ARTICLE IN PRESS
JID:LAA AID:16272 /FLA [m1L; v1.330] P.1 (1-19)
Linear Algebra and its Applications ••• (••••) •••–•••

Contents lists available at ScienceDirect

Linear Algebra and its Applications

journal homepage: www.elsevier.com/locate/laa

An extended trace formula for vertex operators ☆

Mohammad Reza-Rahmati [a], Gerardo Flores [a,*]

[a] Laboratorio de Percepción y Robótica [LAPyR], Centro de Investigaciones en Óptica, León, Guanajuato, 37150, Mexico

ARTICLE INFO

Article history:
Received 21 January 2022
Accepted 4 December 2022
Available online xxxx
Submitted by P. Semrl

MSC:
17B10
20C32
20G05
22E60
22E65
22E70

Keywords:
Fock space
Infinite wedge representation
Infinite dimensional Lie algebras
Vertex operator
Character formula

ABSTRACT

We present an extension of the trace of a vertex operator and explain a representation-theoretic interpretation of the trace. Specifically, we consider a twist of the vertex operator with infinitely many Casimir operators and compute its trace as a character formula. To do this, we define the Fock space of infinite level $\mathfrak{F}^\infty$. Then, we prove a duality between $\mathfrak{gl}_\infty$ and $\mathfrak{a}_\infty = \widehat{\mathfrak{gl}}_\infty$ of Howe type, which provides a decomposition of $\mathfrak{F}^\infty$ into irreducible representations with joint highest weight vector for $\mathfrak{gl}_\infty$ and $\mathfrak{a}_\infty$. The decomposition of the Fock space $\mathfrak{F}^\infty$ into highest weight representations provides a method to calculate and interpret the extended trace.

1. Introduction

The infinite-dimensional Lie algebras representation theory can express many questions in mathematics and physics. In particular, the infinite wedge representation is a

☆ This paper has not been received any particular financial support.
* Corresponding author.
E-mail addresses: rahmati@cio.mx (M. Reza-Rahmati), gflores@cio.mx (G. Flores).

https://doi.org/10.1016/j.laa.2022.12.001

Please cite this article in press as: M. Reza-Rahmati, G. Flores, An extended trace formula for vertex operators, Linear Algebra Appl. (2022), https://doi.org/10.1016/j.laa.2022.12.001

fundamental discrete structure on which many problems in theoretical quantum physics can be modeled. Moreover, it provides a general theoretic representation framework that supports different sampling problems in quantum theory. The infinite wedge representation is sometimes referred to as *the fermionic Fock space*. Infinite wedge representation provides examples of the representation of infinite-dimensional Lie algebras that produces interesting character formulas. In this sense, characters of representations of infinite-dimensional Lie algebras have been one of the most profound research subjects in the last few decades. Some exciting character formulas can be extracted from specific operators acting on the Fock space trace, providing a powerful study tool. For instance, generating series, Feynman integrals, and probability amplitudes in mathematical physics, [1–13].

A vertex operator is an operator of an infinite-dimensional Lie algebra. Vertex operators present a formalism for the linear action on specific infinite-dimensional vector spaces, such as the fermionic Fock space. The trace of the vertex operators is related to Schur functions and symmetric polynomials. In this context, the symmetric functions play a prominent role in connecting combinatorics to the infinite-dimensional Lie algebras representation theory. The theory of vertex operators involves the representation theory of infinite symmetric group S_∞ and the theory of symmetric polynomials; there are well-known formulas for the trace of vertex operators in terms of symmetric polynomials, [2], [14], [15], [1], [16], [6], [7].

The computation of the trace of the vertex operators $Tr(q^{L_0}\exp(\sum_n A_n\alpha_{-n}) \times \exp(\sum_n B_n\alpha_n))$, is crucial in representation theory. The transformation beneath the trace is acting on common Fock space $\mathfrak{F} = \mathfrak{F}^1$, also called the infinite wedge representation. One can consider higher Casimir operators $L_j,\ j > 0$ acting on the Fock space $\mathfrak{F}$. We follow a computation of Bloch-Okounkov [14] for the character of the infinite wedge representation, where a product formula is established for the character. Vertex operators appear in the context of string theory partition functions of CY 3-folds. In this case, the vertex operator is twisted by one or more Casimir operators, where the interest is to calculate its trace. We give a natural extension of the trace to the case where infinitely many Casimir operators appear in the trace function. Our idea is to use a decomposition of the Fock space of level infinity into irreducible highest weight representations of Lie superalgebra $\mathfrak{a}_\infty = \hat{\mathfrak{gl}}_\infty$. The decomposition breaks the trace into a sum of the traces on the irreducible components, [3,17].

1.1. Contributions

Based on the computation of the trace formulas for $Tr(q^{L_0}\exp(\sum_n A_n\alpha_{-n}) \times \exp(\sum_n B_n\alpha_n))$ and the Bloch-Okounkov result on the character formula for $Tr\left(\exp(\sum_{j\geq 0} 2\pi i L_j)\right)$, we propose to compute the following trace:

$$Trace = Tr\left(\exp(\sum_{j\geq 0} 2\pi i L_j)\exp(\sum_{n>0} A_n\alpha_{-n})\exp(\sum_{n>0} B_n\alpha_n)\right). \tag{1.1}$$

Our approach is to define the Fock space of level ∞ denoted by $\mathfrak{F}^{\infty}$, as the natural generalization of the Fock space of finite level l. Then, we show the existence of a decomposition,

$$\mathfrak{F}^{\infty} = \bigoplus_{\lambda} L(\mathfrak{gl}_{\infty}, \lambda) \otimes L(\mathfrak{a}_{\infty}, \Lambda(\lambda)) \tag{1.2}$$

where λ runs overall generalized partitions, from which a character formula can be calculated. The λ summand is a joint-highest weight representation of $\mathfrak{gl}_{\infty}$ and $\mathfrak{a}_{\infty}$. We shall interpret the trace in (1.1) as the trace of an operator acting on the Fock space $\mathfrak{F}^{\infty}$. The decomposition above in (1.2) gives a way to express the trace formula as a sum of traces over highest weight representations of $\mathfrak{gl}_{\infty} \times \mathfrak{a}_{\infty}$, where we can conduct the computation directly.

1.2. Organization of the text

The remainder of this paper is as follows. Section 2 provides basic definitions on Fock spaces and infinite wedge representation. We also introduce the vertex operators and character formulas for the vertex operators. Section 3 explains the problem that we are going to solve in the paper together with its motivation from physics. Section 4 contains the main contributions of the paper. Further, we add an application given in Section 5. Some conclusions are given in Section 6. Finally, the appendix contains the definition of several infinite-dimensional Lie algebras that are important in this context.

2. Preliminaries

2.1. Infinite wedge representation

The Fock space is an infinite-dimensional vector space representing certain infinite-dimensional Lie algebras. It provides a systematic framework of importance in Physics by the trace of vertex operators [acting on the Fock space]. It also plays a crucial role in string theory to explain the probabilistic amplitudes.

Definition 2.1. The half infinite wedge or fermionic Fock space $\mathfrak{F}$ defined by:

$$\mathfrak{F} = \bigwedge^{\frac{\infty}{2}} V = \bigoplus_{i_r \in 1/2+\mathbb{Z}} \mathbb{C}.\ v_{i_1} \wedge v_{i_2} \wedge ..., \qquad i_j = i_{j-1} - 1/2,\ j \gg 0, \tag{2.1}$$

is the vector space spanned by the semi-infinite wedge product of a fixed basis of the infinite-dimensional vector space,

$$V = \sum_{i \in 1/2+\mathbb{Z}} \mathbb{C}.v_i, \tag{2.2}$$

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i.e., the monomials $v_{i_1} \wedge v_{i_2} \wedge ...$ such that:

- $i_1 > i_2 > ...$
- $i_j = i_{j-1} - 1/2$ for $j \gg 0$.

Besides, we have the creation and annihilation operators defined by:

$$\begin{aligned} \psi_k : & v_{i_1} \wedge v_{i_2} \wedge ... \mapsto v_k \wedge v_{i_1} \wedge v_{i_2} \wedge ... \\ \psi_k^* : & v_{i_1} \wedge v_{i_2} \wedge ... \mapsto (-1)^l v_{i_1} \wedge v_{i_2} \wedge ... \wedge \widehat{v_{i_l=k}} \wedge ... \end{aligned} \tag{2.3}$$

The monomials can be parametrized in terms of partition

$$|\lambda\rangle = v_\lambda = \lambda_1 - 1/2 \wedge \lambda_2 - 3/2 \wedge ... \tag{2.4}$$

The Fock space $\mathfrak{F}$ is almost a Hilbert space, with respect to the inner product $\langle\ v_\lambda,\ v_\mu\ \rangle = \delta_{\lambda\mu}$. Its completion with respect to the norm of the inner product is a Hilbert space. We can also write this using the Frobenius coordinates of partitions:

$$|\lambda\rangle = \prod_{i=1}^{l} \psi_{a_i}^* \psi_{b_i} |0\rangle, \qquad a_i = \lambda_i - i + \frac{1}{2},\ b_i = \lambda_i^t - i + \frac{1}{2}, \tag{2.5}$$

where $(a_1, ..., a_l | b_1, ..., b_l)$ are called the Frobenius coordinates of λ. The operator

$$C = \sum_{k \in 1/2 + \mathbb{Z}} : \psi_k \psi_k^* : \tag{2.6}$$

is called the charge operator, whose action on $\mathfrak{F}$ is:

$$C\,(v_{i_1} \wedge v_{i_2} \wedge ...) = [(\sharp \text{ present positive } v_i) - (\sharp \text{ missing negative } v_i)] v_{i_1} \wedge v_{i_2} \wedge ... \tag{2.7}$$

The vectors of 0-charge are characterized by being annihilated by C. Besides, the energy operator H (Hamiltonian) is defined by

$$H = \sum_{k \in \frac{1}{2} + \mathbb{Z}} k : \psi_k \psi_k^* : \tag{2.8}$$

and satisfies $H v_\lambda = |\lambda| v_\lambda$. Let us define the Bosonic operators as follows:

$$\alpha_n = \sum_{k \in \frac{1}{2} + \mathbb{Z}} \psi_{k+n} \psi_k^*. \tag{2.9}$$

They satisfy the commutation relations $[\alpha_n, \psi_k] = \psi_{k+n},\ [\alpha_n, \psi_k^*] = -\psi_{k-n}$.

The fermionic Fock space $\mathfrak{F}$ is the Hilbert space generated by a pair of fermions $\psi^{\pm}(z)$ with components $\psi_r^{\pm},\ r \in \frac{1}{2} + \mathbb{Z}$ satisfying the Clifford commutation relations,

$$[\psi_i^+, \psi_j^-] = \delta_{i,-j}, \qquad [\psi_i^+, \psi_j^+] = 0, \qquad [\psi_i^-, \psi_j^-] = 0. \tag{2.10}$$

2.2. Bosonic Fock space

The Hilbert space $\mathfrak{F}$ can be constructed in two primary isomorphic forms (Boson-Fermion correspondence). They are called Fermionic and Bosonic Fock spaces. In this regard, Definition 2.1 is usually referred to as the fermionic Fock space. The Bosonic Fock space is a representation defined on the polynomial ring $\mathbb{C}[x_1, x_2, ...; q, q^{-1}]$, see [13]. It is not hard to write a specific isomorphism between $\mathfrak{F}$ and the mentioned polynomial ring. In this context, the vertex operators act as certain differential operators, and the trace of vertex operator appears as generating a series of the ring of symmetric functions on infinitely many variables. We review some features of this below.

Consider the coordinate ring of the affine variety $Sym^k(\mathbb{C})$, namely $B_k{=}\mathbb{C}[x_1, ..., x_n]/S_n$. Write the Hilbert series of B_k as $H_{B_k}(q) = \sum_n q^n h_n(B_k)$, where $h_n(B_k) = \natural\{\text{monomials in } B_k \text{ of charge } k\}$. Let also q act as $\mathbb{C}^\times$ on the other variables. We understand that B_k is the ring of symmetric functions in variables $x_1, ..., x_k$. It is also generated by Schur functions, i.e., $B_k = \langle\ S_\mu(x_1, ..., x_k),\ |\mu| \leq k\ \rangle$. In string theory B_k arises as the Hilbert space $\mathcal{H}_k$ generated by the Boson oscillators up to charge k, with commutation relations, $[\alpha_n, \alpha_m] = n\delta_{n+m,0}$.

Let us associate monomials to partitions as follows:

$$\lambda = 1^{\lambda_1} 2^{\lambda_2} .... \longmapsto \alpha_{-1}^{-\lambda_1} \alpha_{-2}^{-\lambda_2} ....|0\rangle, \tag{2.11}$$

thus,

$$B_k \cong \mathcal{H}_k = \left\langle\ \alpha_{-1}^{m_1} ... \alpha_{-k}^{m_k} |0\rangle \mid m_1, ...m_k \geq 0\ \right\rangle, \tag{2.12}$$

and we have the inclusions $\mathcal{H}_0 \subset \mathcal{H}_1 \subset ...$ corresponding to the nested sequence of Young diagrams of increasing number or rows. The $\mathbb{C}^\times$ induces an action of $Sym^*(\mathbb{C})$ such that the functions $S_\mu(x_1, ..., x_k)$ are eigen-functions with eigenvalues $q^{|\mu|}$ of the action of q^{L_0} on $\mathcal{H}$, where

$$L_0 = \sum_{n>0} \alpha_{-n}\alpha_n. \tag{2.13}$$

Then, we can write:

$$H_{B_k}(q) = Tr_{\mathcal{H}_k} q^{L_0} = \sum_{|\mu| \leq k} q^{|\mu|} = \prod_{n=1}^{k} (1 - q^n)^{-1}. \tag{2.14}$$

The generating function of these series becomes:

$$G(t, q) = \sum_{n=0}^{\infty} t^k H(B_k)(q) = \sum_k t^k Tr_{\mathcal{H}_k} q^{L_0} = \sum_\mu s_\mu(t) s_\mu(1, q, ...). \tag{2.15}$$

We can generalize this argument by considering the coordinate ring of the affine variety $B_{k_1,...,k_n} = Sym^{k_1}(\mathbb{C}) \times ... \times Sym^{k_n}(\mathbb{C})$, and define the generating series $G(t_1, ..., t_n; q) = \sum_{k_1,...,k_n} t_1^{k_1} ... t_n^{k_n} H_{B_{k_1,...,k_n}}(q)$. The ring $B_{k_1,...,k_n}$ is generated by the product of the Schur polynomials,

$$B_{k_1,...,k_n} = \big\langle S_{\mu_1}(x_{1,1}, ..., x_{1,k_1}). \ldots . S_{\mu_n}(x_{n,1}, ..., x_{n,k_n});\ l(\mu_j) \leq k_j \big\rangle. \tag{2.16}$$

The above ring is isomorphic to the Hilbert space spanned by the Bosonic operators,

$$\mathcal{H}_{k_1,...,k_n} = \left\langle \prod_{j=1}^{n} (\alpha_{-1}^{j})^{n_{j,1}} ... (\alpha_{-k_n}^{j})^{n_{j,k_n}} |0\rangle \right\rangle. \tag{2.17}$$

In the new terminology, the Hilbert series can be written as:

$$H_{B_{k_1,...,k_n}}(q) = Tr_{\mathcal{H}_{k_1,...,k_n}} q^{L_0}, \tag{2.18}$$

where $L_0 = \sum_{j=1}^{n} \sum_{r>0} \alpha_{-r}^{j} \alpha_r^{j}$ is the charge operator. We obtain $G(t_1, ..., t_n, q) = \prod_{j=1}^{n} G(t_j, q) = \prod_{j=1}^{n} \prod_i (1 - q^{-i} t_j)$. The classical Boson-Fermion correspondence is an isomorphism between two representations of the Heisenberg algebra, namely *the Bosonic Fock space* and *the fermionic Fock space*. The Boson-Fermion correspondence is a basic result in mathematical physics. There are various applications of this correspondence. It provides an explicit way of comparing expressions for q-dimensions of representations, through which new combinatorial identities are derived by computing characters of representations in two different ways.

2.3. The Fock space of level l

The Fock space of level l denoted by $\mathfrak{F}^l$ is the fermionic Fock space on l pairs of fermions $\psi_r^{\pm,j}, r \in \frac{1}{2} + \mathbb{Z},\ j = 1, ..., l$. Let us denote by $\widehat{C}^l$ the Clifford algebra on $\psi_r^{\pm,j}, r \in \frac{1}{2} + \mathbb{Z}$ for $j = 1, ..., l$. $\mathfrak{F}^l$ is a simple $\widehat{C}^l$-module generated by $|0\rangle$, such that $\psi_r^{\pm,j}|0\rangle = 0,\ r > 0$. The $\frac{1}{2}\mathbb{Z}_+$-gradation of $\mathfrak{F}^l$ is given by the eigenvalues of the degree operator d such that $d|0\rangle = 0,\ [d, \psi_{-r}^{\pm,j}] = r\psi_{-r}^{\pm,j}$, where any graded subspace is finite-dimensional. We shall use normal ordering notations defined as follows:

$$\begin{aligned}
&: \psi_r^{+,j} \psi_s^{-,k} := \begin{cases} -\psi_s^{-,k} \psi_r^{+,j}, & s = -r < 0 \\ \psi_r^{+,j} \psi_s^{-,k} \end{cases} \\
&: \psi_r^{-,j} \psi_s^{+,k} := \begin{cases} -\psi_s^{+,k} \psi_r^{-,j}, & s = -r < 0 \\ \psi_r^{-,j} \psi_s^{+,k} \end{cases} \\
&: \psi_r^{+,j} \psi_s^{+,k} := \psi_r^{+,j} \psi_s^{+,k} \\
&: \psi_r^{-,j} \psi_s^{-,k} := \psi_r^{-,j} \psi_s^{-,k} \\
&: \psi_r^{\pm,j} \phi_s^{+,k} := \psi_r^{\pm,j} \phi_s^{+,k}.
\end{aligned} \tag{2.19}$$

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Define the operators e_*^{ij} and e_{ij}^* by:

$$e_*^{ij} = \sum_{u\in \frac{1}{2}+\mathbb{Z}} :\psi_{-u}^{+,i}\psi_u^{-,j}:, \qquad e_{ij}^* = \sum_{k=1}^{l} :\psi_{\frac{1}{2}-i}^{+,k}\psi_{j-\frac{1}{2}}^{-,k}: \tag{2.20}$$

With the above set-up, we have the following:

- The map $\mathfrak{gl}_l \longrightarrow End(\mathfrak{F}^l),\ E^{ij} \longmapsto e_*^{ij}$ is a representation of $\mathfrak{gl}_l$.
- The map $\mathfrak{a}_\infty \longrightarrow End(\mathfrak{F}^l),\ E_{ij} \longmapsto e_{ij}^*$ defines a representation of $\mathfrak{a}_\infty$.
- The above actions of $\mathfrak{gl}_l$ and $\mathfrak{a}_\infty$ on $\mathfrak{F}^l$ commute, [see [17] sec. 5.4]. Thus, we have the relation:

$$[e_*^{ij}, e_{rs}^*] = \left[\sum_{u\in \frac{1}{2}+\mathbb{Z}} :\psi_{-u}^{+,i}\psi_u^{-,j}: ,\ \sum_{l=1}^{\infty} :\psi_{\frac{1}{2}-r}^{+,l}\psi_{s-\frac{1}{2}}^{-,l}:\right] = 0. \tag{2.21}$$

We use the generalized partitions as the weights of representations of $\mathfrak{gl}_l$, i.e., the partitions λ of the form $\lambda = (\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_{i-1} \geq \lambda_i = ... = \lambda_{j-1} = 0 \geq \lambda_j \geq ... \geq \lambda_r)$. Also, for the weight of representations of $\mathfrak{a}_\infty$ we denote $\Lambda(\lambda) = \Lambda_{\lambda_1} + ... + \Lambda_{\lambda_l} = l\Lambda_0 + \sum_i \lambda_i' \epsilon_i$, where,

$$\lambda_i' = \begin{cases} |\ \{\ j\ |\lambda_j \geq i,\ i \geq 1\ \}\ | \\ -|\ \{\ j\ |\lambda_j < i,\ i \leq 0\ \}\ | \end{cases}. \tag{2.22}$$

Notice that we use the notation in the Appendix, in particular, (A.4). Let's define,

$$\begin{aligned} \varpi_m^{+,j} &= \psi_{-m+\frac{1}{2}}^{+,j}...\psi_{-\frac{3}{2}}^{+,j}\psi_{-\frac{1}{2}}^{+,j} \\ \varpi_m^{-,s} &= \psi_{-m+\frac{1}{2}}^{-,s}...\psi_{-\frac{3}{2}}^{-,s}\psi_{-\frac{1}{2}}^{-,s} \end{aligned}, \qquad \varpi_0^{+,s} = 1. \tag{2.23}$$

A joint highest weight vector in $\mathfrak{F}^l$ associated to λ with respect to the standard Borel $\mathfrak{gl}(l) \times \mathfrak{a}_\infty$ is

$$v_\lambda = \varpi_{\lambda_1}^{+,s}\varpi_{\lambda_2}^{+,s}...\varpi_{\lambda_i}^{+,s}\varpi_{-\lambda_j}^{-,s}\varpi_{-\lambda_{j+1}}^{+,s}...\varpi_{-\lambda_l}^{+,s}, \tag{2.24}$$

whose weight with respect to $\mathfrak{gl}(l)$ is λ, and with respect to $\mathfrak{a}_\infty$ is $\Lambda(\lambda)$, [see [17] sec. 5.4].

Theorem 2.2 *(See [17] sec. 5.4). We have the decomposition*

$$\mathfrak{F}^l = \bigoplus_\lambda L(\mathfrak{gl}(l), \lambda) \otimes L(\mathfrak{a}_\infty, \Lambda(\lambda)). \tag{2.25}$$

One may state other kind of dualities by considering different vertex operators associated to the pairs $(\mathfrak{sp}(\infty), \mathfrak{c}_\infty)$ and $(O(\infty), \mathfrak{d}_\infty)$.

Remark 2.3. In order to illustrate the fermionic operators, let $V = (\mathbb{C}^l \otimes \mathbb{C}^\infty) \bigoplus (\mathbb{C}^{l*} \otimes \mathbb{C}^{\infty *})$, where $\mathbb{C}^\infty$ is a vector space with basis $w_r, r \in -\frac{1}{2} - \mathbb{Z}_+$, the dual indexed is given by w_{-r}, and $\mathbb{C}^l$ has basis $v^{+,i}$ with dual $v^{-,i}$. Then, we may illustrate $\psi_r^{\pm,i} = v^{\pm,i} \otimes w_r$.

2.4. The character of vertex operators

The operators of the form:

$$\Gamma_+(x) = \exp(\sum_{n\geq 1} \frac{x^n}{n} \alpha_n), \qquad \Gamma_-(x) = \exp(\sum_{n>0} \frac{x^n}{n} \alpha_{-n}) \tag{2.26}$$

are called *vertex operators*. They are adjoint with respect to the natural inner product. We have a commutation relation:

$$\Gamma_+(x)\Gamma_-(y) = (1 - xy)\Gamma_-(y)\Gamma_+(x). \tag{2.27}$$

Also, we have,

$$\Gamma_+(x)v_\mu = \sum_{\lambda \supset \mu} S_{\lambda/\mu}(x)v_\lambda. \tag{2.28}$$

Vertex operators provide powerful tools to express partitions. We give some examples below.

- Vertex operators simplify expressions on generating series over partitions. For example, we can write,

$$\Gamma_+(1)|\mu\rangle = \sum_{\lambda \supset \mu} |\lambda\rangle, \qquad \Gamma_-(1)|\mu\rangle = \sum_{\lambda \subset \mu} |\lambda\rangle. \tag{2.29}$$

- As another example, we may write the well-known McMahon function as follows:

$$\begin{aligned} Z = \sum_{\text{3-dim partitions}} q^{\natural\ boxes} &= \langle (\prod_{t=0}^{\infty} q^{L_0}\Gamma_+(1))q^{L_0}(\prod_{t=-\infty}^{-1} \Gamma_-(1)q^{L_0})\rangle \\ &= \langle \prod_{n>0} \Gamma_+(q^{n-\frac{1}{2}}) \prod_{n>0} \Gamma_-(q^{-n-\frac{1}{2}})\rangle. \end{aligned} \tag{2.30}$$

By employing innovative combinatorial tools over Young diagrams, one can obtain more complicated formulas involving generating series. Let us give an example. One may divide a 3-dimensional partition into slices of two-dimensional partitions along the diagonals. In this way, the vertex operator divides into the multiplication of many vertex operators of the slices,

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$$Z(\{x_m^{\pm}\}) = \langle \dots \prod_{u_i<m<v_{i+1}} \Gamma_+(x_m^-) \prod_{v_i<m<u_i} \Gamma_-(x_m^+) \dots \rangle = \langle \prod_{u_0<m<u_n} \Gamma_{-\epsilon(m)}(x_m^{\epsilon(m)})\rangle. \tag{2.31}$$

In this way, one can obtain product formulas such as:

$$Z(\{x_m^{\pm}\}) = \prod_{m_1<m_2} (1 - x_{m_1}^- x_{m_2}^+). \tag{2.32}$$

For example, we may choose

$$\begin{aligned} \{x_m^+\} &= \{t^i q^{v_i} | i = 1, 2, \dots\} \\ \{x_m^-\} &= \{t^{j-1} q^{-v_i^t} | j = 1, 2, \dots\}, \end{aligned} \tag{2.33}$$

and we get,

$$Z_{\lambda\mu\nu}(t, q) = \langle \prod_{u_0<m<u_n} \Gamma_{-\epsilon(m)}(x_m^{\epsilon(m)})\rangle. \tag{2.34}$$

The partition function can also be read by putting a wall on the distance M along with one of the axis. Then, using commutation (2.27) we have expressions of the form,

$$\begin{aligned} Z &= \langle \prod_{0<m<\infty} \Gamma_-(x_m^+) \prod_{-M<m<0} \Gamma_+(x_m^-)\rangle \\ &= \prod_{l_1=1}^{\infty} \prod_{l_2=1}^{M} (1 - x_{l_1-\frac{1}{2}}^+ x_{-l_2+\frac{1}{2}}^-)^{-1} (\langle \prod_{-M<m<0} \Gamma_+(x_m^-) \prod_{0<m<\infty} \Gamma_-(x_m^+)\rangle, \end{aligned} \tag{2.35}$$

where the last factor in parentheses is equal to 1, and we obtain a product formula.

The following theorem is another form of generating series arising from the vertex operator's trace.

Theorem 2.4. *[13] We have the following formula for the trace of a vertex operator acting on $\mathfrak{F} = \bigwedge^{\frac{\infty}{2}} V$:*

$$Tr\left(q^{L_0} \exp(\sum_n A_n \alpha_{-n}) \exp(\sum_n B_n \alpha_n)\right) = \prod_n \sum_k \sum_{l=0}^{k} \frac{n^l A_n^l B_n^l}{l!} q^{nk} \binom{k}{l}, \tag{2.36}$$

where L_0 is the charge operator, and $q^{L_0}|\lambda\rangle = |\lambda||\lambda\rangle$.

Proof. Denote the operator in the trace by T. We have the isomorphism:

$$\bigwedge^{\frac{\infty}{2}} V = \bigotimes_{n=1}^{\infty} \bigoplus_{k=0}^{\infty} \alpha_{-n}^k |0\rangle, \tag{2.37}$$

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which implies:

$$\begin{aligned} Tr(T) &= \prod_{n=1}^{\infty} Tr\left(T|_{\bigoplus_{k=0}^{\infty}\alpha_{-n}^{k}|0\rangle}\right) \\ &= \prod_{n}\sum_{k}\langle 0|\alpha_n^k\Big|q^{L_0}e^{A_n\alpha_{-n}}e^{B_n\alpha_n}\Big||\alpha_n^k|0\rangle\rangle \\ &= \prod_{n}\sum_{k,l,m}\frac{A_n^l B_n^m}{l!m!}q^{n(l-m+k)}\langle 0|\alpha_n^k\Big|\alpha_{-n}^l\alpha_n^m\Big||\alpha_n^k|0\rangle\rangle \\ &= \prod_{n}\sum_{k,l}\frac{A_n^l B_n^l}{l!l!}q^{nk}\langle 0|\alpha_n^k\Big|\alpha_{-n}^l\alpha_n^l\Big||\alpha_n^k|0\rangle\rangle \\ &= \prod_{n}\sum_{k}\sum_{l=0}^{k}\frac{n^l A_n^l B_n^l}{l!}q^{nk}\binom{k}{l}. \quad \square \end{aligned} \tag{2.38}$$

In the following, we present another calculation due to Bloch and Okounkov [14] of the trace of a representation on the infinite wedge space defined in 2.1. This is the foundation of our main result presented in Theorem 4.3 given in Section 4.

Theorem 2.5. *[14] Let $\mathfrak{F} = \bigwedge^{\frac{\infty}{2}} V$ be the Fock space on a fixed basis of $V = \bigoplus_{j\in\mathbb{Z}} \mathbb{C}v_j$. Then, the character of $\mathfrak{F}$ is given by*

$$ch(\mathfrak{F}) = \prod_{n\geq 0}(1 + y_0 y_1^{n+\frac{1}{2}} y_2^{(n+\frac{3}{2})^2}\ldots.)(1 + y_0^{-1} y_1^{n-\frac{1}{2}} y_2^{-(n-\frac{1}{2})^2}\ldots.), \tag{2.39}$$

where $y_j = e^{2\pi i\tau_j}$.

Proof. The associated character is the trace of the operator,

$$y_0^{L_0} y_1^{L_1}\ldots. y_n^{L_n}\ldots = \exp(\sum_j 2\pi i\tau_j L_j), \tag{2.40}$$

where L_i are commuting operators which act on $\mathfrak{F}$ by

$$L_j \longmapsto \sum_{n\in\frac{1}{2}+\mathbb{Z}}\left(n - \frac{1}{2}\right)^j E_{n,n}. \tag{2.41}$$

The operators L_j mutually define a representation of

$$H = \mathbb{C}L_0 \oplus \mathbb{C}L_1 \oplus \ldots \longrightarrow End(\mathfrak{F}). \tag{2.42}$$

We compute the action of the operators $\exp(2\pi i\tau_j L_j)$ on the basis elements. Thus, we have,

$$L_j.\left(\psi_{-i_1}\psi_{-i_2}...\psi_{-i_l}\psi^*_{-j_1}...\psi^*_{-j_k}|0\rangle\right) = \Big(\sum_{a=1}^{l}(i_a - 1/2)^j - \sum_{b=1}^{k}(-j_b - \frac{1}{2})^j\Big) \times \left(\psi_{-i_1}\psi_{-i_2}...\psi_{-i_l}\psi^*_{-j_1}...\psi^*_{-j_k}|0\rangle\right). \tag{2.43}$$

We also have,

$$\begin{aligned} \exp(2\pi i\tau_j L_j)\psi_{-n}|0\rangle &= \left(y_0 y_1^{n-\frac{1}{2}} y_2^{(n-\frac{1}{2})^2}...\right)\psi_{-n}|0\rangle \\ \exp(2\pi i\tau_j L_j)\psi^*_{-n}|0\rangle &= \left(y_0^{-1} y_1^{n-\frac{1}{2}} y_2^{-(n-\frac{1}{2})^2}...\right)\psi^*_{-n}|0\rangle. \end{aligned} \tag{2.44}$$

The claim of the theorem follows from the isomorphism:

$$\mathfrak{F} = \bigwedge^{\frac{\infty}{2}} V = \bigotimes_n (1+\psi_{-n})(1+\psi^*_{-n})|0\rangle. \quad \square \tag{2.45}$$

3. Problem statement

Motivated by the two calculations of the trace of vertex operator presented in the previous section, i.e.,

$$Tr\left(q^{L_0}\exp(\sum_n A_n\alpha_{-n})\exp(\sum_n B_n\alpha_n)\right), \qquad Tr\left(\exp(\sum_{j\geq 0} 2\pi i L_j)\right), \tag{3.1}$$

where $\alpha_n,\ n\in\mathbb{Z}$ are Boson operators; L_0 is the energy operator; and $L_j,\ j>0$ are certain Casimir operators, [13,14], we propose to compute the following trace:

$$Trace = Tr\left(\exp(\sum_{j\geq 0} 2\pi i L_j)\exp(\sum_{n>0} A_n\alpha_{-n})\exp(\sum_{n>0} B_n\alpha_n)\right). \tag{3.2}$$

Therefore, we pose the following questions:

- How can one compute the trace in terms of the former traces?
- What is the representation theory interpretation of that?
- If the coefficients A_n, B_n are suitably chosen, what is the trace's physical interpretation in terms of string theory partition functions?

The character can be studied from different points of view. A direct way to calculate it could be to expand the exponentials inside the trace, apply basic commutation rules between the operators, and then compute the matrix elements. Some formulas in Lie theory, such as the Backer-Campbell-Hausdorff formula or the Wick formula, can also be helpful for calculation. Although this method can bring computational insights toward

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the above question, it hits with the ad hoc complexities and difficulties. One may expand the operators in the trace both in terms of Bosonic operators $\alpha_{\pm n}$, and also fermionic operators $\psi_j,\ j \in \frac{1}{2} + \mathbb{Z}$.

4. Main results

In order to interpret and compute a trace formula for (3.2) we make the following definition that is a natural generalization of the Fock space of level l, defined in 2.3.

Definition 4.1 *(Fock space $\mathfrak{F}^\infty$).* Consider the fermionic fields $\psi_r^{\pm,j},\ r \in \frac{1}{2} + \mathbb{Z},\ j \in \mathbb{Z}$ satisfying the natural Clifford commutation relations,

$$\begin{aligned} [\psi_r^{+,i}, \psi_s^{-,j}] &= \delta_{i,j}\delta_{r,-s} I \\ [\psi_r^{+,i}, \psi_r^{+,j}] &= [\psi_r^{-,i}, \psi_s^{-,i}] = 0. \end{aligned} \tag{4.1}$$

Set $\widehat{C}^\infty$ the Clifford algebra on these fields. By definition $\mathfrak{F}^\infty$ is a simple $\widehat{C}^\infty$-module generated by $|0\rangle$, such that $\psi_r^{\pm,j}|0\rangle = 0,\ r > 0$.

Next, we express a duality of Howe-type for the pair $(\mathfrak{gl}_\infty, \mathfrak{a}_\infty)$. In other words, the Fock space $\mathfrak{F}^\infty$ is a representation of both the Lie algebras $\mathfrak{gl}_\infty$ and $\mathfrak{a}_\infty$. Moreover, $\mathfrak{F}^\infty$ decomposes to the sum of their irreducible representation. Next, we are ready to present our first main result.

Theorem 4.2 *(Main Result. $(\mathfrak{gl}_\infty, \mathfrak{a}_\infty)$-Howe duality). There exists a decomposition,*

$$\mathfrak{F}^\infty = \bigoplus_\lambda L(\mathfrak{gl}_\infty, \lambda) \otimes L(\mathfrak{a}_\infty, \Lambda(\lambda)), \tag{4.2}$$

where λ runs over all generalized partitions. Besides, there is a character formula,

$$ch(\mathfrak{F}^\infty) = \prod_i^\infty \prod_j^\infty (1 + y_j x_i)(1 + y_j^{-1} x_i^{-1}), \tag{4.3}$$

where $x_i, y_j,\ i, j \in \mathbb{N}$ are variables.

Proof. In (2.20), we replace the operators e_{ij}^* by,

$$e_{ij}^* = \sum_{k=-\infty}^{\infty} :\psi_{\frac{1}{2}-i}^{+,k} \psi_{j-\frac{1}{2}}^{-,k}: , \qquad i, j \in \mathbb{Z}. \tag{4.4}$$

The map $\mathfrak{a}_\infty \longrightarrow End(\mathfrak{F}^\infty),\ E_{ij} \longmapsto e_{ij}^*$ defines a representation of $\mathfrak{a}_\infty$. Also, let us define the operators $e_*^{ij}(n)$ by:

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$$e_*^{ij} = \sum_{r\in\frac{1}{2}+\mathbb{Z}} :\psi^{+,i}_{-r}\psi^{-,j}_{r}:, \qquad (r \in \frac{1}{2}+\mathbb{Z},\ i,j\in\mathbb{Z}). \tag{4.5}$$

The map $\mathfrak{gl}_\infty \longrightarrow End(\mathfrak{F}^\infty),\ E^{ij} \longmapsto e_*^{ij}$ is a representation of $\mathfrak{gl}_\infty$. We need to check that the action of $\mathfrak{gl}_\infty$ and $\mathfrak{a}_\infty$ commute. That is,

$$[e_*^{ij}, e_{rs}^*] = \left[\left(\sum_{u\in\frac{1}{2}+\mathbb{Z}} :\psi^{+,i}_{-u}\psi^{-,j}_{u}:\right),\ \left(\sum_{l=1}^{\infty} :\psi^{+,l}_{\frac{1}{2}-r}\psi^{-,l}_{s-\frac{1}{2}}:\right)\right] = 0. \tag{4.6}$$

A joint highest weight vector in $\mathfrak{F}^\infty$, associated to a generalized partition $\lambda = (\lambda_1, ..., \lambda_j)$ w.r.t. the standard Borel of $\mathfrak{gl}_\infty \times \mathfrak{a}_\infty$, is $v_\lambda = \varpi^{+,s}_{\lambda_1}\varpi^{+,s}_{\lambda_2}...\varpi^{+,s}_{\lambda_i}\varpi^{-,s}_{-\lambda_j}\varpi^{+,s}_{-\lambda_{j+1}}...\varpi^{+,s}_{-\lambda_l}$, with weight λ w.r.t. $\mathfrak{gl}_\infty$ and weight $\Lambda(\lambda)$ w.r.t. $\mathfrak{a}_\infty$. By applying any root vector of $\mathfrak{gl}_\infty$ and $\mathfrak{a}_\infty$ to v_λ, it produces two identical $\psi^{\bullet,\bullet}_{\bullet}$ in the resulting monomial. As in the finite case any irreducible representation of $\mathfrak{gl}_\infty$ appears in $\mathfrak{F}^\infty$ and the multiplicity free decomposition $\mathfrak{F}^\infty = \bigoplus_\lambda L(\mathfrak{gl}_\infty, \lambda) \otimes L(\mathfrak{a}_\infty, \Lambda(\lambda))$ follows.

The character of $\mathfrak{gl}_\infty$ on $\mathfrak{F}^\infty$ is the trace of the operator $\prod_i x_i^{\epsilon_*^{ii}}$. The character of $\mathfrak{a}_\infty$ on $\mathfrak{F}^\infty$ is the trace of $\prod_j y_j^{\epsilon_{jj}^*}$. Therefore, $ch(\mathfrak{F}^\infty)$ is the trace of the product of the two operators. Calculating the trace of the operator $\prod_i x_i^{\epsilon_*^{ii}} \prod_j y_j^{\epsilon_{jj}^*}$ on both sides of (4.2) we get:

$$ch(\mathfrak{F}^\infty) = \bigoplus_\lambda ch\left(L(\mathfrak{gl}_\infty, \lambda)\right) \otimes ch\left(L(\mathfrak{a}_\infty, \Lambda(\lambda))\right) = \prod_i\prod_j (1+y_jx_i)(1+y_j^{-1}x_i^{-1}). \quad \square \tag{4.7}$$

The duality in Theorem 4.2 enables us to compute the trace formula (3.2) by computing it on each summand. Specifically, we have the following.

Theorem 4.3 *(Main result). We have the following formula for the trace in* (3.2)

$$\begin{aligned} Trace = \sum_\lambda \prod_{n\geq 0} (1+y_0y_1^{n+\frac{1}{2}}y_2^{(n+\frac{3}{2})^2}....)(1+y_0^{-1}y_1^{n-\frac{1}{2}}y_2^{-(n-\frac{1}{2})^2}....) \\ \prod_{r\geq 1} y_r^{p_r(\lambda)} \sum_{\mu\prec\lambda} S^{(A_n)}_{\lambda/\mu}(x_1,x_2,...)S^{(B_n)}_{\lambda^t/\mu}(x_1,x_2,...), \end{aligned} \tag{4.8}$$

where x_i and y_i are independent variables, and

$$p_r(\lambda) = \sum_l (\lambda_l - l + \frac{1}{2})^r + (-1)^{r+1}(l-\frac{1}{2})^r = \sum_l (m_l+\frac{1}{2})^r + (-1)^{r+1}(n_l+\frac{1}{2})^r \tag{4.9}$$

holds, where $(m_1, ..., m_s|n_1, ..., n_s)$ are Frobenius coordinates of λ. The effect of the coefficients A_n, B_n is absorbed in the variables $x_1, x_2, ...$, [see the proof for the explanation on dependence to the coefficients A_n, B_n].

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Remark 4.4. The dependence of the above trace to the coefficients A_n and B_n is somehow formal. The contribution to the trace coming from $Tr\big(\exp(\sum_{n>0} A_n\alpha_{-n}) \times \exp(\sum_{n>0} B_n\alpha_n)\big)$ is given in the last sum appearing in (4.8), where the effect of A_n, B_n is absorbed in the variables $x_1, x_2, \ldots$, [see [13] page 11 and 24, or [6] pages 25 and 70 for the notation].

Proof of Theorem 4.3. Let us denote,

$$\mathcal{L} = \exp(\sum_j 2\pi i L_j), \qquad T = \exp(\sum_n A_n\alpha_{-n})\exp(\sum_n B_n\alpha_n). \tag{4.10}$$

According to Theorem 4.2, we need to compute:

$$\sum_\lambda Tr\left(\mathcal{L}\big|_{L(\mathfrak{gl}_\infty,\lambda)}\right) Tr\left(T\big|_{L(\mathfrak{a}_\infty,\Lambda^{\mathfrak{a}_\infty}(\lambda))}\right). \tag{4.11}$$

We first compute the factor relevant to $\mathcal{L}$. Consider $v_\lambda = |\lambda\rangle$, the vector of weight λ. We have the formula:

$$\mathfrak{F}^\lambda = \bigotimes_n (1+\psi_{-n})(1+\psi^*_{-n})|\lambda\rangle. \tag{4.12}$$

By lemma 5.1 in [14], we also have:

$$\begin{aligned} \exp(2\pi i L_j)\psi_{-n}|\lambda\rangle &= \left(y_0 y_1^{n-\frac{1}{2}} y_2^{(n-\frac{1}{2})^2} \ldots \prod_{r\geq 1} y_r^{p_r(\lambda)}\right)\psi_{-n}|\lambda\rangle \\ \exp(2\pi i L_j)\psi^*_{-n}|\lambda\rangle &= \left(y_0^{-1} y_1^{n-\frac{1}{2}} y_2^{-(n-\frac{1}{2})^2} \ldots \prod_{r\geq 1} y_r^{p_r(\lambda)}\right)\psi^*_{-n}|\lambda\rangle, \end{aligned} \tag{4.13}$$

therefore

$$Tr\left(\mathcal{L}\big|_{L(\mathfrak{gl}_\infty,\lambda)}\right) = \prod_{n\geq 0}(1+y_0 y_1^{n+\frac{1}{2}} y_2^{(n+\frac{3}{2})^2}\ldots.)(1+y_0^{-1}y_1^{n-\frac{1}{2}}y_2^{-(n-\frac{1}{2})^2}\ldots.)\prod_{r\geq 1} y_r^{p_r(\lambda)}. \tag{4.14}$$

On the other hand, it is well known that,

$$Tr\left(T\big|_{L(\mathfrak{a}_\infty,\Lambda^{\mathfrak{a}_\infty}(\lambda))}\right) = \sum_{\mu\prec\lambda} S^{(A_n)}_{\lambda/\mu}(x_1,x_2,\ldots)S^{(B_n)}_{\lambda^t/\mu}(x_1,x_2,\ldots), \tag{4.15}$$

where $S^{(A_n)}_{\lambda/\mu}$ is the skew Schur function $S_{\lambda/\mu}$ specialized to the case in which the symmetric power functions p_n equal nA_n. Similarly, $S^{(B_n)}_{\lambda/\mu}$ is the skew Schur function $S_{\lambda/\mu}$ specialized to the case in which the symmetric power functions p_n equals nB_n, [see [13] page 11 and 24 for the notation]. □

5. Application

The string theory partition function of toric CY 3-folds can be formulated from their tropical diagram by basic combinatorial rules of the topological vertex, [18], [19], [20], [21], [22], [23], [24]. These partition functions can also be described combinatorially by specific vector fields acting on the tropical diagram that fixes the vertices. The symmetry group of the action is a unitary group $U(N)$ and is called the gauge group. One may consider tropical diagrams that are more complicated, especially when there are many cells. Thus, one may ask what happens if we apply infinitely many twists in the vertex operator. Below we explain this motivating question.

The partition function of the $U(1)$ theory can be written in the form:

$$Z(\tau, m, \epsilon) = Tr\left(Q_\tau^{L_0} \exp\left(\sum_{n\geq 1} \frac{Q_m^n - 1}{n(q^{\frac{n}{2}} - q^{-\frac{n}{2}})}\alpha_n\right) \exp\left(\sum_{n\geq 1} \frac{Q_{-m}^n - 1}{n(q^{\frac{n}{2}} - q^{-\frac{n}{2}})}\alpha_{-n}\right)\right). \tag{5.1}$$

Using the commutation relation of $\alpha_\pm$ it can be written as follows, cf. [20–22],

$$Z(\tau, m, \epsilon) = \prod_k (1 - Q_\tau^k)^{-1} \prod_{i,j} \frac{(1 - Q_\tau^k Q_m^{-1} q^{i+j-1})(1 - Q_\tau^k Q_m q^{i+j-1})}{(1 - Q_\tau^k q^{i+j-1})}. \tag{5.2}$$

Besides, the partition function in (5.1) can be generalized to

$$\begin{aligned} &Z(\tau, m, \epsilon, t) \\ &= Tr\left(Q_\tau^{L_0} e^{\sum_n t_n L_n} \exp\left(\sum_{n\geq 1} \frac{Q_m^n - 1}{n(q^{\frac{n}{2}} - q^{-\frac{n}{2}})}\alpha_n\right) \exp\left(\sum_{n\geq 1} \frac{Q_{-m}^n - 1}{n(q^{\frac{n}{2}} - q^{-\frac{n}{2}})}\alpha_{-n}\right)\right). \end{aligned} \tag{5.3}$$

Also, in the limit $m \mapsto 0$ we obtain,

$$Z(\tau, m = 0, \epsilon, t) = Tr\left(Q_\tau^{L_0} e^{\sum_n t_n L_n}\right). \tag{5.4}$$

We can write the partition function in terms of the Gromov-Witten potentials:

$$Z(\tau, m, \epsilon) = \exp\left(\sum_{g\geq 0} \epsilon^{2g-2} F_g\right), \tag{5.5}$$

where

$$e^{F_1} = \prod_k (1 - Q_\tau^k)^{-1} \left(\frac{((1 - Q_\tau^k)^2 Q_m^{-1})(1 - Q_\tau^k Q_m)^2}{(1 - Q_\tau^k)^4}\right)^{\frac{1}{24}}. \tag{5.6}$$

Remark 5.1. The case in (5.3) with finitely many L_j appears in the topological string partition functions associated with toric CY 3-folds where the web diagram has a $M \times N$ cell structure, where $M, N \in \mathbb{N}$; see [25,5,26–28,9–11] for details. However, the computation of the trace in the presence of infinitely many Casimirs L_j becomes extremely difficult when one tries to expand all the exponential operators.

6. Conclusions

An extension of the calculation of the trace of the vertex operator with infinitely many Casimirs is presented based on a duality of Howe type for the pair $(\mathfrak{a}_\infty, \mathfrak{gl}_\infty)$. The trace formula applies to an extension of the topological string theory partition function of CY 3-folds.

Declaration of competing interest

There is no competing interest.

Data availability

No data was used for the research described in the article.

Acknowledgements

A previous version of this work was presented in the "Focus Program on Analytic Function Spaces and their Applications July 1 - December 31, 2021", which took place at the Fields Institute for Research in Mathematical Sciences during the pandemic. The authors would like to thank the organizers for their support during this program.

The authors would like to thank the Isaac Newton Institute for Mathematical Sciences, Cambridge, for support and hospitality during the program [Dispersive hydrodynamics: mathematics, simulation, and experiments, with applications in nonlinear waves HYD2, July 2022 to December 2022] where the final part of the work on this paper was carried out. This work was supported by EPSRC grant number EP/K032208/1.

The authors thank Amer Iqbal, Antonio Moro, Yuji Kodama, and Muteeb Nouman for valuable discussions.

Appendix A. The Lie algebras $\mathfrak{a}_\infty$, $\mathfrak{c}_\infty$ and $\mathfrak{d}_\infty$

We introduce the three infinite-dimensional Lie algebras whose representations are crucial in string theory. The reference for this appendix is [17] Section 5.4, where all the materials discussed can be found in more detail.

(1) Lie Algebra $\mathfrak{a}_\infty$: Let $\mathfrak{a}_\infty = \widehat{\mathfrak{gl}}_\infty = \mathfrak{gl}_\infty \oplus \mathbb{C}K$ with the bracket,

$$[X + cK, Y + dK] = [X, Y]' + \tau(X, Y)K. \tag{A.1}$$

The function $\tau(X, Y) = Tr([J, X]Y)$ is called a cocycle function, where $J = \sum_{j\leq 0} E_{jj}$ and $[.,.]'$ is the bracket of $\mathfrak{gl}_\infty$. We have the degree gradation $\mathfrak{gl}_\infty = \bigoplus_j \mathfrak{gl}_\infty^j$, where j runs over integers and it is called $\mathbb{Z}$-principal gradation. The degree of E_{ij} would be $j - i$. Besides, we have a decomposition:

$$\mathfrak{a}_\infty = \mathfrak{a}_\infty^+ \oplus \mathfrak{a}_\infty^0 \oplus \mathfrak{a}_\infty^-, \qquad \mathfrak{a}_\infty^\pm = \bigoplus_{j>0} \mathfrak{gl}_\infty^{\pm j},\ \mathfrak{a}_\infty^0 = \mathfrak{gl}_\infty^0 \oplus \mathbb{C}K. \tag{A.2}$$

The root system of $\mathfrak{a}_\infty$ is

$$R = \{\epsilon_i - \epsilon_j \ |i \neq j\}, \qquad \epsilon_i(E_{ii}) = \delta_{ij},\ \epsilon_i(K) = 0. \tag{A.3}$$

The set $\Pi = \{\epsilon_i - \epsilon_{i+1} \ |i \in \mathbb{Z}\}$ is a fundamental system for $\mathfrak{a}_\infty$ with corresponding co-roots $\{H_i^{\mathfrak{a}} = E_{ii} - E_{i+1,i+1} + \delta_{i,0}K\}$. We denote by $\Lambda_j^{\mathfrak{a}}$ the j-th fundamental weight of $\mathfrak{a}_\infty$, i.e. $\Lambda_j^{\mathfrak{a}}(H_i^{\mathfrak{a}}) = \delta_{ij},\ (i \in \mathbb{Z}),\ \Lambda_j^{\mathfrak{a}}(K) = 1$. A straightforward computation gives:

$$\begin{aligned} \Lambda_j^{\mathfrak{a}} &= \Lambda_0^{\mathfrak{a}} - \sum_{i=j+1}^{0} \epsilon_i \qquad j < 0 \\ \Lambda_j^{\mathfrak{a}} &= \Lambda_0^{\mathfrak{a}} + \sum_{i=1}^{j} \epsilon_i \qquad j \geq 1. \end{aligned} \tag{A.4}$$

The irreducible highest weight representation of $\mathfrak{a}_\infty$ of the highest weight Λ is denoted by $L(\mathfrak{a}_\infty, \Lambda)$.

(2) Lie algebra $\mathfrak{c}_\infty$: Let $V = \bigoplus_{j\in\mathbb{Z}} \mathbb{C}v_j$ be the vector space generated by the vectors v_j, where $E_{ij}v_j = v_i$. Consider the symmetric bilinear form,

$$C(v_i, v_j) = (-1)^i \delta_{i,1-j}, \forall i, j \in \mathbb{Z}. \tag{A.5}$$

Set $\mathfrak{c}_\infty = \bar{\mathfrak{c}}_\infty \oplus \mathbb{C}K$, where

$$\bar{\mathfrak{c}}_\infty = \{X \in \mathfrak{gl}_\infty \mid C(X(u), v) + C(u, X(v)) = 0\}. \tag{A.6}$$

A fundamental system for $\mathfrak{c}_\infty$ is given by $\{-2\epsilon_1, \epsilon_i - \epsilon_{i+1};\ i \geq 0\}$ with simple co-roots:

$$\begin{aligned} H_i^{\mathfrak{c}} &= E_{ii} + E_{-i,-i} = E_{i+1,i+1} - E_{1-i,1-i}, \\ H_0^{\mathfrak{c}} &= E_{00} - E_{11} + K. \end{aligned} \tag{A.7}$$

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The j-th fundamental weight for $\mathfrak{c}_\infty$ is defined by the same as the case for $\mathfrak{a}_\infty$ and explicitly written as follows,

$$\Lambda_j^{\mathfrak{c}} = \Lambda_0^{\mathfrak{c}} + \sum_{i=1}^{j} \epsilon_i, \qquad j \geq 1. \tag{A.8}$$

(3) Lie Algebra $\mathfrak{d}_\infty$: Define the Lie algebra $\mathfrak{d}_\infty = \bar{\mathfrak{d}}_\infty \oplus \mathbb{C}K$ where,

$$\bar{\mathfrak{d}}_\infty = \{X \in \mathfrak{gl}_\infty | D(X(u), v) = D(u, X(v))\}, \tag{A.9}$$

and where $D(v_i, v_j) = \delta_{i,1-j}$. It has the fundamental system $\{\pm\epsilon_1 - \epsilon_2, \epsilon_i - \epsilon_{i+1},\ i \geq 2\}$ with simple co-roots:

$$\begin{aligned} H_i &= E_{ii} - E_{-i,-i} - E_{i+1,i+1} - E_{1-i,1-i}, \\ H_0 &= E_{00} - E_{-1,-1} - E_{22} - E_{11} + 2K. \end{aligned} \tag{A.10}$$

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