A high-gain nonlinear observer for extended bilinear systems in block form

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Mohammad Reza-Rahmati[1,+] **and Gerardo Flores**[2,*,+]

[1]Universidad De La Salle Bajío, Engineering department, León, Guanajuato, 37150, Mexico.
[2]Centro de Investigaciones en Óptica, Laboratorio de Percepción y Robótica [LAPyR], León, 37150, Mexico.
[*]gflores@cio.mx
[+]These authors contributed equally to this work.

ABSTRACT

This paper presents the design of a high-gain nonlinear observer for bilinear systems in the block form. We prove the uniform exponential stability of the observer error by finding a concrete exponential bound. Our method proves the uniform boundedness of the observer error that is stronger than state-of-the-art methods. In particular, we extend the result of O. I. Goncharov on the exponential stability of high-gain bilinear observers into observers of block forms. We also study and simulate two examples of observers for bioreactor systems in two and three dimensions. Results show the effectiveness of the proposed approach.

1 Introduction

Among the different nonlinear observers' approaches, perhaps the high gain observers (HGO) are among the most important in the control community. High gain observers have been extensively used in the design of output feedback control. The structure of an HGO involves a positive parameter design, namely, observer gain θ that has to be taken high enough to guarantee stability. This paper follows certain hierarchies of high gain observer systems to study the uniform observation problem for an HGO in block matrix form when the blocks are all the exact sizes.

A high gain observer is a dynamical system with a corrective term involving the observer gain factor, $\theta > 0$, which must be taken big enough to guarantee the convergence of observation error. Usually, this parameter appears by positive exponents on different lines or subspaces of the state space, where its highest power is the dimension of the system. In this case, the gain factor θ enters in numerical estimates problem for the system when a relatively high value is under demand.

There can be a classification of high gain systems into two branches. The first is the class of uniformly observable systems, i.e., high gain systems observable from any point. The second class is non-uniform observer systems, in which the input is required to satisfy excitation conditions to ensure observability. An example of the latter kind is considered in[27] in which a class of Multi-Input/Multi-Output observers where the gain factor varies with time.

A photobioreactor is a biological system where a biochemical process is carried out using a living organism. The system is sensitive to change under ambient conditions, and the optimal conditions of the microalgae culture have to be satisfied. The system is designed to control the pH, temperature, and oxygen. We will present examples in dimensions 2 and 3 with experimental simulations. The asymptotic stability of the systems is studied in the final part of the article.

1.1 Related work

Recently some solutions has been made toward the stability of nonlinear systems,[2],[3],[4],[5],[6],[7],[8],[9],[10],[11],[12],[13],[14],[15]. Bilinear systems are a particular case of nonlinear systems, prevalent for modeling biological systems. In some of these bilinear systems, various kinds of observation problems are comparatively complicated. Among these is the uniform observation problem for extended bilinear systems in block form. The problem here is to find a uniform estimate for the state vector.

The uniform observability question for bilinear systems is of intense interest from a practical point of view. When the problem is solvable, we say the system is uniformly observable. One way to attain this is to add sufficient constraints to the system. Different criteria for the uniform observability has been studied in[16],[17],[18],[19],[20],[21],[22],[23],[24],[25],[26],[53]. One can put the constraints on the input $u = u(t)$ like to be sufficiently smooth or even linear, bounded or having known estimates, to obtain the uniform observation.

This paper is motivated by the two works[1],[27] on solving the observer problem for specific bilinear systems in dimension n. The reference[27] studies the error dynamics of the extended high gain bilinear system in block form in the presence of perturbation noise. The authors prove the existence of a bound on the high gain observer's error dynamics. However, the bound is complicated, and their proof is very long. In[1], the author proves the uniform and exponential stability of the bilinear system in the scalar variables. In general, uniform stability is a stronger condition for the observer. The proof in[1] is solid and organized to present the theoretical insight into the high gain observer designs.

1.1.1 Contribution

We generalize the method of[1] to prove the uniform observation problem for the bilinear system in a block form when the blocks have the same size.

On the other hand, in[27], a form of boundedness stability for observers of systems in the form $\dot{x} = F(u,x)x + \phi(u,x)$ has been proved. We prove a stronger statement with more abstract methods. We demonstrate a nonlinear observer's uniform exponential stability by finding a more concrete exponential bound (by a different approach). Moreover, our method proves the uniform boundedness of the error, a stronger result than the one in[27]. We use the feedback linearization of observer systems,[14],[28],[29],[12] to prove a similar uniform exponential stability theorem in nonlinear observer systems. Besides, to investigate our method's behavior, we apply the observer to two versions of the bioreactor model; the bioreactor explains a bacteria growth model and population distribution,[30]. Simulations are presented for SI-SO and MI-SO bioreactor systems.

1.2 Content

The remainder of this paper is organized as follows. Section 2 presents observer design for systems in bilinear form. Section 3 demonstrates the exponential stability of the proposed observer for bilinear systems of the form,

$$\dot{x} = Ax + (Bu)x, \qquad y = Cx.$$

Then, in Section 4 results are extended for bilinear systems in the form,

$$\dot{x} = F(u,x)x + \phi(u,x), \qquad y = Cx.$$

We present two cases of study in Section 5: SI-SO and MI-SO versions of the bioreactor system, where the proposed observer is applied and simulated. Finally, some conclusions and future work are given in Section 6.

2 Problem setting

In[1], O. I. Goncharov poses the problem of uniform observation for a scalar bilinear system:

$$\dot{x} = Ax + u(Bx + D), \qquad y = Cx \tag{1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^1$, $A, B \in \mathbb{R}^{n\times n}$, $C \in \mathbb{R}^{1\times n}$, and $D \in \mathbb{R}^{n\times 1}$ are constant matrices. One may assume without loss of generality that A, B and C are in canonical controllability forms[3], that is,

$$A = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ 0 & 0 & 0 & 1 & \\ & . & .. & & \\ & . & .. & & 1 \\ a_1 & a_2 & & \dots & a_q \end{pmatrix}, \ B = \begin{pmatrix} b_{11} & 0 & 0 & \dots & 0 \\ b_{21} & b_{22} & 0 & \dots & 0 \\ b_{31} & b_{32} & b_{33} & \dots & \\ & . & \dots & & \\ & . & \dots & & \\ b_{q1} & b_{q2} & & \dots & b_{qq} \end{pmatrix}, \ C = [1\ 0\ 0\ \dots\ 0]. \tag{2}$$

An observer for system (1) is

$$\dot{\hat{x}} = A\hat{x} + uB\hat{x} + K(y - C\hat{x}), \qquad \hat{x}, y \in \mathbb{R}^n. \tag{3}$$

where $K = [k_1, \dots, k_q]^T$ is a column block matrix of feedback coefficients, and $u(t)$ is bounded[1]. Error $e = x - \hat{x}$ satisfies

$$\dot{e} = A_0 e + uBe \tag{4}$$

where

$$A_0 = A - KC = \begin{pmatrix} -k_1 & 1 & 0 & \dots & 0 \\ -k_2 & 0 & 1 & \dots & 0 \\ -k_3 & 0 & 0 & 1 & \\ & . & .. & & \\ & . & .. & & 1 \\ -k_q + a_1 & a_2 & & \dots & a_q \end{pmatrix} \tag{5}$$

One uses the standard hierarchy of high-gain observers such that the eigenvalues of the matrix A_0 can be chosen to be proportional to a high gain factor [see[1] lemma 1]. The hierarchy states that if the feedback coefficients k_i is big enough, then eigenvalues of the matrix A_0 are all real and negative (A is Hurwitz), and the contribution of the coefficients a_i in the characteristic polynomial of A can be neglected, see[31],[32]. Goncharov proves that the error $e(t)$ is exponentially bounded, that is, there exist constants $M, a_0 > 0$ and a polynomial $P(\theta)$ such that

$$\| e \| \leqslant P(\theta) \exp\left(-(\theta - a_0)t\right), \qquad (\theta > M). \tag{6}$$

where θ is called *the high gain factor*. We generalize this result for similar systems where the matrices A, B, and C are in block forms with blocks of the same size. We state that in the following problem statement.

2.1 Problem statement

This section presents the problem statement related to the design of observers, divided into two cases: the linear version and a nonlinear version of a dynamic system. Let us consider first the case of the linear one.

Problem 1 (Linear case)**.** *Design an observer for a bilinear system of the form,*

$$\dot{x} = Ax + (Bu)x, \qquad y = Cx, \tag{7}$$

where $x = [x_1,\ x_2,\ \ldots,\ x_q]$, with $x_i \in \mathbb{R}^r, u \in \mathbb{R}^r$, and $A,\ B \in \mathbb{R}^{qr \times qr}$ and $qr = n$. Notice that the system above has the same shape as (2) but with entries given by block matrices of size $r \times r$. Besides, $u = diag[u_1 I_r,\ \ldots,\ u_q I_r]$, $C = [I_r,\ 0,\ \ldots,\ 0]$ hold.

In addition, we consider the nonlinear multi-input single-output (MI-SO) system, for which one has to design an observer. The problem is as follows.

Problem 2 (Nonlinear case)**.** *Consider the nonlinear multi-input single-output (MI-SO) system,*

$$\dot{x} = F(u,x)x + \phi(u,x), \qquad y = Cx, \tag{8}$$

where $x \in \mathbb{R}^{qr}$, and

$$F(u,x) = \begin{pmatrix} 0 & F_1 & 0 & \ldots & 0 \\ 0 & 0 & F_2 & \ldots & 0 \\ 0 & 0 & 0 & F_3 & \\ & . & .. & & \\ & . & .. & & F_{q-1} \\ 0 & 0 & & \ldots & 0 \end{pmatrix}, \qquad \phi(u,x) = \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \ldots \\ \phi_q \end{bmatrix},$$

where $F_k = F_k(u, x_1, \ldots, x_q)$, $\phi_k = \phi_k(u, x_1, \ldots, x_k)$, $u \in \mathbb{R}^r$, $x_i \in \mathbb{R}^r$ and $C = [I_r\ \ 0 \ldots\ 0]$. The problem is to find an observer for the aforementioned bilinear MI-SO system.

3 Solution to Problem 1: A high gain observer for bilinear systems of the form $\dot{x} = Ax + (Bu)x$

In this section, we present a solution to the observation problem 1 for a bilinear system of the form,

$$\dot{x} = Ax + (Bu)x, \quad y = Cx \tag{9}$$

where

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_q \end{bmatrix},\ x_i \in \mathbb{R}^r,\ qr = n$$
$$A = \begin{pmatrix} 0 & I_r & 0 & \dots & 0 \\ 0 & 0 & I_r & \dots & 0 \\ 0 & 0 & 0 & I_r & \\ & . & .. & & \\ & . & .. & & I_r \\ A_1 & A_2 & & \dots & A_q \end{pmatrix},\ A_k \in \mathbb{R}^{r\times r} \tag{10}$$
$$B = \begin{pmatrix} B_{11} & 0 & 0 & \dots & 0 \\ B_{21} & B_{22} & 0 & \dots & 0 \\ B_{31} & B_{32} & B_{33} & \dots & \\ & . & \dots & & \\ & . & \dots & & \\ B_{q1} & B_{q2} & & \dots & B_{qq} \end{pmatrix},\ B_{ij} \in \mathbb{R}^{r\times r}$$

with that aim, we consider the following assumptions.

Assumption 1. *The matrices A_i pairwise commute, i.e., $A_iA_j = A_jA_i,\ i,j = 1,\dots,q$, and $C = [I_r,\ 0,\ \dots\ ,\ 0]$.*

Assumption 2. *The control input is structured as $u = diag[u_0I_r, u_1I_r, \dots, u_{q-1}I_r]$, where $u_i \in \mathbb{R}$ are assumed to be bounded.*

Considering assumptions 1 and 2 the proposed observer for (9) is given by

$$\dot{\hat{x}} = A\hat{x} + (Bu)\hat{x} + K(y - C\hat{x}), \qquad \hat{x}, y \in \mathbb{R}^n \tag{11}$$

where $K = [K_1I_r,\ \dots\ , K_qI_r]^T$ is a column block matrix of feedback coefficients. The error $e = x - \hat{x}$ has dynamics, given by

$$\dot{e} = A_0e + (Bu)e \tag{12}$$

where

$$A_0 = A - KC = \begin{pmatrix} -K_1I_r & I_r & 0 & \dots & 0 \\ -K_2I_r & 0 & I_r & \dots & 0 \\ -K_3I_r & 0 & 0 & I_r & \\ & . & .. & & \\ & . & .. & & I_r \\ -K_qI_r + A_1 & A_2 & & \dots & A_q \end{pmatrix}. \tag{13}$$

The matrix A_0 can be decomposed as,

$$A_0 = \bar{A} + \hat{A},\ \text{with}\quad \bar{A} = \begin{pmatrix} -K_1I_r & I_r & 0 & \dots & 0 \\ -K_2I_r & 0 & I_r & \dots & 0 \\ -K_3I_r & 0 & 0 & I_r & \\ & . & .. & & \\ & . & .. & & I_r \\ -K_qI_r & 0 & & \dots & 0 \end{pmatrix}, \hat{A} = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ & . & .. & & \\ & . & .. & \dots & \\ A_1 & A_2 & & \dots & A_q \end{pmatrix}. \tag{14}$$

In the following, we present and prove the theorem that is the spirit of the main result. It generalizes Lemma 1 in the work of O. I. Goncharov[1]. The theorem states that the feedback coefficients K_i above can be chosen high enough such that the eigenvalues of A_0, (denoted by Sp(A_0)) are all negative, distinct, and proportional to the high-gain factor θ,[16],[53]. Besides, the effect of the matrix $\hat{A}$ on the sign of the eigenvalues of A_0 can be neglected.

Theorem 3.1. *Assume we start from a set of n real numbers,*

$$Sp(\bar{A}) = \{\bar{\alpha}_1, \dots, \bar{\alpha}_n\},\ \bar{\alpha}_i \neq \bar{\alpha}_j,\ i \neq j \tag{15}$$

and consider the feedback coefficients as

$$K_i(\theta) = \overline{K_i}\theta^i + o(i) \tag{16}$$

where θ is the gain factor and the coefficients $\overline{K_i}$ are defined via

$$\bar{\rho}(s) := \prod_{i=1}^{n}(s - \bar{\alpha}_i) = |s^q I_r + \overline{K_1} s^{q-1} I_r + \ldots + \overline{K_q} I_r|; \tag{17}$$

and $o(i)$ are the terms of degree strictly less than i in θ. Then, the characteristic polynomial of $A_0 = A - KC$ in (13) corresponding to the error observer (12) is of the form

$$\rho(s) = |s^q I_r + K_1(\theta) s^{q-1} I_r + \ldots + K_q(\theta) I_r| \tag{18}$$

where $K_i(\theta)$ was defined above.

Before presenting the proof of Theorem 3.1, we present the following proposition required for the proof.

Proposition 3.1 ([33],[34],[35])**.** *If*

$$A = \begin{pmatrix} A_{11} & . & \ldots & A_{1q} \\ A_{21} & . & \ldots & \\ & . & .. & \\ & . & ..... & \\ A_{q1} & A_{q2} & \ldots & A_{qq} \end{pmatrix} \tag{19}$$

is a $q \times q$ block matrix of $A_{ij} \in \mathbb{R}^{r \times r}$ such that $A_{ij}A_{kl} = A_{kl}A_{ij}$ for all entries, then [1]

$$|A| = \left| \sum_{\pi \in S_k} A_{1\pi(1)} \ldots A_{k\pi(k)} \right|. \tag{20}$$

Remark 1. *The above formula has been proved by induction and the axiomatic definition of determinant functions in the different references. It can also be proved using the method of Schur complements in matrix analysis, see for instance references* [36],[37],[38],[39],[40],[41],[42].

We are ready to present the proof of Theorem 3.1.

Proof of Theorem 3.1. Let us consider the matrix,

$$sI - A_0 = \begin{pmatrix} (K_1 + sI_r) & -I_r & 0 & \ldots & 0 \\ K_2 I_r & sI_r & -I_r & \ldots & 0 \\ K_3 I_r & 0 & sI_r & -I_r & \\ & . & .. & & \\ & . & .. & & -I_r \\ K_q I_r - A_1 & -A_2 & & \ldots & sI_r - A_q \end{pmatrix}. \tag{21}$$

Expanding and grouping the block entries as the Proposition 3.1, similar to the argument in[1], we obtain

$$|sI - A_0| = \left| sD(N_1(1)) + \sum_{k=1}^{q-1} (-1)^{k-1} D(N_k(s)) + (-1)^q A_1 (-1)^{q-1} D(N_q)(s) \right| \tag{22}$$

where

$$N_p = \begin{pmatrix} sI_r & -I_r & 0 & \ldots & 0 \\ 0 & sI_r & -I_r & \ldots & 0 \\ 0 & 0 & sI_r & -I_r & \\ & . & .. & & \\ & . & .. & & -I_r \\ -A_{p+1} & -A_{p+2} & & \ldots & sI_r - A_q \end{pmatrix} \tag{23}$$

[1]We denote the sum inside the above determinant as $D(A) = \sum_{\pi \in S_k} A_{1,\pi(1)} \ldots A_{k,\pi(k)}$, associated to the block matrix A in (19), where S_k is the symmetric group on k elements.

is the $(n-k)$ minor in the lowest left corner. We note that the number of the blocks producing the exponent of s in $D(N_k(s))$ reduces as k grows. Therefore, we have

$$D(N_k(s)) = s^{q-k} I_r + o(s^{q-k}). \tag{24}$$

Replacing (24) in (22) we obtain, $|sI - A_0| = \left|s^q I_r + \Sigma_k (-1)^{q-k}(s^{n-k} I_r + o(s^{n-k})\right|$. The proof of the Theorem is complete. $\square$

Corollary 3.1.1. *Assume the setting of Theorem 3.1 holds. Then in the decomposition of A_0 as (14) (on the matter relevant to stability), the effect of the second factor, i.e matrices A_j, $j = 1, \dots, q$ in (**??**) what is this equation? on $Sp(A_0)$, can be neglected subject to the condition that the high gain factor θ is big enough. In this case*

$$Sp(A_0) = \{\theta\alpha_1(\theta),\ \theta\alpha_2(\theta),\ \dots,\ \theta\alpha_n(\theta)\} \tag{25}$$

such that $\lim_{\theta\to\infty} \alpha_i(\theta) = \bar{\alpha}_i$.

Proof. The proof is mainly that of Goncharov in the scalar case (Lemma 1 in reference[1]), plus the extended determinant formula used in Theorem 3.1. The elements $\theta\alpha_1(\theta),\ \theta\alpha_2(\theta),\ \dots,\ \theta\alpha_n(\theta)$ are the roots of the characteristic polynomial of A_0. Therefore, the characteristic polynomial of A_0 is as follows,

$$\rho(s) = \prod_{i=1}^{n}(s - \theta\alpha_i) = \theta^n \prod_{i=1}^{n}\left(\frac{s}{\theta} - \alpha_i\right) = \theta^n \left|\bar{s}^q I_r + \frac{K_1(\theta)}{\theta}\bar{s}^{q-1} I_r + \dots\right|, \tag{26}$$

where we have used the simple relation $\bar{s} = s/\theta$. Then, we have $\lim_{\theta\to\infty} \frac{K_i(\theta)}{\theta^i} = \overline{K_i}$ for all i. In other words $\frac{\rho(s)}{\theta^n} \to \bar{\rho}(s)$. It follows that the eigenvalues satisfy $\lim \alpha_i(\theta) \to \overline{\alpha}_i$ as $\theta \to +\infty$. $\square$

Remark 2. *From the previous discussion, one can conclude that with the above choice of gain factors K_i of the feedback system, the spectrum of A_0 tends to $-\infty$ and is proportional by a gain factor θ. Thus, in the remainder, we always assume that the condition above on feedback coefficients is satisfied.*

Now, we are ready to study the stability of the dynamics of the observer's error (12). For that, first, notice that a general solution for a bilinear system,

$$\dot{x} = Ax + (Bu)x \tag{27}$$

is given by

$$x(t) = \exp(At)x_0 + \int_0^t u(s)\exp(A(t-s))Bx(s)ds. \tag{28}$$

A similar formula holds for the error dynamics. If θ is high enough, the eigenvalues of A_0 satisfy $\lambda_i \le -\theta,\ 1 \le i \le n$. Therefore,

$$\|\exp(At)\| \le P(\theta)\exp(-\theta t) \tag{29}$$

where P is a polynomial that tends to ∞ with θ,[1]. By Lagrange interpolation for matrix functions ([43] Chapter 5) one can find an expression

$$\exp(At) = \sum_{l=0}^{n-1} A^l P_l(t) \tag{30}$$

for specific polynomials P_l. Therefore, we must analyze the matrix $\exp(At)B$. We do that in the following.

Theorem 3.2. *Assume the hierarchy of Theorem 3.1 with $\bar{\alpha}_i < -1$. Then the error dynamics in (12) is,*

$$\dot{e} = A_0 e + (Bu)e \tag{31}$$

associated to the observer (11) is exponentially stable, i.e. there exists constants $M, a_0 > 0$ and a polynomial $P(\theta)$ such that,

$$\| e \| \leqslant P(\theta)e^{-(\theta - a_0)t}, \qquad (\theta > M). \tag{32}$$

Our strategy to prove Theorem 3.2 is to extend Lemmas 1-4 in reference[1] to block matrices of the form in (10). For that, first let $A(\theta)=[a_{ij}(\theta)]$ and $B(\theta)=[b_{ij}(\theta)]$ be two $n\times m$ matrices with polynomial entries in θ, and define the relation,

$$A(\theta)\lesssim B(\theta) \tag{33}$$

if there exists non-negative constants K and M such that the entries of these matrices satisfy

$$a_{ij}(\theta)\leq Kb_{ij}(\theta),\qquad i=1,\dots,n,\ j=1,\dots,m.\qquad \forall\theta\geq M. \tag{34}$$

Such a condition essentially means that $\deg a_{ij}(\theta)\leq\deg b_{ij}(\theta)$ pairwise for all i and j. Thus, the following properties hold [see Definition 1 in reference [1] page 1602]:

1) the relation $\lesssim$ is transitive and reflexive,
2) if $A(\theta)$ and $B(\theta)$ are in block form then $A(\theta)\lesssim B(\theta)$ iff the blocks of the matrices satisfy the same relation,
3) if $A(\theta)\lesssim B(\theta)$ and $C(\theta)\lesssim D(\theta)$, then $A(\theta)+C(\theta)\lesssim B(\theta)+D(\theta)$,
4) if $A(\theta)\lesssim B(\theta)$ and $c\in\mathbb{R}$, then $cA(\theta)\lesssim B(\theta)$,
5) if $A(\theta)\lesssim B(\theta)$ and $C(\theta)\lesssim D(\theta)$, then $AC(\theta)\lesssim BD(\theta)$.

The proofs of the properties 1-5 are given in reference [1]. We employ the relation $\lesssim$ and the properties 1-5 in the following two lemmas.

Lemma 3.1. *Let the matrix A_0 has the form given in (14), and let the feedback coefficients $K_i(\theta)$ be chosen from Theorem 3.1, then*

$$A_0^p\lesssim\begin{pmatrix}\theta^pI_r & \theta^{p-1}I_r & \dots & \theta^{p-n+1}I_r\\ \theta^{p+1}I_r & \theta^pI_r & \dots & \theta^{p-n+2}I_r\\ \dots & & .. & \dots\\ \dots & & .. & \\ \theta^{p+n-1}I_r & \theta^{p+n-2}I_r & \dots & \theta^pI_r\end{pmatrix}=:D^p. \tag{35}$$

Moreover we have

$$A_0^pB\lesssim\begin{pmatrix}\theta^pI_r & \theta^{p-1}I_r & \dots & \theta^{p-n+1}I_r\\ \theta^{p+1}I_r & \theta^pI_r & \dots & \theta^{p-n+2}I_r\\ \dots & & .. & \dots\\ \dots & & .. & \\ \theta^{p+n-1}I_r & \theta^{p+n-2}I_r & \dots & \theta^pI_r\end{pmatrix}\begin{pmatrix}I_r & 0 & \dots & 0\\ I_r & I_r & \dots & 0\\ \dots & & .. & \dots\\ \dots & & .. & \\ I_r & I_r & \dots & I_r\end{pmatrix}\lesssim D^p. \tag{36}$$

Proof of Lemma 3.1. To prove this lemma, we use the properties given above. The proof of the first inequality is by induction on p. The case $p=1$ is trivial from the general form of the matrix A_0 in (13). For the inductive step, one uses the property 5) to pass from $A_0^{p-1}\lesssim D^{p-1}$ to $A_0^p\lesssim D^p$. According to property 2) above, one compares the matrices in block form in the same way as the scalar case. To prove the second inequality, we use the first inequality and again with property 5), and we have

$$A_0^pB\lesssim D^p\begin{pmatrix}I_r & 0 & \dots & 0\\ I_r & I_r & \dots & 0\\ \dots & & .. & \dots\\ \dots & & .. & \\ I_r & I_r & \dots & I_r\end{pmatrix}=\begin{pmatrix}\theta^pI_r & \theta^{p-1}I_r & \dots & \theta^{p-n+1}I_r\\ \theta^{p+1}I_r & \theta^pI_r & \dots & \theta^{p-n+2}I_r\\ \dots & & .. & \dots\\ \dots & & .. & \\ \theta^{p+n-1}I_r & \theta^{p+n-2}I_r & \dots & \theta^pI_r\end{pmatrix}\begin{pmatrix}I_r & 0 & \dots & 0\\ I_r & I_r & \dots & 0\\ \dots & & .. & \dots\\ \dots & & .. & \\ I_r & I_r & \dots & I_r\end{pmatrix}\lesssim D^p. \tag{37}$$

The proof is complete. □

The inequalities in Lemma 3.1 can be used to construct estimates for the exponential e^{A_0t}.

Lemma 3.2. *Assume the matrices A_0 and B are given as in (13) and (10) respectively, and let the feedback coefficients $K_i(\theta)$ be chosen in accordance with Theorem 3.1, then*

$$A_0^pB\lesssim\begin{pmatrix}I_r & \theta^{-1}I_r & \dots & \theta^{p-n+1}I_r\\ \theta^{+1}I_r & I_r & \dots & \theta^{-n+2}I_r\\ \dots & & .. & \dots\\ \dots & & .. & \\ \theta^{+n-1}I_r & \theta^{+n-2}I_r & \dots & \theta^pI_r\end{pmatrix}e^{-\theta t}. \tag{38}$$

Proof. (sketch) The proof is based on lemma 4 of reference [1]. We apply the Lagrange interpolation formula in matrix form to the matrix A_0 (one notes that the eigenvalues are distinct) [please see reference [43] page 108, and the proof of lemma 4 in [1]]. By the Lagrange interpolation formula [see [1] page 1604], it follows that, [2]

$$e^{A_0 t} = \sum_l \frac{(A_0 - \theta\lambda_1(\theta)I)\dots[l]\dots(A_0 - \theta\lambda_n(\theta)I)}{(\theta\lambda_k(\theta) - \theta\lambda_1(\theta))\dots[l]\dots(\theta\lambda_k(\theta) - \theta\lambda_n(\theta))} e^{\theta\lambda_l(\theta)t} = \sum_l e^{\theta\lambda_l(\theta)t} \sum_p G_{lp} \frac{A_0{}^p B}{\theta^p}(\theta) \tag{39}$$

where the functions $G_{lp}(\theta)$ are the same as in the proof of lemma 4 in [1], page 1604. They satisfy $G_{lp}(\theta) \to G_{lp}(\overline{\lambda}_1, ..., \overline{\lambda}_n)$ as $\theta \to +\infty$. Using this formula with the inequalities in Lemma 3.1, it follows that

$$\begin{aligned} e^{A_0}B = \sum_l e^{\theta\lambda_l(\theta)t} \sum_p G_{lp} \frac{A_0{}^p B}{\theta^p}(\theta) &\lesssim \sum_l e^{\theta\lambda_l(\theta)t} \sum_p G \frac{D^p}{\theta^p}(\theta) \\ &\lesssim \begin{pmatrix} I_r & \theta^{-1} I_r & \dots & \theta^{p-n+1} I_r \\ \theta^{+1} I_r & I_r & \dots & \theta^{-n+2} I_r \\ \dots & & .. & \dots \\ \dots & & .. & \\ \theta^{+n-1} I_r & \theta^{+n-2} I_r & \dots & \theta^p I_r \end{pmatrix} e^{\theta t}. \end{aligned} \tag{40}$$

The factor G is a constant such that $|G_{lp}(\theta)| < G$ for large θ. □

We are ready to present the proof of Theorem 3.2.

Proof of Theorem 3.2. We sketch the method of theorem 1 in [1] for block matrices. We use the following change of coordinates:

$$e = (e_1, ..., e_q) = (\varepsilon_1, \theta\varepsilon_2, ..., \theta^{q-1}\varepsilon_q). \tag{41}$$

Set $\Delta_\theta = \text{diag}[I_r, \theta I_r, ..., \theta^{q-1} I_r]$. By Lemma 3.2 one has the inequality

$$\Delta_{1/\theta} e^{A_0(t-\tau)} B \Delta_\theta \lesssim \Delta_{1/\theta} \begin{pmatrix} I_r & \theta^{-1} I_r & \dots & \theta^{p-n+1} I_r \\ \theta^{+1} I_r & I_r & \dots & \theta^{-n+2} I_r \\ \dots & & .. & \dots \\ \dots & & .. & \\ \theta^{+n-1} I_r & \theta^{+n-2} I_r & \dots & \theta^p I_r \end{pmatrix} \Delta_\theta e^{-\theta(t-\tau)} = \begin{pmatrix} I_r & I_r & \dots & I_r \\ I_r & I_r & \dots & I_r \\ \dots & & .. & \dots \\ \dots & & .. & \\ I_r & I_r & \dots & I_r \end{pmatrix} e^{-\theta(t-\tau)}. \tag{42}$$

Thus, from definition of $\lesssim$ we have $\Delta_{1/\theta} e^{A_0(t-\tau)} B \Delta_\theta \leq K e^{-\theta(t-\tau)}$ in the extended form. Following the ideas from the theorem above, the proof follows as the scalar case presented in[1]. We obtain the inequality $e^{\theta t} \parallel \varepsilon(t) \parallel \leq P(\theta) \parallel e_0 \parallel + u_0 K \int_0^t e^{\theta\tau} \parallel \varepsilon(t) \parallel d\tau$. Thus by Gronwall-Bellman lemma, we get,

$$\parallel \varepsilon(t) \parallel \leqslant P(\theta) e^{-\theta t} \parallel e_0 \parallel + u_0 K e^{-(\theta - u_0 K)t} \int_0^t P(\theta) \parallel e_0 \parallel e^{-u_0 K \tau} d\tau, \tag{43}$$

in the block form, where u_0 and K are constants independent of θ, and P is a polynomial. □

4 Solution to Problem 2: A high gain observer for bilinear systems in the form $\dot{x} = F(u,x)x + \phi(u,x)$

In this section, we consider the following nonlinear multi-input multi-output (MI-MO) system

$$\dot{x} = F(u,x)x + \phi(u,x), \qquad y = Cx \tag{44}$$

where

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_q \end{bmatrix}, \text{ with } x_i \in \mathbb{R}^r, \qquad u \in \mathbb{R}, \quad F = \begin{pmatrix} 0 & F_1 & 0 & \dots & 0 \\ 0 & 0 & F_2 & \dots & 0 \\ 0 & 0 & 0 & F_3 & \\ & . & .. & & \\ & . & .. & & F_{q-1} \\ 0 & 0 & & \dots & 0 \end{pmatrix}, \qquad \phi(u,x) = \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \dots \\ \phi_q \end{bmatrix}, \text{ where } qr = n, \tag{45}$$

[2] Here, the symbol $[l]$ indicates that the l-th element in a product or a sum is omitted.

and $F_k = F_k(u, x_1, ..., x_q)$, $\phi_k = \phi_k(u, x_1, ..., x_k)$ and $C = [I_r \ \ 0 \ \dots \ 0]$. More generally, we discuss the case $y = h(u, x)$. The system has been studied in [27],[9],[8] in slightly different setups. Denote

$$f(u,x) = F(u,x)x + \phi(u,x), \tag{46}$$

and introduce new input variables

$$u_0 = u, \qquad \dot{u}_i = u_{i+1}. \tag{47}$$

Let consider the change of variables,

$$\begin{aligned} z_1 &= \psi_1(u,x) = h(u_0, x) \\ z_i &= \psi_i(u,x) = L_f \psi_{i-1}. \end{aligned} \tag{48}$$

We shall assume that the following regularity condition is satisfied for system (44).

Assumption 3 ([8] page 7)**.** *The canonical flag of system (44) is uniform, i.e. the family of n distributions*

$$D_i(u): \ x \longmapsto \ker\left[\frac{\partial \psi_i}{\partial x}(x,u)\right] \tag{49}$$

have dimension $n-i$ regardless of u.

According to Theorem 1.3 in [8], the above change of variables will transform the equation (44) into

$$\dot{z} = \begin{bmatrix} z_2 \\ z_3 \\ \dots \\ z_q \\ L_f \psi_{q-1} \end{bmatrix} + \begin{bmatrix} b_1(u_0, z_1) \\ b_2(u_0, u_1, z_1, z_2) \\ \dots \\ \dots \\ b_q(u_0, \dots, u_{q-1}, z_1, \dots, z_q) \end{bmatrix}. \tag{50}$$

Besides, Assumption 3, implies that the exact linearization problem is solvable for system (44) or the same for (50) [see[8], theorem 1.3]. However, there is a little difference. In our setup, the variables are vector variables of size r; this does not affect the formulas in the coordinate changes and is still valid in the vector variable form. Therefore, the proof of Theorem 1.3 in [8] also carries over to this case without any changes.

Now, we are ready to present our second main result. We propose an observer of system (44) of the form:

$$\begin{aligned} \dot{\hat{x}} &= F(u,\hat{x})\hat{x} + \phi(u,\hat{x}) - \theta \Delta_\theta P(t) C(\hat{x} - y) \\ \dot{\varepsilon} &= \theta(-\varepsilon + \theta E^T \varepsilon + C^T(C\hat{x} - y)), \qquad \varepsilon(0) = 0 \end{aligned} \tag{51}$$

where $\Delta_\theta = \mathrm{diag}[I_r, \ \theta^{-1} I_r, \ \dots, \ \theta^{-q+1} I_r]$, and $P(t)$ is a solution to the Ricatti equation

$$\dot{P}(t) = \theta[F(u,\hat{x})P(t) + P(t)F(u,\hat{x})^T - P(t)C^T C P(t)], \tag{52}$$

where $P(0) = P(0)^T > 0$, and

$$\varepsilon = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \dots \\ \varepsilon_q \end{bmatrix} \text{ with } \varepsilon_i \in \mathbb{R}^r, \qquad E = \begin{pmatrix} 0 & I_r & 0 & \dots & 0 \\ 0 & 0 & I_r & \dots & 0 \\ 0 & 0 & 0 & I_r & \\ & . & .. & & \\ & . & .. & & I_r \\ 0 & 0 & & \dots & 0 \end{pmatrix}. \tag{53}$$

The scalar θ is the gain factor. The above observer stability is proved in the following theorem.

Theorem 4.1. *Let $e = x - \hat{x}$ be the error of the observer (51). Then there exists constants $a_0, M > 0$ and a polynomial $C(\theta)$ such that,*

$$\| e \| \leqslant C(\theta) e^{-(\theta - a_0)t}, \qquad (\theta > M). \tag{54}$$

Proof. As mentioned above, the Assumption 3 and from the Theorem 1.3 in [8], imply that the exact linearization problem for system (44) or (50) is solvable. Because of the triangular coordination appearing, the local linearization of system (44) is conjugate to a bilinear system of the form (9). In fact, according to the standard method of change of variables in feedback control (see[3] pages 142-143, see also [8] chapter 1), the vector functions b_j in (50) are linear in $u_j,\ j \geq 1$, and also the variables are sorted in a triangular form. The only variable in the u-part, which possibly appears nonlinear, is u_0 [see for example [8] sec. 1]. On the other hand, if we consider the canonical form in [[8], theorem 1.3] where the variables z_j may have higher dimensions, then we can solve the last coordinate of the equation for the control function u to make the whole system linear. Thus, the feedback canonical linear form of (44) is in the form (9). The change of coordinates in the systems above also applies to the corresponding observer. Using the form (44) we may assume the system is already given as $\dot{x} = Ax + \phi(u,x)$. In this case the error can be written as $\dot{e} = A_0 e + \theta(\hat{\phi}(u,\hat{x}) - \phi(u,x))$, where A_0 is as in (13), [see for instance [8] page 14]. When the exact linearization problem is solvable for (44), the error associated to the observer (51) after linearization of the system can be written as $\dot{e} = A_0 e + (Bu)e$, i.e. in the form that was used in Theorem 3.2. Thus, one can first change coordinates so that the equation (50) becomes linear, i.e., in the form (9). Then in the new coordinates, the observer and the error get the desired form. Thus, the observer for system (44) transforms to the observer (11) under the same change of coordinates. It follows that the error dynamics of observers (11) and (51) are also conjugate by the change of coordinates. This proves that the estimates in Theorem 3.2 are satisfied in the nonlinear case. Therefore, our theorem is a consequence of Theorem 3.2 in the linear case. □

Remark 3. *Various examples of system (44) has been considered in [4],[2],[5],[6],[7],[8],[9],[10],[11],[12],[13],[14],[15] with different set up, where asymptotic stability of the observer error has been demonstrated. The equation (44) in its general form (where the blocks may have different sizes) is related to differential systems on Siegel upper half spaces and flag varieties. Their stability analysis produces one of the significant interactions between dynamical systems and algebraic geometry,[44].*

5 A study case: the bioreactor bilinear system

In this section, two bioreactors' models are investigated to estimate their states utilizing the observer presented in previous sections. The bioreactor system is a model of bacteria growth or measurement population density that can be modeled in various dimensions and with different inputs and outputs,[30]. In our case, we study two bioreactor models of dimension two[45] and three[27], the first with a single input and the latter with multiple inputs. Both are bilinear systems. These systems classify as non-minimal systems [see[2] Ch. 6] since their zero dynamics show no convergence. The bioreactor models of the following two examples are taken from[45] and[27], respectively; however, our analysis is different.

5.1 *Example 1*: Bioreactor as a SI-SO system

We consider the system [45]

$$\begin{aligned} \dot{x}_1 &= \frac{a_1 x_1 x_2}{a_2 x_1 + x_2} + u x_1 \\ \dot{x}_2 &= -\frac{a_3 a_1 x_1 x_2}{a_2 x_1 + x_2} - u x_2 + u a_4 \\ y &= h(x) = x_1 \end{aligned} \tag{55}$$

where $x = (x_1, x_2) \in \mathbb{R}^2,\ u \in \mathbb{R},\ a_1, a_2, a_3 \in \mathbb{R}^+$. Set

$$\mu(x) = \frac{a_1 x_1 x_2}{a_2 x_1 + x_2},\ g(x) = \begin{bmatrix} x_1 \\ -x_2 + a_4 \end{bmatrix},\ f(x) = \begin{bmatrix} \mu(x) \\ -a_3 \mu(x) \end{bmatrix}. \tag{56}$$

We can write system (55) in the form of (44) as

$$\dot{x} = \begin{pmatrix} 0 & \mu(x)/x_2 \\ 0 & 0 \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} u x_1 \\ -a_3 \mu(x) + u(-x_2 + a_4) \end{bmatrix}. \tag{57}$$

The observer of this system is in the general form of (51)

$$\dot{\hat{x}} = \begin{pmatrix} 0 & \mu(\hat{x})/\hat{x}_2 \\ 0 & 0 \end{pmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} + \begin{bmatrix} u \hat{x}_1 \\ -a_3 \mu(\hat{x}) + u(-\hat{x}_2 + a_4) \end{bmatrix} + \theta \Delta_\theta (C\hat{x} - y) \tag{58}$$

where $C = [1\ \ 0]$ and $\Delta_\theta = [1\ \theta^{-1}]$,[45]. It can be written as

$$\dot{\hat{x}} = \begin{bmatrix} \mu(\hat{x}) \\ -a_3 \mu(\hat{x}) \end{bmatrix} + u \begin{bmatrix} \hat{x}_1 \\ -\hat{x}_2 + a_4 \end{bmatrix} + \theta \Delta_\theta (C\hat{x} - y). \tag{59}$$

$a_1 = 1$	$a_2 = 1$	$a_3 = 1$	$a_4 = 0.1$	$\theta = 1$

Table 1. The simulation parameters of the Example 1.

For the simulation, the system's initial conditions are $x_1(0) = 0.9$; $x_1(0) = 6$, while the observer's initial conditions are $\hat{x}_1(0) = 0.1$; $\hat{x}_1(0) = 0.1$. The system's parameters are given in Tab. 1. As we are dealing with the estimation problem and not the control problem, the control input u is chosen as a function of time. The results of the simulations are depicted in Fig. 1. Notice how the observer states exponentially converge to the real system states, as shown in Fig. 2, where the error convergence is shown.

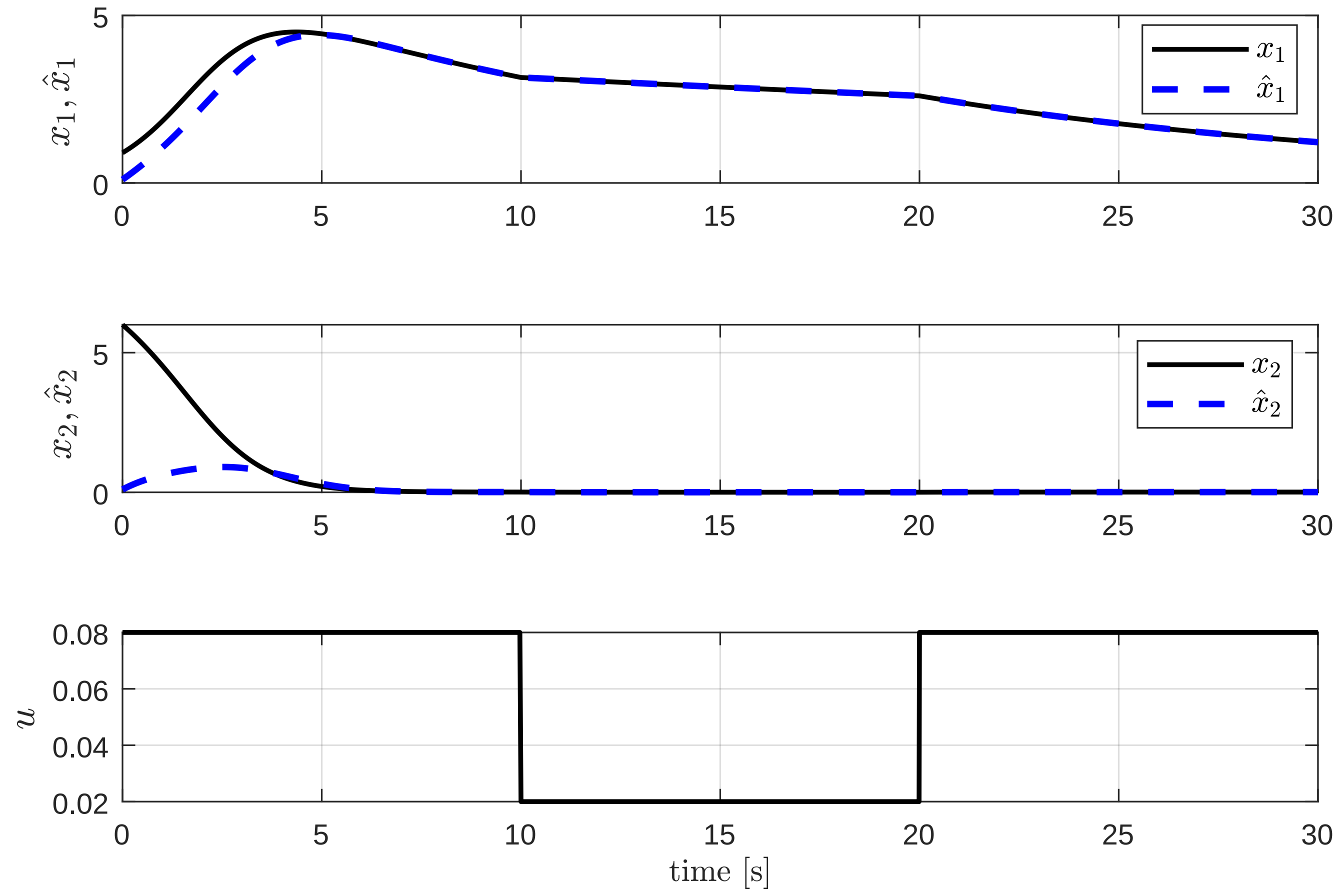

Figure 1. The trajectory convergence of Example 1: state (solid); state estimate (dashed blue line). The control input u was chosen as depicted above.

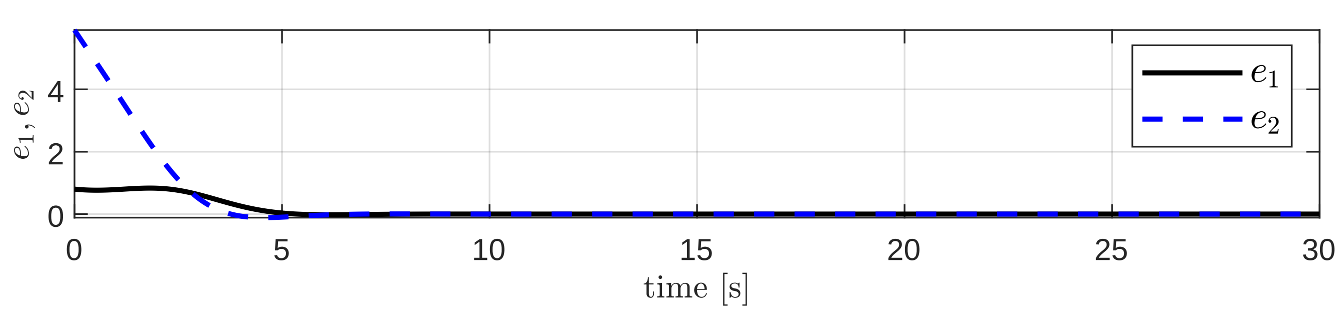

Figure 2. Error observer convergence for the Example 1.

5.1.1 Example 2: Bioreactor as a MI-SO system

Next, consider the multi-input single-output bioreactor model studied in[27]

$$\begin{aligned}
\dot{x}_1 &= a_1 x_1^2 x_2 - a_1 x_1 x_2 u_1 \\
\dot{x}_2 &= \bar{\mu}(x)\nu(x)x_2 - x_2 u_2 \\
\dot{x}_3 &= -a_2\bar{\mu}(x)\nu(x)x_2 - (x_3 - a_3)u_2 \\
y_1 &= x_1
\end{aligned} \tag{60}$$

where

$$\bar{\mu}(x_1,x_2) = a_4 \frac{x_1 x_3}{(a_5 + x_1)(a_6 + x_3)}, \; \nu(x_1,x_2) = a_7 \frac{x_2}{a_8 + x_1}. \tag{61}$$

The coefficients a_i are kinetic parameters that for the case of the present simulation take the values shown in Tab. 2. We set $\kappa(x) = \bar{\mu}(x)\nu(x)$ and

$$f(x) = \begin{bmatrix} a_1 x_1^2 x_2 \\ x_2\kappa(x) \\ -a_2\kappa(x)x_2 \end{bmatrix}, \qquad g_1(x) = \begin{bmatrix} -a_1 x_1 x_2 \\ 0 \\ 0 \end{bmatrix}, \; g_2(x) = \begin{bmatrix} 0 \\ -x_2 \\ -(x_2 - a_3) \end{bmatrix}, \tag{62}$$

and then (60) finds the standard form

$$\dot{x} = f(x) + g_1(x)u_1 + g_2(x)u_2. \tag{63}$$

One can write this system in the of (44) as

$$\dot{x} = \begin{pmatrix} 0 & a_1 x_1^2 & 0 \\ 0 & 0 & \frac{x_2}{x_3}\kappa(x) \\ 0 & 0 & 0 \end{pmatrix} x(t) + \begin{bmatrix} a_1 x_1 x_2 u_1 \\ -x_2 u_2 \\ a_2\kappa(x)x_2 - (x_2 - a_3)u_2 \end{bmatrix}. \tag{64}$$

Notice that the system is already in the form (50). An observer for system (60) is

$$\begin{aligned}
\dot{\hat{x}} = &\begin{pmatrix} 0 & a_1 \hat{x}_1^2 & 0 \\ 0 & 0 & \frac{\hat{x}_2}{\hat{x}_3}\kappa(\hat{x}) \\ 0 & 0 & 0 \end{pmatrix} \hat{x}(t) + \begin{bmatrix} a_1 \hat{x}_1 \hat{x}_2 u_1 \\ -\hat{x}_2 u_2 \\ a_2\kappa(\hat{x})\hat{x}_2 - (\hat{x}_2 - a_3)u_2 \end{bmatrix} \\
&- \theta\Delta_\theta^{-1} P(t) C^T (C \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \end{bmatrix} - y(t))
\end{aligned} \tag{65}$$

where $C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$, $\Delta_\theta = \mathrm{diag}[1 \;\; \theta^{-1} \;\; \theta^{-2}]$, and $P(t)$ is a symmetric positive matrix given by the Ricatti equation (52)

$$\begin{aligned}
\dot{P}(t) = \theta(P(t) + &\begin{pmatrix} 0 & a_1 x_1^2 & 0 \\ 0 & 0 & \frac{x_2}{x_3}\kappa(x) \\ 0 & 0 & 0 \end{pmatrix} P(t) + \\
&+ P(t) \begin{pmatrix} 0 & a_1 x_1^2 & 0 \\ 0 & 0 & \frac{x_2}{x_3}\kappa(x) \\ 0 & 0 & 0 \end{pmatrix}^T - P(t)C^T C P(t)
\end{aligned} \tag{66}$$

$P(t) > 0$,[27]. As we mentioned, the error dynamics is conjugated to a system of the form (31) where the inequality (32) can be proved.

For the simulation, the system's initial conditions are $x_1(0) = 0.1$, $x_1(0) = 0.5$, $x_3(0) = 1$; while the observer's initial conditions are $\hat{x}_1(0) = 0$, $\hat{x}_2(0) = 0$, and $\hat{x}_1(0) = 0$; and $P(0) = \mathrm{diag}(0.1)$ for the Riccati differential equation. The parameters of this MI-SO bioreactor model are depicted in Tab. 2. Once again, the control inputs (u_1, u_2) are chosen as depicted in Fig. 4. The observer states and the real system states are shown in Fig. 3. Notice the convergence of all the observer states to the real states. To visualize more clearly this, in Fig. 5 we have plotted the observer's errors.

Remark 4. *The reader can consult references [46],[47],[48],[49],[50],[51],[52],[54] for more example of bioreactor models. The bioreactor model is one of the examples of non-minimal bilinear systems.*

$a_1 = 3$	$a_2 = 2$	$a_3 = 3$	$a_4 = 100$	$a_5 = 12$
$a_6 = 9.82$	$a_7 = 0.5$	$a_8 = 0.1$	$\theta = 3.5$	

Table 2. Parameters of the simulation for the MI-SO system.

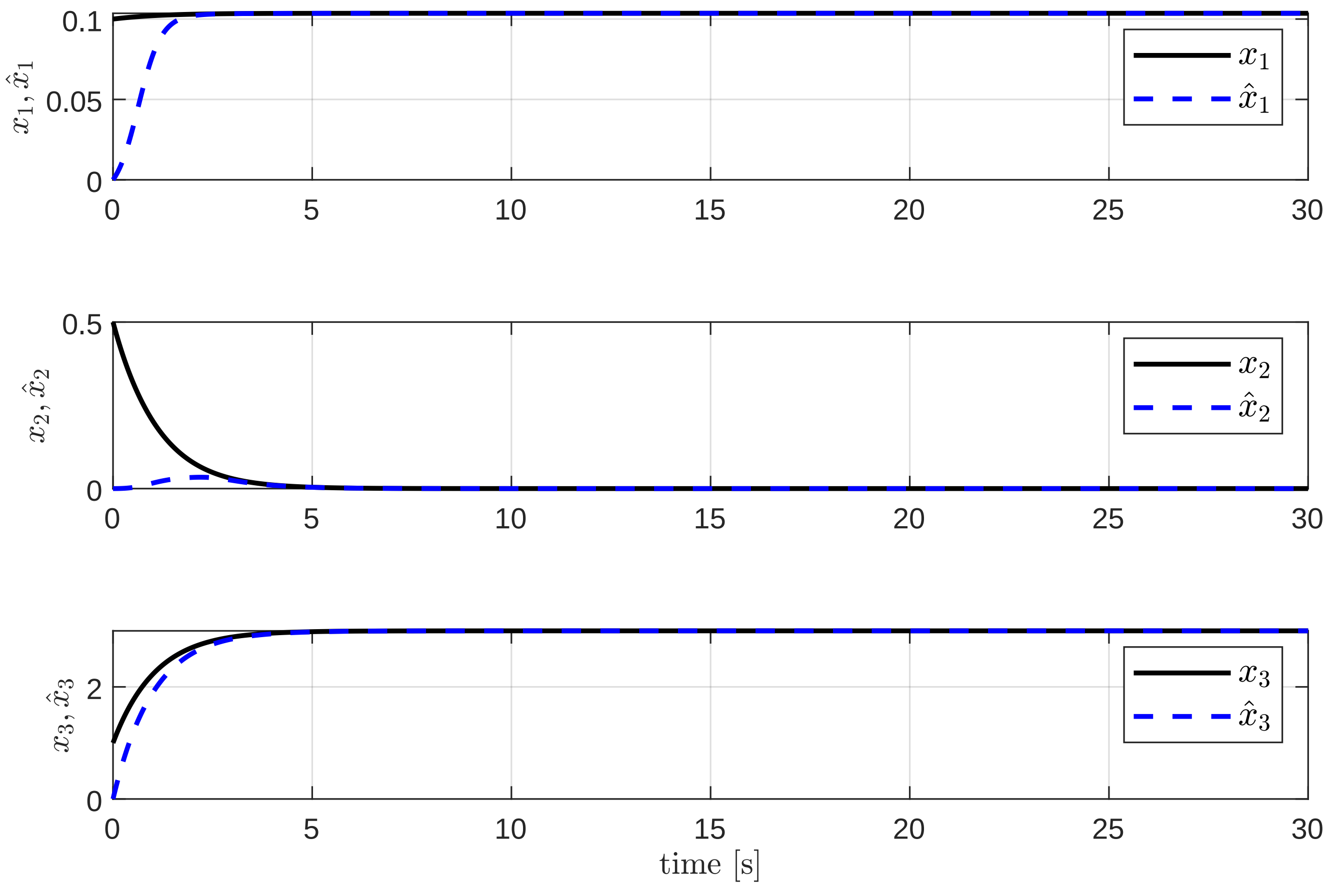

Figure 3. Original states and their estimates for Example 2.

6 Conclusion

The exponential stability of bilinear systems in block form has been proved. Moreover, under a regularity condition on the canonical flag of the system, a similar statement has been proved for the extended observer in the nonlinear case. This provides a uniform exponential bound for the error dynamics, stronger than the result of [27]. Two bilinear systems are presented as an application, for which two corresponding observers are proposed and simulated. This shows the effectiveness of the proposed approach.

In future work, we intend to apply this observer in real systems, such as mechanical underactuated systems. Also, by using our nonlinear observer, it is possible to propose output feedback controllers.

Author contributions statement

M. Reza-Rahmati and G. Flores reviewed the manuscript and contributed equally to this paper.

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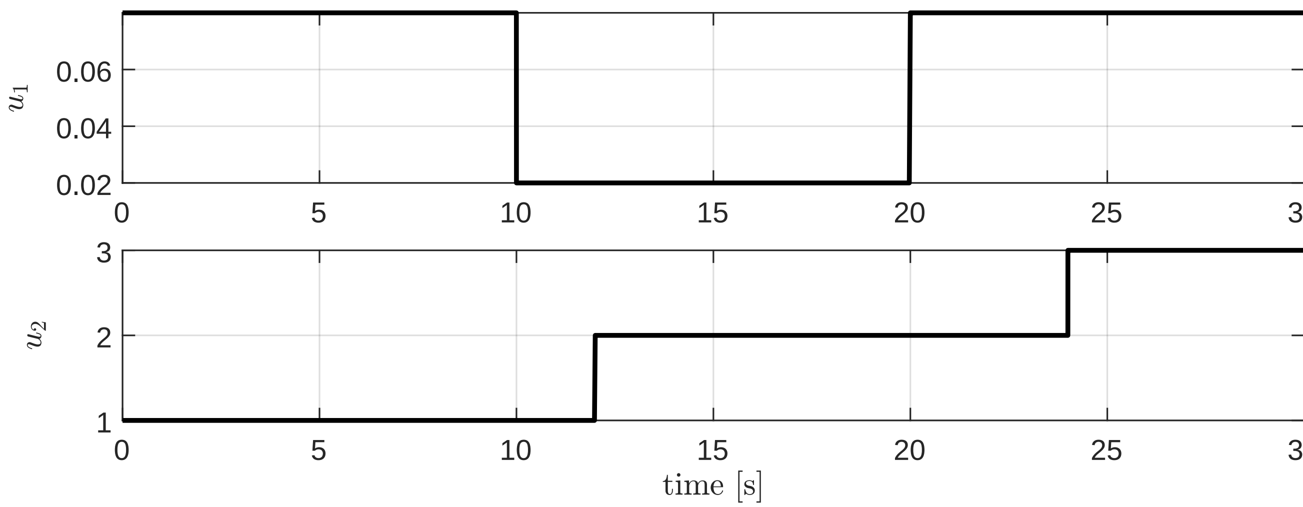

Figure 4. Control inputs for the Example 2.

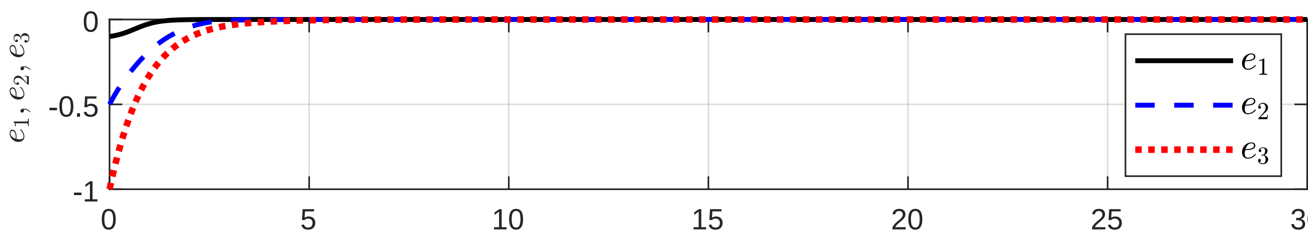

Figure 5. Error observer convergence for the Example 2.

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