On Brunosky numbers and observability indices in nonlinear MIMO systems

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Abstract

When the exact linearization problem is solvable for a nonlinear multi-input multi-output systems, it is possible to conduct the linearization in two standard ways. The first way employs a sequence of integrable distributions defined by the vector fields involved in the system. The second way uses the output functions and the codistributions defined as the kernels of codistributions. In both cases, one ends up with a change of coordinates which transforms the system to canonical block forms, where the sizes of the blocks are invariants of the system. One can associate two sets of indices which are canonical invariants of the system, called *Brunovsky indices*. In this work, we compare these two sets of invariants that are obtained in a system of dimension n. The indices are classically called the Brunovsky controllability indices and Brunovsky observability indices. We prove that the two sets of invariants give transpose partitions of n. That is, if $(h_1 \geq ... \geq h_N)$ are the controllability indices and $(h'_1 \geq ... \geq h'_M)$ are observability indices of the same nonlinear system, then the two partitions

$$n = h_1 + h_2 + ... + h_N = h'_1 + \cdots + h'_{M-1} + h'_M \tag{0.1}$$

are transpose to each other. In other words, the sizes of blocks that appear in the above two canonical forms are not only in general identical, but also they may have the different number of blocks. Therefore, they generally determine two different partitions of the dimension of the system. Besides, we present several sets of conditions that characterize the Brunovsky canonical forms, both in the controllable and in the observable cases, and prove their mutual equivalence.

Keywords: Brunovsky canonical form, Brunovsky numbers, Observability indices, Controllability indices, Feedback linearization.

1. Introduction

This text is devoted to comparing the two different ways of linearizations which apply to a MIMO system. Assuming the exact linearization problem is solvable for a MIMO system, there are two canonical ways to linearize it. One uses only the integrability condition on a sequence of distributions defined by the vector fields involved in the differential equation, while the second employs differential forms defined by the outputs of the system, [15], see also [2, 3, 8, 13, 12, 17, 19, 20, 21, 22, 23, 25]. One can arrange the coordinates in each method to obtain a unique canonical form. However, the two canonical block forms obtained at the end are not identical. Although both are linearizations of the same systems, the block forms that appear in the linearizations may have different sizes. The two linear forms of the MIMO system are canonical, and the blocks' sizes [see equations (2.11) and (2.37) below] appearing in the case, are called Brunovsky indices, [6, 7, 11].

The two Brunovsky canonical forms of a linearizable MIMO system are classically known in the theoretical literature. Each Brunovsky form determines a set of natural numbers which give a partition of the dimension

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of the system. However, as to our knowledge, nobody has compared the two sets of invariants. The significance of this relationship between two sets of indices appearing in the above two feedback linearization problems encouraged us to write this work. Certain geometric conditions characterize the exact linearization property for a control system. In [16], two sets of conditions, namely **Conditions A and B**, [see Section 2.1 below] have been presented, which characterize when a control system is equivalent to a Brunovsky canonical form. In this text, we present the third set of conditions, namely **Conditions C** that characterizes the Brunovsky observability form. We prove that the above sets of conditions are mutually equivalent. We also slightly generalize some existing results in nonlinear systems, which appear in our contributions.

We shall consider a nonlinear system as follows

$$\dot{x} = f(x) + \sum_{j=1}^{r} u_j g_j(x), \qquad x \in \mathbb{R}^n \tag{1.1}$$

where $f, g_1, ..., g_r$ are smooth vector fields on an open subset of $\mathbb{R}^n$ and $u_j : \mathbb{R}^n \to \mathbb{R}$ are input control functions. In this text, we assume u_j are smooth on an open subset of $\mathbb{R}^n$. However, in general, these functions can be considered more general [see [15] for instance]. The classical control provides a systematic approach to determine when the system (1.1) is equivalent to a linear form, [15, 16], see Section 2.1]. In this case, one says that the exact linearization problem is solvable (or the system is controllable). Then the exact linearization problem provides the following form,

$$\dot{z}_1^{(j)} = z_2^{(j)}, \quad \dot{z}_2^{(j)} = z_3^{(j)}, \quad \ldots, \quad \dot{z}_{b_j}^{(j)} = v^{(j)}, \qquad j = 1, \ldots, N.\ N \geq 1 \tag{1.2}$$

of the system (1.1), where $z_i^{(j)}$ are the new coordinate of the system. The latter form plays a crucial role in controls. For example, when the system (1.1) can be written in the form (1.2) we say it is controllable, i.e., there exists a change of coordinates and also a choice of the input functions that transforms it to a particular block diagonal linear form. The system's total dimension is divided into several smaller loops by possibly permuting the new coordinates. After obtaining the form (1.2) one defines a new vector coordinates

$$\begin{aligned} y_1 &:= (z_1^{(1)}, \ldots, z_j^{(h)}), \\ \dot{y}_j &= y_{j+1}, \qquad j = 1, 2, \ldots \end{aligned} \tag{1.3}$$

by grouping the variables. Differentiating with respect to t provides a sequence of blocks with the same entries as in (1.2) but possibly permuting the coordinates. As a result, the system can be written in a simple block form. The latter form (1.3) of the system (1.1) is canonical, i.e. it is unique. In particular, the block matrices that appear in the canonical form (1.3) are invariants of (1.1) called the Brunovsky controllability indices. We denote them by $(h_1, \ldots, h_N)$. Obviously we have $n = h_1 + \cdots + h_N$.

The system (1.1) can also be studied as a MIMO system with outputs as follows,

$$\begin{aligned} \dot{x} &= f(x) + \sum_{j=1}^{r} u_j g_j(x), \qquad & x \in \mathbb{R}^n \\ y &= \lambda(x) & y \in \mathbb{R}^r, \end{aligned} \tag{1.4}$$

and considering the same assumption as in (1.1) for the functions f and u_j, where the functions $\lambda_j : \mathbb{R}^n \to \mathbb{R}$ are the systems' outputs. The exact linearization problem for (1.4) is also referred to as *the observability problem*, [see [15] chap. 5]. In general, one can define certain set of indices called relative degrees associated to (1.4), [[15] chapter 5, see also Section 2.2]. In the case that the observability problem is solvable for (1.4) the relative degrees determine a form of (1.4) in the shape of (1.2). Again by grouping the variables as (1.3) we reach a canonical form, called Brunovsky canonical form of (1.4), and the sizes of the blocks determine a second set of indices $(h'_1, \ldots, h'_M)$ which are also invariants of the MIMO system (1.4). However, this canonical form and the associated indices are in general different from the former canonical form for the control system (1.1) and the controllability indices.

This article aims to compare the observability indices with the controllable ones. Both of the sets of Brunovsky indices are partitions of the dimension n. Moreover, we show that these two partitions of n are transposed to each other. The result can be regarded as an effort to understand the invariants mentioned above.

1.1. Contribution

Consider the system (1.1) with the setup explained above. The vector fields f and g_j and their brackets determine a sequence of integrable distributions $\mathfrak{g}_0 = \langle g_1, ..., g_r\rangle_{\mathbb{R}}$, $\mathfrak{g}_k = \langle [f + \mathfrak{g}_0, \mathfrak{g}_{k-1}]\rangle_{\mathbb{R}}$, $k \geq 1$, in an inductive way whose dimensions are constant on an open subset of the ambient space [see Section 2.1 for precise definition]. Moreover, $\dim \mathfrak{g}_N = n$ for some N. The integrability conditions on the distributions $\mathfrak{g}_k$ provides a standard method to determine certain conditions under which the system (1.1) is transformable into a Brunovsky canonical form.

Conditions A:

(A1) The submodules $\mathfrak{g}_k$ are closed under bracket operation of vector fields.

(A2) The numbers $b_j(x)$ are constant.

(A3) $b_N(x) = n$ for some N, where N is the smallest such number.

According to [16] the set of these conditions can be equivalently replaced by the following conditions B.

Conditions B:

(B1) There are smooth functions a_{ij} such that for each $i \leq j$

$$[ad^i(f)g_r, ad^j(f)g_s] = \sum_{1\leq i\leq r}\sum_{l\leq j} a_{il}ad^l(f)g_i. \tag{1.5}$$

(B2) The numbers $\dim \operatorname{span}\{ad^j(f)g_i(x) \mid j \leq k\} = b'_k(x)$ are constant.

(B3) $\dim \operatorname{span}\{ad^j(f)g_i(x) \mid j \leq n-1\} = n$

The linearization problem of (1.4) can also be approached by defining the sequence of codistributions $\Omega_0 = \langle d\lambda_1, ..., d\lambda_r\rangle_{\mathbb{R}}$, $\Omega_k = \langle \Omega_{k-1} + \sum_j L_{g_j}\Omega_{k-1} + L_f\Omega_{k-1}\rangle_{\mathbb{R}}$,. We present a new and a third set of conditions, namely

Conditions C:

(C1) The codistributions Ω_j have constant dimensions, $j = 1, 2, ..., M$.

(C2) Ω_M has dimension n, where the number M is the smallest number with this property.

(C3) For each j the codistribution Ω_j is invariant under f, g_l, for all $l \leq r$.

that determines whether the system (1.4) can be transformed to a Brunovsky canonical form. The set of **Conditions C** employs the sequence of codistributions defined by the output function λ. In other words, **Conditions C** describe the observability indices analogous to **Conditions A, B**, which describe the controllability indices. We show that the three sets of conditions **A, B** and **C** are mutually equivalent. The theorem and its proof appear as our first contribution [Theorem 4.1 below]. It also provides a method of how the controllability and observability indices are related. It follows that the MIMO system (1.4) can be linearized in two canonical ways.

Based on the above observation, assume the set of **A, B, C** are satisfied. In particular, the two systems (1.1) and (1.4) are completely linearizable. There are two standard approaches to linearize the two systems above. In the first case, there exists a change of coordinates and a choice of the input control functions u_j such that the differential equation (1.1) gets transformed to the Brunovsky canonical controllable form,

$$\dot{x} = Ax + Bu, \qquad A = \begin{pmatrix} 0 & E_1 & ... & 0 & 0 \\ 0 & 0 & E_2... & 0 & 0 \\ 0 & 0 & ... & 0 & 0 \\ ... & ... & ... & & \\ 0 & 0 & ... & 0 & E_{N-1} \\ 0 & 0 & .... & 0 & 0 \end{pmatrix}_{n\times n}, \; B = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ ... \\ E_N \end{pmatrix}_{n\times m}, \; u = \begin{bmatrix} u_1 & u_2 & ... & u_r \end{bmatrix}^T \quad (1.6)$$

where $E_j = [I_{h_j} \;\; 0]^T$,
$j = 1, 2, ..., N$. In the second case, the system (1.4) is equivalent to the canonical form

$$\begin{aligned} &\dot{x} = A'x + B'u, \qquad A' = \begin{pmatrix} 0 & 0 & ... & 0 & 0 \\ E'_2 & 0 & ... & 0 & 0 \\ 0 & E'_3 & ... & 0 & 0 \\ ... & ... & ... & & \\ 0 & 0 & ... & E'_M & 0 \end{pmatrix}, \; B' = \begin{pmatrix} E'_1 \\ 0 \\ 0 \\ ... \\ 0 \end{pmatrix}, \; u = \begin{bmatrix} u_1 & u_2 & ... & u_r \end{bmatrix}^T \\ &y = C.x, \qquad C = [E'_1 \;\; 0 \;\; 0 \;\; ... \; 0] \end{aligned} \quad (1.7)$$

where $E'_j = [I_{h'_i} \;\; 0]$, $\;j = 1, 2, ..., M$. As explained, we obtain two Brunovsky canonical forms of the same system. The two set of indices $(h_1, \ldots, h_N)$ and $(h'_1, \ldots, h'_M)$ are invariants of the systems (1.1) and (1.4). The observability indices of the system (1.4), due to the choice of outputs, $\lambda_1, ..., \lambda_r$ give a transpose partition of n with respect to the Brunovsky controllability indices. In other words, the two partitions

$$n = h_1 + h_2 + ... + h_N = h'_1 + \cdots + h'_{M-1} + h'_M \quad (1.8)$$

are transpose to each other, see Definition 4.3.

The paper has two main contributions; the first involves the proof of the equivalence of the set of **Conditions C** with the sets of **Conditions A, B**, mentioned above. It appears in Theorem 4.1 in Section 4. The second result is the proof of (1.8), which appears in Theorem 4.4. We also check this result in an example.

1.2. Content

The remainder of this paper is given as follows. In section 2.1, we define controllability indices for a system that is feedback linearizable. Section 2.2 defines the dual concept of observability indices for the same system with specific outputs. We had stated the proofs when the exact result did not exist in the literature. The problem statement is presented in section 3. The contributions are given in Section 4. Specifically, Theorem 4.1 is the new contribution, and Proposition 4.2 has a partial contribution. In section 4, we express and prove the above claim as the main contribution [Theorem 4.4]. We also give a complete example. Finally, in section 5 we give some final remarks and conclusions.

2. Preliminaries

Through the text, the vector fields f, g_j, and the input functions are assumed to be defined over $\mathbb{R}^n$. So let us assume that our system is defined on an open subset of $\mathbb{R}^n$, which is the intersection of the domains where all these functions are well-defined.

2.1. Controllability indices

We shall use the language of distributions on a differentiable manifold to analyze the system in question. A distribution is identified by a set of vector fields on a smooth manifold, denoted by $\mathfrak{g} = \langle X_1, ..., X_r \rangle_{\mathbb{R}}$, where the angles mean the local span of the vector fields in $\mathbb{R}^n$. We say the distribution is smooth if the vector fields X_j defining it are smooth vector fields. The dimension of a distribution at a point x is the dimension of the span of the vectors, $X_1(x), ..., X_r(x)$ regarded as vectors in $\mathbb{R}^n$. The point x is called a non-singular point for the distribution $\mathfrak{g}$ if in a neighborhood of x the dimension is constant and equal to r, the number of vector fields. The operations on the vector fields on a differentiable manifold naturally extend over the distributions, [15]. An important notion is that of involutiveness of a distribution, meaning that the bracket of any two vector fields in the distribution is still a member of the distribution. We sometimes refer to this notion by saying the distribution is closed under a bracket. By Frobenius theorem, this is equivalent to that the distribution is integrable, meaning that the system of differential equations defined by the vector fields in the distribution is locally solvable by integral foliations, see [15].

Consider the nonlinear control system of the form (1.1). One applies three kinds of coordinate changes to the above system:

(1) The first is the usual change of coordinates, i.e., the diffeomorphisms

$$\phi : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0), \tag{2.1}$$

which transforms the differential equation (1.1) by $f \longmapsto D\phi(f),\ g_j \longmapsto D\phi(g_j),\ (j \geq 1)$.

(2) The second is coordinate changes in the input space, $\mathbb{R}^r$ which are non-linear in x. That is, transformations of the form

$$g_j \longmapsto \sum_l a_{jl} g_l, \qquad j = 1, 2, ..., r, \tag{2.2}$$

where the matrix $A(x) = (a_{ij}(x))$ is smooth.

(3) In control systems, one studies the system's linearization of (1.1) under a specific choice of the functions u_j, called feedback control. Thus, there is a third kind of transformations applicable,

$$f \longmapsto f + \sum_j \beta_j g_j, \qquad \beta_j : \mathbb{R}^n \to \mathbb{R},\ \beta_j(0) = 0. \tag{2.3}$$

We denote the group generated by the above three kinds of transformations by G. We say the system (1.1) is G-linearizable or G-equivalent to a controllable system if a sequence of transformations in G transforms it to a bilinear system.

On the other hand, let us denote the Lie algebra of smooth vector fields on $\mathbb{R}^n$ by $\mathfrak{X}^\infty(\mathbb{R}^n)$. Define the following submodules of $\mathfrak{X}^\infty(\mathbb{R}^n)$,

$$\mathfrak{g}_0 = \langle g_1, ..., g_r \rangle_{\mathbb{R}}, \qquad \mathfrak{g}_k = \langle [f + \mathfrak{g}_0, \mathfrak{g}_{k-1}] \rangle_{\mathbb{R}} \tag{2.4}$$

generated over $C^\infty(\mathbb{R}^n)$ associated to the system (1.1), and let

$$b_j(x) = \dim \mathfrak{g}_j(x), \tag{2.5}$$

where we use the language of distributions on manifolds. In particular, we denote $\mathfrak{g}_k(x) = \big\langle X(x), X \in \mathfrak{g}_k \big\rangle_{\mathbb{R}} \subset T_x\mathbb{R}^n$. The submodules $\mathfrak{g}_k$, $k \geq 0$ are invariant under the action of the group of transformation G, or sometimes called G-invariant.

Conditions A:

(A1) The submodules $\mathfrak{g}_k$ are closed under bracket operation of vector fields.

(A2) The numbers $b_j(x)$ are constant.

(A3) $b_N(x) = n$ for some N, where N is the smallest such number.

Because the Lie bracketing is invariant under the G-action, the above conditions are independent of the aforementioned coordinate changes. The conditions A are invariant under the group of transformations generated by the elements in (1)-(2)-(3) mentioned above. According to [16] the set of these conditions can be equivalently replaced by the following conditions B.

Conditions B:

(B1) There are smooth functions a_{ij} such that for each $i \leq j$

$$[ad^i(f)g_r, ad^j(f)g_s] = \sum_{1\leq i\leq r}\sum_{l\leq j} a_{il}ad^l(f)g_i. \tag{2.6}$$

(B2) The numbers $\dim \operatorname{span}\{ad^j(f)g_i(x) \mid j \leq k\} = b'_k(x)$ are constant.

(B3) $\dim \operatorname{span}\{ad^j(f)g_i(x) \mid j \leq n-1\} = n$

The number N is the smallest natural number such that $b_N(x) = n$. Its existence is guaranteed by the condition (A3). One easily sees that $b_N(x) \geq b_{N-1}(x) \geq ... \geq b_0$, and similarly $b'_N(x) \geq b'_{N-1}(x) \geq ... \geq b'_0$. By (B1) we have $b_j(x) = b'_j(x)$. We may also define the invariants,

$$h_0(x) = b_0(x), \qquad h_j(x) = b_j(x) - b_{j-1}(x) = b'_j(x) - b'_{j-1}(x). \tag{2.7}$$

It is easy to see that $h_1(x) \geq h_2(x) \geq ...$ [cf. [16]]. The set of conditions A or its equivalent set B are closely related to the linearization problem for the equation (1.1) in control systems. In this regard, we have the following important definition.

Definition 2.1. [16] (Brunovsky canonical form) Assume that the numbers $b_j(x)$ are independent of x (as it is the same for $b'_j(x)$ or $h'_j(x)$). Write the coordinate $x = (x_0, ..., x_N)$ such that $\dim x_j = h_j$. A system of the form (1.1) such that,

$$f = \begin{bmatrix} 0 \\ .... \\ 0 \\ \tilde{x}_1 \\ \tilde{x}_2 \\ .... \\ \tilde{x}_{N-1} \end{bmatrix}, \ g_1 = \begin{bmatrix} 1 \\ 0 \\ ... \\ 0 \end{bmatrix}, \ g_2 = \begin{bmatrix} 0 \\ 1 \\ ... \\ 0 \end{bmatrix}, \ ...., g_{b_0} = \begin{bmatrix} 0 \\ 0 \\ .. \\ 1 \\ .. \\ 0 \end{bmatrix}, \ g_j = \begin{bmatrix} 0 \\ 0 \\ ... \\ 0 \end{bmatrix}, \ j > b_0, \tag{2.8}$$

where the 1 in g_{b_0} is in the b_0 place, is called a system in Brunovsky canonical form.

The following theorem relates the set of conditions A and B to the exact linearization problem of the system (1.1) and the Brunovsky canonical form.

Theorem 2.2. *[16] The following conditions are equivalent locally near $0 \in \mathbb{R}^n$.*

- *The system (1.1) is G-equivalent to a Brunovsky canonical form system.*
- *The set of conditions A are satisfied.*
- *The set of conditions B are satisfied.*
- *The system (1.1) is G-linearizable to a controllable system.*

Proof. The proof of the above Theorem is based on the following. If a sequence of distributions (defined over $\mathbb{R}$),

$$\mathfrak{g}_0 \subset \mathfrak{g}_1 \subset ... \subset \mathfrak{g}_N \tag{2.9}$$

on a manifold M of dimension n have constant dimensions $b_0 \leq b_1 \leq ... \leq b_N$. Then, there exists a coordinate system $(x_0, ..., x_n)$ on M such that the integral manifolds of $\mathfrak{g}_j$ are of the form

$$x_j = C_j, \qquad j = h_j + 1, ..., n, \; C_j \text{ constant} \tag{2.10}$$

■

Definition 2.3. (Brunovsky controllability indices) [16, 7, 11] The numbers h_j defined in (2.7) (or the same in Definition 2.1) are called Brunovsky controllability indices of the system (1.1).

The controllability indices characterize a unique canonical block form, which is equivalent to (1.1). It follows that these indices are invariants of the equation (1.1). We express this in the following proposition.

Proposition 2.4 (see [16, 15, 7, 11, 4, 9, 18, 5])**.** Assume any one of the sets of conditions A or B is satisfied for the control system (1.1). Then, there exists a change of coordinates and a choice of the input control functions u_j such that the differential equation (1.1) gets transformed to the Brunovsky canonical controllable form,

$$\dot{x} = Ax + Bu, \qquad A = \begin{pmatrix} 0 & E_1 & ... & 0 & 0 \\ 0 & 0 & E_2 ... & 0 & 0 \\ 0 & 0 & ... & 0 & 0 \\ ... & ... & ... & & \\ 0 & 0 & ... & 0 & E_{N-1} \\ 0 & 0 & .... & 0 & 0 \end{pmatrix}_{n\times n}, \; B = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ ... \\ E_N \end{pmatrix}_{n\times m}, \; u = \begin{bmatrix} u_1 & u_2 & ... & u_r \end{bmatrix}^T \tag{2.11}$$

where $E_j = [I_{h_j} \quad 0]^T, \; j = 1, 2, ..., N$. The size of the block matrices E_j are the Brunovsky controllability indices.

Proof. See the references above. ■

Remark 2.5. To obtain the Brunovsky indices, it is not sufficient that the system just is linearized. It is also necessary that the system is in the Brunovsky canonical form. For example, the simple system,

$$\dot{x}_1 = a_1 x_1 + b_1 u_1 \quad , \quad \dots \quad , \quad \dot{x}_n = a_n x_n + b_n u_n$$

is not in controller canonical form. The above sort of feedback linearization is different from the one in ordinary differentiable dynamics [see Theorem 2.2 and the explanation before that in the next section]. After the linearization, the system can be solved by algebraic equations for the functions u_j in a way that it finds the form (1.2). The Brunovsky form will then obtain by exchanging the order of variables so that the block forms will appear correctly.

Example 2.6. (Linear system) [15] When the functions $f(x)$ and $g_j(x)$ in (1.1) are already given by linear matrices, the aforementioned distributions can be easily determined. Consider the system,

$$\dot{x} = Ax + Bu, \qquad x \in \mathbb{R}^n, \; A \in M_{n\times n}, \; B \in M_{n\times m}, \; u \in \mathbb{R}^m. \tag{2.12}$$

The distributions $\mathfrak{g}_k$ get the form

$$\mathfrak{g}_k = Image[B \;\; AB \;\; A^2B \;\; ... \;\; A^kB] \tag{2.13}$$

meaning the span of the *columns* of all the matrices in the bracket. The matrix

$$\mathfrak{g}_n(A, B) = [B \;\; AB \;\; A^2B \;\; ... \;\; A^{n-1}B] \in \mathbb{C}^{n\times nm} \tag{2.14}$$

is called the controllability matrix of the pair (A, B). Thus, we have

$$b_1 = \operatorname{rank}(B), \qquad b_j = \operatorname{rank}(\mathfrak{g}_{j-1}) - \operatorname{rank}(\mathfrak{g}_{j-2}) \tag{2.15}$$

where $\mathfrak{g}_j$ is given by (2.13). The aforementioned canonical form of Brunovsky is used in order to classify the linear system (2.12), [see [7] as well as the explanation below].

Definition 2.7. [7] Two pairs of matrices (A, B) and (C, D) in the form of equation (2.12) are block equivalent if there exist matrices $R_{n\times n}, S_{m\times m}, T_{m\times n}$ such that

$$[C \;\; D] = R[A \;\; B] \begin{pmatrix} R^{-1} & 0 \\ T & S \end{pmatrix} \tag{2.16}$$

where R, S are invertible.

Remark 2.8. [7] Assume that A, B are constant matrices and that the system (2.12) is completely controllable, i.e.,

$$\operatorname{rank}(B, AB, ..., A^n B) = n. \tag{2.17}$$

To formulate the preceding definition more precisely, we translate it into some feedback control statements. By adding linear feedback to (A, B), we mean that in (2.12), we substitute

$$u = Tx + v \tag{2.18}$$

where T is a $m \times n$ constant matrix. As a result of this transformation, we obtain a system $(C", D")$, with

$$C" = A + BT, \qquad D" = B. \tag{2.19}$$

We say $(C", D")$ behaves like (C', D') if there are non-singular matrices R and S of type $n \times n, m \times m$ respectively, such that,

$$C' = R^{-1}A"R, \qquad D' = R^{-1}BS \tag{2.20}$$

Summarizing, the definition asks whether for given systems $(A, B), (C', D')$ there are matrices $R_{m\times n}$, $T_{m\times n}$, $S_{m\times m}$ with, R, S being non-singular, such that

$$C' = R^{-1}(A + BT)R, \qquad D' = R^{-1}BS. \tag{2.21}$$

If the answer is positive, we shall say that (A, B) and (C', D') are feedback (or briefly, F-) equivalent. By a straightforward computation, it can be checked that F-equivalence is an equivalence relation, i.e., it is symmetric, reflexive, and transitive.

The following proposition relates the above definition to the Brunovsky numbers.

Proposition 2.9 (see [14, 24, 8, 11, 12, 1, 10])**.** Two matrices are block equivalent if and only if they have the same (controllability) Brunovsky numbers.

Proof. See the references above. ∎

By considering Definition 2.7 and Proposition 2.9, we understand that the Brunovsky numbers are invariants of the bilinear control systems. In the following, we present another way to interpret the Brunovsky numbers for analytic matrices in one variable. The analysis of this section can also be expressed in terms of global analytic block similarity of a pair of matrices $(A(z), B(z))$ for bilinear systems. In case that the equation 1.1 could be defined by the matrices $A(z)$ and $B(z)$, then we have the following format of Theorem 2.2.

Theorem 2.10. *[11] Let $A : X \to \mathbb{C}^n,\ B : X \to \mathbb{C}^{n\times n}$ be analytic matrices, where X is connected and open in $\mathbb{C}$. Then the pair $(A(z), B(z)$ is (globally) block equivalent to a pair in Brunovsky canonical form*

$$[G(z) \quad J(z)] := R(z)[A(z) \quad B(z)] \begin{pmatrix} R(z)^{-1} & 0 \\ T(z) & S(z) \end{pmatrix}, \tag{2.22}$$

where P, Q are invertible, if and only if the following holds

- *The Brunovsky numbers of $(A(z), B(z))$ namely $h_1(z) \geq h_2(z) \geq ... \geq h_N(z)$ are independent of $z \in X$.*
- *The size and the numbers of the Jordan blocks in the Jordan part of the Brunovsky form of $(A(z), B(z))$ are independent of z.*
- *There are s different analytic eigenfunctions $\alpha_i : X \to \mathbb{C}$, where s is the number of eigenvalues of A having eigenvector in $\ker(B)$, denoted by*

  $$\sigma(A(z), B(z)) = \{\alpha_1, ..., \alpha_s\} \tag{2.23}$$

  such that if

  $$Z = \{z \in X | \alpha_i(x) = \alpha_j(x)\}, \tag{2.24}$$

  then, the sum of generalized eigenspace

  $$N(z) = \lim_{x\to z} (R_{\lambda_1} + ... + R_{\lambda_s}) \tag{2.25}$$

  is direct and $\dim N(z) = n - \sum h_i$.

Remark 2.11. Theorem 2.10 is stated for a pair of matrices whose entries are analytic functions defined on a complex plane domain. However, it is possible to modify the arguments in the proofs in [11] such that it also works for functions of several complex variables. We have mentioned the theorem as an alternative way or approach to the Brunovsky canonical form and controllability indices. One may consider the Brunovsky canonical form as just associated to the pairs of matrices $(A(z), B(z))$ where the group of symmetries is defined as the beginning of this section, [the same in the remark 2.8]. We will not enter the details of this theorem in this text and refer the interested reader to [11] and the references therein.

2.2. Observability indices

The language of codistributions may also describe the integrability conditions for a collection of vector fields on a manifold. This notion is dual to that of distributions. This problem originally goes back to different formulations of the Frobenius theorem on a smooth manifold's integrability of differential systems. By definition, a codistribution is identified by the linear span of 1-forms on (open subset of) a manifold, denoted $\mathfrak{g}' = \langle w_1, ..., w_{n-r}\rangle_{\mathbb{R}}$, ($\langle ., .\rangle$ means span of co-vectors). The codistribution is called smooth if the 1-forms in a generator are smooth differential forms. The annihilator $\mathfrak{g}'^{\perp}$ of a codistribution $\mathfrak{g}'$ at a point x are the vectors in the tangent space to x that are killed by all the elements of $\mathfrak{g}'$. The dual notion of coditribution can state the Frobenius theorem's integrability criteria. We say a system of differential equations given by a distribution $\mathfrak{g}$ of dimension r is integrable if there can be found $(n - r)$ smooth functions $\lambda_1, ..., \lambda_{n-r}$ such that $\mathfrak{g}' := \langle d\lambda_1, ..., d\lambda_{n-r}\rangle_{\mathbb{R}} = \mathfrak{g}^{\perp}$, [see [15] chapter 1 for the details on the terminology].

On the other hand, *the Brunovsky observability indices* are closely related to the notion of relative degrees for the multi-input multi-output observer systems. Thus we first recall the definition of relative degrees in this case. In general, relative degrees can be defined for any observer system with an arbitrary algebraic observer. The Brunovsky indices refer to the choice of specific observer equations so that the exact linearization problem is solvable for (1.4). We define the observability indices through a maximal set of codistributions defined by the system 1.4 itself. The two notions of relative degrees and observability indices are closely related. Now, let us consider the Multi-Input Multi-Output System (1.4). To this system, one classically assigns relative degrees as follows.

Definition 2.12. (Relative degrees) [[15] Chapter 5] We say the system (1.4) has relative degrees $(a_1, ..., a_M)$ at a point x if the two following conditions hold:

- $L_{g_i} L_f^j \lambda_l = 0$ for $j < a_l - 1$ in a neighborhood of x.
- The matrix
$$D(x) = \begin{pmatrix} L_{g_1} L_f^{a_1-1}\lambda_1(x) & ... & L_{g_r} L_f^{a_1-1}\lambda_1(x) \\ L_{g_1} L_f^{a_2-1}\lambda_2(x) & ... & L_{g_r} L_f^{a_2-1}\lambda_2(x) \\ .... & & \\ & & \\ L_{g_1} L_f^{a_r-1}\lambda_r(x) & ... & L_{g_r} L_f^{a_r-1}\lambda_r(x) \end{pmatrix} \tag{2.26}$$
is non-singular at x.

Relative degrees play a crucial role in the analysis of the systems (1.4). The following theorem provides a canonical way to compute relative degrees.

Theorem 2.13. *[15] Assume that the system* (1.4) *has relative degrees* $(a_1, ..., a_M)$*, then* $a = a_1 + a_2 + ... + a_M \leq n$*. Define the functions*
$$\phi_j^i(x) = L_f^{j-1} h_i(x). \tag{2.27}$$
Then, it is always possible to find $n - a$ *functions* $(\phi_{a+1}, ..., \phi_n)$ *such that the change of coordinates*
$$\begin{aligned} &\phi : \mathbb{R}^n \longrightarrow \mathbb{R}^n \\ &\phi = (\phi_1^1(x), ..., \phi_{a_1}^1, ..., \phi_{a+1}, ..., \phi_n) \end{aligned} \tag{2.28}$$
transforms (1.4) *to the system of the form*
$$\begin{aligned} \dot{\phi}_j^{a_l} &= \phi_{j+1}^i, \qquad j < a_i - 1,\ l = 1, 2, ..., r \\ \dot{\phi}_{a_l-1}^{a_l} &= L_f^{a_l}\lambda_l + \sum_{s=1}^{r} L_{g_s} L_f^{a_l-1}\lambda_l u_s \\ \dot{\phi}_j &= Z(x) = L_f \phi_j, \qquad j > n - a. \end{aligned} \tag{2.29}$$
If the distribution $\mathfrak{g}_0 = \langle g_1, ..., g_r\rangle$ *is closed under bracket operation then it is possible to choose* $\phi_j,\ j > n-a$ *such that*
$$L_{g_i}\phi_j = 0, \qquad j > n - a,\ i = 1, ..., r \tag{2.30}$$

Proof. See [15] Chapter 5. ■

We consider the case when the inequality in Theorem 2.13 is an equality, i.e. when $a_1 + a_2 + \cdots + a_M = n$. In this case, one says that the exact linearization problem is solvable for the system (1.4). We have the following theorem.

Theorem 2.14 ([15] Theorem 5.2.3)**.** *There exists outputs* $\lambda_1, ..., \lambda_r$ *such that*
$$a_1 + a_2 + ... + a_r = n \tag{2.31}$$
if and only if the following conditions are satisfied,

- *The distributions* $\mathfrak{g}_j$ *have constant dimensions,* $j = 1, 2, N-1$*.*
- $\mathfrak{g}_N$ *has dimension* n*.*
- $\mathfrak{g}_k$ *are closed under bracket.*

Proof. See [15] Theorem 5.2.3. ■

The set of conditions in Theorem 2.14 is the same as the set of conditions A (thus, also conditions B). We have to say that the indices $a_1, \dots, a_r$ are invariants of the system (1.4) when the linearization problem is solvable. However, these invariants may not be canonical in this form. In order to make the index set $\{a_i \mid 1 \leq i \leq r\}$ canonical, one needs to group the variables to obtain the Brunovsky form. We explain this as follows [see also the proof of Theorem 4.1]. Assume we are in the situation of the Theorem 2.14, i.e., the exact linearization problem is solvable for (1.4). Define the codistributions,

$$\Omega_0 = \langle d\lambda_1, ..., d\lambda_r \rangle_{\mathbb{R}}, \qquad \Omega_k = \langle \Omega_{k-1} + \sum_j L_{g_j} \Omega_{k-1} + L_f \Omega_{k-1} \rangle_{\mathbb{R}}, \tag{2.32}$$

and denote

$$h'_0 = \operatorname{rank}(\Omega_0), \qquad h'_k = \operatorname{rank}(\Omega_k) - a'_{k-1} \tag{2.33}$$

where we have $h'_1 + h'_2 + ... + h'_M = \dim \Omega_M = n$. Also, we have

$$\Omega_0^{\perp} \supset \Omega_1^{\perp} \supset .... \supset \Omega_M^{\perp}. \tag{2.34}$$

In the notation at the beginning of this section, we have set $\mathfrak{g}'_{M-k} = \Omega_k^{\perp}$ [we use both of the notations $\mathfrak{g}'_{M-k}$ and $\Omega_k^{\perp}$ in the following]. We have the simple relation $\operatorname{rank}(\Omega_k) + \operatorname{rank}(\Omega_k)^{\perp} = n$. Theorem 2.14 is the base of the following definition.

Definition 2.15. (Brunovsky observability indices) Assume that the exact observation problem is solvable for the system (1.4). The indices defined by

$$h'_1, h'_2, \dots, h'_M, \qquad h'_i = \operatorname{rank}(\Omega_i) - \operatorname{rank}(\Omega_{i-1}) \tag{2.35}$$

are called observability indices of the system (1.4) $[\Omega_{-1} = 0]$. We have

$$h'_1 + h'_2 + ... + h'_M = n. \tag{2.36}$$

The observability indices characterize a unique block canonical form of (1.4) which plays a crucial role in control systems, see Remark 2.8. It also follows that these indices are invariants of (1.4). We mention this in the following proposition.

Proposition 2.16 (see [16, 15, 7, 11, 4, 9, 18, 5])**.** Assume the set of conditions in Theorem 2.14 is satisfied (Conditions A or B), then, the system (1.4) is equivalent to the Brunovsky canonical observable form

$$\begin{aligned} &\dot{x} = A'x + B'u, \qquad A' = \begin{pmatrix} 0 & 0 & ... & 0 & 0 \\ E'_2 & 0 & ... & 0 & 0 \\ 0 & E'_3 & ... & 0 & 0 \\ ... & ... & ... & & \\ 0 & 0 & ... & E'_M & 0 \end{pmatrix}, \ B' = \begin{pmatrix} E'_1 \\ 0 \\ 0 \\ ... \\ 0 \end{pmatrix}, \ u = \begin{bmatrix} u_1 & u_2 & ... & u_r \end{bmatrix}^T \\ &y = C.x, \qquad C = [E'_1 \ \ 0 \ \ 0 \ \ ... \ 0] \end{aligned} \tag{2.37}$$

where $E'_j = [I_{h'_i} \ \ 0], j = 1, 2, ..., M$. The size of the block matrices E'_j are the Brunovsky observability indices.

Proof. See [16, 15, 7, 11, 4, 9, 18, 5, 6, 7]. ∎

By Theorem 2.13 and Theorem 2.14, there exists a change of coordinates that will transform the system (1.4) to the form explained by (2.29) while (2.31) holds. In this case, the observability problem for (1.4) is solvable. Say also (1.4) is observable. We obtain a set of indices that fulfill the total dimension n. We also call this case a maximal case. According to Theorem 2.13, the desired change of variable, in this case, is given by

$$\psi = \bigoplus_i \begin{bmatrix} \lambda_i \\ L_f \lambda_i \\ ... \\ L_f^{a_i - 1} \lambda_i \end{bmatrix} : \mathbb{R}^n \to \mathbb{R}^n. \tag{2.38}$$

Example 2.17. (Linear case) [15] Example 2.6 has an analogous version for MIMO systems. In this case, we work with codistributions and their kernels. Consider the linear system,

$$\begin{aligned} &\dot{x} = Ax + Bu, \qquad && x \in \mathbb{R}^n, \ A \in M_{n\times n}, \ B \in M_{n\times m}, \ u \in \mathbb{R}^m \\ &y = Cx, && C = \begin{pmatrix} c_1 & c_2 & \dots & c_m \end{pmatrix}, \ c_i \in \mathbb{R}^1. \end{aligned} \tag{2.39}$$

In this case, the codistributions defined by (2.32) find the following form,

$$\Omega_k = \langle \ C, \ CA, \ \dots \ , \ CA^{k-1} \ \rangle_{\mathbb{R}} \tag{2.40}$$

where the angles mean the subspace of span of the *rows* of all matrices. Consequently

$$\Omega_0^{\perp} = \ker(C), \qquad \Omega_k^{\perp} = \ker \begin{bmatrix} C \\ CA \\ \dots. \\ CA^{k-1} \end{bmatrix} \tag{2.41}$$

therefore we have

$$h'_k = \operatorname{rank} \begin{bmatrix} C \\ CA \\ \dots. \\ CA^{k-1} \end{bmatrix}. \tag{2.42}$$

If one defines $Z = \bigcap_i \ker(CA^i)$ the maximal integral submanifolds of Z are of the form $z_0 + Z$. Again, similar to the controllability canonical form, the canonical observability form can be used to classify MIMO linear systems.

3. Problem Statement

To the nonlinear control system (generally multi-input system) (1.1), one can associate a set of indices that divide n as a partition and provide a canonical form of the exact linearization of (1.1). There is a change of coordinates and a choice of control functions u_j such that the above differential equation gets transformed to a special block form called Brunovsky canonical controllable form. The block matrices' size provides invariants of the controllable system (1.1) called the Brunovsky controllability indices. We denote them by $(h_1, ..., h_M)$ [see Section 2.1]. These indices are defined by certain invariants of distributions constructed from the vector fields f and g_j and the bracket operations. In [16] two equivalent set of conditions, namely **Conditions A** and **Conditions B** have been settled that characterize the Brunovsky controllability indices [see Section 2.1]. The controllability indices $(h_1, h_2, ..., h_N)$ give a partition of n, i.e.

$$n = h_1 + ... + h_N. \tag{3.1}$$

Consider the system (1.4) with the same conditions above. In an alternative method, by using the outputs, λ_j, one can define another set of indices that are defined somehow in a dual manner to the previous procedure. Again the block's size of matrices appearing, provide invariants of the system (1.4); they are called the Brunovsky observability indices, denoted by $(h'_1, ..., h'_M)$ [see Section 2.2]. The observability indices $(h'_1, ..., h'_M)$ also give a partition of n, that is,

$$n = h'_1 + ... + h'_M. \tag{3.2}$$

Observability indices give a canonical form of the relative degrees when the variables are appropriately grouped. The main problem is as follows.

Problem 3.1. How the two set of indices $(h_1, ..., h_N)$ and $(h'_1, ..., h'_M)$ are related? In other words, how the controllability indices of the system (1.1) are related to the observability indices of the system (1.4)? Do we have the equality $M = N$?

The above two canonical ways of linearizing (1.4) are not identical. So we have to say the two sets of indices are not generally the same, nor do we have $M = N$, i.e., the number of the blocks in the above two linearization processes are not equal [see the example at the end of text].

4. Main Results

According to the explanation given in Sections 2 and 3, there are two different ways of linearizing a system given by (1.1) or a system in the form of (1.4), when they are exactly linearizable. The nontrivial expectation is that these two linearizations may not be the same, and the two-block forms we mentioned in the previous sections can be different. Naturally, one finds that the two canonical sets of indices we encountered are not equal. In general, the two sets of indices are far from being identical, although they both give a partition of n; they even may not have an equal number of elements. This section investigates the relationship between two sets of indices defined in Sections 2.1 and 2.2. Because the controllability and observability indices given in Definitions 2.3 and 2.15, respectively, characterize a particular form of exact linearity of the same systems of differential equations, it is natural to expect a simple relation between them. We note that the ways these two series of numbers are defined use dual concepts of distributions vs. codistributions on $\mathbb{R}^n$. Thus, the connections between the invariants above should also be related to duality. Below we state another set of conditions which can be compared with the set of conditions **A** and **B** given above and also in [16]. This also appears to be one of our contributions to the text.

Conditions C:

(C1) The codistributions Ω_j have constant dimensions, $j = 1, 2, ..., M$.

(C2) Ω_M has dimension n, where the number M is the smallest number with this property.

(C3) For each j the codistribution Ω_j is invariant under f, g_l, for all $l \leq r$.

Recall that a codistribution Ω is invariant under a vector field f if $L_f w \in \Omega$ for all forms $w \in \Omega$. The notion is dual to the one of distributions, so that we call a distribution $\mathfrak{g}$ invariant under a vector field f if $[f, X] \in \mathfrak{g}$ for all $X \in \mathfrak{g}$. One can think of the set of **Condition C** to be dual to the sets **A, B**. The following Theorem explains this relation and gives part of our main result.

Theorem 4.1. *The conditions in Theorem 2.14 can be equivalently replaced by the set of* ***Conditions C****. The three sets of conditions* ***A, B, C*** *each one is equivalent to the other.*

Proof. By Theorem 4.1.2 of [15] the action of the codistributions Ω_j on the distributions $\mathfrak{g}_i$ can be presented in a $a \times a$ block matrix with blocks given by

$$\bigoplus_i^{\text{rows}} \begin{bmatrix} d\lambda_i \\ dL_f\lambda_i \\ \ldots \\ dL_f^{a_i-1}\lambda_i \end{bmatrix} \left(\bigoplus_j^{\text{columns}} \begin{pmatrix} g_j & ad(f)g_j & \ldots & ad(f)^{a_i-1}g_j \end{pmatrix} \right) = \bigoplus_i^{\text{rows}} \left[\bigoplus_j^{\text{columns}} \begin{pmatrix} 0 & \ldots & L_{ad(f)^{a_i-1}g_j}\lambda_i \\ 0 & \ldots & * \\ & \ldots & * \\ L_{g_j}L_f^{a_i-1}\lambda_i & * & * \end{pmatrix} \right] \tag{4.1}$$

where the last matrix is non-singular. The same as the argument in [15], Lemma 4.1.2, the two following sets of conditions (identities) are equivalent (by the same proof as in [15]),

(a) $L_g\psi(x) = L_gL_f\psi(x) = \cdots = L_gL_f^k\psi(x) = 0$.

(b) $L_g\psi(x) = L_{ad(f)g}\psi(x) = \cdots = L_{ad(f)^kg}\psi(x) = 0$.

with

$$\psi = \bigoplus_i \begin{bmatrix} \lambda_i \\ L_f\lambda_i \\ \ldots \\ L_f^{a_i-1}\lambda_i \end{bmatrix}, \qquad g = [g_1, \ldots, g_r]. \tag{4.2}$$

Notice that

$$L_{ad(f)^jg}\psi(x) = D\psi(ad(f)^jg) = \bigoplus_i \begin{bmatrix} d\lambda_i \\ dL_f\lambda_i \\ \ldots \\ dL_f^{a_i-1}\lambda_i \end{bmatrix} (ad(f)^jg). \tag{4.3}$$

The primary application of the lemma 4.2.1 of [15] (and the vector form we stated above) is that in order to find the component coordinates of ψ one can use the second list of equations instead of the ordinary one that is given in the first line of identities. By Definition 2.12, we know that the codistributions Ω_j are defined via the first list of equations. By the equivalence of Conditions A and B, we claim the following. $\mathfrak{g}_k$ are closed under the bracket for all k, if and only if Ω_j are invariant under f, g_l, for all j. This is because if $\mathfrak{g}_k$ are closed under bracket, that is also the Condition B holds, then we have the second line of equations defining their annihilators. However, the annihilator of the involutive distribution is invariant under f and g (by Frobenius theorem, cf. [15] theorem 1.4.1). On the other hand, the first line's identities are the same as the codistributions Ω_j. This argument is reversible also. If the codistributions Ω_j are invariant under f and g_j, we can make the argument in the other direction and conclude that $\mathfrak{g}_j$ are involutive [again by the Frobenius theorem].

The equivalence of having constant dimensions in Condition sets A (or B) and C can be checked similarly. Notice that the distribution has a constant dimension if its annihilating co-distribution (denoted by upper perp. symbol above) has a constant dimension. Therefore, the other two conditions are trivial to be compared. One notes that the items in **Conditions C** are G-invariant. Therefore, one may check the criteria by its G-equivalent linearized form. The block multiplication in the identity (4.1) gets the following form in the linear case,

$$\begin{bmatrix} c_i \\ c_i A \\ \dots \\ c_i A^{a_i-1} \end{bmatrix} \begin{pmatrix} B_j & AB_j & \dots & A^{a_i-1}B_j \end{pmatrix} = \begin{pmatrix} 0 & \dots & c_i A^{a_i-1} B_j \\ 0 & \dots & * \\ & \dots & * \\ c_i A^{a_i-1} B_j & * & * \end{pmatrix}, \tag{4.4}$$

where the claim in the proof of Theorem can be checked by linear spaces [see also the example below], the conditions for having a constant dimension of distributions and codistributions can be seen in the linear case concretely. ■

The equivalence of the set of **Condition C** and those in **Conditions A, B** does not imply that the two series of indices, i.e., the Brunovsky controllability and observability indices coincide. Nevertheless, the relation between the set of indices appears to be our paper's main result. Therefore, we first state the following proposition.

Proposition 4.2. In the set-up of Theorems 2.13 and 2.14 assume $\lambda_1, ..., \lambda_r$ is a set of outputs such that (2.31) holds. Then, for any other choice of these functions, $\tilde{\lambda}_1, ..., \tilde{\lambda}_r$ the associated relative degrees $\tilde{a}_1, ..., \tilde{a}_M$ satisfy:

$$\tilde{a}_i \leq a_i, \qquad i = 1, 2, ..., M. \tag{4.5}$$

Proof. The claim of the proposition is a MIMO analog of Theorem 4.8.2 in [15], where a similar statement is proved for a SISO system. Because multi-input multi-output systems can be considered as several blocks of single-input single-output systems, proposition 4.2 results as an application of theorem 4.8.2 in [15] to several blocks of coordinates. By the way, the claim can also be understood from the block matrices illustration in (4.1). The relation (4.1) implies that the length of the first column matrix can not exceed the length of the second-row matrix, which is independent of λ_i and depends on g_j, [see also [15] theorem 4.8.2]. ■

The proposition 4.2 is a generalization of theorem 4.8.2 in [15] to a MIMO system. For our next result, we recall the following definition.

Definition 4.3. (Transpose partition) The pairs of partitions for a single number, whose Ferrers diagrams (Young tableau) transform into each other when reflected about the line $y = -x$, with the coordinates of the upper left dot taken as $(0, 0)$, are called conjugate (or transpose) partitions. In this case, the corresponding Young tableau is transposed to each other in its rows and columns. An example is illustrated in the following picture.

$$\begin{array}{|c|c|c|} \hline a & d & f \\ \hline b & e & g \\ \hline c & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array} \qquad 3 + 3 + 1 = 3 + 2 + 2 \qquad \begin{array}{|c|c|c|} \hline a & b & c \\ \hline d & e & \multicolumn{1}{c}{} \\ \cline{1-2} f & g & \multicolumn{1}{c}{} \\ \cline{1-2} \end{array} \tag{4.6}$$

The operation is called transposition.

Another example is

$$\begin{array}{|c|c|c|c|c|}\hline a & d & f & h & i \\ \hline b & e & g \\ \cline{1-3} c \\ \cline{1-1}\end{array} \qquad 5+3+1 = 3+2+2+1+1 \qquad \begin{array}{|c|c|c|}\hline a & b & c \\ \hline d & e \\ \cline{1-2} f & g \\ \cline{1-2} h \\ \cline{1-1} i \\ \cline{1-1}\end{array} \tag{4.7}$$

The following theorem explains the relation between Brunovsky controllability indices and the Brunovsky observability indices introduced in Sections 2.1 and 2.2.

Theorem 4.4. *Assume the exact linearization problem is solvable for the system* (1.1) *and the system* (1.4) *due to a choice of outputs* $\lambda_1, ..., \lambda_r$*. The observability indices of* (1.4) *give a transpose partition of* n *concerning the Brunovsky controllability indices for* (1.1)*. In other words, the two partitions*

$$n = h_1 + h_2 + ... + h_N = h'_1 + \cdots + h'_{M-1} + h'_M \tag{4.8}$$

are transpose to each other.

In terms of the corresponding Young tableau, we get a picture like below,

N rows  M rows (4.9)

where the j-th row of the left-hand tableau has h_j boxes, and the j-th row of the right-hand tableau has h'_j boxes. The number of columns in each tableau equals the number of rows in the other one by transposition. The above phenomenon for the SISO systems is not an exciting example; however, it gives exciting examples in MIMO systems.

Proof. (proof of Theorem 4.4) We first consider the Brunovsky observability indices. According to Theorem 2.13 the change of coordinates

$$\phi = \bigoplus_i \begin{bmatrix} \lambda_i \\ L_f \lambda_i \\ ... \\ L_f^{a_i - 1} \lambda_i \end{bmatrix} : \mathbb{R}^n \to \mathbb{R}^n \tag{4.10}$$

transforms system (1.4) into

$$\begin{aligned} \dot{x} &= A\text{"}x + B\text{"}u, \qquad A\text{"} = \begin{pmatrix} A_1 & 0 & ... & 0 & 0 \\ 0 & A_2 & ... & 0 & 0 \\ 0 & 0 & ... & 0 & 0 \\ ... & ... & ... & & \\ 0 & 0 & ... & 0 & A_M \end{pmatrix}, \ u = \begin{bmatrix} u_1 & u_2 & ... & u_r \end{bmatrix}^T \\ y &= C\text{"}x, \end{aligned} \tag{4.11}$$

where the matrices A_j (of size $a_j \times a_j$) are in Brunovsky canonical form in a SISO system. We may put $x = (x_1, ..., x_M)$ where x_j has dimension a_j. We can group the coordinate's content so that in the first group, set all the first coordinate of $x_1, ..., x_M$ and in the second group all the second coordinates. If the sizes of A_j were different, we would fill the coordinate gaps with zeros. Because of their canonical forms of

A_j, differentiating the first block gives the second block, and so on. This permutation of coordinates gives the Brunovsky canonical form (2.37),

$$\begin{aligned} &\dot{x} = A'x + B'u, \qquad A' = \begin{pmatrix} 0 & 0 & ... & 0 & 0 \\ E'_2 & 0 & ... & 0 & 0 \\ 0 & E'_3 & ... & 0 & 0 \\ ... & ... & ... & & \\ 0 & 0 & ... & E'_M & 0 \end{pmatrix}, \; B' = \begin{pmatrix} E'_1 \\ 0 \\ 0 \\ ... \\ 0 \end{pmatrix}, \; u = \begin{bmatrix} u_1 & u_2 & ... & u_r \end{bmatrix}^T \\ &y = Cx, \qquad C = [E'_1 \;\; 0 \;\; 0 \;\; ... \; 0]. \end{aligned} \tag{4.12}$$

Therefore the observability indices can be obtained as

$$h'_k = \sharp\{a_j \geq k, \; j \geq 0\}, \tag{4.13}$$

where $\sharp$ means *number of elements*, considering our new arrangement. The equation (4.1) and the set of equations after that shows that there exists another way to arrange the coordinates; that is we use the set of equations (b). The equation $L_{ad(f)^j g}\psi(x) = D\psi(ad(f)^j g)$ tells that in these coordinates we differentiate first in the direction of flows of g_j for different j. On the other hand, if we consider the set of **Conditions B** we find that a_j also defines the controllability indices. Now the theorem's claim follows from the fact that (4.13) defines the transpose or conjugate partition to the one given by a_j. This, for instance, can be seen by noting that the transpose partition is the reflection of the Young diagram under the line $y = -x$. The identity (4.1) implies that $n = h_1 + h_2 + ... + h_N = h'_1 + \cdots + h'_{M-1} + h'_M$. ■

Remark 4.5. The point to compare the Brunovsky indices both in the controllable and observable cases occurs to be essential to make the relation (4.8) clear. The identity (4.8) shows a transposition duality between these two sets of indices. We claim that it is not clearly attended in the literature despite its simplicity.

Remark 4.6. Because the vectors

$$\mathfrak{d}_i = \begin{bmatrix} d\lambda_i \\ dL_f\lambda_i \\ ... \\ dL_f^{a_i-1}\lambda_i \end{bmatrix}, \qquad i = 1, ..., r \tag{4.14}$$

are independent [cf. [15] lemma 5.1.1] one has

$$\dim\ker\mathfrak{d}_i = n - h_1 - ... - \widehat{h_i} - ... - h_r \tag{4.15}$$

where the hat ($\widehat{.}$) means deletion.

Next, we give a simple example of checking the relation (4.8) and also the computation of Brunovsky indices.

Example 4.7 ([15] Example 5.2.6)**.** We consider the following differential equation near $0 \in \mathbb{R}^5$,

$$\dot{x} = \begin{bmatrix} x_2 + x_2^2 \\ x_3 - x_1x_4 + x_4x_5 \\ x_2x_4 + x_1x_5 - x_5^2 \\ x_5 \\ x_2^2 \end{bmatrix} + u_1 \begin{bmatrix} 0 \\ 0 \\ \cos(x_1 - x_2) \\ 0 \\ 0 \end{bmatrix} + u_2 \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}. \tag{4.16}$$

Then, we define the following distributions on $\mathbb{R}^5$

$$\begin{aligned} \mathfrak{g}_0 &= \langle g_1, g_2 \rangle \\ \mathfrak{g}_1 &= \langle g_1, g_2, ad_f g_1, ad_f g_2 \rangle \\ \mathfrak{g}_2 &= \left\langle g_1, g_2 . ad_f g_1, ad_f^2 g_1, ad_f g_2, ad_f^2 g_2 \right\rangle. \end{aligned} \tag{4.17}$$

We calculate the following:

$$ad_f g_1 = \begin{bmatrix} 0 \\ -\cos(x_1 - x_5) \\ -x_2 \sin(x_1 - x_5) \\ 0 \\ 0 \end{bmatrix}, ad_f g_2 = \begin{bmatrix} 0 \\ -1 \\ -(x_1 - x_5) \\ -1 \\ 0. \end{bmatrix} \tag{4.18}$$

One can quickly check the following relations:

$$\begin{aligned} &[g_1, ad_f g_1] = [g_2, ad_f g_1] = [g_2, ad_f g_1] = [g_1, ad_f g_2] = 0 \\ &[ad_f g_1, ad_f g_2] = \tan(x_1 - x_5) g_1(x). \end{aligned} \tag{4.19}$$

Using the above relations, each distributor's $\mathfrak{g}_j$ are closed under bracket and have constant ranks 2, 4, and 5 near $0 \in \mathbb{R}^5$. Therefore the Brunovsky controllability indices for this system are $h_1 = 2, h_2 = 4-2 = 2, h_3 = 5-4 = 1$.

On the other hand $\dim \mathfrak{g}_1^\perp = 1$. We may easily find a function λ_1 such that $\langle d\lambda_1 \rangle = \mathfrak{g}_1^\perp$. A trivial check shows that $y_1 = \lambda_1(x) = x_1 - x_5$ is such a function. Now, $\dim \mathfrak{g}_0^\perp = 3$. Thus, we have

$$d\lambda_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & -1 \end{bmatrix}, \qquad dL_f \lambda_1 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \end{bmatrix}. \tag{4.20}$$

We may also easily guess a function λ_2 such that $\langle d\lambda_1,\ dL_f\lambda_1, d\lambda_2 \rangle = \mathfrak{g}_0^\perp$. A simple choice is $\lambda_2(x) = x_2$. ThereforeTherefore, methods in control to find the functions λ_1, λ_2. Thus, we have

$$\begin{aligned} &L_{g_1}\lambda_1 = L_{g_2}\lambda_1 = L_{g_1} L_f \lambda_1 = L_{g_2} L_f \lambda_1 = 0 \\ &L_{g_1}\lambda_2 = L_{g_2}\lambda_2 = 0, \end{aligned} \tag{4.21}$$

and the matrix

$$\begin{pmatrix} L_{g_1} L_f^2 \lambda_1 & L_{g_2} L_f^2 \lambda_1 \\ L_{g_1} L_f \lambda_1 & L_{g_1} L_f \lambda_1 \end{pmatrix} \tag{4.22}$$

is non-singular. The Brunovsky observability indices are $h'_1 = 3, h'_2 = 2$. Thus, these numbers are maximal, i.e., they fulfill the total dimension $n = 5$. Now one sees that the two partitions of the system dimension by Brunovsky indices obtained by the above two ways are transposed to each other:

$$\begin{aligned} &h = (h_1 = 2, h_2 = 2, h_3 = 1) \\ &h' = (h'_1 = 3, h'_2 = 2) \\ &5 = 2 + 2 + 1 = 3 + 2. \end{aligned} \tag{4.23}$$

We can also see this in the Young diagrams

$$\begin{array}{|c|c|} \hline a & b \\ \hline d & e \\ \hline f & \\ \cline{1-1} \end{array} \qquad 2 + 2 + 1 = 3 + 2 \qquad \begin{array}{|c|c|c|} \hline a & d & f \\ \hline b & e & \\ \cline{1-2} \end{array} \tag{4.24}$$

5. Conclusion

When the exact linearization problem is solvable for the general system (1.1), it can be linearized in two different canonical ways. In this case, one obtains two different sets of Brunovsky indices, which are invariants of the system. We have compared the two sets of Brunovsky indices in a controllable system of the canonical form (1.1) and the associated MIMO system (1.4) of dimension n. Several new results have been obtained in the way of the proof. For instance, we provide a new set of criteria that are equivalent to the ones in [16]. In this regard, our results are a complement to [16], which this work has been partially motivated by their main results: Theorems 4.1 and 4.4. The conclusion is that the two sets of invariants, namely the Brunovsky controllability indices and the Brunovsky observability indices, give two transpose partitions of n. The result has been analyzed in an example. In general, these two sets of indices are not identical on the same system and may not have an equal number of elements.

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