Comparative asymptotics for discrete semiclassical orthogonal polynomials

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Abstract

We study the ratio $\frac{P_n(x; z)}{\phi_n(x)}$ asymptotically as $n \to \infty$, where the polynomials $P_n(x; z)$ are orthogonal with respect to a discrete linear functional and $\phi_n(x)$ denote the falling factorial polynomials.

We give recurrences that allow the computation of high order asymptotic expansions of $P_n(x; z)$ and give examples for most discrete semiclassical polynomials of class $s \leq 2$.

We show several plots illustrating the accuracy of our results.

Keywords: Semiclassical orthogonal polynomials, asymptotic expansions, ordinary differential equations.

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1 Introduction

Let \( N_0 \) be the set of nonnegative integers
\[
N_0 = \mathbb{N} \cup \{0\} = \{0, 1, 2, \ldots\}.
\]
We will denote by \( \delta_{k,n} \) the Kronecker delta, defined by
\[
\delta_{k,n} = \begin{cases} 
1, & k = n \\
0, & k \neq n 
\end{cases}, \quad k, n \in N_0,
\]
and let \( F \) be the ring of formal power series in the variable \( z \)
\[
F = \mathbb{C}[[z]] = \left\{ \sum_{n=0}^{\infty} c_n z^n : c_n \in \mathbb{C} \right\}.
\]
We consider the differential operator \( \vartheta : F \to F \) defined by [37, 16.8.2]
\[
\vartheta = z \partial_z, \quad (1)
\]
where \( \partial_z \) is the derivative operator
\[
\partial_z = \frac{\partial}{\partial z}.
\]
The action of \( \vartheta \) on the monomials is given by
\[
\vartheta^k x^z = x^k z^x, \quad (2)
\]
where we always assume that \( x \) and \( z \) are independent variables.

Suppose that \( L : F[x] \to F \) is a linear functional (acting on the variable \( x \)), and \( \{\Lambda_n (x)\}_{n \geq 0} \subset \mathbb{C}[x] \) is a sequence of monic polynomials with \( \text{deg} (\Lambda_n) = n \). If the system of linear equations
\[
L [\Lambda_k \Lambda_n] + \sum_{i=0}^{n-1} L [\Lambda_k \Lambda_i] \xi_{n,i} = 0, \quad 0 \leq k \leq n - 1, \quad (3)
\]
has a unique solution \( \{\xi_{n,i}(z)\}_{0 \leq i \leq n-1} \subset \mathbb{F} \), we can define monic polynomials \( P_n (x; z) \) by \( P_0 (x; z) = 1 \) and
\[
P_n (x; z) = \Lambda_n (x) + \sum_{i=0}^{n-1} \xi_{n,i}(z) \Lambda_i (x), \quad n \geq 1. \quad (4)
\]
We say that \( \{P_n(x; z)\}_{n \geq 0} \) is a sequence of (monic) orthogonal polynomials with respect to the functional \( L \), [2], [4], [21], [22], [27], [28], [46].

In this paper, we focus on linear functionals of the form

\[
L[u] = \sum_{x=0}^{\infty} u(x) \frac{(a)_x z^x}{(b + 1)_x x!}, \quad u \in \mathbb{F}[x],
\]

and we use the notation

\[
(a)_n = \prod_{i=1}^{p} (a_i)_n, \quad (b)_n = \prod_{i=1}^{q} (b_i)_n, \quad n \in \mathbb{N}_0,
\]

\[
c + r = (c_1 + r, c_2 + r, \ldots, c_m + r) \in \mathbb{C}^m, \quad r \in \mathbb{C}, \quad c \in \mathbb{C}^m,
\]

where

\[
a = (a_1, \ldots, a_p) \in \mathbb{C}^p, \quad b = (b_1, \ldots, b_q) \in \mathbb{C}^q, \quad p, q \in \mathbb{N}_0,
\]

and the Pochhammer polynomial \((x)_n\) is defined by \((x)_0 = 1\) and [37, 18:12]

\[
(x)_n = \prod_{j=0}^{n-1} (x + j), \quad n \in \mathbb{N}.
\]

If \(\mu_n(z) \in \mathbb{F}\) denote the standard moments of \(L\) on the monomial basis

\[
\mu_n(z) = L[x^n], \quad n \in \mathbb{N}_0,
\]

it follows from (2) and (5) that

\[
\mu_{n+1} = \partial \mu_n = \partial^n \mu_0, \quad n \in \mathbb{N}_0.
\]

Moreover, using (5) we can see that [15]

\[
L[\sigma(x) u(x)] = L[z \tau(x) u(x + 1)], \quad u \in \mathbb{C}[x],
\]

where

\[
\sigma(x) = x (x + b)_1, \quad \tau(x) = (x + a)_1.
\]

Because of (9), we say that the functional \(L\) is of Toda-type [3], [14], [38], [47], and because of (10) we also call \(L\) discrete semiclassical [1], [16], [18], [33], [36], [49]. The class of the functional \(L\) is defined by

\[
s = \max \{\deg(\sigma) - 1, \deg(\tau) - 1\} = \max \{p - 1, q\},
\]
and semiclassical functional of class $s = 0$ are called classical.

Our objective is to obtain comparative asymptotics (also called relative asymptotics) [5], [23], [24], [25], [29], [30], [31], [32], [34], [39], [40], [41], [42], [43], [44], for the polynomials $P_n(x; z)$ with respect to the basis of falling factorial polynomials defined by $\phi_0(x) = 1$ and

$$\phi_n(x) = \prod_{k=0}^{n-1} (x - k), \quad n \in \mathbb{N}. \quad (11)$$

In other words, we want to study the limit

$$\lim_{n \to \infty} \frac{P_n(x; z)}{\phi_n(x)} = O(1), \quad x \not\in \mathbb{N}_0,$$

where $z$ is a fixed number, and $x$ belongs to a compact subset of the complex plane containing the origin. We already considered this type of limits in [10], [12] (Charlier and Meixner polynomials), and in [13] (Krawtchouk polynomials).

Since the functional $L$ is supported on the lattice $\mathbb{N}_0$, the zeros of the polynomial $P_n(x; z)$ will converge to non-negative integer values as $n \to \infty$. Thus, it is natural to approximate $P_n(x; z)$ with a monic polynomial having zeros at $x = 0, 1, \ldots, n - 1$.

The organization of the paper is as follows: in Section 2, we review some of our results from [14]. The polynomials $P_n(x; z)$ have different asymptotic approximations depending on the relation between the parameters $p$ and $q$ defined in (6). Thus, we consider the cases $p = q$ (Section 3.1), $p = q - 1$ (Section 3.2), $p < q - 1$ (Section 3.3), and $p = q + 1$ (Section 3.4). In Section 4, we describe the functions that we use in our plots, and make some observations on the difficulties in computing polynomials $P_n(x; z)$ numerically.

Finally, in the conclusions’ section we summarize the results and discuss future directions.

## 2 Preliminary material

In [14], we studied families of polynomials (that we said to be of Toda type), orthogonal with respect to a linear functional $L : \mathbb{F}[x] \to \mathbb{F}$ satisfying

$$D_z L [u] = L [x u], \quad u \in \mathbb{F} [x],$$
where \( D_z : \mathbb{F} \to \mathbb{F} \) is a fixed derivation (on the variable \( z \)) associated to \( L \).

In this section, we review some of the results that we obtained, and apply them to the particular cases:

(i) \( D_z = \vartheta \), where the operator \( \vartheta \) was defined in (1).

(ii) The variable transformation \( D_w = w (1 - w) \partial_w \), \( w = \frac{z}{z-1} \).

2.1 Toda-type orthogonal polynomials

The linear system (3) can be written as

\[
L [ \Lambda_k P_n ] = h_n \delta_{k,n}, \quad 0 \leq k \leq n,
\]

and we see that the sequence \( \{ P_n (x; z) \} \) satisfies the orthogonality conditions

\[
L [ P_k P_n ] = h_n \delta_{k,n}, \quad 0 \leq k \leq n,
\]

where \( h_n (z) \in \mathbb{F} \setminus \{ 0 \} \) is the norm of \( P_n (x; z) \).

From (12), we see that

\[
L [ x P_k P_n ] = 0, \quad k \neq n, n \pm 1,
\]

and therefore the polynomials \( P_n (x; z) \) satisfy the three term recurrence relation

\[
x P_n (x; z) = P_{n+1} (x; z) + \beta_n (z) P_n (x; z) + \gamma_n (z) P_{n-1} (x; z)
\]

with \( P_{-1} = 0, \ P_0 = 1 \). The coefficients \( \beta_n (z), \gamma_n (z) \in \mathbb{F} \) are given by [8]

\[
\beta_0 = \frac{L[x]}{L[1]}, \quad \gamma_0 = 0,
\]

and

\[
\beta_n = \frac{L[x P_n^2]}{h_n}, \quad \gamma_n = \frac{L[xP_n P_{n-1}]}{h_{n-1}}, \quad n \in \mathbb{N}.
\]

If we define \( \sigma_n (z) \in \mathbb{F} \) by

\[
P_n (x; z) = x^n - \sigma_n (z) x^{n-1} + u_n (x; z), \quad \deg (u_n) \leq n - 2,
\]
we have $\sigma_0 = 0$, and using (13) we get

$$x^{n+1} - \sigma_n x^n + x u_n = x^{n+1} - \sigma_{n+1} x^n + u_{n+1} + \beta_n (x^n - \sigma_n x^{n-1} + u_n) + \gamma_n P_{n-1}.$$ 

Comparing coefficients of $x^n$, we obtain $-\sigma_n = -\sigma_{n+1} + \beta_n$, or

$$\beta_n = \sigma_{n+1} - \sigma_n. \quad (17)$$

Our next result relates $\sigma_n, h_n, \beta_n$ and $\gamma_n$.

**Proposition 1** Let $\vartheta$ be defined by (1), $h_n$ be defined by (12), $\beta_n, \gamma_n$ be defined by (15), and $\sigma_n$ be defined by (16). Then, we have

$$\vartheta \sigma_n = \gamma_n \quad (18)$$

and

$$\vartheta \ln h_n = \beta_n. \quad (19)$$

**Proof.** From (16) we have

$$\vartheta P_n (x; z) = -\vartheta \sigma_n (z) x^{n-1} + \vartheta u_n (x; z),$$

and using (12) we get

$$L [P_{n-1} \vartheta P_n] = - (\vartheta \sigma_n) L [x^{n-1} P_{n-1}] = - (\vartheta \sigma_n) h_{n-1}. \quad (20)$$

On the other hand, since $L [P_n P_{n-1}] = 0$ and $\deg (\vartheta P_{n-1}) = n - 2$,

$$0 = \vartheta L [P_n P_{n-1}] = L [P_{n-1} \vartheta P_n] + L [P_n \vartheta P_{n-1}] + L [x P_n P_{n-1}]$$

$$= - (\vartheta \sigma_n) h_{n-1} + \gamma_n h_{n-1},$$

and we obtain (18). Since $\deg (\vartheta P_n) = n - 1$ we have

$$\vartheta h_n = \vartheta L [P_n^2] = L [2 P_n \vartheta P_n] + L [x P_n^2] = L [x P_n^2] = \beta_n h_n,$$

and (19) follows. ■

As a direct consequence, we see that $(\beta_n, \gamma_n)$ are solutions of the *Toda equations* [47].

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Corollary 2 The coefficients of the 3-term recurrence relation (13) are solutions of the differential-difference equations

\[ \vartheta \beta_n = \Delta \gamma_n, \quad \vartheta \ln \gamma_n = \nabla \beta_n, \]  

(21)

with initial conditions (14), where

\[ \Delta f (n) = f (n + 1) - f (n), \quad \nabla f (n) = f (n) - f (n - 1). \]  

(22)

Essential for our work in this paper is the following theorem.

Theorem 3 The polynomials \( P_n (x; z) \) defined by (12) satisfy the recurrence

\[ \vartheta P_n = -\gamma_n P_{n-1}, \]  

(23)

and the ODE

\[ [\vartheta^2 + (x - \beta_n) \vartheta + \gamma_n] P_n = 0. \]  

(24)

Proof. If we write

\[ \vartheta P_n = \sum_{k=1}^{n-1} v_k P_k, \]

then (20) and (18) give

\[ v_{n-1} = \frac{1}{h_{n-1}} L [P_{n-1} \vartheta P_n] = -\vartheta \sigma_n = -\gamma_n. \]

Moreover, for all \( k = 0, 1, \ldots, n - 2 \)

\[ 0 = \vartheta L [P_n P_k] = L [P_k \vartheta P_n] + L [P_n \vartheta P_k] + L [x P_n P_k] = L [P_k \vartheta P_n] = h_k v_k, \]

and therefore we obtain (23).

From (13) and (23), we have

\[ \vartheta P_n = -\gamma_n P_{n-1} = P_{n+1} + (\beta_n - x) P_n. \]

Using (17), we get

\[ \vartheta^2 P_n = \vartheta P_{n+1} + P_n \vartheta \beta_n + (\beta_n - x) \vartheta P_n \]

\[ = -\gamma_n P_n + (\gamma_{n+1} - \gamma_n) P_n + (\beta_n - x) \vartheta P_n \]

and (24) follows. ■
Since \( \vartheta = z \partial_z \), we have

\[
    z \partial_z P_n = -\gamma_n P_{n-1},
\]

and

\[
    z (z \partial_z^2 P_n + \partial_z P_n) + (x - \beta_n) z \partial_z P_n + \gamma_n P_n = 0. \tag{25}
\]

As we will see in (34), \( \gamma_n (0) = 0 \). If we define \( g_n (z) \in F \) by

\[
    \gamma_n (z) = zg_n (z), \tag{26}
\]

then

\[
    P'_n = -g_n P_{n-1}, \tag{27}
\]

and (25) becomes

\[
    zP''_n + (x + 1 - \beta_n) P'_n + g_n P_n = 0, \tag{28}
\]

where we will always use the notation

\[
    P'_n = \partial_z P_n.
\]

### 2.2 The function \( \sigma_n (z) \)

A fundamental quantity in our studies is \( \sigma_n (z) \) defined in (16).

**Theorem 4** The coefficients in the power series expansion

\[
    \sigma_n (z) = \sum_{k=0}^{\infty} s_k (n) z^k, \tag{29}
\]

are given by

\[
    s_0 (n) = \frac{n (n - 1)}{2}, \quad s_1 (n) = n \frac{(n - 1 + a)}{(n + b)}, \tag{30}
\]

and

\[
    s_k (n) = \frac{1}{k (k - 1)} \sum_{j=1}^{k-1} (k - j) s_{k-j} (n) \Delta \nabla [s_j (n)], \quad k \geq 2, \tag{31}
\]

\( \Delta, \nabla \) are the finite difference operators (acting on \( n \)) defined in (22).
Proof. From (17), (18), and (21) we get
\[ \vartheta \ln(\vartheta \sigma_n) = \vartheta \ln(\gamma_n) = \beta_n - \beta_{n-1} = \sigma_{n+1} - 2\sigma_n + \sigma_{n-1}. \]
Using the difference operators (22), we can write
\[ \sigma_{n+1} - 2\sigma_n + \sigma_{n-1} = \nabla \Delta \sigma_n, \]
and hence
\[ \sigma''_n(z) = \sigma'_n(z) \frac{\nabla \Delta \sigma_n(z) - 1}{z}. \] (32)
Since
\[ \nabla \Delta s_{n,0} = \nabla \Delta \frac{n(n-1)}{2} = 1, \]
we see that from (29) that
\[ \frac{\nabla \Delta \sigma_n - 1}{z} = \sum_{k=1}^{\infty} \nabla \Delta s_{n,k} z^{k-1} = \sum_{k=0}^{\infty} \nabla \Delta s_{n,k+1} z^k. \]
Also,
\[ \sigma'_n(z) = \sum_{k=1}^{\infty} k s_{n,k} z^{k-1} = \sum_{k=0}^{\infty} (k+1) s_{n,k+1} z^k, \]
and
\[ \sigma''_n(z) = \sum_{k=2}^{\infty} k(k-1) s_{n,k} z^{k-2} = \sum_{k=0}^{\infty} (k+2)(k+1) s_{n,k+2} z^k. \]
Comparing coefficients of \( z \) in (32) gives
\[ (k+2)(k+1) s_{n,k+2} = \sum_{j=0}^{k} (k-j+1) s_{n,k-j+1} \nabla \Delta s_{n,j+1}, \]
and (31) follows after shifting \( k \rightarrow k-2 \) and \( j \rightarrow j-1. \) ■
Using (17) and (18), we obtain the following result.

Corollary 5 The coefficients of the 3-term recurrence relation (13) admit the formal power series
\[ \beta_n(z) = \sum_{k=0}^{\infty} \Delta s_k(n) z^k, \quad \gamma_n(z) = \sum_{k=1}^{\infty} k s_k(n) z^k, \] (33)
where the coefficients \( s_k(n) \) are defined by (29). In particular,
\[ \beta_n(0) = n, \quad \gamma_n(0) = 0. \] (34)
Remark 6 From (26) and (33), we have

\[ g_n(z) = \sum_{k=0}^{\infty} (k + 1) s_{k+1}(n) z^k. \]  
(35)

From (30), we see that

\[ s_1(n) = n^{\theta} \frac{(1 - n^{-1} + n^{-1}a)}{(1 + n^{-1}b)}. \]

where

\[ \theta = p + 1 - q. \]  
(36)

If we write

\[ s_1(n) = n^{\theta} \sum_{k=0}^{\infty} r_k n^{-k}, \]  
(37)

we get

\[ \sum_{k=0}^{k} e_{k-j}(b) r_j = e_k(a - 1), \]

where the elementary symmetric polynomials \( e_n(c) \) are defined by the generating function [37, 19.19.4]

\[ \sum_{n=0}^{\infty} e_n(c) t^n = \prod_{i=1}^{m} (1 + t c_i), \quad c \in \mathbb{C}^m. \]  
(38)

Since \( e_0 = 1 \), we obtain the recurrence

\[ r_k = e_k(a - 1) - \sum_{j=0}^{k-1} e_{k-j}(b) r_j, \quad r_0 = 1. \]  
(39)

The first two coefficients \( r_k \) are

\[ r_1 = e_1(a - 1) - e_1(b), \]
\[ r_2 = e_2(a - 1) - e_2(b) - e_1(a - 1) e_1(b) + e_1^2(b). \]

To study the asymptotic behavior of the coefficients \( s_k(n) \) as \( n \to \infty \), we need to consider 2 cases: \( \theta < 2 \) and \( \theta = 2 \). We will analyze the case \( \theta < 2 \) in the next Theorem, and the case \( \theta = 2 \) in Section 2.4.
Theorem 7 Let
\[ \Theta_k = (\theta - 2)k + \eta(\theta), \]
with
\[ \eta(\theta) = \begin{cases} 
0, & \theta = 1 \\
1, & \theta = 0 \\
2, & \theta \neq 0, 1 
\end{cases} \]
We have:
(i) If \( \theta < 0 \), then
\[ s_k(n) \sim A_k(\theta)n^{\Theta_k}, \quad n \to \infty, \quad (40) \]
where \( A_1 = 1 \) and for \( k \geq 2 \)
\[ A_k = \frac{1}{k(k-1)} \sum_{j=1}^{k-1} (k-j) \Theta_j(\Theta_j-1) A_j A_{k-j}. \quad (41) \]
(ii) If \( \theta = 0 \), then as \( n \to \infty \),
\[ s_1(n) \sim 1, \quad s_k(n) \sim r_1 C(k-1) n^{-2k+1}, \quad k \geq 2, \]
where \( C(k) \) is the \( k \)th Catalan number \([37, 26.5(i)]\)
\[ C(k) = \frac{1}{k+1} \binom{2k}{k}. \]
(iii) If \( \theta = 1 \), then as \( n \to \infty \),
\[ s_1(n) \sim n, \quad s_k(n) \sim r_2 n^{-k}, \quad k \geq 2. \]
Proof. See [14]. \( \blacksquare \)

Remark 8 Using induction, we can see that the solution of (41) is given by
\[ A_k(\theta) = -\theta \left(\frac{1-\theta}{k-1}\right)^k (1 + k - \theta k)_{k-3}. \]
As a direct application of (31), we can illustrate the results of Theorem 7 for some particular cases.
Example 9 Let $\theta = 1$. As $n \to \infty$, we have
\[ s_2 = r_2 n^{-2} + (r_1 r_2 + 3r_3) n^{-3} + O\left(n^{-4}\right), \]
\[ s_3 = r_2 n^{-3} + 3(r_1 r_2 + 2r_3) n^{-4} + O\left(n^{-5}\right), \]
and we see that $s_k(n) \sim r_2 n^{-k}$, $n \geq 2$, as expected. Also,
\[
\sigma_n(z) = \frac{n^2}{2} + \left(z - \frac{1}{2}\right) n + n_1 z + r_2 n^{-1} + (r_3 + r_2 z) z n^{-2}
+ [r_4 + (r_1 r_2 + 3r_3) z + r_2 z^2] z n^{-3} + O\left(n^{-4}\right),
\]
\[
\beta_n(z) = n + z - r_2 z n^{-2} + [(1 - 2z) r_2 - 2r_3] z n^{-3} + O\left(n^{-4}\right), \quad (42)
\]
and
\[
g_n(z) = n + r_1 + r_2 n^{-1} + (2zr_2 + r_3) z n^{-2} + O\left(n^{-3}\right). \quad (43)
\]

Example 10 Let $\theta = 0$. As $n \to \infty$, we have
\[ s_2 = r_1 n^{-3} + (r_1^2 + 3r_2) n^{-4} + O\left(n^{-5}\right), \]
\[ s_3 = 2r_1 n^{-5} + 2(3r_1^2 + 5r_2) n^{-6} + O\left(n^{-7}\right), \]
and we see that $s_k(n) \sim C(k-1) \cdot r_1 n^{-2k+1}$, $n \geq 2$, as expected. Also,
\[
\sigma_n(z) = \frac{n^2}{2} - \frac{1}{2} n + z + r_1 z n^{-1} + r_2 z n^{-2} + (r_3 + r_2 z) z n^{-3} + O\left(n^{-4}\right),
\]
\[
\beta_n(z) = n - r_1 z n^{-2} + (r_1 - 2r_2) z n^{-3} - [r_1 (3z + 1) - 3 (r_2 - r_3)] z n^{-4} + O\left(n^{-5}\right), \quad (44)
\]
and
\[
g_n(z) = 1 + r_1 n^{-1} + r_2 n^{-2} + (2zr_2 + r_3) z n^{-3} + O\left(n^{-4}\right). \quad (45)
\]

Example 11 Let $\theta = -1$. As $n \to \infty$, we have
\[ s_2 = n^{-4} + 4r_1 n^{-5} + (1 + 3r_1^2 + 7r_2) n^{-6} + O\left(n^{-7}\right), \]
\[ s_3 = 4n^{-7} + 28r_1 n^{-8} + (20 + 51r_1^2 + 61r_2) n^{-9} + O\left(n^{-10}\right), \]
and we see that $s_k(n) \sim A(k) \cdot r_1 n^{-3k+2}$, $n \geq 2$, as expected. Also,
\[
\sigma_n(z) = \frac{n^2}{2} - \frac{1}{2} n + z n^{-1} + r_1 z n^{-2} + r_2 z n^{-3} + (z + r_3) z n^{-4} + O\left(n^{-5}\right),
\]
\[
\beta_n(z) = n - z n^{-2} + (1 - 2r_1) z n^{-3} - [1 + 3 (r_2 - r_1)] z n^{-4} + O\left(n^{-5}\right), \quad (46)
\]
and
\[
g_n(z) = n^{-1} + r_1 n^{-2} + r_2 n^{-3} + (2z + r_3) z n^{-4} + O\left(n^{-5}\right). \quad (47)
\]
2.3 The function $\Phi_n (z; x)$

Sometimes, the falling factorial polynomials $\phi_n (x)$ defined in (11), are called binomial polynomials, since we have

$$\frac{\phi_n (x)}{n!} = \binom{x}{n}, \quad n \in \mathbb{N}_0. \quad (48)$$

From the definition (11), we see that

$$\phi_{n+1} (x) = (x - n) \phi_n (x) = x \phi_n (x - 1), \quad n \geq 0, \quad (49)$$

and from (7) it follows that the falling factorial polynomials and the Pochhammer polynomials are related by

$$\phi_n (x) = (-1)^n (-x)_n = (x + 1 - n)_n.\quad (50)$$

Using (34) in (13), we obtain

$$P_{n+1} (x; 0) = (x - n) P_n (x; 0), \quad P_0 (x; 0) = 1,$$

and comparing with the recurrence satisfied by the falling factorial polynomials (49), we conclude that

$$P_n (x; 0) = \phi_n (x). \quad (50)$$

Note that from (27) and (50), we see that

$$P'_n (x; 0) = -g_n (0) \phi_{n-1} (x). \quad (51)$$

If we define $\Phi_n (z; x)$ by

$$P_n (x; z) = \phi_n (x) \Phi_n (z; x), \quad (52)$$

then (49) and (51) give the recurrence

$$\Phi'_n (z; x) = -\frac{g_n (z)}{x+1-n} \Phi_{n-1} (z; x). \quad (53)$$

It also follows from (28) and (50) that $\Phi_n (z; x)$ is the solution of the ODE

$$z \Phi''_n + (x + 1 - \beta_n) \Phi'_n + g_n \Phi_n = 0, \quad (54)$$
with initial condition
\[ \Phi_n(0; x) = 1. \] (55)

Note that setting \( z = 0 \) in (54) and using (34) gives
\[ \Phi_n'(0; x) = -\frac{g_n(0)}{x + 1 - n} \]
in agreement with (53).

**Proposition 12** Suppose that
\[ \Phi_n(z; x) = \sum_{k=0}^{\infty} \frac{\alpha_k(n)}{(x + 1 - n)_k} \frac{z^k}{k!}, \quad \alpha_0(n) = 1. \] (56)

Then, the coefficients \( \alpha_k(n) \) satisfy the recurrence
\[ \alpha_{k+1}(n) = -\sum_{j=0}^{k} s_{j+1}(n) \alpha_{k-j}(n-1) (x + 2 - n + k - j). \] (57)

In particular,
\[ \alpha_1(n) = -s_1(n). \] (58)

**Proof.** Taking a derivative in (56), we have
\[
\Phi_n'(z; x) = \sum_{k=0}^{\infty} k \frac{\alpha_k(n)}{(x + 1 - n)_k} \frac{z^{k-1}}{k!} = \frac{1}{x + 1 - n} \sum_{k=0}^{\infty} \frac{\alpha_{k+1}(n) z^k}{(x + 2 - n)_k k!},
\]
since from (7) we see that
\[ (x)_{k+1} = x (x + 1)_k. \]

From (53), we conclude that
\[
\sum_{k=0}^{\infty} \frac{\alpha_{k+1}(n) z^k}{(x + 2 - n)_k k!} = -g_n(z) \sum_{k=0}^{\infty} \frac{\alpha_k(n-1) z^k}{(x + 2 - n)_k k!},
\]
and using (35), we get
\[
\frac{\alpha_{k+1}(n)}{(x + 2 - n)_k} = -\sum_{j=0}^{k} s_{j+1}(n) \frac{\alpha_{k-j}(n-1)}{(x + 2 - n)_{k-j}}. \] (59)
The result follows after using the identity
\[ \frac{(x)_n}{(x)_m} = (x + m)_{n-m}, \quad m \leq n. \]

\[ \text{Remark 13 Suppose that } \theta < 2. \text{ It follows from (59) that to find the leading term in the asymptotic expansion of } \alpha_k(n) \text{ as } n \to \infty, \text{ one needs to consider only the term with } j = 0. \text{ Thus,} \]
\[ \alpha_{k+1}(n) \sim -s_1(n) \alpha_k(n - 1), \quad n \to \infty \]
and we conclude that
\[ \alpha_k(n) \sim (-1)^k \prod_{j=0}^{k-1} s_1(n - j), \quad n \to \infty. \]

Using (37), we get
\[ \alpha_k(n) = (-1)^k n^{k \theta} \left[ 1 + k \left( r_1 - \frac{k-1}{2} \theta \right) n^{-1} + O\left(n^{-2}\right) \right], \quad n \to \infty. \]

\[ \text{Example 14 Let } \theta = 1. \text{ As } n \to \infty, \text{ we have} \]
\[ \frac{\alpha_k(n)}{(x+1-n)_k} = 1 + \frac{x+1+r_1}{n} k + O\left(n^{-2}\right), \]
and therefore
\[ \Phi_n(z;x) = e^z \left[ 1 + \frac{x+1+r_1}{n} z + O\left(n^{-2}\right) \right], \quad n \to \infty. \quad (60) \]

\[ \text{2.4 The variable } w \]
If we use (31) with \( \theta = 2, \) we get
\[ s_1 = n^2 + r_1 n + r_2 + r_3 n^{-1} + O\left(n^{-2}\right), \]
\[ s_2 = n^2 + r_1 n + r_2 + 2r_3 n^{-1} + O\left(n^{-2}\right), \]
\[ s_3 = n^2 + r_1 n + r_2 + 3r_3 n^{-1} + O\left(n^{-2}\right), \]
and this is clearly not an asymptotic sequence. As we showed in [14], what we need is to change variables from \( z \) to
\[ w = \frac{z}{z - 1}. \quad (61) \]
Theorem 15  Let $\sigma_n(z)$ defined by (16). If we write

$$\sigma_n(w) = \sum_{k=0}^{\infty} \xi_k(n) w^k,$$

we have

$$\xi_0(n) = \frac{n(n-1)}{2}, \quad \xi_1(n) = -n\frac{(n-1+a)_1}{(n+b)_1}, \quad (62)$$

and

$$\xi_k = \xi_{k-1} + \frac{1}{k(k-1)} \sum_{j=1}^{k-1} (k-j) \xi_{k-j} \nabla \xi_j, \quad k \geq 2. \quad (63)$$

Proof. See [14].

Remark 16 If we use (37) in (62), we get

$$\xi_1(n) = -n^2 \sum_{k=0}^{\infty} r_k n^{-k}, \quad (64)$$

where the coefficients $r_k$ can be computed using (39).

The asymptotic behavior of the coefficients $\xi_k(n)$ is given in the following result.

Theorem 17 For all $k \geq 2$, we have

$$\xi_k(n) = O\left(n^{-k+1}\right), \quad n \to \infty. \quad (65)$$

Proof. See [14].

Remark 18 For the first few $\xi_k(n)$, we can use (63) and (64), and obtain

$$\xi_2(n) = \frac{r_3}{n} + \frac{r_1 r_3 + 3r_4}{n^2} + O\left(n^{-3}\right),$$

$$\xi_3(n) = -\frac{r_1 r_3 + 2r_4}{n^2} + O\left(n^{-3}\right), \quad (66)$$

$$\xi_4(n) = \frac{(1 + r_1^2 + r_2) r_3 + 5(r_1 r_4 + r_5)}{n^3} + O\left(n^{-4}\right),$$

as $n \to \infty$, in agreement with (65).
Note that we have
\[ \gamma_n = z\sigma'_n(z) = w(1 - w)\hat{\sigma}_n(w), \]
where we will always use the notation
\[ \Phi_n = \partial_\nu \Phi_n. \]
Therefore, in this case we define
\[ \gamma_n(w) = w(1 - w)g_n(w), \quad \text{(67)} \]
with
\[ g_n(w) = \sum_{k=0}^\infty (k + 1)\xi_{n,k+1}w^k. \]

**Example 19** Using (64) and (66), we can compute the first terms in the asymptotic expansions of \( \sigma_n(w), \beta_n(w), \) and \( g_n(w) : \)
\[ \sigma_n(w) = \left( \frac{1}{2} - w \right) n^2 - \left( \frac{1}{2} + r_1 w \right) n - r_2 w + r_3 (w - 1) wn^{-1} + O(n^{-2}), \]
\[ \beta_n(w) = (1 - 2w) n - (1 + r_1) w - r_3 (w - 1) wn^{-2} + O(n^{-3}), \quad \text{(68)} \]
and
\[ g_n(w) = -n^2 - r_1 n - r_2 + r_3 (2w - 1) n^{-1} + O(n^{-2}), \quad \text{(69)} \]
as \( n \to \infty. \)

### 3 Asymptotic analysis

In this section, we will obtain asymptotic approximations for \( P_n(x; z) \) as \( n \to \infty, \) with \( x = O(1) \) and all other parameters fixed. Because of the moments’ recurrence (9), the analyticity of all the moments \( \mu_n(z) \) (and in consequence the polynomials \( P_n \) themselves) as functions of \( z \) will agree with that of the first moment \( \mu_0(z). \)

But since \( \mu_0(z) \) is a hypergeometric function,
\[ \mu_0(z) = {}_pF_q \left( \begin{array}{c} a \\ b \end{array} ; z \right) = \sum_{x=0}^\infty \frac{(a)_x}{(b+1)_x} \frac{z^x}{x!}, \quad a \in \mathbb{C}^p, b \in \mathbb{C}^q, \]
its domain of analyticity depends on the parameters $p, q$. We have three cases to consider:

(i) If $p < q + 1$, then $\mu_0(z)$ is an entire function of $z$. From (36), we see that this corresponds to the case $\theta < 2$.

(ii) If $p = q + 1$ ($\theta = 2$), then $\mu_0(z)$ is analytic inside the unit circle, $|z| < 1$, and can be extended by analytic continuation to the cut plane $\mathbb{C} \setminus [1, \infty)$.

(iii) If $p > q + 1$ ($\theta > 2$), then $\mu_0(z)$ diverges for all $z \neq 0$, except when one of the numerator parameters is a negative integer, and $\mu_0(z)$ becomes a polynomial (in $z$) of degree $N$. We will not study this situation in this paper, since in this case we need to scale $n$ in terms of $N$ and consider the limit as $N \to \infty$ (see [13] for the Krawtchouk polynomials).

We will divide the first case (i) in 3 subcases:

(a) When $p = q$ ($\theta = 1$), $\mu_0(z)$ is entire (but barely!) and the asymptotic expansion of $P_n(x; z)$ will contain an exponential multiple $e^z$.

(b) When $p = q - 1$ ($\theta = 0$), $P_n(x; z)$ will have a regular asymptotic expansion.

(c) When $p < q - 1$ ($\theta < 0$), some of the first terms in the asymptotic expansion of $P_n(x; z)$ will be missing.

If $p = q + 1$ ($\theta = 2$), then $\mu_0(z)$ will have a logarithmic singularity at $z = 1$. Thus, we expect that the asymptotic expansion of $P_n(x; z)$ will have a factor of the form $(1 - z)^\varsigma$, where the power could depend on $n$ (and $x$). In this case, it is better to perform a change of variables and work with $w$ defined in (61).

**Notation 20** We say that a family of polynomials is of type $(p, q)$, if it’s orthogonal with respect to the functional (5) with $a \in \mathbb{C}^p$ and $b \in \mathbb{C}^q$.

### 3.1 Case $p = q$ ($\theta = 1$)

From (60), we see that in this case we should ”peel off” an exponential term from $\Phi_n(z; x)$. Thus, if

$$\Phi_n(z; x) = e^z \Lambda_n(z; x),$$

we have

$$\Phi'_n = e^z (\Lambda_n + \Lambda'_n), \quad \Phi''_n = e^z (\Lambda_n + 2\Lambda'_n + \Lambda''_n).$$

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and (54) becomes
\[ z\Lambda''_n + (2z + x + 1 - \beta_n) \Lambda'_n + (z + x + 1 - \beta_n + g_n) \Lambda_n = 0. \] (71)

From (42) and (43), we see that
\[ \beta_n = n + \tilde{\beta}_n, \quad g_n = n + \tilde{g}_n, \quad \tilde{\beta}_n = O(1), \quad \tilde{g}_n = O(1), \quad n \to \infty, \]
and hence
\[ z\Lambda''_n + \left(2z + x + 1 - \beta_n\right) \Lambda'_n + \left(z + x + 1 + \tilde{g}_n - \tilde{\beta}_n\right) \Lambda_n = 0. \] (72)

Thus, we shall have \( \Lambda_n = O(1), \quad n \to \infty. \) Replacing
\[ \tilde{\beta}_n (z) = \sum_{k=0}^{\infty} v_k (z) n^{-k}, \quad \tilde{g}_n (z) = \sum_{k=0}^{\infty} u_k (z) n^{-k}, \]
and
\[ \Lambda_n (z; x) = \sum_{k=0}^{\infty} \lambda_k (z; x) n^{-k}, \]
in (72) and comparing coefficients of \( n^{-k} \), we obtain the recurrence
\[ \lambda'_{k+1} = z\lambda''_k + (2z + x + 1) \lambda'_k + (z + x + 1) \lambda_k + \sum_{j=0}^{k} \left[ (u_{k-j} - v_{k-j}) \lambda_j - v_{k-j} \lambda'_j \right]. \] (73)

From (55) and (70) we have \( \Lambda_n (0; x) = \Phi_n (0; x) = 1 \), and therefore
\[ \lambda_k (0; x) = \delta_{0,k}, \quad k \geq 0. \] (74)

Note that from (42) and (43) we see that
\[ u_0 = r_1, \quad u_1 = r_2, \quad u_2 = 2zr_2 + r_3, \]
\[ v_0 = z, \quad v_1 = 0, \quad v_2 = -r_2 z. \]

When \( k = -1, \) (73) and (74) give
\[ \lambda'_0 = 0, \quad \lambda_0 (0; x) = 1, \]
and thus
\[ \lambda_0(z; x) = 1. \] (75)

Using (75) in (73), we get
\[ \lambda'_1 = z + x + 1 + u_0 - v_0 = x + 1 + r_1, \]
and since \( \lambda_1(0; x) = 0 \), we obtain
\[ \lambda_1(z; x) = (x + 1 + r_1) z. \] (76)

Similarly, using (75) and (76) in (73), we get after some simplification
\[ \lambda'_2 = \lambda'_1 (x + 1 + z) + \lambda_1 \lambda'_1 + r_2, \]
and since \( \lambda_2(0; x) = 0 \), we conclude that
\[ \lambda_2 = \lambda'_1 \left(x + \frac{z}{2} + 1\right) z + \frac{1}{2} (\lambda_1)^2 + r_2 z, \]
or
\[ \lambda_2(z; x) = [(x + 1) (x + 1 + r_1) + r_2] z + (x + 1 + r_1) (x + 2 + r_1) \frac{z^2}{2}. \] (77)

### 3.1.1 Polynomials of type \((0, 0)\) (Charlier polynomials).

The Charlier polynomials were introduced by Carl Vilhelm Ludwig Charlier (1862–1934) in his paper [7] and have the hypergeometric representation
\[ P_n(x; z) = (-z)^n _2F_0 \left[ \begin{array}{c} -n, -x \end{array} ; -\frac{1}{z} \right]. \]

For this family, we have \( r_k = 0, k \geq 1 \), and therefore
\[ \beta_n = n + z, \quad g_n = n. \]

Replacing in (71), we get
\[ z \Lambda''_n + (z + x + 1 - n) \Lambda'_n + (x + 1) \Lambda_n = 0. \] (78)

Therefore, the recurrence (73) becomes
\[ \lambda'_{k+1} = z \lambda''_k + (z + x + 1) \lambda'_k + (x + 1) \lambda_k, \]
or
\[
\lambda_{k+1} (z) = z (\lambda_k' + \lambda_k) + x \left[ \lambda_k (z) - \lambda_k (0) \right] + x \int_0^z \lambda_k (t) \, dt.
\]
Starting with \( \lambda_0 (z) = 1 \), we obtain
\[
\begin{align*}
\lambda_1 (z) &= (x + 1) z, \\
\lambda_2 (z) &= (x + 1)^2 z + (x + 1) z^2, \\
\lambda_3 (z) &= (x + 1)^3 z + (x + 1)^2 (2x + 3) \frac{z^2}{2} + (x + 1)^3 \frac{z^3}{6}.
\end{align*}
\] (79)

However, in this case the ODE satisfied by \( \Lambda_n (z; x) \) (78) has the exact solution [12]
\[
\Lambda_n (z; x) = {}_1 F_1 \left( \frac{x + 1}{x + 1 - n}; -z \right),
\]
where we have used the initial value \( \Lambda_n (0; x) = 1 \). Therefore,
\[
\Lambda_n (z; x) = \sum_{k=0}^\infty \frac{(x + 1)_k}{(x + 1 - n)_k} \frac{(-z)^k}{k!}
\] (80)
and using the first few terms we obtain
\[
\sum_{k=0}^3 \frac{(x + 1)_k}{(x + 1 - n)_k} \frac{(-z)^k}{k!} = 1 + \frac{(x + 1)}{n} z + \left[ (x + 1)^2 z + (x + 1) \frac{z^2}{2} \right] n^{-2}
\]
\[
+ \left[ (x + 1)^3 z + (x + 1)^2 (2x + 3) \frac{z^2}{2} + (x + 1)^3 \frac{z^3}{6} \right] n^{-3} + O \left( n^{-4} \right)
\]
as \( n \to \infty \), in agreement with (79).

### 3.1.2 Polynomials of type \((1, 1)\) (generalized Meixner)

For this family, we have
\[
\frac{s_1 (n)}{n} = \frac{n + a - 1}{n + b} = 1 + \frac{a - b - 1}{n + b} = 1 + (a - b - 1) \sum_{k=1}^\infty \frac{(-b)^{k-1}}{n^k},
\]
and therefore
\[
r_k = (a - b - 1) (-b)^{k-1}, \quad k \geq 1.
\] (81)

Using (81) in (75)–(77), we get \( \lambda_0 (z; x) = 1 \),
\[
\lambda_1 (z; x) = (x + a - b) z,
\] (82)
\[ \lambda_2(z; x) = [(x + a)(x + 1 - b) + b^2] z + (x + a - b + 1)(x + a - b)\frac{z^2}{2}. \quad (83) \]

For additional information on these polynomials, see [6], [9], [15], [16], [17], [19].

### 3.1.3 Polynomials of type \((2, 2)\)

For this family, we have
\[
\frac{s_1(n)}{n} = \frac{(n + a_1 - 1)(n + a_2 - 1)}{(n + b_1)(n + b_2)} = 1 + \frac{(a_1 - b_2 - 1)(a_2 - b_2 - 1)}{(b_1 - b_2)(n + b_2)} - \frac{(a_1 - b_1 - 1)(a_2 - b_1 - 1)}{(b_1 - b_2)(n + b_1)}
\]

and therefore
\[
\tau_k = \frac{\tau_k^{(1)}(b_2) - \tau_k^{(1)}(b_1)}{b_1 - b_2}, \quad k \geq 1, \quad (84)
\]

with
\[
\tau_k^{(1)}(b) = (b - a_1 + 1)(b - a_2 + 1)(-b)^{k-1}.
\]

In particular,
\[
\begin{align*}
r_1 &= a_1 + a_2 - b_1 - b_2 - 2, \\
r_2 &= 1 - a_1 - a_2 - (a_1 + a_2 - 2)(b_1 + b_2) + b_1^2 + b_2^2 + b_1 b_2 + a_1 a_2.
\end{align*}
\]

Using (84) in (75)–(77), we get \(\lambda_0(z; x) = 1\),
\[
\lambda_1(z; x) = (x + a_1 + a_2 - b_1 - b_2 - 1) z, \quad (85)
\]

and
\[
\begin{align*}
\lambda_2(z; x) &= [(x + 1)(x + a_1 + a_2 - b_1 - b_2 - 1) + r_2] z \\
&\quad + (x + a_1 + a_2 - b_1 - b_2 - 1)(x + a_1 + a_2 - b_1 - b_2)\frac{z^2}{2}. \quad (86)
\end{align*}
\]

For additional information on these polynomials, see [15] and [17].
3.2 Case $p = q - 1 \ (\theta = 0)$

From (44) and (45), we see that

$$\beta_n = n + n^{-2} \tilde{\beta}_n, \quad \tilde{\beta}_n = O(1), \quad g_n = O(1), \quad n \to \infty,$$

and replacing in (54), we get

$$z \Phi''_n + \left( x + 1 - n - n^{-2} \tilde{\beta}_n \right) \Phi'_n + g_n \Phi_n = 0. \quad (87)$$

Thus, we shall have $\Phi_n = O(1), \ n \to \infty$ with $\Phi_n(0; x) = 1$. Replacing

$$\tilde{\beta}_n(z) = \sum_{k=0}^{\infty} v_k(z) n^{-k}, \quad g_n(z) = \sum_{k=0}^{\infty} u_k(z) n^{-k},$$

and

$$\Phi_n(z; x) = \sum_{k=0}^{\infty} \varphi_k(z; x) n^{-k}, \quad \varphi_k(0; x) = \delta_{0,k}, \quad k \geq 0,$$

in (87) and comparing coefficients of $n^{-k}$, we obtain the recurrence

$$\varphi'_{k+1} = z \varphi''_k + (x + 1) \varphi'_k + \sum_{j=0}^{k} \varphi_j u_{k-j} - \sum_{j=0}^{k-2} \varphi'_j v_{k-2-j}. \quad (88)$$

Replacing $\varphi_0 = 1$ in (88) with $k = 0$, we have

$$\varphi_1' = u_0 = 1,$$

and therefore

$$\varphi_1(z; x) = z. \quad (89)$$

Using $\varphi_0 = 1, \varphi_1 = z$ in (88) with $k = 1$, we get

$$\varphi'_2 = x + 1 + u_1 + z u_0 = x + 1 + r_1 + z,$$

and hence

$$\varphi_2(z; x) = (x + 1 + r_1) z + \frac{z^2}{2}. \quad (90)$$

Similarly, we have

$$\varphi'_3 = z + (x + 1) \varphi'_2 + \varphi_0 u_2 + \varphi_1 u_1 + \varphi_2 u_0 - \varphi'_0 v_0$$

$$= z + (x + 1) \varphi'_2 + r_2 + r_1 z + \varphi_2,$$

and we conclude that

$$\varphi_3(z; x) = [(x + 1) (x + 1 + r_1) + r_2] z + [2 (x + 1 + r_1) + 1] \frac{z^2}{2} + \frac{z^3}{6}. \quad (91)$$
3.2.1 Polynomials of type \((0, 1)\) (generalized Charlier)

For this family, we have
\[
s_1(n) = \frac{n}{n+b} = \sum_{k=0}^{\infty} \frac{(-b)^k}{n^k},
\]
and therefore
\[
r_k = (-b)^k, \quad k \geq 0.
\]

Using (92) in (89)–(91), we get
\[
\Phi_n(z; x) \sim 1 + \frac{z}{n} + \frac{(x+1-b) z + \frac{z^2}{2}}{n^2} + \frac{[(x+1)(x+1-b)+b^2] z + [2(x+1-b)+1] \frac{z^2}{2} + \frac{z^3}{6}}{n^3}
\]
as \(n \to \infty\).

For additional information on these polynomials, see [9], [15], [16], [17], [26], [45], [48].

3.2.2 Polynomials of type \((1, 2)\)

For this family, we have
\[
s_1(n) = \frac{n(n+a-1)}{(n+b_1)(n+b_2)} = 1 + \frac{(a-1-b_1)b_1}{(b_1-b_2)(n+b_1)} - \frac{(a-1-b_2)b_2}{(b_1-b_2)(n+b_2)},
\]
and therefore
\[
r_k = \frac{(b_1+1-a)(-b_1)^k + (a-1-b_2)(-b_2)^k}{b_1-b_2}, \quad k \geq 0.
\]

In particular,
\[
r_0 = 1, \quad r_1 = a - b_1 - b_2 - 1, \quad r_2 = (1-a)(b_1+b_2) + b_1^2 + b_2^2 + b_1 b_2.
\]

Using (93) in (89)–(91), we get
\[
\Phi_n(z; x) = 1 + zn^{-1} + \left[ (x+a-b_1-b_2) z + \frac{z^2}{2} \right] n^{-2} + \left[ (x+1)(x+a-b_1-b_2)+r_2 \right] zn^{-3} + \left[ \left( x+a-b_1-b_2 + \frac{1}{2} \right) z^2 + \frac{z^3}{6} \right] n^{-3} + O(n^{-4})
\]
3.3 Case $p < q - 1$ ($\theta < 0$)

Looking at (46) and (47), suggests that as $n \to \infty$, 
\[ \beta_n = n + n^{\theta - 1} \tilde{\beta}_n, \quad \tilde{\beta}_n = O(1), \quad g_n = n^\theta \tilde{g}_n, \quad \tilde{g}_n = O(1), \]
and replacing in (54), we get
\[ z \Phi''_n + \left( x + 1 - n - n^{\theta - 1} \tilde{\beta}_n \right) \Phi'_n + n^\theta \tilde{g}_n \Phi_n = 0. \] (94)

Thus, we expect that
\[ \Phi_n(z; x) = 1 + n^{\theta - 1} \tilde{\Phi}_n(z; x), \quad \tilde{\Phi}_n = O(1), \quad n \to \infty \]
with $\tilde{\Phi}_n(0; x) = 0$, and therefore the ODE (94) becomes
\[ z n^{\theta - 1} \tilde{\Phi}''_n + \left( x + 1 - n - n^{\theta - 1} \tilde{\beta}_n \right) n^{\theta - 1} \tilde{\Phi}'_n + n^\theta \tilde{g}_n + n^{2\theta - 1} \tilde{g}_n \tilde{\Phi}_n = 0, \]
or
\[ z \tilde{\Phi}''_n + \left( x + 1 - n - n^{\theta - 1} \tilde{\beta}_n \right) \tilde{\Phi}'_n + n \tilde{g}_n + n^\theta \tilde{g}_n \tilde{\Phi}_n = 0. \] (95)

Replacing
\[ \tilde{\beta}_n(z) = \sum_{k=0}^{\infty} u_k(z) n^{-k}, \quad g_n(z) = \sum_{k=0}^{\infty} u_k(z) n^{-k}, \]
and
\[ \tilde{\Phi}_n(z; x) = \sum_{k=0}^{\infty} \varphi_k(z; x) n^{-k}, \quad \varphi_k(0; x) = 0, \quad k \geq 0 \]
in (95) and comparing coefficients of $n^{-k}$, we obtain the recurrence
\[ \varphi'_k = u_k + z \varphi''_{k-1} + (x + 1) \varphi'_{k-1} + \sum_{j=0}^{k-1} \varphi_j u_{k-1-j} - \sum_{j=0}^{k+1} \varphi'_j v_{k+\theta - 2-j}. \] (96)

Setting $k = 0$ in (96), we get
\[ \varphi'_0 = u_0 = 1, \]
and therefore
\[ \varphi_0 (z; x) = z. \] (97)

For \( k = 1 \), we have
\[ \varphi'_{1} = u_1 + z\varphi''_0 + (x + 1) \varphi'_0 \]
\[ + \sum_{j=0}^{\theta} \varphi_j u_{\theta-j} - \sum_{j=0}^{\theta-1} \varphi'_j u_{\theta-1-j}, \]
but since \( \theta < 0 \) and \( \varphi_0 = z \),
\[ \varphi'_{1} = u_1 + x + 1 \]
and hence
\[ \varphi_{1} (z; x) = (x + 1 + r_1) z. \] (98)

Continuing this way, we see that
\[ \varphi'_{k} = u_k + z\varphi''_{k-1} + (x + 1) \varphi'_{k-1}, \quad 1 \leq k < 1 - \theta, \]
and for \( k = 1 - \theta \)
\[ \varphi'_{1-\theta} = u_{1-\theta} + z\varphi''_{-\theta} + (x + 1) \varphi'_{-\theta} + \varphi_0 u_0. \]

Thus,
\[ \varphi_k (z; x) = \int_{0}^{z} u_k (t) dt + z\varphi'_{k-1} (z; x) + x\varphi_{k-1} (z; x), \quad 1 \leq k < 1 - \theta, \] (99)

and
\[ \varphi_{1-\theta} (z; x) = \int_{0}^{z} u_{1-\theta} (t) dt + z\varphi'_{-\theta} (z; x) + x\varphi_{-\theta} (z; x) + \frac{z^2}{2}. \] (100)

### 3.3.1 Polynomials of type \((0, 2)\)

For this family, we have
\[ \frac{s_1 (n)}{n^{-1}} = \frac{n^2}{(n + b_1) (n + b_2)} = 1 + \frac{b_2^2}{(b_1 - b_2) (n + b_2)} - \frac{b_1^2}{(b_1 - b_2) (n + b_1)}, \]
and therefore
\[ r_k = \frac{(-b_2)^{k+1} - (-b_1)^{k+1}}{b_1 - b_2}, \quad k \geq 0. \]

In particular,
\[ r_0 = 1, \quad r_1 = -(b_1 + b_2), \quad r_2 = b_1 b_2 + b_1^2 + b_2^2. \quad (101) \]

Using (101) in (98) and (100), we get
\[ \varphi_1 (z; x) = (x + 1 - b_1 - b_2) z, \]
\[ \varphi_2 = \int_0^z u_2 (t) dt + z \varphi_1' + x \varphi_1 + \frac{z^2}{2} = \int_0^z r_2 dt + (x + 1) (x + 1 - b_1 - b_2) z + \frac{z^2}{2}, \]
and hence
\[ \varphi_2 (z; x) = (b_1 b_2 + b_1^2 + b_2^2) z + (x + 1) (x + 1 - b_1 - b_2) z + \frac{z^2}{2}. \]

Combining the results above and recalling that \( \varphi_0 = z \), we obtain
\[ \Phi_n (z; x) = 1 + zn^{-2} + (x + 1 - b_1 - b_2) zn^{-3} \\
+ \left[ (b_1 b_2 + b_1^2 + b_2^2) z + (x + 1) (x + 1 - b_1 - b_2) z + \frac{z^2}{2} \right] n^{-4} + O (n^{-5}). \]

For additional information on these polynomials, see [15] and [17].

3.4 Case \( p = q + 1 \) (\( \theta = 2 \))

Let \( w \) be defined by (61). Using
\[ \partial_z = -(w - 1)^2 \partial_w, \quad \partial_z^2 = (w - 1)^4 \partial_w^2 + 2 (w - 1)^3 \partial_w, \]
in (25), we get
\[ w^2 (1 - w)^2 \partial_w^2 \Phi_n + (x + 1 - \beta_n - 2w) w (1 - w) \partial_w \Phi_n + \gamma_n \Phi_n = 0, \]
and from (67) we have
\[ w (1 - w) \Phi_n + (x + 1 - \beta_n - 2w) \Phi_n + g_n \Phi_n = 0. \quad (102) \]
Based on the case $\theta = 1$ (Section 3.1), we expect that $\Phi_n(w; x)$ will contain an exponential term. Replacing
\[
\Phi_n(w; x) = \exp[\Upsilon_n(w; x)], \quad \Upsilon_n(0; x) = 0,
\]
in (102), we obtain
\[
w(1 - w) \left[ \ddot{\Upsilon}_n + \left( \dot{\Upsilon}_n \right)^2 \right] + (x + 1 - \beta_n - 2w) \dot{\Upsilon}_n + g_n = 0. \tag{103}
\]
From (68)–(69), we have
\[
\beta_n = (1 - 2w) n - (1 + r_1) w + \bar{\beta}_n, \quad \bar{\beta}_n = O(n^{-2}), \quad n \to \infty,
\]
\[
g_n = -n^2 - r_1 n + \bar{g}_n, \quad \bar{g}_n = O(1), \quad n \to \infty, \tag{104}
\]
and replacing in (103) gives, to leading order,
\[
w(1 - w) \left( \dot{\Upsilon}_n \right)^2 \sim (1 - 2w) n \dot{\Upsilon}_n + n^2, \quad n \to \infty
\]
and therefore
\[
\dot{\Upsilon}_n \sim \frac{n}{w}, \quad \text{or} \quad \dot{\Upsilon}_n \sim \frac{n}{w - 1}, \quad n \to \infty.
\]
Since we want $\Upsilon_n(w; x)$ to be analytic in a neighborhood of $w = 0$, we choose
\[
\Upsilon_n(w; x) \sim \ln(1 - w) n, \quad n \to \infty,
\]
and set
\[
\Upsilon_n(w; x) = \ln(1 - w) n + \sum_{k=0}^{\infty} \epsilon_k(w; x) n^{-k}, \quad \epsilon_k(0; x) = 0, \quad k \geq 0, \tag{105}
\]
\[
\bar{\beta}_n(w) = \sum_{k=2}^{\infty} v_k(w; x) n^{-k}, \quad \bar{g}_n(w) = \sum_{k=0}^{\infty} u_k(w; x) n^{-k}, \tag{106}
\]
where from (68)–(69) we see that
\[
v_2 = r_3 (1 - w) w, \quad u_0 = -r_2, \quad u_1 = r_3 (2w - 1). \tag{107}
\]
Using (105)–(106) in (103) and comparing powers of $n$, we get
\[
\dot{\epsilon}_0 = \frac{x + 1 + r_1}{w - 1}.
\]
Thus, since $\epsilon_0 (0; x) = 0$,

$$\epsilon_0 (w; x) = (x + 1 + r_1) \ln (1 - w).$$

We could proceed in this manner, but instead we consider $\Psi_n (w; x)$ defined by

$$\Phi_n (w; x) = (1 - w)^{n+x+1+r_1} \Psi_n (w; x), \quad (108)$$

so that

$$\Psi_n (w; x) = \exp \left[ \sum_{k=1}^{\infty} \epsilon_k (w; x) n^{-k} \right] = O (1), \quad n \to \infty.$$

Using (104) and (108) in (102), we get

$$w (1 - w)^2 \ddot{\Psi}_n + (1 - w) \left[ x + 1 - w (r_1 + 2x + 3) - \tilde{\beta}_n - n \right] \dot{\Psi}_n + \left[ (n + x + 1 + r_1) \tilde{\beta}_n + (1 - w) (\tilde{g}_n - (x + 1) (x + 1 + r_1)) \right] \Psi_n = 0. \quad (109)$$

Replacing (106) and

$$\Psi_n (w; x) = \sum_{k=0}^{\infty} \psi_k (w; x) n^{-k}, \quad \psi_k (0; x) = \delta_{0,k}, \quad k \geq 0$$

in (109), we obtain the recurrence

$$(1 - w) \dot{\psi}_{k+1} = w (1 - w)^2 \ddot{\psi}_k + (1 - w) \left[ x + 1 - (r_1 + 2x + 3) w \right] \dot{\psi}_k + (x + 1) (x + 1 + r_1) (w - 1) \psi_k + (1 - w) \sum_{j=0}^{k} \psi_j u_{k-j} + \sum_{j=0}^{k-1} \psi_j v_{k+1-j} + \sum_{j=0}^{k-2} \left[ (x + 1 + r_1) \psi_j - \dot{\psi}_j \right] v_{k-j} = 0. \quad (110)$$

Setting $k = 0$ and $\psi_0 = 1$ in (110), we obtain

$$\dot{\psi}_1 = -(x + 1) (x + 1 + r_1) + u_0,$$

and since $u_0 = -r_2$ and $\psi_1 (0; x) = 0$, we conclude that

$$\psi_1 (w; x) = - [(x + 1) (x + 1 + r_1) + r_2] w. \quad (111)$$
Replacing \( k = 1 \) and \( \psi_0 = 1 \) in (110), we have
\[
(1 - w) \dot{\psi}_2 = (1 - w) [x + 1 - (r_1 + 2x + 3)w] \dot{\psi}_1 \\
+ (x + 1) (x + 1 + r_1) (w - 1) \dot{\psi}_1 + (1 - w) (u_1 + \psi_1 u_0) + v_2,
\]
and using (107) and \( \psi_1 = w \dot{\psi}_1 \), we get
\[
(1 - w) \dot{\psi}_2 = (1 - w) [x + 1 - (r_1 + 2x + 3)w] \dot{\psi}_1 \\
+ (x + 1) (x + 1 + r_1) (w - 1) w \dot{\psi}_1 \\
+ (1 - w) \left( r_3 (2w - 1) - r_2 w \dot{\psi}_1 \right) + r_3 (1 - w) w,
\]
or
\[
\dot{\psi}_2 = [x + 1 - ((x + 2) (x + 2 + r_1) + r_2) w] \dot{\psi}_1 + r_3 (3w - 1).
\]
Since \( \psi_2 (0; x) = 0 \), we conclude that
\[
\psi_2 (w; x) = \left[ (x + 1) w - ((x + 2) (x + 2 + r_1) + r_2) \frac{w^2}{2} \right] \dot{\psi}_1 + \frac{r_3}{2} w (3w - 2),
\]
and noting from (111) that
\[
- [(x + 2) (x + 2 + r_1) + r_2] w = \psi_1 (w; x + 1),
\]
we can write
\[
\psi_2 (w; x) = \left[ x + 1 + \frac{1}{2} \psi_1 (w; x + 1) \right] \psi_1 (w; x) + \frac{r_3}{2} w (3w - 2). \tag{112}
\]

### 3.4.1 Polynomials of type \((1, 0)\) (Meixner polynomials)

The Meixner polynomials were introduced by Josef Meixner (1908 – 1994) in his paper [35] and have the representation
\[
P_n (x; z) = (a)_n \left( 1 - \frac{1}{z} \right)^{-n} \binom{-n, -x}{a; 1 - \frac{1}{z}}, \quad z \in \mathbb{C} \setminus [1, \infty).
\]

For this family, we have
\[
- \frac{\xi_1 (n)}{n^2} = \frac{n + a - 1}{n},
\]


and therefore
$$r_0 = 1, \quad r_1 = a - 1, \quad r_k = 0, \quad k \geq 2,$$
(113)

and
$$\beta_n(w) = (1 - 2w)n - aw, \quad g_n(w) = -n^2 - (a - 1)n.$$  
(114)

Thus, in this case $\tilde{\beta}_n = \tilde{g}_n = 0$, and using (113) in (109), we obtain

$$w (1 - w) \ddot{\Psi}_n + [x + 1 - (2x + 2 + a)w - n] \dot{\Psi}_n - (x + 1) (x + a) \Psi_n = 0,$$
(115)

while the recurrence (110) becomes

$$\dot{\psi}_{k+1} = w (1 - w) \dot{\psi}_k + [x + 1 - (2x + 2 + a)w] \dot{\psi}_k - (x + 1) (x + a) \psi_k.$$  

It follows that, as $n \to \infty$,

$$\Psi_n(w; x) \sim 1 - (x + 1) (x + a) wn^{-1} - [x + 1 - \frac{1}{2} (x + 2) (x + 1 + a) w] (x + 1) (x + a) wn^{-2}.$$  
(116)

However, the ODE (115) can be solved exactly, and we have [12]

$$\Psi_n(w; x) = \text{2F1} \left( \frac{x + 1, x + a}{x + 1 - n}; w \right),$$

and using the first couple of terms, we get

$$\Psi_n(w; x) \sim \sum_{k=0}^2 \frac{(x + 1)_k (x + a)_k}{(x + 1 - n)_k} \frac{w^k}{k!} \sim - (x + 1) (x + a) wn^{-1} - (x + 1) (x + a) w \left[ x + 1 - \frac{1}{2} (x + 2) (x + 1 + a) w \right] n^{-2}, \quad n \to \infty,$$

in agreement with (116).

### 3.4.2 Polynomials of type (2, 1) (generalized Hahn polynomials of type I)

For this family, we have

$$- \xi_1(n) = \frac{(n + a_1 - 1) (n + a_2 - 1)}{n (n + b)}$$

$$= 1 + \frac{(a_1 - 1) (a_2 - 1)}{bn} - \frac{(b + 1 - a_1)(b + 1 - a_2)}{b(n + b)}.$$
and therefore

\[ r_0 = 1, \quad r_1 = a_1 + a_2 - 2 - b, \]
\[ r_k = (b + 1 - a_1) (b + 1 - a_2) (-b)^{k-2}, \quad k \geq 2. \]  

(117)

Using (117) in (111)–(112), we get

\[ \psi_1 (w; x) = -\left[(x + 1) (x + a_1 + a_2 - 1 - b) + (b - a_1 + 1) (b - a_2 + 1)\right] w \]

(118)

and

\[ \psi_2 (w; x) = \left[x + 1 + \frac{1}{2} \psi_1 (w; x + 1)\right] \psi_1 (w; x) \]
\[ -\frac{1}{2} (b - a_1 + 1) (b - a_2 + 1) bw (3w - 2). \]

(119)

For additional information on these polynomials, see [11], [15], [16], [17], [20].

3.4.3 Polynomials of type \((3, 2)\)

For this family, we have

\[ -\frac{\xi_1 (n)}{n^2} = \frac{(n + a_1 - 1) (n + a_2 - 1) (n + a_3 - 1)}{n (n + b_1) (n + b_2)} \]

and using the elementary symmetric polynomials defined by (38), we can write

\[ r_0 = 1, \quad r_1 = e_1 (A) - e_1 (b), \]
\[ r_2 = e_2 (A) - e_1 (A) e_1 (b) + e_1^2 (b) - e_2 (b) \]
\[ r_3 = 2e_1 (b) e_2 (b) + e_1 (a) [e_1^2 (b) - e_2 (b)] - e_2 (a) e_1 (b) + e_3 (a) - e_1^3 (b) \]

(120)

where

\[ A = a - 1. \]

At this point, we truly reach the limit of being able to type expressions in a compact way. For the first terms in the asymptotic expansion of these polynomials, we refer to the general formulas (111)–(112) with \(r_1, r_2\) given by (120).

For additional information on these polynomials, see [15] and [17].
4 Numerical results

Since we can write the falling factorial polynomials in terms of factorials (48), we can use the reflection formula for the Gamma function [37, 5.5.3]

\[ \Gamma(z) \Gamma(1 - z) = \frac{\pi}{\sin(\pi z)}, \]

and obtain

\[ \phi_n(x) = \frac{x!}{\Gamma(x + 1 - n)} \frac{x! \sin[\pi (n - x)]}{\pi} \Gamma(n - x). \]

But

\[ \sin(\pi(n - x)) = -\cos(\pi n) \sin(\pi x) = (-1)^{n+1} \sin(\pi x), \]

and therefore

\[ \phi_n(x) = (-1)^{n+1} x! \frac{\sin(\pi x)}{\pi} \Gamma(n - x). \]

Let \( \hat{\Phi}_n(z; x) \) denote an asymptotic approximation for the function \( \Phi_n(z; x) \) defined by (52). In order to plot the different asymptotic approximations for \( P_n(x; z) \), we will consider two cases:

i) On the negative real axis, we shall graph

\[ \frac{P_n(x; z)}{\Gamma(n - x)} \quad \text{and} \quad (-1)^{n+1} x! \frac{\sin(\pi x)}{\pi} \hat{\Phi}_n(z; x), \]

(121)

since both functions are analytic, nonzero, and bounded in this region.

ii) On the positive real axis (with \( x < n \)), we shall graph

\[ \frac{P_n(x; z)}{x!\Gamma(n - x)} \quad \text{and} \quad (-1)^{n+1} \frac{\sin(\pi x)}{\pi} \hat{\Phi}_n(z; x), \]

(122)

since both functions are analytic and bounded in this region.

To compute the polynomials \( P_n(x; z) \), we first compute the moments of \( L \) on the monomial basis (8) to a very high order of accuracy (with error less than \( \varepsilon = 10^{-100} \)), solve the system of equations (3)

\[ \mu_{n+k} + \sum_{i=0}^{n-1} \mu_{k+i} \xi_{n,i} = 0, \quad 0 \leq k \leq n - 1, \]
and construct the polynomials using (4),

\[ P_n(x; z) = x^n + \sum_{i=0}^{n-1} \xi_{n,i}(z) x^i. \]

After that, we double-check that

\[ |L[x^k P_n]| < \varepsilon, \quad 0 \leq k \leq n - 1, \quad |L[x^n P_n]| > \varepsilon. \]

We have tried other methods (using Hankel determinants, recurrences, or the Toda equations and the 3-term recurrence relation), but found them unsatisfactory from a numerical point of view.

We will now present some graphs of the examples studied in the previous sections, showing the accuracy of our asymptotic approximations in a neighborhood of \( x = 0 \).

In Figure 1, we plot the functions (121)–(122) for the generalized Meixner polynomials, with

\[ \hat{\Phi}_n(z; x) = e^z \left[ 1 + \lambda_1(z; x) n^{-1} + \lambda_2(z; x) n^{-2} \right], \]

where \( \lambda_1(z; x) \) was defined in (82), \( \lambda_2(z; x) \) was defined in (83), \( n = 10, \quad a = 0.2479357, \quad b = 0.7146983, \quad \text{and} \quad z = 0.3974126. \)

![Figure 1](image1.png)

(a) \( x < 0 \) \hspace{2cm} (b) \( x > 0 \)

Figure 1: A plot of the scaled generalized Meixner polynomial \( P_{10}^{(1,1)}(x; z) \) and its approximation.

In Figure 2, we plot the functions (121)–(122) for the polynomials of type \( (2, 2) \), with

\[ \hat{\Phi}_n(z; x) = e^z \left[ 1 + \lambda_1(z; x) n^{-1} + \lambda_2(z; x) n^{-2} \right], \]
where $\lambda_1(z;x)$ was defined in (85), $\lambda_2(z;x)$ was defined in (86), $n = 10$, $a_1 = 0.2479357$, $a_2 = 0.1963478$, $b_1 = 0.7146983$, $b_2 = 0.5712349$, and $z = 0.3974126$.

In Figure 2, we plot the functions (121)–(122) for the polynomials of type $(1, 2)$, with

$$\hat{\Phi}_n(z) = 1 + zn^{-1} + \left( (x + 1 - b) z + \frac{z^2}{2} \right) n^{-2},$$

where $n = 10$, $b = 0.7146983$, and $z = 0.3974126$.

In Figure 3, we plot the functions (121)–(122) for the polynomials of type $(1, 2)$, with

$$\hat{\Phi}_n(z) = 1 + zn^{-1} + \left( (x + 1 - b) z + \frac{z^2}{2} \right) n^{-2},$$

where $n = 10$, $b = 0.7146983$, and $z = 0.3974126$.

In Figure 4, we plot the functions (121)–(122) for the polynomials of type $(1, 2)$, with

$$\hat{\Phi}_n(z) = 1 + zn^{-1} + \left( (x + 1 - b) z + \frac{z^2}{2} \right) n^{-2},$$

where $n = 10$, $b = 0.7146983$, and $z = 0.3974126$. 

Figure 2: A plot of the scaled polynomial $P_{10}^{(2,2)}(x; z)$ and its approximation.

Figure 3: A plot of the scaled generalized Charlier polynomial $P_{10}^{(0,1)}(x; z)$ and its approximation.
where \( n = 10, a = 0.2479357, b_1 = 0.7146983, b_2 = 0.5712349, \) and \( z = 0.3974126. \)

![Figure 4: A plot of the scaled polynomial \( P_{10}^{(1,2)} (x; z) \) and its approximation.](image)

In Figure 5, we plot the functions (121)–(122) for the polynomials of type \((0, 2)\), with

\[
\widehat{\Phi}_n (z; x) = 1 + n^{-2} \left[ z + (x + 1 - b_1 - b_2) z n^{-1} \right],
\]

where \( n = 10, b_1 = 0.7146983, b_2 = 0.5712349, \) and \( z = 0.3974126. \)

![Figure 5: A plot of the scaled polynomial \( P_{10}^{(0,2)} (x; z) \) and its approximation.](image)

In Figure 6, we plot the functions (121)–(122) for the generalized Hahn polynomials of type I, with

\[
\widehat{\Phi}_n (w; x) = (1 - w)^{n+x+1+r_1} \left[ 1 + \psi_1 (w; x) n^{-1} + \psi_2 (w; x) n^{-2} \right],
\]
Figure 6: A plot of the scaled generalized Hahn polynomial $P_{10}^{(2,1)}(x; z)$ and its approximation.

where $\psi_1(w; x)$ was defined in (118), $\psi_2(w; x)$ was defined in (119), $r_1 = a_1 + a_2 - 2 - b$, $n = 10$, $a_1 = 0.2479357$, $a_2 = 0.1963478$, $b = 0.7146983$, $z = -0.01574126$, and $w = 0.0154973$.

Finally, in Figure 7, we plot the functions (121)–(122) for the polynomials of type $(3, 2)$, with

$$\Phi_n(w; x) = (1 - w)^{n+x+1+r_1} \left[ 1 + \frac{\psi_1(w; x)}{n} + \frac{\psi_2(w; x)}{n^2} \right],$$

where $\psi_1(w; x)$ was defined in (111), $\psi_2(w; x)$ was defined in (112), $r_1, r_2, r_3$ are given by (120), $n = 10$, $a_1 = 0.2479357$, $a_2 = 0.1963478$, $a_3 = 0.3614782$, $b_1 = 0.7146983$, $b_2 = 0.5712349$, $z = -0.01574126$, and $w = 0.0154973$.

Figure 7: A plot of the scaled polynomial $P_{10}^{(3,2)}(x; z)$ and its approximation.
5 Conclusions

We have given asymptotic expansions for the ratio

\[ \frac{P_n(x; z)}{\phi_n(x)}, \quad x = O(1), \quad x \not\in \mathbb{N}_0, \]

as \( n \to \infty \), where \( z \) (and any other parameters) is fixed. The polynomials \( P_n(x; z) \) are orthogonal with respect to the linear functional

\[ L[u] = \sum_{x=0}^{\infty} u(x) \frac{(a)_x}{(b+1)_x} x! \]

and depending on the value of the parameter \( \theta = p + 1 - q \), we have the following cases:

(i) If \( \theta < 1 \), then

\[ \frac{P_n(x; z)}{\phi_n(x)} = 1 + zn^{\theta-1} \left[ 1 + \frac{x + 1 + r_1}{n} + O\left(n^{-2}\right) \right], \quad n \to \infty, \]

where

\[ \frac{(1 - n^{-1} + an^{-1})_1}{(1 + bn^{-1})_1} = \sum_{k=0}^{\infty} r_k n^{-k}. \]

(ii) If \( \theta = 1 \), then as \( n \to \infty \)

\[ \frac{P_n(x; z)}{\phi_n(x)} = e^z \left[ 1 + \frac{x + 1 + r_1}{n} z O\left(n^{-2}\right) \right]. \]

This result extends our previous work on the Charlier polynomials, [10], [12].

(iii) If \( \theta = 2 \), then as \( n \to \infty \)

\[ \frac{P_n(x; w)}{\phi_n(x)} = (1 - w)^{n+x+1+r_1} \left[ 1 - \frac{(x + 1)(x + 1 + r_1) + r_2 w + O\left(n^{-2}\right)}{n} \right], \]

where \( w = \frac{z}{z-1} \). This result extends our previous work on the Meixner polynomials, [10], [12].

(iv) If \( \theta > 2 \), then the polynomials \( P_n(x; w) \) depend on a parameter \( N \), with \( -N \in \mathbb{N} \). We have not analyzed this case, since it will require scaling \( N \) in terms of \( n \). For some related work on the Krawtchouk polynomials, see [13]. We plan to study this case in a forthcoming paper.
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