Hosking integral in nonhelical Hall cascade

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(28 November 2022)

The Hosking integral, which characterizes magnetic helicity fluctuations in subvolumes, is known to govern the decay of magnetically dominated turbulence. Here we show that, when the evolution of the magnetic field is controlled by the motion of electrons only, as in neutron star crusts, the decay of the magnetic field is still controlled by the Hosking integral, but it has now effectively different dimensions than in ordinary magnetohydrodynamic (MHD) turbulence. This causes the correlation length to increase with time t like $t^{4/13}$ instead of $t^{4/9}$ in MHD. The magnetic energy decreases like $t^{-10/13}$, which is slower than in MHD, where it decays like $t^{-10/9}$. This agrees with earlier numerical results for the nonhelical Hall cascade.

1. Introduction

The x-ray emission from neutron stars during the first hundreds of years is believed to be powered by magnetic dissipation within their outer crusts. Since the ions are immobile in neutron star crusts, electric currents are transported by electrons alone. Their velocity is $\boldsymbol{u} = -\boldsymbol{J}/en_e$, where $\boldsymbol{J} = \boldsymbol{\nabla} \times \boldsymbol{B}/\mu_0$ is the current density, $\boldsymbol{B}$ the magnetic field, e is the elementary charge, n_e is the electron density, and μ_0 is the permeability. The evolution of $\boldsymbol{B}$ is then governed by the induction equation where the electromotive force $\boldsymbol{u} \times \boldsymbol{B}$ is given by $-\boldsymbol{J} \times \boldsymbol{B}/en_e$. The induction equation therefore takes the form (Goldreich & Reisenegger 1992)

$$\frac{\partial \boldsymbol{B}}{\partial t} = \boldsymbol{\nabla} \times \left(-\frac{1}{en_e} \boldsymbol{J} \times \boldsymbol{B} - \eta\mu_0 \boldsymbol{J} \right), \tag{1.1}$$

where η is the magnetic diffusivity. The nonlinearity in this equation leads to a cascade toward smaller scales—similar to the turbulent cascade in hydrodynamics turbulence (Goldreich & Reisenegger 1992). This model is therefore referred to what is called the Hall cascade. There has been extensive work trying to quantify the amount of dissipation that occurs (Gourgouliatos *et al.* 2016, 2020; Gourgouliatos & Hollerbach 2018; Igoshev *et al.* 2021; Anzuini *et al.* 2022). Idealized simulations in Cartesian geometry resulted in power law scaling for the resistive Joule dissipation (Brandenburg 2020, hereafter B20). It depends on the typical length scale of the turbulence, the electron density, the magnetic field strength, and possibly the magnetic helicity. Denoting volume averages by angle brackets, the decay of the magnetic energy $\mathcal{E} = \langle \boldsymbol{B}^2 \rangle / 2\mu_0$ with time t tends to follow power law behavior, $\mathcal{E} \propto t^{-p}$, where the exponent p is smaller than in magnetohydrodynamic (MHD) turbulence. In the helical case, it was found that $p = 2/5$, while for the nonhelical case, B20 reported $p \approx 0.9$. The correlation length of the turbulence, ξ, increases with time like $\xi \propto t^q$, where $q = 2/5$ in the helical case, i.e., $q = p$, and $q \approx 0.3$ in the nonhelical case. In the helical case, the exponent $2/5$ was possible to explain on dimensional grounds by noting that the magnetic field does not correspond to a speed (the Alfvén speed, as in MHD) with dimensions $\mathrm{m\,s^{-1}}$ in SI units, but to a diffusivity with dimensions $\mathrm{m^2\,s^{-1}}$.

The decay properties of the nonhelical Hall cascade were not yet theoretically understood at the time. In the last one to two years, however, significant progress has been made in describing the decay of magnetically dominated turbulence, where a new conserved quantity has been identified, which is now called the Hosking integral. The purpose of the present paper is to propose the scaling of the Hall cascade under the assumption that it is governed by the constancy of the Hosking integral, which now has different dimensions than in MHD.

2. Hosking integral and scaling for the Hall cascade

The Hosking integral I_H is defined as the asymptotic limit of the magnetic helicity density correlation integral $\mathcal{I}_\mathrm{H}(R)$ for scales R large compared to the correlation length of the turbulence, ξ, but small compared to the system size L. The original work on this integral is that by Hosking & Schekochihin (2021), who subsequently applied it to the magnetic field decay in the early universe (Hosking & Schekochihin 2022), see also Brandenburg *et al.* (2015) and Brandenburg & Kahniashvili (2017) for earlier work were inverse cascading of magnetically dominated nonhelical turbulence was first reported. The function $\mathcal{I}_\mathrm{H}(R)$ is given by

$$\mathcal{I}_\mathrm{H}(R)=\int_{V_R}\mathrm{d}^3r\langle h(\boldsymbol{x})h(\boldsymbol{x}+\boldsymbol{r})\rangle, \tag{2.1}$$

where V_R is the volume of a ball of radius R, $h=\boldsymbol{A}\cdot\boldsymbol{B}$ is the magnetic helicity density, and $\boldsymbol{A}$ is the magnetic vector potential, so $\boldsymbol{B}=\boldsymbol{\nabla}\times\boldsymbol{A}$.

What matters here is the fact that the dimensions of h are $[h]=[B]^2[x]$, and therefore the dimensions of $\mathcal{I}_\mathrm{H}$ and I_H are

$$[I_\mathrm{H}]=[B]^4[x]^5. \tag{2.2}$$

However, as already noted in B20, using $e=1.6\times10^{-19}\,\mathrm{A\,s}$, $\mu_0=4\pi\times10^{-7}\,\mathrm{T\,m\,A^{-1}}$, and $n_e\approx2.5\times10^{40}\,\mathrm{m^{-3}}$ for neutron star crusts, we have $en_e\mu_0\approx5\times10^{15}\,\mathrm{T\,s\,m^{-2}}$, and therefore

$$\frac{B}{en_e\mu_0}=\frac{B}{5\times10^{15}\,\mathrm{T}}\,\frac{\mathrm{m}^2}{\mathrm{s}}, \tag{2.3}$$

which is why we say B has dimensions of $\mathrm{m^2\,s^{-1}}$.† Therefore, the dimensions of I_H are

$$[I_\mathrm{H}]=[x]^{13}[\mathrm{s}]^{-4}. \tag{2.4}$$

Thus, given that $I_\mathrm{H}=\mathrm{const}$ in the limit of small magnetic diffusivity, a self-similar evolution must imply that all relevant length scales, and in particular the magnetic correlation length $\xi(t)$, must increase with time like $\xi\sim t^{4/13}$. Since $4/13\approx0.31$, this is indeed close to the behavior $\xi\sim t^q$ with $q\approx0.3$ found in Sec. 3.2 of B20 (their Run B), as already highlighted in the introduction of the present paper.

To demonstrate that the spectra at different times are indeed self-similar, we collapse them on top of each other by plotting them versus $k\xi(t)$. This ensures that their maxima are always approximately near unity. In addition, we must also compensate for the decay

† In MHD, by comparison, the ion density ρ is a relevant quantity. Using $\rho=10^3\,\mathrm{kg\,m^{-3}}$ for solar surface plasma, and the identity $1\,\mathrm{T}=1\,\mathrm{kg\,s^{-2}\,A^{-1}}$, we have $\mu_0=4\pi\times10^{-7}\,\mathrm{T^2\,s^2\,m\,kg^{-1}}$, and therefore $\rho\mu_0\approx3.5\times10^{-2}\,\mathrm{T\,s\,m^{-1}}$, or $B/\sqrt{\rho_0\mu_0}=(B/3.5\times10^{-2}\,\mathrm{T})\,\mathrm{m\,s^{-1}}$, which is why we say that in MHD, B has the dimensions of $\mathrm{m\,s^{-1}}$.

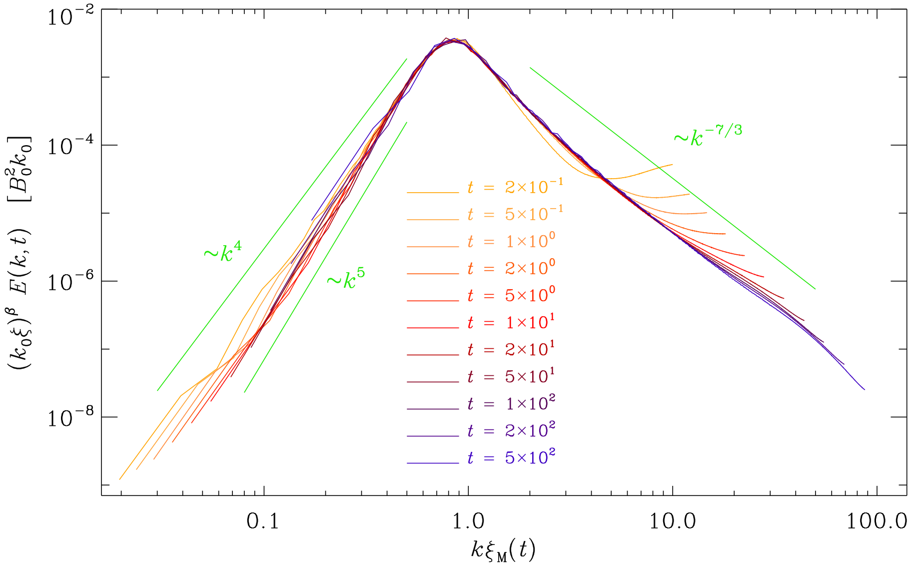

FIGURE 1. Compensated spectra for Run B of B20, which corresponds to Run B1 in the present paper. Here, $\beta = 1.7$ has been used as the best empirical fit parameter.

in amplitude by multiplying the spectra by a time-dependent function, e.g., $\xi(t)^\beta$, where β is a suitable exponent, so that the compensated spectra all have the same height. In this way, we find a universal spectral function by plotting

$$[\xi(t)]^\beta\, E\big(k\xi(t), t\big) \equiv \phi\big(k\xi(t)\big). \tag{2.5}$$

As an example, we show in Figure 1 the compensated spectra for Run B of B20, which we discuss in more detail below. At small k, the spectrum steepens from k^4 to k^5. Beyond the peak, it falls off with a $k^{-7/3}$ inertial range, as was already found by Biskamp *et al.* (1996).

To determine the theoretically expected value of β, we invoke the condition that the compensated spectra be invariant under rescaling, $t \to \tau t'$, $\boldsymbol{x} \to \tau^q \boldsymbol{x}'$, where τ is an arbitrary scaling factor. We recall that the dimensions of $E(k,t)$ are $[x]^5[t]^{-2}$, so rescaling yields a factor τ^{5q-2}. In addition, expressing E' in terms of its universal spectral function $\phi(k\xi)$, the factor ξ^β on the left-hand side of the Eq. (2.5) produces a factor $\xi^{-\beta}$ on the right-hand side of Eq. (2.5), and therefore, after rescaling, a $\tau^{-\beta q}$ factor, i.e.,

$$E(k) \to \tau^{5q-2}\, E'(k) \propto \xi(t)^{-\beta}\tau^{-\beta q}\phi(k\xi). \tag{2.6}$$

Therefore, $5q-2 = -\beta q$ must be satisfied in order that the compensated spectra remain invariant under rescaling. Thus, $\beta = 2/q - 5$, as already found in B20. Inserting now $q = 4/13$ yields $\beta = 3/2$. The total magnetic energy is therefore

$$\mathcal{E} = \int \xi^{-\beta}\phi(k\xi)\,\mathrm{d}k = \xi^{-(\beta+1)}\int \phi(k\xi)\,\mathrm{d}(k\xi) \propto t^{-(\beta+1)\,q}, \tag{2.7}$$

and since $\mathcal{E} \propto t^{-p}$, we have $p = (\beta+1)\,q$. Using $\beta = 2/q - 5$, we have $p = 2\,(1-2q)$; see Eq. (28) of B20. For $q = 4/13 \approx 0.31$, we have $p = 2\,(1 - 8/13) = 10/13 \approx 0.78$, which is not quite as close to the value reported in B20 as that of q, but this could be ascribed to the lack of scale separation and also the magnetic field no longer being strong enough

Run	Lu	$t_1/[t]$	$t_2/[t]$	$\tilde{\eta}$	$\tilde{B}_{\rm rms}$	$\tilde{\epsilon}$	p	q	β	$p_{\rm H}$
B1	650	0.2	500	0.024	600	3×10^6	0.8 ± 0.1	0.3 ± 0.1	1.7 ± 0.1	0.16
B2	1300	3	200	0.011	700	6×10^6	0.78 ± 0.05	0.31 ± 0.05	1.6 ± 0.05	0.11

Table 1. Summary of runs discussed in this paper.

so that the Lundquist number,†

$$\mathrm{Lu}=B_{\rm rms}/en_e\mu_0\eta, \tag{2.8}$$

is no longer in the asymptotic regime. This also resulted in the empirical value of β being slightly larger than the theoretical one, as we will see next.

3. Comparison with simulations for different diffusivities

In B20, various simulations of the Hall cascade have been presented, including forced and decaying simulations, helical and nonhelical ones, with constant and time-varying magnetic diffusivities, with and without stratification, etc. The main purpose of that work was to understand the dissipative losses that would lead to resistive heating in the crust of a neutron star. One of those simulations is particularly relevant for the present paper: his Run B, which had a relatively strong initial magnetic field, no helicity, large scale-separation, and a magnetic diffusivity that decreased with time in a power law fashion, allowing the simulation to retain a higher Lundquist number as the magnetic field decreases.

In the present paper, we analyze his Run B, which is here called Run B1. It is actually a new run, because we now have calculated the Hosking integral during run time. We also compare with another run (Run B2), where we decreased the magnetic diffusivity by a factor of two. As in B20, η is assumed to decrease with time proportional to $t^{-3/7}$. We kept, however, the same resolution of 1024^3 mesh points for both runs, but we must keep in mind that this can lead to artifacts resulting from a poorly resolved diffusive subrange for Run B2.

In Table 1, we compare several characteristic parameters: the start and end times, t_1 and t_2, respectively, of the interval for which averaged data have been accumulated, nondimensional measures of the magnetic diffusivity, the magnetic field strength, and the dissipation, $\tilde{\eta}$, $\tilde{B}_{\rm rms}$, and $\tilde{\epsilon}$, respectively, and the instantaneous scaling exponents p, q, and β. For $\tilde{\eta}$, $\tilde{B}_{\rm rms}$, and $\tilde{\epsilon}$, we compute the following averaged ratios:

$$\tilde{\eta}\equiv\langle t\eta/\xi^2\rangle,\quad \tilde{B}_{\rm rms}\equiv\langle B_{\rm rms}/(en_e\mu_0\eta)\rangle,\quad\text{and}\quad \tilde{\epsilon}\equiv\langle\epsilon/(e^2n_e^2\mu_0\eta^3/\xi^2)\rangle, \tag{3.1}$$

where $\xi(t)=\mathcal{E}^{-1}\int k^{-1}E(k,t)\,\mathrm{d}k$ is the correlation length and $\epsilon=\eta\mu_0\langle\boldsymbol{J}^2\rangle$ is the magnetic dissipation with $\eta=\eta(t)$, as noted above. These were also computed in B20. Time is given in diffusive units, $[t]=(\eta k_0^2)^{-1}$. In the runs of series B of B20, the value of k_0 is 180 times larger than the lowest wavenumber $k_1\equiv 2\pi/L$ of our cubic domain of size L^3.

It turns out that a lower resistivity is important for obtaining the expected scaling. We therefore now consider Run B2, where $\mathrm{Lu}\approx1300$. The result is shown in Figure 2, where we used $\beta=1.6$ as the best fit, which is still slightly larger than the expected value of $3/2$, but it goes in the right direction. Therefore, we show in Figure 2 the resulting

† Note that, unlike the case of MHD, in the present case of Hall cascade, no wavenumber factor enters in the definition of the Lundquist number.

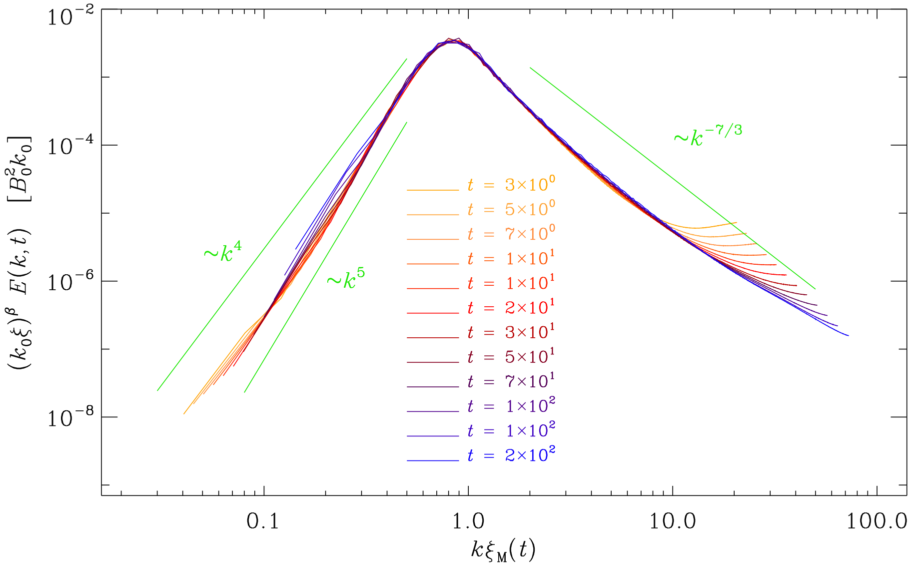

FIGURE 2. Compensated spectra for Run B2.

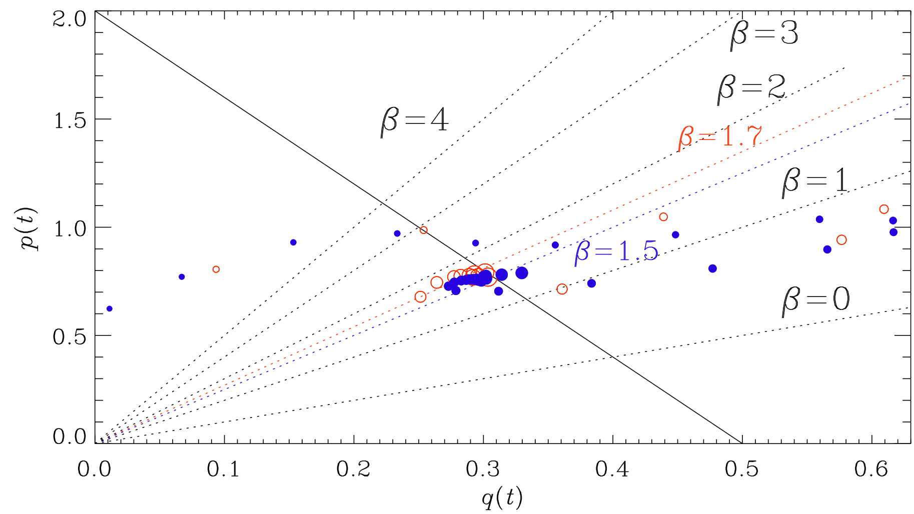

FIGURE 3. *pq* diagrams for Runs B1 (open red symbols) and B2 (closed blue symbols). Larger symbols denote later times.

compensated spectra for Run B2, where the magnetic diffusivity is half that of Run B1 and Lu is now twice as large a before; see Table 1.

Another comparison between Runs B1 and B2 is shown in Figure 3, where we compare their evolution in the *pq*–diagram. While Run B1 clearly evolves along the $\beta \approx 1.7$ line, Run B2 tends to be closer to the $\beta \approx 3/2$ line. Note also that both runs settle near the $p = 2(1 - 2q)$ self-similarity line (B20), although we begin to see departures near the end of the run, which is due to the finite size of the domain.

Finally, we show in Figure 4 the scaling of $I_\mathrm{H}(t)$, where we see that the decay exponent $p_\mathrm{H} \equiv -\mathrm{d}\ln I_\mathrm{H}/\mathrm{d}\ln t$ is about $p_\mathrm{H} \approx 0.16$ for Run B1 and about 0.11 for Run B2. Earlier work by Zhou *et al.* (2022) showed that p_H decreases as the Lundquist number increases,

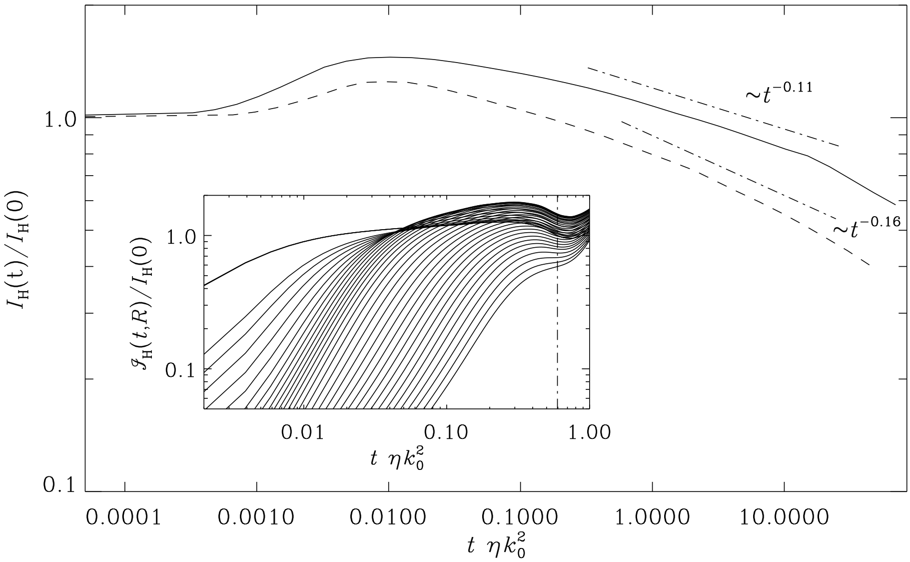

FIGURE 4. Evolution of $I_{\mathrm{H}}(t)$, showing first a slight increase and then a decline proportional to $t^{-0.16}$ for Run B1 and proportional to $t^{-0.11}$ for Run B2. The inset shows the evolution of $\mathcal{I}_{\mathrm{H}}(R;t)$ for selected values of R for Run B2.

MHD	$[B]=[x]/[t]$	$[\langle h\rangle]=[x]^3/[t]^2$	$q=2/3$ (helical)
		$[I_{\mathrm{H}}]=[x]^9/[t]^4$	$q=4/9$ (nonhel)
	$[E(k,t)]=[x^3][t]^{-2}$	$\beta=2/q-3$	$\beta=2\cdot 9/4-3=3/2$
		$p=2\,(1-q)$	$p=2\,(1-4/9)=10/9$
Hall	$[B]=[x]^2/[t]$	$[\langle h\rangle]=[x]^5/[t]^2$	$q=2/5$ (helical)
		$[I_{\mathrm{H}}]=[x]^{13}/[t]^4$	$q=4/13$ (nonhel)
	$[E(k,t)]=[x^5][t]^{-2}$	$\beta=2/q-5$	$\beta=2\cdot 13/4-5=3/2$
		$p=2\,(1-2q)$	$p=2\,(1-2\cdot 4/13)=10/13$

TABLE 2. Comparison of the scalings for MHD and the Hall cascade.

and is, in MHD, around 0.2 for $\mathrm{Lu}\approx 10^3$, and decreases to $p_{\mathrm{H}}\approx 0.01$ for $\mathrm{Lu}\approx 4\times 10^7$. Such large values can currently only be obtained with magnetic hyper-diffusivity (Hosking & Schekochihin 2021; Zhou *et al.* 2022), but this has not been attempted in the present work.

As already noted by Zhou *et al.* (2022), there is an initial increase in $I_{\mathrm{H}}(t)$. This is explained by the fact that the magnetic field obeys Gaussian statistics initially, but not during the later evolution. The inset shows the R dependence of $\mathcal{I}_{\mathrm{H}}(R;t)$ for different t. The relevant value of R is deemed to be at the location where the local slope of $\mathcal{I}_{\mathrm{H}}(R)$ is minimum at late times.

4. Conclusion

The present work has highlighted the power of dimensional arguments, which were here applied to the case of the Hall cascade without helicity, where the magnetic field is naturally represented as a quantity with units of a magnetic diffusivity. The magnetic helicity density has units of $\mathrm{m}^5\,\mathrm{s}^{-2}$ and the Hosking integral has units of $\mathrm{m}^{13}\,\mathrm{s}^{-4}$, which yields $q = 4/13$, $\beta = 3/2$, and $p = 10/13$. Comparing with standard MHD, where the magnetic field has units of $\mathrm{m\,s}^{-1}$, our exponents p and q are now smaller, but β is still the same in both cases; see Table 2 for a comparison between Hall cascade and MHD. The empirically determined value of β is somewhat larger, but this can be explained by finite scale separation and small Lundquist numbers.

The decay properties of the Hall cascade are important in understanding resistive heating in neutron stars while producing at the same time larger scale magnetic fields at a certain speed (B20). Such simulations have already been done in spherical geometry (Gourgouliatos *et al.* 2020), but the magnetic field in those simulations did not yet exhibit clear forward or inverse cascading. This is presumably due to their initial magnetic field being strongly localized at intermediate length scales. Using an initial broken power law, as done here, would help producing the expected forward or inverse cascadings, but this may also require much larger resolution than what is currently possible. Similarly, of course, the values of n_e and η are depth dependent in real neutron stars, but the work of B20 showed that this did not affect the scaling behavior of the magnetic decay. Therefore, importance of the Hosking integrals may well carry over to real neutron stars.

Acknowledgements

I would like to thank the Isaac Newton Institute for Mathematical Sciences, Cambridge, for support and hospitality during the programme "Frontiers in dynamo theory: from the Earth to the stars" where work on this paper was undertaken.

Funding

This work was supported by EPSRC grant no EP/R014604/1 and the Swedish Research Council (Vetenskapsrådet, 2019-04234). Nordita is sponsored by Nordforsk. We acknowledge the allocation of computing resources provided by the Swedish National Allocations Committee at the Center for Parallel Computers at the Royal Institute of Technology in Stockholm and Linköping.

Declaration of Interests

The authors report no conflict of interest.

Data availability statement

The data that support the findings of this study are openly available on Zenodo at doi:10.5281/zenodo.7357799 (v2022.11.24).

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