Recurrence relations for the moments of discrete semiclassical orthogonal polynomials.

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November 5, 2022

Abstract

We study recurrence relations satisfied by the moments $\nu_n(z)$ of a linear functional L whose first moment satisfies a differential equation (in z) with polynomial coefficients.

Dedicated to Dick Askey (1933 – 2019), Grandmaster of Special Functions!

Subject Classification Codes: 33C47 (primary), 33C20, 11B37 (secondary).
Keywords: Orthogonal polynomials, moments, recurrences.

1 Introduction

Let $\mathbb{K}$ be a field (we mostly think of $\mathbb{K}$ as the set of complex numbers $\mathbb{C}$) and $\mathbb{N}_0$ be the set of nonnegative integers

$$\mathbb{N}_0 = \mathbb{N} \cup \{0\} = \{0, 1, 2, \ldots\}.$$

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We will denote by $\delta_{k,n}$ the *Kronecker delta*, defined by

$$\delta_{k,n}=\begin{cases}1, & k=n\\ 0, & k\neq n\end{cases}, \quad k,n\in\mathbb{N}_0,$$

and say that $\{\Lambda_n(x)\}_{n\geq 0}\subset\mathbb{K}[x]$ is a *monic basis* if $\Lambda_n(x)$ is monic and $\deg(\Lambda_n)=n$ for all $n\in\mathbb{N}_0$.

Suppose that $\{\Lambda_n(x)\}_{n\geq 0}$ is a monic basis and $L:\mathbb{K}[x]\to\mathbb{K}$ is a *linear functional* (acting on the variable x) satisfying

$$h_n=L\left[\Lambda_n^2\right]\neq 0, \quad n\in\mathbb{N}_0.$$

If the system of linear equations

$$\sum_{i=0}^{n}L\left[\Lambda_k\Lambda_i\right]c_{n,i}=h_n\delta_{k,n}, \quad 0\leq k\leq n, \quad c_{n,n}=1, \tag{1}$$

has a **unique solution** $\{c_{n,i}\}_{0\leq i\leq n}$, we can define a **monic** polynomial $P_n(x)$ by

$$P_n(x)=\sum_{i=0}^{n}c_{n,i}\Lambda_i(x),$$

and say that $\{P_n(x)\}_{n\geq 0}$ is an *orthogonal polynomial sequence* with respect to the functional L.

The system (1) can be written as

$$L\left[\Lambda_k P_n\right]=h_n\delta_{k,n}, \quad 0\leq k\leq n,$$

and using linearity we see that the sequence $\{P_n(x)\}_{n\geq 0}$ satisfies the *orthogonality conditions*

$$L\left[P_k P_n\right]=h_n\delta_{k,n}, \quad 0\leq k\leq n. \tag{2}$$

If we define the (symmetric) matrix of moments $\mathbf{G}$ by

$$\mathbf{G}_{i,k}=L\left[\Lambda_i\Lambda_k\right], \quad i,k\in\mathbb{N}_0, \tag{3}$$

one can show [14] that the condition

$$\det_{0\leq i,k\leq n}\left(\mathbf{G}_{i,k}\right)\neq 0, \quad n\geq 0,$$

is equivalent to the existence of a unique family of orthogonal polynomial satisfying (2) and $\deg(P_n) = n$.

The theory of orthogonal polynomials is vast and rich, extending all the way back to the groundbreaking work of Legendre [42], where he introduced the family of polynomials that now bears his name. We direct the interested reader to (some of!) the fundamental treatises on the field [8], [10], [29], [31], [33], [40], [70].

A particular fruitful approach that has received a lot of attention in recent years, is to work with the (infinite) matrix (3) acting on the (infinite) vector $\overrightarrow{P} = (P_0, P_1, \ldots)$. One can then view orthogonal polynomial sequences as elements of an infinite dimensional vector space [17], [22], [30], [45], [74], [75], [76], [77].

Of course, in its full generality, it's difficult to get results that apply to *any* family of orthogonal polynomials. Thus, one chooses, for example:

i.) an operator (difference, differential, functional, integral) that annihilates $P_n(x)$.

ii.) a degree-reducing operator relating $P_n(x)$ and $P_{n-1}(x)$ (Sheffer classification, umbral calculus, generating functions).

iii.) a particular form of the linear functional L (continuous, discrete, matrix valued, q-series).

iv.) a particular domain of L ($\mathbb{C}$, $\mathbb{N}_0$, $\mathbb{R}$, quadratic lattices, unit circle).

Another possibility, is to ask L to satisfy a relation of the form

$$L[\sigma p] = L[\tau U[p]], \quad p \in \mathbb{K}[x],$$

where $\sigma(x), \tau(x)$ are *fixed* polynomials, and $U : \mathbb{K}[x] \to \mathbb{K}[x]$ is a *degree reducing* linear operator satisfying $U[1] = 0$ and

$$\deg U[x^n] = n - 1, \quad n \in \mathbb{N}.$$

In this case, we say that L is a *semiclassical functional* with respect to U. The *class* of the functional L is defined by

$$s = \max\{\deg(\sigma) - 2, \deg(\tau) - 1\},$$

and semiclassical functional of class $s = 0$ are called *classical*.

This type of functionals was introduced by Shohat [67], and studied in detail by P. Maroni and collaborators [53], [55], [56], particularly when

$U[p] = \partial_x p$ is the derivative operator [47], [51], [52], and also for the operator

$$U_\omega [p] = \frac{p(x+\omega) - p(x)}{\omega},$$

which contains the finite difference operators Δ, ∇ as special cases $(\omega = \pm 1)$, and the derivative operator as a limiting case [1]. Other examples include the $q-$semiclassical polynomials [38], [57], associated with the operator

$$U_q [p] = \frac{p(qx) - p(x)}{(q-1)x}, \quad q \neq 1.$$

In this paper, we will focus on the so-called *discrete semiclassical orthogonal polynomials* [7], [27], [50], [58], [79], where U is the shift operator $U[p] = p(x+1)$. In this case, the linear functional L is of the form

$$L[p] = \sum_{x=0}^{\infty} p(x)\rho(x), \quad p \in \mathbb{K}[x],$$

where $\rho(x)$ is a given *weight function*. The traditional starting point is the *Pearson equation* satisfied by $\rho(x)$

$$U[\sigma\rho] = \tau(x)\rho(x), \tag{4}$$

but after trying this approach in [24], we found it very dissatisfying, especially when one considers spectral transformations of L.

For example, applying an Uvarov transformation to L at a point ω (see Section 3.3) will lead to the Pearson equation

$$\frac{\widetilde{\rho}(x+1)}{\widetilde{\rho}(x)} = \frac{(x-\omega)(x+1-\omega)\tau(x)}{(x-\omega)(x+1-\omega)\sigma(x+1)},$$

and this begs the question of when one is allowed (or not) to simplify the above expression. A possibility to avoid this problem is to study the difference equation satisfied by the *Stieltjes transform* of L

$$S(t) = L\left[\frac{1}{t-x}\right], \quad t \notin \mathbb{N}_0,$$

and we did this in [25], where we classified the discrete semiclassical orthogonal polynomials of class $s \leq 2$.

Now suppose that the weight function $\rho(x)$ also contains an *independent variable* z, $\rho = \widetilde{\rho}(x; z)$. Although this may seem like an extra assumption, we note that one could always introduce such a variable as a *Toda deformation* [9], [62], [72],

$$\widetilde{\rho}(x; z) = \rho(x)\, e^{xf(z)}, \quad f(z_0) = 0,$$

and recover the original functional L by setting $z = z_0$. We studied this type of weight functions in [23], and observed that the operator ϑ defined by

$$\vartheta[u] = z\frac{du}{dz}$$

is naturally associated to the shift operator.

As we will see in Section 2, this allows us to replace the Pearson equation (4) with the ODE satisfied by the first moment $\lambda_0(z) = L[1]$,

$$\sigma(\vartheta)[\lambda_0] = z\tau(\vartheta)[\lambda_0]. \tag{5}$$

We note in passing that the ODE (5) is the true starting point of the theory, and by considering alternative equations satisfied by $\lambda_0(z)$, one could study semiclassical orthogonal polynomials associated with different operators U.

The structure of the paper is as follows: in Section 2, we introduce the operator ϑ and the ODE satisfied by the moments of a discrete linear functional

$$\sigma(\vartheta)\Lambda_n(\vartheta)[\lambda_0] = z\tau(\vartheta)\Lambda_n(\vartheta+1)[\lambda_0], \quad n \in \mathbb{N}_0. \tag{6}$$

This will naturally lead to the class of functionals whose first moment $\lambda_0(z)$ can be represented as a (generalized) hypergeometric function.

Since the ODE (6) contains a shift, we need to choose a convenient basis $\{\Lambda_n(x)\}_{n\geq 0}$. In Section 2.1, we study the monomial basis and derive a linear recurrence <u>of order $n+s+1$</u> for the (standard) moments $\mu_n(z)$. We also find a representation for $\mu_n(z)$ as a linear combination involving a family of polynomials that satisfies a differential-difference equation.

In Section 2.2, we consider the basis of falling factorial polynomials defined by $\phi_0(x) = 1$,

$$\phi_{n+1}(x) = x\phi_n(x-1), \quad n \in \mathbb{N},$$

which allows us to easily work on the lattice $\mathbb{N}_0$. We use Newton's interpolation formula and obtain a linear recurrence <u>of order $s+1$</u> for the (modified) moments $\nu_n(z)$. The linear functionals of class $s=1$ are particularly interesting, since in this case the moments $\nu_n(z)$ are themselves a family of orthogonal polynomials. This is an area that has been studied in detail by M. Ismail and D. Stanton, see [34], [35], and [36].

Both the monomials and the falling factorial polynomials are examples of *Newton basis polynomials* defined by $\mathfrak{n}_0(x)=1$ and

$$\mathfrak{n}_k(x)=\prod_{j=0}^{k-1}(x-\kappa_j),$$

where $\{\kappa_j\}_{j\geq 0}$ is a fixed sequence. This type of polynomials satisfy $2-$term recurrence relations, which we study in Section 2.3. Among other results, we look at the connection between the monomial and falling factorial bases (through Stirling numbers), and find the (formal) representation for the Stieltjes transform

$$S(\omega;z)=\sum_{k=0}^{\infty}\frac{\lambda_k(z)}{\Lambda_{k+1}(\omega)}. \tag{7}$$

In [26], we used (7) to derive recurrence relations for the modified moments $\nu_n(z)$.

In Section 3, we consider transformations Ω_β^α between different families of discrete semiclassical orthogonal polynomials. We introduce a uniform notation to label objects belonging to different families, and show how the recurrence relations for the moments change as we apply a transformation.

In Sections 3.1, 3.2, 3.3, and (3.4) we consider the special cases $\alpha=\beta+1$ (*Christoffel transformation*) [12], [28], [66], $\alpha=\beta-1$ (*Geronimus transformation)* [19], [20], [41], [54], their composition (*Uvarov transformation)* [5], [6], [15], [39], [49], and $\alpha=\beta=-N$, $N\in\mathbb{N}$ (*truncation transformation).* These *rational spectral transformations* have been studied by many authors, [4], [43], [61], [81]. The relation between these transformations and the so-called *Darboux transformation*, has also been considered [13], [48], [80].

2 Differential operators and moment functionals

Let $\mathbb{F}$ denote the ring of *formal power series* in the variable z

$$\mathbb{F} = \mathbb{K}[[z]] = \left\{ \sum_{n=0}^{\infty} c_n z^n : \quad c_n \in \mathbb{K} \right\},$$

and $\vartheta : \mathbb{F} \to \mathbb{F}$ be the differential operator defined by [59, 16.8.2]

$$\vartheta = z\partial_z, \tag{8}$$

where ∂_z is the *derivative operator* $\partial_z = \frac{\partial}{\partial z}$. The operator ϑ has the following properties.

Proposition 1 *Let the differential operator ϑ be defined by (8). Then, for all $u, v \in \mathbb{K}[x]$ we have:*

(i) *The action of ϑ on the monomials is given by*

$$u(\vartheta)[z^x] = u(x) z^x, \tag{9}$$

*where we **always** assume that x and z are* independent variables.

(ii) *ϑ is multiplicative*

$$(uv)(\vartheta) = u(\vartheta) v(\vartheta). \tag{10}$$

(iii) *For all $k \in \mathbb{N}_0$,*

$$u(\vartheta)\left[z^k v(\vartheta)\right] = z^k \mathcal{S}_\vartheta^k [u] v(\vartheta), \tag{11}$$

where $\mathcal{S}_r$ denotes the shift operator *defined by*

$$\mathcal{S}_r[f] = f(r+1). \tag{12}$$

Proof. (i) Iterating (8), we get

$$\vartheta^k [z^x] = x^k z^x, \quad k \in \mathbb{N}_0.$$

Using linearity, the result follows.

(ii) Using (9), we have

$$\vartheta^{n+m}\left[z^{x}\right]=x^{n+m}z^{x}=x^{n}x^{m}z^{x}=x^{n}\vartheta^{m}\left[z^{x}\right]=\vartheta^{m}\left[x^{n}z^{x}\right]=\vartheta^{m}\vartheta^{n}\left[z^{x}\right],$$

for all $m,n\in\mathbb{N}_0$. The result follows from linearity.

(iii) Using (9) and (10), we see that

$$\begin{aligned}u\left(\vartheta\right)\left[z^{k}v\left(\vartheta\right)\left[z^{x}\right]\right]&=u\left(\vartheta\right)\left[z^{k}v\left(x\right)z^{x}\right]=u\left(\vartheta\right)\left[v\left(x\right)z^{x+k}\right]\\&=v\left(x\right)u\left(\vartheta\right)\left[z^{x+k}\right]=v\left(x\right)u\left(x+k\right)z^{x+k}\\&=z^{k}v\left(x\right)u\left(x+k\right)z^{x}=z^{k}v\left(\vartheta\right)u\left(\vartheta+k\right)\left[z^{x}\right],\end{aligned}$$

and the result follows. ■

Let $L:\mathbb{K}[x]\rightarrow\mathbb{F}$ be the *linear functional* (acting on the variable x) defined by

$$L\left[u\right]=\sum_{x=0}^{\infty}u\left(x\right)\rho\left(x\right)z^{x},\quad u\in\mathbb{K}[x],\tag{13}$$

where $\rho:\mathbb{N}_0\rightarrow\mathbb{K}$ is a given function.

Remark 2 *If $f\in\mathbb{K}\left[[x]\right]$, we can extend (9) to*

$$f\left(\vartheta\right)\left[z^{x}\right]=\sum_{n=0}^{\infty}c_{n}\vartheta^{n}\left[z^{x}\right]=z^{x}\sum_{n=0}^{\infty}c_{n}x^{n}=f\left(x\right)z^{x},$$

and therefore we can consider L as a functional on $\mathbb{K}\left[[x]\right]$, satisfying

$$L\left[uf\right]=\sum_{x=0}^{\infty}u(x)f\left(x\right)\rho\left(x\right)z^{x}=f\left(\vartheta\right)\left[\sum_{x=0}^{\infty}u(x)\rho\left(x\right)z^{x}\right]=f\left(\vartheta\right)\left[L\left[u\right]\right],\tag{14}$$

for all $u\in\mathbb{K}[x],\ f\in\mathbb{K}\left[[x]\right]$.

Let $\left\{\Lambda_n\right\}_{n\geq0}$ be a monic polynomial basis. If we define a sequence of *moments* [2], [3], [68] by

$$\lambda_n\left(z\right)=L\left[\Lambda_n\right]\in\mathbb{F},$$

then from (14) we obtain

$$f\left(\vartheta\right)\left[\lambda_0\right]=f\left(\vartheta\right)\left[L\left[1\right]\right]=L\left[f\right],\quad f\in\mathbb{K}\left[[x]\right],\tag{15}$$

and in particular

$$\lambda_n\left(z\right)=L\left[\Lambda_n\right]=\Lambda_n\left(\vartheta\right)\left[\lambda_0\right].\tag{16}$$

Using (15), we can obtain a generating function for the moments of L.

Proposition 3 *Let $E_{\Lambda}(t,x)$ denote the* exponential generating function *[78] of the polynomials $\Lambda_n(x)$*

$$E_{\Lambda}(t;x)=\sum_{n=0}^{\infty}\Lambda_n(x)\frac{t^n}{n!}. \tag{17}$$

Then, the exponential generating function *of the moments $\lambda_n(z)$ is given by*

$$\epsilon_{\lambda}(t;z)=\sum_{n=0}^{\infty}\lambda_n(z)\frac{t^n}{n!}=L\left[E_{\Lambda}(t;x)\right],$$

where it's always understood that L is ***only*** *acting on the variable x.*

In particular, if $E_{\Lambda}(t;x)=[f(t)]^x$ is an exponential function, we have

$$\epsilon_{\lambda}(t;z)=\lambda_0\left[zf(t)\right]. \tag{18}$$

Proof. Using (16) and (17), we get

$$\sum_{n=0}^{\infty}\lambda_n(z)\frac{t^n}{n!}=\sum_{n=0}^{\infty}\Lambda_n(\vartheta)\left[\lambda_0\right]\frac{t^n}{n!}=E_{\Lambda}(t;\vartheta)\left[\lambda_0\right],$$

and from (15) we see that $E_{\Lambda}(t;\vartheta)\left[\lambda_0\right]=L\left[E_{\Lambda}(t;x)\right].$

If $E_{\Lambda}(t;x)=[f(t)]^x$, then

$$L\left[E_{\Lambda}(t;x)\right]=L\left[f^x\right]=\sum_{x=0}^{\infty}\rho(x)\frac{[zf(t)]^x}{x!}=\lambda_0\left[zf(t)\right].$$

■

Up to this point, $\rho(x)$ is an arbitrary weight function. We will now characterize it by imposing a condition on the first moment $\lambda_0(z)$.

Theorem 4 *If the first moment $\lambda_0(z)$ satisfies the* differential equation with polynomial coefficients

$$\left[\sigma(\vartheta)-z\tau(\vartheta)\right]\left[\lambda_0\right]=0, \quad \sigma,\tau\in\mathbb{K}[x], \tag{19}$$

then

(i) L is a semiclassical functional

$$L\left[\sigma u\right]=L\left[z\tau\mathcal{S}_x\left[u\right]\right], \quad u\in\mathbb{K}[x] \tag{20}$$

with respect to the shift operator $\mathcal{S}_x$ defined in (12).

(ii) If $\sigma(0) = 0$, then $\rho(x)$ satisfies the Pearson equation [60]

$$\frac{\rho(x+1)}{\rho(x)} = \frac{\tau(x)}{\sigma(x+1)}, \quad x \in \mathbb{N}_0. \tag{21}$$

(iii) If we set $\rho(0) = 1$, then

$$\rho(x) = \prod_{k=0}^{x-1} \frac{\tau(k)}{\sigma(k+1)}, \quad x \in \mathbb{N}. \tag{22}$$

Proof. (i) Let $u \in \mathbb{K}[x]$. Using (11) in (19), we see that

$$u(\vartheta)\,\sigma(\vartheta)[\lambda_0] = u(\vartheta)[z\tau(\vartheta)[\lambda_0]] = z\tau(\vartheta)\,u(\vartheta+1)[\lambda_0], \tag{23}$$

and using (15), we conclude that

$$L[\sigma(x)\,u(x)] = L[z\tau(x)\,u(x+1)].$$

(ii) If $\sigma(0) = 0$, we can use (20) and obtain

$$\sum_{x=1}^{\infty} \sigma(x)\,u(x)\,\rho(x)\,z^x = L[\sigma u] = L[z\tau u(x+1)]$$

$$= \sum_{x=0}^{\infty} \tau(x)\,u(x+1)\,\rho(x)\,z^{x+1} = \sum_{x=1}^{\infty} \tau(x-1)\,u(x)\,\rho(x-1)\,z^x.$$

Comparing powers of z, (21) follows.

(iii) Using (21), we get

$$\prod_{k=0}^{x-1} \frac{\tau(k)}{\sigma(k+1)} = \prod_{k=0}^{x-1} \frac{\rho(k+1)}{\rho(k)} = \frac{\rho(x)}{\rho(0)},$$

and (22) follows if we define $\rho(0) = 1$. ■

The *Pochhammer symbol* $(c)_x$ is defined by [63]

$$(c)_x = \lim_{k\to\infty} k^x \prod_{j=0}^{k} \frac{c+j}{c+x+j}, \quad -(c+x) \notin \mathbb{N}_0,$$

and when $n \in \mathbb{N}_0$, $(c)_n$ becomes a *polynomial* in c of degree n

$$(c)_n = \prod_{j=0}^{n-1} (c+j), \quad n \in \mathbb{N}, \quad (c)_0 = 1. \tag{24}$$

We will use the notation [59, 16.1]

$$(\mathbf{c})_n = (c_1)_n \cdots (c_m)_n, \quad \mathbf{c} \in \mathbb{K}^m,$$

and also

$$(x + \mathbf{c}) = (x + c_1) \cdots (x + c_m), \quad \mathbf{c} \in \mathbb{K}^m.$$

In the special case $m = 0$, we understand that

$$\mathbb{K}^0 = \emptyset, \quad (\emptyset)_n = 1, \quad n \in \mathbb{N}_0,$$

while for $m = \infty$ we have

$$\mathbb{K}^\infty = \left\{ \{c_k\}_{k \geq 0} : c_k \in \mathbb{K} \right\}.$$

Let $p, q \in \mathbb{N}_0$ be some fixed numbers. In the remainder of the paper, we will **always** have $\mathbf{a} \in \mathbb{K}^p, \mathbf{b} \in \mathbb{K}^q$ and

$$\sigma(x) = x(x + \mathbf{b}), \quad \tau(x) = (x + \mathbf{a}). \tag{25}$$

Using (24), we can rewrite (22) as

$$\rho(x) = \frac{(\mathbf{a})_x}{(\mathbf{b}+1)_x} \frac{1}{x!},$$

and using (25) in (19), we have

$$[\vartheta(\vartheta + \mathbf{b}) - z(\vartheta + \mathbf{a})][\lambda_0] = 0. \tag{26}$$

The ODE (26) is the (generalized) *hypergeometric differential equation* [59, 16.8.3] of order

$$\mathrm{o} = \max\{p, q+1\},$$

and the first moment $\lambda_0(z)$ can be represented as

$$\lambda_0(z) = {}_pF_q\left(\begin{matrix} \mathbf{a} \\ \mathbf{b}+1 \end{matrix} ; z \right),$$

where the (generalized) *hypergeometric function* ${}_pF_q$ is defined by [59, 16.2.1], [69],

$$ {}_pF_q\left(\begin{array}{c} \mathbf{a} \\ \mathbf{b} \end{array} ; z \right) = \sum_{x=0}^{\infty} \frac{(\mathbf{a})_x}{(\mathbf{b})_x} \frac{z^x}{x!}. $$

We define the *class* s of the semiclassical functional L by

$$ s = \mathrm{o} - 1 = \max\{p-1, q\}, $$

and functionals of class $s = 0$ are called *classical*.

Multiplying (26) by $\Lambda_n(\vartheta)$ and using (23), we conclude that

$$ [\vartheta(\vartheta+\mathbf{b})\Lambda_n(\vartheta) - z(\vartheta+\mathbf{a})\Lambda_n(\vartheta+1)][\lambda_0] = 0, \quad n \in \mathbb{N}_0, \tag{27} $$

and expanding the polynomials coefficients on the basis $\{\Lambda_n\}_{n\geq 0}$,

$$ x(x+\mathbf{b})\Lambda_n(x) = \sum_{k=0}^{n+q+1} c_{n,k}\Lambda_k(x), $$

$$ (x+\mathbf{a})\Lambda_n(x+1) = \sum_{k=0}^{n+p} \widetilde{c}_{n,k}\Lambda_k(x), $$

we get a recurrence relation of order $n+s+1$ for the moments $\lambda_n(z)$

$$ \sum_{k=-n}^{q+1} c_{n,n+k}\lambda_{n+k} - z\sum_{k=-n}^{p} \widetilde{c}_{n,n+k}\lambda_{n+k} = 0. \tag{28} $$

The question is: can we do better than this? In other words, can one choose a convenient basis Λ_n so that the recurrence (28) will have minimal order $s+1$? The answer is yes, as we will see in Section 2.2. In the meantime, we study the simplest basis: the monomials.

2.1 Standard moments

To simplify the formulas, in the remainder of the paper we will use the *umbral notation [65]*

$$ \psi^k \leftrightarrow \psi_k, \quad \psi \in \mathbb{K}^{\infty}. $$

So, for example, the equation

$$ (\psi+b)\psi^{n+1} - z(\psi+a)(\psi+1)^n = 0 $$

can be written in extended form as

$$\psi_{n+2} + b\psi_{n+1} - z\sum_{k=0}^{n} \binom{n}{k} (\psi_{k+1} + a\psi_k) = 0.$$

Let $\mu_n(z) \in \mathbb{F}$ denote the *standard moments* of L on the monomial basis $\Lambda_n(x) = x^n$

$$\mu_n(z) = L[x^n], \quad n \in \mathbb{N}_0.$$

Using $\Lambda_n(x) = x^n$ in (27), we get

$$\left[(\vartheta + \mathbf{b})\,\vartheta^{n+1} - z(\vartheta + \mathbf{a})(\vartheta + 1)^n\right][\mu_0] = 0. \tag{29}$$

The polynomials $(x + \mathbf{c})$ can be written in the monomial basis as

$$(x + \mathbf{c}) = \sum_{k=0}^{m} e_{m-k}(\mathbf{c})\, x^k, \quad \mathbf{c} \in \mathbb{K}^m, \tag{30}$$

where the *elementary symmetric polynomials* $e_n(\mathbf{c})$ are defined by the generating function [46]

$$\sum_{n=0}^{\infty} e_n(\mathbf{c})\, t^n = \prod_{i=1}^{m} (1 + tc_i), \quad \mathbf{c} \in \mathbb{K}^m. \tag{31}$$

Using these formulas, we can write a recurrence for $\mu_n(z)$.

Theorem 5 *(i) The standard moments of L satisfy the recurrence*

$$(\mu + \mathbf{b})\,\mu^{n+1} - z(\mu + \mathbf{a})(\mu + 1)^n = 0. \tag{32}$$

(ii) We have the explicit recurrence

$$\sum_{k=0}^{q} e_{q-k}(\mathbf{b})\,\mu_{n+k+1} - z\sum_{k=0}^{n} \binom{n}{k} \sum_{j=0}^{p} e_{p-j}(\mathbf{a})\,\mu_{k+j} = 0. \tag{33}$$

In particular, for $n = 0$

$$\sum_{k=0}^{q} e_{q-k}(\mathbf{b})\,\mu_{k+1} - z\sum_{j=0}^{p} e_{p-j}(\mathbf{a})\,\mu_j = 0. \tag{34}$$

Proof. (i) Using (15) in (29), we have

$$\left[(\vartheta+\mathbf{b})\,\vartheta^{n+1}\right][\mu_0] = (\mu+\mathbf{b})\,\mu^{n+1},$$
$$(\vartheta+\mathbf{a})\,(\vartheta+1)^n\,[\mu_0] = (\mu+\mathbf{a})\,(\mu+1)^n\,,$$

and the result follows.

(ii) Using (30) in (32), we get

$$(\mu+\mathbf{b})\,\mu^{n+1} = \sum_{k=0}^{q} e_{q-k}\,(\mathbf{b})\,\mu^{n+k+1},$$
$$(\mu+\mathbf{a})\,(\mu+1)^n = \sum_{k=0}^{n}\binom{n}{k}(\mu+\mathbf{a})\,\mu^k = \sum_{k=0}^{n}\binom{n}{k}\sum_{j=0}^{p} e_{p-j}\,(\mathbf{a})\,\mu^{k+j},$$

and the result follows. ■

It is clear from (33) that elements of the set

$$\{\mu_k : \quad k>s\}\,, \quad s=\max\{p-1,q\}\,,$$

are linear combinations of the first $s+1$ standard moments. Thus, we have a representation of the form

$$\mu_n\,(z) = \sum_{k=0}^{s} g_{n,k}\,(z)\,\mu_k\,(z)\,, \quad n\in\mathbb{N}_0, \tag{35}$$

where the coefficients must satisfy

$$g_{n,k}\,(z) = \delta_{n,k}, \quad 0\le n,k\le s. \tag{36}$$

If we introduce the vectors $\overrightarrow{\mu}, \overrightarrow{g}_n \in \mathbb{F}^{s+1}$ defined by

$$(\overrightarrow{\mu})_k = \mu_k, \quad (\overrightarrow{g}_n)_k = g_{n,k}, \quad 0\le k\le s,$$

we can write (35) as an inner product

$$\mu_n = \overrightarrow{g}_n \cdot \overrightarrow{\mu}. \tag{37}$$

To satisfy the initial conditions (36), we need

$$\overrightarrow{g}_n = \overrightarrow{\varepsilon}_n, \quad 0\le n\le s,$$

where the *standard unit vectors* $\overrightarrow{\varepsilon}_n \in \mathbb{K}^{s+1}$ are defined by

$$(\overrightarrow{\varepsilon}_n)_k = \delta_{n,k}, \quad 0\le k\le s, \quad n\in\mathbb{N}_0.$$

Theorem 6 *With the previous definitions, let the matrix M be given by*

$$M = \left(\overrightarrow{\varepsilon}_1, \overrightarrow{\varepsilon}_2, \cdots, \overrightarrow{\varepsilon}_s, \overrightarrow{g}_{s+1}\right) \in \mathbb{F}^{(s+1)\times(s+1)},$$

where the vectors form the columns *of M. Then, $\overrightarrow{g}_n(z)$ satisfies the* differential-difference equation

$$\overrightarrow{g}_{n+1} = (\vartheta + M)\,\overrightarrow{g}_n, \quad n \geq 0, \quad \overrightarrow{g}_0 = \overrightarrow{\varepsilon}_0. \tag{38}$$

Proof. From (37). we get

$$\mu_{n+1} = \vartheta\left[\mu_n\right] = \vartheta\left[\overrightarrow{g}_n \cdot \overrightarrow{\mu}\right] = \vartheta\left[\overrightarrow{g}_n\right] \cdot \overrightarrow{\mu} + \overrightarrow{g}_n \cdot \vartheta\left[\overrightarrow{\mu}\right]$$

and since

$$\vartheta\left[\overrightarrow{\mu}\right] = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_{s+1} \end{pmatrix} = \begin{pmatrix} \overrightarrow{g}_1 \\ \overrightarrow{g}_2 \\ \vdots \\ \overrightarrow{g}_{s+1} \end{pmatrix} \begin{pmatrix} \mu_0 \\ \mu_1 \\ \vdots \\ \mu_s \end{pmatrix},$$

we have $\vartheta\left[\overrightarrow{\mu}\right] = M^T \overrightarrow{\mu}$, with

$$M^T = \begin{pmatrix} \overrightarrow{g}_1 \\ \overrightarrow{g}_2 \\ \vdots \\ \overrightarrow{g}_{s+1} \end{pmatrix} \in \mathbb{F}^{(s+1)\times(s+1)},$$

where vectors form the *rows* of the matrix M^T. Thus,

$$\overrightarrow{g}_{n+1} \cdot \overrightarrow{\mu} = \mu_{n+1} = \vartheta\left[\overrightarrow{g}_n\right] \cdot \overrightarrow{\mu} + \overrightarrow{g}_n \cdot \left(M^T \overrightarrow{\mu}\right) = \left(\vartheta\left[\overrightarrow{g}_n\right] + M \overrightarrow{g}_n\right) \cdot \overrightarrow{\mu}$$

from which the result follows. ■

Remark 7 *In [21], we derived (38) using a different method.*

From (33), we see that we have three cases to consider.

Corollary 8 *(i) If $p > q + 1$, then the vector polynomials*

$$\overrightarrow{Q}_n(z) = z^n\ \overrightarrow{g}_n(z) \in \left(\mathbb{K}[z]\right)^{s+1}, \quad n \geq 0,$$

satisfy the differential-difference equation

$$\overrightarrow{Q}_{n+1} = z\left(\vartheta + M - nI\right)\overrightarrow{Q}_n, \quad n \geq 0, \quad \overrightarrow{Q}_0 = \overrightarrow{\varepsilon}_0, \tag{39}$$

where I is the $(s+1 \times s+1)$ identity matrix.

(ii) If $p = q+1$, then the vector polynomials

$$\overrightarrow{Q}_n(z) = (1-z)^n \ \overrightarrow{g}_n(z) \in (\mathbb{K}[z])^{s+1}, \quad n \geq 0,$$

satisfy the differential-difference equation

$$\overrightarrow{Q}_{n+1} = \left[(1-z)(\vartheta + M) + nzI\right]\overrightarrow{Q}_n, \quad n \geq 0, \quad \overrightarrow{Q}_0 = \overrightarrow{\varepsilon}_0. \tag{40}$$

(iii) If $p < q+1$, then $\overrightarrow{g}_n(z)$ is a vector polynomial.

Proof. (i) If $p > q+1$, then the standard moments will satisfy a recurrence of the form

$$z\mu_{n+p} = \sum_{k=0}^{n+p-1} c_{n,k}(z)\,\mu_k,$$

and setting $\overrightarrow{g}_n(z) = z^{-n}\overrightarrow{Q}_n(z)$ in (38), we get (39).

(ii) If $p = q+1$, then the standard moments will satisfy a recurrence of the form

$$(1-z)\,\mu_{n+p} = \sum_{k=0}^{n+p-1} c_{n,k}(z)\,\mu_k,$$

and if we set $\overrightarrow{g}_n(z) = (1-z)^{-n}\,\overrightarrow{Q}_n(z)$ in (38), we get (40).

(iii) If $p < q+1$, then the standard moments will satisfy a recurrence of the form

$$\mu_{n+q+1} = z\sum_{k=0}^{n+q} c_{n,k}\mu_k,$$

and it follows that the functions $\overrightarrow{g}_n(z)$ are polynomials in z. ■

Finally, we will study the exponential generating function of the standard moments.

Proposition 9 *(i) The exponential generating function of the standard moments*

$$\epsilon_\mu(t;z) = \sum_{n=0}^{\infty}\mu_n(z)\,\frac{t^n}{n!}$$

is given by

$$\epsilon_{\mu}(t;z)=\mu_0\left(ze^{t}\right). \tag{41}$$

(ii) The function $\epsilon_{\mu}(t;z)$ is a solution of the linear ODE (in the t variable)

$$\left[\sigma(\partial_t)-ze^{t}\tau(\partial_t)\right][y]=0. \tag{42}$$

Proof. (i) The exponential generating function of the monic basis is the exponential function

$$\sum_{n=0}^{\infty}x^{n}\frac{t^{n}}{n!}=e^{xt},$$

and using (18) we obtain (41).

(ii) Since

$$\partial_t\left[y\left(ze^{t}\right)\right]=ze^{t}y'\left(ze^{t}\right)=z\partial_z\left[y\left(ze^{t}\right)\right]=\vartheta\left[y\left(ze^{t}\right)\right],$$

it follows from (19) that $\mu_0(ze^{t})$ satisfies (42). ∎

Remark 10 *If we define*

$$G_k(t,z)=\sum_{n=0}^{\infty}g_{n,k}(z)\frac{t^{n}}{n!},\quad 0\leq k\leq s,$$

it follows from (35) that

$$\mu_0\left(ze^{t}\right)=\sum_{k=0}^{s}G_k(t,z)\,\mu_k(z),$$

and therefore the functions $G_k(t,z)$, $0\leq k\leq s$ form a basis of solutions of the ODE (42) with initial conditions

$$\left[\partial_t^{n}G_k\right]_{t=0}=\delta_{n,k},\quad 0\leq n,k\leq s,$$

since from (41) we see that

$$\left[\partial_t^{n}\mu_0\left(ze^{t}\right)\right]_{t=0}=\mu_n(z).$$

2.2 Modified moments

Let $\phi_n(x)$ denote the *falling factorial polynomials* defined by $\phi_0(x) = 1$ and

$$\phi_n(x) = \prod_{k=0}^{n-1} (x-k), \quad n \in \mathbb{N}. \tag{43}$$

Sometimes, the polynomials $\phi_n(x)$ are called "binomial polynomials", since

$$\frac{\phi_n(x)}{n!} = \binom{x}{n}, \quad n \in \mathbb{N}_0. \tag{44}$$

From the definition (43), we see that

$$\phi_{n+1}(x) = (x-n)\,\phi_n(x) = x\phi_n(x-1), \quad n \geq 0, \tag{45}$$

and from (24) it follows that the falling factorial polynomials and the Pochhammer polynomials are related by

$$\phi_n(x) = (-1)^n (-x)_n = (x+1-n)_n .$$

The falling factorial polynomials are eigenfuncions of the differential operator $z^n \partial_z^n$ since

$$z^n \partial_z^n [z^x] = z^n \phi_n(x)\, z^{x-n} = \phi_n(x)\, z^x. \tag{46}$$

Remark 11 *Caution must be exercised when using the operators $z^n \partial_z^n$ and ϑ^n since*

$$\vartheta^n = (z\partial_z)^n \neq z^n (\partial_z)^n, \quad n > 1.$$

Proposition 12 *Let the* modified moments *be defined by*

$$\nu_n(z) = L[\phi_n], \quad n \in \mathbb{N}_0. \tag{47}$$

Then, for all $n \in \mathbb{N}_0$,

$$\nu_n(z) = z^n \frac{(\mathbf{a})_n}{(\mathbf{b}+1)_n} \, {}_pF_q\left(\begin{matrix} \mathbf{a}+n \\ \mathbf{b}+n+1 \end{matrix} ; z \right).$$

Proof. The results follows from (46) and the formula [59, 16.3.1]

$$\partial_z^n \left[{}_pF_q \left(\begin{matrix} \mathbf{a} \\ \mathbf{b}+1 \end{matrix} ; z \right) \right] = \frac{(\mathbf{a})_n}{(\mathbf{b}+1)_n} \, {}_pF_q \left(\begin{matrix} \mathbf{a}+n \\ \mathbf{b}+n+1 \end{matrix} ; z \right).$$

■

Using (27) with $\Lambda_n(\vartheta) = \phi_n(\vartheta - 1)$, we get

$$\vartheta(\vartheta + \mathbf{b}) \phi_n(\vartheta - 1) \sigma(\vartheta) [\nu_0] = z(\vartheta + \mathbf{a}) \phi_n(\vartheta) \tau(\vartheta) [\nu_0],$$

and from (45) we conclude that

$$[(\vartheta + \mathbf{b}) \phi_{n+1}(\vartheta) - z(\vartheta + \mathbf{a}) \phi_n(\vartheta)] [\nu_0] = 0. \tag{48}$$

Unlike the monomial case, there is no immediate formula that would express products of the form $(\vartheta + \mathbf{c}) \phi_n(\vartheta)$ in terms of the polynomials $\phi_n(\vartheta)$. Thus, we will find one next.

Any polynomial $u(x)$ can be represented in the basis of falling factorials using *Newton's interpolation formula [18]*

$$u(x) = \sum_{k=0}^{\deg(u)} \frac{\Delta^k [u] (c)}{k!} \phi_k(x - c), \tag{49}$$

where the *forward difference operator* Δ^n (acting on x) is defined by

$$\Delta^n [f] (x) = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} f(x + k). \tag{50}$$

We start with a result that may be already known, but we have not been able to find in the literature.

Lemma 13 *For any function $f(x)$, we have*

$$\Delta^j [f \phi_n] (0) = 0, \quad 0 \leq j < n, \tag{51}$$

and

$$\frac{\Delta^{n+j} [f \phi_n] (0)}{(n+j)!} = \frac{\Delta^j [f] (n)}{j!}, \quad n, j \geq 0. \tag{52}$$

Proof. Using the definition (50),

$$\Delta^{j}\left[f\phi_{n}\right](0)=\sum_{i=0}^{j}\binom{j}{i}(-1)^{j-i}f(i)\,\phi_{n}(i)\,,$$

and since $\phi_{n}(i)=0$, for $i<n$, we see that

$$\Delta^{j}\left[f\phi_{n}\right](0)=0,\quad 0\leq j<n,$$

If $j\geq 0$, then

$$\begin{aligned}\Delta^{n+j}\left[f\phi_{n}\right](0)&=\sum_{i=n}^{n+j}\binom{n+j}{i}(-1)^{n+j-i}f(i)\,\phi_{n}(i)\\&=\sum_{i=0}^{j}\binom{n+j}{n+i}(-1)^{j-i}f(n+i)\,\phi_{n}(n+i)\,.\end{aligned}$$

Using (44), we have

$$\binom{n+j}{n+i}\phi_{n}(n+i)=\frac{(n+j)!}{j!}\binom{j}{i},$$

and therefore

$$\begin{aligned}\Delta^{n+j}\left[f\phi_{n}\right](0)&=\frac{(n+j)!}{j!}\sum_{i=0}^{j}\binom{j}{i}(-1)^{j-i}f(n+i)\\&=\frac{(n+j)!}{j!}\Delta^{j}\left[f\right](n)\,.\end{aligned}$$

■

Using (52), we obtain the following Corollary.

Corollary 14 *If $u(x)$ is a polynomial of degree k, then*

$$u(x)\,\phi_{n}(x)=\sum_{j=0}^{k}\frac{\Delta^{j}\left[u\right](n)}{j!}\phi_{n+j}(x)\,. \tag{53}$$

Proof. Using (49) and (51), we have

$$u(x)\phi_n(x) = \sum_{j=0}^{n+k} \frac{\Delta^j [u\phi_n](0)}{j!} \phi_j(x) = \sum_{j=n}^{n+k} \frac{\Delta^j [u\phi_n](0)}{j!} \phi_j(x)$$
$$= \sum_{j=0}^{k} \frac{\Delta^{n+j} [u\phi_n](0)}{(n+j)!} \phi_{n+j}(x),$$

and the results follows from (52). ■

From the previous Corollary, we obtain an explicit recurrence of order $s+1 = \max\{p, q+1\}$ for the modified moments $\nu_n(z)$.

Proposition 15 *Let $\nu_n(z)$ be defined by (47). Then,*

$$\sum_{j=0}^{q} \frac{\Delta^j [(x+\mathbf{b})](n+1)}{j!} \nu_{n+1+j} - z \sum_{j=0}^{p} \frac{\Delta^j [(x+\mathbf{a})](n)}{j!} \nu_{n+j} = 0. \tag{54}$$

Proof. Using (53), we have

$$(x+\mathbf{c})\phi_n(x) = \sum_{j=0}^{m} \frac{\Delta^j [(x+\mathbf{c})](n)}{j!} \phi_{n+j}(x), \quad \mathbf{c} \in \mathbb{K}^m,$$

and therefore we can write (48) as

$$\left[\sum_{j=0}^{q} \frac{\Delta^j [(x+\mathbf{b})](n+1)}{j!} \phi_{n+1+j}(x)\right]_{x=\vartheta} [\nu_0]$$
$$= z \left[\sum_{j=0}^{p} \frac{\Delta^j [(x+\mathbf{a})](n)}{j!} \phi_{n+j}(x)\right]_{x=\vartheta} [\nu_0].$$

Using (15), the result follows. ■

Finally, we will study the exponential generating function of the standard moments.

Proposition 16 *(i) The exponential generating function of the modified moments*

$$\epsilon_\nu(t; z) = \sum_{n=0}^{\infty} \nu_n(z) \frac{t^n}{n!}$$

is given by

$$\epsilon_{\nu}(t;z) = \nu_0(z+zt). \tag{55}$$

(ii) The function $\epsilon_{\nu}(t;z)$ is a solution of the linear ODE (in the t variable)

$$[\sigma((t+1)\partial_t) - z\tau((t+1)\partial_t)][y] = 0. \tag{56}$$

Proof. (i) Using (44) and the binomial theorem, we obtain the exponential generating function of the falling factorial polynomials

$$\sum_{n=0}^{\infty}\phi_n(x)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\binom{x}{n}t^n = (1+t)^x,$$

and using (18), we get (55).

(ii) Since

$$\begin{aligned}(1+t)\partial_t[y((1+t)z)] &= z(1+t)y'((1+t)z)\\ &= z\partial_z[y((1+t)z)] = \vartheta[y((1+t)z)],\end{aligned}$$

it follows from (19) that $\nu_0(z+zt)$ is a solution of (56). ■

Remark 17 *The differential equation (56) needs to be understood in an operational sense, since the coefficients are not constant. For instance, we have*

$$(1+t)\partial_t[(1+t)\partial_t] = (1+t)\left(\partial_t + (1+t)\partial_t^2\right) = (1+t)^2\partial_t^2 + (1+t)\partial_t,$$

and therefore

$$[(1+t)\partial_t + a_1][(1+t)\partial_t + a_2] = (1+t)^2\partial_t^2 + (1+a_1+a_2)(1+t)\partial_t + a_1a_2.$$

It is clear from (54) that the elements of the set $\{\nu_k : k \geq s+1\}$, are linear combinations of the first $s+1$ modified moments. Thus, we have a representation of the form

$$\nu_n(z) = \sum_{k=0}^{s} f_{n,k}(z)\,\nu_k(z), \tag{57}$$

where the coefficients must satisfy the initial conditions

$$f_{j,k}(z) = \delta_{j,k}, \quad 0 \leq j,k \leq s.$$

If we define

$$F_k(t,z) = \sum_{n=0}^{\infty} f_{n,k}(z) \frac{t^n}{n!}, \quad 0 \le k \le s,$$

where $f_{n,k}(z)$ are the coefficients in (57), we see that

$$\nu_0(z+zt) = \sum_{k=0}^{s} F_k(t,z)\, \nu_k(z),$$

and therefore the functions $F_k(t,z)$, $0 \le k \le s$ form a basis of solutions of the ODE (56) with initial conditions

$$[\partial_t^n F_k]_{t=0} = \delta_{n,k}, \quad 0 \le n,k \le s,$$

since from (55) we see that

$$[\partial_t^n \nu_0(z+zt)]_{t=0} = \nu_n(z).$$

In the next section, we will look at more general polynomial bases that contain the monomials and falling factorial as particular cases.

2.3 Two-term recurrence relations

Both the monomial polynomials and the falling factorial polynomials satisfy a *2-term recurrence relation* of the form

$$x\Lambda_n(x) = \Lambda_{n+1}(x) + \kappa_n \Lambda_n(x), \tag{58}$$

where for the monomials $\kappa_n = 0$ and for the falling factorial polynomials $\kappa_n = n$.

Theorem 18 *Let the* Stieltjes transform *of the functional L [71] be defined by*

$$S(\omega; z) = L\left[\frac{1}{\omega - x}\right], \tag{59}$$

where (as always) L is acting on the variable x. Suppose that $\{\Lambda_n(x)\}_{n\ge0}$ is a monic basis satisfying (58), and $\lambda_n(z) = L[\Lambda_n]$. Then, for all $n \in \mathbb{N}$

$$S(\omega; z) = \frac{1}{\Lambda_n(\omega)} L\left[\frac{\Lambda_n}{\omega - x}\right] + \sum_{k=0}^{n-1} \frac{\lambda_k(z)}{\Lambda_{k+1}(\omega)}.$$

Proof. From (58), we have

$$x\Lambda_n(x)\Lambda_n(\omega) = \Lambda_{n+1}(x)\Lambda_n(\omega) + \kappa_n\Lambda_n(x)\Lambda_n(\omega)$$
$$\omega\Lambda_n(x)\Lambda_n(\omega) = \Lambda_n(x)\Lambda_{n+1}(\omega) + \kappa_n\Lambda_n(x)\Lambda_n(\omega),$$

and therefore

$$(x-\omega)\Lambda_n(x)\Lambda_n(\omega) = \Lambda_{n+1}(x)\Lambda_n(\omega) - \Lambda_n(x)\Lambda_{n+1}(\omega).$$

Dividing by $\Lambda_n(\omega)\Lambda_{n+1}(\omega)$,

$$(x-\omega)\frac{\Lambda_n(x)}{\Lambda_{n+1}(\omega)} = \frac{\Lambda_{n+1}(x)}{\Lambda_{n+1}(\omega)} - \frac{\Lambda_n(x)}{\Lambda_n(\omega)},$$

and summing from 0 to $n-1$, we obtain

$$\begin{aligned}(x-\omega)\sum_{k=0}^{n-1}\frac{\Lambda_k(x)}{\Lambda_{k+1}(\omega)} &= \sum_{k=0}^{n-1}\left[\frac{\Lambda_{k+1}(x)}{\Lambda_{k+1}(\omega)} - \frac{\Lambda_k(x)}{\Lambda_k(\omega)}\right] \\ &= \frac{\Lambda_n(x)}{\Lambda_n(\omega)} - \frac{\Lambda_0(x)}{\Lambda_0(\omega)}.\end{aligned}$$

Hence,

$$\frac{1}{\Lambda_n(\omega)}\frac{\Lambda_n(x)}{x-\omega} = \frac{1}{x-\omega} + \sum_{k=0}^{n-1}\frac{\Lambda_k(x)}{\Lambda_{k+1}(\omega)}, \tag{60}$$

since $\Lambda_0(x) = 1$.

Applying L to (60), we see that

$$\frac{1}{\Lambda_n(\omega)}L\left[\frac{\Lambda_n}{x-\omega}\right] = L\left[\frac{1}{x-\omega}\right] + \sum_{k=0}^{n-1}\frac{\lambda_k(z)}{\Lambda_{k+1}(\omega)},$$

and the result follows. ■

Remark 19 *Since*

$$\lim_{n\to\infty}\frac{\Lambda_n(x)}{\Lambda_n(\omega)} = 1,$$

we have (at least formally)

$$S(\omega; z) = \sum_{k=0}^{\infty}\frac{\lambda_k(z)}{\Lambda_{k+1}(\omega)}.$$

The falling factorial case was already considered in [11].

Next, we relate an arbitrary monomial basis to the basis of monomials.

Proposition 20 *Suppose that $\{\Lambda_n\}_{n\geq 0}$ is a monic basis satisfying (58).*

(i) If

$$x^n = \sum_{i=0}^{n} \xi_{n,i} \Lambda_i (x), \tag{61}$$

then, the coefficients $\xi_{n,i}$ satisfy the recurrence

$$\xi_{n+1,i} = \xi_{n,i-1} + \kappa_i \xi_{n,i}, \quad \xi_{n,n} = 1,$$

with boundary conditions

$$\xi_{n,i} = 0, \quad i \notin [0, n].$$

(ii) If

$$\Lambda_n (x) = \sum_{i=0}^{n} \overline{\xi}_{n,i} x^i, \tag{62}$$

then, the coefficients $\overline{\xi}_{n,i}$ satisfy the recurrence

$$\overline{\xi}_{n+1,i} = \overline{\xi}_{n,i-1} - \kappa_n \overline{\xi}_{n,i},$$

with boundary conditions

$$\overline{\xi}_{n,n} = 1, \quad \overline{\xi}_{n,i} = 0, \quad i \notin [0, n].$$

Proof. (i) Since $\Lambda_n (x)$ is monic, we need $\xi_{n,n} = 1$. Using (58), we get

$$\sum_{i=0}^{n+1} \xi_{n+1,i} \Lambda_i (x) = x^{n+1} = \sum_{i=0}^{n} \xi_{n,i} x \Lambda_i (x)$$
$$= \sum_{i=0}^{n} \xi_{n,i} \left[\Lambda_{i+1} (x) + \kappa_i \Lambda_i (x)\right] = \sum_{i=1}^{n+1} \xi_{n,i-1} \Lambda_i (x) + \sum_{i=0}^{n} \xi_{n,i} \kappa_i \Lambda_i (x).$$

Comparing coefficients, we obtain the result.

(ii) In a similar way, we have

$$\sum_{i=0}^{n+1} \overline{\xi}_{n+1,i} x^i + \sum_{i=0}^{n} \kappa_n \overline{\xi}_{n,i} x^i = \Lambda_{n+1} (x) + \kappa_n \Lambda_n (x)$$
$$= x \Lambda_n (x) = \sum_{i=0}^{n} \overline{\xi}_{n,i} x^{i+1} = \sum_{i=1}^{n+1} \overline{\xi}_{n,i-1} x^i,$$

and the result follows. ■

Example 21 *If* $\Lambda_n(x) = \phi_n(x)$, *we get*

$$\xi_{n+1,i} = \xi_{n,i-1} + i\xi_{n,i}, \quad \xi_{n,n} = 1,$$
$$\overline{\xi}_{n+1,i} = \overline{\xi}_{n,i-1} - n\overline{\xi}_{n,i}, \quad \overline{\xi}_{n,n} = 1.$$

In this case, the coefficients $\xi_{n,i}$ *are known as* Stirling numbers of the second kind, *and the coefficients* $\overline{\xi}_{n,i}$ *are known as* Stirling numbers of the first kind [64].

Using Newton's interpolation formula (49), we have

$$x^n = \sum_{k=0}^{n} \frac{\Delta^k [x^n](0)}{k!} \phi_k(x),$$

and therefore the Stirling numbers of the second kind have the representation [59, 26.8.6]

$$\left\{ {n \atop k} \right\} = \frac{\Delta^k [x^n](0)}{k!} = \frac{1}{k!} \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} i^n.$$

Applying L to (61) and (62), we see that

$$\mu_n = \sum_{i=0}^{n} \xi_{n,i} \lambda_i, \quad \lambda_n = \sum_{i=0}^{n} \overline{\xi}_{n,i} \mu_i,$$

and in particular

$$\mu_n = \sum_{k=0}^{n} \left\{ {n \atop k} \right\} \nu_k. \tag{63}$$

3 Transformations of functionals

Let $m \in \mathbb{N}_0$, $\mathbf{c} \in \mathbb{K}^m$, and $\varepsilon \in \mathbb{K}$. If we define the the recurrence operator $\Theta_n(\mathbf{c};\varepsilon)$ by

$$\Theta_n(\mathbf{c};\varepsilon)[\psi] = (\psi + \mathbf{c})(\psi + \varepsilon)^n, \quad \psi \in \mathbb{K}^\infty, \tag{64}$$

we can write (32) as

$$[\Theta_{n+1}(\mathbf{b};0) - z\Theta_n(\mathbf{a};1)][\mu] = 0. \tag{65}$$

Similarly, let the recurrence operator $\Upsilon_n(\mathbf{c})$ be defined by

$$\Upsilon_n(\mathbf{c})[\psi] = \sum_{j=0}^{m} \frac{\Delta^j [(x+\mathbf{c})](n)}{j!} \psi_{n+j}, \quad \psi \in \mathbb{K}^{\infty}. \tag{66}$$

Then, using (54) and (66), we see that the modified moments $\nu_n(z)$ satisfy the recurrence

$$[\Upsilon_{n+1}(\mathbf{b}) - z\Upsilon_n(\mathbf{a})][\nu] = 0. \tag{67}$$

We have $\Upsilon_n(\emptyset)[\psi] = \psi_n$, and from (66), we get

$$\Upsilon_n(c)[\psi] = \psi_{n+1} + (n+c)\psi_n.$$

In general, we have the following result.

Proposition 22 *The recurrence operators Υ_n satisfy the basic recurrence*

$$\Upsilon_n(\mathbf{c}, \gamma) = \Upsilon_{n+1}(\mathbf{c}) + (n+\gamma)\Upsilon_n(\mathbf{c}). \tag{68}$$

Proof. From the definition of Υ_n, we have

$$\Upsilon_n(\mathbf{c}, \gamma)[\psi] = \sum_{j=0}^{m+1} \frac{\Delta^j [(x+\mathbf{c})(x+\gamma)](n)}{j!} \psi_{n+j}.$$

If we use Leibniz rule [37]

$$\Delta^j [uv](n) = \sum_{i=0}^{j} \binom{j}{i} \Delta^{j-i}[u](n+i)\Delta^i[v](n),$$

we get

$$\Delta^j [(x+\mathbf{c})(x+\gamma)](n) = (n+\gamma)\Delta^j[(x+\mathbf{c})](n) + j\Delta^{j-1}[(x+\mathbf{c})](n+1).$$

Since

$$\sum_{j=0}^{m+1} \frac{j\Delta^{j-1}[(x+\mathbf{c})](n+1)}{j!} \psi_{n+j}$$
$$= \sum_{j=1}^{m+1} \frac{\Delta^{j-1}[(x+\mathbf{c})](n+1)}{(j-1)!} \psi_{n+j} = \sum_{j=0}^{m} \frac{\Delta^j[(x+\mathbf{c})](n+1)}{j!} \psi_{n+j+1},$$

we conclude that

$$\sum_{j=0}^{m+1} \frac{\Delta^j \left[(x+\mathbf{c})(x+\gamma) \right](n)}{j!} \psi_{n+j}$$
$$= (n+\gamma) \sum_{j=0}^{m} \frac{\Delta^j \left[(x+\mathbf{c}) \right](n)}{j!} \psi_{n+j} + \sum_{j=0}^{m} \frac{\Delta^j \left[(x+\mathbf{c}) \right](n+1)}{j!} \psi_{n+1+j}$$

and the result follows. ■

If $m = 2$, (68) gives

$$\begin{aligned} \Upsilon_n(c_1, c_2)[\psi] &= \Upsilon_{n+1}(c_1)[\psi] + (n + c_2)\Upsilon_n(c_1)[\psi] \\ &= \psi_{n+2} + (n+1+c_1)\psi_{n+1} + (n+c_2)\left[\psi_{n+1} + (n+c_1)\psi_n\right], \end{aligned}$$

and hence

$$\Upsilon_n(c_1, c_2) = \mathcal{S}_n^2 + (2n + c_1 + c_2 + 1)\mathcal{S}_n + (n + c_1)(n + c_2).$$

Note that

$$\Upsilon_n(c_1, c_2) = (\mathcal{S}_n + n + c_1)(\mathcal{S}_n + n + c_2) = \Upsilon_n(c_1) \circ \Upsilon_n(c_2),$$

where clearly

$$\Upsilon_n(c_1) \circ \Upsilon_n(c_2) = \Upsilon_n(c_2) \circ \Upsilon_n(c_1).$$

Using induction, it follows that

$$\Upsilon_n(\mathbf{c}) = (\mathcal{S}_n + n + \mathbf{c}), \quad \mathbf{c} \in \mathbb{K}^m,$$

and

$$\Upsilon_n(\mathbf{c}) = \Upsilon_n(c_1) \circ \Upsilon_n(c_2) \circ \cdots \circ \Upsilon_n(c_m), \quad \mathbf{c} \in \mathbb{K}^m. \tag{69}$$

Remark 23 *We have*

$$\begin{aligned} &(a_1\mathcal{S}_n + b_1 n + c_1)(a_2\mathcal{S}_n + b_2 n + c_2) - (a_2\mathcal{S}_n + b_2 n + c_2)(a_1\mathcal{S}_n + b_1 n + c_1) \\ &= (a_1 b_2 - a_2 b_1)\mathcal{S}_n, \end{aligned}$$

so in general caution must be exercised when composing linear terms involving $\mathcal{S}_n$.

In the remaining of the paper, we will use the notation

$$\Phi_n = \Theta_{n+1}(\mathbf{b};0) - z\Theta_n(\mathbf{a};1), \tag{70}$$

and

$$\Psi_n = \Upsilon_{n+1}(\mathbf{b}) - z\Upsilon_n(\mathbf{a}), \tag{71}$$

which allow us to write the recurrences for the standard and modified moments as $\Phi_n[\mu] = 0$ and $\Psi_n[\nu] = 0$ respectively.

For $\widetilde{p}, \widetilde{q} \in \mathbb{N}_0$ and $\alpha \in \mathbb{K}^{\widetilde{p}},\ \beta \in \mathbb{K}^{\widetilde{q}}$,we define the *moment transformation* Ω_β^α by

$$\Omega_\beta^\alpha[\lambda_0] = C(\alpha,\beta)\ {}_{p+\widetilde{p}}F_{q+\widetilde{q}}\left(\begin{matrix}\mathbf{a},\alpha\\ \mathbf{b}+1,\beta\end{matrix};z\right), \tag{72}$$

where $C(\alpha,\beta)$ is a constant. Clearly, $\Omega_\beta^\alpha[\lambda_0]$ is a solution of the hypergeometric ODE

$$[\vartheta(\vartheta+\beta-1)(\vartheta+\mathbf{b}) - z(\vartheta+\alpha)(\vartheta+\mathbf{a})][y] = 0. \tag{73}$$

From (32) and (73), we see that the transformed standard moments $\Omega_\beta^\alpha[\mu]$ satisfy the recurrence

$$[\Theta_{n+1}(\mathbf{b},\beta-\mathbf{1};0) - z\Theta_n(\mathbf{a},\alpha;1)][\psi] = 0 \tag{74}$$

while (67) and (73) give a recurrence for the transformed modified moments $\Omega_\beta^\alpha[\nu]$

$$[\Upsilon_{n+1}(\mathbf{b},\beta-\mathbf{1}) - z\Upsilon_n(\mathbf{a},\alpha)][\psi] = 0. \tag{75}$$

Remark 24 *It may seem that the definition of Ω_β^α (72) is ambiguous, because the constant $C(\alpha,\beta)$ is not fixed. But since the recurrences (74) and (75) are homogeneous, they are not affected by a multiplicative constant.*

Comparing (74) with (70), we can define

$$\Omega_\beta^\alpha[\Phi_n] = \Theta_{n+1}(\mathbf{b},\beta-\mathbf{1};0) - z\Theta_n(\mathbf{a},\alpha;1), \tag{76}$$

in the sense that

$$\Omega_\beta^\alpha[\Phi_n]\left[\Omega_\beta^\alpha[\mu]\right] = 0.$$

Similarly, from (75) and (71) we conclude that the operator

$$\Omega_\beta^\alpha[\Psi_n] = \Upsilon_{n+1}(\mathbf{b},\beta-\mathbf{1}) - z\Upsilon_n(\mathbf{a},\alpha) \tag{77}$$

satisfies

$$\Omega_\beta^\alpha[\Psi_n]\left[\Omega_\beta^\alpha[\nu]\right] = 0.$$

Proposition 25 *Let $\mathcal{S}_n$ be the shift operator defined in (12). If $c \in \mathbb{K}$, we have:*

(i)
$$\Omega_c^c [\Phi_n] = (\mathcal{S}_n + c - 1) \circ \Phi_n. \tag{78}$$

(ii)
$$\Omega_c^{c+1} [\Phi_n] \circ (\mathcal{S}_n + c) = (\mathcal{S}_n + c - 1) (\mathcal{S}_n + c) \circ \Phi_n. \tag{79}$$

(iii)
$$\Omega_{c+1}^c [\Phi_n] = \Phi_n \circ (\mathcal{S}_n + c) . \tag{80}$$

(iv)
$$\Omega_c^c [\Psi_n] = (\mathcal{S}_n + n + c) \circ \Psi_n. \tag{81}$$

(v)
$$\Omega_c^{c+1} [\Psi_n] \circ (\mathcal{S}_n + n + c) = (\mathcal{S}_n + n + c + 1) (\mathcal{S}_n + n + c) \circ \Psi_n. \tag{82}$$

(vi)
$$\Omega_{c+1}^c [\Psi_n] = \Psi_n \circ (\mathcal{S}_n + n + c) . \tag{83}$$

Proof. (i) If we consider the composition $(\mathcal{S}_n + c) \circ \Phi_n$, we see that

$$\begin{gathered}(\mathcal{S}_n + c) \circ \Phi_n [\psi] = (\mathcal{S}_n + c) \left[(\psi + \mathbf{b}) \psi^{n+1} - z (\psi + \mathbf{a}) (\psi + 1)^n\right] \\ = (\psi + \mathbf{b}) \psi^{n+2} - z (\psi + \mathbf{a}) (\psi + 1)^{n+1} + c (\psi + \mathbf{b}) \psi^{n+1} - zc (\psi + \mathbf{a}) (\psi + 1)^n \\ = (\psi + c) (\psi + \mathbf{b}) \psi^{n+1} - z (\psi + c + 1) (\psi + \mathbf{a}) (\psi + 1)^n ,\end{gathered}$$

and comparing with (76) we obtain

$$(\mathcal{S}_n + c) \circ \Phi_n [\psi] = \left(\Omega_{c+1}^{c+1} [\Phi_n]\right) [\psi] .$$

The result follows after shifting c.

(ii) Using (78), we have

$$\begin{aligned}&(\mathcal{S}_n + c - 1) (\mathcal{S}_n + c) \circ \Phi_n [\psi] = \left(\Omega_{c,c+1}^{c,c+1} [\Phi_n]\right) [\psi] \\ &= (\psi + c - 1) (\psi + c) (\psi + \mathbf{b}) \psi^{n+1} - z (\psi + c) (\psi + c + 1) (\psi + \mathbf{a}) (\psi + 1)^n \\ &= \left[(\psi + c - 1) (\psi + \mathbf{b}) \psi^{n+1} - z (\psi + c + 1) (\psi + \mathbf{a}) (\psi + 1)^n\right] (\psi + c) ,\end{aligned}$$

and we obtain (79).

(iii) Note that

$$\Omega_{c+1}^{c}\left[\Phi_{n}\left[\psi\right]\right] = \left(\psi+c\right)\left(\psi+\mathbf{b}\right)\psi^{n+1} - z\left(\psi+c\right)\left(\psi+\mathbf{a}\right)\left(\psi+1\right)^{n}$$
$$= \left[\left(\psi+\mathbf{b}\right)\psi^{n+1} - z\left(\psi+\mathbf{a}\right)\left(\psi+1\right)^{n}\right]\left(\psi+c\right),$$

and therefore (80) is true.

(iv) Similarly, we have

$$\left(\mathcal{S}_{n}+n+c\right)\circ\Psi_{n} = \Upsilon_{n+2}\left(\mathbf{b}\right) - z\Upsilon_{n+1}\left(\mathbf{a}\right) + \left(n+c\right)\Upsilon_{n+1}\left(\mathbf{b}\right) - z\left(n+c\right)\Upsilon_{n}\left(\mathbf{a}\right)$$
$$= \Upsilon_{n+1}\left(\mathbf{b},c-1\right) - z\Upsilon_{n}\left(\mathbf{a},c\right),$$

and comparing with (77) we obtain (81).

(v) Using (69), we get

$$\left(\mathcal{S}_{n}+n+c+1\right)\left(\mathcal{S}_{n}+n+c\right)\circ\Psi_{n} = \Omega_{c,c+1}^{c,c+1}\left[\Psi_{n}\right]$$
$$= \Upsilon_{n+1}\left(\mathbf{b},c-1,c\right) - z\Upsilon_{n}\left(\mathbf{a},c,c+1\right) = \left[\Upsilon_{n+1}\left(\mathbf{b},c-1\right) - z\Upsilon_{n}\left(\mathbf{a},c+1\right)\right]\circ\Upsilon_{n}\left(\mathbf{a},c\right),$$

and (82) follows.

(vi) Finally,

$$\Omega_{c+1}^{c}\left[\Psi_{n}\right] = \Upsilon_{n+1}\left(\mathbf{b},c\right) - z\Upsilon_{n}\left(\mathbf{a},c\right) = \left[\Upsilon_{n+1}\left(\mathbf{b}\right) - z\Upsilon_{n}\left(\mathbf{a}\right)\right]\circ\Upsilon_{n}\left(c\right),$$

and we see that (83) is true. ∎

It follows that the special cases $\alpha = \beta$ and $\alpha = \beta \pm 1$ lead to some interesting transformations. We will study them in detail in the next sections.

3.1 The Christoffel transformation

The *Christoffel transformation* is defined by

$$\lambda_{0}^{C} = \Omega_{-\omega}^{-\omega+1}\left[\lambda_{0}\right].$$

From (73), we see that $\lambda_{0}^{C}\left(z;\omega\right)$ is a solution of the ODE

$$\left[\left(\vartheta-\omega-1\right)\vartheta\left(\vartheta+\mathbf{b}\right) - z\left(\vartheta-\omega+1\right)\left(\vartheta+\mathbf{a}\right)\right]\left[y\right] = 0, \qquad (84)$$

and admits the hypergeometric representation

$$\lambda_{0}^{C}\left(z;\omega\right) = -\omega\ {}_{p+1}F_{q+1}\left(\begin{matrix}\mathbf{a},-\omega+1\\ \mathbf{b}+1,-\omega\end{matrix};z\right). \qquad (85)$$

The reason for choosing this particular solution is the identity

$$-\omega\frac{(-\omega+1)_x}{(-\omega)_x}=x-\omega, \tag{86}$$

which shows that the linear functional L^C associated to λ_0^C is given by

$$L^C[u]=L[(x-\omega)u], \quad u\in\mathbb{K}[x]. \tag{87}$$

This transformation was introduced by Elwin Bruno Christoffel (1829–1900) in his pioneering work [16].

Clearly we must have

$$\lambda_0^C= \ L^C[x-\omega]=(\vartheta-\omega)[\lambda_0]\neq 0,$$

and since the operator $\vartheta-\omega$ annihilates any multiple of z^ω, we need

$$\lambda_0(z;\omega)\neq \eta z^\omega, \quad \eta\in\mathbb{K}.$$

From (58) and (87), we get

$$\lambda_n^C= \ L^C[\Lambda_n]=L[(x-\omega)\Lambda_n]=\lambda_{n+1}+(\kappa_n-\omega)\lambda_n,$$

and in particular

$$\mu_n^C=\mu_{n+1}^C-\omega\mu_n^C, \tag{88}$$

and

$$\nu_n^C=\nu_{n+1}^C+(n-\omega)\nu_n^C. \tag{89}$$

Note that,

$$\lambda_0^C=\mu_1^C-\omega\mu_0^C=\nu_1^C-\omega\nu_0^C. \tag{90}$$

From (76), we see that the standard moments μ_n^C satisfy the recurrence $\Phi_n^C\left[\mu_n^C\right]=0$, where

$$\Phi_n^C[\mu]=(\mu-\omega-1)(\mu+\mathbf{b})\mu^{n+1}-z(\mu-\omega+1)(\mu+\mathbf{a})(\mu+1)^n,$$

and from (77), we see that the modified moments ν_n^C satisfy the recurrence $\Psi_n^C\left[\nu_n^C\right]=0$, where

$$\Psi_n^C=\Upsilon_{n+2}(\mathbf{b})+(n-\omega)\Upsilon_{n+1}(\mathbf{b})-z\Upsilon_{n+1}(\mathbf{a})-z(n-\omega+1)\Upsilon_n(\mathbf{a}). \tag{91}$$

Remark 26 *Using (79), we obtain*

$$\left(\mathcal{S}_n-\omega-1\right)\left(\mathcal{S}_n-\omega\right)\circ\Phi_n^C=\ \Phi_n^C\circ\left(\mathcal{S}_n-\omega\right),$$

and therefore

$$\begin{gathered}\Phi_n^C\left[\mu_{n+1}^C-\omega\mu_n^C\right]=\ \Phi_n^C\circ\left(\mathcal{S}_n-\omega\right)[\mu]\\ =\left(\mathcal{S}_n-\omega-1\right)\left(\mathcal{S}_n-\omega\right)\circ\Phi_n[\mu]=0=\Phi_n^C[\mu],\end{gathered}$$

in agreement with (88).

Similarly, using (82), we see that

$$\left(\mathcal{S}_n+n-\omega+1\right)\left(\mathcal{S}_n+n-\omega\right)\circ\Psi_n=\Psi_n^C\circ\left(\mathcal{S}_n+n-\omega\right),$$

and hence

$$\begin{aligned}&\Psi_n^C\left[\nu_{n+1}+(n-\omega)\,\nu_n\right]=\Psi_n^C\circ\left(\mathcal{S}_n+n-\omega\right)[\nu]\\ &=\left(\mathcal{S}_n+n-\omega+1\right)\left(\mathcal{S}_n+n-\omega\right)\circ\Psi_n[\nu]=0=\Psi_n^C\left[\nu^C\right],\end{aligned}$$

in agreement with (91).

Using (41) and (88), we obtain the exponential generating function of the transformed standard moments

$$\sum_{n=0}^{\infty}\mu_n^C(z;\omega)\,\frac{t^n}{n!}=\mu_0^C\left(ze^t;\omega\right)=\left(\mu_1-\omega\mu_0\right)\left(ze^t\right),$$

while from (55) and (89) we get the exponential generating function of the transformed modified moments

$$\sum_{n=0}^{\infty}\nu_n^C(z;\omega)\,\frac{t^n}{n!}=\nu_0^C(z+zt;\omega)=\left(\nu_1-\omega\nu_0\right)(z+zt).$$

3.2 The Geronimus transformation

The *Geronimus transformation* is defined by

$$\lambda_0^G=\Omega_{-\omega+1}^{-\omega}\left[\lambda_0\right]\ ,\quad \omega\notin\mathbb{N}_0.$$

From (73), we see that $\lambda_0^G(z;\omega)$ is a solution of the ODE

$$\vartheta\left(\vartheta+\mathbf{b}\right)\left(\vartheta-\omega\right)[y]=z\left(\vartheta+\mathbf{a}\right)\left(\vartheta-\omega\right)[y],\tag{92}$$

and admits the hypergeometric representation

$$\lambda_0^G(z;\omega)=-\omega^{-1}\ {}_{p+1}F_{q+1}\left(\begin{matrix}\mathbf{a},-\omega\\ \mathbf{b}+1,-\omega+1\end{matrix};z\right).\tag{93}$$

Remark 27 *The function z^{ω} is also a solution of (92), and therefore we could define (as some authors do)*

$$\lambda_0^G(z;\omega) = -\omega^{-1}\ {}_{p+1}F_{q+1}\left(\begin{array}{c}\mathbf{a}, -\omega \\ \mathbf{b}+1, -\omega+1\end{array}; z\right) + \eta z^{\omega}$$

where η is an arbitrary constant.

The identity (86) shows that the linear functional L^G associated to λ_0^G is given by

$$L^G[u] = L\left[\frac{u(x)}{x-\omega}\right], \quad u \in \mathbb{K}[x], \tag{94}$$

and

$$\lambda_0^G(z;\omega) = L\left[\frac{1}{x-\omega}\right](z) = -S(\omega;z), \tag{95}$$

where $S(\omega;z)$ is the Stieltjes transform of the functional L defined in (59). Since

$$(\vartheta-\omega)\left[\lambda_0^G\right] = L\left[(x-\omega)\frac{1}{x-\omega}\right] = L[1] = \lambda_0,$$

we need

$$S(\omega;z) \neq \eta z^{\omega}, \quad \eta \in \mathbb{K}.$$

This transformation was introduced by Yakov Lazarevich Geronimus (1898–1984) in his groundbreaking article [32].

Proposition 28 *The moments of the linear transformation L^G defined by (94) have the integral representation*

$$\lambda_n^G(z;\omega) = \int_0^1 t^{-\omega-1}\lambda_n(zt)\,dt, \quad n \in \mathbb{N}_0. \tag{96}$$

Proof. If we use the integral representation [59, 16.5.2]

$${}_{p+1}F_{q+1}\left(\begin{array}{c}\mathbf{a}, \alpha \\ \mathbf{b}, \beta\end{array}; z\right)$$
$$= \frac{\Gamma(\beta)}{\Gamma(\alpha)\Gamma(\beta-\alpha)}\int_0^1 t^{\alpha-1}(1-t)^{\beta-\alpha-1}\ {}_pF_q\left(\begin{array}{c}\mathbf{a} \\ \mathbf{b}\end{array}; zt\right)dt,$$

in (93), we obtain

$$\lambda_0^G(z;\omega) = \int_0^1 t^{-\omega-1}\lambda_0(zt)\,dt. \tag{97}$$

Extending (97), we obtain (96). ■

Remark 29 *Note that if we use (13) in (96) and formally integrate term by term, we get*

$$\begin{aligned}\lambda_n^G(z;\omega) &= \sum_{x=0}^{\infty}\Lambda_n(x)\frac{(\mathbf{a})_x}{(\mathbf{b}+\mathbf{1})_x}\frac{z^x}{x!}\int_0^1 t^{x-\omega-1}dt \\ &= \sum_{x=0}^{\infty}\frac{\Lambda_n(x)}{x-\omega}\frac{(\mathbf{a})_x}{(\mathbf{b}+\mathbf{1})_x}\frac{z^x}{x!},\end{aligned}$$

in agreement with (94).

From (58) and (94), we see that

$$\lambda_{n+1}^G + (\kappa_n - \omega)\lambda_n^G = L^G[(x-\omega)\Lambda_n(x)] = L[\Lambda_n(x)] = \lambda_n,$$

and in particular

$$\mu_{n+1}^G - \omega\mu_n^G = \mu_n, \tag{98}$$

and

$$\nu_{n+1}^G + (n-\omega)\nu_n^G = \nu_n. \tag{99}$$

Using (60), we get

$$\lambda_n^G(z;\omega) = \Lambda_n(\omega)\left[\lambda_0^G(z;\omega) + \sum_{k=0}^{n-1}\frac{\lambda_k(z)}{\Lambda_{k+1}(\omega)}\right],$$

where care needs to be exercised if $\Lambda_k(\omega) = 0$ for some k.

Remark 30 *From (80), we have*

$$\Phi_n^G\ [\mu] = \Phi_n \circ (\mathcal{S}_n - \omega)[\mu]\ , \tag{100}$$

in agreement with (98), since

$$\Phi_n\left[\mu_{n+1}^G - \omega\mu_n^G\right] = \Phi_n \circ (\mathcal{S}_n - \omega)\left[\mu^G\right] = \Phi_n^G\left[\mu^G\right] = 0 = \Phi_n[\mu].$$

From (83), we get $\Psi_n^G \left[\nu^G\right] = 0$*, where*

$$\Psi_n^G = \Psi_n \circ (\mathcal{S}_n + n - \omega), \tag{101}$$

in agreement with (99), since

$$\Psi_n \left[\nu_{n+1}^G + (n-\omega)\,\nu_n^G\right] = \Psi_n \circ (\mathcal{S}_n + n - \omega) \left[\nu^G\right] = \Psi_n^G \left[\nu^G\right] = 0 = \Psi_n [\nu].$$

Using (41) and (95), we obtain the exponential generating function of μ_n^G

$$\sum_{n=0}^{\infty} \mu_n^G (z;\omega) \frac{t^n}{n!} = \lambda_0^G \left(ze^t;\omega\right) = -S\left(\omega; ze^t\right),$$

and for the transformed modified moments ν_n^G we get

$$\sum_{n=0}^{\infty} \nu_n^G (z;\omega) \frac{t^n}{n!} = \lambda_0^G (z+zt;\omega) = -S(\omega; z+zt).$$

3.3 The Uvarov transformation

Let's consider the composite transformations (Christoffel-Geronimus)

$$\left(\Omega_{1-\omega}^{-\omega} \circ \Omega_{-\omega}^{1-\omega}\right)[\lambda_0],$$

and (Geronimus-Christoffel)

$$\left(\Omega_{-\omega}^{1-\omega} \circ \Omega_{1-\omega}^{-\omega}\right)[\lambda_0].$$

We see that in either case, the transformed first moment is a solution of the ODE

$$(\vartheta-\omega)(\vartheta-\omega-1)\,\vartheta\,(\vartheta+\mathbf{b})[y] = z\,(\vartheta-\omega)(\vartheta-\omega+1)(\vartheta+\mathbf{a})[y], \tag{102}$$

which can be written as

$$(\vartheta-\omega)(\vartheta-\omega-1)\left[\sigma(\vartheta) - z\tau(\vartheta)\right][y] = 0. \tag{103}$$

Lemma 31 *The linear combination*

$$\lambda_0^U (z;\omega) = \lambda_0(z) + \eta z^\omega, \quad \eta \in \mathbb{K}, \tag{104}$$

is a solution of (103).

Proof. Clearly, λ_0 is a solution of (103). If we set $y(z) = z^{\omega}$, we have

$$[\sigma(\vartheta) - z\tau(\vartheta)][z^{\omega}] = \sigma(\omega) z^{\omega} - \tau(\omega) z^{\omega+1}, \tag{105}$$

and therefore

$$\begin{aligned}&(\vartheta-\omega)(\vartheta-\omega-1)[\sigma(\vartheta) - z\tau(\vartheta)][z^{\omega}]\\ &= (\vartheta-\omega)(\vartheta-\omega-1)\left[\sigma(\omega) z^{\omega} - \tau(\omega) z^{\omega+1}\right] = 0.\end{aligned}$$

Thus, (104) is a solution of (103). ■

We define the *Uvarov transformation* by

$$L^{U}[u] = L[u] + \eta u(\omega) z^{\omega}, \quad u \in \mathbb{K}[x],$$

which is well defined as long as

$$\lambda_0(z) \neq -\eta z^{\omega}.$$

This transformation was introduced by Vasiliĭ Borisovich Uvarov (1929–1997) in his monumental paper [73].

From (78), we see that

$$\Phi_n^{U} = (\mathcal{S}_n - \omega - 1)(\mathcal{S}_n - \omega) \circ \Phi_n, \tag{106}$$

and from (81), we have

$$\Psi_n^{U} = (\mathcal{S}_n + n - \omega + 1)(\mathcal{S}_n + n - \omega) \circ \Psi_n. \tag{107}$$

If $\sigma(\omega) = 0$ or $\tau(\omega) = 0$, we obtain some reduced cases.

Proposition 32 *Suppose that $\sigma(\omega) = 0$. Then,*

(i) The transformed moment λ_0^{U} satisfies the reduced ODE

$$(\vartheta - \omega - 1)[\sigma(\vartheta) - z\tau(\vartheta)]\left[\lambda_0^{U}\right] = 0. \tag{108}$$

(ii) The transformed first moment λ_0^{U} is given by

$$\lambda_0^{U} = \Omega_{-\omega}^{-\omega}[\lambda_0]. \tag{109}$$

(iii) The transformed standard moments μ_n^{U} satisfy the reduced recurrence $\Phi_n^{U}[\psi] = 0$, *where*

$$\Phi_n^{U} = (\mathcal{S}_n - \omega - 1) \circ \Phi_n. \tag{110}$$

(iv) The transformed modified moments ν_n^{U} satisfy the reduced recurrence $\Psi_n^{U}[\psi] = 0$, *where*

$$\Psi_n^{U} = (\mathcal{S}_n + n - \omega) \circ \Psi_n. \tag{111}$$

Proof. (i) If $\sigma(\omega) = 0$, then we see from (105) that

$$[\sigma(\vartheta) - z\tau(\vartheta)][z^{\omega}] = -\tau(\omega) z^{\omega+1}$$

and (108) follows.

(ii) Comparing (108) with (73), we can interpret λ_0^U as (109).

(iii) From (78) and (109), we get (110).

(iv) Using (81) in (109) gives (111). ■

Proposition 33 *Suppose that $\tau(\omega) = 0$. Then,*

(i) The transformed first moment λ_0^U satisfies the reduced ODE

$$(\vartheta - \omega)[\sigma(\vartheta) - z\tau(\vartheta)]\left[\lambda_0^U\right] = 0. \tag{112}$$

(ii) The transformed first moment λ_0^U is given by

$$\lambda_0^U = \Omega_{1-\omega}^{1-\omega}[\lambda_0]. \tag{113}$$

(iii) The transformed standard moments μ_n^U satisfy the reduced recurrence $\Phi_n^U[\psi] = 0$, *where*

$$\Phi_n^U = (\mathcal{S}_n - \omega) \circ \Phi_n. \tag{114}$$

(iv) The transformed modified moments ν_n^U satisfy the reduced recurrence $\Psi_n^U[\psi] = 0$, *where*

$$\Psi_n^U = (\mathcal{S}_n + n - \omega + 1) \circ \Psi_n. \tag{115}$$

Proof. (i) If $\tau(\omega) = 0$, then we see from (105) that

$$[\sigma(\vartheta) - z\tau(\vartheta)][z^{\omega}] = \sigma(\omega) z^{\omega},$$

and (112) follows.

(ii) Comparing (112) with (73), we can interpret λ_0^U as (113).

(iii) From (78) and (113), we get (110).

(iv) Using (81) in (113) gives (111). ■

Finally, we have

$$\lambda_n^U = L^U[\Lambda_n] = \lambda_n + \eta\Lambda_n(\omega) z^{\omega}, \tag{116}$$

from which we obtain the exponential generating functions of $\mu_n^U(z;\omega)$

$$\sum_{n=0}^{\infty} \mu_n^U(z;\omega)\frac{t^n}{n!} = \mu_0\left(ze^t\right) + \eta\left(ze^t\right)^{\omega},$$

and $\nu_n^U(z;\omega)$

$$\sum_{n=0}^{\infty} \nu_n^U(z;\omega)\frac{t^n}{n!} = \nu_0(z + zt) + \eta(z + zt)^{\omega}.$$

3.4 Truncated linear functionals

Let $N \in \mathbb{N}_0$ and the *truncated functional* L^T be defined by

$$L^T[u] = \sum_{x=0}^{N} u(x) \frac{(\mathbf{a})_x}{(\mathbf{b}+1)_x} \frac{z^x}{x!}, \quad u \in \mathbb{K}[x], \tag{117}$$

as long as

$$\lambda_0^T(z) = \sum_{x=0}^{N} \frac{(\mathbf{a})_x}{(\mathbf{b}+1)_x} \frac{z^x}{x!} \neq 0.$$

Remark 34 *If $\tau(N) = 0$, then the functional (13) is already a truncated functional, since*

$$(-N)_x = 0, \quad x > N.$$

Therefore, we assume that $\tau(N) \neq 0$.

Lemma 35 *The first transformed moment $\lambda_0^T(z)$ satisfies the ODE*

$$(\vartheta - N - 1)[\sigma(\vartheta) - z\tau(\vartheta)][y] = 0. \tag{118}$$

Proof. Using the Pearson equation (21), we have

$$[\sigma(\vartheta) - z\tau(\vartheta)]\left[\lambda_0^T\right] = \sum_{x=0}^{N} \left[\sigma(x)\rho(x)z^x - \tau(x)\rho(x)z^{x+1}\right]$$

$$= \sum_{x=0}^{N} \sigma(x)\rho(x)z^x - \sum_{x=1}^{N+1} \tau(x-1)\rho(x-1)z^x = -\tau(N)\rho(N)\frac{z^{N+1}}{N!},$$

and since the operator $\vartheta - N - 1$ annihilates any multiple of z^{N+1}, the result follows. ■

Using (11) in (118), we obtain

$$(\vartheta - N - 1)\sigma(\vartheta)\left[\lambda_0^T\right] = z(\vartheta - N)\tau(\vartheta)\left[\lambda_0^T\right],$$

and therefore we have

$$\lambda_0^T = \Omega_{-N}^{-N}[\lambda_0], \quad N \in \mathbb{N}_0. \tag{119}$$

Proposition 36 *The first transformed moment $\lambda_0^T(z)$ can be represented as a Laplace transform*

$$\lambda_0^T(z) = \frac{z^{N+1}}{N!} \frac{(\mathbf{a})_N}{(\mathbf{b}+1)_N} \int_0^\infty {}_{q+1}F_p \left(\begin{array}{c} -N, -\mathbf{b}-N \\ 1-\mathbf{a}-N \end{array} ; (-1)^{q+p+1} t \right) e^{-zt} dt. \tag{120}$$

Proof. If we use the formula [59, 16.2.4]

$$\sum_{k=0}^{N} \frac{(\mathbf{a})_k}{(\mathbf{b})_k} \frac{z^k}{k!} = \frac{z^N}{N!} \frac{(\mathbf{a})_N}{(\mathbf{b})_N} {}_{q+2}F_p \left(\begin{array}{c} -N, 1-\mathbf{b}-N, 1 \\ 1-\mathbf{a}-N \end{array} ; \frac{(-1)^{q+p+1}}{z} \right), \tag{121}$$

we obtain the hypergeometric representation

$$\lambda_0^T = \frac{z^N}{N!} \frac{(\mathbf{a})_N}{(\mathbf{b}+1)_N} {}_{q+2}F_p \left(\begin{array}{c} -N, -\mathbf{b}-N, 1 \\ 1-\mathbf{a}-N \end{array} ; \frac{(-1)^{q+p+1}}{z} \right). \tag{122}$$

Using the integral representation [59, 16.5.3]

$${}_{p+1}F_q \left(\begin{array}{c} \mathbf{a}, \alpha \\ \mathbf{b} \end{array} ; \frac{x}{z} \right) = \frac{z^\alpha}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} {}_pF_q \left(\begin{array}{c} \mathbf{a} \\ \mathbf{b} \end{array} ; xt \right) e^{-zt} dt \tag{123}$$

with $\alpha = 1$, we obtain (120). ∎

From (78) and (119), we get

$$\Phi_n^T = (\mathcal{S}_n - N - 1) \circ \Phi_n, \tag{124}$$

while (81) gives

$$\Psi_n^T = (\mathcal{S}_n + n - N) \circ \Psi_n. \tag{125}$$

Proposition 37 *The transformed modified moments $\nu_n^T(z)$ have the integral representation*

$$\nu_n^T(z) = \frac{(\mathbf{a})_N}{(\mathbf{b}+1)_N} \frac{z^{N+1}}{(N-n)!} \int_0^\infty {}_{q+1}F_p \left(\begin{array}{c} n-N, -\mathbf{b}-N \\ 1-\mathbf{a}-N \end{array} ; (-1)^{q+p+1} t \right) e^{-zt} dt. \tag{126}$$

Proof. Note that since

$$\sum_{x=0}^{N} \phi_n(x) \frac{(\mathbf{a})_x}{(\mathbf{b}+1)_x} \frac{z^x}{x!} = \sum_{x=n}^{N} \frac{(\mathbf{a})_x}{(\mathbf{b}+1)_x} \frac{z^x}{(x-n)!} = \sum_{x=0}^{N-n} \frac{(\mathbf{a})_{x+n}}{(\mathbf{b}+1)_{x+n}} \frac{z^{x+n}}{x!},$$

we have

$$\nu_n^T(z) = z^n \frac{(\mathbf{a})_n}{(\mathbf{b}+1)_n} \sum_{x=0}^{N-n} \frac{(\mathbf{a}+n)_x}{(\mathbf{b}+1+n)_x} \frac{z^x}{x!}. \tag{127}$$

Thus, we can use (121) and obtain

$$\nu_n^T(z) = \frac{(\mathbf{a})_N}{(\mathbf{b}+1)_N} \frac{z^N}{(N-n)!}\ {}_{q+2}F_p\left(\begin{matrix} n-N, -\mathbf{b}-N, 1 \\ 1-\mathbf{a}-N \end{matrix} ; \frac{(-1)^{q+p+1}}{z} \right). \tag{128}$$

In particular,

$$\nu_N^T(z) = \frac{(\mathbf{a})_N}{(\mathbf{b}+\mathbf{1})_N} z^N, \quad \nu_n^T(z) = 0, \quad n > N.$$

■

Remark 38 *Using (123) and (128), we get the integral representation (126).*

4 Conclusion

We have studied the linear functionals characterized by the hypergeometric differential equation satisfied by the first moment $\lambda_0(z)$

$$[\vartheta q(\vartheta) - zp(\vartheta)][\lambda_0] = 0, \quad p, q \in \mathbb{K}[x].$$

We obtained recurrence relations for the moments on the monomial and falling factorial polynomial bases.

We note that one could use the generating function (41) and the ODE it satisfies (42), as a different way of analyzing the standard moments $\mu_n(z)$. Similarly, one could study the modified moments $\nu_n(z)$ using (55) and (56).

We are currently working on further applications of our results to study some properties of the orthogonal polynomials themselves (representations, recurrence-relation coefficients, generating functions, etc).

5 Acknowledgements

This work supported by the strategic program "Innovatives OÖ– 2010 plus" from the Upper Austrian Government. We wish to thank Professor Veronika Pillwein for her generous sponsorship and our colleagues at JKU for their continuous help.

We thank the Isaac Newton Institute for Mathematical Sciences, Cambridge, for support and hospitality during the programme "Applicable resurgent asymptotics: towards a universal theory" (ARA2), where this paper was completed. The work was supported by EPSRC grant no EP/R014604/1.

We also wish to express our gratitude to the anonymous referees, who provided us with invaluable suggestions and comments that greatly improved our first draft of the paper.

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