mathsf{k}[s, s^{-1}][u, v, t]/(u^2, v^2, uv, t^{2q}).$$

Since $HH^*C^*BG \cong HH^*C_*\Omega BG_2^\wedge$, this also computes $HH^*C^*BG[s^{-1}]$.

2.9. A differential graded model

Throughout this section, we work in Case 2.7.1, where G has dihedral Sylow 2-subgroups and one conjugacy class of involutions. As in [**22**], we produce a differential graded model Q for the A_∞ algebra $H_*\Omega BG_2^\wedge$. The proofs are similar to the ones in that paper, but we spell out the details because there are some minor differences. One is that we are in characteristic two, so we don't need to be careful about signs; another is that a polynomial ring in one variable has been replaced by the noncommutative ring $\mathsf{k}\langle\alpha, \beta\rangle/(\alpha^2, \beta^2)$.

Recall from Theorems 2.8.3 and 2.8.10 that

$$H_*\Omega BG_2^\wedge \cong \Lambda(\tau) \otimes \mathsf{k}\langle\alpha, \beta\rangle/(\alpha^2, \beta^2)$$

with m_{2q+1} determined by $m_{2q+1}(\tau, \dots, \tau) = s^q$, where $s = \alpha\beta + \beta\alpha$, and with all other m_i zero for $i > 2$.

The generators of Q are elements $\tau_1, \dots, \tau_{2q}, \alpha, \beta$, where τ_1 will eventually be seen to correspond to the element $\tau \in H_*\Omega BG_2^\wedge$. The relations and differential are as follows:

$$\begin{aligned} \alpha\tau_i &= \tau_i\alpha \\ \beta\tau_i &= \tau_i\beta \\ \alpha^2 &= \beta^2 = 0 \\ d\alpha &= d\beta = 0 \end{aligned}$$

$$\sum_{j+k=i} \tau_j\tau_k = \begin{cases} d\tau_i & 1 \leqslant i \leqslant 2q \\ s^q & i = 2q+1 \\ 0 & 2q+2 \leqslant i \leqslant 4q. \end{cases}$$

where $s = \alpha\beta + \beta\alpha$. The antipode is the algebra anti-automorphism given by $S(\tau_i) = \tau_i$, $S(\alpha) = \alpha$, $S(\beta) = \beta$ (we are in characteristic two, so there are no signs), and the comultiplication is given by

$$\Delta(\tau_i) = \tau_i \otimes 1 + 1 \otimes \tau_i, \qquad \Delta(\alpha) = \alpha \otimes 1 + 1 \otimes \alpha, \qquad \Delta(\beta) = \beta \otimes 1 + 1 \otimes \beta.$$

The degrees are given by $|\tau_i| = (2i-1, iq, iq)$, $|\alpha| = (2, q+1, q)$, $|\beta| = (2, q, q+1)$, and $|s| = (4, 2q+1, 2q+1)$. We shall see that this algebra Q is quasi-isomorphic to $C_*\Omega BG_2^\wedge$.

EXAMPLE 2.9.1. If $q = 1$, the algebra Q is generated by $\tau_1, \tau_2, \alpha, \beta$ with

$$\begin{array}{lll} d(\alpha) = 0 & \alpha^2 = 0 & \tau_1\tau_2 + \tau_2\tau_1 = s = \alpha\beta + \beta\alpha \\ d(\beta) = 0 & \beta^2 = 0 & \tau_2^2 = 0 \\ d(\tau_1) = 0 & \alpha\tau_i = \tau_i\alpha & \\ d(\tau_2) = \tau_1^2 & \beta\tau_i = \tau_i\beta & \end{array}$$

with $|\tau_1| = (1, 1, 1)$, $|\tau_2| = (3, 2, 2)$, $|\alpha| = (2, 2, 1)$, $|\beta| = (2, 1, 2)$ and $|s| = (4, 3, 3)$.

Lemma 2.9.2. *In the algebra Q, every element has a unique expression of the form*

$$f(\tau_1,\dots,\tau_{2q-1})+\tau_{2q}g(\tau_1,\dots,\tau_{2q-1})$$

with coefficients in $\mathsf{k}\langle\alpha,\beta\rangle/(\alpha^2,\beta^2)$.

Proof. The algebra relations (ignoring the differential) say first that the elements $\tau_1,\dots,\tau_{2q}$ commute with α and β; and the remaining relations can be rewritten in the form

$$\tau_i\tau_{2q}=\tau_{2q}\phi_i(\tau_1,\dots,\tau_{2q-1})$$

with $1\leqslant i\leqslant 2q$ (note that $\phi_{2q}=0$). Thus all occurrences of τ_{2q} may be moved to the beginning, and $\tau_{2q}^2=0$. There are no relations among $\tau_1,\dots,\tau_{2q-1}$. □

Definition 2.9.3. We shall refer to a monomial in $\tau_1,\dots,\tau_{2q-1}$, or τ_{2q} times such a monomial, as a *standard monomial* in the variables $\tau_1,\dots,\tau_{2q}$. By the lemma, these monomials form a basis for Q over $\mathsf{k}\langle\alpha,\beta\rangle/(\alpha^2,\beta^2)$.

Lemma 2.9.4. *In the algebra Q, we have $d^2=0$.*

Proof. The differential is given by

$$d(f+\tau_{2q}g)=(df+(\tau_1\tau_{2q-1}+\dots+\tau_{2q-1}\tau_1)g)+\tau_{2q}dg.$$

For $1\leqslant i\leqslant 2q-1$, se see that $dd(\tau_i)$ has two terms for each way of writing i as a sum of three positive integers, and they cancel. So we have $d^2=0$ on the subalgebra they generate. Thus we have

$$\begin{aligned}d^2(f+\tau_{2q}g)&=d(df+(\tau_1\tau_{2q-1}+\dots+\tau_{2q-1}\tau_1)g)+\tau_{2q}dg)\\&=d^2f+(\tau_1\tau_{2q-1}+\dots+\tau_{2q-1}\tau_1)dg+(\tau_1\tau_{2q-1}+\dots+\tau_{2q-1}\tau_1)dg\\&=0.\end{aligned}$$

□

Proposition 2.9.5. *The definitions above make Q into a cocommutative DG Hopf algebra.*

Proof. The above lemmas show that Q is a DG bialgebra. It is easy to check that the antipode satisfies the identity $S(x_{(1)})x_{(2)}=x_{(1)}S(x_{(2)})=0$ in Sweedler notation, for elements of non-zero degree this only needs checking on the generators, where it is clear. Cocommutativity also only needs checking on generators. □

Theorem 2.9.6. *There is a quasi-isomorphism from the A_∞ algebra $H_*\Omega BG_2^{\wedge}$ to the DG algebra Q, sending α to α, β to β, and τ to τ_1.*

Proof. First, we show that H_*Q is isomorphic to $H_*\Omega BG_2^{\wedge}$ as an algebra over the noncommutative ring $\mathsf{k}\langle\alpha,\beta\rangle/(\alpha^2,\beta^2)$. We define a $\mathsf{k}\langle\alpha,\beta\rangle/(\alpha^2,\beta^2)$-module homomorphism $\delta\colon Q\to Q$ sending a monomial of the form $\tau_1\tau_i f$ to $\tau_{i+1}f$ for $1\leqslant i\leqslant 2q-1$, and all other standard monomials to zero. Thus $\delta(f+\tau_{2q}g)=\delta(f)$. Then we have

$$\begin{aligned}\delta d(\tau_1\tau_i f)&=\delta(\tau_1(\tau_1\tau_{i-1}+\dots+\tau_{i-1}\tau_1)f+\tau_1\tau_i df)\\&=(\tau_2\tau_{i-1}+\dots+\tau_i\tau_1)f+\tau_{i+1}df\\d\delta(\tau_1\tau_i f)&=d(\tau_{i+1}f)=(\tau_1\tau_i+\dots+\tau_i\tau_1)f+\tau_{i+1}df\\(\delta d+d\delta)(\tau_1\tau_i f)&=\tau_1\tau_i f\end{aligned}$$

while for $j > 1$ we have

$$\begin{aligned} \delta d(\tau_j f) &= \delta((\tau_1\tau_{j-1} + \cdots + \tau_{j-1}\tau_1)f + \tau_j df) = \tau_j f \\ d\delta(\tau_j f) &= d(0) = 0 \\ (\delta d + d\delta)(\tau_j f) &= \tau_j f. \end{aligned}$$

Thus $\delta d + d\delta$ is the identity on all monomials except those in the $\mathsf{k}\langle\alpha,\beta\rangle/(\alpha^2,\beta^2)$-submodule spanned by 1 and τ_1, where it is zero. So δ defines a homotopy from the identity map of Q to the projection onto this submodule. It follows that H_*Q is isomorphic to $H_*\Omega BG_2^\wedge$ as an algebra over $\mathsf{k}\langle\alpha,\beta\rangle/(\alpha^2,\beta^2)$, with τ_1 corresponding to τ.

We have an A_∞ morphism $f\colon A \to Q$ given by $f_1(\alpha) = \alpha$, $f_1(\beta) = \beta$, and

$$f_i(\tau,\dots,\tau) = \tau_i, \qquad 1 \leqslant i \leqslant 2q.$$

The computation above shows that f_1 is a quasi-isomorphism, and hence by definition so is f. This computation is a practical illustration of Kadeishvili's theorem [**151**]. □

COROLLARY 2.9.7. *The bounded derived categories $\mathsf{D^b}(Q)$, $\mathsf{D^b}(C_*\Omega BG_2^\wedge)$ and $\mathsf{D^b}(C^*BG)$ are equivalent as triangulated categories.*

PROOF. This follows from Theorem 1.9.2, together with Theorem 2.9.6 above. □

The element $s = \alpha\beta + \beta\alpha$ is central in Q, so it makes sense to invert it in the A_∞ algebra $H_*\Omega BG_2^\wedge$.

COROLLARY 2.9.8. *We have equivalences of triangulated categories*

$$\mathsf{D^b}(Q[s^{-1}]) \simeq \mathsf{D^b}(C_*\Omega BG_2^\wedge[s^{-1}]) \simeq \mathsf{D_{csg}}(C_*\Omega BG_2^\wedge) \simeq \mathsf{D_{sg}}(C^*BG).$$

PROOF. Since $H_*\Omega BG_2^\wedge$ is periodic, with periodicity generator s, the effect on $\mathsf{D^b}(Q)$ of inverting s is to quotient out the thick subcategory generated by k. So this corollary again follows from Theorem 1.9.2. □

2.10. Duality for $Q[s^{-1}]$-modules

In this section, we continue to work in Case 2.7.1, where G has dihedral Sylow 2-subgroups and one conjugacy class of involutions.

DEFINITION 2.10.1. We write K for $\mathsf{k}\langle\alpha,\beta\rangle/(\alpha^2,\beta^2)[s^{-1}]$, where $s = \alpha\beta + \beta\alpha$.

LEMMA 2.10.2. *The graded algebra K is simple. The trace form $K \otimes_{\mathsf{k}[s,s^{-1}]} K \to \mathsf{k}[s,s^{-1}]$ induces an isomorphism of K-modules*

$$K \cong \mathsf{Hom}_{\mathsf{k}[s,s^{-1}]}(K, \mathsf{k}[s,s^{-1}]).$$

PROOF. This is the algebra of endomorphisms of a graded vector space of dimension two over the graded field $\mathsf{k}[s,s^{-1}]$, with a basis element u in degree zero and a basis element v in degree one. The element α sends u to v and v to zero, while β sends v to su and u to zero. Thinking in terms of matrices over $\mathsf{k}[s,s^{-1}]$ this can be visualised as

$$\alpha \mapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad \beta \mapsto \begin{pmatrix} 0 & s \\ 0 & 0 \end{pmatrix}, \qquad s \mapsto \begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix},$$

giving an isomorphism

$$K \cong \mathsf{Mat}_2(\mathsf{k}[s,s^{-1}]).$$

The trace form is given by multiplying matrices and taking the trace. It takes $\alpha\otimes\beta$ and $\beta\otimes\alpha$ both to s. It therefore induces an isomorphism of K-modules $K \cong \mathsf{Hom}_{\mathsf{k}[s,s^{-1}]}(K, \mathsf{k}[s, s^{-1}])$ sending α to the homomorphism sending α to zero and β to s and sending β to the homomorphism sending α to s and β to zero. $\square$

If X is any K-module, we write $X^* = \mathsf{Hom}_{\mathsf{k}[s,s^{-1}]}(X, \mathsf{k}[s, s^{-1}])$. Then using the lemma, we have

$$\begin{aligned} X^* &= \mathsf{Hom}_{\mathsf{k}[s,s^{-1}]}(X, \mathsf{k}[s, s^{-1}]) \\ &\cong \mathsf{Hom}_{\mathsf{k}[s,s^{-1}]}(K \otimes_K X, \mathsf{k}[s, s^{-1}]) \\ &\cong \mathsf{Hom}_K(X, \mathsf{Hom}_{\mathsf{k}[s,s^{-1}]}(K, \mathsf{k}[s, s^{-1}])) \\ &\cong \mathsf{Hom}_K(X, K), \end{aligned}$$

and so we can just as well regard X^* as $\mathsf{Hom}_K(X, K)$.

Proposition 2.10.3. *There is a quasi-isomorphism of $Q[s^{-1}]$-bimodules*

$$Q[s^{-1}] \to \Sigma Q[s^{-1}]^*.$$

Proof. The standard monomials form a free basis for $Q[s^{-1}]$ as a K-module. We construct a K-module homomorphism $Q[s^{-1}] \to \Sigma^{|\tau|}Q[s^{-1}]^*$ as follows. It takes all standard monomials to zero except 1 and τ_1. It takes 1 to the element of $Q[s^{-1}]^*$ taking value 1 on τ_1 and zero on all other monomials, and it takes τ_1 to the element of $Q[s^{-1}]^*$ taking value 1 on 1 and value zero on all other standard monomials. It is easy to check that this is a map of $Q[s^{-1}]$-bimodules, and a quasi-isomorphism. $\square$

Proposition 2.10.4. *If X is a left $Q[s^{-1}]$-module and Y is a right $Q[s^{-1}]$-module, then there is a natural isomorphism of K-modules*

$$\mathsf{Hom}_{Q[s^{-1}]}(X, \mathsf{Hom}_K(Y, K)) \cong \mathsf{Hom}_K(Y \otimes_{Q[s^{-1}]} X, K).$$

If Y is a $Q[s^{-1}]$-bimodule, this is an isomorphism of left $Q[s^{-1}]$-modules.

Proof. This is standard. $\square$

Corollary 2.10.5. *If X is a homotopically projective $Q[s^{-1}]$-module then we have a quasi-isomorphism*

$$\mathsf{Hom}_{Q[s^{-1}]}(X, Q[s^{-1}]) \simeq \Sigma\,\mathsf{Hom}_K(X, K).$$

Proof. We have

$$\begin{aligned} \mathsf{Hom}_{Q[\tau^{-1}]}(X, Q[s^{-1}]) &\simeq \mathsf{Hom}_{Q[s^{-1}]}(X, \Sigma Q[s^{-1}]^*) \\ &\cong \Sigma\,\mathsf{Hom}_{Q[s^{-1}]}(X, \mathsf{Hom}_K(Q[s^{-1}], K)) \\ &\cong \Sigma\,\mathsf{Hom}_K(Q[s^{-1}] \otimes_{Q[s^{-1}]} X, K) \\ &\cong \Sigma\,\mathsf{Hom}_K(X, K). \end{aligned}$$ $\square$

Theorem 2.10.6. *Let X and Y be $Q[s^{-1}]$-modules, such that X homotopically projective, and its image in $\mathsf{D^b}(Q[s^{-1}])$ is compact. Then we have a duality*

$$\mathsf{Hom}_{Q[s^{-1}]}(X, Y)^* \cong \mathsf{Hom}_{Q[s^{-1}]}(Y, \Sigma^{-1}X).$$

PROOF. Since X is homotopically projective with compact image in $\mathsf{D}^{\mathsf{b}}(Q[s^{-1}])$, we have quasi-isomorphisms

$$\mathsf{Hom}_{Q[s^{-1}]}(X,Y) \simeq \mathsf{Hom}_{Q[s^{-1}]}(X,Q[s^{-1}]) \otimes_{Q[s^{-1}]} Y$$

and

$$\mathsf{Hom}_{Q[s^{-1}]}(\mathsf{Hom}_{Q[s^{-1}]}(X,Q[s^{-1}]),Q[s^{-1}]) \simeq X.$$

Combining the second of these with Corollary 2.10.5, we have

$$\mathsf{Hom}_{Q[s^{-1}]}(X,Q[s^{-1}])^* \simeq \Sigma^{-1}X.$$

Hence using Proposition 2.10.4, we have

$$\begin{aligned}\mathsf{Hom}_{Q[s^{-1}]}(X,Y)^* &= \mathsf{Hom}_K(\mathsf{Hom}_{Q[s^{-1}]}(X,Y),K)\\ &\simeq \mathsf{Hom}_K(\mathsf{Hom}_{Q[s^{-1}]}(X,Q[s^{-1}]) \otimes_{Q[s^{-1}]} Y,K)\\ &\cong \mathsf{Hom}_{Q[s^{-1}]}(Y,\mathsf{Hom}_K(\mathsf{Hom}_{Q[s^{-1}]}(X,Q[s^{-1}]),K))\\ &\simeq \mathsf{Hom}_{Q[s^{-1}]}(Y,\Sigma^{-1}X).\end{aligned}$$

□

2.11. Some indecomposables

Let G be a finite group with dihedral Sylow 2-subgroups and a single conjugacy class of involutions. Consider first A_∞ modules over the A_∞ algebra $B = H^*BG$. The quotient $B/(t^{2q+1})$ is formal, so ordinary modules over this ring pull back to A_∞ modules over B. For $1 \leqslant i \leqslant 2q$, let X_i be the module $B/(\eta,t^i)$ and X'_i be the module $B/(\xi,t^i)$. Thus X_i has periodic resolution

$$\cdots \xrightarrow{\left(\begin{smallmatrix}\xi & t^i\\ 0 & \eta\end{smallmatrix}\right)} B\oplus B \xrightarrow{\left(\begin{smallmatrix}\eta & t^i\\ 0 & \xi\end{smallmatrix}\right)} B\oplus B \xrightarrow{\left(\begin{smallmatrix}\xi & t^i\\ 0 & \eta\end{smallmatrix}\right)} B\oplus B \xrightarrow{(\eta,t^i)} B \to X_i \to 0$$

and swapping η and ξ gives a resolution of X'_i. In $\mathsf{D}^{\mathsf{b}}(B)$, the residue field k sits in a triangle

$$B/(t) \to B/(\eta,t) \oplus B/(\xi,t) \to \mathsf{k}.$$

Furthermore, $B/(t)$ sits in a triangle

$$\Sigma^{-2}B \xrightarrow{t} B \to B/(t).$$

So in $\mathsf{D}_{\mathsf{sg}}(B)$, $B/(t)$ is isomorphic to zero, and k decomposes as $B/(\eta,t)\oplus B/(\xi,t) = X_1 \oplus X'_1$. The minimal resolutions of X_i and X'_i are as follows.

$$\cdots\to\Sigma^{-9}B\oplus\Sigma^{-6-2i}B \xrightarrow{\left(\begin{smallmatrix}\eta & t^i\\ 0 & \xi\end{smallmatrix}\right)} \Sigma^{-6}B\oplus\Sigma^{-3-2i}B \xrightarrow{\left(\begin{smallmatrix}\xi & t^i\\ 0 & \eta\end{smallmatrix}\right)} \Sigma^{-3}B\oplus\Sigma^{-2i}B \xrightarrow{(\eta,t^i)} B \to X_i \to 0,$$

$$\cdots\to\Sigma^{-9}B\oplus\Sigma^{-6-2i}B \xrightarrow{\left(\begin{smallmatrix}\xi & t^i\\ 0 & \eta\end{smallmatrix}\right)} \Sigma^{-6}B\oplus\Sigma^{-3-2i}B \xrightarrow{\left(\begin{smallmatrix}\eta & t^i\\ 0 & \xi\end{smallmatrix}\right)} \Sigma^{-3}B\oplus\Sigma^{-2i}B \xrightarrow{(\xi,t^i)} B \to X'_i \to 0.$$

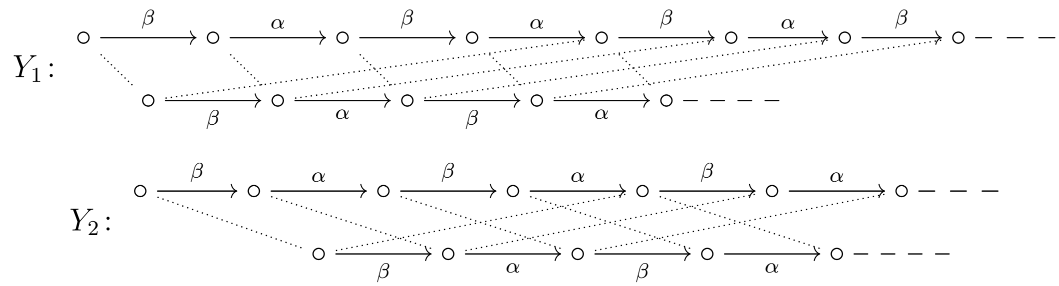
It follows that $\Sigma^2 X_i \cong X'_i$ and $\Sigma^2 X'_i \cong X_i$ in $\mathsf{D}_{\mathsf{sg}}(B)$. The category $\mathsf{D}_{\mathsf{sg}}(B)$ is periodic of period four, with periodicity generator $s = \alpha\beta + \beta\alpha$.

Let $A = B^!$ be the A_∞ algebra $H_*\Omega BG^\wedge_2$. For $1 \leqslant i \leqslant 2q$, let $Y_i = \mathsf{Ext}^*_B(\mathsf{k},X_i)$, the indecomposable A-module with generators u and v satisfying $\alpha u = 0$, $\alpha v = 0$, and

$$\begin{aligned}m_{i+1}(\tau,\dots,\tau,u) &= v,\\ m_{2q+2-i}(\tau,\dots,\tau,v) &= (\alpha\beta)^q u.\end{aligned}$$

Then in $\mathsf{D}_{\mathsf{csg}}(A)$, we have $\Sigma^{2i-1}Y_i \cong Y_{2q+1-i}$, so this gives q isomorphism classes up to shift, all periodic with period four, for a total of $4q$ isomorphism classes. Note that the ring A itself, as an object in $\mathsf{D}_{\mathsf{csg}}(A)$, decomposes as $Y_1 \oplus \Sigma^2 Y_1$.

Here they are for $q = 2$, with the $m_j(\tau, \ldots, \tau, -)$ represented by dotted lines:



Removing a finite number of nodes from the beginning of one of these diagrams does not alter the isomorphism class in $\mathsf{D}_{\mathsf{csg}}(A)$.

2.12. Classification of indecomposables

We continue to work in Case 2.7.1 with $q \geqslant 1$, and write B for the A_∞ algebra H^*BG and $A = B^!$ for the Koszul dual A_∞ algebra $H_*\Omega BG^\wedge_2$. The way we classify the indecomposables in $\mathsf{D}_{\mathsf{csg}}(A) \cong \mathsf{D}_{\mathsf{sg}}(B)$ is via Morita equivalence, reducing to the classification theorem of [**22**].

Let Y_i, $1 \leqslant i \leqslant 2q$, be the modules described in the previous section. Then the regular representation of A decomposes as $Y_1 \oplus \Sigma^2 Y_1$.

Let E be the A_∞ algebra $\mathsf{Hom}^*_A(Y_1, Y_1)$. This is the algebra with $m_i = 0$ for $i \neq 2, 2q+1$, defined as follows. The multiplication m_2 defines the k-algebra structure as $\mathsf{k}[s] \otimes \Lambda(\tau)$, with generators s and τ satisfying $|s| = (4, 2q+1, 2q+1)$, $|\tau| = (1, q, q)$. We have

$$m_{2q+1}(s^{i_1}\tau, \ldots, s^{i_{2q+1}}\tau) = s^{i_1+\cdots+i_{2q+1}+q},$$

and m_{2q+1} vanishes on all other tuples of monomials.

There is a right action of E on Y_1 given by $m_2(u, \tau) = v$, $m_{2q+1}(v, \tau, \ldots, \tau) = m_2(u, s^q)$. This makes Y_1 into an A-E-bimodule, and $\mathsf{Hom}^*_A(Y_1, -)$ induces an equivalence of derived categories $\mathsf{D}^{\mathsf{b}}(A) \simeq \mathsf{D}^{\mathsf{b}}(E)$ that sends A to $E \oplus \Sigma^2 E$ and Y_1 to E. It therefore also induces equivalences $\mathsf{D}_{\mathsf{csg}}(A) \simeq \mathsf{D}_{\mathsf{csg}}(E) \simeq \mathsf{D}^{\mathsf{b}}(E[s^{-1}])$. Theorem 1.1 of [**22**] (with $a = 1$, $b = 2$, $h = 2q + 1$, $\ell = q$) therefore gives the following.

THEOREM 2.12.1. *The triangulated categories*

$$\mathsf{D}_{\mathsf{sg}}(B) \simeq \mathsf{D}_{\mathsf{csg}}(A) \simeq \mathsf{D}^{\mathsf{b}}(A[s^{-1}]) \simeq \mathsf{D}_{\mathsf{csg}}(E) \simeq \mathsf{D}^{\mathsf{b}}(E[s^{-1}])$$

satisfy the Krull–Schmidt theorem, and have $4q$ isomorphism classes of indecomposable objects, in q orbits of the shift functor Σ. The Auslander–Reiten quiver is isomorphic to $\mathbb{Z}A_{2q}/T^2$, where T is the translation functor Σ^{-2}. This is a cylinder of height $2q$ and circumference 2. The functor Σ switches the two ends of the cylinder. $\square$

Here is a picture of the Auslander–Reiten quiver in the case $q = 4$; the left and right side should be identified to form a cylinder:

$$\begin{tikzcd}[column sep=small, row sep=small]
Y_1 \arrow[dr] \arrow[dd, dotted, no head] & & \Sigma^2 Y_1 \arrow[dr] & & Y_1 \arrow[dd, dotted, no head] \\
& Y_2 \arrow[ur] \arrow[dr] & & \Sigma^2 Y_2 \arrow[ur] \arrow[dr] & \\
\Sigma^2 Y_3 \arrow[ur] \arrow[dr] \arrow[dd, dotted, no head] & & Y_3 \arrow[ur] \arrow[dr] & & \Sigma^2 Y_3 \arrow[dd, dotted, no head] \\
& \Sigma^2 Y_4 \arrow[ur] \arrow[dr] & & Y_4 \arrow[ur] \arrow[dr] & \\
\Sigma^3 Y_4 \arrow[ur] \arrow[dr] \arrow[dd, dotted, no head] & & \Sigma Y_4 \arrow[ur] \arrow[dr] & & \Sigma^3 Y_4 \arrow[dd, dotted, no head] \\
& \Sigma Y_3 \arrow[ur] \arrow[dr] & & \Sigma^3 Y_3 \arrow[ur] \arrow[dr] & \\
\Sigma Y_2 \arrow[ur] \arrow[dr] \arrow[d, dotted, no head] & & \Sigma^3 Y_2 \arrow[ur] \arrow[dr] & & \Sigma Y_2 \arrow[d, dotted, no head] \\
{} & \Sigma^3 Y_1 \arrow[ur] & & \Sigma Y_1 \arrow[ur] & {}
\end{tikzcd}$$

Remark 2.12.2. In contrast with Theorem 2.12.1, the category $\mathsf{D}_{\mathsf{sg}}(A) \simeq \mathsf{D}_{\mathsf{csg}}(B)$ has infinite representation type. This can be seen by examining the quotient $H_*\Omega BG^\wedge_2/(\tau, s^q)$. By Theorem 2.8.10, this is the formal A_∞ algebra

$$\mathsf{k}\langle \alpha, \beta \,|\, \alpha^2 = 0, \beta^2 = 0, (\alpha\beta)^q + (\beta\alpha)^q = 0\rangle,$$

which has tame representation type (Ringel [**195**]). It would be interesting to know whether $\mathsf{D}_{\mathsf{sg}}(A)$ also has tame representation type.

2.13. Loops on $BG^\wedge_2$: two classes of involutions

We now turn to Case 2.7.2. This is the case where G has a dihedral Sylow 2-subgroup D of order $4q$ with $q \geqslant 2$, and two conjugacy classes of involution. In this case, G has exactly one subgroup of index two, and it has two isomorphism classes of simple modules in the principal block.

Remark 2.13.1. It follows from the work of Holm [**141**] that the derived equivalence classes of algebras of dihedral type with two isomorphism classes of simple modules are determined by two parameters, namely a positive integer $k \geqslant 1$ and a field element $c \in \{0, 1\}$. For a block of a finite group with dihedral defect group of order $4q$, the parameter k is equal to q. Theorem 6.8 of Eisele [**64**] shows that the case $c = 1$ cannot occur for a block of a finite group, so we have $c = 0$. Note that by Corollary 2.3 of Generalov and Romanova [**116**], the cases $c = 0$ and $c = 1$ have different Hochschild cohomology rings, even in degree one.

By Holm [**141**] and Proposition 2.7.2, for the purposes of studying $BG^\wedge_2$ we may assume that $G = PGL(2, p)$ for a suitable prime $p \equiv 1 \pmod 4$. In this case, the principal block B_0 of $\mathsf{k}G$ belongs to Erdmann's class $\mathrm{D}(2\mathcal{A})$. It has two simple modules k and M, whose Ext^1 quiver is as follows:

$$\begin{tikzcd}
\mathsf{k} \arrow[loop left, "e_3"] \arrow[r, bend left, "e_2"] & \mathsf{M} \arrow[l, bend left, "e_1"]
\end{tikzcd}.$$

Using Remark 2.13.1, the relations are

$$e_2e_1 = 0, \qquad e_3^2 = 0, \qquad (e_1e_2e_3)^q = (e_3e_1e_2)^q.$$

We put an internal grading on the basic algebra in this case by assigning degree $(\frac{1}{2}, 0)$ to e_1 and e_2 and $(0, 1)$ to e_3.

We have $H^*BG = \mathsf{k}[\xi, y, t]/(\xi y)$ where

$$|\xi| = -(3, q+1, q), \qquad |y| = -(1, 0, 1), \qquad |t| = -(2, q, q).$$

The restrictions to D are given by $\mathrm{res}_{\mathrm{D}}(\xi) = xt$, $\mathrm{res}_{\mathrm{D}}(y) = y$, and $\mathrm{res}_{\mathrm{D}}(t) = t$. Massey products are determined by

$$\langle \xi, y, \dots, \xi, y\rangle = \langle y, \xi, \dots, y, \xi\rangle = t^{q+1}.$$

The computation of Hochschild cohomology is again very similar, and we omit the details. The A_∞ structure on H^*BG again follows the same lines as in Theorem 2.4.2. This time, we only replace x by ξ, and again adjust the powers of t. So we have

$$m_{2q}(\xi^i, y, \xi, y, \dots, \xi, y^j) = m_{2q}(y^j, \xi, y, \xi, \dots, y, \xi^i) = \xi^{i-1}y^{j-1}t^{q+1}.$$

The value of $m_{\ell(2q-2)+2}$ on the tuples at the end of the theorem is replaced by $\xi^{i-1}y^{j-1}t^{\ell(q+1)}$.

Theorem 2.13.2. *Let G be a finite group with dihedral Sylow 2-subgroups of order $4q$ with $q \geqslant 2$ a power of two, and two classes of involutions, and let k be a field of characteristic two. Then we have*

$$H_*\Omega BG^\wedge_2 = \Lambda(\tau) \otimes \mathsf{k}\langle \alpha, Y \mid \alpha^2 = 0,\ Y^2 = 0\rangle$$

with

$$|\tau| = (1, q, q), \qquad |\alpha| = (2, q+1, q), \qquad |Y| = (0, 0, 1).$$

In homological degree $2n$ we have monomials $(\alpha Y)^n$, $(Y\alpha)^n$, $(\alpha Y)^{n-1}\alpha$ and $(Y\alpha)^nY$, and in odd degrees we have τ times all of these.

Proof. The Eilenberg–Moore spectral sequence has as its E_2 page

$$\mathsf{Ext}^{*,*}_{H^*BG}(\mathsf{k}, \mathsf{k}) = \Lambda(\tau) \otimes \mathsf{k}\langle \alpha, Y \mid \alpha^2 = 0,\ Y^2 = 0\rangle$$

where the generators have degrees

$$|\tau| = (-1, 2, q, q), \qquad |\alpha| = (-1, 3, q+1, q), \qquad |Y| = (-1, 1, 0, 1).$$

The Eisenbud operator for the relation $\eta y = 0$ is $s = \alpha Y + Y\alpha$ in degree $(-2, 4, q+1, q+1)$. Again there is no room for non-zero differentials, and no ungrading problems, so $E_2 = E_\infty = H_*\Omega BG^\wedge_2$. □

Lemma 2.13.3. *For any A_∞ structure on H^*BG that preserves internal degrees, we have $m_n = 0$ unless $n-2$ is divisible by $q-2$. In particular, for $2 < n < q$ we have $m_n = 0$.*

Proof. The proof is essentially the same as the proof of Lemma 2.4.1. □

Proposition 2.13.4. *Let G be a group with a dihedral Sylow 2-subgroup* D *of order $4q$ with $q \geqslant 2$ a power of two, and two conjugacy classes of involutions, and let k be a field of characteristic two. The Hochschild cohomology HH^*H^*BG has generators s, t, τ, ξ, y, u, v with*

$$|s| = (-2, 4, q+1, q+1)$$

$$
\begin{aligned}
|t| &= -(0,2,q,q) & |\tau| &= (-1,2,q,q)\\
|\xi| &= -(0,3,q+1,q) & |y| &= -(0,1,0,1)\\
|u| &= -(1,0,0,0) & |v| &= -(1,0,0,0).
\end{aligned}
$$

The relations are given by $u^2 = v^2 = uv = \tau^2 = 0$, $yu = \xi v = 0$, $\xi s = ys = 0$, and $us = vs$. The non-zero monomials and their degrees are as follows, with $i_1, i_2 \geqslant 0$, $\varepsilon_1, \varepsilon_2 \in \{0,1\}$.

$$
\begin{aligned}
|s^{i_1}t^{i_2}\tau^{\varepsilon_1}u^{\varepsilon_2}| &= (-2i_1-\varepsilon_1-\varepsilon_2, 4i_1-2i_2+2\varepsilon_1, i_1+q(i_1-i_2+\varepsilon_1), i_1+q(i_1-i_2+\varepsilon_1)),\\
|s^{i_1}t^{i_2}\tau^{\varepsilon_1}v^{\varepsilon_2}| &= (-2i_1-\varepsilon_1-\varepsilon_2, 4i_1-2i_2+2\varepsilon_1, i_1+q(i_1-i_2+\varepsilon_1), i_1+q(i_1-i_2+\varepsilon_1)),\\
|\xi^{i_1}t^{i_2}\tau^{\varepsilon_1}u^{\varepsilon_2}| &= (-\varepsilon_1-\varepsilon_2, -3i_1-2i_2+2\varepsilon_1, -i_1+q(-i_1-i_2+\varepsilon_1), q(-i_1-i_2+\varepsilon_1))\\
|y^{i_1}t^{i_2}\tau^{\varepsilon_1}v^{\varepsilon_2}| &= (-\varepsilon_1-\varepsilon_2, -i_1-2i_2+2\varepsilon_1, q(-i_2+\varepsilon_1), -i_1+q(-i_2+\varepsilon_1))
\end{aligned}
$$

There is only one monomial in degree $(-i, i-2, 0, 0)$ with $i > 2$, namely

$$|s^q t^{q+1}| = (-q, q-2, 0, 0).$$

Proof. As in Theorem 2.3.2, we use the approach of Theorems 1.11.5 and 1.12.2. Thus HH^*H^*BG is the homology of the complex

$$(H^*BG \otimes H_*\Omega BG^{\wedge}_2, \partial),$$

where the generators t, ξ and η are in homological degree zero, the generators τ, α and Y are in homological degree -1, and the differential is given by $\partial = [e, -]$ where $e = t\otimes\tau + \xi\otimes\alpha + y\otimes Y$. Thus setting $s = \alpha Y + Y\alpha$, we have $\partial(t) = 0$, $\partial(\xi) = 0$, $\partial(y) = 0$, $\partial(\alpha) = ys$, $\partial(\tau) = 0$, $\partial(Y) = \xi s$. The generators and relations for the homology of this complex are therefore as given, with $u = \xi\alpha$ and $v = yY$.

For the last statement, the computation is similar to the corresponding part of the proof of Theorem 2.3.2. $\square$

Theorem 2.13.5. *The A_∞ structure on H^*BG is given as follows. The m_n are $\mathsf{k}[t]$-multilinear maps with $m_n = 0$ for n not congruent to 2 modulo $2q-2$, and for $i, j \geqslant 1$*

$$m_{2q}(\xi^i, y, \xi, y, \dots, \xi, y^j) = m_{2q}(y^j, \xi, y, \xi, \dots, y, \xi^i) = \xi^{i-1}y^{j-1}t^{q+1}$$

where the arguments alternate between ξ and y, and the right hand side is zero unless either $i = 1$ or $j = 1$; m_{2q} is zero on all other tuples of monomials not involving t. The maps $m_{\ell(2q-2)+2}$ with $\ell > 1$ similarly vanish on all tuples of monomials not involving t, except the ones which look as above, but for some choice of indices in the tuple:

$$
\begin{aligned}
1 \leqslant e_1 \leqslant e_2 \leqslant \cdots \leqslant e_{\ell-1} < e_{\ell-1} + (2q-2) + 1 \leqslant e_{\ell-2} + 2(2q-2) + 1&\\
\leqslant \cdots \leqslant e_1 + (\ell-1)(2q-2) + 1 \leqslant \ell(2q-2) + 2.&
\end{aligned}
$$

the exponents on the terms are increased by one (or correspondingly more if an index is repeated). The value on these tuples is $\xi^{i-1}y^{j-1}t^{\ell(q+1)}$. Thus

$$m_{\ell(2q-2)+2}(x^{i+\alpha_1}, y^{\alpha_2}, x^{\alpha_3}, \dots, x^{\alpha_{\ell(2q-2)+1}}, y^{j+\alpha_{\ell(2q-2)+2}}) = x^{i-1}y^{j-1}t^{\ell(q+1)}$$

where each α_σ is one plus the number of indices in the list above that are equal to σ.

Proof. This is similar to the proof of Theorem 2.4.2, but using Lemma 2.13.3 and Proposition 2.13.4 instead of Lemma 2.4.1 and Theorem 2.3.2. $\square$

LEMMA 2.13.6. *For any A_∞ structure on $H_*\Omega BG_2^\wedge$ that preserves internal degrees, we have $m_n = 0$ unless $n-2$ is divisible by $q-1$. In particular, for $2 < n < q+1$ we have $m_n = 0$.*

PROOF. Looking at the degrees of the generators τ, α and β, for any monomial ζ in $H_*\Omega BG_2^\wedge$ we have $a \equiv b \pmod{q-1}$. So for any n-tuple $(\zeta_1, \dots, \zeta_n)$, the degree of $m_n(\zeta_1, \dots, \zeta_n)$ satisfies $a \equiv b + n - 2 \pmod{q-1}$. So for this expression to be non-zero we must have $n - 2 \equiv 0 \pmod{q-1}$. □

PROPOSITION 2.13.7. *The Hochschild cohomology $HH^*H_*\Omega BG_2^\wedge$ has generators s, t, τ, ξ, η, u, and v in degrees*

$$\begin{aligned} &|s| = (0, 2, q+1, q+1), && \\ &|t| = -(1, 1, q, q), && |\tau| = (0, 1, q, q), \\ &|\xi| = -(1, 2, q+1, q), && |y| = -(1, 1, 0, 1), \\ &|u| = -(1, 0, 0, 0), && |v| = -(1, 0, 0, 0). \end{aligned}$$

The relations are given by $\xi\eta = 0$, $u^2 = v^2 = uv = \tau^2 = 0$, $\eta u = \xi v = 0$, $\xi s = ys = 0$, and $us = vs$. The non-zero monomials and their degrees are given as follows, with $i_1, i_2 \geqslant 0$, $\varepsilon_1, \varepsilon_2 \in \{0, 1\}$.

$$\begin{aligned} |s^{i_1}t^{i_2}\tau^{\varepsilon_1}u^{\varepsilon_2}| &= (-i_2 - \varepsilon_2, 2i_1 - i_2 + \varepsilon_1, (i_1 - i_2 + \varepsilon_1)q + i_1, (i_1 - i_2 + \varepsilon_1)q + i_1), \\ |s^{i_1}t^{i_2}\tau^{\varepsilon_1}v^{\varepsilon_2}| &= (-i_2 - \varepsilon_2, 2i_1 - i_2 + \varepsilon_1, (i_1 - i_2 + \varepsilon_1)q + i_1, (i_1 - i_2 + \varepsilon_1)q + i_1), \\ |\xi^{i_1}t^{i_2}\tau^{\varepsilon_1}u^{\varepsilon_2}| &= (-i_1 - i_2 - \varepsilon_2, -2i_1 - i_2 + \varepsilon_1, -i_1 + (-i_1 - i_2 + \varepsilon_1)q, (-i_1 - i_2 + \varepsilon_1)q), \\ |y^{i_1}t^{i_2}\tau^{\varepsilon_1}v^{\varepsilon_2}| &= (-i_1 - i_2 - \varepsilon_2, -i_1 - i_2 + \varepsilon_1, (-i_2 + \varepsilon_1)q, -i_1 + (-i_2 + \varepsilon_1)q). \end{aligned}$$

Thus there is only one monomial with degree $(-i, i-2, 0, 0)$ with $i > 2$, namely

$$|s^q t^{q+1}| = (-q-1, q-1, 0, 0).$$

PROOF. Again we use the approach of Theorem 1.12.2. This time, $HH^*H_*\Omega BG_2^\wedge$ is the homology of the complex

$$(H_*\Omega BG_2^\wedge \otimes H^*BG, \partial),$$

where the generators τ, α and Y of $H_*\Omega BG_2^\wedge$ are in homological degree zero, the generators t, ξ, y of H^*BG are in homological degree -1, and the differential is given by $\partial = [e, -]$ where $e = \tau \otimes t + \alpha \otimes \xi + Y \otimes y$. So the answer is the same as in Proposition 2.13.4 but with the degrees changed.

For the last statement, the computation is again similar to the corresponding part of the proof of Theorem 2.3.2. □

THEOREM 2.13.8. *In Case 2.7.2, the A_∞ structure on $H_*\Omega BG_2^\wedge$ is determined by*

$$m_{q+1}(\tau, \tau, \dots, \tau) = s^q,$$

where $s = \alpha Y + Y\alpha$. This implies that

$$m_{q+1}(f_1(\alpha, Y)\tau, f_2(\alpha, Y)\tau, \dots, f_{q+1}(\alpha, Y)\tau) = f_1(\alpha, Y) \dots f_{q+1}(\alpha, Y)s^q,$$

and all m_n for $n > 2$ on all other n-tuples of monomials give zero.

Proof. This is similar to the proof of Theorem 2.8.10, but using Lemma 2.13.6 and Proposition 2.13.7 in place of Lemma 2.8.8 and Proposition 2.8.9. ☐

Everything from this point on is very similar to Case 2.7.1, so we simply state the relevant results.

Remark 2.13.9. In the spectral sequence $HH^*H_*\Omega BG^\wedge_2 \Rightarrow HH^*C_*\Omega BG^\wedge_2$, we have $d_q(\tau) = s^q t^q$. This implies that after inverting s we have

$$HH^*C^*BG[s^{-1}] \cong HH^*C_*\Omega BG^\wedge_2[s^{-1}] \cong \mathsf{k}[s, s^{-1}][u, v, t]/(u^2, v^2, uv, t^q).$$

The differential graded model Q for $C_*\Omega BG^\wedge_2$ is essentially the same as that described in Section 2.9, except that β is replaced by Y in degree $(0, 0, 1)$, and the element $s = \alpha Y + Y\alpha$ is in degree $(2, q+1, q+1)$. So the generators for Q are $\tau_1, \dots, \tau_q, \alpha, Y$, and the relations between the τ_i are given by

$$\sum_{j+k=i} \tau_j \tau_k = \begin{cases} d\tau_i & 1 \leqslant i \leqslant q \\ s^q & i = q+1 \\ 0 & q+2 \leqslant i \leqslant 2q. \end{cases}$$

The final theorem in Case 2.7.2 is as follows.

Theorem 2.13.10. *The triangulated categories*

$$\mathsf{D}_{\mathsf{sg}}(C^*BG) \simeq \mathsf{D}_{\mathsf{csg}}(C_*\Omega BG^\wedge_2) \simeq \mathsf{D}^{\mathsf{b}}(C_*\Omega BG^\wedge_2[s^{-1}])$$

satisfy the Krull–Schmidt theorem, and have $2q$ isomorphism classes of indecomposable objects, in q orbits of the shift functor Σ. The Auslander–Reiten quiver is isomorphic to $\mathbb{Z}A_{2q}/T$, where T is the translation functor Σ^{-2}. This is a cylinder of height $2q$ and circumference one. The functor Σ switches the two ends of the cylinder. ☐

Remark 2.13.11. Again, and for the same reason as in Remark 2.12.2, in contrast with Theorem 2.13.10 the category $\mathsf{D}_{\mathsf{sg}}(C_*\Omega BG^\wedge_2) \simeq \mathsf{D}_{\mathsf{csg}}(C^*BG)$ has infinite representation type. This time, the formal quotient is

$$H_*\Omega BG^\wedge_2/(\tau, s^q) = \mathsf{k}\langle \alpha, Y \mid \alpha^2 = Y^2 = (\alpha Y)^q + (Y\alpha)^q = 0\rangle.$$

2.14. A related symmetric tensor category

The first non-semisimple symmetric tensor category in characteristic two discussed in Benson and Etingof [**20**] is the category denoted $\mathcal{C}_3$ and discussed in Section 5.2.3 of that paper. This has a basic algebra that is of dihedral type $\mathrm{D}(2\mathcal{A})$ with $c = 0$ and $k = 1$ in Erdmann's classification [**74**], given by a quiver and relations

$$\begin{tikzcd} \mathsf{k} \arrow[loop left, "a"] \arrow[r, bend left, "b"] & \mathsf{M} \arrow[l, bend left, "c"] \end{tikzcd}$$

with relations

$$a^2 = 0, \qquad bc = 0, \qquad cba = acb.$$

This is not equivalent to a block of the group algebra of any finite group, but it is quite similar in behaviour.

This algebra admits a $\mathbb{Z} \times \mathbb{Z}$-grading with $|a| = (1, 0)$, $|b| = |c| = (0, \frac{1}{2})$. The minimal resolution over this algebra is the total complex of the following double complex:

$$\begin{tikzcd}
 & & & \vdots \arrow[d] & & \\
 & & P_{\mathsf{M}} \arrow[d, "\bar{b}"'] & P_{\mathsf{k}} \arrow[l, "\bar{a}"'] \arrow[d, "\bar{b}\bar{c}"'] & \cdots & \\
 & & P_{\mathsf{k}} \arrow[d, "\bar{b}\bar{c}"'] & P_{\mathsf{k}} \arrow[l, "\bar{a}"'] \arrow[d, "\bar{b}\bar{c}"'] & \cdots & \\
 & P_{\mathsf{M}} \arrow[d, "\bar{b}"'] & P_{\mathsf{k}} \arrow[l, "\bar{a}"'] \arrow[d, "\bar{b}\bar{c}"'] & P_{\mathsf{k}} \arrow[l, "\bar{a}"'] \arrow[d, "\bar{b}\bar{c}"'] & \cdots & \\
 & P_{\mathsf{k}} \arrow[d, "\bar{b}\bar{c}"'] & P_{\mathsf{k}} \arrow[l, "\bar{a}"'] \arrow[d, "\bar{b}\bar{c}"'] & P_{\mathsf{k}} \arrow[l, "\bar{a}"'] \arrow[d, "\bar{b}\bar{c}"'] & \cdots & \\
P_{\mathsf{M}} \arrow[d, "\bar{b}"'] & P_{\mathsf{k}} \arrow[l, "\bar{c}\bar{a}"'] \arrow[d, "\bar{b}\bar{c}"'] & P_{\mathsf{k}} \arrow[l, "\bar{a}"'] \arrow[d, "\bar{b}\bar{c}"'] & P_{\mathsf{k}} \arrow[l, "\bar{a}"'] \arrow[d, "\bar{b}\bar{c}"'] & P_{\mathsf{k}} \arrow[l, "\bar{a}"'] \arrow[d, "\bar{b}\bar{c}"'] & \cdots \\
P_{\mathsf{k}} & P_{\mathsf{k}} \arrow[l, "\bar{a}"'] & P_{\mathsf{k}} \arrow[l, "\bar{a}"'] & P_{\mathsf{k}} \arrow[l, "\bar{a}"'] & P_{\mathsf{k}} \arrow[l, "\bar{a}"'] & \cdots
\end{tikzcd}$$

The cohomology is

$$H^*\mathcal{C}_3 = \mathsf{Ext}^*_{\mathcal{C}_3}(\mathsf{k}, \mathsf{k}) \cong \mathsf{k}[x, y, z]/(xz + y^2)$$

with $|x| = (-1, -1, 0)$, $|y| = (-2, -1, -1)$, $|z| = (-3, -1, -2)$. The elements x, y and z are given by shifts of degrees $(-1, 0)$, $(-1, -1)$ and $(-1, -2)$ in this diagram, killing the copies of P_{M} and given by the identity map on all copies of P_{k}. These maps commute, not just up to homotopy, and the relation $xz + y^2 = 0$ holds at the level of cocycles. So this Ext algebra is formal as an A_∞ algebra. It is a Koszul algebra with Koszul dual

$$H^*\mathcal{C}_3{}^! \cong \mathsf{k}[\eta]\langle \xi, \zeta \rangle / (\xi^2, \zeta^2, \xi\zeta + \zeta\xi + \eta^2),$$

with η central, degrees $|\xi| = (0, 1, 0)$, $|\eta| = (1, 1, 1)$, $|\zeta| = (2, 1, 2)$, and again formal as an A_∞ algebra. As a module over $k[\eta]$ it is free of rank four, with basis 1, ξ, ζ, $\xi\zeta$.

Using Theorem 1.11.5, and setting $u = x\xi + z\zeta$, we have

$$HH^*H^*\mathcal{C}_3 = k[x, y, z, \eta, u]/(x\eta^2, z\eta^2, u^2),$$

with $|x| = (0, -1, -1, 0)$, $|y| = (0, -2, -1, -1)$, $|z| = (0, -3, -1, -2)$, $|\eta| = (-1, 2, 1, 1)$, $|u| = (-1, 0, 0, 0)$. Then $HH^*C^*\mathcal{C}_3$ is the same ring, but with the first two degrees added, so $|x| = (-1, -1, 0)$, $|y| = (-2, -1, -1)$, $|z| = (-3, -1, -2)$, $|\eta| = (1, 1, 1)$, $|u| = (-1, 0, 0)$.

CHAPTER 3

The semidihedral case

3.1. Introduction

In this chapter, we study the A_∞ algebras H^*BG and $H_*\Omega BG^\wedge_2$ with coefficients in a field k of characteristic two, in the case where G is a finite group with semidihedral Sylow 2-subgroup. These groups were classified by Alperin, Brauer and Gorenstein [**1**]. The simple groups of this type are the projective special linear groups $PSL(3, p^m)$ with $p^m \equiv 3 \pmod 4$, the projective special unitary groups $PSU(3, p^m)$ with $p^m \equiv 1 \pmod 4$, and the sporadic Mathieu group M_{11} of order 7920.

We begin with the semidihedral group SD of order $8q$ itself. The group algebra in this case was analysed by Bondarenko and Drozd [**33**], who gave a presentation as a quiver with relations, but with a socle ambiguity. We resolve that ambiguity in Theorem 3.2.1, where we prove that for suitable radical generators X and Y we have

$$\mathsf{k}\mathrm{SD} = \langle X, Y \mid X^2 = 0,\ Y^2 = X(YX)^{2q-1} + (YX)^{2q}\rangle.$$

We then recall the structure of $H^*B\mathrm{SD}$ and compute $\mathsf{Ext}^*_{H^*B\mathrm{SD}}(\mathsf{k}, \mathsf{k})$, and show how the Eilenberg–Moore sequence with this as E^2 page converges to $\mathsf{k}\mathrm{SD}$.

There are four cases for the possible fusion in SD, leading to four types for cochains on the classifying space of a finite group with this fusion. Probably the most interesting is the case where G has no normal subgroup of index two. In that case, it turns out that the basic algebra of the principal block admits a grading, that endows the cohomology with a second, internal grading. Also, the cohomology rings of these groups have the structure of a complete intersection, which allows for easy computation of the Hochschild cohomology HH^*H^*BG. These facts together are what allows us to analyse the A_∞ structure. The following theorem is proved in Sections 3.6 to 3.11.

Theorem 3.1.1. *Let G be a finite group with semidihedral Sylow 2-subgroups of order $8q$ ($q \geqslant 2$ a power of two), and with no normal subgroup of index two, and let k be a field of characteristic two. Then the principal block B of $\mathsf{k}G$ has an essentially unique grading. This makes the cohomology ring*

$$H^*BG = \mathsf{k}[x, y, z]/(x^2y + z^2)$$

doubly graded, with $|x| = (-3, -q-1)$, $|y| = (-4, -4q)$ and $|z| = (-5, -3q-1)$. The cochain algebra C^*BG is formal as an A_∞ algebra. We have

$$H_*\Omega BG^\wedge_2 = \Lambda(\hat{x}, \hat{y}) \otimes \mathsf{k}[\hat{z}]$$

with $|\hat{x}| = (2, q+1)$, $|\hat{y}| = (3, 4q)$ and $|\hat{z}| = (4, 3q+1)$. This is not formal, but the A_∞ structure is given up to quasi-isomorphism by the $\mathsf{k}[\hat{z}]$-multilinear maps

$$m_3(\hat{x}, \hat{y}, \hat{x}) = \hat{z}^2, \qquad m_3(\hat{x}, \hat{x}\hat{y}, \hat{x}) = \hat{x}\hat{z}^2, \qquad m_3(\hat{y}, \hat{x}, \hat{x}\hat{y}) = m_3(\hat{x}\hat{y}, \hat{x}, \hat{y}) = \hat{y}\hat{z}^2,$$

and all m_i with $i \geqslant 3$ vanish on all other triples of monomials not involving $\hat{z}$.

As part of this computation, we also compute Hochschild cohomology.

THEOREM 3.1.2. *Let G be a finite group with semidihedral Sylow 2-subgroups of order $8q$ ($q \geqslant 2$ a power of two), and with no normal subgroups of index two, and let k be a field of characteristic two. Then*

$$HH^*H^*BG = H^*BG[\hat{x}, \hat{z}]/(\hat{x}^2 + y\hat{z}^2, x^2\hat{z}^2),$$

with

$$|x| = (0, -3, -q-1), \qquad |y| = (0, -4, -4q), \qquad |z| = (0, -5, -3q-1),$$
$$|\hat{x}| = (-1, 3, q+1), \qquad |\hat{z}| = (-1, 5, 3q+1).$$

The algebra $HH^*C^*BG = HH^*C_*\Omega BG^\wedge_2$ is the same, but with

$$|x| = (-3, -q-1), \qquad |y| = (-4, -4q), \qquad |z| = (-5, -3q-1),$$
$$|\hat{x}| = (2, q+1), \qquad |\hat{z}| = (4, 3q+1).$$

It should be possible to classify the indecomposable modules in the singularity category $\mathsf{D_{sg}}(C^*BG)$ in this case, given that C^*BG is formal. After all, the singularity category of graded modules over H^*BG, which is equivalent to the category of maximal Cohen–Macaulay modules, is well understood. The obstruction is that we don't know whether every object in $\mathsf{D_{sg}}(C^*BG)$ is equivalent to an object with zero differential. We make further comments on this situation in Section 3.11.

The second case in which we are able to make essentially complete computations is where the Sylow 2-subgroups of G are semidihedral, G has a normal subgroup K of index two with generalised quaternion Sylow 2-subgroups, and K has no normal subgroups of index two. This case is very similar to the case discussed above. In particular, again it turns out that the basic algebra of the principal block admits a grading, that endows the cohomology with a second, internal grading. The computations are similar, except that the degrees of various elements have changed. Again the cochain algebra C^*BG is formal as an A_∞ algebra. The corresponding theorems can be found in Sections 3.12 to 3.14. And again, it should be possible to classify the indecomposable modules in $\mathsf{D_{sg}}(C^*BG)$ in this case, with the same obstruction as in the previous case.

The remaining case is the one where the Sylow 2-subgroups of G are semidihedral, G has a normal subgroup K of index two with dihedral Sylow 2-subgroups, and K has no normal subgroups of index two. In this case, C^*BG is not formal, but we compute $H_*(\Omega BG^\wedge_2)$ (Theorem 3.15.3) using the method of squeezed resolutions from [**16**], since the Eilenberg–Moore spectral sequence is difficult to ungrade directly. The information in this case remains rather incomplete.

3.2. Semidihedral groups

The semidihedral group of order $8q$, $q \geqslant 2$ a power of two, is given by the presentation

$$\mathrm{SD} = \langle g, h \mid g^{4q} = 1, h^2 = 1, hgh^{-1} = g^{2q-1} \rangle.$$

Let k be a field of characteristic two. A modified version of the formulas of Bondarenko and Drozd [**33**] describes the group algebra $\mathsf{k}\mathrm{SD}$ as follows. Set

$$X = 1 + h, \qquad Y = (1+h)\left(\sum_{i=0}^{q/2-1} g^{4i+1} + \sum_{i=q/2+1}^{q} g^{4i-1}\right) + g^{2q} + hg^{4q-1}.$$

THEOREM 3.2.1. *With this choice for X and Y, $\mathsf{k}\mathrm{SD}$ has the presentation*

$$\mathsf{k}\mathrm{SD} = \langle X, Y \mid X^2 = 0,\ Y^2 = X(YX)^{2q-1} + (YX)^{2q}\rangle.$$

PROOF. The elements X and Y are in $J(\mathsf{k}\mathrm{SD})$, are independent modulo $J^2(\mathsf{k}\mathrm{SD})$, so they generate $\mathsf{k}\mathrm{SD}$. We have $X^2 = (1+h)^2 = 0$, so we must check the other relation. Set

$$u = \sum_{i=0}^{q/2-1} g^{4i+1} + \sum_{i=q/2+1}^{q} g^{4i-1}$$

so that $Y = (1+h)u+g^{2q}+hg^{-1}$. Write N_1, N_2 and N_4 for the norm elements for $\langle g\rangle$, $\langle g^2\rangle$ and $\langle g^4\rangle$ respectively. Then we have $u^2 = N_4g^2$, $uh+hu = N_2gh$, $(1+h)u(1+h) = N_2g(1+h)$, $((1+h)u)^2 = 0$, $ug = gu$, $g^{2q}h = hg^{2q}$, and $u(g^{-1}+g^{2q+1}) = 1+g^{2q}$. So in the expression for Y, the first and second terms commute, as do the second and third. So squaring Y, we have square terms and cross terms between the first and third term:

$$\begin{aligned} Y^2 &= 0 + 1 + g^{2q} + (1+h)uhg^{-1} + hg^{-1}(1+h)u \\ &= 1 + g^{2q} + uhg^{-1} + ug^{-1} + N_2 + hg^{-1}u + g^{2q+1}u \\ &= 1 + g^{2q} + u(g^{-1}+g^{2q+1}) + (uh+hu)g^{-1} + N_2 \\ &= N_2(1+h). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} YX &= (1+h)u(1+h) + (g^{2q}+hg^{-1})(1+h) \\ &= (N_2g + g^{2q} + g^{2q+1})(1+h). \end{aligned}$$

Since

$$\begin{aligned} (1+h)(N_2g+g^{2q}+g^{2q+1})(1+h) &= (1+h)g^{2q+1}(1+h) \\ &= (g^{-1}+g^{2q+1})(1+h), \end{aligned}$$

by induction on $m \geqslant 1$ we have

$$(YX)^m = (N_2g+g^{2q}+g^{2q+1})(g^{-1}+g^{2q+1})^{m-1}(1+h).$$

We have $N_2g(g^{-1}+g^{2q+1}) = 0$, so this simplifies for $m \geqslant 2$ to

$$(YX)^m = (g^{2q}+g^{2q+1})(g^{-1}+g^{2q+1})^{m-1}(1+h).$$

We also have $(g^{-1}+g^{2q+1})^{2q-2} = (g^{-2}+g^2)^{q-1} = g^2N_4$, and so

$$(YX)^{2q-1} = (g^{2q}+g^{2q+1})g^2N_4(1+h) = (g^2+g^3)N_4(1+h),$$

and

$$\begin{aligned} X(YX)^{2q-1} &= (1+h)(g^2+g^3)N_4(1+h) \\ &= (g^2+g^3+g^{2q-2}+g^{2q-3})N_4(1+h) \end{aligned}$$

$$= gN_2(1+h).$$

Similarly, we have

$$(g^{2q}+g^{2q+1})(g^{-1}+g^{2q+1})^{2q-1} = (g^{2q}+g^{2q+1})(g^{-1}+g^{2q+1})g^2N_4 = N_1,$$

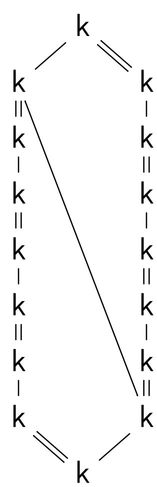
and so $(YX)^{2q} = N_1(1+h)$. Thus

$$\begin{aligned} Y^2 &= N_2(1+h) \\ &= (gN_2+N_1)(1+h) \\ &= X(YX)^{2q-1} + (YX)^{2q}. \end{aligned}$$

We thus have a surjective map from the algebra with the given presentation to $\mathsf{k}SD$. The relations $X^2 = 0$ and $Y^2 = X(YX)^{2q-1} + (YX)^{2q}$ imply that $Y^2X = XY^2 = 0$, and that the element $Y^3 = (YX)^{2q} = (XY)^{2q}$ is killed by X and Y, and is therefore in the socle. Thus the $8q$ alternating words in X and Y, beginning with 1, X, Y, and ending with $(XY)^{2q} = (YX)^{2q}$ span the algebra with the given presentation. The surjective map to $\mathsf{k}SD$ is therefore an isomorphism, and these alternating words form a basis. □

Remark 3.2.2. The reference [**33**] uses a more complicated choice of generators, and gets the same relations, but only modulo the socle element $(XY)^{2q} = (YX)^{2q}$. It is erroneously stated without proof in Section 15 of Benson and Carlson [**17**], and in the papers of Generalov (page 530 of [**95**], page 164 of [**99**], page 279 of [**100**], and page 507 of [**114**]) that the group algebra of the semidihedral group is as given here, but without the extra term $(YX)^{2q}$ in the expression for Y^2. See also Theorem VIII.3 of Erdmann [**74**], where these two possibilities are given, labelled III.1 (d) and III.1 (d′), but without deciding which is true. In Corollary 7.2 of Erdmann [**71**], and the tables at the back of [**74**] the incorrect choice is given. Theorem 3.2.1 shows that the correct answer is III.1 (d′), whereas these sources state it as III.1 (d). It is shown in Proposition 5.1 of Białkowski, Erdmann, Hajduk, Skowroński and Yamagata [**26**] that these two algebras are not isomorphic.

Here is a diagram of the case $q = 2$ (only accurate modulo the extra socle term in the expression for X^2).



This algebra has tame representation type, and its modules were classified by Bondarenko and Drozd [**33**], Crawley-Boevey [**56, 57**]. The cohomology ring was computed first by

Munkholm [**185**] and later also by Evens and Priddy [**80**], and is as follows.

$$H^*B\mathrm{SD} = \mathsf{k}[x, y, z, w]/(xy, y^3, yz, z^2 + x^2w), \tag{3.2.3}$$

with $|x| = |y| = -1$, $|z| = -3$ and $|w| = -4$. Here, x and y are dual to X and Y.

This is not formal as an A_∞ algebra (see Theorem 5.2.1). In the next section we compute a few of the higher multiplications.

REMARK 3.2.4. The subalgebra $\mathcal{A}_1$ of the Steenrod algebra generated by Sq^1 and Sq^2 is closely related to kSD, with presentation

$$\mathsf{k}\langle \mathsf{Sq}^1, \mathsf{Sq}^2 \mid (\mathsf{Sq}^1)^2 = 0, (\mathsf{Sq}^2)^2 = \mathsf{Sq}^1\mathsf{Sq}^2\mathsf{Sq}^1\rangle.$$

This is like a (nonexistent) semidihedral group of order eight, but without the socle element in the second relation. So it has type III.(d) rather than III.(d)′ in Erdmann's classification [**74**]. The cohomology is the same ring as above (3.2.3), but with a different A_∞ structure.

3.3. Resolutions for kSD

In this section, we write out the minimal resolution of k over kSD. Since it is no extra work, we compute the minimal resolution of k over an algebra of type III.I(d) or III.I(d′) in Erdmann's classification [**74**] of algebras of semidihedral type in characteristic two, given by the presentation

$$\Lambda = \mathsf{k}\langle X, Y \mid X^2 = 0,\ Y^2 = X(YX)^{k-1} + \lambda(YX)^k\rangle$$

with $\lambda \in \mathsf{k}$ and $k \geqslant 2$. The case of the group algebra kSD of a semidihedral group of order $8q$ with q a power of two is then recovered by setting $\lambda = 1$ and $k = 2q$.

The minimal resolution of k is the total complex of the following double complex, where we have written v for $\bar{Y}(\bar{X}\bar{Y})^{k-1}$ and w for $(\bar{Y}\bar{X})^{k-1}(1 + \lambda\bar{Y})$.

$$\begin{tikzcd}
 & & & & \Lambda \arrow[d, "\bar{Y}"] & \Lambda \arrow[l, "\bar{Y}"'] \arrow[d, "w"] & \cdots \arrow[l] \\
 & & & & \Lambda \arrow[d, "v"] & \Lambda \arrow[l, "\bar{X}"'] \arrow[d, "v"] & \cdots \arrow[l] \\
 & & \Lambda \arrow[d, "\bar{Y}"] & \Lambda \arrow[l, "\bar{Y}"'] \arrow[d, "w"] & \Lambda \arrow[l, "\bar{Y}\bar{X}"'] \arrow[d, "v"] & \Lambda \arrow[l, "\bar{X}"'] \arrow[d, "v"] & \cdots \arrow[l] \\
 & & \Lambda \arrow[d, "v"] & \Lambda \arrow[l, "\bar{X}"'] \arrow[d, "v"] & \Lambda \arrow[l, "\bar{X}"'] \arrow[d, "v"] & \Lambda \arrow[l, "\bar{X}"'] \arrow[d, "v"] & \cdots \arrow[l] \\
\Lambda \arrow[d, "\bar{Y}"] & \Lambda \arrow[l, "\bar{Y}"'] \arrow[d, "w"] & \Lambda \arrow[l, "\bar{Y}\bar{X}"'] \arrow[d, "v"] & \Lambda \arrow[l, "\bar{X}"'] \arrow[d, "v"] & \Lambda \arrow[l, "\bar{X}"'] \arrow[d, "v"] & \Lambda \arrow[l, "\bar{X}"'] \arrow[d, "v"] & \cdots \arrow[l] \\
\Lambda & \Lambda \arrow[l, "\bar{X}"'] & \Lambda \arrow[l, "\bar{X}"'] & \Lambda \arrow[l, "\bar{X}"'] & \Lambda \arrow[l, "\bar{X}"'] & \Lambda \arrow[l, "\bar{X}"'] & \cdots \arrow[l]
\end{tikzcd}$$

Here $\bar{X}$ and $\bar{Y}$ are the elements of $\mathsf{End}_\Lambda(\Lambda) \cong \Lambda^{op}$ corresponding to X and Y in Λ.

The cohomology element x is represented by left shift composed with

$$\begin{array}{ccccc}
 & & & & \vdots \\
 & & (\bar{X}\bar{Y})^{k-2}\bar{X}(1+\lambda\bar{Y}) & \bar{Y}(1+\lambda\bar{Y}) & \cdots \\
 & & 1 & 1 & \\
(\bar{X}\bar{Y})^{k-2}\bar{X}(1+\lambda\bar{Y}) & \bar{Y}(1+\lambda\bar{Y}) & 1 & 1 & \\
1 & 1 & 1 & 1 & \cdots
\end{array}$$

The element y is represented by a map which is zero on most of the copies of Λ, and non-zero on the upper boundary:

$$\begin{tikzcd}
 & & & & \bullet \arrow[d, "1"] & \cdots \arrow[l, "1"'] \\
 & & & & \bullet \arrow[d, "(\bar{Y}\bar{X})^{k-1}"] & \\
 & & \bullet \arrow[d, "1"] & \bullet \arrow[l, "1"'] & \bullet \arrow[l, "\bar{X}"'] & \\
 & & \bullet \arrow[d, "(\bar{Y}\bar{X})^{k-1}"] & & & \\
\bullet \arrow[d, "1"] & \bullet \arrow[l, "1"'] & \bullet \arrow[l, "\bar{X}"'] & & & \\
\bullet & & & & &
\end{tikzcd}$$

The element z is represented by a shift two to the left and one down, composed with

$$\begin{array}{ccccc}
 & & (\bar{X}\bar{Y})^{k-1} & \bar{Y}+\lambda(\bar{X}\bar{Y})^{k-1}\bar{X} & \cdots \\
 & & 1 & 1 & \\
(\bar{X}\bar{Y})^{k-1} & \bar{Y}+\lambda(\bar{X}\bar{Y})^{k-1}\bar{X} & 1 & 1 & \\
1 & 1 & 1 & 1 & \cdots
\end{array}$$

Finally, the element w is represented by a shift two to the left and two down. This strictly commutes with x, y and z.

In particular, we can read off from the structure and minimal resolution of $\mathsf{k}\mathrm{SD}$ that part of the A_∞ structure on $H^*B\mathrm{SD}$ is given over the central subalgebra $\mathsf{k}[w]$ by

$$m_4(y,x,y,z) = w, \qquad m_{2k-1}(x,y,x,\ldots,y,x) = y^2.$$

3.4. Loops on $B\mathrm{SD}_2^\wedge$

Since SD is a finite 2-group, we have $\Omega B\mathrm{SD}_2^\wedge \simeq \mathrm{SD}$. So we should expect to see the Eilenberg–Moore spectral sequence converging to $\mathsf{k}\mathrm{SD}$.

Theorem 3.4.1. *We have*

$$\mathsf{Ext}^{*,*}_{H^*B\mathrm{SD}}(\mathsf{k},\mathsf{k}) = \Lambda(\hat{w}) \otimes \mathsf{k}\langle \hat{x},\hat{y},\hat{z},\eta \mid \hat{x}^2 = \hat{y}^2 = 0, \hat{x}\hat{z} = \hat{z}\hat{x}, \eta\hat{y} = \hat{y}\eta\rangle$$

where $|\hat{x}| = (-1,1)$, $|\hat{y}| = (-1,1)$, $|\hat{z}| = (-1,3)$, $|\hat{w}| = (-1,4)$, $|\eta| = (-2,3)$, *and* η *is the Massey triple product* $\langle \hat{y},\hat{y},\hat{y}\rangle$. *The Poincaré series is*

$$\sum_{i,j=0}^{\infty} t^i u^j \dim_{\mathsf{k}} \mathsf{Ext}^{i,-j}_{H^*B\mathrm{SD}}(\mathsf{k},\mathsf{k}) = \frac{(1+tu^4)(1+tu)}{1-tu-tu^3-t^2u^3}.$$

Note that $\mathsf{Ext}^{i,-j}$ *is homologically indexed* $(-i,j)$, *so that the coefficient of* t^iu^j *is the dimension of the space of elements of degree* $(-i,j)$.

Proof. The element w is a regular element, and its appearance in the relations is in terms that are at least cubic, so we have an algebra isomorphism

$$\mathsf{Ext}^{*,*}_{H^*B\mathrm{SD}}(\mathsf{k},\mathsf{k}) \cong \Lambda(\hat{w}) \otimes \mathsf{Ext}^{*,*}_{R}(\mathsf{k},\mathsf{k}).$$

where

$$R = H^*B\mathrm{SD}/(w) = \mathsf{k}[x,y,z]/(xy,y^3,yz,z^2).$$

This algebra R is the fibre product of $\mathsf{k}[x,z]/(z^2) \to \mathsf{k}$ and $\mathsf{k}[y]/(y^3) \to \mathsf{k}$. So by Theorem A of Moore [**182**], $\mathsf{Ext}^{*,*}_R(\mathsf{k},\mathsf{k})$ is the coproduct over k of the algebras

$$\mathsf{Ext}^{*,*}_{\mathsf{k}[x,z]/(z^2)}(\mathsf{k},\mathsf{k}) = \mathsf{k}[\hat{x},\hat{z}]/(\hat{x}^2)$$
$$\mathsf{Ext}^{*,*}_{\mathsf{k}[y]/(y^3)}(\mathsf{k},\mathsf{k}) = \mathsf{k}[\hat{y},\eta]/(\hat{y}^2),$$

where η is the Massey triple product $\langle \hat{y},\hat{y},\hat{y}\rangle$. So we have

$$\mathsf{Ext}^{*,*}_R(\mathsf{k},\mathsf{k}) = \mathsf{k}\langle \hat{x},\hat{y},\hat{z},\eta \mid \hat{x}^2 = \hat{y}^2 = 0, \hat{x}\hat{z} = \hat{z}\hat{x}, \eta\hat{y} = \hat{y}\eta\rangle$$

which has Poincaré series

$$\sum_{i,j=0}^{\infty} t^i u^j \dim_{\mathsf{k}} \mathsf{Ext}^{i,-j}_{R}(\mathsf{k},\mathsf{k}) = \frac{1+tu}{1-tu-tu^3-t^2u^3}.$$

Finally, tensoring with $\Lambda(\hat{w})$ multiplies the Poincaré series by $(1+tu^4)$. □

The differentials in the Eilenberg–Moore spectral sequence

$$\mathsf{Ext}^{*,*}_{H^*B\mathrm{SD}}(\mathsf{k},\mathsf{k}) \Rightarrow \mathsf{k}\mathrm{SD}$$

are given by $d^2(\hat{z}) = \eta\hat{x} + \hat{x}\eta$,

$$E^3 = \Lambda(\hat{w}) \otimes \mathsf{k}[\eta] \otimes \mathsf{k}\langle \hat{x},\hat{y} \mid \hat{x}^2 = \hat{y}^2 = 0\rangle,$$

then $d^3(\hat{w}) = \eta^2$,

$$E^4 = E^{4q-2} = \Lambda(\eta) \otimes \mathsf{k}\langle \hat{x},\hat{y} \mid \hat{x}^2 = \hat{y}^2 = 0\rangle,$$

and finally $d^{4q-2}(\eta) = (\hat{x}\hat{y})^{2q} + (\hat{y}\hat{x})^{2q}$. So

$$E^{4q-1} = E^{\infty} = \mathsf{k}\langle \hat{x},\hat{y} \mid \hat{x}^2 = \hat{y}^2 = 0, (\hat{x}\hat{y})^{2q} = (\hat{y}\hat{x})^{2q}\rangle,$$

which is the associated graded of the group algebra $\mathsf{k}\mathrm{SD}$.

3.5. Groups with semidihedral Sylow 2-subgroups

Groups with semidihedral Sylow 2-subgroups were classified by Alperin, Brauer and Gorenstein [**1**], see also Wong [**212**, **213**]. By Section VIII of Brauer [**37**], or Proposition 1.1 of [**1**], there are four possibilities for the 2-fusion in a finite group G with semidihedral Sylow 2-subgroups, which are distinguished by the numbers of conjugacy classes of involutions and of elements of order four. By Theorem 1.1 of Craven and Glesser [**54**], these represent the only possible fusion systems on semidihedral 2-groups.

To describe these, we first describe some particular finite groups with semidihedral Sylow 2-subgroups. First, we describe the groups $SL^{\pm}(2,p^m)$ and $SU^{\pm}(2,p^m)$. These are the subgroups of $GL(2,p^m)$, respectively $GU(2,p^m)$, consisting of elements of determinant ± 1. If $p^m \equiv 3 \pmod 4$ then $SL^{\pm}(2,p^m)$ has semidihedral Sylow 2-subgroups, while if $p^m \equiv 1 \pmod 4$ then $SU^{\pm}(2,p^m)$ has semidihedral Sylow 2-subgroups. We remark that $SL(2,p^m)$ and $SU(2,p^m)$ are isomorphic.

Next, we describe the group denoted $PGL^*(2,p^{2m})$ in Section II.2 of [**1**]. For p odd, the group $P\Gamma L(2,p^{2m})$ is a semidirect product of $PGL(2,p^{2m})$ by a cyclic group of order $2m$ acting as Galois automorphisms. The group $PGL(2,p^{2m})$ has $PSL(2,p^{2m})$ as a normal subgroup of index two. Thus $P\Gamma L(2,p^{2m})$ contains three distinct subgroups, each having $PSL(2,p^{2m})$ as a subgroup of index two. One of these is $PGL(2,p^{2m})$, one is a semidirect product of $PSL(2,p^{2m})$ by the Galois automorphism of order two, and the third one is the group we denote by $PGL^*(2,p^{2m})$. For example, $PGL^*(2,9)$ is isomorphic to the stabiliser of a point in the Mathieu group M_{11}. It is proved in Lemma 2.3 of Gorenstein [**127**] that the Sylow 2-subgroups of $PGL^*(2,p^{2m})$ are semidihedral.

Case 3.5.1. *G has one class of involutions and one class of elements of order four.* In this case, G has no normal subgroup of index two. The group $G/O(G)$ has a simple normal subgroup with odd index, isomorphic to $PSL(3,p^m)$ with $p^m \equiv 3 \pmod 4$, $PSU(3,p^m)$ with $p^m \equiv 1 \pmod 4$, or the Mathieu group M_{11}. The principal block of $\mathsf{k}G$ has three isomorphism classes of simple modules.

Case 3.5.2. *G has two classes of involutions and one class of elements of order four.* In this case, G has a normal subgroup K of index two with generalised quaternion Sylow 2-subgroups, and K has no normal subgroups of index two. The group $G/O(G)$ is either isomorphic to a subroup of $\Gamma L(2,p^m)$ containing $SL^{\pm}(2,p^m)$ with odd index, for some prime power $p^m \equiv 3 \pmod 4$, or it is isomorphic to a subgroup of $\Gamma U(2,p^m)$ containing $SU^{\pm}(2,p^m)$ with odd index, for some prime power $p^m \equiv 1 \pmod 4$. The principal block of $\mathsf{k}G$ has two isomorphism classes of simple modules.

Case 3.5.3. *G has one class of involutions and two classes of elements of order four.* In this case, G a normal subgroup K of index two with dihedral Sylow 2-subgroups, and K has no normal subgroups of index two. The group $G/O(G)$ is isomorphic to a subgroup of $P\Gamma L(2,p^{2m})$ containing $PGL^*(2,p^{2m})$ with odd index, for some odd prime p and positive integer m. The principal block of $\mathsf{k}G$ has two isomorphism classes of simple modules.

Case 3.5.4. *G has two classes of involutions and two classes of elements of order four.* In this case, $O(G)$ is a normal complement to a Sylow 2-subgroup SD, so that $G/O(G) \cong \mathrm{SD}$ and $H^*BG \cong H^*B\mathrm{SD}$. The principal block of $\mathsf{k}G$ is isomorphic to $\mathsf{k}\mathrm{SD}$, and has one isomorphism class of simple modules, namely the trivial module.

Representation theory and cohomology of groups with semidihedral Sylow 2-subgroups, and more generally, blocks with semidihedral defect groups and finite dimensional algebras of semidihedral type, are discussed in Erdmann [**67**, **71**, **73**–**75**, **77**], as well as Benson and Carlson [**17**], Bogdanic [**29**, **30**], Brauer [**37**] (Section VIII), Carlson, Mazza and Thévenaz [**47**], Chin [**50**], Evens and Priddy [**80**], Generalov et al. [**4**, **87**, **89**, **91**–**93**, **95**, **99**–**106**, **114**, **119**–**121**], Hayami [**136**, **137**], Holm [**141**, **142**], Holm and Zimmermann [**145**], Kawai and Sasaki [**153**], Koshitani, Lassueur, and Sambale [**162**], Martino and Priddy [**179**], Müller [**183**], Olsson [**189**], Sasaki [**198**], Taillefer [**210**], Zhou and Zimmermann [**214**]. The homology of $\Omega BG^\wedge_2$ was computed by Levi [**167**].

PROPOSITION 3.5.1. *Suppose that G has a semidihedral Sylow 2-subgroup* SD. *Then the homotopy type of $BG^\wedge_2$ is determined by* $|\mathrm{SD}|$ *and the number of classes of involutions and of elements of order four. In particular, if G has no normal subgroup of index two, then the homotopy type of $BG^\wedge_2$ is determined by* $|\mathrm{SD}|$.

PROOF. This follows from Theorem 1.7.5 and the main theorem of [**1**] described above. □

We end this section with a table of the various cases of algebras of semidihedral type in characteristic two. Note that the definition of semidihedral type in [**74**] is slightly broader than in [**71**, **73**]. In each case except $SD(3\mathcal{K})$, there is a positive integer parameter k, which in our context is equal to $2q$, and in some cases there are also further parameters. In the case of $SD(3\mathcal{K})$ there are three integer parameters $a \geqslant b \geqslant c$, $a \geqslant 2$.

Erdmann [**74**]	[**71**, **73**]	Case	Group	H^*	HH^*
III.I(d)		—	—	[**95**]	[**99**]
III.I(d′)		3.5.4	semidihedral	[**80**, **95**, **185**]	[**114**, **143**]
$SD(2\mathcal{A})_1$	[**71**] II	3.5.3	$SU^\pm(2, p^m)$ $p^m \equiv 1 \pmod 4$	[**50**, **91**]	
$SD(2\mathcal{A})_2$	[**71**] III	3.5.2	$PGL^*(2, p^{2m})$,	[**50**, **91**]	
$SD(2\mathcal{B})_1$	[**71**] IV	3.5.2	— [$B_1(3M_{10})$]	[**4**]	[**120**]
$SD(2\mathcal{B})_2$	[**71**] I	3.5.3	$SL^\pm(2, p^m)$ $p^m \equiv 3 \pmod 4$	[**50**, **102**]	[**103**, **104**, **106**]
$SD(2\mathcal{B})_3$	[**71**] V	3.5.2	—	[**4**]	
$SD(3\mathcal{A})_1$	[**73**] II, §5	3.5.1	$PSU(3, p^m)$, $p^m \equiv 1 \pmod 4$	[**87**]	[**143**]
$SD(3\mathcal{A})_2$	[**73**] VII, §3	—	—	[**91**]	
$SD(3\mathcal{B})_1$	[**73**] IV, §7		—	[**92**]	
$SD(3\mathcal{B})_2$	[**73**] I, §7		—	[**93**]	
$SD(3\mathcal{C})_1$	[**73**] VI, §3	—	—		
$SD(3\mathcal{C})_2$	(excluded)	—	—		
$SD(3\mathcal{D})$	[**73**] III, §6	3.5.1	$PSL(3, p^m)$, $p^m \equiv 3$ (mod 4), M_{11}	[**87**]	[**143**]
$SD(3\mathcal{F})$	[**73**] VIII, §10	—	—		
$SD(3\mathcal{H})$	[**73**] IX, §10		—		
$SD(3\mathcal{K})$	[**73**] V, §9	—	—	[**89**]	[**121**]

Remarks 3.5.2. The types with three simple modules are all derived equivalent to an algebra in the family $SD(3\mathcal{K})$ with uniquely determined values of $a \geqslant b \geqslant c$, by Theorem 4.8 of Holm [**142**]. For blocks with semidihedral defect group of order $8q$ and three simple modules, these parameters are $2q \geqslant 2 \geqslant 1$, so they are all derived equivalent.

Note that by Rickard [**192**,**193**], for self-injective algebras, a derived equivalence induces a stable equivalence of Morita type. By a theorem of Happel (see for example Proposition 2.21.9 of Linckelmann [**176**]), for symmetric algebras, derived equivalence also induces an isomorphism in Hochschild cohomology.

Unfortunately, there are are some copying errors in [**73**, **74**], and an incorrect correction in [**53**]. It is erroneously reported in statement (11.15) (c) of [**73**] (incorrectly labelled (11.5) (c)) that the principal block of M_{11} belongs to family IV. In Table 1 of [**73**], for family IV, P_2 should be "as in I" and not "as in III"; the conditions for it to be a block should be $t = 1$ and $k = 2^{n-2}$, not the other way round. In the tables at the back of [**74**], the principal blocks of $PSL(3, p^m)$ with $p^m \equiv 3 \pmod 4$ are incorrectly assigned to $SD(3\mathcal{B})_1$ rather than $SD(3\mathcal{D})$. In case $SD(3\mathcal{K})$, the parameters should be $a \geqslant b \geqslant c \geqslant 1$, $a \geqslant 2$ rather than $a \geqslant b \geqslant c \geqslant 2$. On pages 143–144 of [**53**], the correction there incorrectly states that both M_{11} and $PSL(3, p^m)$ with $p^m \equiv 3 \pmod 4$ belong to family $SD(3\mathcal{B})_1$, and that there is only one simple module with a non-trivial self-extension; in fact, the family is $SD(3\mathcal{D})$, and there are two such simple modules.

3.6. One class of involutions, one of order four

We begin with Case 3.5.1, where G has one class of involutions, and one class of elements of order four. In this case, G has no normal subgroup of index two, and Proposition 2.2 of [**1**] implies that $G/O(G)$ contains a simple normal subgroup with odd index. By the main theorem of that paper, the simple groups with semidihedral Sylow 2-subgroups are as follows.

(a) The projective special linear groups $PSL(3, p^m)$ with $p^m \equiv 3 \pmod 4$.

(b) The projective special unitary groups $PSU(3, p^m)$ with $p^m \equiv 1 \pmod 4$.

(c) The sporadic Mathieu group M_{11} of order 7920.

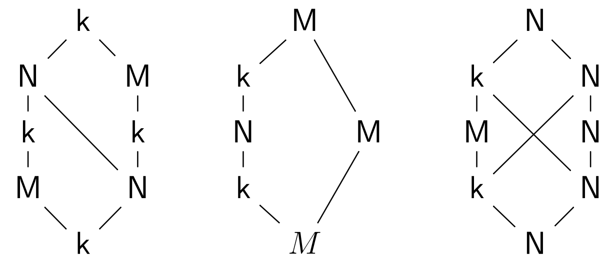
Let G be a finite group with semidihedral Sylow 2-subgroups of order $8q$ and no normal subgroups of index two, and let k be a field of characteristic two. Let B be the principal block of $\mathsf{k}G$. The structure of the projective indecomposable B-modules was determined by Erdmann [**67**].

Remark 3.6.1. The one case not treated in [**67**] is $G = M_{11}$, which was treated in the thesis of Schneider [**199**], and also in unpublished work of Alperin.

The principal blocks of M_{11} and $PSL(3, p^m)$ with $p^m \equiv 3 \pmod 4$ are in family III of [**73**], which is $SD(3\mathcal{D})$ of [**74**]. The principal blocks $PSU(3, p^m)$ with $p^m \equiv 1 \pmod 4$ are in family II of [**73**], which is $SD(3\mathcal{A})_1$ of [**74**].

Remark 3.6.2. Fortunately, Proposition 3.5.1 allows us to do the analysis for just one group for each size $8q$ of semidihedral Sylow 2-subgroup. We choose to examine $PSL(3, p^m)$, where the 2-part of $p^m + 1$ is $2q$.

Let us look first at the cases of $PSL(3,3)$ and M_{11}, whose principal blocks are Morita equivalent. There are three isomorphism classes of simple B-modules, all self-dual, denoted k, M and N. These have dimensions 1, 12 and 26 in the case of $PSL(3,3)$, and dimensions 1, 44 and 10 in the case of M_{11}. Their projective covers are given by the following diagrams.



Note that N is periodic with period four, while k and M are not periodic. The quiver for B is

(3.6.3)
$$\begin{tikzcd} \mathsf{N} \arrow[loop left, "d"] \arrow[r, bend left, "c"] & \mathsf{k} \arrow[l, bend left, "a"] \arrow[r, bend left, "b"] & \mathsf{M} \arrow[l, bend left, "e"] \arrow[loop right, "f"] \end{tikzcd}$$

with relations

$$ef = 0, \qquad be = 0, \qquad fb = 0, \qquad da = aeb, \qquad cd = ebc, \qquad f^2 = bcae, \qquad ac = d^3.$$

This gives a presentation for the basic algebra of B. This corresponds to the case discussed in Theorem VIII.9.12 (with $k = 1$, $s = 4$, $t = 2$) and Proposition IX.6.6 (ii) (with $n = 4$) of Erdmann [**74**],

The unique self-dual grading (up to scalar multiples) on this quiver algebra is given by

$$|a| = |c| = \tfrac{3}{2}, \quad |b| = |e| = \tfrac{1}{2}, \quad |d| = 1, \quad |f| = 2.$$

We choose not to double these degrees, as the choice above makes the degrees in $H^*BG \cong \mathsf{Ext}^*_B(\mathsf{k},\mathsf{k})$ into integers with no common factor.

The principal blocks of the simple groups $PSL(3,p^m)$ with $p^m \equiv 3 \pmod 4$ are very similar, see Erdmann [**67**]. The only difference is that if the 2-part of $p^m + 1$ is $4q$ (with q a power of two) then there are more repetitions of the simple module N in its projective cover. The case treated above is $q = 2$, and in the general case there are $2q-1$ copies of N in the unserial module on the right hand side of the diagram instead of three. So the relation $ac = d^3$ is replaced by $ac = d^{2q-1}$. The Morita type of the principal block only depends on q, and not on p^m. So for example, the principal blocks of M_{11}, $PSL(3,3)$, $PSL(3,11)$ and $PSL(3,19)$ are Morita equivalent, with Sylow 2-subgroups of order 16, and the principal blocks of $PSL(3,7)$ and $PSL(3,23)$ and $PSL(3,71)$ are all Morita equivalent, with Sylow 2-subgroups of order 32. The grading also needs to be adjusted, as follows.

THEOREM 3.6.4. *Let $G = PSL(3,p^m)$, where the 2-part of $p^m + 1$ is $2q$ $(q \geqslant 2)$, or $G = M_{11}$ with $q = 2$, and let k be a field of characteristic two. Then the basic algebra of the principal block is given by the quiver* (3.6.3), *with relations*

$$ef = 0, \qquad be = 0, \qquad fb = 0, \qquad da = aeb, \qquad cd = ebc, \qquad f^2 = bcae, \qquad ac = d^{2q-1}.$$

The unique self dual grading, up to scalar multiples, on this algebra is given by

$$|a| = |c| = q - \tfrac{1}{2}, \qquad |b| = |e| = \tfrac{1}{2}, \qquad |d| = 1, \qquad |f| = q.$$

Proof. The quiver with relations follows from the work of Erdmann [**67**]. Given the relations, the uniqueness of the grading up to scalars is easy linear algebra. Self duality just means that $|a| = |c|$ and $|b| = |e|$. $\square$

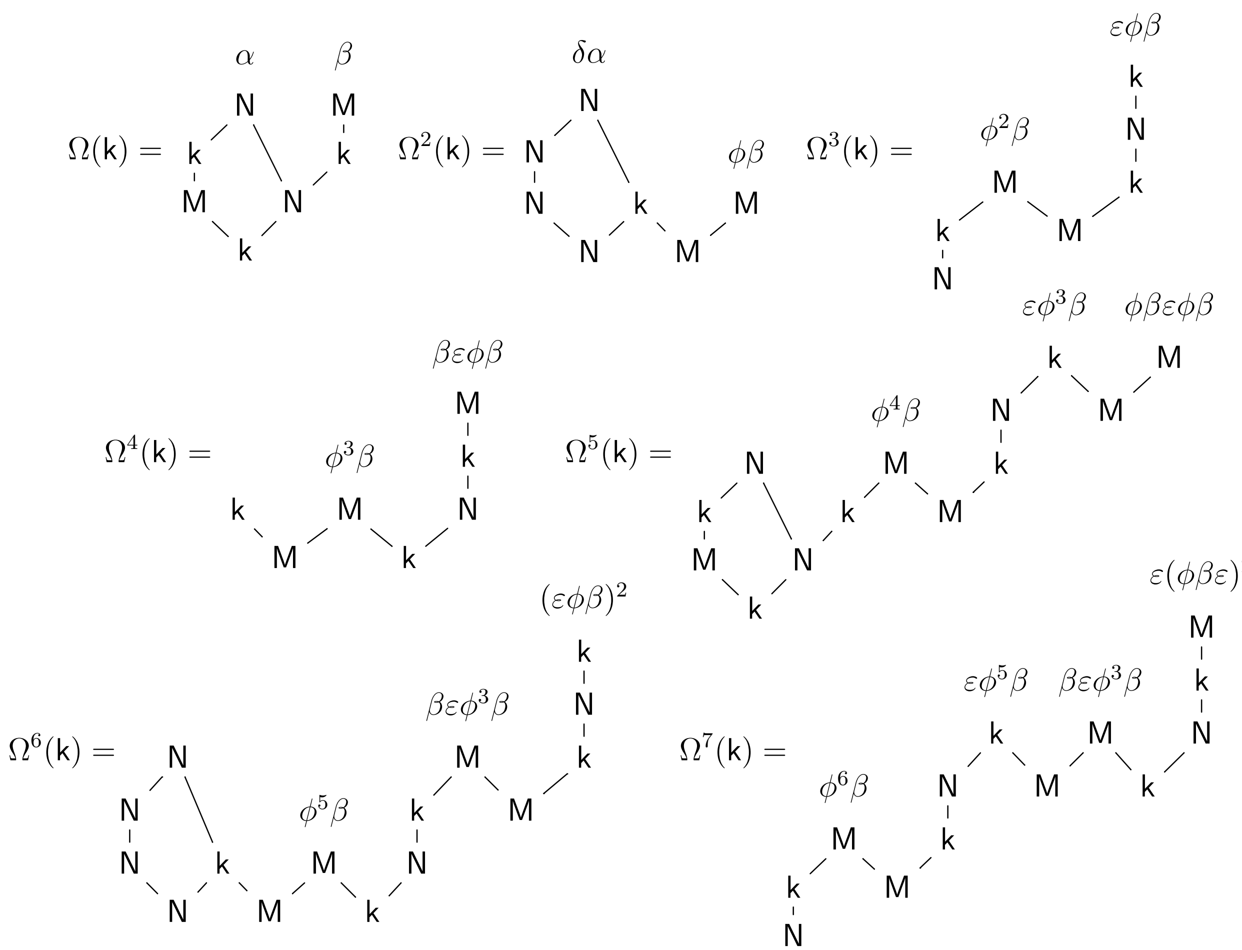
Remark 3.6.5. The given relations imply that

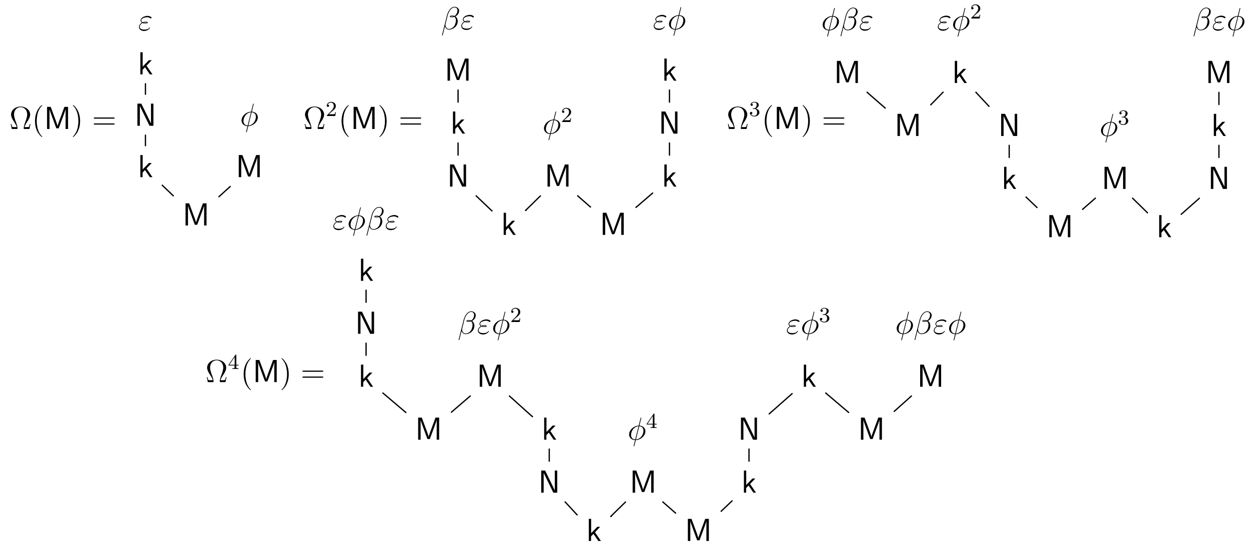
$$\begin{aligned} f^3 &= bcaef = 0, \\ cac &= cd^{2q-1} = ebcd^{2q-2} = ebebcd^{2q-3} = 0, \\ aca &= d^{2q-1}a = d^{2q-2}aeb = d^{2q-3}aebeb = 0, \\ d^{2q+1} &= acd^2 = aebcd = aebebc = 0. \end{aligned}$$

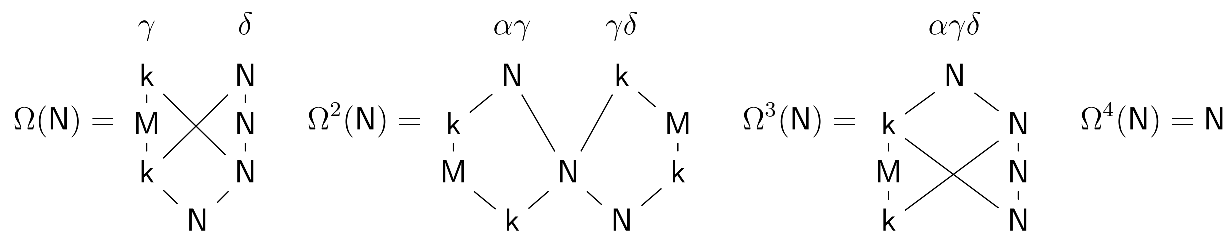
Let α, β, γ, δ, ε, ϕ in $\mathsf{Ext}^1_B(\mathsf{k} \oplus \mathsf{M} \oplus \mathsf{N}, \mathsf{k} \oplus \mathsf{M} \oplus \mathsf{N})$ be the elements dual to a, b, c, d, e, f. These have degrees

$$|\alpha| = |\gamma| = (-1, -q + \tfrac{1}{2}), \qquad |\beta| = |\varepsilon| = (-1, -\tfrac{1}{2}), \qquad |\delta| = (-1, -1), \qquad |\phi| = (-1, -q),$$

We can compute minimal resolutions of the simple modules as in [**17**], and the result is as follows when $q = 2$. For larger values of q, the only difference is that the chains of copies of N in the resolutions of k and N are longer.







For all values of q, the minimal resolution of k takes the form

$$\cdots \to P_{\mathsf{M}} \oplus P_{\mathsf{k}} \oplus P_{\mathsf{M}} \oplus P_{\mathsf{k}} \xrightarrow{\begin{pmatrix} \bar c\bar b & 0 & 0 & 0 \\ \bar f & \bar e & 0 & 0 \\ 0 & \bar e\bar a\bar c & \bar f & 0 \\ 0 & 0 & \bar a\bar c\bar b & \bar b \end{pmatrix}} P_{\mathsf{N}} \oplus P_{\mathsf{M}} \oplus P_{\mathsf{M}} \oplus P_{\mathsf{k}} \xrightarrow{\begin{pmatrix} \bar d & 0 & 0 & 0 \\ \bar e\bar a & \bar f & 0 & 0 \\ 0 & \bar a\bar c\bar b & \bar b & 0 \\ 0 & 0 & \bar f & \bar e \end{pmatrix}} P_{\mathsf{N}} \oplus P_{\mathsf{M}} \oplus P_{\mathsf{k}} \oplus P_{\mathsf{M}}$$

$$\xrightarrow{\begin{pmatrix} \bar a & \bar b & 0 & 0 \\ 0 & \bar f & \bar e & 0 \\ 0 & 0 & \bar e\bar a\bar c & \bar f \end{pmatrix}} P_{\mathsf{k}} \oplus P_{\mathsf{M}} \oplus P_{\mathsf{M}} \xrightarrow{\begin{pmatrix} \bar e\bar a\bar c & \bar f & 0 \\ 0 & \bar a\bar c\bar b & \bar b \end{pmatrix}} P_{\mathsf{M}} \oplus P_{\mathsf{k}} \xrightarrow{\begin{pmatrix} \bar c\bar b & 0 \\ \bar f & \bar e \end{pmatrix}} P_{\mathsf{N}} \oplus P_{\mathsf{M}} \xrightarrow{\begin{pmatrix} \bar d & 0 \\ \bar e\bar a & \bar f \end{pmatrix}} P_{\mathsf{N}} \oplus P_{\mathsf{M}} \xrightarrow{(\bar a \ \bar b)} P_{\mathsf{k}}$$

This is the total complex of the following double complex:

(3.6.6)
$$\begin{tikzcd}
 & & & & P_{\mathsf{N}} \arrow[d, "\bar a"] & P_{\mathsf{N}} \arrow[l, "\bar d"'] \arrow[d, "\bar e\bar a"] & \\
 & & & & P_{\mathsf{k}} \arrow[d, "\bar e\bar a\bar c"] & P_{\mathsf{M}} \arrow[l, "\bar b"'] \arrow[d, "\bar f"] & \cdots \\
 & & P_{\mathsf{N}} \arrow[d, "\bar a"] & P_{\mathsf{N}} \arrow[l, "\bar d"'] \arrow[d, "\bar e\bar a"] & P_{\mathsf{M}} \arrow[l, "\bar c\bar b"'] \arrow[d, "\bar f"] & P_{\mathsf{M}} \arrow[l, "\bar f"'] \arrow[d, "\bar a\bar c\bar b"] & \\
 & & P_{\mathsf{k}} \arrow[d, "\bar e\bar a\bar c"] & P_{\mathsf{M}} \arrow[l, "\bar b"'] \arrow[d, "\bar f"] & P_{\mathsf{M}} \arrow[l, "\bar f"'] \arrow[d, "\bar a\bar c\bar b"] & P_{\mathsf{k}} \arrow[l, "\bar e"'] \arrow[d, "\bar e\bar a\bar c"] & \cdots \\
P_{\mathsf{N}} \arrow[d, "\bar a"] & P_{\mathsf{N}} \arrow[l, "\bar d"'] \arrow[d, "\bar e\bar a"] & P_{\mathsf{M}} \arrow[l, "\bar c\bar b"'] \arrow[d, "\bar f"] & P_{\mathsf{M}} \arrow[l, "\bar f"'] \arrow[d, "\bar a\bar c\bar b"] & P_{\mathsf{k}} \arrow[l, "\bar e"'] \arrow[d, "\bar e\bar a\bar c"] & P_{\mathsf{M}} \arrow[l, "\bar b"'] \arrow[d, "\bar f"] & \\
P_{\mathsf{k}} & P_{\mathsf{M}} \arrow[l, "\bar b"'] & P_{\mathsf{M}} \arrow[l, "\bar f"'] & P_{\mathsf{k}} \arrow[l, "\bar e"'] & P_{\mathsf{M}} \arrow[l, "\bar b"'] & P_{\mathsf{M}} \arrow[l, "\bar f"'] & \cdots
\end{tikzcd}$$

Here, $\bar{a}$ is the element of $\mathsf{Hom}_B(P_{\mathsf{N}}, P_{\mathsf{k}})$ opposite to a, and so on, so that the barred variables satisfy the reverse of the relations in the quiver.

The extensions $\alpha, \ldots, \phi$ satisfy the following relations, which are easy to verify using the grading and the minimal resolutions above:

$$\begin{gathered}
\alpha\varepsilon = 0, \qquad \beta\gamma = 0, \qquad \gamma\alpha = 0, \qquad \delta^2 = 0, \qquad \varepsilon\beta = 0, \\
\beta\varepsilon\phi^2 = \phi^2\beta\varepsilon, \qquad \varepsilon\phi^2\beta = 0, \qquad \alpha\gamma\delta = \delta\alpha\gamma, \qquad \gamma\delta\alpha = 0, \\
m_3(\alpha,\varepsilon,\beta) = \delta\alpha, \qquad m_3(\beta,\gamma,\alpha) = 0, \qquad m_3(\gamma,\alpha,\varepsilon) = 0, \\
m_{2q-1}(\delta,\ldots,\delta) = \alpha\gamma, \qquad m_3(\varepsilon,\beta,\gamma) = \gamma\delta, \qquad m_3(\gamma,\delta\alpha,\varepsilon) = \varepsilon\phi^2, \\
m_3(\beta,\gamma\delta,\alpha) = \phi^2\beta, \qquad m_4(\beta,\gamma,\alpha,\varepsilon) = \phi^2, \\
m_4(\gamma,\alpha,\varepsilon,\phi^2\beta) = m_4(\varepsilon\phi^2,\beta,\gamma,\alpha).
\end{gathered} \tag{3.6.7}$$

REMARK 3.6.8. The relation $\gamma\delta\alpha = 0$ follows from the remaining relations in two ways:

$$\begin{aligned}
\gamma\delta\alpha &= \gamma m_3(\alpha,\varepsilon,\beta) = m_3(\gamma,\alpha,\varepsilon)\beta = 0, \\
\gamma\delta\alpha &= m_3(\varepsilon,\beta,\gamma)\alpha = \varepsilon m_3(\beta,\gamma,\alpha) = 0.
\end{aligned}$$

The last relation describes the unlabelled copy of k at the top of the left end of $\Omega^4(\mathsf{k})$. When postmultiplied by ε or premultiplied by β, this relation follows from the remaining relations:

$$\begin{aligned}
m_4(\gamma,\alpha,\varepsilon,\phi^2\beta)\varepsilon &= m_4(\gamma,\alpha,\varepsilon,\phi^2\beta\varepsilon) = m_4(\gamma,\alpha,\varepsilon,\beta\varepsilon\phi^2) = m_3(\gamma,m_3(\alpha,\varepsilon,\beta),\varepsilon\phi^2) \\
&= m_3(\gamma,\delta\alpha,\varepsilon\phi^2) = m_3(\gamma,\delta\alpha,\varepsilon)\phi^2 = \varepsilon\phi^4 \\
&= \varepsilon\phi^2 m_4(\beta,\gamma,\alpha,\varepsilon) = m_4(\varepsilon\phi^2,\beta,\gamma,\alpha)\varepsilon, \\
\beta m_4(\gamma,\alpha,\varepsilon,\phi^2\beta) &= m_4(\beta,\gamma,\alpha,\varepsilon)\phi^2\beta = \phi^4\beta = \phi^2 m_3(\beta,\gamma\delta,\alpha) \\
&= m_3(\phi^2\beta,\gamma\delta,\alpha) = m_3(\phi^2\beta,m_3(\varepsilon,\beta,\gamma),\alpha) = m_4(\phi^2\beta\varepsilon,\beta,\gamma,\alpha) \\
&= m_4(\beta\varepsilon\phi^2,\beta,\gamma,\alpha) = \beta m_4(\varepsilon\phi^2,\beta,\gamma,\alpha).
\end{aligned}$$

THEOREM 3.6.9. *Let $G = PSL(3,p^m)$, where the 2-part of p^m+1 is $2q$ ($q \geqslant 2$), or $G = M_{11}$ with $q = 2$. The cohomology ring $H^*BG = \mathsf{Ext}^*_B(\mathsf{k},\mathsf{k})$ is generated by the commuting elements*

$$x = \varepsilon\phi\beta, \qquad y = m_4(\gamma,\alpha,\varepsilon,\phi^2\beta) = m_4(\varepsilon\phi^2,\beta,\gamma,\alpha), \qquad z = \varepsilon\phi^3\beta,$$

subject to one relation:

$$H^*BG = \mathsf{Ext}^*_B(\mathsf{k},\mathsf{k}) = \mathsf{k}[x,y,z]/(x^2y + z^2)$$

where $|x| = (-3,-q-1)$, $|y| = (-4,-4q)$ and $|z| = (-5,-3q-1)$.

PROOF. The structure of the cohomology ring of M_{11} was computed in [**17**], and is as above, if we ignore the internal degrees. The principal blocks of $PSL(3,p^m)$ with $p^m \equiv 3 \pmod 8$ are Morita equivalent to that of M_{11}, and therefore give the same answer. The analogous computation with possibly larger values of q gives exactly the same answer for $PSL(3,p^m)$ with $p^m \equiv 3 \pmod 4$. We show that the given elements satisfy these relations, using the relations (3.6.7). We begin by observing (as in Remark 3.6.8) that

$$\begin{aligned}
\beta y &= \beta m_4(\gamma,\alpha,\varepsilon,\phi^2\beta) = m_4(\beta,\gamma,\alpha,\varepsilon)\phi^2\beta = \phi^4\beta, \\
y\varepsilon &= m_4(\varepsilon\phi^2,\beta,\gamma,\alpha)\varepsilon = \varepsilon\phi^2 m_4(\beta,\gamma,\alpha,\varepsilon) = \varepsilon\phi^4,
\end{aligned}$$

and so

$$x^2y = (\varepsilon\phi\beta\varepsilon\phi)(\beta y) = (\varepsilon\phi\beta\varepsilon\phi)(\phi^4\beta) = \varepsilon\phi(\beta\varepsilon\phi^2)\phi^3\beta = \varepsilon\phi(\phi^2\beta\varepsilon)\phi^3\beta = (\varepsilon\phi^3\beta)(\varepsilon\phi^3\beta) = z^2.$$

Commutativity is automatic for elements of H^*BG, but also follows from the relations above:

$$\begin{aligned} yx &= (y\varepsilon)(\phi\beta) = (\varepsilon\phi^4)(\phi\beta) = (\varepsilon\phi)(\phi^4\beta) = (\varepsilon\phi)(\beta y) = xy \\ zx &= (\varepsilon\phi^3\beta)(\varepsilon\phi\beta) = (\varepsilon\phi)(\phi^2\beta\varepsilon)(\phi\beta) = (\varepsilon\phi)(\beta\varepsilon\phi^2)(\phi\beta) = (\varepsilon\phi\beta)(\varepsilon\phi^3\beta) = xz, \\ zy &= (\varepsilon\phi^3)(\beta y) = (\varepsilon\phi^3)(\phi^4\beta) = (\varepsilon\phi^4)(\phi^3\beta) = (y\varepsilon)(\phi^3\beta) = yz. \end{aligned}$$

□

REMARK 3.6.10. Since the homotopy type of $BG_2^\wedge$ only depends on the Sylow 2-subgroup and the fusion, the cohomology ring is the same for $G = PSU(3, p^m)$ where the 2-part of $p^m - 1$ is $2q$ ($q \geqslant 2$).

It would be possible, but not necessary for the currrent purposes, to do a similar analysis for $PSU(3, p^m)$ to that contained in this section. The quiver in that case is as follows

$$\begin{tikzcd} \mathsf{N} \arrow[r, bend left, "c"] & \mathsf{k} \arrow[l, bend left, "a"] \arrow[r, bend left, "b"] & \mathsf{M} \arrow[l, bend left, "e"] \end{tikzcd}$$

with relations

$$be = 0, \qquad aca = a(ebca)^{2q-1}eb, \qquad cac = (ebca)^{2q-1}ebc, \qquad acaca = 0, \qquad cacac = 0.$$

This admits a self-dual grading given by $|a| = |c| = q$, $|b| = |e| = 1 - q$. The problem here, though, is that the method of [**17**] for computing with projective resolutions doesn't really apply, and this makes the details of the computations quite tedious.

3.7. Ext and Hochschild cohomology over H^*BG

Throughout this section, we are still working in Case 3.5.1. So we let G be a finite group with a semidihedral Sylow 2-subgroup of order $8q$ and no normal subgroup of index two, and k a field of characteristic two. Our next task is to compute $\mathsf{Ext}^{**}_{H^*BG}(\mathsf{k}, \mathsf{k})$ and HH^*H^*BG by applying Theorems 1.11.2 and 1.11.5. Recall that by Theorem 3.6.9 and Remark 3.6.10 we have $H^*BG = \mathsf{k}[x, y, z]/(x^2y + z^2)$ with $|x| = (-3, -q-1)$, $|y| = (-4, -4q)$ and $|z| = (-5, -3q-1)$. Let $f = x^2y + z^2 \in \mathsf{k}[x, y, z]$. Then we have

$$\begin{aligned} &\frac{\partial f}{\partial x} = 0, && \frac{\partial f}{\partial y} = x^2, && \frac{\partial f}{\partial z} = 0, \\ &\frac{\partial^{(2)} f}{\partial x^2} = y, && \frac{\partial^{(2)} f}{\partial y^2} = 0, && \frac{\partial^{(2)} f}{\partial z^2} = 1, \\ &\frac{\partial^2 f}{\partial x \partial y} = 0, && \frac{\partial^2 f}{\partial x \partial z} = 0, && \frac{\partial^2 f}{\partial y \partial z} = 0. \end{aligned}$$

Plugging these into Definition 1.11.1, for the algebra $\mathsf{Cliff}(\mathsf{q})$ we have variables $\hat{x}$, $\hat{y}$, $\hat{z}$ dual to x, y and z and s dual to f. These have degrees $|\hat{x}| = (-1, 3, q+1)$, $|\hat{y}| = (-1, 4, 4q)$, $|\hat{z}| = (-1, 5, 3q+1)$, $|s| = (-2, 10, 6q+2)$. Here, the first is the Ext degree, the second comes from the homological degree in H^*BG, and the third is the internal degree coming from the grading on the algebra B. So the degrees of the generators of H^*BG come out as $|x| = (0, -3, -q-1)$, $|y| = (0, -4, -4q)$ and $|z| = (0, -5, -3q-1)$. Then s is central, and we have relations $\hat{x}^2 = ys$, $\hat{y}^2 = 0$, $\hat{z}^2 = s$, $\hat{x}\hat{y} + \hat{y}\hat{x} = 0$, $\hat{x}\hat{z} + \hat{z}\hat{x} = 0$, $\hat{y}\hat{z} + \hat{z}\hat{y} = 0$. The relation $\hat{z}^2 = s$ makes s a redundant generator, and we end up with

$$\mathsf{Cliff}(\mathsf{q}) = H^*BG[\hat{x}, \hat{y}, \hat{z}]/(\hat{x}^2 + y\hat{z}^2, \hat{y}^2). \tag{3.7.1}$$

The differential is given by

$$d\hat{x} = 0, \qquad d\hat{y} = x^2\hat{z}^2, \qquad d\hat{z} = 0. \tag{3.7.2}$$

THEOREM 3.7.3. *We have*

$$\mathsf{Ext}^{**}_{H^*BG}(\mathsf{k}, \mathsf{k}) = \Lambda(\hat{x}, \hat{y}) \otimes \mathsf{k}[\hat{z}].$$

with degrees given by $|\hat{x}| = (-1, 3, q + 1)$, $|\hat{y}| = (-1, 4, 4q)$ *and* $|\hat{z}| = (-1, 5, 3q + 1)$.

PROOF. This follows from Theorem 1.11.2 and the computation (3.7.1) of $\mathsf{Cliff}(\mathsf{q})$. □

THEOREM 3.7.4. *We have*

$$H_*\Omega BG_2^\wedge = \Lambda(\hat{x}, \hat{y}) \otimes \mathsf{k}[\hat{z}]$$

with $|\hat{x}| = (2, q + 1)$, $|\hat{y}| = (3, 4q)$ *and* $|\hat{z}| = (4, 3q + 1)$.

PROOF. Theorem 3.7.3 gives the E_2 page of the spectral sequence

$$\mathsf{Ext}^{**}_{H^*BG}(\mathsf{k}, \mathsf{k}) \Rightarrow H_*\Omega BG_2^\wedge.$$

There is no room for differentials, and there are no ungrading problems. □

REMARK 3.7.5. This agrees with the answer given in Proposition II.4.2.6 of Levi [**167**].

REMARK 3.7.6. When we compute the spectral sequence

$$\mathsf{Ext}^{**}_{H_*\Omega BG_2^\wedge}(k, k) \Rightarrow H^*BG$$

we get $E_2 = \mathsf{k}[x, y] \otimes \Lambda(z)$ with $|x| = (-1, -2, -q - 1)$, $|y| = (-1, -3, -4q)$ and $|z| = (-1, -4, -3q - 1)$. There are no differentials, but the relation $z^2 = 0$ then ungrades to give $z^2 = x^2y$.

THEOREM 3.7.7. *The Hochschild cohomology ring of* H^*BG *is given by*

$$HH^*H^*BG = H^*BG[\hat{x}, \hat{z}]/(\hat{x}^2 + y\hat{z}^2, x^2\hat{z}^2),$$

with

$$|x| = (0, -3, -q - 1), \qquad |y| = (0, -4, -4q), \qquad |z| = (0, -5, -3q - 1),$$
$$|\hat{x}| = (-1, 3, q + 1), \qquad |\hat{z}| = (-1, 5, 3q + 1).$$

PROOF. This follows from (3.7.1) and (3.7.2), using Theorem 1.11.5. □

PROPOSITION 3.7.8. *There are no non-zero elements of degree* $(-n, n-2, 0)$ *in the Hochschild cohomology* HH^*H^*BG *with* $n > 2$.

PROOF. By Theorem 3.7.4, we have a k-basis for HH^*H^*BG consisting of the monomials $x^{i_1}y^{i_2}z^{\varepsilon_3}\hat{x}^{\varepsilon_1}\hat{z}^{i_3}$ with either $i_1 \leqslant 1$ or $i_3 \leqslant 1$. Suppose that such a monomial has degree $(-n, n - 2, 0)$. Comparing degrees, we have

$$-n = -\varepsilon_1 - i_3 \tag{3.7.9}$$

$$n - 2 = -3i_1 - 4i_2 - 5\varepsilon_3 + 3\varepsilon_1 + 5i_3, \tag{3.7.10}$$

$$0 = -(q+1)i_1 - 4qi_2 - (3q+1)\varepsilon_3 + (q+1)\varepsilon_1 + (3q+1)i_3. \tag{3.7.11}$$

We shall show that there are no solutions in non-negative integers with $n > 2$.

First we deal with the case $q = 2$. In this case, equation (3.7.11) becomes

$$0 = -3i_1 - 8i_2 - 7\varepsilon_3 + 3\varepsilon_1 + 7i_3. \tag{3.7.12}$$

Adding equations (3.7.10) and (3.7.12), we get

$$n - 2 = -6i_1 - 12i_2 - 12\varepsilon_3 + 6\varepsilon_1 + 12i_3, \tag{3.7.13}$$

and so

$$n \equiv 2 \pmod 6.$$

If instead, we add equations (3.7.9) and (3.7.10) and subtract equation (3.7.12), we get $-2 = 4i_2 + 2\varepsilon_3 - \varepsilon_1 - 3i_3$, or

$$4i_2 + 2\varepsilon_3 = \varepsilon_1 + 3i_3 - 2. \tag{3.7.14}$$

So ε_1 and i_3 determine i_2 and ε_3, and then i_1. Let $n = 6a + 2$, so that equation (3.7.13) becomes

$$a = -i_1 - 2i_2 + \varepsilon_1 + 2i_3 \geqslant 1. \tag{3.7.15}$$

From equation (3.7.9), we have $i_3 = 6a + 2 - \varepsilon_1$. Then equation (3.7.14) gives $4i_2 + 2\varepsilon_3 = \varepsilon_1 + 18a + 6 - 3\varepsilon_1 - 2$, so

$$2i_2 = 9a + 2 - \varepsilon_1 - \varepsilon_3.$$

Finally, plugging these values of i_2 and i_3 into equation (3.7.15) gives

$$\begin{aligned} i_1 &= a - 2i_2 - 2\varepsilon_3 + \varepsilon_1 + 2i_3 \\ &= a - 9a - 2 + \varepsilon_1 + \varepsilon_3 - 2\varepsilon_3 + \varepsilon_1 + 12a + 4 - 2\varepsilon_1 \\ &= 4a + 2 - \varepsilon_3. \end{aligned}$$

Since $a \geqslant 1$, we see that both i_1 and i_3 are greater than one, which is a contradiction. This completes the case $q = 2$.

Now suppose that $q > 2$. Reading equations (3.7.9), (3.7.10), and (3.7.11) modulo four, we see that $\varepsilon_3 + i_1$ and $n = \varepsilon_1 + i_3$ are both even, and are congruent modulo four. So if $n > 2$ then $n \geqslant 4$, $i_3 \geqslant 3$, and hence $i_1 \leqslant 1$. So either $i_1 = \varepsilon_3 = 0$ or $i_1 = \varepsilon_3 = 1$. Adding equations (3.7.9) and (3.7.10), we get

$$-2 = -3i_1 - 4i_2 - 5\varepsilon_3 + 2\varepsilon_1 + 4i_3.$$

Since $-3i_1 - 5\varepsilon_3$ is divisible by four, we deduce that $\varepsilon_1 = 1$.

In the case $i_1 = \varepsilon_3 = 0$, $\varepsilon_1 = 1$ we get $i_3 = i_2 - 1$, $n = i_2$. Equation (3.7.11) becomes

$$\begin{aligned} 0 &= -4qi_2 + (q+1) + (3q+1)(i_2 - 1) \\ &= (-q+1)i_2 - 2q \end{aligned}$$

so i_2 is not an integer, which is a contradiction.

In the case $i_1 = \varepsilon_3 = 1$, $\varepsilon_1 = 1$ we get $i_3 = i_2 + 1$, $n = i_2 + 2$. Equation (3.7.11) becomes

$$\begin{aligned} 0 &= -(q+1) - 4qi_2 - (3q+1) + (q+1) + (3q+1)(i_2 + 1) \\ &= (-q+1)i_2 \end{aligned}$$

and so $i_2 = 0$, $n = 2$, again a contradiction. So for $q > 2$ there are no monomials of this form. □

Theorem 3.7.16. *In Case 3.5.1, with the grading inherited from the internal grading on the basic algebra of* $\mathsf{k}G$*, the* A_∞ *structure of* H^*BG *is intrinsically formal.*

Proof. This follows from Propositions 1.4.2 and 3.7.8. □

Remark 3.7.17. Another proof of formality, but which does not give intrinsic formality, in Theorem 3.7.16 is to notice that there are endomorphisms of the resolution (3.6.6) representing x, y and z, and strictly satisfying the relation $x^2y = z^2$. The endomorphism representing y just moves the whole diagram two places down and two places to the left. For x, we move three places to the left, but then we have to compose with the maps

$$\begin{array}{ccccccc}
 & & & & \bar{c}\bar{b} & \bar{c} & \\
 & & & & 1 & 1 & \cdots \\
 & & \bar{c}\bar{b} & \bar{c} & 1 & 1 & \\
 & & 1 & 1 & 1 & 1 & \cdots \\
\bar{c}\bar{b} & \bar{c} & 1 & 1 & 1 & 1 & \\
1 & 1 & 1 & 1 & 1 & 1 & \cdots
\end{array}$$

Similarly, for z we move one place down and four to the left, and compose with the same maps. This defines a quasi-isomorphism from the cohomology ring $\mathsf{Ext}^*_{\mathsf{k}G}(\mathsf{k},\mathsf{k})$ to the DG algebra $\mathsf{End}^*_{\mathsf{k}G}(P_\mathsf{k})$, which in turn is quasi-isomorphic to C^*BG.

Corollary 3.7.18. *In Case 3.5.1, we have*

$$HH^*H^*BG \cong HH^*C^*BG \cong HH^*C_*\Omega BG^\wedge_2.$$

Proof. The first isomorphism follows from Theorem 3.7.16, while the second is true for every group. □

3.8. A_∞ structure of $H_*\Omega BG^\wedge_2$

We continue to work in Case 3.5.1. So G is a finite group with a semidihedral Sylow 2-subgroup of order $8q$ and no normal subgroup of index two, and k is a field of characteristic two.

Theorem 3.8.1. *We have*

$$HH^*H_*\Omega BG^\wedge_2 = \mathsf{k}[x,y,\hat{z}] \otimes \Lambda(\hat{x},\hat{y},z)$$

with

$$\begin{aligned}
&|x| = (-1,-2,-q-1), && |y| = (-1,-3,-4q), && |z| = (-1,-4,-3q-1),\\
&|\hat{x}| = (0,2,q+1), && |\hat{y}| = (0,3,4q), && |\hat{z}| = (0,4,3q+1).
\end{aligned}$$

Proof. This is a routine computation using Theorems 1.11.5 and 3.7.4. □

Theorem 3.8.2. *In Case 3.5.1, up to quasi-isomorphism, the maps* m_i *in the* A_∞ *structure on* $H_*\Omega BG^\wedge_2$ *may be taken to be the* $\mathsf{k}[\hat{z}]$*-multilinear maps determined by*

$$m_3(\hat{x},\hat{y},\hat{x}) = \hat{z}^2, \qquad m_3(\hat{x},\hat{x}\hat{y},\hat{x}) = \hat{x}\hat{z}^2, \qquad m_3(\hat{y},\hat{x},\hat{x}\hat{y}) = m_3(\hat{x}\hat{y},\hat{x},\hat{y}) = \hat{y}\hat{z}^2,$$

and all m_i *with* $i \geqslant 3$ *vanish on all other triples of monomials not involving* $\hat{z}$*. We have* $m_3 \circ m_3 = 0$ *(Gerstenhaber's circle product).*

This is the unique A_∞ algebra structure on this algebra, such that the map m_3 represents the class $x^2y\hat{z}^2$ of degree $(-3,1,0)$ in the Hochschild cohomology $HH^*H_*\Omega BG_2^\wedge$.

Proof. Comparing Theorem 3.7.4 with Theorem 3.8.1, we see that in the spectral sequence

$$HH^*H_*\Omega BG_2^\wedge \Rightarrow HH^*C_*\Omega BG_2^\wedge$$

we have $d^2(\hat{y}) = x^2\hat{z}^2$, and no further differentials, and

$$E^3 = E^\infty = \mathsf{k}[x,y,\hat{z}]/(x^2\hat{z}^2)\otimes\Lambda(\hat{x},z). \tag{3.8.3}$$

The relation $\hat{x}^2=0$ ungrades to $\hat{x}^2 = y\hat{z}^2$, while the relation $z^2=0$ ungrades to $z^2=x^2y$. It follows that m_3 on $\hat{x}$, $\hat{x}$ and $\hat{y}$ in $H_*\Omega BG_2^\wedge$ in some order is a non-zero multiple of $\hat{z}^2$.

In degree $(-3,1,0)$, the Hochschild cohomology $HH^*H_*\Omega BG_2^\wedge$ is one dimensional, and is spanned by $x^2y\hat{z}^2$. Since the Hochschild cocycle m_3 is only well defined modulo coboundaries, we examine the values of the coboundary of a 2-cochain f_2 on these elements. For degree reasons, we have $f_2(\hat{x},\hat{x}) = f_2(\hat{x},\hat{y}) = f_2(\hat{y},\hat{x}) = 0$. Let $f_2(\hat{x},\hat{x}\hat{y}) = \lambda\hat{z}^2$ and $f_2(\hat{x}\hat{y},\hat{x}) = \mu\hat{z}^2$. Then we have

$$\delta f_2(\hat{x},\hat{x},\hat{y}) = \lambda\hat{z}^2, \qquad \delta f_2(\hat{x},\hat{y},\hat{x}) = (\lambda+\mu)\hat{z}^2, \qquad \delta f_2(\hat{y},\hat{x},\hat{x}) = \mu\hat{z}^2.$$

Now everything is defined over $\mathbb{F}_2$. So working modulo these coboundaries, any assignment with

$$m_3(\hat{x},\hat{x},\hat{y}) + m_3(\hat{x},\hat{y},\hat{x}) + m_3(\hat{y},\hat{x},\hat{x}) = \hat{z}^2$$

is valid. For symmetry we take $m_3(\hat{x},\hat{y},\hat{x}) = \hat{z}^2$ and $m_3(\hat{x},\hat{x},\hat{y}) = m_3(\hat{y},\hat{x},\hat{x}) = 0$.

Using the fact that m_3 is a Hochschild cocycle, and $\hat{x}\hat{y} = \hat{y}\hat{x}$, we then have

$$\begin{gathered}
m_3(\hat{x},\hat{x},\hat{x}\hat{y}) = 0, \qquad m_3(\hat{x}\hat{y},\hat{x},\hat{x}) = 0, \qquad m_3(\hat{x}\hat{y},\hat{y},\hat{y}) = 0, \qquad m_3(\hat{y},\hat{y},\hat{x}\hat{y}) = 0,\\
m_3(\hat{x},\hat{x}\hat{y},\hat{y}) = 0, \qquad m_3(\hat{y},\hat{x}\hat{y},\hat{x}) = 0, \qquad m_3(\hat{y},\hat{x},\hat{x}\hat{y}) = m_3(\hat{y}\hat{x},\hat{x},\hat{y}),\\
m_3(\hat{x},\hat{y},\hat{x}\hat{y}) = m_3(\hat{x}\hat{y},\hat{y},\hat{x}), \qquad m_3(\hat{x},\hat{x}\hat{y},\hat{x}) = \hat{x}\, m_3(\hat{x},\hat{y},\hat{x}) = \hat{x}\hat{z}^2,\\
m_3(\hat{x},\hat{y},\hat{x}\hat{y}) + m_3(\hat{x}\hat{y},\hat{x},\hat{y}) = \hat{y}\hat{z}^2, \qquad m_3(\hat{y}\hat{x},\hat{y},\hat{x}) + m_3(\hat{y},\hat{x},\hat{y}\hat{x}) = \hat{y}\hat{z}^2,
\end{gathered}$$

The 2-cochain f_2 with $f_2(\hat{x}\hat{y},\hat{x}\hat{y}) = \hat{y}\hat{z}^2$, and $f_2=0$ on other monomials, has coboundary $\delta f_2(\hat{x},\hat{y},\hat{x}\hat{y}) = \hat{y}\hat{z}^2$. So adding a multiple of δf_2 to m_3, we can assume that $m_3(\hat{x},\hat{y},\hat{x}\hat{y}) = 0$. It then follows that

$$\begin{gathered}
m_3(\hat{x},\hat{y},\hat{x}\hat{y}) = m_3(\hat{x}\hat{y},\hat{y},\hat{x}) = 0, \qquad m_3(\hat{y},\hat{x},\hat{x}\hat{y}) = m_3(\hat{x}\hat{y},\hat{x},\hat{y}) = \hat{y}\hat{z}^2,\\
m_3(\hat{x}\hat{y},\hat{x}\hat{y},\hat{x}) = m_3(\hat{x},\hat{x}\hat{y},\hat{x}\hat{y}) = 0, \qquad m_3(\hat{x}\hat{y},\hat{x}\hat{y},\hat{y}) = m_3(\hat{y},\hat{x}\hat{y},\hat{x}\hat{y}) = 0,\\
m_3(\hat{x}\hat{y},\hat{x},\hat{x}\hat{y}) = m_3(\hat{x}\hat{y},\hat{y},\hat{x}\hat{y}) = 0, \qquad m_3(\hat{x}\hat{y},\hat{x}\hat{y},\hat{x}\hat{y}) = 0.
\end{gathered}$$

Now, it is straightforward to compute directly that the Gerstenhaber circle product $m_3\circ m_3$ is the zero Hochschild cochain. Mostly all terms are zero, but there are a few cases that involve some cancellation, such as for example

$$\begin{aligned}
&(m_3\circ m_3)(\hat{x},\hat{y},\hat{x},\hat{x}\hat{y},\hat{x})\\
&\qquad = m_3(m_3(\hat{x},\hat{y},\hat{x})\hat{x}\hat{y},\hat{x}) + m_3(\hat{x},m_3(\hat{y},\hat{x},\hat{x}\hat{y}),\hat{x}) + m_3(\hat{x},\hat{y},m_3(\hat{x},\hat{x}\hat{y},\hat{x})\\
&\qquad = m_3(\hat{z}^2,\hat{x}\hat{y},\hat{x}) + m_3(\hat{x},\hat{y}\hat{z}^2,\hat{x}) + m_3(\hat{x},\hat{y},\hat{x}\hat{z}^2) = 0 + \hat{z}^4 + \hat{z}^4 = 0,\\
&(m_3\circ m_3)(\hat{y},\hat{x},\hat{x}\hat{y},\hat{x},\hat{x}\hat{y})\\
&\qquad = m_3(m_3(\hat{y},\hat{x},\hat{x}\hat{y}),\hat{x},\hat{x}\hat{y}) + m_3(\hat{y},m_3(\hat{x},\hat{x}\hat{y},\hat{x}),\hat{x}\hat{y}) + m_3(\hat{y},\hat{x},m_3(\hat{x}\hat{y},\hat{x},\hat{x}\hat{y}))
\end{aligned}$$

$$= m_3(\hat{y}\hat{z}^2, \hat{x}, \hat{x}\hat{y}) + m_3(\hat{y}, \hat{x}\hat{z}^2, \hat{x}\hat{y}) + m_3(\hat{y}, \hat{x}, 0) = \hat{y}\hat{z}^2 + \hat{y}\hat{z}^2 + 0 = 0.$$

By Proposition 1.5.4, we have $\delta m_4 = m_3 \circ m_3$, so m_4 is a Hochschild cocycle. Since there are no non-zero Hochschild classes in degree $(-4, 2, 0)$, this makes m_4 a coboundary, and so we can take $m_4 = 0$. Then $m_4 \circ m_3 + m_3 \circ m_4$ vanishes, and so m_5 is a Hochschild cocycle. Since there are no non-zero classes in degree $(-5, 3, 0)$, it follows that m_5 is a coboundary, and may hence be taken to be zero. We could continue this way, but eventually there are non-zero elements of Hochschild cohomology in degree $(-n, n-2, 0)$. So instead, define an A_∞ algebra $\mathfrak{a}$ with with $H_*\mathfrak{a} \cong H_*\Omega BG_2^\wedge$, the same structure maps as $H_*\Omega BG_2^\wedge$ up to m_3, and $m_i = 0$ for $i \geqslant 4$. Then the Koszul dual A_∞ algebra $\mathfrak{b} = \mathsf{Hom}_{\mathsf{D^b}(\mathfrak{a})}(\mathsf{k}, \mathsf{k})$ has homology isomorphic to H^*BG as an associative algebra. To see this, we compute the spectral sequence $\mathsf{Ext}^*_{H_*\mathfrak{a}}(\mathsf{k}, \mathsf{k}) \Rightarrow H_*\mathfrak{b}$. The map m_3 determines the d^2 differential in this spectral sequence, and so the E^3 page is given by (3.8.3). There is no room for further non-zero differentials or for ungrading problems, so this is also $H_*\mathfrak{b}$.

By Theorem 3.7.16, H^*BG is intrinsically formal, and so $\mathfrak{b}$ is quasi-isomorphic to C^*BG as an A_∞ algebra. This implies that

$$\mathfrak{a} \simeq \mathsf{Hom}_{\mathsf{D^b}(\mathfrak{b})}(\mathsf{k}, \mathsf{k}) \simeq \mathsf{Hom}_{\mathsf{D^b}(H^*BG)}(\mathsf{k}, \mathsf{k}) \simeq H_*\Omega BG_2^\wedge. \qquad \square$$

3.9. A differential graded model

We continue to work with Case 3.5.1. Theorem 3.8.2 suggests that there may be a nice DG algebra quasi-isomorphic to $C_*\Omega BG_2^\wedge$. Since C^*BG is formal, in order to produce such an algebra, we look at endomorphisms of the minimal resolution of k over H^*BG. This resolution is eventually periodic of period one, and takes the following form.

$$\cdots \to (H^*BG)^4 \xrightarrow{\left(\begin{smallmatrix} z & xy & & \\ x & z & & \\ y & & z & xy \\ & y & x & z \end{smallmatrix}\right)} (H^*BG)^4 \xrightarrow{\left(\begin{smallmatrix} z & xy & & \\ x & z & & \\ y & & z & xy \\ & y & x & z \end{smallmatrix}\right)} (H^*BG)^4$$

$$\xrightarrow{\left(\begin{smallmatrix} x & z & & \\ y & & z & xy \\ & y & x & z \end{smallmatrix}\right)} (H^*BG)^3 \xrightarrow{\left(\begin{smallmatrix} y & x & z \end{smallmatrix}\right)} H^*BG \to \mathsf{k}.$$

The map $\hat{z}$ shifts one to the right by the 4×4 identity matrix, except at the right hand end:

$$\cdots, \qquad \left(\begin{smallmatrix} 1&0&0&0 \\ 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \end{smallmatrix}\right), \qquad \left(\begin{smallmatrix} 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \end{smallmatrix}\right), \qquad \left(\begin{smallmatrix} 0&0&1 \end{smallmatrix}\right).$$

This endomorphism is in the centre of the endomorphism ring of the resolution, and so we can regard everything as defined over $\mathsf{k}[\hat{z}]$.

Similarly, $\hat{y}$ is given by shifting to the right and using the matrices

$$\cdots, \qquad \left(\begin{smallmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 1&0&0&0 \\ 0&1&0&0 \end{smallmatrix}\right), \qquad \left(\begin{smallmatrix} 0&0&0&0 \\ 1&0&0&0 \\ 0&1&0&0 \end{smallmatrix}\right), \qquad \left(\begin{smallmatrix} 1&0&0 \end{smallmatrix}\right),$$

and $\hat{x}$ is given by shifting to the right and using the matrices

$$\cdots, \qquad \left(\begin{smallmatrix} 0&y&0&0 \\ 1&0&0&0 \\ 0&0&0&y \\ 0&0&1&0 \end{smallmatrix}\right), \qquad \left(\begin{smallmatrix} 1&0&0&0 \\ 0&0&0&y \\ 0&0&1&0 \end{smallmatrix}\right), \qquad \left(\begin{smallmatrix} 0&1&0 \end{smallmatrix}\right).$$

These matrices commute, and satisfy $\hat{y}^2 = 0$, but $\hat{x}^2$ is not zero, but rather $y\hat{z}^2$. So we find a homotopy ξ from $\hat{x}^2$ to zero:

$$\cdots, \qquad \begin{pmatrix} 0&0&1&0\\ 0&0&0&1\\ 0&0&0&0\\ 0&0&0&0 \end{pmatrix}, \qquad \begin{pmatrix} 0&0&0&1\\ 0&0&0&0\\ 0&0&0&0 \end{pmatrix}, \qquad \begin{pmatrix} 0&0&0 \end{pmatrix}.$$

Then we have $\xi^2 = 0$, $d\xi = \hat{x}^2$, $\xi\hat{x} = \hat{x}\xi$, and $\xi\hat{y} + \hat{y}\xi = \hat{z}^2$.

THEOREM 3.9.1. *Let Q be the DG algebra over $\mathsf{k}[\hat{z}]$ generated by elements $\hat{x}$, $\hat{y}$ and ξ with*

$$d\hat{x} = 0, \quad d\hat{y} = 0, \quad \hat{y}^2 = 0, \quad \hat{x}\hat{y} = \hat{y}\hat{x},$$
$$d\xi = \hat{x}^2, \quad \xi\hat{x} = \hat{x}\xi, \quad \xi^2 = 0, \quad \xi\hat{y} + \hat{y}\xi = \hat{z}^2,$$

and with degrees

$$|\hat{x}| = (2, q+1), \quad |\hat{y}| = (3, 4q), \quad |\hat{z}| = (4, 3q+1), \quad |\xi| = (5, 2q+2).$$

Then Q is quasi-isomorphic to $C_*\Omega BG_2^\wedge$.

PROOF. The algebra relations imply that this has a free $\mathsf{k}[\hat{z}]$-basis consisting of the elements $\hat{x}^i\hat{y}^{\varepsilon_1}\xi^{\varepsilon_2}$ with $i \geqslant 0$, $\varepsilon_1, \varepsilon_2 \in \{0, 1\}$. The differential sends the basis elements with $\varepsilon_2 = 1$ bijectively to the basis elements with $i \geqslant 2$ and $\varepsilon_2 = 0$. So H_*Q is the algebra $\mathsf{k}[\hat{z}] \otimes \Lambda(\hat{x}, \hat{y})$, which is isomorphic to $H_*\Omega BG_2^\wedge$. The A_∞ structure on H_*Q is not formal. Indeed, it is easy to check that m_3 represents the Hochschild class $x^2y\hat{z}^2$. By Theorem 3.8.2, there is a unique A_∞ structure on this algebra such that m_3 represents this class. It follows that Q is quasi-isomorphic to $C_*\Omega BG_2^\wedge$. □

REMARK 3.9.2. We can give an explicit quasi-isomorphism $H_*\Omega BG_2^\wedge \to Q$ as follows. The map f_1 is the $k[\hat{z}]$-module homomorphism which sends each monomial 1, $\hat{x}$, $\hat{y}$, $\hat{x}\hat{y}$ in $H_*\Omega BG_2^\wedge$ to the monomial with the same name in Q. The map f_2 is given by

$$f_2(\hat{x}, \hat{x}) = \xi, \qquad f_2(\hat{x}, \hat{x}\hat{y}) = \xi\hat{y}, \qquad f_2(\hat{x}\hat{y}, \hat{x}) = \hat{y}\xi,$$

and f_2 is zero on all other pairs of monomials. All higher f_i are the zero map. From this information, we can inductively compute the higher multiplications m_i on $H_*\Omega BG_2^\wedge$, and they agree with those given in Theorem 3.8.2. For example, we have

$$m_2(f_1 \otimes f_2 - f_2 \otimes f_1)(\hat{x}, \hat{y}, \hat{x}) = 0,$$
$$f_2(1 \otimes m_2 - m_2 \otimes 1)(\hat{x}, \hat{y}, \hat{x}) = \xi\hat{y} + \hat{y}\xi = \hat{z}^2,$$

and so $m_3(\hat{x}, \hat{y}, \hat{x}) = \hat{z}^2$.

3.10. Duality for $Q[\hat{z}^{-1}]$-modules

Continuing with Case 3.5.1, by Theorem 3.9.1, $\hat{z}$ is central in Q. It therefore makes sense to invert it and examine $Q[\hat{z}^{-1}]$ as an algebra over the graded field $\mathsf{k}[\hat{z}, \hat{z}^{-1}]$. This parallels Section 2.10, so we give fewer details.

If X is any $\mathsf{k}[\hat{z}, \hat{z}^{-1}]$-module, we write

$$X^* = \mathsf{Hom}_{\mathsf{k}[\hat{z},\hat{z}^{-1}]}(X, \mathsf{k}[\hat{z}, \hat{z}^{-1}]) \cong \mathsf{Hom}_{\mathsf{k}}(X, \mathsf{k}).$$

PROPOSITION 3.10.1. *There is a quasi-isomorphism of $Q[\hat{z}^{-1}]$-bimodules*

$$Q[\hat{z}^{-1}] \simeq \Sigma Q[\hat{z}^{-1}]^*.$$

PROOF. The proof is similar to the proof of Proposition 2.10.3. Consider the basis of $Q[\hat{z}^{-1}]$ as a $\mathsf{k}[\hat{z},\hat{z}^{-1}]$-module given by the monomials $\hat{x}^i\hat{y}^{\varepsilon_1}\xi^{\varepsilon_2}$. The map of $\mathsf{k}[\hat{z},\hat{z}^{-1}]$-modules $Q[\hat{z}^{-1}] \to \Sigma^{|\hat{x}\hat{y}|}Q[\hat{z}^{-1}]^*$ sending all monomials to zero except that

$$1 \mapsto (\hat{x}\hat{y} \mapsto 1), \qquad \hat{x} \mapsto (\hat{y} \mapsto 1), \qquad \hat{y} \mapsto (\hat{x} \mapsto 1), \quad \hat{x}\hat{y} \mapsto (1 \mapsto 1)$$

and these maps take all other monomials to zero, is easily checked to be a quasi-isomorphism of $Q[\hat{z}^{-1}]$-bimodules. Now $|\hat{x}\hat{y}| = 5$, and $\hat{z}$ is a periodicity generator of degree four, so $\Sigma^{|\hat{x}\hat{y}|}Q[\hat{z}^{-1}]^* \cong \Sigma Q[\hat{z}^{-1}]^*$. ☐

COROLLARY 3.10.2. *If X is a homotopically projective $Q[\hat{z}^{-1}]$-module then we have a quasi-isomorphism*

$$\mathsf{Hom}_{Q[\hat{z}^{-1}]}(X, Q[\hat{z}^{-1}]) \simeq \Sigma\mathsf{Hom}_{\mathsf{k}[\hat{z},\hat{z}^{-1}]}(X, \mathsf{k}[\hat{z},\hat{z}^{-1}]).$$

PROOF. The proof is essentially the same as that of Corollary 2.10.5. ☐

THEOREM 3.10.3. *Let X and Y be $Q[\hat{z}^{-1}]$-modules, such that X is homotopically projective, and its image in $\mathsf{D}^{\mathsf{b}}(Q[\hat{z}^{-1}])$ is compact. Then we have a duality*

$$\mathsf{Hom}_{Q[\hat{z}^{-1}]}(X,Y)^* \cong \mathsf{Hom}_{Q[\hat{z}^{-1}]}(Y, \Sigma^{-1}X).$$

PROOF. The proof is essentially the same as that of Theorem 2.10.6. ☐

3.11. The singularity category of C^*BG

We examine the singularity category of C^*BG in Case 3.5.1. By Theorem 3.7.16, the A_∞ structure on C^*BG is formal. This implies that the singularity category is given by $\mathsf{D}_{\mathsf{sg}}(C^*BG) \simeq \mathsf{D}_{\mathsf{sg}}(H^*BG)$ and the cosingularity category is given by $\mathsf{D}_{\mathsf{csg}}(C^*BG) \simeq \mathsf{D}_{\mathsf{csg}}(H^*BG)$. We shall discuss $\mathsf{D}_{\mathsf{sg}}(H^*BG)$. If we were looking as just graded modules, this would be equivalent to the category of maximal Cohen–Macaulay modules over $H^*BG = \mathsf{k}[x,y,z]/(x^2y+z^2)$. In characteristic zero, this algebra was part of the classification theorem of Knörrer [**158**] and Buchweitz, Greuel and Schreyer [**43**]. But in fact the proof goes through for this algebra in arbitrary characteristic, see for example Proposition 14.19 of Leuschke and Wiegand [**166**], and gives a category of bounded Cohen–Macaulay type (D_∞). They state the theorem for the complete local ring, but the arguments work just as well for graded modules over the graded ring.

THEOREM 3.11.1. *Let k be a field of arbitrary characteristic. The following matrix factorisations describe all indecomposable MCM modules over $\mathsf{k}[x,y,z]/(x^2y+z^2)$ with $|x| = -3$, $|y| = -4$, $|z| = -5$. We take the module to be the cokernel of the first matrix, which is the image of the second matrix.*

(1) $\begin{pmatrix} z & y \\ -x^2 & z \end{pmatrix}\begin{pmatrix} z & -y \\ x^2 & z \end{pmatrix}$, *two generators in degrees n, $n-1$;*

(2) $\begin{pmatrix} z & xy \\ -x & z \end{pmatrix}\begin{pmatrix} z & -xy \\ x & z \end{pmatrix}$, *two generators in degrees n, $n+2$;*

(3) $\begin{pmatrix} z & xy & 0 & 0 \\ -x & z & 0 & 0 \\ -y^j & 0 & z & xy \\ 0 & y^j & -x & z \end{pmatrix}\begin{pmatrix} z & -xy & 0 & 0 \\ x & z & 0 & 0 \\ y^j & 0 & z & -xy \\ 0 & -y^j & x & z \end{pmatrix}$ *for some $j \geqslant 1$, four generators in degrees n, $n+2$, $n+5-4j$, $n+7-4j$;*

(4) $\begin{pmatrix} z & xy & 0 & 0 \\ -x & z & 0 & 0 \\ 0 & -y^{j+1} & z & xy \\ -y^j & 0 & -x & z \end{pmatrix} \begin{pmatrix} z & -xy & 0 & 0 \\ x & z & 0 & 0 \\ 0 & y^{j+1} & z & -xy \\ y^j & 0 & x & z \end{pmatrix}$ *for some* $j \geqslant 1$, *four generators in degrees* n, $n+2$, $n+3-4j$, $n+5-4j$.

We write $(1)_n$, $(2)_n$, $(3)_{n,j}$ and $(4)_{n,j}$ for these modules. For each of these modules M, we have $\Omega(M) \cong \Sigma^{-5,-3q-1}(M)$, so in $\mathsf{D}_{\mathsf{sg}}(M)$ we have $\Sigma^{4,3q+1}(M) \cong M$. This periodicity is induced by the degree four element $\hat{z} \in HH^*C^*BG$, whose square is the Eisenbud operator for the relation $x^2y + z^2$.

The Auslander–Reiten quiver of this singularity category has type $\mathbb{Z}D_\infty/\Sigma^4$, and the modules above fit into this as follows.

$$\vdots$$

$$\begin{tikzcd}[column sep=small, row sep=small]
(4)_3 \arrow[dr] & & (4)_3 \arrow[dr] & & (4)_3 \\
& (3)_2 \arrow[ur] \arrow[dr] & & (3)_2 \arrow[ur] \arrow[dr] & \\
(4)_2 \arrow[ur] \arrow[dr] & & (4)_2 \arrow[ur] \arrow[dr] & & (4)_2 \\
& (3)_1 \arrow[ur] \arrow[dr] & & (3)_1 \arrow[ur] \arrow[dr] & \\
(4)_1 \arrow[ur] \arrow[r] \arrow[dr] & (2) \arrow[r] & (4)_1 \arrow[ur] \arrow[r] \arrow[dr] & (2) \arrow[r] & (4)_1 \\
& (1) \arrow[ur] & & (1) \arrow[ur] &
\end{tikzcd}$$

The remaining question is this: Is every object in the singularity category of differential graded modules equivalent to a module with zero differential? If so, the classification above applies to $\mathsf{D}_{\mathsf{sg}}(C^*BG)$.

We have $\mathsf{D}_{\mathsf{sg}}(C^*BG) \simeq \mathsf{D}_{\mathsf{csg}}(C_*\Omega BG^\wedge_2)$. Now the element $\hat{z} \in H_*\Omega BG^\wedge_2$ is the Eisenbud operator for the relation $x^2y = z^2$ in H^*BG. It comes from an element of $HH^*C_*\Omega BG^\wedge_2$ with the same name. It follows that $\hat{z}$ is central, and we may invert it to obtain an equivalence

$$\mathsf{D}_{\mathsf{sg}}(C^*BG) \simeq \mathsf{D}_{\mathsf{csg}}(C_*\Omega BG^\wedge_2) \simeq \mathsf{D}^{\mathsf{b}}(C_*\Omega BG^\wedge_2[\hat{z}^{-1}]).$$

3.12. Two classes of involutions, one of elements of order four

We now turn to the Case 3.5.2 of a finite group G with semidihedral Sylow 2-subgroup SD of order $8q$, $q \geqslant 2$, with two classes of involutions and one class of elements of order four. In this case, G has a normal subgroup K of index two with generalised quaternion Sylow 2-subgroups, and K has no normal subgroups of index two.

The principal blocks of this type are all in Erdmann's classes $SD(2\mathcal{A})_1$ and $SD(2\mathfrak{B})_2$. Class $SD(2\mathfrak{B})_2$ turns out to be the easier to deal with, so we take $G = SL^{\pm}(2, p^m)$ with

$p^m \equiv 3 \pmod 4$. The basic algebra of the principal block is given by the quiver

(3.12.1)
$$\begin{tikzcd} \mathsf{k} \arrow[loop left, "a"] \arrow[r, bend left, "b"] & \mathsf{M} \arrow[l, bend left, "c"] \arrow[loop right, "d"] \end{tikzcd}$$

with relations

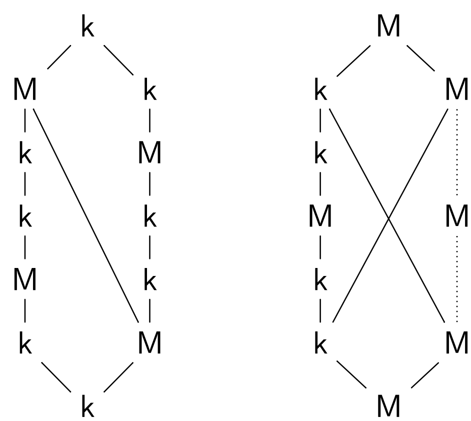
$$db = bacba, \qquad cd = acbac, \qquad bc = d^{2q-1}, \qquad a^2 = 0.$$

These imply that

$$\begin{aligned} d^2b &= dbacba = bacba^2cba = 0, \\ cd^2 &= acbacd = acba^2cbac = 0, \\ bcb &= d^{2q-1}b = d^{2q-3}(d^2b) = 0, \\ cbc &= cd^{2q-1} = (cd^2)d^{2q-3} = 0. \end{aligned}$$

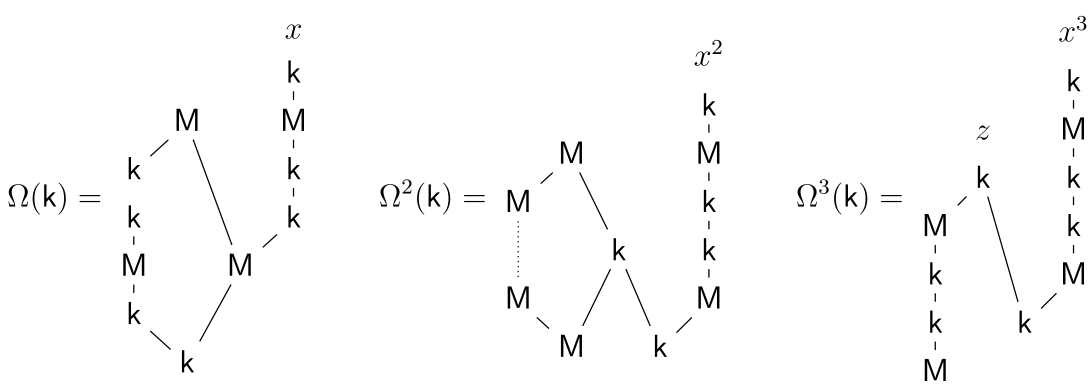
This admits a $\mathbb{Z}$-grading with $|a| = 1-q$, $|b| = |c| = q - \frac{1}{2}$, $|d| = 1$.

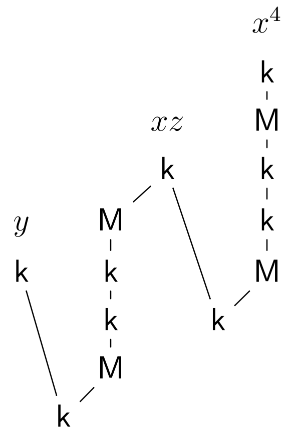
The structures of the projective indecomposables are as follows:

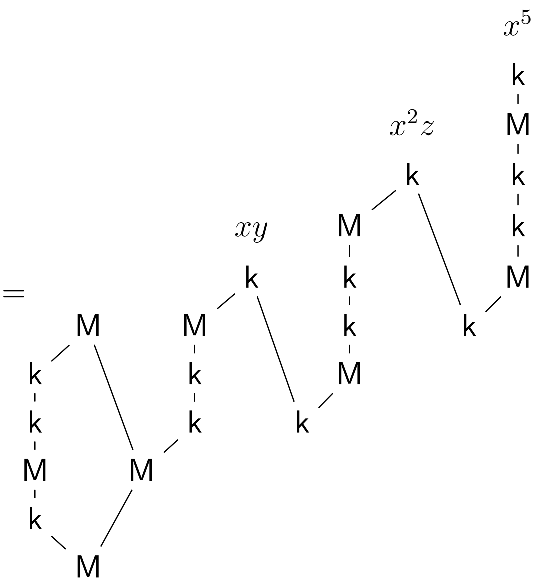


Here, the case $q = 2$ is as shown, and the dotted lines indicate that for $q > 2$ there are more copies of M in the right arm of P_{M}.

The minimal resolution of the trivial module may be computed using the method of [**17**], and the result is as follows.



$\Omega^4(\mathsf{k}) =$ 

$\Omega^5(\mathsf{k}) =$ 

The minimal resolution is the total complex of the following double complex:

(3.12.2)
$$
\begin{tikzcd}
 & & & & P_{\mathsf{M}} \arrow[d, "\bar{b}"] & P_{\mathsf{M}} \arrow[l, "\bar{d}"'] \arrow[d, "\bar{b}\bar{c}\bar{a}\bar{b}"] & \\
 & & & & P_{\mathsf{k}} \arrow[d, "\bar{b}\bar{c}\bar{a}\bar{b}\bar{c}"] & P_{\mathsf{k}} \arrow[l, "\bar{a}"'] \arrow[d, "\bar{b}\bar{c}\bar{a}\bar{b}\bar{c}"] & \cdots \\
 & & P_{\mathsf{M}} \arrow[d, "\bar{b}"] & P_{\mathsf{M}} \arrow[l, "\bar{d}"'] \arrow[d, "\bar{b}\bar{c}\bar{a}\bar{b}"] & P_{\mathsf{k}} \arrow[l, "\bar{c}\bar{a}"'] \arrow[d, "\bar{b}\bar{c}\bar{a}\bar{b}\bar{c}"] & P_{\mathsf{k}} \arrow[l, "\bar{a}"'] \arrow[d, "\bar{b}\bar{c}\bar{a}\bar{b}\bar{c}"] & \\
 & & P_{\mathsf{k}} \arrow[d, "\bar{b}\bar{c}\bar{a}\bar{b}\bar{c}"] & P_{\mathsf{k}} \arrow[l, "\bar{a}"'] \arrow[d, "\bar{b}\bar{c}\bar{a}\bar{b}\bar{c}"] & P_{\mathsf{k}} \arrow[l, "\bar{a}"'] \arrow[d, "\bar{b}\bar{c}\bar{a}\bar{b}\bar{c}"] & P_{\mathsf{k}} \arrow[l, "\bar{a}"'] \arrow[d, "\bar{b}\bar{c}\bar{a}\bar{b}\bar{c}"] & \cdots \\
P_{\mathsf{M}} \arrow[d, "\bar{b}"] & P_{\mathsf{M}} \arrow[l, "\bar{d}"'] \arrow[d, "\bar{b}\bar{c}\bar{a}\bar{b}"] & P_{\mathsf{k}} \arrow[l, "\bar{c}\bar{a}"'] \arrow[d, "\bar{b}\bar{c}\bar{a}\bar{b}\bar{c}"] & P_{\mathsf{k}} \arrow[l, "\bar{a}"'] \arrow[d, "\bar{b}\bar{c}\bar{a}\bar{b}\bar{c}"] & P_{\mathsf{k}} \arrow[l, "\bar{a}"'] \arrow[d, "\bar{b}\bar{c}\bar{a}\bar{b}\bar{c}"] & P_{\mathsf{k}} \arrow[l, "\bar{a}"'] \arrow[d, "\bar{b}\bar{c}\bar{a}\bar{b}\bar{c}"] & \\
P_{\mathsf{k}} & P_{\mathsf{k}} \arrow[l, "\bar{a}"'] & P_{\mathsf{k}} \arrow[l, "\bar{a}"'] & P_{\mathsf{k}} \arrow[l, "\bar{a}"'] & P_{\mathsf{k}} \arrow[l, "\bar{a}"'] & P_{\mathsf{k}} \arrow[l, "\bar{a}"'] & \cdots
\end{tikzcd}
$$

The cohomology ring in this case is therefore

$$H^*BG = \mathsf{k}[x,y,z]/(x^2y + z^2)$$

with

$$|x| = (-1, q-1), \qquad |y| = (-4, -4q), \qquad |z| = (-3, -q-1).$$

The situation is therefore very similar to Case 3.5.1.

3.13. Ext and Hochschild cohomology

Continuing with Case 3.5.2, the proofs of the following theorems are exactly as in the corresponding computations in Section 3.7 for Case 3.5.1.

THEOREM 3.13.1. *We have*

$$\mathsf{Ext}^{**}_{H^*BG}(\mathsf{k},\mathsf{k}) = \Lambda(\hat{x},\hat{y}) \otimes \mathsf{k}[\hat{z}]$$

with degrees given by $|\hat{x}| = (-1, 1, 1-q)$, $|\hat{y}| = (-1, 4, 4q)$ *and* $|\hat{z}| = (-1, 3, q+1)$. $\square$

THEOREM 3.13.2. *We have*

$$H_*\Omega BG_2^\wedge = \Lambda(\hat{x}, \hat{y}) \otimes \mathsf{k}[\hat{z}]$$

with $|\hat{x}| = (0, 1-q)$, $|\hat{y}| = (3, 4q)$ *and* $|\hat{z}| = (2, q+1)$. $\square$

THEOREM 3.13.3. *We have*

$$HH^*H^*BG = H^*BG[\hat{x}, \hat{z}]/(\hat{x}^2 + y\hat{z}^2, x^2\hat{z}^2)$$

with $|x| = (0, -1, q-1)$, $|y| = (0, -4, -4q)$, $|z| = (0, -3, -q-1)$, $|\hat{x}| = (-1, 1, 1-q)$, $|\hat{z}| = (-1, 3, q+1)$. $\square$

The proof of the following proposition follows along the same lines as the proof of Proposition 3.7.8, but the details are different, so we spell them out.

PROPOSITION 3.13.4. *There are no non-zero elements of degree* $(-n, n-2, 0)$ *in the Hochschild cohomology* HH^*H^*BG *with* $n > 2$.

PROOF. We have a k-basis for HH^*H^*BG consisting of the monomials $x^{i_1}y^{i_2}z^{\varepsilon_3}\hat{x}^{\varepsilon_1}\hat{z}^{i_3}$ with either $i_1 \leqslant 1$ or $i_3 \leqslant 1$. Suppose that such a monomial has degree $(-n, n-2, 0)$. Comparing degrees, we have

$$-n = -\varepsilon_1 - i_3 \tag{3.13.5}$$

$$n - 2 = -i_1 - 4i_2 - 3\varepsilon_3 + \varepsilon_1 + 3i_3 \tag{3.13.6}$$

$$0 = (q-1)i_1 - 4qi_2 - (q+1)\varepsilon_3 - (q-1)\varepsilon_1 + (q+1)i_3. \tag{3.13.7}$$

We shall show that there are no solutions in non-negative integers with $n > 2$.

First we deal with the case $q = 2$. In this case, equation (3.13.7) becomes

$$0 = i_1 - 8i_2 - 3\varepsilon_3 - \varepsilon_1 + 3i_3. \tag{3.13.8}$$

Adding equations (3.13.6) and (3.13.8), we get

$$n - 2 = -12i_2 - 6\varepsilon_3 + 6i_3, \tag{3.13.9}$$

and so

$$n \equiv 2 \pmod 6.$$

If instead, we add equations (3.13.5) and (3.13.6) and subtract equation (3.13.8), we get $-2 = 4i_2 + \varepsilon_1 - i_3$, or

$$4i_2 = -\varepsilon_1 + i_3 - 2. \tag{3.13.10}$$

So i_3 determines ε_1 and i_2.

Let $n = 6a + 2$, so that equation (3.13.9) gives

$$a = -2i_2 - \varepsilon_3 + i_3 \geqslant 1. \tag{3.13.11}$$

Equation (3.13.5) implies $i_3 = 6a + 2 - \varepsilon_1$. Then equation (3.13.10) gives $i_2 = (3a - \varepsilon_1)/2$. Plugging these values for i_2 and i_3 into equation (3.13.11) gives

$$a = -3a + \varepsilon_1 - \varepsilon_3 + 6a + 2 - \varepsilon_1 = 3a + 2 - \varepsilon_3,$$

and so $\varepsilon_3 = 2a + 2$ is bigger than one. This contradiction completes the case $q = 2$.

Now suppose that $q > 2$. Reading equations (3.13.5), (3.13.6), and (3.13.7) modulo four, we see that $\varepsilon_3 + i_1$ and $n = \varepsilon_1 + i_3$ are both even, and are congruent modulo four. Since $n > 2$, we have $n \geqslant 4$, $i_3 \geqslant 3$, and hence $i_1 \leqslant 1$. So either $i_1 = \varepsilon_3 = 0$ or $i_1 = \varepsilon_3 = 1$. Adding the equations (3.13.5) and (3.13.6), we get

$$-2 = -i_1 - 4i_2 - 3\varepsilon_3 + 2i_3. \tag{3.13.12}$$

Since $-i_1 - 3\varepsilon_3$ is divisible by four, we deduce that i_3 is odd, and hence $\varepsilon_1 = 1$, and $n = 1 + i_3$.

Since $i_1 = \varepsilon_3$ and $\varepsilon_1 = 1$, equation (3.13.7) becomes

$$0 = -4qi_2 - 2\varepsilon_3 - (q-1) + (q+1)i_3$$

and equation (3.13.12) gives

$$2i_2 = 1 - 2\varepsilon_3 + i_3.$$

Substituting, we get

$$\begin{aligned} 0 &= -2q(1 - 2\varepsilon_3 + i_3) - 2\varepsilon_3 - (q-1) + (q+1)i_3 \\ &= (1-q)i_3 + (4q-2)\varepsilon_3 + 1 - 3q \end{aligned}$$

and so

$$(4q-2)\varepsilon_3 = (q-1)i_3 + (3q-1).$$

This is bigger than zero, so $\varepsilon_3 = 1$, which then gives $i_3 = 1$. Then by equation (3.13.5), $n = \varepsilon_1 + i_3 = 2$, which is a contradiction. □

Theorem 3.13.13. *In Case 3.5.2, with the grading inherited from the internal grading on the basic algebra of* $\mathsf{k}G$*, the* A_∞ *structure of* H^*BG *is formal.*

Remark 3.13.14. Another proof of formality, but which does not give intrinsic formality, in Theorem 3.13.13 is to notice that there are endomorphisms of the resolution (3.12.2) representing x, y and z, and strictly satisfying the relation $x^2y = z^2$. The endomorphism representing y just moves the whole diagram two places up and two places to the right. For x, we move one place to the right, but then we have to compose with the maps

$$\begin{array}{ccccccc}
 & & & & \bar{c}\bar{a}\bar{b} & \bar{c} & \\
 & & & & 1 & 1 & \cdots \\
 & & \bar{c}\bar{a}\bar{b} & \bar{c} & 1 & 1 & \\
 & & 1 & 1 & 1 & 1 & \cdots \\
\bar{c}\bar{a}\bar{b} & \bar{c} & 1 & 1 & 1 & 1 & \\
1 & 1 & 1 & 1 & 1 & 1 & \cdots
\end{array}$$

Similarly, for z we move one place up and two to the right, and compose with the same maps. This defines a quasi-isomorphism from the cohomology ring $\mathsf{Ext}^*_{\mathsf{k}G}(\mathsf{k},\mathsf{k})$ to the DG algebra $\mathsf{End}^*_{\mathsf{k}G}(P_{\mathsf{k}})$, which in turn is quasi-isomorphic to C^*BG.

Corollary 3.13.15. *In Case 3.5.2, we have*

$$HH^*H^*BG \cong HH^*C^*BG \cong HH^*C_*\Omega BG^\wedge_2.$$

Proof. The first isomorphism follows from Theorem 3.13.13, while the second is true for every group. □

3.14. A_∞ structure, a DG model, and duality

We continue to work in Case 3.5.2, and because the details are similar to those in Case 3.5.1, we skip some details. So G is a finite group with a semidihedral Sylow 2-subgroup of order $8q$, and has a normal subgroup K of index two with generalised quaternion Sylow 2-subgroups, and K has no normal subgroup of index two.

THEOREM 3.14.1. *We have*

$$HH^*H_*\Omega BG_2^\wedge = \mathsf{k}[x,y,\hat{z}] \otimes \Lambda(\hat{x},\hat{y},z)$$

with

$$|x| = (-1,0,q-1), \qquad |y| = (-1,-3,-4q), \qquad |z| = (-1,-2,-q-1),$$
$$|\hat{x}| = (0,0,1-q), \qquad |\hat{y}| = (0,3,4q), \qquad |\hat{z}| = (0,2,q+1).$$

PROOF. This is a routine computation using Theorems 1.11.5 and 3.13.2. □

THEOREM 3.14.2. *In Case 3.5.2, up to quasi-isomorphism, the maps m_i in the A_∞ structure on $H_*\Omega BG_2^\wedge$ may be taken to be the $\mathsf{k}[\hat{z}]$-multilinear maps determined by*

$$m_3(\hat{x},\hat{y},\hat{x}) = \hat{z}^2, \qquad m_3(\hat{x},\hat{x}\hat{y},\hat{x}) = \hat{x}\hat{z}^2, \qquad m_3(\hat{y},\hat{x},\hat{x}\hat{y}) = m_3(\hat{x}\hat{y},\hat{x},\hat{y}) = \hat{y}\hat{z}^2,$$

and all m_i with $i \geqslant 3$ vanish on all other triples of monomials not involving $\hat{z}$. We have $m_3 \circ m_3 = 0$ (Gerstenhaber's circle product).

This is the unique A_∞ algebra structure on this algebra, such that the map m_3 represents the class $x^2y\hat{z}^2$ of degree $(-3,1,0)$ in the Hochschild cohomology $HH^*H_*\Omega BG_2^\wedge$.

PROOF. This is the same as the proof of Theorem 3.8.2. □

THEOREM 3.14.3. *Let Q be the DG algebra over $\mathsf{k}[\hat{z}]$ generated by elements $\hat{x}$, $\hat{y}$ and ξ with*

$$d\hat{x} = 0, \quad d\hat{y} = 0, \quad \hat{y}^2 = 0, \quad \hat{x}\hat{y} = \hat{y}\hat{x},$$
$$d\xi = \hat{x}^2, \quad \xi\hat{x} = \hat{x}\xi, \quad \xi^2 = 0, \quad \xi\hat{y} + \hat{y}\xi = \hat{z}^2,$$

and with degrees

$$|\hat{x}| = (0,q+1), \quad |\hat{y}| = (3,4q), \quad |\hat{z}| = (2,3q+1), \quad |\xi| = (1,2q+2).$$

Then Q is quasi-isomorphic to $C_*\Omega BG_2^\wedge$.

PROOF. This is proved in the same way as Theorem 3.9.1. □

Since $\hat{z}$ is central in Q, we may invert it. Let $Q[\hat{z}^{-1}]$ be the resulting DG algebra over $\mathsf{k}[\hat{z},\hat{z}^{-1}]$.

PROPOSITION 3.14.4. *There is a quasi-isomorphism of $Q[\hat{z}^{-1}]$-bimodules*

$$Q[\hat{z}^{-1}]^* \cong \Sigma Q[\hat{z}^{-1}].$$

COROLLARY 3.14.5. *If X is a homotopically projective $Q[\hat{z}^{-1}]$-module then we have a quasi-isomorphism*

$$\mathsf{Hom}_{Q[\hat{z}^{-1}]}(X, Q[\hat{z}^{-1}]) \simeq \Sigma\mathsf{Hom}_{\mathsf{k}[\hat{z},\hat{z}^{-1}]}(X, \mathsf{k}[z,z^{-1}]).$$

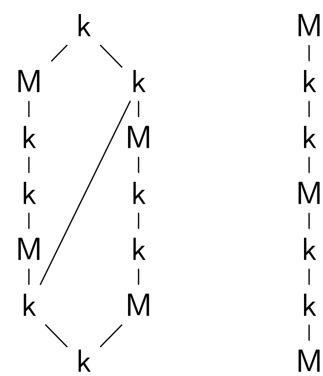
Theorem 3.14.6. *Let X and Y be $Q[\hat{z}^{-1}]$-modules, such that X is homotopically projective, and its image in $\mathsf{D}^{\mathsf{b}}(Q[\hat{z}^{-1}])$ is compact. Then we have a duality*

$$\mathsf{Hom}_{Q[\hat{z}^{-1}]}(X,Y)^* \cong \mathsf{Hom}_{Q[\hat{z}^{-1}]}(Y,\Sigma^{-1}X).$$

3.15. One class of involutions, two of elements of order four

Now we consider Case 3.5.3, of a finite group G with semidihedral Sylow 2-subgroup SD of order $8q$, $q \geqslant 2$, with one class of involutions and two classes of elements of order four. In this case, G has a normal subgroup K of index two with dihedral Sylow 2-subgroups, and K has no normal subgroups of index two. The group K is therefore in Case 2.7.1 of the classification of groups with dihedral Sylow 2-subgroups.

The principal blocks of this type are all in Erdmann's class $SD(2\mathcal{A})_2$. This causes a problem with socle relations. To see this, let us look at the principal block B_0 of the group $PGL^*(2,p^{2m})$. Let k and M be the two simple modules. In the case $q=2$, their projective covers are given by the following diagrams.



The quiver for B_0 is

$$\begin{tikzcd} \mathsf{k} \arrow[loop left, "a"] \arrow[r, bend left, "b"] & \mathsf{M} \arrow[l, bend left, "c"] \end{tikzcd}$$

with relations

$$bc = 0, \qquad (cba)^{2q} = (acb)^{2q}, \qquad a^2 = cb(acb)^{2q-1} + \lambda(cba)^{2q}$$

with $\lambda \in \mathsf{k}$ unknown at this point. If $\lambda \neq 0$ then any non-trivial grading on this algebra has $|a| = 0$ and $|b| + |c| = 0$, which then induces the trivial grading on cohomology.

We begin with the case $q = 2$. In this case, we can choose $G = M_{10} = PGL^*(2,9)$. Running the following Magma code, we find that the socle constant in the relations for the algebra of type $SD(2\mathcal{A})_2$ is equal to one.

```
SetSeed(1441119655);
M11:=Group("M11");
M10:=Stabiliser(M11,1);
A:=BasicAlgebraOfPrincipalBlock(M10,GF(2));
e:=IdempotentGenerators(A)[1];
f:=IdempotentGenerators(A)[2];
a:=NonIdempotentGenerators(A)[1];
b:=NonIdempotentGenerators(A)[2];
```

```
c:=NonIdempotentGenerators(A)[3];
cc:=c*(1+a^5);
aa:=a+b*cc;
bb:=b;
cc*bb eq 0;
(aa*bb*cc)^4 eq (bb*cc*aa)^4;
aa^2 eq (bb*cc*aa)^3*bb*cc + (bb*cc*aa)^4;
```

It follows that there is no useful grading on the basic algebra, so we are going to have to resort to other means.

Martino [**178**] computed the cohomology ring for groups in Case 3.5.3 to be

$$H^*BG = \mathsf{k}[y, z, w, v]/(y^3, vy, yz, v^2 + z^2w)$$

with $|y| = -1$, $|z| = -3$, $|w| = -4$, $|v| = -5$. Part of the A_∞ structure is given by

$$m_3(z, y, y^2) = v, \qquad m_4(y^2, y, y^2, y) = w,$$
$$m_{4q-1}(v, y, v, y, \dots, y, v) = w^{2q}y^2.$$

The computation of the Ext ring is similar to the case of the semidihedral group.

Theorem 3.15.1. *In Case 3.5.3 we have*

$$\mathsf{Ext}^{*,*}_{H^*BG}(\mathsf{k},\mathsf{k}) = \Lambda(\hat{w}) \otimes \mathsf{k}\langle \hat{y}, \hat{z}, \hat{v}, \hat{\eta} \mid \hat{y}^2 = \hat{z}^2 = 0, \hat{v}\hat{z} = \hat{z}\hat{v}, \eta\hat{y} = \hat{y}\eta\rangle$$

with $\eta = \langle \hat{y}, \hat{y}, \hat{y}\rangle$, $|\hat{y}| = (-1, 1)$, $|\hat{z}| = (-1, 3)$, $|\hat{w}| = (-1, 4)$, $|\hat{v}| = (-1, 5)$, $|\eta| = (-2, 3)$.

In the Eilenberg–Moore spectral sequence

$$\mathsf{Ext}^{*,*}_{H^*BG}(\mathsf{k},\mathsf{k}) \Rightarrow H_*\Omega BG^\wedge_2$$

we have $d^2(\hat{v}) = \eta\hat{z} + \hat{z}\eta$,

$$E^3 = \Lambda(\hat{w}) \otimes \mathsf{k}[\eta] \otimes \mathsf{k}\langle \hat{y}, \hat{z} \mid \hat{y}^2 = \hat{z}^2 = 0\rangle,$$

then $d^3(\hat{w}) = \eta^2$,

$$E^4 = E^\infty = \Lambda(\eta) \otimes \mathsf{k}\langle \hat{y}, \hat{z} \mid \hat{y}^2 = \hat{z}^2 = 0\rangle.$$

Ungrading, we have $|\hat{y}| = 0$, $|\eta| = 1$, $|\hat{z}| = 2$, and since there are no lower terms in the filtration, we have $\hat{y}^2 = 0$, and $\eta^2 = 0$. However, the relation $\hat{z}^2 = 0$ is harder to ungrade.

To compute the ring structure of $H_*\Omega BG^\wedge_2$, and in particular the square of $\hat{z}$, we resort to the method of squeezed resolutions described in Benson [**16**]. It is easy to compute the minimal squeezed projective resolution for G, which is as follows.

$$\cdots \xrightarrow{\left(\begin{smallmatrix}(\bar{c}\bar{a}\bar{b})^{2q} & 0 \\ 0 & (\bar{c}\bar{a}\bar{b})^{2q}\end{smallmatrix}\right)} P_{\mathsf{M}} \oplus P_{\mathsf{M}} \xrightarrow{\left(\begin{smallmatrix}\bar{c}\bar{a}\bar{b} & 0 \\ 0 & \bar{c}\bar{a}\bar{b}\end{smallmatrix}\right)} P_{\mathsf{M}} \oplus P_{\mathsf{M}} \xrightarrow{\left(\begin{smallmatrix}(\bar{c}\bar{a}\bar{b})^{2q} & 0 \\ 0 & (\bar{c}\bar{a}\bar{b})^{2q}\end{smallmatrix}\right)} P_{\mathsf{M}} \oplus P_{\mathsf{M}} \xrightarrow{(\bar{b}\ \bar{a}\bar{b})} P_{\mathsf{k}} \to 0.$$

After the first step, this repeats with period two. Indeed, after the first step, it decomposes as a direct sum of two copies of the two-periodic complex

$$\cdots \xrightarrow{(\bar{c}\bar{a}\bar{b})^{2q}} P_{\mathsf{M}} \xrightarrow{\bar{c}\bar{a}\bar{b}} P_{\mathsf{M}} \xrightarrow{(\bar{c}\bar{a}\bar{b})^{2q}} P_{\mathsf{M}}.$$

The point here is that $C_*\Omega BG^\wedge_2$ is quasi-isomorphic to the $\mathsf{k}G$-endomorphism DG algebra of this squeezed resolution.

REMARK 3.15.2. There is an error in [**16**], which only shows up if $\Omega BG^\wedge_p$ is not connected, namely when $G/O^p(G)$ is not trivial. Namely, the augmentation in Theorem 3.4 of that paper should be to $\mathsf{k}G/O^p(G)$ rather than to k. This affects the computation of products in Section 4.6 of the paper, which we are using here.

The element $\hat{y} \in H_*\Omega BG^\wedge_2$ is represented by the map of complexes

$$\begin{tikzcd}
\cdots \arrow[r] & P_{\mathsf{M}} \oplus P_{\mathsf{M}} \arrow[r] \arrow[d, "\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)"] & P_{\mathsf{M}} \oplus P_{\mathsf{M}} \arrow[r] \arrow[d, "\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)"] & P_{\mathsf{k}} \arrow[d, "(\bar{a})"] \\
\cdots \arrow[r] & P_{\mathsf{M}} \oplus P_{\mathsf{M}} \arrow[r] & P_{\mathsf{M}} \oplus P_{\mathsf{M}} \arrow[r] & P_{\mathsf{k}}
\end{tikzcd}$$

The square of this map is not zero, but is null homotopic, with homotopy u given by

$$\begin{tikzcd}
\cdots \arrow[r] & P_{\mathsf{M}} \oplus P_{\mathsf{M}} \arrow[r] \arrow[d, "\left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)"] \arrow[dl, "(\bar{c}\bar{a}\bar{b})^{2q-1}\left(\begin{smallmatrix} 0 & 1 \\ 0 & \lambda \end{smallmatrix}\right)"'] & P_{\mathsf{M}} \oplus P_{\mathsf{M}} \arrow[r] \arrow[d, "\left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)"] \arrow[dl, "\left(\begin{smallmatrix} 0 & 1 \\ 0 & \lambda \end{smallmatrix}\right)"'] & P_{\mathsf{k}} \arrow[d, "(\bar{a}^2)"] \arrow[dl, "(\bar{c}\bar{a}\bar{b})^{2q-1}\bar{c}\left(\begin{smallmatrix} 1 \\ \lambda \end{smallmatrix}\right)"'] \\
\cdots \arrow[r] & P_{\mathsf{M}} \oplus P_{\mathsf{M}} \arrow[r] & P_{\mathsf{M}} \oplus P_{\mathsf{M}} \arrow[r] & P_{\mathsf{k}}
\end{tikzcd}$$

The element $\langle \hat{y}, \hat{y}, \hat{y} \rangle$ is represented by the map of complexes $u\hat{y} + \hat{y}u$:

$$\begin{tikzcd}
\cdots \arrow[r] & P_{\mathsf{M}} \oplus P_{\mathsf{M}} \arrow[r] \arrow[d, "\left(\begin{smallmatrix} 1 & 0 \\ \lambda & 1 \end{smallmatrix}\right)"] & P_{\mathsf{M}} \oplus P_{\mathsf{M}} \arrow[r] \arrow[d, "(\bar{c}\bar{a}\bar{b})^{2q-1}\left(\begin{smallmatrix} 1 & 0 \\ \lambda & 1 \end{smallmatrix}\right)"] & P_{\mathsf{M}} \oplus P_{\mathsf{M}} \arrow[r] \arrow[d, "\left(\begin{smallmatrix} 1 & 0 \\ \lambda & 1 \end{smallmatrix}\right)"] & P_{\mathsf{k}} \arrow[d, "(\bar{c}\bar{a}\bar{b})^{2q-1}\bar{c}\left(\begin{smallmatrix} \bar{a} \\ 1+\lambda\bar{a} \end{smallmatrix}\right)"] \\
\cdots \arrow[r] & P_{\mathsf{M}} \oplus P_{\mathsf{M}} \arrow[r] & P_{\mathsf{M}} \oplus P_{\mathsf{M}} \arrow[r] & P_{\mathsf{M}} \oplus P_{\mathsf{M}} \arrow[r] & P_{\mathsf{M}} \oplus P_{\mathsf{M}} \arrow[r] & P_{\mathsf{k}} \oplus P_{\mathsf{k}}
\end{tikzcd}$$

and the element $\hat{z}$ is represented by the map

$$\begin{tikzcd}
\cdots \arrow[r] & P_{\mathsf{M}} \oplus P_{\mathsf{M}} \arrow[r] \arrow[d, "\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)"] & P_{\mathsf{M}} \oplus P_{\mathsf{M}} \arrow[r] \arrow[d, "\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)"] & P_{\mathsf{k}} \arrow[d, "\left(\begin{smallmatrix} \bar{c} \\ 0 \end{smallmatrix}\right)"] \\
\cdots \arrow[r] & P_{\mathsf{M}} \oplus P_{\mathsf{M}} \arrow[r] & P_{\mathsf{M}} \oplus P_{\mathsf{M}} \arrow[r] & P_{\mathsf{M}} \oplus P_{\mathsf{M}} \arrow[r] & P_{\mathsf{M}} \oplus P_{\mathsf{M}} \arrow[r] & P_{\mathsf{k}}
\end{tikzcd}$$

Now $u\hat{y} + \hat{y}u$ does not commute with $\hat{z}$, but $(u\hat{y} + \hat{y}u)(1 + \lambda\hat{y})$ does, so it is more convenient to set

$$\eta = (u\hat{y} + \hat{y}u)(1 + \lambda\hat{y}).$$

This is represented by the map

$$\begin{tikzcd}
\cdots \arrow[r] & P_{\mathsf{M}} \oplus P_{\mathsf{M}} \arrow[r] \arrow[d, "\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)"] & P_{\mathsf{M}} \oplus P_{\mathsf{M}} \arrow[r] \arrow[d, "(\bar{c}\bar{a}\bar{b})^{2q-1}\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)"] & P_{\mathsf{M}} \oplus P_{\mathsf{M}} \arrow[r] \arrow[d, "\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)"] & P_{\mathsf{k}} \arrow[d, "(\bar{c}\bar{a}\bar{b})^{2q-1}\bar{c}\left(\begin{smallmatrix} \bar{a} \\ 1 \end{smallmatrix}\right)"] \\
\cdots \arrow[r] & P_{\mathsf{M}} \oplus P_{\mathsf{M}} \arrow[r] & P_{\mathsf{M}} \oplus P_{\mathsf{M}} \arrow[r] & P_{\mathsf{M}} \oplus P_{\mathsf{M}} \arrow[r] & P_{\mathsf{M}} \oplus P_{\mathsf{M}} \arrow[r] & P_{\mathsf{k}} \oplus P_{\mathsf{k}}
\end{tikzcd}$$

Thus $\hat{z}^2 = 0$, and $\hat{y}\hat{z} + \hat{z}\hat{y}$ is the periodicity generator of degree two, central in the endomorphism algebra:

$$\begin{tikzcd}
\cdots \arrow[r] & P_{\mathsf{M}} \oplus P_{\mathsf{M}} \arrow[r] \arrow[d, "\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)"] & P_{\mathsf{M}} \oplus P_{\mathsf{M}} \arrow[r] \arrow[d, "\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)"] & P_{\mathsf{k}} \arrow[d, "\left(\begin{smallmatrix}\bar{c}\bar{a}\\ \bar{c}\end{smallmatrix}\right)"] & & \\
\cdots \arrow[r] & P_{\mathsf{M}} \oplus P_{\mathsf{M}} \arrow[r] & P_{\mathsf{M}} \oplus P_{\mathsf{M}} \arrow[r] & P_{\mathsf{M}} \oplus P_{\mathsf{M}} \arrow[r] & P_{\mathsf{M}} \oplus P_{\mathsf{M}} \arrow[r] & P_{\mathsf{k}}
\end{tikzcd}$$

THEOREM 3.15.3. *In Case 3.5.3, we have*

$$H_*\Omega BG_2^{\wedge} = \Lambda(\eta) \otimes \mathsf{k}\langle \hat{y}, \hat{z} \mid \hat{y}^2 = \hat{z}^2 = 0\rangle$$

with $|\eta| = 1$, $|\hat{y}| = 0$ *and* $|\hat{z}| = 2$. □

It is interesting to note that the degree zero element $\hat{y}$ is not central in $H_*\Omega BG_2^{\wedge}$, while $\hat{y}\hat{z} + \hat{z}\hat{y}$ is the central periodicity generator. This is very similar to what happens for groups with dihedral Sylow 2-subgroups in Case 2.7.2.

Part of the A_∞ structure is given by $m_3(\hat{y}, \hat{y}, \hat{y}) = \eta(1 + \lambda\hat{y})$, $m_{2q}(\eta, \ldots, \eta) = (\hat{y}\hat{z} + \hat{z}\hat{y})^{2q}$.

CHAPTER 4

The generalised quaternion case

4.1. Introduction

In this chapter, we discuss the case of finite groups with generalised quaternion Sylow 2-subgroups.

We begin with the generalised quaternion group Q of order $8q$ itself. The group algebra in this case was analysed by Dade [**59**], and we describe a modified version of his presentation as a quiver with relations. If $q = 1$ and k contains $\mathbb{F}_4$ then for suitable radical generators we have

$$\mathsf{k}\mathrm{Q} = \mathsf{k}\langle X, Y \mid X^2 = YXY, \quad Y^2 = XYX, \quad X^4 = Y^4 = 0\rangle.$$

If $q \geqslant 2$, and k is any field of characteristic two, for suitable radical generators X and Y we have

$$\mathsf{k}\mathrm{Q} = \mathsf{k}\langle X, Y \mid X^2 = (YX)^{2q-1}Y + (XY)^{2q},\ Y^2 = (XY)^{2q-1}X + (YX)^{2q},\ X^4 = Y^4 = 0\rangle.$$

See Theorem 4.2.1 for details.

There are three cases for the possible fusion in Q, leading to three types of cochains on the classifying space of a finite group with this fusion. Probably the most interesting is the case where G has no normal subgroup of index two.

THEOREM 4.1.1. *Let G be a finite group with generalised quaternion Sylow 2-subgroup, and let k be a field of characteristic two. Then the following are equivalent:*

1. *the A_∞ algebra C^*BG is formal,*
2. *the A_∞ algebra $H_*\Omega BG^\wedge_2$ is formal,*
3. *G has no normal subgroup of index two.*

This theorem follows from Theorem 4.7.4, Corollary 4.7.5, and Theorem 4.9.2.

4.2. Generalised quaternion groups

The generalised quaternion group of order $8q$, q a power of two, is given by the presentation

$$\mathrm{Q} = \langle g, h \mid g^2 = h^2 = (g^{-1}h)^{2q}\rangle.$$

These relations imply that $g^2 = h^2$ is central, and $g^4 = h^4 = 1$. If $q = 1$, this is the quaternion group of order eight.

THEOREM 4.2.1. *We have the following presentations for $\mathsf{k}\mathrm{Q}$.*

(i) *In the case $q = 1$, suppose that k contains $\mathbb{F}_4 = \{0, 1, \omega, \bar{\omega}\}$, with $1 + \omega + \omega^2 = 0$. Set*

$$X = gh + \omega g + \bar{\omega}h, \qquad Y = gh + \bar{\omega}g + \omega h.$$

Then

$$\mathsf{k}\mathrm{Q} = \mathsf{k}\langle X, Y \mid X^2 = YXY, \quad Y^2 = XYX, \quad X^4 = Y^4 = 0\rangle.$$

The automorphism $g \mapsto h \mapsto gh \mapsto g$ *of* Q *of order three sends* $X \mapsto \bar{\omega}X$ *and* $Y \mapsto \omega Y$.

(ii) *For* $q \geqslant 2$, *and any field* k *of characteristic two, set*

$$u = g + h, \quad v = u^{4q-3} + \sum_{2^i=2}^{q} u^{2q-2^i}, \quad x = (g+1) + u, \quad y = (h+1) + u,$$

and finally, $X = x + (xy)^{2q-1}$, $Y = y + (yx)^{2q-1}$. *Then the group algebra has the following presentation:*

$$\mathsf{k}\mathrm{Q} = \mathsf{k}\langle X, Y \mid X^2 = (YX)^{2q-1}Y + (XY)^{2q},\ Y^2 = (XY)^{2q-1}X + (YX)^{2q},\ X^4 = Y^4 = 0\rangle.$$

These relations imply that $(XY)^{2q} = X^3 = (YX)^{2q} = Y^3$ *is annihilated by* X *and* Y, *and hence lie in* $\mathsf{Soc}(\mathsf{k}\mathrm{Q}) = J^{2q}(\mathsf{k}\mathrm{Q})$.

Proof. This follows Dade [**59**], with a change of variables in the second case. First, in both cases X and Y are in $J(\mathsf{k}\mathrm{Q})$ and are linearly independent modulo $J^2(\mathsf{k}\mathrm{Q})$, and therefore generate $\mathsf{k}\mathrm{Q}$.

(i) A somewhat long computation shows that $X^2 = (1+g^2)(gh+\omega h+\bar{\omega}g) = YXY$, and hence $X^4 = 0$. Applying the automorphism of $\mathbb{F}_4$, we get $Y^2 = XYX$ and $Y^4 = 0$. These relations imply that $\mathsf{k}\mathrm{Q}$ is spanned by 1, X, Y, XY, YX, XYX, YXY and $XYXY = YXYX$, so comparing dimensions, these relations define $\mathsf{k}\mathrm{Q}$.

(ii) By [**59**], the elements x and y in $J(\mathsf{k}\mathrm{Q})$ satisfy

$$\mathsf{k}\mathrm{Q} = \langle x, y \mid x^2 = y^2 = (xy)^{2q-1}x + (yx)^{2q-1}y + (xy)^{2q}, \quad x^4 = y^4 = 0\rangle.$$

These relations imply that $(xy)^{2q} = x^3 = (yx)^{2q} = y^3$ spans $\mathsf{Soc}(\mathsf{k}\mathrm{Q}) = J^{2q}(\mathsf{k}\mathrm{Q})$, $x^2 = y^2$ is central, and $J^{2q+1}(\mathsf{k}\mathrm{Q}) = 0$. Since X and Y are congruent to x and y modulo $J^{2q-1}(\mathsf{k}\mathrm{Q})$, it follows that monomials in X and Y of length at least three are equal to the corresponding monomials in x and y. So X and Y satisfy

$$\begin{aligned} X^2 &= x^2 + (xy)^{2q-1}x = (yx)^{2q-1}y + (xy)^{2q} = (YX)^{2q-1}Y + (XY)^{2q} \\ Y^2 &= y^2 + (yx)^{2q-1}x = (xy)^{2q-1}x + (yx)^{2q} = (XY)^{2q-1}Y + (YX)^{2q}, \end{aligned}$$

and $X^4 = Y^4 = 0$. Note that unlike x^2 and y^2, the elements X^2 and Y^2 are not central. □

Remark 4.2.2. It is erroneously stated on page 303 of [**74**], page 38 of [**94**], and page 518 of [**118**] that the group algebra of the generalised quaternion group is as given here, but without the extra term $(XY)^{2q}$, $(YX)^{2q}$ in the expressions for X^2 and Y^2.

Remark 4.2.3. In the case of the quaternion group of order eight, over a field containing $\mathbb{F}_4$, there is a $\mathbb{Z}/3$-grading on the group algebra given by $|X| = 1$ and $|Y| = -1$. This is the grading induced by the automorphism of order three.

In the case of the generalised quaternion groups of order at least 16, there is no non-trivial grading on the group algebra for which the generators X and Y are homogeneous, because of the socle terms in the relations.

It is known that $\mathsf{k}\mathrm{Q}$ has tame representation type, by embedding Q into a semidihedral group of twice the order. By a theorem of Green [**131**, Theorem 8], as long as k is algebraically closed, inducing an indecomposable $\mathsf{k}\mathrm{Q}$-module gives an indecomposable module for the semidihedral group. On the other hand, although the indecomposables for the semidihedral group are classified, nobody has been able to use this to classify indecomposable modules for generalised quaternion groups.

The cohomology ring for $q = 1$ is

$$H^*BG = \mathsf{k}[u, v, z]/(u^2 + uv + v^2, u^2v + uv^2)$$

with $|u| = |v| = 1$ and $|z| = 4$, and with u and v dual to $U = \bar{\omega}X + \omega Y = gh + h$ and $V = \omega X + \bar{\omega}Y = gh + g$.

For $q \geqslant 2$, we have

$$H^*BG = \mathsf{k}[x, y, z]/(xy, x^3 + y^3),$$

again with $|x| = |y| = 1$ and $|z| = 4$, and with x and y dual to X and Y. See for example Rusin [**197**] or Martino and Priddy [**179**].

Note that if k contains $\mathbb{F}_4$ then the cohomology of the quaternion group of order eight can be made to fit the same pattern by using the elements x, y in H^1BG dual to X and Y in $J(\mathsf{k}\mathrm{Q})$. These are homogeneous with respect to the grading described in Remark 4.2.3, so that the $\mathbb{Z} \times \mathbb{Z}/3$-grading is given by $|x| = (-1, -1)$, $|y| = (-1, 1)$ and $|z| = (-4, 0)$.

4.3. $HH^*H^*B\mathrm{Q}$

The cohomology ring $H^*B\mathrm{Q} = \mathsf{k}[x, y, z]/(xy, x^3 + y^3)$ is a complete intersection of codimension two, so we can calculate $HH^*H^*B\mathrm{Q}$ and $\mathsf{Ext}^{*,*}_{H^*B\mathrm{Q}}(\mathsf{k}, \mathsf{k})$ using Theorems 1.11.5 and 1.11.2. We first compute $\mathsf{Cliff}(\mathsf{q})$.

PROPOSITION 4.3.1. *Let* Q *be a generalised quaternion group of order* $8q$ *with* q *a power of two. Let* k *be a field of characteristic two, and if* $q = 1$*, we suppose that* k *contains* $\mathbb{F}_4$*. Then the algebra* $\mathsf{Cliff}(\mathsf{q})$ *is equal to* $H^*B\mathrm{Q}\langle \hat{x}, \hat{y}, \hat{z}; s_1, s_2\rangle$*, where* s_1 *and* s_2 *are central, and*

$$\hat{x}^2 = \hat{y}^2 = \hat{z}^2 = 0, \qquad \hat{x}\hat{y} + \hat{y}\hat{x} = s_1, \qquad \hat{x}\hat{z} = \hat{z}\hat{x}, \qquad \hat{y}\hat{z} = \hat{z}\hat{y}.$$

The degrees are given by $|\hat{x}| = |\hat{y}| = (-1, 1)$*,* $|\hat{z}| = (-1, 4)$*,* $|s_1| = (-2, 2)$*,* $|s_2| = (-2, 3)$*. The differential* d *on* $\mathsf{Cliff}(\mathsf{q})$ *is given by*

$$d(\hat{x}) = ys_1 + x^2s_2, \qquad d(\hat{y}) = xs_1 + y^2s_2, \qquad d(\hat{z}) = d(s_1) = d(s_2) = 0.$$

PROOF. Let $f_1(x, y, z) = xy$ and $f_2(x, y, z) = x^3 + y^3$, so that $H^*B\mathrm{Q} = \mathsf{k}[x, y, z]/(f_1, f_2)$. Then we have

$$\begin{gathered}
\frac{\partial f_1}{\partial x} = y, \quad \frac{\partial f_1}{\partial y} = x, \quad \frac{\partial f_1}{\partial z} = 0, \quad \frac{\partial f_2}{\partial x} = x^2, \quad \frac{\partial f_2}{\partial y} = y^2, \quad \frac{\partial f_2}{\partial z} = 0, \\
\frac{\partial^{(2)} f_1}{\partial x^2} = 0, \quad \frac{\partial^{(2)} f_1}{\partial y^2} = 0, \quad \frac{\partial^{(2)} f_1}{\partial z^2} = 0, \quad \frac{\partial^{(2)} f_2}{\partial x^2} = x, \quad \frac{\partial^{(2)} f_2}{\partial y^2} = y, \quad \frac{\partial^{(2)} f_2}{\partial z^2} = 0, \\
\frac{\partial^2 f_1}{\partial x \partial y} = 1, \quad \frac{\partial^2 f_1}{\partial x \partial z} = 0, \quad \frac{\partial^2 f_1}{\partial y \partial z} = 0, \quad \frac{\partial^2 f_2}{\partial x \partial y} = 0, \quad \frac{\partial^2 f_2}{\partial x \partial z} = 0, \quad \frac{\partial^2 f_2}{\partial y \partial z} = 0.
\end{gathered}$$

Plugging these into Definition 1.11.1, with s_1 and s_2 the degree -2 generators corresponding to the relations f_1 and f_2, we get the given relations and differential for $\mathsf{Cliff}(\mathsf{q})$. $\square$

REMARK 4.3.2. In the case of the quaternion group of order eight without assuming that k contains $\mathbb{F}_4$, the algebra $\mathsf{Cliff}(\mathsf{q})$ is equal to $H^*B\mathrm{Q}\langle \hat{u}, \hat{v}, \hat{z}; s_1, s_2\rangle$, where s_1 and s_2 are central, and

$$\hat{u}^2 = \hat{v}^2 = \hat{u}\hat{v} + \hat{v}\hat{u} = s_1, \qquad \hat{u}\hat{z} = \hat{z}\hat{u}, \qquad \hat{v}\hat{z} = \hat{z}\hat{v}.$$

The degrees are given by $|\hat{u}| = |\hat{v}| = (-1, 1)$, $|\hat{z}| = (-1, 4)$, $|s_1| = (-2, 2)$, $|s_2| = (-2, 3)$. The differential d on $\mathsf{Cliff}(\mathsf{q})$ is given by

$$d(\hat{u}) = vs_1 + v^2s_2, \qquad d(\hat{v}) = us_1 + u^2s_2, \qquad d(\hat{z}) = d(s_1) = d(s_2) = 0.$$

THEOREM 4.3.3. *The Hochschild cohomology $HH^*H^*B\mathrm{Q}$ is*

$$H^*B\mathrm{Q}[\hat{z}, s_1, s_2, w_1, w_2]/(x^2w_1 + yw_2, y^2w_1 + xw_2,$$
$$xs_1 + y^2s_2, ys_1 + x^2s_2, w_1s_1 + w_2s_2, w_1^2, w_2^2, w_1w_2, \hat{z}^2).$$

where

$$w_1 = x\hat{x} + y\hat{y}, \qquad w_2 = y^2\hat{x} + x^2\hat{y}.$$

The generators have degrees $|\hat{z}| = (-1, 4)$, $|s_1| = (-2, 2)$, $|s_2| = (-2, 3)$, $|w_1| = (-1, 0)$, $|w_2| = (-1, -1)$.

PROOF. This follows from Theorem 1.11.5 and Proposition 4.3.1 □

4.4. Loops on $B\mathrm{Q}_2^\wedge$

Since Q is a finite 2-group, we have $\Omega B\mathrm{Q}_2^\wedge \simeq Q$. So we should expect to see the Eilenberg–Moore spectral sequence converging to $\mathsf{k}\mathrm{Q}$.

THEOREM 4.4.1. *If* Q *is a generalised quaternion group of order $8q$ with either $q \geqslant 2$ or* k *containing $\mathbb{F}_4$, then*

$$\mathsf{Ext}^{*,*}_{H^*B\mathrm{Q}}(\mathsf{k}, \mathsf{k}) \cong \mathsf{k}\langle \hat{x}, \hat{y} \mid \hat{x}^2 = \hat{y}^2 = 0\rangle \otimes \mathsf{k}[\hat{z}, s]/(\hat{z}^2).$$

The degrees are given by $|\hat{x}| = |\hat{y}| = (-1, 1)$, $|\hat{z}| = (-1, 4)$ and $|s| = (-2, 3)$.

If Q *is a quaternion group of order eight then*

$$\mathsf{Ext}^{*,*}_{H^*B\mathrm{Q}}(\mathsf{k}, \mathsf{k}) \cong \mathsf{k}\langle \hat{u}, \hat{v} \mid \hat{u}^2 = \hat{v}^2 = \hat{u}\hat{v} + \hat{v}\hat{u}\rangle \otimes \mathsf{k}[\hat{z}, s]/(\hat{z}^2).$$

The degrees are given by $|\hat{u}| = |\hat{u}| = (-1, 1)$, $|\hat{z}| = (-1, 4)$ and $|s| = (-2, 3)$.

PROOF. In both cases, $H^*B\mathrm{Q}$ is a complete intersection, so we compute the Ext ring using Theorem 1.11.2. The algebra $\mathsf{Cliff}(\mathsf{q})$ is given by Proposition 4.3.1 and Remark 4.3.2. The generator s_1 is redundant, so we eliminate it, and we write s for s_2. □

For $q = 1$, the differentials in the Eilenberg–Moore spectral sequence

$$\mathsf{Ext}^{*,*}_{H^*B\mathrm{Q}}(\mathsf{k}, \mathsf{k}) \Rightarrow \mathsf{k}\mathrm{Q}$$

are given by $d^2(s) = \hat{u}^4 = \hat{v}^4 = (\hat{u}\hat{v} + \hat{v}\hat{u})^2$ and $d^3(\hat{z}) = s^2$.

If k contains $\mathbb{F}_4$ then we can set $\hat{x} = \bar{\omega}\hat{u} + \omega\hat{v}$ and $\hat{y} = \omega\hat{u} + \bar{\omega}\hat{v}$, so that $\mathsf{Ext}^{*,*}_{H^*B\mathrm{Q}}(\mathsf{k}, \mathsf{k})$ becomes

$$\mathsf{k}\langle \hat{x}, \hat{y} \mid \hat{x}^2 = \hat{y}^2 = 0\rangle \otimes \mathsf{k}[\hat{z}, s]/(\hat{z}^2).$$

We have $\hat{u}\hat{v} + \hat{v}\hat{u} = \hat{x}\hat{y} + \hat{y}\hat{x}$, and $d^2(s) = (\hat{x}\hat{y} + \hat{y}\hat{x})^2$, and $d^2(s^2) = 0$. Then

$$E^3 = \mathsf{k}\langle \hat{x}, \hat{y} \mid \hat{x}^2 = \hat{y}^2 = 0, (\hat{x}\hat{y})^2 = (\hat{y}\hat{x})^2\rangle \otimes k[\hat{z}, s^2]/(\hat{z}^2).$$

The differential $d^3(\hat{z}) = s^2$ then gives

$$E^4 = E^\infty = \mathsf{k}\langle \hat{x}, \hat{y} \mid \hat{x}^2 = \hat{y}^2 = 0, (\hat{x}\hat{y})^2 = (\hat{y}\hat{x})^2\rangle.$$

This is the associated graded of kQ with respect to the radical filtration, with $\hat{x}$ representing X and $\hat{y}$ representing Y.

The Eilenberg–Moore spectral sequence in the case $q \geqslant 2$ is similar, but the differentials happen in the opposite order. The first differential is $d^3(\hat{z}) = s^2$, giving

$$E^4 = \mathsf{k}\langle \hat{x}, \hat{y} \mid \hat{x}^2 = \hat{y}^2 = 0\rangle \otimes k[s]/(s^2).$$

Then the next non-zero differential is $d^{4q-2}(s) = (\hat{x}\hat{y} + \hat{y}\hat{x})^{2q}$, so that

$$E^{4q-1} = E^\infty = \mathsf{k}\langle \hat{x}, \hat{y} \mid \hat{x}^2 = \hat{y}^2 = 0, (\hat{x}\hat{y})^{2q} = (\hat{y}\hat{x})^{2q}\rangle.$$

This is again the associated graded of kQ with respect to the radical filtration, with $\hat{x}$ representing X and $\hat{y}$ representing Y.

4.5. Isoclinism

In order to describe the finite groups with generalised quaterion Sylow 2-subgroups in the next section, we first discuss isoclinism and the groups $SL^\circ(2, p^m)$, which are isoclinic to the groups $SL^\pm(2, p^m)$ described in Section 3.5. We restrict ourselves to the situation we need, rather than describing isoclinism in general.

Suppose that G is a finite group with a central subgroup of order two, $Z = \{1, z\}$, contained in a normal subgroup H of index $|G : H| = 2$. Then we can make a new group of the same order as follows. Consider the maps

$$\mathbb{Z}/2 \to G \times \mathbb{Z}/4 \to \mathbb{Z}/2,$$

where $\mathbb{Z}/4 = \langle \gamma \mid \gamma^4 = 1\rangle$, the first map sends the generator of $\mathbb{Z}/2$ to (z, γ^2), and the second map sends G to $\mathbb{Z}/2$ surjectively with kernel H, and $\mathbb{Z}/4$ to $\mathbb{Z}/2$ with kernel $\langle \gamma^2\rangle$. Let $\tilde{G}$ be the kernel of the second map modulo the image of the first. Then $\tilde{G}$ has the same order as G, and is said to be *isoclinic* to G. The group $\tilde{G}$ has a normal subgroup of index two isomorphic to H, and a central subgroup of order two generated by $(z, 1)$ with $\tilde{G}/\langle (z, 1)\rangle \cong G/\langle z\rangle$, but G and $\tilde{G}$ are not in general isomorphic.

If $\rho\colon G \to GL(n, \mathbb{C})$ is a complex representation of G with z represented as minus the identity, then we can obtain a complex representation of $\tilde{G}$ by sending elements of the subgroup of index two isomorphic to H to the same matrices as before, but the elements outside are multiplied by the complex number i. The character table of $\tilde{G}$ therefore looks just like that of G except that the character values of elements outside H on the representations with z acting as minus the identity have been multiplied by i. In particular, the character degrees are the same.

As an example, let G be a semidihedral group of order $8q$, $q \geqslant 2$, with presentation

$$G = \langle g, h \mid g^{4q} = 1, h^2 = 1, hgh^{-1} = g^{2q-1}\rangle$$

as in Section 3.2. The element z is g^{2q}, and the normal subgroup H of index two is the (generalised) quaternion subgroup generated by g^2 and gh. Then $\tilde{G}$ is generated by $\tilde{g} = g\gamma$ and $\tilde{h} = h\gamma$, with g^{2q} identified with $\gamma^2 = \tilde{h}^2$. We have

$$\tilde{h}\tilde{g}\tilde{h}^{-1} = \gamma hgh^{-1} = \gamma g^{2q-1} = \gamma^2\tilde{g}^{2q-1} = \tilde{g}^{-1}.$$

Thus

$$\tilde{G} = \langle \tilde{g}, \tilde{h} \mid \tilde{g}^{4q} = 1, \tilde{h}^2 = \tilde{g}^{2q}, \tilde{h}\tilde{g}\tilde{h}^{-1} = \tilde{g}^{-1} \rangle$$

which is a presentation for the generalised quaternion group of order $8q$. Thus the semidihedral group and generalised quaternion group of the same order are isoclinic.

Now let p^m be an odd prime power, and let us apply the same process to the groups $SL^{\pm}(2, p^m)$ and $SU^{\pm}(2, p^m)$ described in Section 3.5. These have centres of order two, and normal subgroups $SL(2, p^m)$ and $SU(2, p^m)$ of index two with (generalised) quaternion Sylow 2-subgroups. These data allow us to define isoclinic groups, which we shall denote $SL^{\circ}(2, p^m)$ and $SU^{\circ}(2, p^m)$, of the same orders as $SL^{\pm}(2, p^m)$ and $SU^{\pm}(2, p^m)$. The groups $SL^{\circ}(2, p^m)$ with $p^m \equiv 3 \pmod 4$ and $SU^{\circ}(2, p^m)$ with $p^m \equiv 1 \pmod 4$ have generalised quaternion Sylow 2-subgroups, and will appear in Case 4.6.2 in the next section.

4.6. Groups with generalised quaternion Sylow 2-subgroups

Groups with generalised quaternion Sylow 2-subgroups were classified by Brauer and Suzuki [**39**], see also Section VII of Brauer [**36**], as well as Suzuki [**208**], Glauberman [**124**]. The main theorem is that if G has a generalised quaternion Sylow 2-subgroup then the involution in the centre of a Sylow 2-subgroup has central image in $G/O(G)$. So the quotient of $G/O(G)$ by this central involution has dihedral Sylow 2-subgroups of order $4q$, and no odd order normal subgroups. Such groups were analysed by Gorenstein and Walter [**129**]. By Theorem 1.1 of Craven and Glesser [**54**], these also represent the only possible fusions systems on a generalised quaternion 2-group. As a consequence, there are three mutually exclusive possibilities for the fusion in G.

Case 4.6.1. If G has one class of elements of order four then $G/O(G)$ is isomorphic to either the double cover $2A_7$ of the alternating group A_7, or a subgroup of $\Gamma L(2, p^m)$ with p^m a power of an odd prime, containing $SL(2, p^m)$ with odd index. The principal block of $\mathsf{k}G$ has three isomorphism classes of simple modules.

Case 4.6.2. If G has two classes of elements of order four then G has a normal subgroup of index two, but no normal subgroup of index four. In this case, $G/O(G)$ contains a normal subgroup of odd index isomorphic to either $SL^{\circ}(2, p^m)$ with $p^m \equiv 3 \pmod 4$ or $SU^{\circ}(2, p^m)$ with $p^m \equiv 1 \pmod 4$ (see Section 4.5). The principal block of $\mathsf{k}G$ has two isomorphism classes of simple modules.

Case 4.6.3. If G has three classes of elements of order four then $O(G)$ is a normal complement in G to a Sylow 2-subgroup Q, so that $G/O(G) \cong \mathrm{Q}$ and $H^*BG \cong H^*B\mathrm{Q}$. The principal block of $\mathsf{k}G$ is isomorphic to $\mathsf{k}\mathrm{Q}$, and has one isomorphism class of simple modules, namely the trivial module.

Proposition 4.6.1. *Suppose that G has generalised quaternion Sylow 2-subgroup* Q. *Then the homotopy type of $BG^{\wedge}_2$ is determined by* $|\mathrm{Q}|$ *and the number of conjugacy classes of elements of order four.*

Proof. This follows from Theorem 1.7.5 and the classification theorem described above. $\square$

Representation theory and cohomology of groups with generalised quaternion Sylow 2-subgroups, and more generally, blocks with generalised quaternion defect groups and finite dimensional algebras of quaternion type, are discussed in Erdmann [**69**, **70**, **72**, **74**–**76**], as well as Bogdanic [**31**, **32**], Cabanes and Picaronny [**46**], Carlson, Mazza and Thévenaz [**47**], Eisele [**65**], Erdmann and Skowroński [**79**], Generalov et al. [**94**,**96**,**108**,**118**], Hayami [**135**], Holm [**142**], Holm, Kessar and Linckelmann [**144**], Ivanov et al. [**146**–**149**], Kawai and Sasaki [**153**], Kessar and Linckelmann [**156**], Koshitani and Lassueur [**161**], Langer [**164**], Martino and Priddy [**179**], Müller [**184**], Olsson [**189**], Taillefer [**210**], Zhou and Zimmermann [**214**]. The homology of $\Omega BG_2^\wedge$ was computed by Levi [**167**].

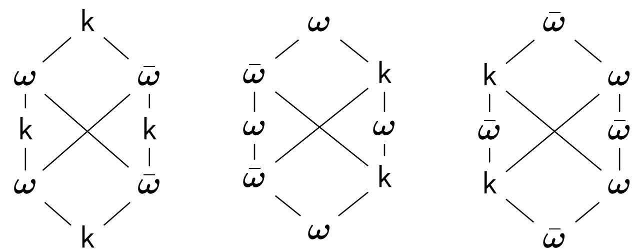
Remark 4.6.2. Let B be the principal block of $\mathsf{k}G$. In Case 4.6.1, one can put a $(\mathbb{Z}\times\mathbb{Z})$-grading on the basic algebra of B, and in Case 4.6.2, one can put a $\mathbb{Z}$-grading on the basic algebra. In Case 4.6.3, there is no nontrivial grading. However, in all cases, these gradings are unhelpful, because they induce the trivial grading on cohomology.

We end this section with a table of the various cases of algebras of quaternion type in characteristic two, in Erdmann's classification. We note some minor misprints. In the appendix to [**70**], the entry $k+2$ in the Cartan matrix for type II should be $k+s$. In the entry for type $\mathrm{Q}(3\mathcal{K})$ in the tables at the end of [**74**], the last column should say $q\equiv 3$ mod 4 rather than $q\equiv 1$ mod 4. It seems unclear what happened to type $\mathrm{Q}(2\mathcal{B})_2$ of [**74**] in the analysis of [**69**].

Erdmann [**74**]	[**69**, **70**]	Case	Group	H^*	HH^*
III.1(e)		—	—		[**94**]
III.1(e′)		4.6.3	Q_{2^n}	[**185**]	[**118**, **135**]
$\mathrm{Q}(2\mathcal{A})$	[**69**] I	4.6.2	$SU^\circ(2,p^m)$, $p^m\equiv 1 \pmod 4$	[**179**]	
$\mathrm{Q}(2\mathcal{B})_1$	[**69**] II	4.6.2	$SL^\circ(2,p^m)$, $p^m\equiv 3 \pmod 4$	[**179**]	[**96**, **108**]
$\mathrm{Q}(2\mathcal{B})_2$	?	—	—		
$\mathrm{Q}(2\mathcal{B})_3$	[**69**] II ($k=1$)	—	—		
$\mathrm{Q}(3\mathcal{A})_1$	[**70**] II	—	—		
$\mathrm{Q}(3\mathcal{A})_2$	[**70**] III	4.6.1	$SL(2,p^m)$, $p^m\equiv 1 \pmod 4$	[**179**]	
$\mathrm{Q}(3\mathcal{B})$	[**70**] IV	4.6.1	$2A_7$	[**179**]	
$\mathrm{Q}(3\mathcal{C})$	[**70**] I	—	—		
$\mathrm{Q}(3\mathcal{D})$	[**70**] V	—	—		
$\mathrm{Q}(3\mathcal{K})$	[**70**] VI	4.6.1	$SL(2,p^m)$, $p^m\equiv 3 \pmod 4$	[**179**]	

4.7. One class of elements of order four

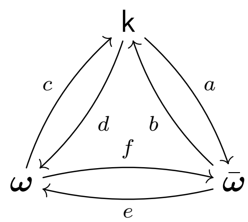
Let G be a finite group with quaternion or generalised quaternion Sylow 2-subgroups of order $8q$, and let k be a field of characteristic two. In this section we shall be interested in Case 4.6.1, and our approach will be to work directly with projective resolutions. Let us look first at the case of $SL(2,3)\cong \mathrm{Q}_8\rtimes\mathbb{Z}/3$, with $q=1$. There are three isomorphism classes of simple B-modules, all of dimension one, which we shall denote k, $\boldsymbol{\omega}$ and $\bar{\boldsymbol{\omega}}$. Their projective covers are given by the following diagrams.



P_{k} $\qquad$ $P_{\boldsymbol{\omega}}$ $\qquad$ $P_{\bar{\boldsymbol{\omega}}}$

Note that k, $\boldsymbol{\omega}$ and $\bar{\boldsymbol{\omega}}$ are all periodic with period four. The quiver for B is

(4.7.1)



with relations

$$\begin{aligned} aba &= fd, & cdc &= bf, & efe &= db, \\ bab &= ce, & dcd &= ea, & fef &= ac, \\ abf &= 0, & cdb &= 0, & efd &= 0. \end{aligned}$$

These relations imply that

(4.7.2) $$acd = bac = bfe = cef = dba = dce = eab = fdc = fea = 0,$$

as well as

(4.7.3) $$cea = bfd, \qquad eac = dbf, \qquad ace = fdb,$$

and that the composite of any five arrows is zero.

For larger values of q, we can choose a prime power $p^m \equiv 3 \pmod 4$ such that the 2-part of p^m+1 is $4q$, and take $G = SL(2,p^m)$. In this case, we label the two non-trivial simple modules M and N rather than $\boldsymbol{\omega}$ and $\bar{\boldsymbol{\omega}}$. By (1.3) of [**76**] (see also Theorem VII.8.8 of [**74**]), the structures of the projectives are similar to the above, but longer. The quiver is the same, but the relations are as follows:

$$\begin{aligned} (ab)^{2q-1}a &= fd, & (cd)^{2q-1}c &= bf, & (ef)^{2q-1}e &= db, \\ (ba)^{2q-1}b &= ce, & (dc)^{2q-1}d &= ea, & (fe)^{2q-1}f &= ac, \\ abf &= 0, & cdb &= 0, & efd &= 0. \end{aligned}$$

Again, these imply relations (4.7.2) and (4.7.3), and that the composite of any $4q+1$ arrows is zero.

We have

$$B \cong \mathsf{End}_B(B)^{\mathsf{op}} = \mathsf{End}_B(P_{\mathsf{k}} \oplus P_{\mathsf{M}} \oplus P_{\mathsf{N}})^{\mathsf{op}},$$

and we write $\bar a$ for the element of $\mathsf{Hom}_B(P_\mathsf{N}, P_\mathsf{k})$ opposite to a, and so on. The relations satisfied by these are obtained by reversing those satisfied by the original elements.

Theorem 4.7.4. *Let G be a finite group with quaternion or generalised quaternion Sylow 2-subgroups and no normal subgroup of index two. Then*

$$H^*BG = \Lambda(y) \otimes k[z]$$

is a formal A_∞ structure. The degrees of the generators are $|y| = -3$ and $|z| = -4$.

Proof. The minimal resolution P_* of the trivial module over B is given by

$$\cdots \xrightarrow{(\bar d,\bar a)} P_\mathsf{k} \xrightarrow{\bar a\bar e\bar c=\bar d\bar f\bar b} P_\mathsf{k} \xrightarrow{\binom{\bar c}{\bar b}} P_\mathsf{M}\oplus P_\mathsf{N} \xrightarrow{\begin{pmatrix}\bar c\bar d & \bar f\\ \bar e & \bar b\bar a\end{pmatrix}} P_\mathsf{M}\oplus P_\mathsf{N} \xrightarrow{(\bar d,\bar a)} P_\mathsf{k}$$

The cohomology $H^*BG \cong \mathsf{Ext}^*_B(\mathsf{k},\mathsf{k})$ as an algebra is easily read off from this, and is as given in the theorem.

One choice of a map of minimal resolutions $\tilde y$ representing y is as follows.

$$\begin{tikzcd}
P_\mathsf{k} \arrow[r, "\binom{\bar c}{\bar b}"] \arrow[d, "(1)"'] & P_\mathsf{M}\oplus P_\mathsf{N} \arrow[r, "\begin{pmatrix}\bar c\bar d & \bar f\\ \bar e & \bar b\bar a\end{pmatrix}"] \arrow[d, "(\bar a\bar e{,}0)"'] & P_\mathsf{M}\oplus P_\mathsf{N} \arrow[r, "(\bar d\ \bar a)"] \arrow[d, "\begin{pmatrix}\bar c\bar d & 0\\ 0 & 0\end{pmatrix}"'] & P_\mathsf{k} \arrow[r, "\bar a\bar e\bar c=\bar d\bar f\bar b"] \arrow[d, "\binom{\bar f\bar b}{0}"'] & P_\mathsf{k} \arrow[r] \arrow[d, "(1)"'] & P_\mathsf{M}\oplus P_\mathsf{N} \arrow[r] & P_\mathsf{M}\oplus P_\mathsf{N} \arrow[r] & P_\mathsf{k} \\
P_\mathsf{k} \arrow[r, "\bar a\bar e\bar c=\bar d\bar f\bar b"'] & P_\mathsf{k} \arrow[r, "\binom{\bar c}{\bar b}"'] & P_\mathsf{M}\oplus P_\mathsf{N} \arrow[r, "\begin{pmatrix}\bar c\bar d & \bar f\\ \bar e & \bar b\bar a\end{pmatrix}"'] & P_\mathsf{M}\oplus P_\mathsf{N} \arrow[r, "(\bar d\ \bar a)"'] & P_\mathsf{k}
\end{tikzcd}$$

The element z lifts to the periodicity generator $\tilde z\colon P_* \to P_*$ of degree -4. We have $\tilde y\tilde z = \tilde z\tilde y$, and we have the following homotopy u from $\tilde y \circ \tilde y$ to zero.

$$\begin{tikzcd}
P_\mathsf{k} \arrow[r, "\binom{\bar c}{\bar b}"] \arrow[d, "\binom{\bar f\bar b}{0}"] & P_\mathsf{M}\oplus P_\mathsf{N} \arrow[r, "\begin{pmatrix}\bar c\bar d & \bar f\\ \bar e & \bar b\bar a\end{pmatrix}"] \arrow[d, "(\bar a\bar e\ 0)"] \arrow[dl, "\begin{pmatrix}\bar e & 0\\ 0 & 0\end{pmatrix}"] & P_\mathsf{M}\oplus P_\mathsf{N} \arrow[r, "(\bar d\ \bar a)"] \arrow[d, "(0\ 0)"] \arrow[dl, "(0\ 0)"] & P_\mathsf{k} \arrow[r, "\bar a\bar e\bar c=\bar d\bar f\bar b"] \arrow[d, "\binom{0}{0}"] \arrow[dl, "(0)"] & P_\mathsf{k} \arrow[r, "\binom{\bar c}{\bar b}"] \arrow[d, "\binom{\bar f\bar b}{0}"] \arrow[dl, "\binom{0}{\bar b}"] & P_\mathsf{M}\oplus P_\mathsf{N} \arrow[r] \arrow[d, "(\bar a\bar e\ 0)"] \arrow[dl, "\begin{pmatrix}\bar e & 0\\ 0 & 0\end{pmatrix}"] & \cdots \arrow[dl, "(0\ 0)"] \\
P_\mathsf{M}\oplus P_\mathsf{N} \arrow[r, "(\bar d\ \bar a)"'] & P_\mathsf{k} \arrow[r, "\bar a\bar e\bar c=\bar d\bar f\bar b"'] & P_\mathsf{k} \arrow[r, "\binom{\bar c}{\bar b}"'] & P_\mathsf{M}\oplus P_\mathsf{N} \arrow[r, "\begin{pmatrix}\bar c\bar d & \bar f\\ \bar e & \bar b\bar a\end{pmatrix}"'] & P_\mathsf{M}\oplus P_\mathsf{N} \arrow[r, "(\bar d\ \bar a)"'] & P_\mathsf{k} \arrow[r] & 0
\end{tikzcd}$$

Thus $du = \tilde y \circ \tilde y$. Moreover, it is easy to check that $u\tilde y = 0$ and $\tilde y u = 0$. Using the recipe of Kadeishvili given in the proof of Theorem 1.3.8 for computing the A_∞ structure on H^*BG from the differential graded algebra structure on $\mathsf{End}_B(P_*)$, this implies that for all $n > 2$ we have $m_n(y, \dots, y) = 0$. Explicitly, let $A = \mathsf{End}_B(P_*)$ with m_1 the differential and m_2 the composition of endomorphisms. Then A is quasi-isomorphic to the A_∞ algebra C^*BG. The map f_1 takes $y^\varepsilon z^i$ to $\tilde y^\varepsilon \tilde z^i$, and f_2 takes (yz^{i_1}, yz^{i_2}) to $uz^{i_1+i_2}$ and the remaining monomials to zero. Then $u\tilde y = 0$ implies that $m_2(f_2 \otimes f_1) = 0$ while $\tilde y u = 0$ implies that $m_2(f_1 \otimes f_2) = 0$. We also check that $f_2(1 \otimes m_2 - m_2 \otimes 1) = 0$. Now applying Remark 1.3.10, we may take $f_i = 0$ and $m_i = 0$ for $i \geqslant 3$ to deduce that C^*BG is formal. ☐

Corollary 4.7.5. *Let G be a finite group with quaternion or generalised quaternion Sylow 2-subgroup, and with no normal subgroup of index two.*

(i) *We have* $HH^*H^*BG \cong \Lambda(y,\hat{z}) \otimes \mathsf{k}[z,\hat{y}]$ *with* $|y| = (0,-3)$, $|z| = (0,-4)$, $|\hat{y}| = (-1,3)$, $|\hat{z}| = (-1,4)$.

(ii) *We have* $HH^*C^*BG \cong HH^*C_*\Omega BG_2^\wedge$ *has the same generators and relations, but the degrees are given by* $|y| = -3$, $|z| = -4$, $|\hat{y}| = 2$, $|\hat{z}| = 3$.

(iii) *We have* $H_*\Omega BG_2^\wedge \cong \Lambda(\hat{z}) \otimes \mathsf{k}[\hat{y}]$ *with* $|\hat{z}| = 3$ *and* $|\hat{y}| = 2$. *This is formal as an* A_∞ *algebra.*

(iv) *We have* $HH^*H_*\Omega BG_2^\wedge \cong \Lambda(y,\hat{z}) \otimes \mathsf{k}[z,\hat{y}]$ *with* $|y| = (-1,-2)$, $|z| = (-1,-3)$, $|\hat{y}| = (0,2)$ *and* $|\hat{z}| = (0,3)$.

THEOREM 4.7.6. *The category*

$$\mathsf{D}_{\mathsf{sg}}(H^*BG) \simeq \mathsf{D}_{\mathsf{sg}}(C^*BG) \simeq \mathsf{D}_{\mathsf{csg}}(C_*\Omega BG_2^\wedge) \simeq \mathsf{D}_{\mathsf{csg}}(H_*\Omega BG_2^\wedge) \simeq \mathsf{D}^{\mathsf{b}}(\Lambda(\hat{z}) \otimes \mathsf{k}[\hat{y},\hat{y}^{-1}])$$

has four isomorphism classes of indecomposable objects. As objects in $\mathsf{D}^{\mathsf{b}}(\Lambda(\hat{z}) \otimes \mathsf{k}[\hat{y},\hat{y}^{-1}])$ *they are* $\mathsf{k}[\hat{y},\hat{y}^{-1}]$ *and* $\Lambda(\hat{z}) \otimes \mathsf{k}[\hat{y},\hat{y}^{-1}]$, *with zero differential, and their odd shifts. Both have period two, with* $\hat{y}$ *inducing the periodicity.*

The category

$$\mathsf{D}_{\mathsf{csg}}(H^*BG) \simeq \mathsf{D}_{\mathsf{csg}}(C^*BG) \simeq \mathsf{D}_{\mathsf{sg}}(C_*\Omega BG_2^\wedge) \simeq \mathsf{D}_{\mathsf{sg}}(H_*\Omega BG_2^\wedge) \simeq \mathsf{D}^{\mathsf{b}}(\Lambda(y) \otimes \mathsf{k}[z,z^{-1}])$$

has eight isomorphism classes of indecomposable objects. As objects in $\mathsf{D}^{\mathsf{b}}(\Lambda(y) \otimes \mathsf{k}[z,z^{-1}])$ *they are* $\mathsf{k}[z,z^{-1}]$ *and* $\Lambda(y)\otimes\mathsf{k}[z,z^{-1}]$ *with zero differential, and their shifts. Both have period four, with* z *inducing the periodicity.*

The category

$$\mathsf{D}^{\mathsf{b}}(H^*BG) \simeq \mathsf{D}^{\mathsf{b}}(C^*BG) \simeq \mathsf{D}^{\mathsf{b}}(C_*\Omega BG_2^\wedge) \simeq \mathsf{D}^{\mathsf{b}}(H_*\Omega BG_2^\wedge)$$

has a countable infinity of isomorphism classes of indecomposable objects, as follows. As objects of $\mathsf{D}^{\mathsf{b}}(H^*BG)$ *for* $n \geqslant 0$ *there is an indecomposable module with generators* u *and* v, *with* $z^n u = yv$, *and there is one more indecomposable* $\mathsf{k}[z]$. *These all have zero differential.*

4.8. Two classes of elements of order four

In this section, we examine Case 4.6.2, of a finite group G with generalised quaternion Sylow 2-subgroups of order $8q$ and two classes of elements of order four. This implies that $q \geqslant 2$.

THEOREM 4.8.1. *Suppose that* G *is a finite group with generalised quaternion Sylow* 2*-subgroups and two classes of elements of order four. Then*

$$H^*BG = \mathsf{k}[y,z]/(y^4),$$

with $|y| = -1$, $|z| = -4$.

PROOF. Without loss of generality, assume that $O(G) = 1$. Then G has a central involution s, and the class of the central extension of $G/\langle s\rangle$ by $\langle s\rangle$, in the notation of Section 2.13, is $t + y^2$. So in the spectral sequence of the central extension, we have $d_2(w) = t + y^2$, $d_3(w) = \mathsf{Sq}^1(t + y^2) = \xi + yt$, and $H^*BG = \mathsf{k}[\xi, y, t]/(\xi y, t + y^2, \xi + yt) \otimes \mathsf{k}[w^4]$. In this ring, we have $t = y^2$, $\xi = yt = y^3$, and $y^4 = \xi y = 0$. So letting z be a representative of w^4 in H^*BG, the structure is as given. $\square$

Theorem 4.8.2. *The* Ext *ring of H^*BG is given by*

$$\mathsf{Ext}^{*,*}_{H^*BG}(\mathsf{k},\mathsf{k}) = \Lambda(\hat{y},\hat{z}) \otimes \mathsf{k}[\eta]$$

with $\eta = \langle \hat{y},\hat{y},\hat{y},\hat{y}\rangle$, $|\hat{y}| = (-1,1)$, $|\hat{z}| = (-1,4)$, and $|\eta| = (-2,4)$.

Proof. The algebra H^*BG is a complete intersection. The second partial derivatives of the relation all vanish, so this follows by an easy application of Theorem 1.11.2. $\square$

Corollary 4.8.3. *We have $H_*\Omega BG^{\wedge}_2 = \Lambda(\hat{y},\hat{z}) \otimes \mathsf{k}[\eta]$ with $\eta = \langle \hat{y},\hat{y},\hat{y},\hat{y}\rangle$, $|\hat{y}| = 0$, $|\hat{z}| = 3$ and $|\eta| = 2$.*

Proof. There is no room for differentials in the Eilenberg–Moore spectral sequence

$$\mathsf{Ext}^{*,*}_{H^*BG} \Rightarrow H_*\Omega BG^{\wedge}_2.$$

For the ungrading, the only issue is to choose the correct representative for $\hat{y}$ so that it squares to zero. This is possible, because the group of connected components is $\mathbb{Z}/2$, and the group algebra of this has a non-zero element that squares to zero. $\square$

Theorem 4.8.4. *The Hochschild cohomology of H^*BG is given by*

$$HH^*H^*BG = \Lambda(\hat{y},\hat{z}) \otimes \mathsf{k}[y,z,\eta]/(y^4)$$

with $|y| = (0,-1)$, $|z| = (0,-4)$, $|\hat{y}| = (-1,1)$, $|\hat{z}| = (-1,4)$ and $|\eta| = (-2,4)$.

Proof. By Theorem 4.8.1, H^*BG is a complete intersection, so this follows from Theorem 1.11.5. $\square$

Theorem 4.8.5. *The Hochschild cohomology of $H_*\Omega BG^{\wedge}_2$ is given by*

$$HH^*H_*\Omega BG^{\wedge}_2 = \Lambda(\hat{y},\hat{z},\hat{\eta}) \otimes \mathsf{k}[y,z,\eta]$$

with $|\hat{y}| = (0,0)$, $|\hat{z}| = (0,3)$, $|\hat{\eta}| = (-1,-2)$, $|y| = (-1,0)$, $|z| = (-1,-3)$ and $|\eta| = (0,2)$.

Proof. By Corollary 4.8.3, $H_*\Omega BG^{\wedge}_2$ is a complete intersection, so this follows from Theorem 1.11.5. $\square$

The fact that in $\mathsf{Ext}^{*,*}_{H^*BG}(\mathsf{k},\mathsf{k})$ we have $\eta = \langle \hat{y},\hat{y},\hat{y},\hat{y}\rangle$ implies that in the spectral sequence

$$HH^*H_*\Omega BG^{\wedge}_2 \Rightarrow HH^*C_*\Omega BG^{\wedge}_2$$

we have a differential $d^3(\hat{\eta}) = y^4$.

4.9. Non-formality

Our goal in this section is to prove that in Case 4.6.2, C^*BG is not formal as an A_∞ algebra. To do so, we shall show that the Massey triple product $\langle y^2, y^2, y^2\rangle$ vanishes, but $\langle y^2, y^2, y^2, y^2\rangle$ is equal to y^2z.

If G is $SL^{\circ}(2,p^m)$ with $p^m \equiv 3 \pmod 4$ then G is an example of Case 4.6.2, and the principal block belongs to Erdmann's [**74**] class $Q(2\mathcal{B})_1$, which is labelled I in the Appendix to [**69**]. If G is $SU^{\circ}(2,p^m)$ with $p^m \equiv 1 \pmod 4$ then G is also an example of Case 4.6.2, and the principal block belongs to Erdmann's class $Q(2\mathcal{A})$, which is labelled II in [**69**]. The two types are derived equivalent.

Type $Q(2\mathcal{B})_1$ is the easier to handle, so we assume that we are in the case of $SL^\circ(2,p^m)$ with $p^m \equiv 3 \pmod 4$. The quiver for the principal block B is

$$\begin{tikzcd} \mathsf{k} \arrow[loop left, "a"] \arrow[r, bend left, "b"] & \mathsf{M} \arrow[l, bend left, "c"] \arrow[loop right, "d"] \end{tikzcd}$$

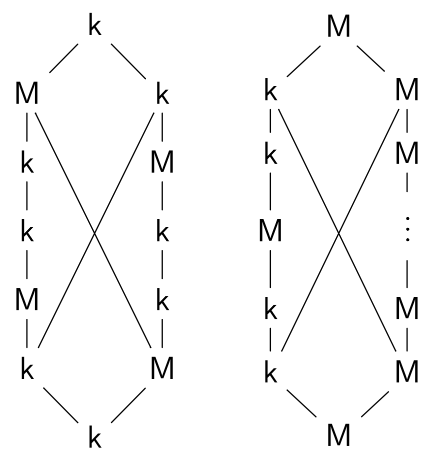
with relations

$$bc = d^{2q-1}, \quad db = bacba, \quad cd = acbac, \quad a^2 = cbacb + \lambda(acb)^2, \quad ba^2 = 0$$

for some value of the parameter $\lambda \in \mathsf{k}$ that has not been determined. Note that these relations imply that

$$\begin{aligned} bcb &= d^{2q-1}b = d^{2q-2}bacba = d^{2q-3}bacba^2cba = 0,\\ cbc &= cd^{2q-1} = acbacd^{2q-2} = acba^2cbacd^{2q-3} = 0,\\ a^2c &= cbacbc + \lambda acbacbc = 0. \end{aligned}$$

The projective covers of the simple modules k and M have the following diagrams (beware of the extra socle term in the expression for a^2):



where the number of copies of M down the right hand side of the projective cover of M is $2q-1$.

The minimal resolution of k is periodic with period four, and has the following form:

$$\cdots \to P_{\mathsf{k}} \oplus P_{\mathsf{M}} \xrightarrow{(\bar a, \bar b)} P_{\mathsf{k}} \xrightarrow{(\bar a^3)} P_{\mathsf{k}} \xrightarrow{\begin{pmatrix} \bar a + \lambda \bar b\bar c\bar a\bar b\bar c \\ \bar c \end{pmatrix}} P_{\mathsf{k}} \oplus P_{\mathsf{M}} \xrightarrow{\begin{pmatrix} \bar a & \bar b\bar c\bar a\bar b \\ \bar c\bar a\bar b\bar c + \lambda\bar c\bar a\bar b\bar c\bar a & \bar d \end{pmatrix}} P_{\mathsf{k}} \oplus P_{\mathsf{M}} \xrightarrow{(\bar a, \bar b)} P_{\mathsf{k}}$$

We lift $y \in \mathsf{Ext}^1_{\mathsf{kQ}}(\mathsf{k},\mathsf{k})$ to a map $\tilde y$ of resolutions as follows:

$$\begin{tikzcd}[column sep=large]
P_{\mathsf{k}} \oplus P_{\mathsf{M}} \arrow[r, "(\bar a, \bar b)"] \arrow[d, "(1{,}0)"] & P_{\mathsf{k}} \arrow[r, "(\bar a^3)"] \arrow[d, "(\bar a^2)"] & P_{\mathsf{k}} \arrow[r, "\begin{pmatrix} \bar a + \lambda \bar b\bar c\bar a\bar b\bar c \\ \bar c \end{pmatrix}"] \arrow[d, "\begin{pmatrix} 1 \\ \lambda\bar c \end{pmatrix}"] & P_{\mathsf{k}} \oplus P_{\mathsf{M}} \arrow[r, "\begin{pmatrix} \bar a & \bar b\bar c\bar a\bar b \\ \bar c\bar a\bar b\bar c + \lambda\bar c\bar a\bar b\bar c\bar a & \bar d \end{pmatrix}"] \arrow[d, "\begin{pmatrix} 1 & 0 \\ 0 & \bar c\bar a\bar b \end{pmatrix}"] & P_{\mathsf{k}} \oplus P_{\mathsf{M}} \arrow[r, "(\bar a, \bar b)"] \arrow[d, "(1{,}0)"] & P_{\mathsf{k}} \\
P_{\mathsf{k}} \arrow[r, "(\bar a^3)"'] & P_{\mathsf{k}} \arrow[r, "\begin{pmatrix} \bar a + \lambda \bar b\bar c\bar a\bar b\bar c \\ \bar c \end{pmatrix}"'] & P_{\mathsf{k}} \oplus P_{\mathsf{M}} \arrow[r, "\begin{pmatrix} \bar a & \bar b\bar c\bar a\bar b \\ \bar c\bar a\bar b\bar c + \lambda\bar c\bar a\bar b\bar c\bar a & \bar d \end{pmatrix}"'] & P_{\mathsf{k}} \oplus P_{\mathsf{M}} \arrow[r, "(\bar a, \bar b)"'] & P_{\mathsf{k}} &
\end{tikzcd}$$

The composite $\tilde{y}^8$ is zero, and we have the following homotopy u_1 from $\tilde{y}^4$ to zero:

$$\begin{tikzcd}
P_{\mathsf{k}} \arrow[r, "(\bar{a}^3)"] \arrow[d, "(\bar{a}^2)"'] & P_{\mathsf{k}} \arrow[r, "\begin{pmatrix} \bar{a}+\lambda\bar{b}\bar{c}\bar{a}\bar{b}\bar{c} \\ \bar{c} \end{pmatrix}"] \arrow[d, "(\bar{a}^2)"] \arrow[dl, "(0)"] & P_{\mathsf{k}} \oplus P_{\mathsf{M}} \arrow[r, "\begin{pmatrix} \bar{a} & \bar{b}\bar{c}\bar{a}\bar{b} \\ \bar{c}\bar{a}\bar{b}\bar{c}+\lambda\bar{c}\bar{a}\bar{b}\bar{c}\bar{a} & \bar{d} \end{pmatrix}"] \arrow[d, "\begin{pmatrix} \bar{a}^2 & 0 \\ 0 & 0 \end{pmatrix}"] \arrow[dl, "{(0,\bar{b}\bar{c}\bar{a}\bar{b} + \lambda\bar{a}\bar{b}\bar{c}\bar{a}\bar{b})}"] & P_{\mathsf{k}} \oplus P_{\mathsf{M}} \arrow[r, "{(\bar{a},\bar{b})}"] \arrow[d, "\begin{pmatrix} \bar{a}^2 & 0 \\ 0 & 0 \end{pmatrix}"] \arrow[dl, "\begin{pmatrix} 0 & \bar{b} \\ 0 & \lambda\bar{c}\bar{b} \end{pmatrix}"] & P_{\mathsf{k}} \arrow[r, "(\bar{a}^3)"] \arrow[d, "(\bar{a}^2)"] \arrow[dl, "\begin{pmatrix} \bar{a} \\ 0 \end{pmatrix}"] & P_{\mathsf{k}} \arrow[r] & \cdots \\
P_{\mathsf{k}} \arrow[r, "(\bar{a}^3)"'] & P_{\mathsf{k}} \arrow[r, "\begin{pmatrix} \bar{a}+\lambda\bar{b}\bar{c}\bar{a}\bar{b}\bar{c} \\ \bar{c} \end{pmatrix}"'] & P_{\mathsf{k}} \oplus P_{\mathsf{M}} \arrow[r, "\begin{pmatrix} \bar{a} & \bar{b}\bar{c}\bar{a}\bar{b} \\ \bar{c}\bar{a}\bar{b}\bar{c}+\lambda\bar{c}\bar{a}\bar{b}\bar{c}\bar{a} & \bar{d} \end{pmatrix}"'] & P_{\mathsf{k}} \oplus P_{\mathsf{M}} \arrow[r, "{(\bar{a},\bar{b})}"'] & P_{\mathsf{k}}
\end{tikzcd}$$

This satisfies $u_1^2 = 0$ and $u_1\tilde{y}^4 = \tilde{y}^4 u_1$.

The composite $u_1\tilde{y}$ is given by the matrices

$$(0), \qquad (\lambda\bar{b}\bar{c}\bar{a}\bar{b}\bar{c} + \lambda^2\bar{a}\bar{b}\bar{c}\bar{a}\bar{b}\bar{c}), \qquad \begin{pmatrix} 0 & \bar{b}\bar{c}\bar{a}\bar{b} \\ 0 & 0 \end{pmatrix}, \qquad \begin{pmatrix} \bar{a} & 0 \\ 0 & 0 \end{pmatrix}$$

while $\tilde{y}u_1$ is given by the matrices

$$(\bar{a}), \qquad (0), \qquad \begin{pmatrix} 0 & \bar{b}\bar{c}\bar{a}\bar{b} + \lambda\bar{a}\bar{b}\bar{c}\bar{a}\bar{b} \\ 0 & \lambda^2\bar{c}\bar{a}\bar{b}\bar{c}\bar{a}\bar{b} \end{pmatrix}, \qquad \begin{pmatrix} 0 & \bar{b} \\ 0 & 0 \end{pmatrix}$$

Since $u_1\tilde{y} + \tilde{y}u_1$ is non-zero, we need to find a homotopy from it to zero. The following is such a homotopy u_2:

$$(0), \qquad (0, \lambda\bar{b}\bar{c}\bar{a}\bar{b} + \lambda^2\bar{a}\bar{b}\bar{c}\bar{a}\bar{b}), \qquad \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Then $\tilde{y}u_2 + u_2\tilde{y}$ is a homotopy from $\tilde{y}^2u_1 + u_1\tilde{y}^2$ to zero, $\tilde{y}^2u_2 + \tilde{y}u_2\tilde{y} + u_2\tilde{y}^2$ is a homotopy from $\tilde{y}^3u_1 + u_1\tilde{y}^3$ to zero, and $\tilde{y}^3u_2 + \tilde{y}^2u_2\tilde{y} + \tilde{y}u_2\tilde{y}^2 + u_2\tilde{y}^3$ is a homotopy from $\tilde{y}^4u_1 + u_1\tilde{y}^4 = 0$ to zero.

At the next stage, the relevant composites are given by the matrices

$$\begin{array}{lcccc}
\tilde{y}^3u_2\colon & \begin{pmatrix} 0 \\ 0 \end{pmatrix}, & (0, \lambda\bar{b}\bar{c}\bar{a}\bar{b} + \lambda^2\bar{a}\bar{b}\bar{c}\bar{a}\bar{b}), & (0,0), & \begin{pmatrix} \bar{a}^2 \\ 0 \end{pmatrix} \\
\tilde{y}^2u_2\tilde{y}\colon & \begin{pmatrix} \lambda^3\bar{a}\bar{b}\bar{c}\bar{a}\bar{b}\bar{c} \\ 0 \end{pmatrix}, & (0,0), & (\bar{a}^2, 0), & \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\
\tilde{y}u_2\tilde{y}^2\colon & \begin{pmatrix} 0 \\ 0 \end{pmatrix}, & (1,0), & (0,0), & \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\
u_2\tilde{y}^3\colon & \begin{pmatrix} 1 \\ 0 \end{pmatrix}, & (0,0), & (0,0), & \begin{pmatrix} 0 \\ 0 \end{pmatrix}.
\end{array}$$

Thus $\tilde{y}^3u_2 + \tilde{y}^2u_2\tilde{y} + \tilde{y}u_2\tilde{y}^2 + u_2\tilde{y}^3 + u_1^2$ is given by the matrices

$$\begin{pmatrix} 1 + \lambda^3\bar{a}\bar{b}\bar{c}\bar{a}\bar{b}\bar{c} \\ 0 \end{pmatrix}, \qquad (1, \lambda\bar{b}\bar{c}\bar{a}\bar{b} + \lambda^2\bar{a}\bar{b}\bar{c}\bar{a}\bar{b}), \qquad (\bar{a}^2, 0), \qquad \begin{pmatrix} \bar{a}^2 \\ 0 \end{pmatrix}.$$

This is homotopic to $\tilde{y}^2\tilde{z}$, which is given by the matrices

$$\begin{pmatrix} 1 \\ \lambda\bar{c}\bar{a}\bar{b}\bar{c} \end{pmatrix}, \qquad (1,0), \qquad (\bar{a}^2, 0) \qquad \begin{pmatrix} \bar{a}^2 \\ 0 \end{pmatrix}.$$

The following is such a homotopy u_3:

$$\begin{pmatrix} \lambda^2\bar{a} & \lambda^2\bar{b}\bar{c}\bar{a}\bar{b}+\lambda^3\bar{a}\bar{b}\bar{c}\bar{a}\bar{b} \\ \lambda^2\bar{c}\bar{a}\bar{b}\bar{c}\bar{a} & \lambda\bar{c}\bar{a}\bar{b} \end{pmatrix}, \qquad (0,0), \qquad (0), \qquad \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

So we set

$$\begin{aligned} f_1(y) &= \tilde{y}, \\ f_2(y,y^3) &= f_2(y^3,y) = f_2(y^2,y^2) = u_1, \\ f_3(y,y^3,y) &= u_2, \\ f_3(y^2,y^2,y^2) &= \tilde{y}u_2 + u_2\tilde{y}, \\ f_3(y^3,y,y^3) &= \tilde{y}^2u_2 + \tilde{y}u_2\tilde{y} + u_2\tilde{y}^2 + \tilde{y}^2\tilde{z}, \\ f_4(y^3,y,y^3,y) &= f_4(y^2,y^2,y^2,y^2) = f_4(y,y^3,y,y^3) = u_3 \end{aligned}$$

So we have $m_3 = 0$, $m_1f_3 = m_2(f_2\otimes f_1 + f_1\otimes f_2)$, and

$$m_4(y^3,y,y^3,y) = m_4(y^2,y^2,y^2,y^2) = m_4(y,y^3,y,y^3) = y^2z. \tag{4.9.1}$$

We can now check that the values of $m_1f_4 + f_1m_4$ and $m_2(f_3\otimes f_1 + f_2\otimes f_2 + f_1\otimes f_3)$ on the quadruples (y^3,y,y^3,y), (y^2,y^2,y^2,y^2) and (y,y^3,y,y^3) are all equal to

$$\tilde{y}^3u_2 + \tilde{y}^2u_2\tilde{y} + \tilde{y}u_2\tilde{y}^2 + u_2\tilde{y}^3 + u_1^2.$$

Equation (4.9.1) may now be interpreted in terms of Hochschild cohomology, using Proposition 1.4.2. Since $m_3 = 0$, m_4 is a Hochschild cocycle. It represents the element η^2y^2z in degree $(-4,2)$, which by Theorem 4.8.4 is non-zero.

At the next stage, the expression $m_2(f_4\otimes f_1 + f_3\otimes f_2 + f_2\otimes f_3 + f_1\otimes f_4)$ sends the 5-tuple (y^2,y^2,y^2,y^2,y^2) to $u_3\tilde{y}^2 + (\tilde{y}u_2+u_2\tilde{y})u_1 + u_1(\tilde{y}u_2+u_2\tilde{y}) + \tilde{y}^2u_3$, which is

$$(0), \qquad \begin{pmatrix} \lambda^2\bar{a}^3 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \qquad \begin{pmatrix} \lambda^2\bar{a}^3 \\ 0 \end{pmatrix}.$$

This is homotopic to zero, with homotopy u_4 given by

$$(0), \qquad \begin{pmatrix} \lambda^2 & \lambda^3\bar{b}\bar{c}\bar{a}\bar{b} \end{pmatrix}, \qquad \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \qquad \begin{pmatrix} \lambda^2 \\ 0 \end{pmatrix}.$$

Thus we can take $m_5 = 0$ and $f_5(y^2,y^2,y^2,y^2,y^2) = u_4$, to obtain

$$m_1f_5 + f_1m_5 = m_2(f_4\otimes f_1 + f_3\otimes f_2 + f_2\otimes f_3 + f_1\otimes f_4).$$

THEOREM 4.9.2. *If G is a finite group with generalised quaternion Sylow 2-subgroups and two classes of elements of order four, then C^*BG and $C_*\Omega BG^\wedge_2$ are not formal as A_∞ algebras.*

PROOF. We have just shown that H^*BG is not formal, since we can choose m_3 to be zero, but then m_4 cannot be chosen to be zero. The fact that $H_*\Omega BG^\wedge_2$ is not formal follows from the relation $m_4(\tilde{y},\tilde{y},\tilde{y},\tilde{y}) = \eta$. This in turn follows from the fact that the Massey product $\langle\tilde{y},\tilde{y},\tilde{y},\tilde{y}\rangle = \eta$ in Corollary 4.8.3 has no indeterminacy, see Theorem 1.4.5. $\square$

This finally allows us to compute $HH^*C^*BG \cong HH^*C_*\Omega BG^\wedge_2$.

THEOREM 4.9.3. *We have $HH^*C^*BG \cong HH^*C_*\Omega BG^\wedge_2 \cong \Lambda(\hat{y})\otimes \mathsf{k}[y,z,\eta]/(y^4, y^2\eta^2)$ with $|y| = -1$, $|z| = -4$, $|\hat{y}| = 0$, $\eta = 2$.*

Proof. The computation above of m_3 and m_4 in H^*BG, together with Corollary 4.8.3 show that in the spectral sequence

$$HH^*H^*BG \Rightarrow HH^*C^*BG$$

we have $d^2 = 0$ and $d^3(\hat{z}) = y^2\eta^2$. This then implies that η is a universal cycle, and $E_4 = E_\infty$.

Ungrading $y^4 = 0$ in E_∞, we see that y^4 has to be a linear combination of the elements $y^2z\eta$ and $z^i\eta^{2i-2}$ with $i \geqslant 2$. Since $\tilde{y}^8 = 0$, the relation ungrading $y^4 = 0$ has to satisfy $y^8 = 0$, so it cannot involve the elements $z^i\eta^{2i-2}$. So y^4 is some multiple of $y^2z\eta$. But if it's a non-zero multiple then y^8 is a non-zero multiple of $y^2z^3\eta^3$. This contradiction shows that $y^4 = 0$ in HH^*H^*BG.

Ungrading $y^2\eta^2 = 0$ in E_∞, we see that $y^2\eta^2$ is a linear combination of the elements $z^i\eta^{2i+1}$ with $i \geqslant 1$. Again, since $\tilde{y}^8 = 0$, $y^2\eta^2$ is nilpotent, and so we have $y^2\eta^2 = 0$ in HH^*H^*BG. $\square$

It follows from this, that in the spectral sequence

$$HH^*H_*\Omega BG^\wedge_2 \Rightarrow HH^*C_*\Omega BG^\wedge_2 \cong HH^*C^*BG$$

we are forced to have $d^2(\hat{z}) = y^2\eta^2$, $d^3(\hat{\eta}) = y^4$, to give the same answer for HH^*C^*BG.

CHAPTER 5

Beyond tame

5.1. Introduction

In this chapter we discuss C^*BG and $C_*\Omega BG^\wedge_p$ beyond the tame case. We've seen in our discussion of tame representation type, that $H_*\Omega BG^\wedge_p$ always has polynomial growth. Furthermore, there is always a finitely generated central subalgebra over which $H_*\Omega BG^\wedge_p$ is finitely generated as a module.

This is not always the case for finite groups. There is a dichotomy, discovered by Ran Levi, between polynomial and exponential growth for $H_*\Omega BG^\wedge_p$, and we discuss this in Section 5.3. We give examples both of exponential growth and of polynomial growth beyond tame representation type. For example, groups of Lie type in non-defining characteristic are always of polynomial growth, as we shall explain in Section 5.7.

The other aspect revealed by our discussion of tame representation type is that in some unexpected cases it turns out that C^*BG is formal as an A_∞ algebra. We begin with a discussion of this phenomenon.

5.2. Formality

One of the surprising aspects of our work is the discovery that C^*BG is formal in two of the cases with semidihedral Sylow 2-subgroups, see Theorems 3.7.16 and 3.13.13, and also when G has generalised quaternion Sylow subgroups and no normal subgroup of index two, see Theorem 4.7.4. In this section, we discuss formality in general for the A_∞ algebra C^*BG. We begin with finite p-groups.

Theorem 5.2.1. *Let G be a finite p-group. Then the following are equivalent.*

(i) *The comultiplication on a minimal resolution of k as a $\mathsf{k}G$-module is strictly coassociative.*
(ii) *The A_∞ algebra C^*BG is formal.*
(iii) *H^*BG is a polynomial ring.*
(iv) *$p = 2$ and G is an elementary abelian 2-group.*

Proof. (i) ⇒ (ii): Let P_* be a minimal resolution of k as a $\mathsf{k}G$-module. Then the differential on $\mathsf{Hom}_{\mathsf{k}G}(P_*, \mathsf{k})$ is zero, and so $H^*BG \cong \mathsf{Hom}_{\mathsf{k}G}(P_*, \mathsf{k})$ with the multiplication induced from the comultiplication on P_*. Since the comultiplication is coassociative, $\mathsf{Hom}_{\mathsf{k}G}(P_*, \mathsf{k})$ is a DG algebra with zero differential, and is therefore formal.

(ii) ⇒ (iii): If C^*BG is formal then the Eilenberg–Moore spectral sequence gives an isomorphism $\mathsf{Ext}^{*,*}_{H^*BG}(\mathsf{k}, \mathsf{k}) \cong \mathsf{k}G$. In particular, $\mathsf{Ext}^{*,*}_{H^*BG}(\mathsf{k}, \mathsf{k})$ has finite total dimension over k, so H^*BG has finite global dimension. It follows that it is regular, and hence a polynomial ring.

(iii) ⇔ (iv) is proved in Corollary 6.6 of Benson and Carlson [**18**].

(iv) $\Rightarrow$ (i): If G is cyclic of order two then the reduced bar resolution is minimal, so G satisfies (i). If G is an elementary abelian 2-group then we express G as a direct product of cyclic groups of order two, and form the tensor product of their minimal resolutions. The comultiplication resulting from this is strictly coassociative. $\square$

As an illustration of the grading techniques, we prove the following, which also gives another proof of (iv) $\Rightarrow$ (ii) in Theorem 5.2.1.

THEOREM 5.2.2. *Suppose that G is a finite group with elementary abelian Sylow 2-subgroup D, and k is a field of characteristic two. Then there is a grading on $\mathsf{k}G$ such that the A_∞ algebra C^*BG is intrinsically formal. In particular, without reference to grading, C^*BG is formal.*

PROOF. By Theorem 1.8.2 and Remark 1.8.3, we can suppose that G is a semidirect product $D \rtimes H$, with H a p'-subgroup of $\mathsf{Aut}(D)$.

Let $D = \langle g_1, \dots, g_r\rangle \cong (\mathbb{Z}/2)^r$. The group H acts on kD, and this gives a short exact sequene of kH-modules

$$0 \to J^2(kD) \to J(kD) \to J(kD)/J^2(kD) \to 0.$$

Since p does not divide $|H|$, this sequence splits. Let U be an invariant complement to $J^2(kD)$ in $J(kD)$, and let $X_1, \dots, X_r$ be a basis for U. Then

$$\mathsf{k}D = k[X_1, \dots, X_r]/(X_1^2, \dots, X_r^2).$$

We can put a grading on $\mathsf{k}D$ by setting $|X_i| = 1$. Putting elements of H in degree zero then defines a grading on kG. This gives us an H-invariant grading on $H^*BD = k[x_1, \dots, x_r]$ with $|x_i| = (-1, -1)$. Then the ring $H^*BG = (H^*BD)^H$ is doubly graded. The cohomological degrees of elements are equal to their internal degrees. The A_∞ maps $m_i \colon H^*BG \to H^*BG$ have degrees $(i-2, 0)$, see Theorem 1.3.8. So for $i > 2$ we have $m_i = 0$, since either the source or the target is zero. $\square$

REMARK 5.2.3. A discussion of formality for C^*BG in the case of a compact Lie group G can be found in Benson and Greenlees [**21**]. The last section of this paper has an discussion of the literature.

5.3. Polynomial versus exponential growth

DEFINITION 5.3.1. Let f be a real valued function on the non-negative integers. We say that f grows *at most polynomially* if there exists a polynomial function p such that for all $n \geqslant 0$ we have $|f(n)| \leqslant p(n)$.

In commutative algebra, we have the following theorem, characterising complete intersections.

THEOREM 5.3.2 (Gulliksen [**134**], Theorem 2.3). *Let R be a commutative local ring with residue field k. Then R is a complete intersection if and only if $\mathsf{Ext}^*_R(\mathsf{k}, \mathsf{k})$ has polynomial growth.*

The corresponding theorem for loop space homology of finite complexes is as follows.

Definition 5.3.3. Let f be a real valued function on the non-negative integers. We say that f grows *at least semi-exponentially* if there exists a constant $C > 1$ such that for n large enough $\sum_{i=0}^{n} |f(i)| \geqslant C^{\sqrt{n}}$.

Example 5.3.4. The partition function $p(n)$ satisfies $\log p(n) \sim \pi\sqrt{\frac{2n}{3}}$ as $n \to \infty$, so $p(n)$ has semi-exponential growth.

Theorem 5.3.5 (Felix, Halperin and Thomas [**81**]). *Let X be a simply connected finite CW complex, and p a prime. Then $H_n(\Omega X; \mathbb{F}_p)$ grows either at most polynomially, or at least semi-exponentially.*

Remark 5.3.6. Anick [**3**] found examples where the growth is semi-exponential but not exponential, in the contexts of both Theorem 5.3.2 and Theorem 5.3.5.

Definition 5.3.7. A finite CW complex is said to be *elliptic* at p if if $H_*(\Omega X; \mathbb{F}_p)$ has polynomial growth.

Using Theorem 5.3.5, Levi [**169**] proved the following.

Theorem 5.3.8. *For a finite group G, the loop space homology $H_*\Omega BG^\wedge_p$ grows either at most polynomially or at least semi-exponentially.*

Examples are given in Levi [**168**, **169**] of groups for which the homology contains a free algebra on two variables, so that the growth is exponential. No examples are currently known where the growth is at least semi-exponential but not exponential.

Remark 5.3.9. A discussion of various derived notions of complete intersections, in the context of polynomial versus semi-exponential growth, can be found in Benson, Greenlees and Shamir [**24**], Greenlees, Hess and Shamir [**132**]. The hope is that for spaces of the form $BG^\wedge_p$ with G a finite group, these notions coincide, and describe when $H_*\Omega BG^\wedge_p$ has at most polynomial growth.

5.4. An exponential compact Lie example

For non-connected compact Lie groups, it is not hard to cook up examples of exponential growth. In this section, we give an example which is not only of exponential growth, but also formal. In Section 5.5 we give a finite group example based on this one.

Let $\mathsf{k} = \mathbb{Q}$, let T be an r-dimensional torus, and let $G = T \rtimes \mathbb{Z}/2$, where the involution inverts every element of T. Then $H^*BT = \mathbb{Q}[x_1, \dots, x_r]$ with $|x_i| = 2$ for $1 \leqslant i \leqslant r$, and H^*BG is the subalgebra generated by $x_{i,j} = x_i x_j$ with $1 \leqslant i \leqslant j \leqslant r$. The relations are

$$x_{i,i}x_{j,j} = x_{i,j}^2, \qquad x_{i,i}x_{j,k} = x_{i,j}x_{i,k}, \qquad x_{i,j}x_{k,\ell} = x_{i,k}x_{j,\ell} = x_{i,\ell}x_{j,k}.$$

Here, distinct letters in the subscripts represent distinct indices. This is a Koszul algebra, so $\mathsf{Ext}^*_{H^*BG}(\mathsf{k}, \mathsf{k})$ is the Koszul dual, which is a non-commutative algebra generated by degree $(-1, 2)$ elements $\hat{x}_{i,j}$ with relations

$$\hat{x}_{i,i}^2 = 0, \qquad [\hat{x}_{i,i}, \hat{x}_{i,j}] = 0, \qquad [\hat{x}_{i,i}, \hat{x}_{j,j}] + \hat{x}_{i,j}^2 = 0,$$
$$[\hat{x}_{i,i}, \hat{x}_{j,k}] + [\hat{x}_{i,j}, \hat{x}_{i,k}] = 0, \qquad [\hat{x}_{i,j}, \hat{x}_{k,\ell}] + [\hat{x}_{i,k}, \hat{x}_{j,\ell}] + [\hat{x}_{i,\ell}, \hat{x}_{j,k}] = 0.$$

Note that here, for elements x, x' of odd degree, $[x, x']$ means $xx' + x'x$.

The Eilenberg–Moore spectral sequence

$$\mathsf{Ext}^*_{H^*BG}(\mathbb{Q},\mathbb{Q}) \Rightarrow H_*\Omega BG^\wedge_{\mathbb{Q}}$$

has no room for differentials or ungrading problems, so $H_*\Omega BG^\wedge_{\mathbb{Q}}$ is the same ring as $\mathsf{Ext}^*_{H^*BG}(\mathbb{Q},\mathbb{Q})$, but where the generators $\hat{x}_{i,j}$ are in degree one. For $r \geqslant 3$, this has exponential growth. For example, when $r = 3$ it is a free module on eight generators over the free subalgebra $\mathbb{Q}\langle \hat{x}_{1,2}, \hat{x}_{1,3}, \hat{x}_{2,3}\rangle$, and the quotient by the ideal generated by $\hat{x}_{1,2}$, $\hat{x}_{1,3}$, $\hat{x}_{2,3}$ is an exterior algebra on $\hat{x}_{1,1}$, $\hat{x}_{2,2}$, $\hat{x}_{3,3}$. The Poincaré series for $r = 3$ is given by

$$\sum_{n=0}^{\infty} t^n \dim_{\mathsf{k}} H_n\Omega BG^\wedge_{\mathbb{Q}} = \frac{(1+t)^3}{1-3t} = 1 + 6t + 21t^2 + 64t^3 + 192t^4 + 576t^5 + \cdots$$

For general r, we have

$$\sum_{n=0}^{\infty} t^n \dim_{\mathsf{k}} H^{2n}BG = \frac{\sum_{i=0}^{\lfloor \frac{r}{2}\rfloor}\binom{r}{2i}t^i}{(1-t)^r}, \qquad \sum_{n=0}^{\infty} t^n \dim_{\mathsf{k}} H_n\Omega BG^\wedge_{\mathbb{Q}} = \frac{(1+t)^r}{\sum_{i=0}^{\lfloor \frac{r}{2}\rfloor}\binom{r}{2i}(-t)^i}.$$

This is an example of the general relation (1.12.5) between the Poincaré series of a Koszul algebra and its dual. To apply the formula literally, the variable t in the first sum is replaced by st^{-2}, and in the second by st.

Remark 5.4.1. It follows from the main theorem of Benson and Greenlees [**21**] that for this family of examples, the A_∞ structure on H^*BG is formal. Then since it is a Koszul algebra, it follows that the Koszul dual $H_*\Omega BG^\wedge_{\mathbb{Q}}$ is also formal.

5.5. An exponential finite group example

The loop space homology in the cases discussed in Chapters 2–4 is of polynomial growth, and almost commutative, in the sense that there is a central subring over which the whole ring is finitely generated as a module. In this section, for contrast, we examine a finite group example where $H_*\Omega BG^\wedge_p$ has exponential growth. We take our cue from what happened in the compact Lie example of Section 5.4. This is related to Levi's example but is somewhat simpler to analyse using our technique of introducing an internal grading on the group algebra.

Let p be an odd prime, k be a field of characteristic p, and let G be the group

$$(\mathbb{Z}/p \times \mathbb{Z}/p) \rtimes \mathbb{Z}/2$$

given by the presentation

$$\langle g, h, s \mid g^p = h^p = s^2 = 1, gh = hg, sg = g^{-1}s, sh = h^{-1}s\rangle.$$

Let H be the subgroup of index two generated by g and h, and let $X = \sum_{i=1}^{p-1} g^i/i$ and $Y = \sum_{i=1}^{p-1} h^i/i$ as elements of $\mathsf{k}H \leqslant \mathsf{k}G$. Then we have the following presentation for the group algebra:

$$\mathsf{k}G = \mathsf{k}\langle X, Y, s \mid X^p = Y^p = 0, XY = YX, sX = -Xs, sY = -Ys, s^2 = 1\rangle.$$

We can put a double grading on this by setting $|X| = (1,0)$, $|Y| = (0,1)$ and $|s| = (0,0)$. Then

$$H^*BH = \mathsf{k}[u,v] \otimes \Lambda(x,y)$$

with $|u| = (-2,-p,0)$, $|v| = (-2,0,-p)$, $|x| = (-1,-1,0)$ and $|y| = (-1,0,-1)$. The cohomology ring H^*BG is equal to the invariants of the action of s, which is the subring generated by $a = u^2$, $b = uv$, $c = v^2$, $\alpha = xu$, $\beta = xv$, $\gamma = yu$, $\delta = yv$, and $\varepsilon = xy$. Regarding a, b and c as polynomial generators and the rest as exterior generators, the further relations are:

$$ac = b^2, \quad a\beta = b\alpha, \quad b\beta = c\alpha, \quad a\delta = b\gamma, \quad b\delta = c\gamma, \quad a\varepsilon = \alpha\gamma, \quad b\varepsilon = \alpha\delta = \beta\gamma,$$
$$c\varepsilon = \beta\delta, \quad \alpha\beta = 0, \quad \gamma\delta = 0, \quad \alpha\varepsilon = 0, \quad \beta\varepsilon = 0, \quad \gamma\varepsilon = 0, \quad \delta\varepsilon = 0.$$

Ignoring the higher multiplications, this is a Koszul algebra, and so its Ext algebra is the Koszul dual, with eight generators and 16 relations:

$$\begin{aligned}\mathsf{Ext}^{*,*}_{H^*BG}(\mathsf{k},\mathsf{k}) = \mathsf{k}\langle \hat a, \hat b, \hat c, \hat\alpha, \hat\beta, \hat\gamma, \hat\delta, \hat\varepsilon \mid {}& \hat a^2 = \hat c^2 = [\hat a, \hat b] = [\hat b, \hat c] = [\hat a, \hat c] + \hat b^2 = 0,\\ & [\hat a, \hat\alpha] = [\hat a, \hat\gamma] = [\hat c, \hat\beta] = [\hat c, \hat\delta] = 0,\\ & [\hat a, \hat\beta] + [\hat b, \hat\alpha] = [\hat a, \hat\delta] + [\hat b, \hat\gamma] = [\hat b, \hat\beta] + [\hat c, \hat\alpha] = [\hat b, \hat\delta] + [\hat c, \hat\gamma] = 0,\\ & [\hat a, \hat\varepsilon] + [\hat\alpha, \hat\gamma] = [\hat b, \hat\varepsilon] + [\hat\alpha, \hat\delta] + [\hat\beta, \hat\gamma] = [\hat c, \hat\varepsilon] + [\hat\beta, \hat\delta] = 0 \,\rangle.\end{aligned}$$

The degrees are

$$|\hat a| = (-1,4,2p,0), \qquad |\hat b| = (-1,4,p,p), \qquad |\hat c| = (-1,4,0,2p), \qquad |\hat\alpha| = (-1,3,p+1,0),$$
$$|\hat\beta| = (-1,3,1,p), \qquad |\hat\gamma| = (-1,3,p,1), \qquad |\hat\delta| = (-1,3,0,p+1), \qquad |\hat\varepsilon| = (-1,2,1,1).$$

Since the differentials in the Eilenberg–Moore spectral sequence preserve the two internal degrees, it is easy to see that there is no room for non-zero differentials. For example, $d^n(\hat a)$ has last degree zero, so it can only involve $\hat a$ and $\hat\alpha$. No monomial in these has the appropriate third degree.

Similarly, when we ungrade the E^∞ page of the spectral sequence, there are no other monomials of the same internal degrees as the quadratic terms in the list, so the ungraded relations are the same as in E^∞. It follows that $H_*\Omega BG^\wedge_p \cong \mathsf{Ext}^{*,*}_{H^*BG}(\mathsf{k},\mathsf{k})$ is as described above, but where the first two degrees have been added:

$$|\hat a| = (3,2p,0), \qquad |\hat b| = (3,p,p), \qquad |\hat c| = (3,0,2p), \qquad |\hat\alpha| = (2,p+1,0),$$
$$|\hat\beta| = (2,1,p), \qquad |\hat\gamma| = (2,p,1), \qquad |\hat\delta| = (2,0,p+1), \qquad |\hat\varepsilon| = (1,1,1).$$

This algebra contains for example the free algebra $\mathsf{k}\langle\hat\beta,\hat\gamma\rangle$, and therefore has exponential growth.

To obtain this Poincaré series, we use the formula (1.12.5). For $R = H^*BG$ we have

$$p_R(s,t) = \frac{1 + st^{-2} + 4st^{-3} + st^{-4} + s^2t^{-6}}{(1-st^{-4})^2}.$$

For $R^! = H_*\Omega BG^\wedge_p$ we have

$$p_{R^!}(s,t) = 1/p_R(-st^{-1},t^{-1}) = \frac{(1+st^3)^2}{1 - st - 4st^2 - st^3 + s^2t^4}.$$

Setting $s = 1$ and cancelling gives the required Poincaré series.

$$\sum_{n=0}^\infty t^n \dim_{\mathsf{k}} H_n\Omega BG^\wedge_p = \frac{(1-t+t^2)^2}{1-3t+t^2} = 1 + t + 5t^2 + 12t^3 + 32t^4 + 84t^5 + 220t^6 + 576t^7 + \cdots$$

which agrees with the answer given in Section 2 of [**16**] in the case $p = 3$.

REMARK 5.5.1. A similar but more complicated analysis holds in larger rank. Let $G = (\mathbb{Z}/p)^r \rtimes \mathbb{Z}/2$ with the involution inverting every element of order p, and let H be the normal subgroup of index two. The group algebra $\mathsf{k}G$ has r internal gradings, one for each factor of H. The cohomology ring H^*BG is equal to the invariants of the involution on H^*BH. This is again a Koszul algebra, with Koszul dual $\mathsf{Ext}^{*;*}_{H^*BG}(\mathsf{k},\mathsf{k})$. There is no room for non-zero differentials in the Eilenberg–Moore spectral sequence, and no ungrading problems, so we have $H_*\Omega BG^\wedge_p = \mathsf{Ext}^{*;*}_{H^*BG}(\mathsf{k},\mathsf{k})$. This again has exponential growth, but the task of writing down the Poincaré series is more complicated.

5.6. Reflection groups

We do not want to give the impression that polynomial growth for $H_*\Omega BG^\wedge_p$ only happens for finite or tame representation type. We therefore mention the following. In the next two sections we give further examples of polynomial growth.

THEOREM 5.6.1. *Suppose that G is a semidirect product $E \rtimes H$ with E an elementary abelian p-group (p odd), and H a p-adic reflection group of order prime to p, acting on E via the reduction modulo p of the reflection representation. Then H^*BG is a polynomial tensor exterior algebra. In this case, $H_*\Omega BG^\wedge_p$ is also usually polynomial tensor exterior, and always has polynomial growth.*

PROOF. We have $H^*BG = (H^*BE)^H$, the invariants of H on H^*BE. It follows from a theorem of Solomon [**203**] that H^*BG is a polynomial algebra tensored with an exterior algebra. So $\mathsf{Ext}^*_{H^*BG}(\mathsf{k},\mathsf{k})$ is also polynomial tensor exterior, and has polynomial growth. It then follows from the Eilenberg–Moore spectral sequence that $H_*\Omega BG^\wedge_p$ has polynomial growth. $\square$

REMARK 5.6.2. In the theorem, if we use a single grading on $\mathsf{k}G$ by powers of the radical, the polynomial generators for H^*BG lie in degrees $(-2n_i, -pn_i)$ and the exterior ones in degrees $(-2n_i+1, -p(n_i-1)-1)$, where n_i runs over the degrees of the fundamental invariants of the reflection group H. So the polynomial generators of $\mathsf{Ext}^*_{H^*BG}(\mathsf{k},\mathsf{k})$ are in degree $(-1, 2n_i-1, p(n_i-1)+1)$ and the exterior generators are in degrees $(-1, 2n_i, pn_i)$. There is no room for non-zero differentials, but it occasionally happens that the exterior relations ungrade to have non-zero squares and commutators in the polynomial part. An example of this is the symmetric group of degree three at the prime three, with $E = \mathbb{Z}/3$ and $H = \mathbb{Z}/2$.

5.7. Groups of Lie type in non-defining characteristic

In this section, we describe why, if G is a finite group of Lie type in non-defining characteristic p, $H_*\Omega BG^\wedge_p$ has polynomial growth. This is a consequence of a construction of Quillen [**191**], elaborated in Friedlander [**84**, **85**], Fiedorowicz and Priddy [**82**], Wilkerson [**211**], and Kleinerman [**157**].

Let G be a connected compact Lie group, and let $G(p^m)$ be the corresponding finite group of Lie type over the finite field $\mathbb{F}_{p^m}$. Let ℓ be a prime different from p, and k a field of characteristic ℓ. Then there is an Adams operation $\psi^{p^m}\colon BG^\wedge_\ell \to BG^\wedge_\ell$ and a homotopy fibre square

$$\begin{tikzcd} BG(p^m)^\wedge_\ell \arrow[r] \arrow[d] & BG^\wedge_\ell \arrow[d, "\mathsf{id}\times\mathsf{id}"] \\ BG^\wedge_\ell \arrow[r, "\mathsf{id}\times\psi^{p^m}"] & BG^\wedge_\ell \times BG^\wedge_\ell. \end{tikzcd}$$

In the case of a twisted group of Lie type, ψ^{p^m} is replaced by its composite with a diagram automorphism, and the same homotopy fibre square results.

We consider the Eilenberg–Moore spectral sequence of this fibre square with coefficients in k:

$$\mathsf{Tor}^{H^*BG\otimes H^*BG}_{*,*}(H^*BG, H^*BG) \Rightarrow H^*BG(p^m)$$

If ℓ is not a torsion prime for G then H^*BG is a polynomial ring. In this case, the spectral sequence stops at the E_2 page, and gives a finite filtration on $H^*BG(p^m)$ whose associated graded is a polynomial algebra tensored with an exterior algebra. The degrees of the polynomial generators are twice the degrees of suitable fundamental invariants of the Weyl group, while the degrees of the exterior generators are one less.

If ℓ is an odd prime then there is no ungrading problem, and this gives the structure of the cohomology as a polynomial tensor exterior algebra. On the other hand, if $\ell = 2$, it can happen that the exterior generators ungrade to give elements whose square is not necessarily zero, but is expressible in terms of the other generators. The exact relations can be difficult to determine. Independently of the exact relations, the answer is always a complete intersection.

We now apply another Eilenberg–Moore spectral sequence (see Remark 1.6.3)

$$\mathsf{Ext}^{*,*}_{H^*BG(p^m)}(\mathsf{k},\mathsf{k}) \Rightarrow H_*\Omega BG(p^m)^\wedge_\ell.$$

So the associated graded of $H_*\Omega BG(p^m)^\wedge_\ell$ is usually a polynomial tensor exterior algebra. If $\ell = 2$, the computation has to be made using Theorem 1.11.2. The result is that $H_*\Omega BG(p^m)$ has polynomial growth. It is finite as a module over its centre, and the centre is finitely generated as a k-algebra.

Example 5.7.1 (Quillen [**191**]). Let $G = U(n)$, of Lie type A_{n-1}. We have

$$H^*BU(n) = \mathsf{k}[c_1, \dots, c_n],$$

where the c_i are the Chern classes of degree $2n$. Then $G(p^m)$ is the general linear group $GL(n, p^m)$.

For ℓ odd, this gives

$$H^*BGL(n, p^m) = \mathsf{k}[c_r, c_{2r}, \dots, c_{tr}] \otimes \Lambda(e_r, e_{2r}, \dots, e_{tr})$$

where r is the order of p^m modulo ℓ, and t is the integer part of n/r. The degrees are $|c_{ir}| = -2ir$, $|e_{ir}| = -2ir + 1$. Then the associated graded of $H_*\Omega BGL(n, p^m)^\wedge_\ell$ is

$$\mathsf{k}[\hat{e}_r, \hat{e}_{2r}, \dots, \hat{e}_{tr}] \otimes \Lambda(\hat{c}_r, \hat{c}_{2r}, \dots, \hat{c}_{tr})$$

with $|\hat{e}_{ir}| = 2ir - 2$, $|\hat{c}_{ir}| = 2ir - 1$. Beware that there is no reason why the answer should be graded commutative, so it is not obvious how to ungrade the square zero relations for the

$\hat{c}_{ir}$. For example, in characteristic three we have

$$H_*\Omega BGL(2,2)^\wedge_3 = \mathsf{k}[\hat{e}_2,\hat{c}_2]/(\hat{c}_2^2+\hat{e}_2^3),$$

see Section 1.13. But in any case, the answer has polynomial growth.

For $\ell = 2$, we have $r = 1$ and $t = n$. In this case, if $p^m \equiv 1 \pmod 4$ we get the same answer as above, but if $p^m \equiv 3 \pmod 4$ then we have

$$e_j^2 = \sum_{a=0}^{j-1} c_a c_{2j-1-a}.$$

Here, c_{2j-1-a} is interpreted as zero if $2j-1-a > n$. This gives a complete intersection with Krull dimension n, with $n + \lfloor \frac{n}{2} \rfloor$ generators $e_1,\dots,e_n,c_2,c_4,\dots$ and $\lfloor \frac{n}{2} \rfloor$ relations. So by Theorem 1.11.2, the E^2 term of the Eilenberg–Moore spectral sequence is finite as a module over a central polynomial subring with $\lfloor \frac{n}{2} \rfloor$ generators. So $H_*\Omega BGL(n,p^m)^\wedge_\ell$ has polynomial growth.

EXAMPLE 5.7.2 (Kleinerman [**157**]). Let $G = G_2$ and $\ell = 2$. Two is a torsion prime for G, and we have

$$H^*BG_2 = \mathsf{k}[d_4,d_6,d_7].$$

The Eilenberg–Moore spectral sequence gives the associated graded of $H^*BG_2(p^m)$ (p an odd prime) to be $\mathsf{k}[d_4,d_6,d_7]\otimes\Lambda(y_3,y_5,y_6)$. Ungrading the relations gives $y_3^2 = y_6$, $y_5^2 = y_3d_7+y_6d_4$ and $y_6^2 = y_5d_7 + y_6d_6$ (Grbić [**130**]). So $H^*BG_2(p^m)$ is the complete intersection

$$\mathsf{k}[d_4,d_6,d_7,y_3,y_5]/(y_5^2+y_3d_7+y_3^2d_4,\ y_3^4+y_5d_7+y_3^2d_6).$$

Using Theorem 1.11.2, we see that the E^2 page of the Eilenberg–Moore spectral sequence for $H_*\Omega BG_2(p^m)^\wedge_2$ is generated over the central subalgebra $\mathsf{k}[s_{10},s_{12}]$ by elements $\hat{d}_4$, $\hat{d}_6$, $\hat{d}_7$, $\hat{y}_3$, $\hat{y}_5$. The relations say that all squares and commutators of the latter elements are zero except for

$$\hat{y}_5^2 = [\hat{d}_7,\hat{y}_3] = s_{10}, \qquad [\hat{d}_7,\hat{y}_5] = s_{12}.$$

There is no room for differentials, but ungrading the E^∞ page requires some work. This is done in the paper of Levi and Seeliger [**172**], where they also compute the coproduct and action of the dual Steenrod algebra. It turns out that E^∞ as given above is isomorphic to $H_*\Omega BG_2(p^m)^\wedge_2$. The degrees are added, so that the elements $\hat{d}_i$ and $\hat{y}_i$ now have degree $i-1$, and the elements s_j have degree $j-2$.

This is of polynomial growth, since it is finitely generated (actually free of rank 2^5) over the central polynomial subalgebra $k[s_{10},s_{12}]$ with Poincaré series

$$\sum_{n=0}^\infty \dim_{\mathsf{k}} H_n\Omega BG_2(p^m)^\wedge_2 = \frac{(1+t^2)(1+t^3)(1+t^4)(1+t^5)(1+t^6)}{(1-t^8)(1-t^{10})} = \frac{(1+t^3)(1+t^6)}{(1-t^2)(1-t^5)}.$$

5.8. An exotic example: $B\mathsf{Sol}(q)$

Let q be an odd prime power, and let $\mathsf{Sol}(q)$ be the exotic Benson–Solomon 2-local finite group. This was originally discussed as a configuration that was proved not to come from a finite group in Solomon [**204**]. Its classifying space $B\mathsf{Sol}(q)$ was then discussed in Benson [**14**], and finally it was constructed as a fusion system and linking system by Levi and Oliver [**171**]. Using the fibre square like the one in the previous section, the associated graded of $H^*B\mathsf{Sol}(q)$ was computed in [**14**] to be a polynomial ring on generators in degrees 8, 12, 14 and 15 tensored with an exterior algebra on generators in degrees 7, 11, 13 and 14. The ungrading was carried out by Grbić [**130**], who computed it to be the codimension three complete intersection

$$H^*B\mathsf{Sol}(q) = \mathsf{k}[u_8, u_{12}, u_{14}, u_{15}, y_7, y_{11}, y_{13}]/(f_{22}, f_{26}, f_{28})$$

where the (homological) degrees are minus the subscripts, and where

$$\begin{aligned} f_{22} &= y_{11}^2 + u_8 y_7^2 + u_{15} y_7, \\ f_{26} &= y_{13}^2 + u_{12} y_7^2 + u_{15} y_{11}, \\ f_{28} &= y_7^4 + u_{14} y_7^2 + u_{15} y_{13}. \end{aligned}$$

Applying Theorem 1.11.2, we find that $\mathsf{Ext}^{**}_{H^*B\mathsf{Sol}(q)}(\mathsf{k},\mathsf{k})$ is generated over a central subalgebra $\mathsf{k}[s_{22}, s_{26}, s_{28}]$ by elements $\hat{u}_8, \hat{u}_{12}, \hat{u}_{14}, \hat{u}_{15}, \hat{y}_7, \hat{y}_{11}, \hat{y}_{13}$. The degrees of the elements $\hat{u}_i$ and $\hat{y}_i$ are $(-1, i)$, while the degrees of the s_j are $(-2, j)$. The relations say that all squares and commutators of the latter elements are zero except for

$$\hat{y}_{11}^2 = [\hat{u}_{15}, \hat{y}_7] = s_{22}, \qquad \hat{y}_{13}^2 = [\hat{u}_{15}, \hat{y}_{11}] = s_{26}, \qquad [\hat{u}_{15}, \hat{y}_{13}] = s_{28}.$$

In the Eilenberg–Moore spectral sequence

$$\mathsf{Ext}^{**}_{H^*B\mathsf{Sol}(q)}(\mathsf{k},\mathsf{k}) \Rightarrow H_*\Omega B\mathsf{Sol}(q)$$

there is no room for non-zero differentials, but ungrading the E^∞ page takes more work. This is done in the paper of Levi and Seeliger [**172**], where they also compute the coproduct and action of the dual Steenrod algebra. It turns out that E^∞ as given above is isomorphic to $H_*\Omega B\mathsf{Sol}(q)$. The degrees are added, so that the $\hat{u}_i$ and $\hat{y}_i$ now have degree $i-1$ and the s_j have degree $j-2$.

This is of polynomial growth, since it is finitely generated (actually free of rank 2^7) as a module over the central polynomial subalgebra $\mathsf{k}[s_{22}, s_{26}, s_{28}]$, with Poincaré series

$$\begin{aligned} \sum_{n=0}^{\infty} t^n \dim_{\mathsf{k}} H_n \Omega B\mathsf{Sol}(q) &= \frac{(1+t^7)(1+t^{11})(1+t^{13})(1+t^{14})(1+t^6)(1+t^{10})(1+t^{12})}{(1-t^{20})(1-t^{24})(1-t^{26})} \\ &= \frac{(1+t^7)(1+t^{11})(1+t^{14})}{(1-t^6)(1-t^{10})(1-t^{13})}. \end{aligned}$$

5.9. Some questions

We end with some questions related to our computations.

QUESTION 5.9.1 (John Greenlees). For a finite group G, is $\mathsf{D^b}(C^*BG)$ generated by C^*BP, where P is a Sylow p-subgroup of G?

In the cases where we have been able to describe the structure of the singularity category, the answer to this question is yes. It is also yes in the case of a finite p-group, by the work of Greenlees and Stevenson [**133**].

QUESTION 5.9.2. The ring H^*BG acts on $\mathsf{D^b}(C^*BG) \simeq \mathsf{D^b}(C_*\Omega BG^\wedge_p)$ and hence on $\mathsf{D_{sg}}(C^*BG) \simeq \mathsf{D_{csg}}(C_*\Omega BG^\wedge_p)$ and $\mathsf{D_{csg}}(C^*BG) \simeq \mathsf{D_{sg}}(C_*\Omega BG^\wedge_p)$. What are the supports of these? In particular, is the support of $\mathsf{D_{sg}}(C^*BG)$ equal to the nucleus of G, as defined in [**19**] and discussed further in [**13**]?

QUESTION 5.9.3. In the examples that have been computed so far for $H_*\Omega BG^\wedge_p$, we have the following.

(1) If the growth is polynomial then there is a central subring over which the homology is finitely generated as a module.
(2) If the growth is semi-exponential then there is a free subalgebra on two generators, which implies exponential growth.

To what extent are these true in general?

Is $H_*\Omega BG^\wedge_p$ always finitely presented as an algebra?

Is the Poincaré series

$$\sum_{n=0}^{\infty} t^n \dim_{\mathsf{k}} H_n\Omega BG^\wedge_p$$

always a rational function of t?

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