Comparative asymptotics for discrete semiclassical orthogonal polynomials

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Abstract

We study the ratio $\frac{P_n(x;z)}{\phi_n(x)}$ asymptotically as $n \to \infty$, where the polynomials $P_n(x;z)$ are orthogonal with respect to a discrete linear functional and $\phi_n(x)$ denote the falling factorial polynomials.

We give recurrences that allow the computation of high order asymptotic expansions of $P_n(x;z)$ and give examples for most discrete semiclassical polynomials of class $s \leq 2$.

We show several plots illustrating the accuracy of our results.

Keywords: Semiclassical orthogonal polynomials, asymptotic expansions, ordinary differential equations.

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1 Introduction

Let $\mathbb{N}_0$ be the set of nonnegative integers

$$\mathbb{N}_0 = \mathbb{N} \cup \{0\} = \{0, 1, 2, \ldots\}.$$

We will denote by $\delta_{k,n}$ the *Kronecker delta*, defined by

$$\delta_{k,n} = \begin{cases} 1, & k = n \\ 0, & k \neq n \end{cases}, \quad k, n \in \mathbb{N}_0,$$

and let $\mathbb{F}$ be the ring of *formal power series* in the variable z

$$\mathbb{F} = \mathbb{C}[[z]] = \left\{ \sum_{n=0}^{\infty} c_n z^n : \quad c_n \in \mathbb{C} \right\}.$$

We consider the differential operator $\vartheta : \mathbb{F} \to \mathbb{F}$ defined by [37, 16.8.2]

$$\vartheta = z \partial_z, \tag{1}$$

where ∂_z is the *derivative operator*

$$\partial_z = \frac{\partial}{\partial z}.$$

The action of ϑ on the monomials is given by

$$\vartheta^k z^x = x^k z^x, \tag{2}$$

where we always assume that x and z are **independent variables**.

Suppose that $L : \mathbb{F}[x] \to \mathbb{F}$ is a *linear functional* (acting on the variable x), and $\{\Lambda_n(x)\}_{n \geq 0} \subset \mathbb{C}[x]$ is a sequence of **monic polynomials** with $\deg(\Lambda_n) = n$. If the system of linear equations

$$L[\Lambda_k \Lambda_n] + \sum_{i=0}^{n-1} L[\Lambda_k \Lambda_i] \xi_{n,i} = 0, \quad 0 \leq k \leq n-1, \tag{3}$$

has a **unique solution** $\{\xi_{n,i}(z)\}_{0 \leq i \leq n-1} \subset \mathbb{F}$, we can define **monic polynomials** $P_n(x; z)$ by $P_0(x; z) = 1$ and

$$P_n(x; z) = \Lambda_n(x) + \sum_{i=0}^{n-1} \xi_{n,i}(z) \Lambda_i(x), \quad n \geq 1. \tag{4}$$

We say that $\{P_n(x;z)\}_{n\geq 0}$ is a sequence of (monic) *orthogonal polynomials* with respect to the functional L, [2], [4], [21], [22], [27], [28], [46].

In this paper, we focus on linear functionals of the form

$$L[u] = \sum_{x=0}^{\infty} u(x) \frac{(\mathbf{a})_x}{(\mathbf{b}+1)_x} \frac{z^x}{x!}, \quad u \in \mathbb{F}[x], \tag{5}$$

and we use the notation

$$(\mathbf{a})_n = \prod_{i=1}^{p} (a_i)_n, \quad (\mathbf{b})_n = \prod_{i=1}^{q} (b_i)_n, \quad n \in \mathbb{N}_0,$$
$$\mathbf{c}+r = (c_1+r, c_2+r, \ldots, c_m+r) \in \mathbb{C}^m, \quad r \in \mathbb{C},\ \mathbf{c} \in \mathbb{C}^m,$$

where

$$\mathbf{a} = (a_1, \ldots, a_p) \in \mathbb{C}^p, \quad \mathbf{b} = (b_1, \ldots, b_q) \in \mathbb{C}^q, \quad p, q \in \mathbb{N}_0, \tag{6}$$

and the *Pochhammer polynomial* $(x)_n$ is defined by $(x)_0 = 1$ and [37, 18:12]

$$(x)_n = \prod_{j=0}^{n-1} (x+j), \quad n \in \mathbb{N}. \tag{7}$$

If $\mu_n(z) \in \mathbb{F}$ denote the *standard moments* of L on the monomial basis

$$\mu_n(z) = L[x^n], \quad n \in \mathbb{N}_0, \tag{8}$$

it follows from (2) and (5) that

$$\mu_{n+1} = \vartheta \mu_n = \vartheta^n \mu_0, \quad n \in \mathbb{N}_0. \tag{9}$$

Moreover, using (5) we can see that [15]

$$L[\sigma(x) u(x)] = L[z\tau(x) u(x+1)], \quad u \in \mathbb{C}[x], \tag{10}$$

where

$$\sigma(x) = x(x+\mathbf{b})_1, \quad \tau(x) = (x+\mathbf{a})_1.$$

Because of (9), we say that the functional L is of *Toda-type* [3], [14], [38], [47], and because of (10) we also call L *discrete semiclassical* [1], [16], [18], [33], [36], [49]. The *class* of the functional L is defined by

$$s = \max\{\deg(\sigma) - 1, \deg(\tau) - 1\} = \max\{p-1, q\},$$

and semiclassical functional of class $s = 0$ are called *classical*.

Our objective is to obtain *comparative asymptotics* (also called *relative asymptotics*) [5], [23], [24], [25], [29], [30], [31], [32], [34], [39], [40], [41], [42], [43], [44], for the polynomials $P_n(x; z)$ with respect to the basis of *falling factorial polynomials* defined by $\phi_0(x) = 1$ and

$$\phi_n(x) = \prod_{k=0}^{n-1} (x - k), \quad n \in \mathbb{N}. \tag{11}$$

In other words, we want to study the limit

$$\lim_{n\to\infty} \frac{P_n(x; z)}{\phi_n(x)}, \quad x = O(1), \quad x \notin \mathbb{N}_0,$$

where z is a fixed number, and x belongs to a compact subset of the complex plane containing the origin. We already considered this type of limits in [10], [12] (Charlier and Meixner polynomials), and in [13] (Krawtchouk polynomials).

Since the functional L is supported on the lattice $\mathbb{N}_0$, the zeros of the polynomial $P_n(x; z)$ will converge to non-negative integer values as $n \to \infty$. Thus, it is natural to approximate $P_n(x; z)$ with a monic polynomial having zeros at $x = 0, 1, \ldots, n-1$.

The organization of the paper is as follows: in Section 2, we review some of our results from [14]. The polynomials $P_n(x; z)$ have different asymptotic approximations depending on the relation between the parameters p and q defined in (6). Thus, we consider the cases $p = q$ (Section 3.1), $p = q-1$ (Section 3.2), $p < q-1$ (Section 3.3), and $p = q+1$ (Section 3.4). In Section 4, we describe the functions that we use in our plots, and make some observations on the difficulties in computing polynomials $P_n(x; z)$ numerically.

Finally, in the conclusions' section we summarize the results and discuss future directions.

2 Preliminary material

In [14], we studied families of polynomials (that we said to be of *Toda type*), orthogonal with respect to a linear functional $L : \mathbb{F}[x] \to \mathbb{F}$ satisfying

$$D_z L[u] = L[xu], \quad u \in \mathbb{F}[x],$$

where $D_z : \mathbb{F} \to \mathbb{F}$ is a fixed *derivation* (on the variable z) associated to L.

In this section, we review some of the results that we obtained, and apply them to the particular cases:

(i) $D_z = \vartheta$, where the operator ϑ was defined in (1).

(ii) The variable transformation

$$D_w = w(1-w)\partial_w, \quad w = \frac{z}{z-1}.$$

2.1 Toda-type orthogonal polynomials

The linear system (3) can be written as

$$L[\Lambda_k P_n] = h_n \delta_{k,n}, \quad 0 \leq k \leq n,$$

and we see that the sequence $\{P_n(x;z)\}_{n\geq 0}$ satisfies the *orthogonality conditions*

$$L[P_k P_n] = h_n \delta_{k,n}, \quad 0 \leq k \leq n, \tag{12}$$

where $h_n(z) \in \mathbb{F}\backslash\{0\}$ is the *norm* of $P_n(x;z)$.

From (12), we see that

$$L[xP_k P_n] = 0, \quad k \neq n, n \pm 1,$$

and therefore the polynomials $P_n(x;z)$ satisfy the *three term recurrence relation*

$$xP_n(x;z) = P_{n+1}(x;z) + \beta_n(z) P_n(x;z) + \gamma_n(z) P_{n-1}(x;z) \tag{13}$$

with $P_{-1} = 0, \ P_0 = 1$. The coefficients $\beta_n(z), \gamma_n(z) \in \mathbb{F}$ are given by [8]

$$\beta_0 = \frac{L[x]}{L[1]}, \quad \gamma_0 = 0, \tag{14}$$

and

$$\beta_n = \frac{L[xP_n^2]}{h_n}, \quad \gamma_n = \frac{L[xP_n P_{n-1}]}{h_{n-1}}, \quad n \in \mathbb{N}. \tag{15}$$

If we define $\sigma_n(z) \in \mathbb{F}$ by

$$P_n(x;z) = x^n - \sigma_n(z) x^{n-1} + u_n(x;z), \quad \deg(u_n) \leq n-2, \tag{16}$$

we have $\sigma_0 = 0$, and using (13) we get

$$x^{n+1} - \sigma_n x^n + x u_n = x^{n+1} - \sigma_{n+1} x^n + u_{n+1} + \beta_n \left(x^n - \sigma_n x^{n-1} + u_n \right) + \gamma_n P_{n-1}.$$

Comparing coefficients of x^n, we obtain $-\sigma_n = -\sigma_{n+1} + \beta_n$, or

$$\beta_n = \sigma_{n+1} - \sigma_n. \tag{17}$$

Our next result relates σ_n, h_n, β_n and γ_n.

Proposition 1 *Let ϑ be defined by (1), h_n be defined by (12), β_n, γ_n be defined by (15), and σ_n be defined by (16). Then, we have*

$$\vartheta \sigma_n = \gamma_n \tag{18}$$

and

$$\vartheta \ln h_n = \beta_n. \tag{19}$$

Proof. From (16) we have

$$\vartheta P_n (x; z) = -\vartheta \sigma_n (z) x^{n-1} + \vartheta u_n (x; z),$$

and using (12) we get

$$L [P_{n-1} \vartheta P_n] = - (\vartheta \sigma_n) L \left[x^{n-1} P_{n-1} \right] = - (\vartheta \sigma_n) h_{n-1}. \tag{20}$$

On the other hand, since $L [P_n P_{n-1}] = 0$ and $\deg (\vartheta P_{n-1}) = n - 2$,

$$\begin{aligned} 0 &= \vartheta L [P_n P_{n-1}] = L [P_{n-1} \vartheta P_n] + L [P_n \vartheta P_{n-1}] + L [x P_n P_{n-1}] \\ &= - (\vartheta \sigma_n) h_{n-1} + \gamma_n h_{n-1}, \end{aligned}$$

and we obtain (18). Since $\deg (\vartheta P_n) = n - 1$ we have

$$\vartheta h_n = \vartheta L \left[P_n^2 \right] = L [2 P_n \vartheta P_n] + L \left[x P_n^2 \right] = L \left[x P_n^2 \right] = \beta_n h_n,$$

and (19) follows. ■

As a direct consequence, we see that (β_n, γ_n) are solutions of the *Toda equations* [47].

Corollary 2 *The coefficients of the 3-term recurrence relation (13) are solutions of the differential-difference equations*

$$\vartheta \beta_n = \Delta \gamma_n, \quad \vartheta \ln \gamma_n = \nabla \beta_n, \tag{21}$$

with initial conditions (14), where

$$\Delta f(n) = f(n+1) - f(n), \quad \nabla f(n) = f(n) - f(n-1). \tag{22}$$

Essential for our work in this paper is the following theorem.

Theorem 3 *The polynomials $P_n(x; z)$ defined by (12) satisfy the recurrence*

$$\vartheta P_n = -\gamma_n P_{n-1}, \tag{23}$$

and the ODE

$$\left[\vartheta^2 + (x - \beta_n)\vartheta + \gamma_n\right] P_n = 0. \tag{24}$$

Proof. If we write

$$\vartheta P_n = \sum_{k=1}^{n-1} v_k P_k,$$

then (20) and (18) give

$$v_{n-1} = \frac{1}{h_{n-1}} L\left[P_{n-1} \vartheta P_n\right] = -\vartheta \sigma_n = -\gamma_n.$$

Moreover, for all $k = 0, 1, \ldots, n-2$

$$0 = \vartheta L\left[P_n P_k\right] = L\left[P_k \vartheta P_n\right] + L\left[P_n \vartheta P_k\right] + L\left[x P_n P_k\right] = L\left[P_k \vartheta P_n\right] = h_k v_k,$$

and therefore we obtain (23).

From (13) and (23), we have

$$\vartheta P_n = -\gamma_n P_{n-1} = P_{n+1} + (\beta_n - x) P_n.$$

Using (17), we get

$$\begin{aligned} \vartheta^2 P_n &= \vartheta P_{n+1} + P_n \vartheta \beta_n + (\beta_n - x)\vartheta P_n \\ &= -\gamma_{n+1} P_n + (\gamma_{n+1} - \gamma_n) P_n + (\beta_n - x)\vartheta P_n \end{aligned}$$

and (24) follows. ■

Since $\vartheta = z\partial_z$, we have

$$z\partial_z P_n = -\gamma_n P_{n-1},$$

and

$$z\left(z\partial_z^2 P_n + \partial_z P_n\right) + (x - \beta_n)\, z\partial_z P_n + \gamma_n P_n = 0. \tag{25}$$

As we will see in (34), $\gamma_n(0) = 0$. If we define $g_n(z) \in \mathbb{F}$ by

$$\gamma_n(z) = z g_n(z), \tag{26}$$

then

$$P_n' = -g_n P_{n-1}, \tag{27}$$

and (25) becomes

$$zP_n'' + (x + 1 - \beta_n)\, P_n' + g_n P_n = 0, \tag{28}$$

where we will **always** use the notation

$$P_n' = \partial_z P_n.$$

2.2 The function $\sigma_n(z)$

A fundamental quantity in our studies is $\sigma_n(z)$ defined in (16).

Theorem 4 *The coefficients in the power series expansion*

$$\sigma_n(z) = \sum_{k=0}^{\infty} s_k(n)\, z^k, \tag{29}$$

are given by

$$s_0(n) = \frac{n(n-1)}{2}, \quad s_1(n) = n\frac{(n-1+\mathbf{a})_1}{(n+\mathbf{b})_1}, \tag{30}$$

and

$$s_k(n) = \frac{1}{k(k-1)} \sum_{j=1}^{k-1} (k-j)\, s_{k-j}(n)\, \Delta\nabla\left[s_j(n)\right], \quad k \geq 2, \tag{31}$$

Δ, ∇ *are the finite difference operators (acting on n) defined in (22).*

Proof. From (17), (18), and (21) we get

$$\vartheta \ln\left(\vartheta\sigma_n\right) = \vartheta \ln\left(\gamma_n\right) = \beta_n - \beta_{n-1} = \sigma_{n+1} - 2\sigma_n + \sigma_{n-1}.$$

Using the difference operators (22), we can write

$$\sigma_{n+1} - 2\sigma_n + \sigma_{n-1} = \nabla\Delta\sigma_n,$$

and hence

$$\sigma_n''\left(z\right) = \sigma_n'\left(z\right)\frac{\nabla\Delta\sigma_n\left(z\right) - 1}{z}. \tag{32}$$

Since

$$\nabla\Delta s_{n,0} = \nabla\Delta\frac{n\left(n-1\right)}{2} = 1,$$

we see that from (29) that

$$\frac{\nabla\Delta\sigma_n - 1}{z} = \sum_{k=1}^{\infty}\nabla\Delta s_{n,k}z^{k-1} = \sum_{k=0}^{\infty}\nabla\Delta s_{n,k+1}z^k.$$

Also,

$$\sigma_n'\left(z\right) = \sum_{k=1}^{\infty}ks_{n,k}z^{k-1} = \sum_{k=0}^{\infty}\left(k+1\right)s_{n,k+1}z^k,$$

and

$$\sigma_n''\left(z\right) = \sum_{k=2}^{\infty}k\left(k-1\right)s_{n,k}z^{k-2} = \sum_{k=0}^{\infty}\left(k+2\right)\left(k+1\right)s_{n,k+2}z^k.$$

Comparing coefficients of z in (32) gives

$$\left(k+2\right)\left(k+1\right)s_{n,k+2} = \sum_{j=0}^{k}\left(k-j+1\right)s_{n,k-j+1}\nabla\Delta s_{n,j+1},$$

and (31) follows after shifting $k \to k-2$ and $j \to j-1$. ■

Using (17) and (18), we obtain the following result.

Corollary 5 *The coefficients of the 3-term recurrence relation (13) admit the formal power series*

$$\beta_n\left(z\right) = \sum_{k=0}^{\infty}\Delta s_k\left(n\right)z^k, \quad \gamma_n\left(z\right) = \sum_{k=1}^{\infty}ks_k\left(n\right)z^k, \tag{33}$$

where the coefficients $s_k\left(n\right)$ are defined by (29). In particular,

$$\beta_n\left(0\right) = n, \quad \gamma_n\left(0\right) = 0. \tag{34}$$

Remark 6 *From (26) and (33), we have*

$$g_n(z) = \sum_{k=0}^{\infty} (k+1)\, s_{k+1}(n)\, z^k. \tag{35}$$

From (30), we see that

$$s_1(n) = n^{\theta} \frac{(1 - n^{-1} + n^{-1}\mathbf{a})_1}{(1 + n^{-1}\mathbf{b})_1},$$

where

$$\theta = p + 1 - q. \tag{36}$$

If we write

$$s_1(n) = n^{\theta} \sum_{k=0}^{\infty} r_k n^{-k}, \tag{37}$$

we get

$$\sum_{j=0}^{k} e_{k-j}(\mathbf{b})\, r_j = e_k(\mathbf{a} - 1),$$

where the *elementary symmetric polynomials* $e_n(\mathbf{c})$ are defined by the generating function [37, 19.19.4]

$$\sum_{n=0}^{\infty} e_n(\mathbf{c})\, t^n = \prod_{i=1}^{m} (1 + t c_i), \quad \mathbf{c} \in \mathbb{C}^m. \tag{38}$$

Since $e_0 = 1$, we obtain the recurrence

$$r_k = e_k(\mathbf{a} - 1) - \sum_{j=0}^{k-1} e_{k-j}(\mathbf{b})\, r_j, \quad r_0 = 1. \tag{39}$$

The first two coefficients r_k are

$$r_1 = e_1(\mathbf{a} - 1) - e_1(\mathbf{b}),$$
$$r_2 = e_2(\mathbf{a} - 1) - e_2(\mathbf{b}) - e_1(\mathbf{a} - 1)\, e_1(\mathbf{b}) + e_1^2(\mathbf{b}).$$

To study the asymptotic behavior of the coefficients $s_k(n)$ as $n \to \infty$, we need to consider 2 cases: $\theta < 2$ and $\theta = 2$. We will analyze the case $\theta < 2$ in the next Theorem, and the case $\theta = 2$ in Section 2.4.

Theorem 7 *Let*

$$\Theta_k = (\theta - 2)\,k + \eta(\theta),$$

with

$$\eta(\theta) = \begin{cases} 0, & \theta = 1 \\ 1, & \theta = 0 \\ 2, & \theta \neq 0, 1 \end{cases}.$$

We have:

(i) If $\theta < 0$, *then*

$$s_k(n) \sim A_k(\theta)\, n^{\Theta_k}, \quad n \to \infty, \tag{40}$$

where $A_1 = 1$ *and for* $k \geq 2$

$$A_k = \frac{1}{k(k-1)} \sum_{j=1}^{k-1} (k-j)\,\Theta_j\,(\Theta_j - 1)\, A_j A_{k-j}. \tag{41}$$

(ii) If $\theta = 0$, *then as* $n \to \infty$,

$$s_1(n) \sim 1, \quad s_k(n) \sim r_1 C(k-1)\;\; n^{-2k+1}, \quad k \geq 2,$$

where $C(k)$ *is the* k^{th} Catalan number [37, 26.5(i)]

$$C(k) = \frac{1}{k+1}\binom{2k}{k}.$$

(iii) If $\theta = 1$, *then as* $n \to \infty$,

$$s_1(n) \sim n, \quad s_k(n) \sim r_2\; n^{-k}, \quad k \geq 2.$$

Proof. See [14]. ■

Remark 8 *Using induction, we can see that the solution of (41) is given by*

$$A_k(\theta) = -\theta \frac{(1-\theta)^k}{(k-1)!}\,(1 + k - \theta k)_{k-3}.$$

As a direct application of (31), we can illustrate the results of Theorem 7 for some particular cases.

Example 9 *Let $\theta = 1$. As $n \to \infty$, we have*

$$s_2 = r_2 n^{-2} + (r_1 r_2 + 3r_3) n^{-3} + O\left(n^{-4}\right),$$
$$s_3 = r_2 n^{-3} + 3(r_1 r_2 + 2r_3) n^{-4} + O\left(n^{-5}\right),$$

and we see that $s_k(n) \sim r_2\, n^{-k}$, $n \geq 2$, as expected. Also,

$$\sigma_n(z) = \frac{n^2}{2} + \left(z - \frac{1}{2}\right) n + r_1 z + r_2 z n^{-1} + (r_3 + r_2 z)\, z n^{-2} + \left[r_4 + (r_1 r_2 + 3r_3) z + r_2 z^2\right] z n^{-3} + O\left(n^{-4}\right),$$

$$\beta_n(z) = n + z - r_2 z n^{-2} + \left[(1 - 2z)\, r_2 - 2r_3\right] z n^{-3} + O\left(n^{-4}\right), \tag{42}$$

and

$$g_n(z) = n + r_1 + r_2 n^{-1} + (2z r_2 + r_3)\, n^{-2} + O\left(n^{-3}\right). \tag{43}$$

Example 10 *Let $\theta = 0$. As $n \to \infty$, we have*

$$s_2 = r_1 n^{-3} + (r_1^2 + 3r_2) n^{-4} + O\left(n^{-5}\right),$$
$$s_3 = 2r_1 n^{-5} + 2(3r_1^2 + 5r_2) n^{-6} + O\left(n^{-7}\right),$$

and we see that $s_k(n) \sim C(k-1)\ r_1 n^{-2k+1}$, $n \geq 2$, as expected. Also,

$$\sigma_n(z) = \frac{n^2}{2} - \frac{1}{2} n + z + r_1 z n^{-1} + r_2 z n^{-2} + (r_1 z + r_3) z n^{-3} + O\left(n^{-4}\right),$$

$$\begin{aligned} \beta_n(z) &= n - r_1 z n^{-2} + (r_1 - 2r_2)\, z n^{-3} \\ &- \left[r_1 (3z + 1) - 3(r_2 - r_3)\right] z n^{-4} + O\left(n^{-5}\right), \end{aligned} \tag{44}$$

and

$$g_n(z) = 1 + r_1 n^{-1} + r_2 n^{-2} + (2z r_1 + r_3)\, n^{-3} + O\left(n^{-4}\right). \tag{45}$$

Example 11 *Let $\theta = -1$. As $n \to \infty$, we have*

$$s_2 = n^{-4} + 4r_1 n^{-5} + \left(1 + 3r_1^2 + 7r_2\right) n^{-6} + O\left(n^{-7}\right),$$
$$s_3 = 4n^{-7} + 28 r_1 n^{-8} + \left(20 + 51 r_1^2 + 61 r_2\right) n^{-9} + O\left(n^{-10}\right),$$

and we see that $s_k(n) \sim A(k)\ r_1 n^{-3k+2}$, $n \geq 2$, as expected. Also,

$$\sigma_n(z) = \frac{n^2}{2} - \frac{1}{2} n + z n^{-1} + r_1 z n^{-2} + r_2 z n^{-3} + (z + r_3) z n^{-4} + O\left(n^{-5}\right),$$

$$\beta_n(z) = n - z n^{-2} + (1 - 2r_1)\, z n^{-3} - \left[1 + 3(r_2 - r_1)\right] z n^{-4} + O\left(n^{-5}\right), \tag{46}$$

and

$$g_n(z) = n^{-1} + r_1 n^{-2} + r_2 n^{-3} + (2z + r_3)\, n^{-4} + O\left(n^{-5}\right). \tag{47}$$

2.3 The function $\Phi_n(z;x)$

Sometimes, the falling factorial polynomials $\phi_n(x)$ defined in (11), are called *binomial polynomials*, since we have

$$\frac{\phi_n(x)}{n!} = \binom{x}{n}, \quad n \in \mathbb{N}_0. \tag{48}$$

From the definition (11), we see that

$$\phi_{n+1}(x) = (x-n)\,\phi_n(x) = x\phi_n(x-1), \quad n \geq 0, \tag{49}$$

and from (7) it follows that the falling factorial polynomials and the Pochhammer polynomials are related by

$$\phi_n(x) = (-1)^n (-x)_n = (x+1-n)_n.$$

Using (34) in (13), we obtain

$$P_{n+1}(x;0) = (x-n)\,P_n(x;0), \quad P_0(x;0) = 1,$$

and comparing with the recurrence satisfied by the falling factorial polynomials (49), we conclude that

$$P_n(x;0) = \phi_n(x). \tag{50}$$

Note that from (27) and (50), we see that

$$P_n'(x;0) = -g_n(0)\,\phi_{n-1}(x). \tag{51}$$

If we define $\Phi_n(z;x)$ by

$$P_n(x;z) = \phi_n(x)\,\Phi_n(z;x), \tag{52}$$

then (49) and (51) give the recurrence

$$\Phi_n'(z;x) = -\frac{g_n(z)}{x+1-n}\Phi_{n-1}(z;x). \tag{53}$$

It also follows from (28) and (50) that $\Phi_n(z;x)$ is the solution of the ODE

$$z\Phi_n'' + (x+1-\beta_n)\,\Phi_n' + g_n\Phi_n = 0, \tag{54}$$

with initial condition

$$\Phi_n(0;x) = 1. \tag{55}$$

Note that setting $z = 0$ in (54) and using (34) gives

$$\Phi_n'(0;x) = -\frac{g_n(0)}{x+1-n}$$

in agreement with (53).

Proposition 12 *Suppose that*

$$\Phi_n(z;x) = \sum_{k=0}^{\infty} \frac{\alpha_k(n)}{(x+1-n)_k} \frac{z^k}{k!}, \quad \alpha_0(n) = 1. \tag{56}$$

Then, the coefficients $\alpha_k(n)$ satisfy the recurrence

$$\alpha_{k+1}(n) = -\sum_{j=0}^{k} s_{j+1}(n)\, \alpha_{k-j}(n-1)\, (x+2-n+k-j)_j\,. \tag{57}$$

In particular,

$$\alpha_1(n) = -s_1(n)\,. \tag{58}$$

Proof. Taking a derivative in (56), we have

$$\Phi_n'(z;x) = \sum_{k=0}^{\infty} \frac{k\alpha_k(n)}{(x+1-n)_k} \frac{z^{k-1}}{k!} = \frac{1}{x+1-n} \sum_{k=0}^{\infty} \frac{\alpha_{k+1}(n)}{(x+2-n)_k} \frac{z^k}{k!},$$

since from (7) we see that

$$(x)_{k+1} = x\,(x+1)_k\,.$$

From (53), we conclude that

$$\sum_{k=0}^{\infty} \frac{\alpha_{k+1}(n)}{(x+2-n)_k} \frac{z^k}{k!} = -g_n(z) \sum_{k=0}^{\infty} \frac{\alpha_k(n-1)}{(x+2-n)_k} \frac{z^k}{k!},$$

and using (35), we get

$$\frac{\alpha_{k+1}(n)}{(x+2-n)_k} = -\sum_{j=0}^{k} s_{j+1}(n)\, \frac{\alpha_{k-j}(n-1)}{(x+2-n)_{k-j}}. \tag{59}$$

The result follows after using the identity

$$\frac{(x)_n}{(x)_m} = (x+m)_{n-m}, \quad m \leq n.$$

■

Remark 13 *Suppose that $\theta < 2$. It follows from (59) that to find the leading term in the asymptotic expansion of $\alpha_k(n)$ as $n \to \infty$, one needs to consider only the term with $j = 0$. Thus,*

$$\alpha_{k+1}(n) \sim -s_1(n)\,\alpha_k(n-1), \quad n \to \infty$$

and we conclude that

$$\alpha_k(n) \sim (-1)^k \prod_{j=0}^{k-1} s_1(n-j), \quad n \to \infty.$$

Using (37), we get

$$\alpha_k(n) = (-1)^k n^{k\theta} \left[1 + k\left(r_1 - \frac{k-1}{2}\theta\right) n^{-1} + O\left(n^{-2}\right)\right], \quad n \to \infty.$$

Example 14 *Let $\theta = 1$. As $n \to \infty$, we have*

$$\frac{\alpha_k(n)}{(x+1-n)_k} = 1 + \frac{x+1+r_1}{n} k + O\left(n^{-2}\right),$$

and therefore

$$\Phi_n(z;x) = e^z \left[1 + \frac{x+1+r_1}{n} z + O\left(n^{-2}\right)\right], \quad n \to \infty. \tag{60}$$

2.4 The variable w

If we use (31) with $\theta = 2$, we get

$$\begin{aligned} s_1 &= n^2 + r_1 n + r_2 + r_3 n^{-1} + O\left(n^{-2}\right), \\ s_2 &= n^2 + r_1 n + r_2 + 2r_3 n^{-1} + O\left(n^{-2}\right), \\ s_3 &= n^2 + r_1 n + r_2 + 3r_3 n^{-1} + O\left(n^{-2}\right), \end{aligned}$$

and this is clearly **not** an asymptotic sequence. As we showed in [14], what we need is to change variables from z to

$$w = \frac{z}{z-1}. \tag{61}$$

Theorem 15 *Let $\sigma_n(z)$ defined by (16). If we write*

$$\sigma_n(w) = \sum_{k=0}^{\infty} \xi_k(n) w^k,$$

we have

$$\xi_0(n) = \frac{n(n-1)}{2}, \quad \xi_1(n) = -n\frac{(n-1+\mathbf{a})_1}{(n+\mathbf{b})_1}, \tag{62}$$

and

$$\xi_k = \xi_{k-1} + \frac{1}{k(k-1)} \sum_{j=1}^{k-1} (k-j)\, \xi_{k-j} \nabla \Delta \xi_j, \quad k \geq 2. \tag{63}$$

Proof. See [14]. ■

Remark 16 *If we use (37) in (62), we get*

$$\xi_1(n) = -n^2 \sum_{k=0}^{\infty} r_k n^{-k}, \tag{64}$$

where the coefficients r_k can be computed using (39).

The asymptotic behavior of the coefficients $\xi_k(n)$ is given in the following result.

Theorem 17 *For all $k \geq 2$, we have*

$$\xi_k(n) = O\left(n^{-k+1}\right), \quad n \to \infty. \tag{65}$$

Proof. See [14]. ■

Remark 18 *For the first few $\xi_k(n)$, we can use (63) and (64), and obtain*

$$\begin{aligned} \xi_2(n) &= \frac{r_3}{n} + \frac{r_1 r_3 + 3r_4}{n^2} + O\left(n^{-3}\right), \\ \xi_3(n) &= -\frac{r_1 r_3 + 2r_4}{n^2} + O\left(n^{-3}\right), \\ \xi_4(n) &= \frac{\left(1 + r_1^2 + r_2\right) r_3 + 5(r_1 r_4 + r_5)}{n^3} + O\left(n^{-4}\right), \end{aligned} \tag{66}$$

as $n \to \infty$, in agreement with (65).

Note that we have

$$\gamma_n = z\sigma_n'(z) = w(1-w)\dot{\sigma}_n(w),$$

where we will **always** use the notation

$$\dot{\Phi}_n = \partial_w \Phi_n.$$

Therefore, in this case we define

$$\gamma_n(w) = w(1-w)\mathfrak{g}_n(w), \tag{67}$$

with

$$\mathfrak{g}_n(w) = \sum_{k=0}^{\infty}(k+1)\xi_{n,k+1}w^k.$$

Example 19 *Using (64) and (66), we can compute the first terms in the asymptotic expansions of* $\sigma_n(w)$, $\beta_n(w)$, *and* $\mathfrak{g}_n(w)$ *:*

$$\sigma_n(w) = \left(\frac{1}{2}-w\right)n^2 - \left(\frac{1}{2}+r_1 w\right)n - r_2 w + r_3(w-1)wn^{-1} + O\left(n^{-2}\right),$$

$$\beta_n(w) = (1-2w)n - (1+r_1)w - r_3(w-1)wn^{-2} + O\left(n^{-3}\right), \tag{68}$$

and

$$\mathfrak{g}_n(w) = -n^2 - r_1 n - r_2 + r_3(2w-1)n^{-1} + O\left(n^{-2}\right), \tag{69}$$

as $n\to\infty$.

3 Asymptotic analysis

In this section, we will obtain asymptotic approximations for $P_n(x;z)$ as $n\to\infty$, with $x=O(1)$ and all other parameters fixed. Because of the moments' recurrence (9), the analyticity of **all** the moments $\mu_n(z)$ (and in consequence the polynomials P_n themselves) as functions of z will agree with that of the first moment $\mu_0(z)$.

But since $\mu_0(z)$ is a hypergeometric function,

$$\mu_0(z) = {}_pF_q\left(\begin{array}{c}\mathbf{a}\\ \mathbf{b}\end{array};z\right) = \sum_{x=0}^{\infty}\frac{(\mathbf{a})_x}{(\mathbf{b}+1)_x}\frac{z^x}{x!}, \quad \mathbf{a}\in\mathbb{C}^p, \mathbf{b}\in\mathbb{C}^q,$$

its domain of analyticity depends on the parameters p, q. We have three cases to consider:

(i) If $p < q+1$, then $\mu_0(z)$ is an entire function of z. From (36), we see that this corresponds to the case $\theta < 2$.

(ii) If $p = q+1$ $(\theta = 2)$, then $\mu_0(z)$ is analytic inside the unit circle, $|z| < 1$, and can be extended by analytic continuation to the cut plane $\mathbb{C} \setminus [1, \infty)$.

(iii) If $p > q+1$ $(\theta > 2)$, then $\mu_0(z)$ diverges for all $z \neq 0$, except when one of the numerator parameters is a negative integer, and $\mu_0(z)$ becomes a polynomial (in z) of degree N. We will not study this situation in this paper, since in this case we need to scale n in terms of N and consider the limit as $N \to \infty$ (see [13] for the Krawtchouk polynomials).

We will divide the first case (i) in 3 subcases:

(a) When $p = q$ $(\theta = 1)$, $\mu_0(z)$ is entire (but barely!) and the asymptotic expansion of $P_n(x; z)$ will contain an exponential multiple e^z.

(b) When $p = q-1$ $(\theta = 0)$, $P_n(x; z)$ will have a regular asymptotic expansion.

(c) When $p < q-1$ $(\theta < 0)$, some of the first terms in the asymptotic expansion of $P_n(x; z)$ will be missing.

If $p = q+1$ $(\theta = 2)$, then $\mu_0(z)$ will have a logarithmic singularity at $z = 1$. Thus, we expect that the asymptotic expansion of $P_n(x; z)$ will have a factor of the form $(1-z)^{\varsigma}$, where the power could depend on n (and x). In this case, it is better to perform a change of variables and work with w defined in (61).

Notation 20 *We say that a family of polynomials is of* type (p, q), *if it's orthogonal with respect to the functional (5) with* $\mathbf{a} \in \mathbb{C}^p$ *and* $\mathbf{b} \in \mathbb{C}^q$.

3.1 Case $p = q$ $(\theta = 1)$

From (60), we see that in this case we should "peel off" an exponential term from $\Phi_n(z; x)$. Thus, if

$$\Phi_n(z; x) = e^z \Lambda_n(z; x), \tag{70}$$

we have

$$\Phi_n' = e^z (\Lambda_n + \Lambda_n'), \quad \Phi_n'' = e^z (\Lambda_n + 2\Lambda_n' + \Lambda_n''),$$

and (54) becomes

$$z\Lambda_n'' + (2z + x + 1 - \beta_n)\,\Lambda_n' + (z + x + 1 - \beta_n + g_n)\,\Lambda_n = 0. \tag{71}$$

From (42) and (43), we see that

$$\beta_n = n + \widetilde{\beta}_n, \quad g_n = n + \widetilde{g}_n, \quad \widetilde{\beta}_n = O\left(1\right), \quad \widetilde{g}_n = O\left(1\right), \quad n \to \infty,$$

and hence

$$z\Lambda_n'' + \left(2z + x + 1 - n - \widetilde{\beta}_n\right)\Lambda_n' + \left(z + x + 1 + \widetilde{g}_n - \widetilde{\beta}_n\right)\Lambda_n = 0. \tag{72}$$

Thus, we shall have $\Lambda_n = O\left(1\right), \quad n \to \infty$. Replacing

$$\widetilde{\beta}_n\left(z\right) = \sum_{k=0}^{\infty} v_k\left(z\right) n^{-k}, \quad \widetilde{g}_n\left(z\right) = \sum_{k=0}^{\infty} u_k\left(z\right) n^{-k},$$

and

$$\Lambda_n\left(z; x\right) = \sum_{k=0}^{\infty} \lambda_k\left(z; x\right) n^{-k},$$

in (72) and comparing coefficients of n^{-k}, we obtain the recurrence

$$\lambda_{k+1}' = z\lambda_k'' + (2z + x + 1)\,\lambda_k' + (z + x + 1)\,\lambda_k + \sum_{j=0}^{k}\left[\left(u_{k-j} - v_{k-j}\right)\lambda_j - v_{k-j}\lambda_j'\right]. \tag{73}$$

From (55) and (70) we have $\Lambda_n\left(0; x\right) = \Phi_n\left(0; x\right) = 1$, and therefore

$$\lambda_k\left(0; x\right) = \delta_{0,k}, \quad k \geq 0. \tag{74}$$

Note that from (42) and (43) we see that

$$u_0 = r_1, \quad u_1 = r_2, \quad u_2 = 2zr_2 + r_3,$$
$$v_0 = z, \quad v_1 = 0, \quad v_2 = -r_2 z.$$

When $k = -1$, (73) and (74) give

$$\lambda_0' = 0, \quad \lambda_0\left(0; x\right) = 1,$$

and thus

$$\lambda_0(z;x) = 1. \tag{75}$$

Using (75) in (73), we get

$$\lambda_1' = z + x + 1 + u_0 - v_0 = x + 1 + r_1,$$

and since $\lambda_1(0;x) = 0$, we obtain

$$\lambda_1(z;x) = (x + 1 + r_1)\, z. \tag{76}$$

Similarly, using (75) and (76) in (73), we get after some simplification

$$\lambda_2' = \lambda_1'(x + 1 + z) + \lambda_1\lambda_1' + r_2,$$

and since $\lambda_2(0;x) = 0$, we conclude that

$$\lambda_2 = \lambda_1'\left(x + \frac{z}{2} + 1\right) z + \frac{1}{2}(\lambda_1)^2 + r_2 z,$$

or

$$\lambda_2(z;x) = [(x+1)(x+1+r_1) + r_2]\, z + (x+1+r_1)(x+2+r_1)\frac{z^2}{2}. \tag{77}$$

3.1.1 Polynomials of type $(0,0)$ (Charlier polynomials).

The Charlier polynomials were introduced by Carl Vilhelm Ludwig Charlier (1862–1934) in his paper [7] and have the hypergeometric representation

$$P_n(x;z) = (-z)^n \; {}_2F_0\left[\begin{array}{c} -n, -x \\ - \end{array}; -\frac{1}{z}\right].$$

For this family, we have $r_k = 0$, $k \geq 1$, and therefore

$$\beta_n = n + z, \quad g_n = n.$$

Replacing in (71), we get

$$z\Lambda_n'' + (z + x + 1 - n)\,\Lambda_n' + (x+1)\,\Lambda_n = 0. \tag{78}$$

Therefore, the recurrence (73) becomes

$$\lambda_{k+1}' = z\lambda_k'' + (z + x + 1)\,\lambda_k' + (x+1)\,\lambda_k,$$

or

$$\lambda_{k+1}(z) = z(\lambda_k' + \lambda_k) + x[\lambda_k(z) - \lambda_k(0)] + x\int_0^z \lambda_k(t)\,dt.$$

Starting with $\lambda_0(z) = 1$, we obtain

$$\begin{aligned} &\lambda_1(z) = (x+1)z, \quad \lambda_2(z) = (x+1)^2 z + (x+1)_2 \tfrac{z^2}{2}, \\ &\lambda_3(z) = (x+1)^3 z + (x+1)_2 (2x+3) \tfrac{z^2}{2} + (x+1)_3 \tfrac{z^3}{6}. \end{aligned} \tag{79}$$

However, in this case the ODE satisfied by $\Lambda_n(z;x)$ (78) has the exact solution [12]

$$\Lambda_n(z;x) = {}_1F_1\left(\begin{matrix} x+1 \\ x+1-n \end{matrix} ; -z \right),$$

where we have used the initial value $\Lambda_n(0;x) = 1$. Therefore,

$$\Lambda_n(z;x) = \sum_{k=0}^{\infty} \frac{(x+1)_k}{(x+1-n)_k} \frac{(-z)^k}{k!} \tag{80}$$

and using the first few terms we obtain

$$\begin{aligned} \sum_{k=0}^{3} \frac{(x+1)_k}{(x+1-n)_k} \frac{(-z)^k}{k!} &= 1 + \frac{(x+1)z}{n} + \left[(x+1)^2 z + (x+1)_2 \frac{z^2}{2} \right] n^{-2} \\ &+ \left[(x+1)^3 z + (x+1)_2 (2x+3) \frac{z^2}{2} + (x+1)_3 \frac{z^3}{6} \right] n^{-3} + O\left(n^{-4}\right) \end{aligned}$$

as $n \to \infty$, in agreement with (79).

3.1.2 Polynomials of type $(1,1)$ (generalized Meixner)

For this family, we have

$$\frac{s_1(n)}{n} = \frac{n+a-1}{n+b} = 1 + \frac{a-b-1}{n+b} = 1 + (a-b-1)\sum_{k=1}^{\infty} \frac{(-b)^{k-1}}{n^k},$$

and therefore

$$r_k = (a-b-1)(-b)^{k-1}, \quad k \geq 1. \tag{81}$$

Using (81) in (75)–(77), we get $\lambda_0(z;x) = 1$,

$$\lambda_1(z;x) = (x+a-b)z, \tag{82}$$

and

$$\lambda_2(z;x) = \left[(x+a)(x+1-b)+b^2\right]z + (x+a-b+1)(x+a-b)\frac{z^2}{2}. \quad (83)$$

For additional information on these polynomials, see [6], [9], [15], [16], [17], [19].

3.1.3 Polynomials of type $(2,2)$

For this family, we have

$$\frac{s_1(n)}{n} = \frac{(n+a_1-1)(n+a_2-1)}{(n+b_1)(n+b_2)} =$$
$$1 + \frac{(a_1-b_2-1)(a_2-b_2-1)}{(b_1-b_2)(n+b_2)} - \frac{(a_1-b_1-1)(a_2-b_1-1)}{(b_1-b_2)(n+b_1)}$$

and therefore

$$r_k = \frac{\tau_k^{(1)}(b_2) - \tau_k^{(1)}(b_1)}{b_1-b_2}, \quad k \geq 1, \quad (84)$$

with

$$\tau_k^{(1)}(b) = (b-a_1+1)(b-a_2+1)(-b)^{k-1}.$$

In particular,

$$r_1 = a_1+a_2-b_1-b_2-2,$$
$$r_2 = 1-a_1-a_2-(a_1+a_2-2)(b_1+b_2)+b_1^2+b_2^2+b_1b_2+a_1a_2.$$

Using (84) in (75)–(77), we get $\lambda_0(z;x) = 1$,

$$\lambda_1(z;x) = (x+a_1+a_2-b_1-b_2-1)z, \quad (85)$$

and

$$\begin{aligned}\lambda_2(z;x) &= \left[(x+1)(x+a_1+a_2-b_1-b_2-1)+r_2\right]z \\ &+ (x+a_1+a_2-b_1-b_2-1)(x+a_1+a_2-b_1-b_2)\tfrac{z^2}{2}.\end{aligned} \quad (86)$$

For additional information on these polynomials, see [15] and [17].

3.2 Case $p = q - 1$ $(\theta = 0)$

From (44) and (45), we see that

$$\beta_n = n + n^{-2}\widetilde{\beta}_n, \quad \widetilde{\beta}_n = O(1), \quad g_n = O(1), \quad n \to \infty,$$

and replacing in (54), we get

$$z\Phi_n'' + \left(x + 1 - n - n^{-2}\widetilde{\beta}_n\right)\Phi_n' + g_n\Phi_n = 0. \tag{87}$$

Thus, we shall have $\Phi_n = O(1)$, $n \to \infty$ with $\Phi_n(0; x) = 1$. Replacing

$$\widetilde{\beta}_n(z) = \sum_{k=0}^{\infty} v_k(z)\, n^{-k}, \quad g_n(z) = \sum_{k=0}^{\infty} u_k(z)\, n^{-k},$$

and

$$\Phi_n(z; x) = \sum_{k=0}^{\infty} \varphi_k(z; x)\, n^{-k}, \quad \varphi_k(0; x) = \delta_{0,k}, \quad k \geq 0,$$

in (87) and comparing coefficients of n^{-k}, we obtain the recurrence

$$\varphi_{k+1}' = z\varphi_k'' + (x+1)\varphi_k' + \sum_{j=0}^{k} \varphi_j u_{k-j} - \sum_{j=0}^{k-2} \varphi_j' v_{k-2-j}. \tag{88}$$

Replacing $\varphi_0 = 1$ in (88) with $k = 0$, we have

$$\varphi_1' = u_0 = 1,$$

and therefore

$$\varphi_1(z; x) = z. \tag{89}$$

Using $\varphi_0 = 1, \varphi_1 = z$ in (88) with $k = 1$, we get

$$\varphi_2' = x + 1 + u_1 + zu_0 = x + 1 + r_1 + z,$$

and hence

$$\varphi_2(z; x) = (x + 1 + r_1)\, z + \frac{z^2}{2}. \tag{90}$$

Similarly, we have

$$\begin{aligned}\varphi_3' &= z + (x+1)\varphi_2' + \varphi_0 u_2 + \varphi_1 u_1 + \varphi_2 u_0 - \varphi_0' v_0 \\ &= z + (x+1)\varphi_2' + r_2 + r_1 z + \varphi_2,\end{aligned}$$

and we conclude that

$$\varphi_3(z; x) = [(x+1)(x+1+r_1) + r_2]\, z + [2(x+1+r_1) + 1]\frac{z^2}{2} + \frac{z^3}{6}. \tag{91}$$

3.2.1 Polynomials of type $(0, 1)$ (generalized Charlier)

For this family, we have

$$s_1(n) = \frac{n}{n+b} = \sum_{k=0}^{\infty} \frac{(-b)^k}{n^k},$$

and therefore

$$r_k = (-b)^k, \quad k \geq 0. \tag{92}$$

Using (92) in (89)–(91), we get

$$\Phi_n(z; x) \sim 1 + \frac{z}{n} + \frac{(x+1-b)z + \frac{z^2}{2}}{n^2} + \frac{\left[(x+1)(x+1-b) + b^2\right] z + \left[2(x+1-b)+1\right] \frac{z^2}{2} + \frac{z^3}{6}}{n^3}$$

as $n \to \infty$.

For additional information on these polynomials, see [9], [15], [16], [17], [26], [45], [48].

3.2.2 Polynomials of type $(1, 2)$

For this family, we have

$$s_1(n) = \frac{n(n+a-1)}{(n+b_1)(n+b_2)} = 1 + \frac{(a-1-b_1)b_1}{(b_1-b_2)(n+b_1)} - \frac{(a-1-b_2)b_2}{(b_1-b_2)(n+b_2)},$$

and therefore

$$r_k = \frac{(b_1+1-a)(-b_1)^k + (a-1-b_2)(-b_2)^k}{b_1-b_2}, \quad k \geq 0.$$

In particular,

$$\begin{aligned} r_0 &= 1, \quad r_1 = a - b_1 - b_2 - 1, \\ r_2 &= (1-a)(b_1+b_2) + b_1^2 + b_2^2 + b_1 b_2. \end{aligned} \tag{93}$$

Using (93) in (89)–(91), we get

$$\begin{aligned} \Phi_n(z; x) = 1 + zn^{-1} + \left[(x+a-b_1-b_2)z + \frac{z^2}{2}\right] n^{-2} \\ + \left[(x+1)(x+a-b_1-b_2) + r_2\right] zn^{-3} \\ + \left[\left(x+a-b_1-b_2+\frac{1}{2}\right) z^2 + \frac{z^3}{6}\right] n^{-3} + O\left(n^{-4}\right) \end{aligned}$$

as $n \to \infty$.

For additional information on these polynomials, see [15] and [17].

3.3 Case $p < q-1$ $(\theta < 0)$

Looking at (46) and (47), suggests that as $n \to \infty$,

$$\beta_n = n + n^{\theta-1}\widetilde{\beta}_n, \quad \widetilde{\beta}_n = O(1), \quad g_n = n^{\theta}\widetilde{g}_n, \quad \widetilde{g}_n = O(1),$$

and replacing in (54), we get

$$z\Phi_n'' + \left(x + 1 - n - n^{\theta-1}\widetilde{\beta}_n\right)\Phi_n' + n^{\theta}\widetilde{g}_n\Phi_n = 0. \tag{94}$$

Thus, we expect that

$$\Phi_n(z;x) = 1 + n^{\theta-1}\widetilde{\Phi}_n(z;x), \quad \widetilde{\Phi}_n = O(1), \quad n \to \infty$$

with $\widetilde{\Phi}_n(0;x) = 0$, and therefore the ODE (94) becomes

$$zn^{\theta-1}\widetilde{\Phi}_n'' + \left(x + 1 - n - n^{\theta-1}\widetilde{\beta}_n\right)n^{\theta-1}\widetilde{\Phi}_n' + n^{\theta}\widetilde{g}_n + n^{2\theta-1}\widetilde{g}_n\widetilde{\Phi}_n = 0,$$

or

$$z\widetilde{\Phi}_n'' + \left(x + 1 - n - n^{\theta-1}\widetilde{\beta}_n\right)\widetilde{\Phi}_n' + n\widetilde{g}_n + n^{\theta}\widetilde{g}_n\widetilde{\Phi}_n = 0. \tag{95}$$

Replacing

$$\widetilde{\beta}_n(z) = \sum_{k=0}^{\infty} v_k(z)\, n^{-k}, \quad g_n(z) = \sum_{k=0}^{\infty} u_k(z)\, n^{-k},$$

and

$$\widetilde{\Phi}_n(z;x) = \sum_{k=0}^{\infty} \varphi_k(z;x)\, n^{-k}, \quad \varphi_k(0;x) = 0, \quad k \geq 0$$

in (95) and comparing coefficients of n^{-k}, we obtain the recurrence

$$\varphi_k' = u_k + z\varphi_{k-1}'' + (x+1)\varphi_{k-1}' + \sum_{j=0}^{k-1+\theta} \varphi_j u_{k-1+\theta-j} - \sum_{j=0}^{k+\theta-2} \varphi_j' v_{k+\theta-2-j}. \tag{96}$$

Setting $k = 0$ in (96), we get

$$\varphi_0' = u_0 = 1,$$

and therefore

$$\varphi_0(z;x) = z. \tag{97}$$

For $k = 1$, we have

$$\varphi_1' = u_1 + z\varphi_0'' + (x+1)\varphi_0' + \sum_{j=0}^{\theta} \varphi_j u_{\theta-j} - \sum_{j=0}^{\theta-1} \varphi_j' v_{\theta-1-j},$$

but since $\theta < 0$ and $\varphi_0 = z$,

$$\varphi_1' = u_1 + x + 1$$

and hence

$$\varphi_1(z;x) = (x + 1 + r_1) z. \tag{98}$$

Continuing this way, we see that

$$\varphi_k' = u_k + z\varphi_{k-1}'' + (x+1)\varphi_{k-1}', \quad 1 \leq k < 1 - \theta,$$

and for $k = 1 - \theta$

$$\varphi_{1-\theta}' = u_{1-\theta} + z\varphi_{-\theta}'' + (x+1)\varphi_{-\theta}' + \varphi_0 u_0.$$

Thus,

$$\varphi_k(z;x) = \int_0^z u_k(t)\,dt + z\varphi_{k-1}'(z;x) + x\varphi_{k-1}(z;x), \quad 1 \leq k < 1 - \theta, \tag{99}$$

and

$$\varphi_{1-\theta}(z;x) = \int_0^z u_{1-\theta}(t)\,dt + z\varphi_{-\theta}'(z;x) + x\varphi_{-\theta}(z;x) + \frac{z^2}{2}. \tag{100}$$

3.3.1 Polynomials of type $(0,2)$

For this family, we have

$$\frac{s_1(n)}{n^{-1}} = \frac{n^2}{(n+b_1)(n+b_2)} = 1 + \frac{b_2^2}{(b_1-b_2)(n+b_2)} - \frac{b_1^2}{(b_1-b_2)(n+b_1)},$$

and therefore

$$r_k = \frac{(-b_2)^{k+1} - (-b_1)^{k+1}}{b_1 - b_2}, \quad k \geq 0.$$

In particular,

$$r_0 = 1, \quad r_1 = -(b_1 + b_2), \quad r_2 = b_1 b_2 + b_1^2 + b_2^2. \tag{101}$$

Using (101) in (98) and (100), we get

$$\varphi_1(z; x) = (x + 1 - b_1 - b_2) z,$$

$$\varphi_2 = \int_0^z u_2(t) \, dt + z\varphi_1' + x\varphi_1 + \frac{z^2}{2} = \int_0^z r_2 dt + (x+1)(x + 1 - b_1 - b_2) z + \frac{z^2}{2},$$

and hence

$$\varphi_2(z; x) = \left(b_1 b_2 + b_1^2 + b_2^2\right) z + (x+1)(x + 1 - b_1 - b_2) z + \frac{z^2}{2}.$$

Combining the results above and recalling that $\varphi_0 = z$, we obtain

$$\Phi_n(z; x) = 1 + zn^{-2} + (x + 1 - b_1 - b_2) zn^{-3}$$
$$+ \left[\left(b_1 b_2 + b_1^2 + b_2^2\right) z + (x+1)(x + 1 - b_1 - b_2) z + \frac{z^2}{2}\right] n^{-4} + O\left(n^{-5}\right).$$

For additional information on these polynomials, see [15] and [17].

3.4 Case $p = q + 1$ $(\theta = 2)$

Let w be defined by (61). Using

$$\partial_z = -(w-1)^2 \partial_w, \quad \partial_z^2 = (w-1)^4 \partial_w^2 + 2(w-1)^3 \partial_w,$$

in (25), we get

$$w^2 (1-w)^2 \partial_w^2 \Phi_n + (x + 1 - \beta_n - 2w) w (1-w) \partial_w \Phi_n + \gamma_n \Phi_n = 0,$$

and from (67) we have

$$w(1-w) \ddot{\Phi}_n + (x + 1 - \beta_n - 2w) \dot{\Phi}_n + \mathfrak{g}_n \Phi_n = 0. \tag{102}$$

Based on the case $\theta = 1$ (Section 3.1), we expect that $\Phi_n(w;x)$ will contain an exponential term. Replacing

$$\Phi_n(w;x) = \exp[\Upsilon_n(w;x)], \quad \Upsilon_n(0;x) = 0,$$

in (102), we obtain

$$w(1-w)\left[\ddot{\Upsilon}_n + \left(\dot{\Upsilon}_n\right)^2\right] + (x+1-\beta_n-2w)\dot{\Upsilon}_n + \mathfrak{g}_n = 0. \tag{103}$$

From (68)–(69), we have

$$\begin{aligned} &\beta_n = (1-2w)n - (1+r_1)w + \widetilde{\beta}_n, \quad \widetilde{\beta}_n = O(n^{-2}), \quad n\to\infty, \\ &\mathfrak{g}_n = -n^2 - r_1 n + \widetilde{\mathfrak{g}}_n, \quad \widetilde{\mathfrak{g}}_n = O(1), \quad n\to\infty, \end{aligned} \tag{104}$$

and replacing in (103) gives, to leading order,

$$w(1-w)\left(\dot{\Upsilon}_n\right)^2 \sim (1-2w)n\dot{\Upsilon}_n + n^2, \quad n\to\infty$$

and therefore

$$\dot{\Upsilon}_n \sim \frac{n}{w}, \quad \text{or} \quad \dot{\Upsilon}_n \sim \frac{n}{w-1}, \quad n\to\infty.$$

Since we want $\Upsilon_n(w;x)$ to be analytic in a neighborhood of $w=0$, we choose

$$\Upsilon_n(w;x) \sim \ln(1-w)n, \quad n\to\infty,$$

and set

$$\Upsilon_n(w;x) = \ln(1-w)n + \sum_{k=0}^{\infty}\epsilon_k(w;x)n^{-k}, \quad \epsilon_k(0;x) = 0, \quad k\geq 0, \tag{105}$$

$$\widetilde{\beta}_n(w) = \sum_{k=2}^{\infty}v_k(w;x)n^{-k}, \quad \widetilde{\mathfrak{g}}_n(w) = \sum_{k=0}^{\infty}u_k(w;x)n^{-k}, \tag{106}$$

where from (68)–(69) we see that

$$v_2 = r_3(1-w)w, \quad u_0 = -r_2, \quad u_1 = r_3(2w-1). \tag{107}$$

Using (105)–(106) in (103) and comparing powers of n, we get

$$\dot{\epsilon}_0 = \frac{x+1+r_1}{w-1}.$$

Thus, since $\epsilon_0(0;x) = 0$,

$$\epsilon_0(w;x) = (x+1+r_1)\ln(1-w).$$

We could proceed in this manner, but instead we consider $\Psi_n(w;x)$ defined by

$$\Phi_n(w;x) = (1-w)^{n+x+1+r_1}\,\Psi_n(w;x), \tag{108}$$

so that

$$\Psi_n(w;x) = \exp\left[\sum_{k=1}^{\infty}\epsilon_k(w;x)\,n^{-k}\right] = O(1), \quad n\to\infty.$$

Using (104) and (108) in (102), we get

$$\begin{aligned} &w(1-w)^2\,\ddot{\Psi}_n + (1-w)\left[x+1-w(r_1+2x+3)-\widetilde{\beta}_n - n\right]\dot{\Psi}_n \\ &+\left[(n+x+1+r_1)\widetilde{\beta}_n + (1-w)(\widetilde{\mathfrak{g}}_n-(x+1)(x+1+r_1))\right]\Psi_n = 0. \end{aligned} \tag{109}$$

Replacing (106) and

$$\Psi_n(w;x) = \sum_{k=0}^{\infty}\psi_k(w;x)\,n^{-k}, \quad \psi_k(0;x) = \delta_{0,k}, \quad k\geq 0$$

in (109), we obtain the recurrence

$$\begin{aligned} (1-w)\,\dot{\psi}_{k+1} &= w(1-w)^2\,\ddot{\psi}_k + (1-w)\left[x+1-(r_1+2x+3)w\right]\dot{\psi}_k \\ &+(x+1)(x+1+r_1)(w-1)\,\psi_k + (1-w)\sum_{j=0}^{k}\psi_j u_{k-j} \\ &+\sum_{j=0}^{k-1}\psi_j v_{k+1-j} + \sum_{j=0}^{k-2}\left[(x+1+r_1)\,\psi_j - \dot{\psi}_j\right]v_{k-j} = 0. \end{aligned} \tag{110}$$

Setting $k=0$ and $\psi_0 = 1$ in (110), we obtain

$$\dot{\psi}_1 = -(x+1)(x+1+r_1) + u_0,$$

and since $u_0 = -r_2$ and $\psi_1(0;x) = 0$, we conclude that

$$\psi_1(w;x) = -\left[(x+1)(x+1+r_1)+r_2\right]w. \tag{111}$$

Replacing $k = 1$ and $\psi_0 = 1$ in (110), we have

$$(1-w)\,\dot{\psi}_2 = (1-w)\left[x+1-(r_1+2x+3)w\right]\dot{\psi}_1 \\ +(x+1)\,(x+1+r_1)\,(w-1)\,\psi_1 + (1-w)\,(u_1+\psi_1 u_0) + v_2,$$

and using (107) and $\psi_1 = w\dot{\psi}_1$, we get

$$(1-w)\,\dot{\psi}_2 = (1-w)\,(x+1-(r_1+2x+3)w)\,\dot{\psi}_1 \\ +(x+1)\,(x+1+r_1)\,(w-1)\,w\dot{\psi}_1 \\ +(1-w)\left(r_3\,(2w-1) - r_2 w\dot{\psi}_1\right) + r_3\,(1-w)\,w,$$

or

$$\dot{\psi}_2 = \left[x+1-((x+2)\,(x+2+r_1)+r_2)\,w\right]\dot{\psi}_1 + r_3\,(3w-1)\,.$$

Since $\psi_2\,(0;x) = 0$, we conclude that

$$\psi_2\,(w;x) = \left[(x+1)\,w - ((x+2)\,(x+2+r_1)+r_2)\,\frac{w^2}{2}\right]\dot{\psi}_1 + \frac{r_3}{2}w\,(3w-2)\,,$$

and noting from (111) that

$$-\left[(x+2)\,(x+2+r_1)+r_2\right]w = \psi_1\,(w;x+1)\,,$$

we can write

$$\psi_2\,(w;x) = \left[x+1+\frac{1}{2}\psi_1\,(w;x+1)\right]\psi_1\,(w;x) + \frac{r_3}{2}w\,(3w-2)\,. \qquad (112)$$

3.4.1 Polynomials of type $(1,0)$ (Meixner polynomials)

The Meixner polynomials were introduced by Josef Meixner (1908 – 1994) in his paper [35] and have the representation

$$P_n\,(x;z) = (a)_n\left(1-\frac{1}{z}\right)^{-n}\,{}_2F_1\left[\begin{matrix}-n,-x\\ a\end{matrix};1-\frac{1}{z}\right], \quad z\in\mathbb{C}\setminus[1,\infty).$$

For this family, we have

$$-\frac{\xi_1\,(n)}{n^2} = \frac{n+a-1}{n},$$

and therefore

$$r_0 = 1, \quad r_1 = a - 1, \quad r_k = 0, \quad k \geq 2, \tag{113}$$

and

$$\beta_n(w) = (1-2w)\, n - aw, \quad \mathfrak{g}_n(w) = -n^2 - (a-1)\, n. \tag{114}$$

Thus, in this case $\widetilde{\beta}_n = \widetilde{\mathfrak{g}}_n = 0$, and using (113) in (109), we obtain

$$\begin{aligned} & w(1-w)\, \ddot{\Psi}_n + [x+1-(2x+2+a)w-n]\, \dot{\Psi}_n \\ & -(x+1)(x+a)\, \Psi_n = 0, \end{aligned} \tag{115}$$

while the recurrence (110) becomes

$$\dot{\psi}_{k+1} = w(1-w)\, \ddot{\psi}_k + [x+1-(2x+2+a)w]\, \dot{\psi}_k - (x+1)(x+a)\, \psi_k.$$

It follows that, as $n \to \infty$,

$$\begin{aligned} & \Psi_n(w;x) \sim 1 - (x+1)(x+a)\, wn^{-1} \\ & -\left[x+1-\tfrac{1}{2}(x+2)(x+1+a)\, w\right](x+1)(x+a)\, wn^{-2}. \end{aligned} \tag{116}$$

However, the ODE (115) can be solved exactly, and we have [12]

$$\Psi_n(w;x) = {}_2F_1\left(\begin{matrix} x+1, x+a \\ x+1-n \end{matrix}; w\right),$$

and using the first couple of terms, we get

$$\Psi_n(w;x) \sim \sum_{k=0}^{2} \frac{(x+1)_k\,(x+a)_k}{(x+1-n)_k}\frac{w^k}{k!} \sim -(x+1)(x+a)\, wn^{-1}$$
$$-(x+1)(x+a)\, w\left[x+1-\frac{1}{2}(x+2)(x+1+a)\, w\right] n^{-2}, \quad n \to \infty,$$

in agreement with (116).

3.4.2 Polynomials of type $(2,1)$ (generalized Hahn polynomials of type I)

For this family, we have

$$\begin{aligned} -\frac{\xi_1(n)}{n^2} &= \frac{(n+a_1-1)(n+a_2-1)}{n(n+b)} \\ &= 1 + \frac{(a_1-1)(a_2-1)}{bn} - \frac{(b+1-a_1)(b+1-a_2)}{b(n+b)}, \end{aligned}$$

and therefore

$$\begin{aligned} &r_0 = 1, \quad r_1 = a_1 + a_2 - 2 - b, \\ &r_k = (b + 1 - a_1)(b + 1 - a_2)(-b)^{k-2}, \quad k \geq 2. \end{aligned} \tag{117}$$

Using (117) in (111)–(112), we get

$$\psi_1(w;x) = -[(x+1)(x + a_1 + a_2 - 1 - b) + (b - a_1 + 1)(b - a_2 + 1)]\, w \tag{118}$$

and

$$\begin{aligned} &\psi_2(w;x) = \left[x + 1 + \tfrac{1}{2}\psi_1(w;x+1)\right]\psi_1(w;x) \\ &-\tfrac{1}{2}(b - a_1 + 1)(b - a_2 + 1)\, bw\,(3w - 2). \end{aligned} \tag{119}$$

For additional information on these polynomials, see [11], [15], [16], [17], [20].

3.4.3 Polynomials of type $(3, 2)$

For this family, we have

$$-\frac{\xi_1(n)}{n^2} = \frac{(n + a_1 - 1)(n + a_2 - 1)(n + a_3 - 1)}{n(n + b_1)(n + b_2)}$$

and using the elementary symmetric polynomials defined by (38), we can write

$$\begin{aligned} &r_0 = 1, \quad r_1 = e_1(\mathbf{A}) - e_1(\mathbf{b}), \\ &r_2 = e_2(\mathbf{A}) - e_1(\mathbf{A})\, e_1(\mathbf{b}) + e_1^2(\mathbf{b}) - e_2(\mathbf{b}) \\ &r_3 = 2e_1(\mathbf{b})\, e_2(\mathbf{b}) + e_1(\mathbf{a})\left[e_1^2(\mathbf{b}) - e_2(\mathbf{b})\right] - e_2(\mathbf{a})\, e_1(\mathbf{b}) + e_3(\mathbf{a}) - e_1^3(\mathbf{b}) \end{aligned} \tag{120}$$

where

$$\mathbf{A} = \mathbf{a} - 1.$$

At this point, we truly reach the limit of being able to type expressions in a compact way. For the first terms in the asymptotic expansion of these polynomials, we refer to the general formulas (111)–(112) with r_1, r_2 given by (120).

For additional information on these polynomials, see [15] and [17].

4 Numerical results

Since we can write the falling factorial polynomials in terms of factorials (48), we can use the reflection formula for the Gamma function [37, 5.5.3]

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)},$$

and obtain

$$\phi_n(x) = \frac{x!}{\Gamma(x+1-n)} = \frac{x!\sin[\pi(n-x)]}{\pi}\Gamma(n-x).$$

But

$$\sin(\pi(n-x)) = -\cos(\pi n)\sin(\pi x) = (-1)^{n+1}\sin(\pi x),$$

and therefore

$$\phi_n(x) = (-1)^{n+1} x! \frac{\sin(\pi x)}{\pi}\Gamma(n-x).$$

Let $\widehat{\Phi}_n(z;x)$ denote an asymptotic approximation for the function $\Phi_n(z;x)$ defined by (52). In order to plot the different asymptotic approximations for $P_n(x;z)$, we will consider two cases:

i) On the negative real axis, we shall graph

$$\frac{P_n(x;z)}{\Gamma(n-x)} \quad \text{and} \quad (-1)^{n+1} x! \frac{\sin(\pi x)}{\pi}\widehat{\Phi}_n(z;x), \tag{121}$$

since both functions are analytic, nonzero, and bounded in this region.

ii) On the positive real axis (with $x < n$), we shall graph

$$\frac{P_n(x;z)}{x!\Gamma(n-x)} \quad \text{and} \quad (-1)^{n+1} \frac{\sin(\pi x)}{\pi}\widehat{\Phi}_n(z;x), \tag{122}$$

since both functions are analytic and bounded in this region.

To compute the polynomials $P_n(x;z)$, we first compute the moments of L on the monomial basis (8) to a **very** high order of accuracy (with error less than $\varepsilon = 10^{-100}$), solve the system of equations (3)

$$\mu_{n+k} + \sum_{i=0}^{n-1}\mu_{k+i}\xi_{n,i} = 0, \quad 0 \le k \le n-1,$$

and construct the polynomials using (4),

$$P_n(x;z) = x^n + \sum_{i=0}^{n-1} \xi_{n,i}(z)\, x^i.$$

After that, we double-check that

$$\left| L\left[x^k P_n\right] \right| < \varepsilon, \quad 0 \leq k \leq n-1, \quad \left| L\left[x^n P_n\right] \right| > \varepsilon.$$

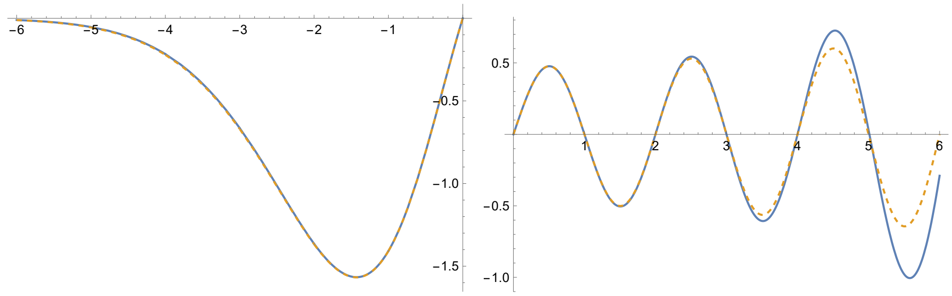
We have tried other methods (using Hankel determinants, recurrences, or the Toda equations and the 3-term recurrence relation), but found them unsatisfactory from a numerical point of view.

We will now present some graphs of the examples studied in the previous sections, showing the accuracy of our asymptotic approximations in a neighborhood of $x = 0$.

In Figure 1, we plot the functions (121)–(122) for the generalized Meixner polynomials, with

$$\widehat{\Phi}_n(z;x) = e^z \left[1 + \lambda_1(z;x)\, n^{-1} + \lambda_2(z;x)\, n^{-2}\right],$$

where $\lambda_1(z;x)$ was defined in (82), $\lambda_2(z;x)$ was defined in (83), $n = 10$, $a = 0.2479357$, $b = 0.7146983$, and $z = 0.3974126$.



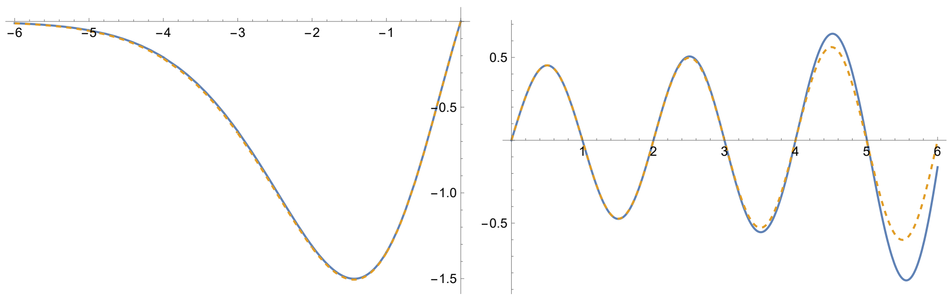
(a) $x < 0$ (b) $x > 0$

Figure 1: A plot of the scaled generalized Meixner polynomial $P_{10}^{(1,1)}(x;z)$ and its approximation.

In Figure 2, we plot the functions (121)–(122) for the polynomials of type $(2, 2)$, with

$$\widehat{\Phi}_n(z;x) = e^z \left[1 + \lambda_1(z;x)\, n^{-1} + \lambda_2(z;x)\, n^{-2}\right],$$

where $\lambda_1(z;x)$ was defined in (85), $\lambda_2(z;x)$ was defined in (86), $n = 10$, $a_1 = 0.2479357$, $a_2 = 0.1963478$, $b_1 = 0.7146983$, $b_2 = 0.5712349$, and $z = 0.3974126$.



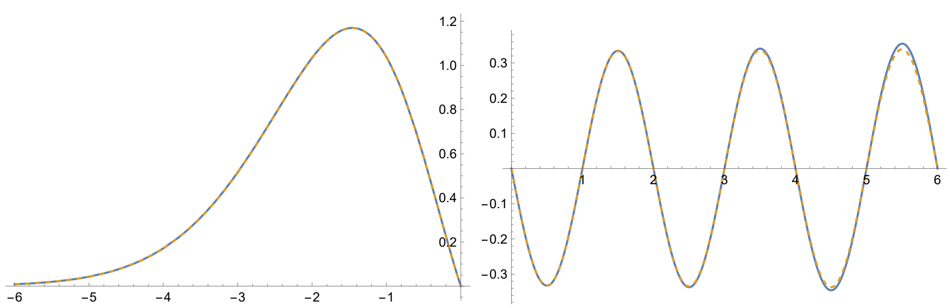
(a) $x < 0$ (b) $x > 0$

Figure 2: A plot of the scaled polynomial $P_{10}^{(2,2)}(x;z)$ and its approximation.

In Figure 3, we plot the functions (121)–(122) for the generalized Charlier polynomials, with

$$\widehat{\Phi}_n(z;x) = 1 + zn^{-1} + \left[(x+1-b)z + \frac{z^2}{2}\right]n^{-2},$$

where $n = 10$, $b = 0.7146983$, and $z = 0.3974126$.



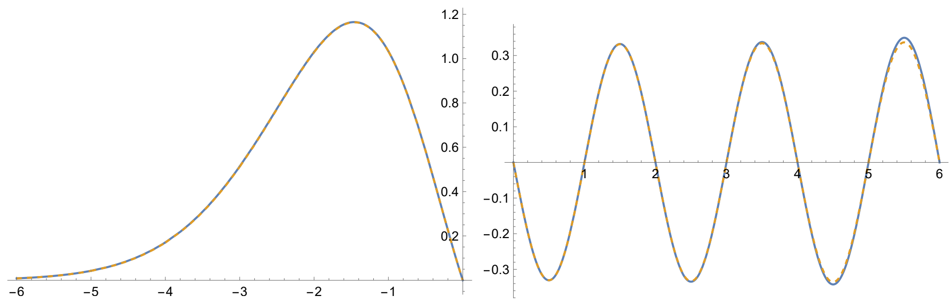
(a) $x < 0$ (b) $x > 0$

Figure 3: A plot of the scaled generalized Charlier polynomial $P_{10}^{(0,1)}(x;z)$ and its approximation.

In Figure 4, we plot the functions (121)–(122) for the polynomials of type $(1,2)$, with

$$\widehat{\Phi}_n(z;x) = 1 + zn^{-1} + \left[(x+a-b_1-b_2)z + \frac{z^2}{2}\right]n^{-2},$$

where $n = 10$, $a = 0.2479357$, $b_1 = 0.7146983$, $b_2 = 0.5712349$, and $z = 0.3974126$.



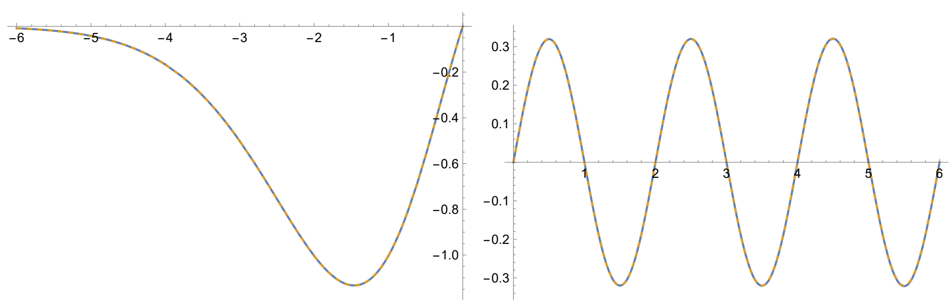
(a) $x < 0$ (b) $x > 0$

Figure 4: A plot of the scaled polynomial $P_{10}^{(1,2)}(x; z)$ and its approximation.

In Figure 5, we plot the functions (121)–(122) for the polynomials of type $(0, 2)$, with

$$\widehat{\Phi}_n(z; x) = 1 + n^{-2}\left[z + (x + 1 - b_1 - b_2) z n^{-1}\right],$$

where $n = 10$, $b_1 = 0.7146983$, $b_2 = 0.5712349$, and $z = 0.3974126$.

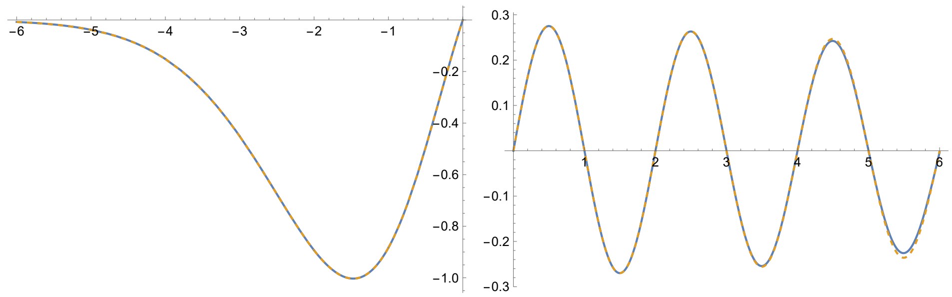


(a) $x < 0$ (b) $x > 0$

Figure 5: A plot of the scaled polynomial $P_{10}^{(0,2)}(x; z)$ and its approximation.

In Figure 6, we plot the functions (121)–(122) for the generalized Hahn polynomials of type I, with

$$\widehat{\Phi}_n(w; x) = (1 - w)^{n+x+1+r_1}\left[1 + \psi_1(w; x) n^{-1} + \psi_2(w; x) n^{-2}\right],$$



(a) $x < 0$ (b) $x > 0$

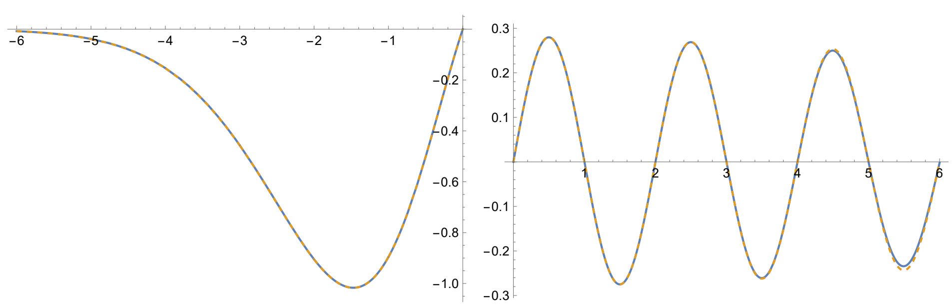
Figure 6: A plot of the scaled generalized Hahn polynomial $P_{10}^{(2,1)}(x; z)$ and its approximation.

where $\psi_1(w; x)$ was defined in (118), $\psi_2(w; x)$ was defined in (119), $r_1 = a_1 + a_2 - 2 - b$, $n = 10$, $a_1 = 0.2479357$, $a_2 = 0.1963478$, $b = 0.7146983$, $z = -0.01574126$, and $w = 0.0154973$.

Finally, in Figure 7, we plot the functions (121)–(122) for the polynomials of type $(3, 2)$, with

$$\Phi_n(w; x) = (1 - w)^{n+x+1+r_1} \left[1 + \frac{\psi_1(w; x)}{n} + \frac{\psi_2(w; x)}{n^2}\right],$$

where $\psi_1(w; x)$ was defined in (111), $\psi_2(w; x)$ was defined in (112), r_1, r_2, r_3 are given by (120), $n = 10$, $a_1 = 0.2479357$, $a_2 = 0.1963478$, $a_3 = 0.3614782$, $b_1 = 0.7146983$, $b_2 = 0.5712349$, $z = -0.01574126$, and $w = 0.0154973$.



(a) $x < 0$ (b) $x > 0$

Figure 7: A plot of the scaled polynomial $P_{10}^{(3,2)}(x; z)$ and its approximation.

5 Conclusions

We have given asymptotic expansions for the ratio

$$\frac{P_n(x;z)}{\phi_n(x)}, \quad x = O(1), \quad x \notin \mathbb{N}_0,$$

as $n \to \infty$, where z (and any other parameters) is fixed. The polynomials $P_n(x;z)$ are orthogonal with respect to the linear functional

$$L[u] = \sum_{x=0}^{\infty} u(x) \frac{(\mathbf{a})_x}{(\mathbf{b}+1)_x} \frac{z^x}{x!}, \quad \mathbf{a} \in \mathbb{C}^p, \mathbf{b} \in \mathbb{C}^q,$$

and depending on the value of the parameter $\theta = p + 1 - q$, we have the following cases:

(i) If $\theta < 1$, then

$$\frac{P_n(x;z)}{\phi_n(x)} = 1 + zn^{\theta-1}\left[1 + \frac{x+1+r_1}{n} + O\left(n^{-2}\right)\right], \quad n \to \infty,$$

where

$$\frac{\left(1 - n^{-1} + \mathbf{a}n^{-1}\right)_1}{\left(1 + \mathbf{b}n^{-1}\right)_1} = \sum_{k=0}^{\infty} r_k n^{-k}.$$

(ii) If $\theta = 1$, then as $n \to \infty$

$$\frac{P_n(x;z)}{\phi_n(x)} = e^z \left[1 + \frac{x+1+r_1}{n} z + O\left(n^{-2}\right)\right].$$

This result extends our previous work on the Charlier polynomials, [10], [12].

(iii) If $\theta = 2$, then as $n \to \infty$

$$\frac{P_n(x;w)}{\phi_n(x)} = (1-w)^{n+x+1+r_1}\left[1 - \frac{(x+1)(x+1+r_1)+r_2}{n} w + O\left(n^{-2}\right)\right],$$

where $w = \frac{z}{z-1}$. This result extends our previous work on the Meixner polynomials, [10], [12].

(iv) If $\theta > 2$, then the polynomials $P_n(x;w)$ depend on a parameter N, with $-N \in \mathbb{N}$. We have not analyzed this case, since it will require scaling N in terms of n. For some related work on the Krawtchouk polynomials, see [13]. We plan to study this case in a forthcoming paper.

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