The full analytic trans-series in integrable field theories

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Abstract

We analyze a family of generalized energy densities in integrable quantum field theories in the presence of an external field coupled to a conserved charge. By using the Wiener-Hopf technique to solve the linear thermodynamic Bethe ansatz equations we derive the full analytic trans-series for these observables in terms of a perturbatively defined basis. We show how to calculate these basis elements to high orders analytically and reveal their complete resurgence structure. We demonstrate that the physical value of the generalized energy densities is obtained by the median resummation of their ambiguity-free trans-series.

1 Introduction

The standard tool to investigate interacting systems is perturbation theory. The perturbative series is typically asymptotic and the factorial growth signals non-perturbative terms. For a complete description one has to build a multiple series, i.e. a trans-series both in the perturbative coupling and in the exponentially suppressed non-perturbative corrections. This trans-series is understood as Borel resummed, and the requirement of being free of ambiguities requires an intricate interplay between the various perturbative and non-perturbative terms. The theory which formulates this is called resurgence, which lives its renaissance now, see [1, 2, 3] for recent reviews. Most of the resurgence applications however, originate from differential equations. In contrast, we would like to report here on a relevant progress in a wide class of integral equations. These linear integral equations appear in integrable systems, when one calculates the ground-state energy density via the thermodynamic limit of the Bethe ansatz equations (TBA).

Systems soluble by the Bethe ansatz are relevant in condensed matter systems, in statistical as well as in particle physics [4, 5]. They provide explicitly soluble toy models, where non-perturbative, strongly interacting phenomena can be analyzed in simplified circumstances. Additionally, some of them also have experimental realizations. Recently there has been great progress in their perturbative as well as leading non-perturbative analysis. On the condensed matter side the groundstate energy density of the Lieb-Liniger, Gaudin-Yang and Hubbard models together with their generalizations were investigated

[6, 7, 8, 9, 10, 11]. The non-pertubative terms were in many cases related to the superconductive gap as well as to renormalon diagrams. On the particle physics side free energies of asymptotically free integrable quantum field theories in the presence of an external field coupled to a conserved charge were analyzed [12, 13, 14, 15, 16, 17, 18]. These included the $O(N)$ non-linear sigma model and its supersymmetric extension, the Gross-Neveu model and the principal chiral field for which the large order behavior of the perturbative series were also investigated [19, 20, 21, 22, 23]. In these quantum theories the non-perturbative terms are related to instantons or renormalons, which were further confirmed by large N calculations [24, 25] and in the $O(3)$ model by introducing a θ-term [22].

In constructing the ambiguity free trans-series the first problem is to efficiently calculate the perturbative terms. This was first achieved for the energy density of $O(N)$ models in [26, 27] by matching the behavior of the resolvent in the central and edge regions. The method was extended for statistical models and for the circular plate capacitor [6, 28] and by combining with the Wiener-Hopf technique to integrable quantum field theories [29]. The first few exponentially suppressed corrections can be extracted from the asymptotics of the perturbative coefficients [6, 29, 7, 8, 30, 31, 20, 21, 23]. A systematic treatment based on the Wiener-Hopf approach was presented in [19], which resulted in the precise structure of the trans-series and explicit calculations of the first few non-perturbative corrections. This was further extended to higher orders and improved by introducing the running coupling for the $O(N)$ models in [23]. The aim of our present paper is to solve completely these models by determining the full trans-series, i.e. all the non-perturbative terms together with their perturbative expansions. We are doing this by expressing these higher perturbative expansions in terms of the original perturbative series of generalized observables, which we also obtain from the known perturbative series of the ground-state energy.

The paper is organized as follows. In section 2 we introduce the integral equation, the generalized observables and the differential equations which relate them to each other. In section 3 we demonstrate how the Wiener-Hopf technique can be used to calculate these generalized observables. This provides a structural result, which we make explicit by introducing a perturbatively calculable basis in section 4 and constructing the full trans-series. In section 5 we present a method for determining the basis and investigate how the various parts of the trans-series are interconnected. We also relate its median resummation to the TBA result. Finally, in section 6 we provide explicit examples and conclude in section 7.

2 Observables and their properties

We investigate linear integral equations of the form

$$\chi_n(\theta) - \int_{-B}^{B} d\theta' K(\theta-\theta')\chi_n(\theta') = r_n(\theta) \quad ; \qquad |\theta| \leq B \quad , \tag{1}$$

where $r_n(\theta) = \cosh n\theta$ and the kernel is a symmetric function, which, in most of the applications, is related to the logarithmic derivative of the scattering matrix [12, 13, 14, 15, 16, 17, 18, 6]. We are interested in the observables

$$\mathcal{O}_{n,m} = \int_{-B}^{B} \frac{d\theta}{2\pi} \chi_n(\theta) r_m(\theta) \tag{2}$$

as functions of B, but we do not indicate this dependence explicitly. This observable is symmetric in n and m, which are not necessarily integers but non-negative. Its B-derivative (which we denote by a dot) can be written in terms of the boundary values of

χ_n-s as [23]

$$\frac{d\mathcal{O}_{n,m}}{dB} \equiv \dot{\mathcal{O}}_{n,m} = \frac{1}{\pi}\chi_n(B)\chi_m(B) \quad . \tag{3}$$

By generalising the manipulation of the integral equation in [11, 10] one can show that these boundary values satisfy the differential equation

$$\frac{\ddot{\chi}_n(B)}{\chi_n(B)} - n^2 = f(B) \quad , \tag{4}$$

where $f(B)$ is an n-independent function, which can be calculated, for instance, from the $n = 1$ case. These equations connect all observables to one of them, say to $\mathcal{O}_{1,1}$, which is the groundstate energy of the integrable model in a magnetic field coupled to a conserved charge. The observable $\mathcal{O}_{n,m}$ for n, m integers can be interpreted as the expectation value of the conserved spin m charge, in the presence of the magnetic field, when the Hamiltonian is given by the conserved spin n charge.

3 Wiener-Hopf integral equation

The standard way to solve the integral equation is the Wiener-Hopf technique [13, 14, 19, 23]. As a first step we extend the source as $r_n(\theta) = \Theta(-\theta + B)\frac{e^{n\theta}}{2} + \Theta(\theta + B)\frac{e^{-n\theta}}{2}$ as well as the integrations, (but not $\chi_n(\theta)$), for the whole line

$$\chi_n(\theta) - \int_{-\infty}^{\infty} d\theta' K(\theta - \theta')\chi_n(\theta') = r_n(\theta) + L(\theta) + R(\theta) \tag{5}$$

by paying the price of introducing an unknown function $R(\theta) = L(-\theta)$, which, however, vanishes for $\theta < B$. In solving the equation in Fourier space the key point is the factorization

$$\frac{1}{1 - \tilde{K}(\omega)} = G_+(\omega)G_-(\omega) \quad ; \qquad G_-(\omega) = G_+(-\omega) \tag{6}$$

into factors analytic in the lower and upper half planes. Implementing the separation of the equation into lower and upper half analytical pieces we arrive at

$$X_n(i\kappa) + \int_{-\infty}^{\infty} \frac{e^{2i\omega B}\sigma(\omega)X_n(\omega)}{\kappa - i\omega}\frac{d\omega}{2\pi} = \frac{1}{n - \kappa} \quad , \tag{7}$$

where $\sigma(\omega) = \frac{G_-(\omega)}{G_+(\omega)}$ and the unknown function $X_n(\omega)$ is related to the Fourier transform of $R(\theta)$ as $X_n(\omega) = \frac{2e^{-(n+i\omega)B}G_+(\omega)\tilde{R}(\omega)}{G_+(in)} + \frac{G_+(\omega)}{G_+(in)}\frac{1}{(n+i\omega)}$. Except for the explicitly introduced pole at $\omega = in$, $X_n(\omega)$ is analytic in the upper half plane. We also assume that $n > 0$. The $n = 0$ case requires special care [19, 23] and we can recover it by solving the differential equations (3,4).

In the typical applications $\sigma(\omega)$ has a cut and poles at $i\kappa_l$, $l = 1, 2, \ldots$, on the positive imaginary line. Additionally, we also have the explicit pole at in coming from X_n. It is advantageous to disentangle the poles from the cut by moving the cut a bit away from the imaginary line in either direction [19, 23]. We then deform the integration contour from the real line surrounding the cut and separately the poles whose residues we collect. We can do it in two different ways, by integrating a bit left or right of the poles. The residues will also depend on this choice, but the final result must be the same. This is the manifestation of a Stokes phenomena and the two different choices will be related to the two lateral resummations. For definiteness, we integrate a bit left of the imaginary line (in κ a bit above the real positive line):

$$X_n(i\kappa)+i\sum_{l=0}^{\infty}\frac{S_l q_{n,\kappa_l}}{\kappa+\kappa_l}e^{-2\kappa_l B}+\int_{C_+}e^{-2B\kappa'}\frac{\delta\sigma(i\kappa')X_n(i\kappa')}{\kappa+\kappa'}\frac{d\kappa'}{\pi}=\frac{1}{n-\kappa}\quad , \tag{8}$$

where $q_{n,\kappa_l}=X_n(i\kappa_l)$ and S_l is the residue of $i\sigma(i\kappa+0)$ at κ_l, while $\delta\sigma(\kappa)=\frac{1}{2i}(\sigma(i\kappa-0)-\sigma(i\kappa+0))$ is the discontinuity of σ. We included in the sum the contribution of the pole of $X_n(i\kappa)$ at $\kappa_0=n$ with residue $S_0=-i\sigma(in+0)=-i\sigma_n^+$ with the convention that $q_{n,n}=1$. Here we assume that all poles κ_l are distinct, including κ_0, i.e. $n\neq\kappa_l$. In the more general case, $\sigma(i\kappa)$ can have higher order poles, which could even coincide with the pole at in. In this case S_l and q_{n,κ_l} are related to the expansion of the functions around the singularity, however, we do not consider these complicated cases in this short letter, see our upcoming paper for further details [32].

Typically, we can always introduce a running coupling v

$$\kappa=vx\quad ;\qquad 2B=\frac{1}{v}+\gamma\log v+L \tag{9}$$

with an arbitrary constant L, such that the integral equation for the rescaled variable $Q_n(x)=X_n(ivx)$ takes the generic form

$$Q_n(x)+i\sum_{l=0}^{\infty}\frac{S_l q_{n,\kappa_l}\nu^{\kappa_l}}{\kappa_l+vx}+\int_{C_+}\frac{e^{-y}\mathcal{A}(y)Q_n(y)}{x+y}\frac{dy}{\pi}=\frac{1}{n-vx}\quad , \tag{10}$$

where $q_{n,\kappa_l}=Q_n(\frac{\kappa_l}{v})$, $q_{n,n}=1$ and $\nu=e^{-2B}=e^{-L}v^{-\gamma}e^{-1/v}$. The model-dependent parameter γ has to be chosen such that $\mathcal{A}(y)=e^{-vy(\gamma\log v+L)}\delta\sigma(vy)$ has a power-series expansion in v without any $\log v$ terms: $\mathcal{A}(y)=\sum_{j=0}^{\infty}v^j\alpha_j(y)$. By appropriately choosing L the linear y-dependence in $\log\mathcal{A}(y)$ can be canceled. Here q_{n,κ_l}-s (except $q_{n,n}=1$) are also unknowns, which have to be calculated by evaluating the integral equation (10) at the positions $xv=\kappa_l$. If $Q_n(x)$ including q_{n,κ_l} are determined, then the observable $\mathcal{O}_{n,m}$ can be written (similarly to [23]) as

$$\mathcal{O}_{nm}=\frac{e^{(n+m)B}}{4\pi}G_+(im)G_+(in)W_{n,m} \tag{11}$$

with

$$W_{n,m}=\frac{1}{n+m}+i\sum_{l=0}^{\infty}\frac{S_l q_{n,\kappa_l}\nu^{\kappa_l}}{m-\kappa_l}+\sigma_m^+\nu^m q_{n,m}+\frac{v}{\pi}\int_{C_+}\frac{e^{-x}\mathcal{A}(x)Q_n(x)}{m-vx}dx\quad , \tag{12}$$

where $q_{n,m}=Q_n(\frac{m}{v})$. Here we assumed that $n\neq\kappa_l$ and $n\neq m$, otherwise we have to calculate the residue of a second order pole. The $n=m$ case can be recovered by taking the $n\to m$ limit. For the boundary value of the field, similarly to [23], we obtain

$$\chi_n(B)=\frac{e^{nB}}{2}G_+(in)w_n \tag{13}$$

with

$$w_n=1+i\sum_{l=0}^{\infty}S_l q_{n,\kappa_l}\nu^{\kappa_l}+\frac{v}{\pi}\int_{C_+}e^{-x}\mathcal{A}(x)Q_n(x)dx \tag{14}$$

for $n\neq 0$. In the case of $n=0$ we need to calculate f from χ_1 in (4) and solve the equation (4) for χ_0 and (3) for $\mathcal{O}_{0,0}$.

4 Trans-series ansatz and its solution

We solve these equations for $Q_n(x)$ in terms of a trans-series ansatz

$$Q_n(x) = \sum_{l=0}^{\infty} \nu^{d_l} \sum_{j=0}^{\infty} Q_{n,j}^{(d_l)}(x) v^j \quad , \tag{15}$$

where the set of nonzero $Q_{n,j}^{(d_l)}$-s is model-dependent. One has to investigate the set $\{\kappa_l\}$, which (up to some isolated cases such as κ_0) can be described as a union of finitely many sets of the form $\{a_i l + b_i\}$ with $l = 1, 2, \dots$. We should introduce d_l such that all non-peturbative corrections are accounted for. We give concrete examples later.

The generic solution to (15) can be calculated iteratively in l. We start with the $l = 0$ perturbative part, i.e. we have to solve perturbatively the following problem:

$$P_\alpha(x) + \int_{C_+} \frac{e^{-y}\mathcal{A}(y) P_\alpha(y)}{x+y} \frac{dy}{\pi} = \frac{1}{\alpha - vx} \quad . \tag{16}$$

This can be done by expanding $\mathcal{A}(y)$ and the source term in power series in v and iteratively solving at any order based on lower order solutions [19, 23]. We will see, however, that the explicit solution is not needed. Observe also that originally we needed P_α for $\alpha > 0$, but the equation and the perturbative solution make perfect sense also for $\alpha < 0$. Using these solutions the unknown $Q_n(x)$ can be written as

$$Q_n(x) = P_n(x) + i \sum_{l=0}^{\infty} S_l q_{n,\kappa_l} \nu^{\kappa_l} P_{-\kappa_l}(x) \quad , \tag{17}$$

where, from the definition of q_{n,κ_s}, we obtain a closed system of linear equations of the form

$$q_{n,\kappa_s} - i \sum_{l=0}^{\infty} S_l q_{n,\kappa_l} \nu^{\kappa_l} A_{-\kappa_l,-\kappa_s} = A_{n,-\kappa_s} \tag{18}$$

with the exception of $q_{n,n} = 1$. Here we introduced the symmetric building block (for $\alpha \neq -\beta$) as

$$A_{\alpha,\beta} = \frac{1}{\alpha+\beta} + \langle P_\alpha \rangle_\beta \quad , \tag{19}$$

which contains the moment

$$\langle Q \rangle_\beta = \int_{C_+} \frac{e^{-x}\mathcal{A}(x) Q(x)}{\beta - vx} \frac{vdx}{\pi} \quad . \tag{20}$$

The symmetric moments $\langle P_\alpha \rangle_\beta$ are understood perturbatively in v and are well-defined for any signs of α and β. We note that the recursive structure for q_{n,κ_l} is the consequence of the integral equation, where the model-specific feature lies in the set κ_l (the non-perturbative nature) as well as in $A_{n,m}$ (the perturbative nature). In the following we solve this linear system of equations. Since $q_{n,n} = 1$ is not an unknown, we regard its contribution as an inhomogeneous source term

$$q_{n,\kappa_s} - i \sum_{l=1}^{\infty} q_{n,\kappa_l} S_l \nu^{\kappa_l} A_{-\kappa_l,-\kappa_s} = s_{n,-\kappa_s} \quad ; \qquad s_{n,-\kappa_s} = A_{n,-\kappa_s} + \sigma_n^+ \nu^n A_{-n,-\kappa_s} \quad . \tag{21}$$

This linear matrix equation $(\mathbf{I} - \mathbf{A})\mathbf{q}_n = \mathbf{s}_n$, with $\mathbf{A}_{s,l} = i S_l \nu^{\kappa_l} A_{-\kappa_s,-\kappa_l}$ can be solved by inversion $\mathbf{q}_n = (\mathbf{I} - \mathbf{A})^{-1}\mathbf{s}_n$, which can be represented by the Neumann series $\mathbf{q}_n =$

$(\mathbf{I}+\mathbf{A}+\mathbf{A^2}+\ldots)\mathbf{s}_n$ and expanded in ν. Alternatively, we can plug back recursively every lower order solution in to the ν-expansion, leading to

$$q_{n,\kappa_s}=\sum_{\rm paths} sA_{n,{\rm path},-\kappa_s}S_{\rm path} \quad , \tag{22}$$

where a path means a sequence starting from n and ending at $-\kappa_s$: $(n,l_1,l_2,\ldots,l_N,-\kappa_s)$. The contribution of such a path is

$$\begin{aligned} sA_{n,{\rm path},-\kappa_s} &= s_{n,-\kappa_{l_1}}A_{-\kappa_{l_1},-\kappa_{l_2}}\ldots A_{-\kappa_{l_{N-1}},-\kappa_{l_N}}A_{-\kappa_{l_N},-\kappa_s} \\ S_{\rm path} &= iS_{l_1}\nu^{\kappa_{l_1}}\ldots iS_{l_N}\nu^{\kappa_{l_N}} \qquad , \end{aligned} \tag{23}$$

where the inner indices take only the values $l_k=1,2,\ldots$. At each non-perturbative order in ν we have only a finite number of terms contributing. With this solution the unknown function can also be written as

$$Q_n\left(\frac{m}{v}\right)=q_{n,m}=A_{n,-m}+i\sum_{l=0}^{\infty}S_l q_{n,\kappa_l}\nu^{\kappa_l}A_{-\kappa_l,-m} \quad . \tag{24}$$

The observables $W_{n,m}$ can be obtained in terms of q_{n,κ_l} as

$$W_{n,m}=s_{m,n}+i\sum_{l=0}^{\infty}S_l q_{n,\kappa_l}\nu^{\kappa_l}s_{m,-\kappa_l}=A_{n,m}+O(\nu) \quad . \tag{25}$$

Clearly, the basic building block $A_{n,m}$ is nothing but the perturbative part of our generic observable $W_{n,m}$. The boundary value of the field can be expressed as

$$w_n=a_n+i\sum_{l=0}^{\infty}S_l q_{n,\kappa_l}\nu^{\kappa_l}a_{-\kappa_l} \quad , \tag{26}$$

where $a_\alpha=\lim_{\beta\to\infty}\beta A_{\alpha,\beta}$.

By this we provided a complete solution of the problem, i.e. we expressed the observables in terms of the perturbatively defined $A_{n,m}$-s. In the following we explain how the perturbative expansion of the building blocks can be calculated.

5 Median resummation and alien derivatives

Let us summarize what we have achieved so far. The observables $\mathcal{O}_{n,m}$ and $\chi_n(B)$ can be written in terms of $W_{n,m}$ and w_n as

$$\mathcal{O}_{nm}=\frac{e^{(n+m)B}}{4\pi}G_+(im)G_+(in)W_{n,m} \quad ; \qquad \chi_n(B)=\frac{e^{nB}}{2}G_+(in)w_n \quad , \tag{27}$$

which satisfy two differential equations

$$(n+m)W_{n,m}+\dot W_{n,m}=w_nw_m \tag{28}$$
$$2n\dot w_n+\ddot w_n=fw_n \tag{29}$$

and have the solutions (25) and (26) in terms of q_{n,κ_s}, which is given by (22).

The perturbative parts $A_{n,m}$ and a_n satisfy the $W\to A, w\to a$ differential equations (28,29). Since Volin's method [26, 27, 29] determines $W_{1,1}$ at the perturbative level it provides $A_{1,1}$. We can then extract the perturbative part of w_1, namely a_1 from (28), and by plugging back to eq. (29) we can extract a_n for any n, not necessarily positive integer. These perturbative series then can be used to calculate the expansion of $A_{n,m}$ to the desired order from the perturbative part of (28). By using these building blocks the all order solution for q_{n,κ_l} , $W_{n,m}$ and w_n can be built up.

The results for $q_{n,m}$, $W_{n,m}$ and w_n are given in terms of trans-series, which is understood as laterally Borel resummed. This prescription does not follow from our derivation, although very plausible from the contour shift, see also [19] for comments about this point. Thus we assume that the lateral Borel resummation of the trans-series solution gives the TBA result. Since the TBA result is free of ambiguities, we can calculate the various alien derivatives of $A_{n,m}$ from the ambiguity cancellations. These quantities differ only by the various source terms, thus we expect that the $(\mathbf{I}-\mathbf{A})^{-1}$ operation guarantees the ambiguity cancellation.

We analyze the behaviour of $q_{n,m}$ using resurgence theory and alien derivatives following [2, 3, 31]. Assuming $m > 0$ and $n > \kappa_1$, the leading singularity of the Borel transform of $A_{n,-m}$ on the positive real line is at κ_1. The corresponding ambiguity, which is encoded in the alien derivative[1] $\Delta_{\kappa_1} A_{n,-m}$ has to be canceled by the leading non-perturbative correction of order ν^{κ_1}, i.e. by $iS_l A_{n,-\kappa_l} A_{-\kappa_l,-m}$ leading to $\Delta_{\kappa_1} A_{n,-m} = 2iS_1 A_{n,-\kappa_1} A_{-\kappa_1,-m}$. By moving iteratively further and subtracting the already known alien derivatives one can show that

$$\Delta_{\kappa_l} A_{n,m} = 2iS_l A_{n,-\kappa_l} A_{-\kappa_l,m} \quad . \tag{30}$$

We can then construct a multi-parameter trans-series for our basic quantity as

$$\hat{q}_{n,m}(\{\sigma\}) = \sum_{\text{paths}} sA_{n,\text{path},-m}\sigma_{\text{path}} \quad , \tag{31}$$

where

$$\sigma_{\text{path}} = \sigma_{l_1}\nu^{\kappa_{l_1}} \dots \sigma_{l_N}\nu^{\kappa_{l_N}} \quad . \tag{32}$$

By using (22) and (30) one can show that the action of the pointed alien derivative $\dot{\Delta}_{\kappa_l}$ on the trans-series is equivalent to $2iS_l$ times differentiation wrt. σ_l:

$$\dot{\Delta}_{\kappa_l}\hat{q}_{n,m}(\{\sigma\}) \equiv \nu^{\kappa_l}\Delta_{\kappa_l}\hat{q}_{n,m}(\{\sigma\}) = 2iS_l\partial_{\sigma_l}\hat{q}_{n,m}(\{\sigma\}) \quad . \tag{33}$$

The Stokes automorphism which relates the two lateral Borel resummations is the exponentiation of all the alien derivatives, which then acts as

$$\begin{aligned} \mathcal{S}\hat{q}_{n,m}(\{\sigma\}) &= e^{\sum_l \dot{\Delta}_{\kappa_l}}\hat{q}_{n,m}(\{\sigma\}) = e^{\sum_l 2iS_l\partial_{\sigma_l}}\hat{q}_{n,m}(\{\sigma\}) \\ &= \hat{q}_{n,m}(\{\sigma_l \to \sigma_l + 2iS_l\}) \quad . \end{aligned} \tag{34}$$

The ambiguity free median resummation

$$\mathcal{S}^{\frac{1}{2}}\hat{q}_{n,m}(\{\sigma\}) = \hat{q}_{n,m}(\{\sigma_k \to \sigma_k + iS_k\}) \tag{35}$$

is then nothing but the TBA result, if we turn off every σ expect the one corresponding to the source (22). Similarly, $W_{n,m} = \mathcal{S}^{\frac{1}{2}}(s_{m,n})$.

6 Examples

In this section we provide some examples. There is a large class of integrable particle-models, where a magnetic field can be coupled to one of the global charges and the energy

[1] The alien derivative is understood in the running coupling v as: $\dot{\Delta}_n = \nu^n\Delta_n$, where $[\dot{\Delta}_n, \partial_B] = 0$. Thus it has an extra $v^{-\gamma}$ factor compared to the standard definition.

density can be investigated by the thermodynamic limit of the Bethe ansatz equations. In these cases the kernel is related to the logarithmic derivative of the scattering matrix and the Wiener-Hopf method leads to a generic structure. We focus here on the bosonic models having

$$\sigma(i\kappa \pm 0) = e^{\gamma\kappa\log\kappa + b\kappa}\frac{H(-\kappa)}{H(\kappa)}\left(\mp i\cos(\frac{\gamma\pi\kappa}{2}) + \sin(\frac{\gamma\pi\kappa}{2})\right) \quad , \tag{36}$$

where γ, b are model-dependent constants, while $H(\kappa)$ is a model-dependent product of gamma-functions. We indicated the signs of the residues depending on the two possible ways how we can shift the cut away from the imaginary line. The poles on the imaginary line are determined by the careful analysis of $H(\kappa)$, which we go through model by model. Clearly, the running coupling can be always introduced with γ and b, together with another linear term coming from the H-s, can be transformed out by an appropriate choice of L. With this choice the kernel in the integral equation takes the form

$$\mathcal{A}(x) = \cos(\frac{\gamma\pi v x}{2})e^{\gamma v x\log x + \sum_{k=1}^{\infty} z_{2k+1}(vx)^{2k+1}} \quad , \tag{37}$$

where z_k is proportional to ζ_k in a model-dependent way.

We start with the observable $A_{1,1}$, which can be calculated by modifying Volin's method to keep track of the ζ-s coming from the kernel. The result is

$$\begin{aligned}2A_{1,1} = {} & 1 + \frac{v}{2} + \left(\frac{5\gamma}{4} + \frac{9}{8}\right)v^2 + \left(\frac{10\gamma^2}{3} + \frac{53\gamma}{8} + \frac{57}{16}\right)v^3 \\ & + \frac{1}{384}v^4\left(-36\gamma^3(21\zeta_3 - 94) + 10924\gamma^2 + 13344\gamma + 9(144z_3 + 625)\right) + O\left(v^5\right) \quad .\end{aligned} \tag{38}$$

We regard this as an input to our analysis and show how all the non-perturbative parts can be determined from this. We can calculate this series analytically up to 50 orders and numerically up to few hundred orders with very high precision [30, 31, 21]. For demonstration, we merely included here the first few terms, and keep doing the same from now on. By using the differential equation $2A_{1,1} + \dot{A}_{1,1} = a_1^2$, i.e. the perturbative part of (28), one can obtain

$$\begin{aligned}a_1 = {} & 1 + \frac{v}{4} + \left(\frac{5\gamma}{8} + \frac{9}{32}\right)v^2 + \left(\frac{5\gamma^2}{3} + \frac{53\gamma}{32} + \frac{75}{128}\right)v^3 \\ & + \frac{v^4\left(-288\gamma^3(21\zeta_3 - 94) + 43696\gamma^2 + 35160\gamma + 9(1152z_3 + 1225)\right)}{6144} + O\left(v^5\right) \quad .\end{aligned} \tag{39}$$

Then using (29) for $n = 1$ we obtain

$$f = -v^2 - 6\gamma v^3 - 26\gamma^2 v^4 + v^5\left(\frac{1}{4}\gamma^3(63\zeta_3 - 386) - 27z_3\right) + O\left(v^6\right) \quad . \tag{40}$$

By solving (29) for other n-s we can get

$$\begin{aligned}a_n = {} & 1 + \frac{v}{4n} + \frac{v^2(20\gamma n + 9)}{32n^2} + \frac{v^3\left(640\gamma^2 n^2 + 636\gamma n + 225\right)}{384n^3} \\ & + \frac{v^4\left(288n^3\left(\gamma^3(94 - 21\zeta_3) + 36z_3\right) + 43696\gamma^2 n^2 + 35160\gamma n + 11025\right)}{6144n^4} + O\left(v^5\right) \quad .\end{aligned} \tag{41}$$

Actually a_n can be obtained directly from a_1 by the $v \to \frac{v}{n}$, $\gamma \to \gamma n$, $z_{2k+1} \to n^{2k+1}z_{2k+1}$ replacements.The exceptional χ_0 is

$$\chi_0 = \frac{1}{\sqrt{v}}\left(1 - \frac{\gamma v}{2} - \frac{5\gamma^2 v^2}{8} + \frac{1}{16}v^3\left(\gamma^3(7\zeta_3 - 15) - 12z_3\right) + O(v^4)\right) \quad . \tag{42}$$

Finally, by solving (28) we obtain the basic building blocks

$$A_{n,m} = \frac{1}{m+n} + \frac{v}{4mn} + \frac{v^2(20\gamma mn + 9m + 9n)}{32m^2n^2} \\ + \frac{v^3\left(m^2\left(640\gamma^2n^2 + 636\gamma n + 225\right) + 6mn(106\gamma n + 39) + 225n^2\right)}{384m^3n^3} + O\left(v^4\right) \quad . \tag{43}$$

In order to get the generic solutions in terms of these A-s as (22,25,26) we need the locations κ_l and Stokes constants S_l, which we analyse model by model.

6.1 $O(N)$ models

The $O(N)$ non-linear sigma models in a magnetic field coupled to one of the $O(N)$ charges [13, 14] can be analyzed by the thermodynamic limit of the Bethe Ansatz equation, which takes the form of the integral equation (1) with the kernel related to the S-matrix [33]

$$S(\theta) = -\frac{\Gamma(\frac{1}{2} - \frac{i\theta}{2\pi})\Gamma(\Delta - \frac{i\theta}{2\pi})\Gamma(1 + \frac{i\theta}{2\pi})\Gamma(\Delta + \frac{1}{2} + \frac{i\theta}{2\pi})}{\Gamma(\frac{1}{2} + \frac{i\theta}{2\pi})\Gamma(\Delta + \frac{i\theta}{2\pi})\Gamma(1 - \frac{i\theta}{2\pi})\Gamma(\Delta + \frac{1}{2} - \frac{i\theta}{2\pi})} \tag{44}$$

as $K(\theta) = \frac{1}{2\pi i}\partial_\theta \log S(\theta)$, where $\Delta = \frac{1}{N-2}$. For the $\cosh n\theta$ source term $\mathcal{O}_{n,m}$ describes the expectation value of the spin m conserved charge with the Hamiltonian being the spin n charge. The energy density analyzed in the literature corresponds to $\mathcal{O}_{11}$. The Wiener-Hopf decomposition gives (36) with

$$\gamma = 2\Delta - 1 \quad ; \quad H(\kappa) = \frac{\Gamma(1 + \Delta\kappa)}{\Gamma(\frac{1}{2} + \frac{\kappa}{2})} \tag{45}$$

and the kernel is described by

$$z_{2k+1} = 2\frac{\zeta_{2k+1}}{2k+1}(\Delta^{2k+1} - 1 + 2^{-2k-1}) \quad . \tag{46}$$

The zeros of $\sigma(i\kappa)$ are located model-independently at the positions $\kappa = 2l - 1$, while its poles are at $\kappa = l(N-2)$, where $l \in \mathbb{N}$. This implies that $\kappa_l = l\kappa_1$ with $\kappa_1 = N - 2$ for N even and $\kappa_1 = 2N - 4$ for N odd [19, 23].

6.1.1 $O(4)$ model

Let us start with the $O(4)$ model, which is the simplest. In this case the running coupling is $v = \frac{1}{2B}$ and $\Delta = \frac{1}{2}$. The poles and the zeros do not interact and $\kappa_l = 2l$; $l = 1, 2, \ldots$, with residues

$$S_l = \frac{((2l-1)!!)^2}{2^{2l-1}l!(l-1)!} \quad . \tag{47}$$

Observe also that $\sigma(in) = 0$ for n odd, i.e. $S_0 = 0$, so in these cases $\kappa_0 = n$ is not singular and we do not have the $l = 0$ term in the sums. With these building blocks the trans-series for the observable w_1 takes the form

$$w_1 = a_1 + \sum_{l_1,l_2,\ldots} e^{-4(l_1+l_2+\ldots)B}(iS_{l_1})(iS_{l_2})\ldots A_{1,-2l_1}A_{-2l_1,-2l_2}\ldots a_{-l_k} \quad . \tag{48}$$

It is free of ambiguities due to the relation

$$\Delta_{2l}A_{\alpha,\beta} = 2iS_lA_{\alpha,-2l}A_{-2l,\beta} \quad . \tag{49}$$

Finally $W_{1,1}$ can be obtained from eq. (28) as

$$W_{1,1}=A_{1,1}+Me^{-2B}+\sum_{l_1,l_2,\dots}e^{-4(l_1+l_2+\dots)B}iS_{l_1}iS_{l_2}\dots A_{1,-2l_1}A_{-2l_1,-2l_2}\dots A_{-l_k,1}\quad, \tag{50}$$

where M is an integration constant, which comes from the zero mode of $2+\partial_B$. By taking the $n\to 1$ and $m\to 1$ limit in (25) it can be calculated explicitly to be $M=-2i$. The first few terms take the form

$$W_{1,1}=A_{1,1}+Me^{-2B}+ie^{-4B}S_2A_{1,-2}^2+e^{-8B}((iS_2)^2A_{1,-2}^2A_{-2,-2}+iS_4A_{1,-4}^2)+\dots\quad. \tag{51}$$

By explicitly investigating the analytic structure of $A_{1,1}$ on the Borel plane we confirmed the perturbative expansion of all these terms up to high orders. We also verified numerically that the median resummation reproduced the TBA result.

These results have a direct extension for $N>4$. The only difference is that κ_1 is N-dependent, otherwise the poles form the lattice $\kappa_l=l\kappa_1$ and the generic solutions in terms of the A-s (22,25,26) applies. The integration constant for $W_{1,1}$ is $M=-2e(\frac{\Delta}{e})^{2\Delta}\frac{\Gamma(1-\Delta)}{\Gamma(1+\Delta)}e^{i\pi\Delta}$.

6.1.2 $O(3)$ model

The $O(3)$ model is the most complicated among the $O(N)$ models. This is due to the fact that $\sigma(i)\neq 0$, and we have to carry the $l=0$ term in the sums for $W_{1,1}$ and w_1. The poles of $\sigma(i\kappa)$ are again located at $\kappa_l=2l$. The building blocks $A_{n,m}$ can be used here with $\Delta=1$. By focusing on w_1 the main difference compared to the $O(4)$ model is that additionally to the $O(4)$ like sums, we also have others starting at ν:

$$\begin{aligned}w_1=a_1+&\sum_{l_1,l_2,\dots}\nu^{2(l_1+l_2+\dots)}iS_{l_1}iS_{l_2}\dots A_{1,-2l_1}A_{-2l_1,-2l_2}\dots a_{-l_k}\\ &iS_0\nu\left(1+\sum_{l_1,l_2,\dots}\nu^{2(l_1+l_2+\dots)}iS_{l_1}iS_{l_2}\dots A_{-1,-2l_1}A_{-2l_1,-2l_2}\dots a_{-l_k}\right)\quad.\end{aligned} \tag{52}$$

What is interesting is that the two parts are not related by any resurgence relations, i.e. the Δ_1 alien derivative of the first line is not related to the second line. This can be also seen by noting that S_0 is imaginary and the real leading term cannot be related to the purely imaginary alien derivative of a real series. The even alien derivatives satisfy the relations as before $\Delta_{2l}A_{n,m}=2iS_lA_{n,-2l}A_{-2l,m}$. The observable $W_{1,1}$ can be obtained by integrating the differential equation (28). Again the constant term should be fixed from taking the $n\to 1$ and $m\to 1$ limit.

6.2 Principal chiral models

The $SU(N)$ principal chiral model can be described by

$$\gamma=0\quad;\quad H(\kappa)=\frac{\Gamma(1+(1-\Delta)\kappa)\Gamma(1+\Delta\kappa)}{\Gamma(1+\kappa)}\quad, \tag{53}$$

where $\Delta=1/N$. In order to use the generic forms we need the replacements

$$z_{2k+1}=2\frac{\zeta_{2k+1}}{2k+1}\left(-1+\Delta^{2k+1}+(1-\Delta)^{2k+1}\right)\quad. \tag{54}$$

The poles of $\sigma(i\kappa)$ again form a lattice $\kappa_l=l\kappa_1$ with $\kappa_1=\frac{N}{N-1}$. This model is very similar to the $O(4)$ model, which is the $SU(2)$ case here.

6.3 Supersymmetric $O(N)$ models

In this model we have

$$\gamma = -1 \quad ; \quad H(\kappa) = \frac{\Gamma(\frac{1}{2} + \frac{(1-2\Delta)\kappa}{2})\Gamma(1+\Delta\kappa)}{\Gamma(\frac{1}{2}+\frac{\kappa}{2})} \quad , \tag{55}$$

where $\Delta = 1/(N-2)$ and

$$z_{2k+1} = 2\frac{\zeta_{2k+1}}{2k+1}\left(\Delta^{2k+1} - 2 + 2^{-2k} + (1-2\Delta)^{2k+1}(1-2^{-2k-1})\right) \quad . \tag{56}$$

The low N cases are similar to the previous cases having only a simple lattice with $\kappa_l = l\kappa_1$ where $\kappa_1 = 2$ for $N = 3, 4$ while $\kappa_1 = 6$ for $N = 5$. For $N > 5$ we have to distinguish between the even and odd cases just as we did for the $O(N)$ models. We actually have the same set as for the $O(N)$ models and additionally $\mu_l = \frac{N-2}{N-4}(2l-1)$, although for odd Ns some of the residues are zero. This is a very complicated pattern and we are planning to investigate these cases in detail in our forthcoming publication [32].

7 Conclusion

In this paper we developed a method to solve completely the integral equations (1) in terms of a trans-series. By taking the perturbative energy density $A_{1,1}$ as an input we determined a set observables $A_{n,m}$ which constitute a complete basis in the trans-series solution. We used these building blocks to construct the full trans-series for various other observables including the generalized energy densities and the boundary values of the Bethe Ansatz densities. We also revealed the analytical structure of all $A_{n,m}$-s on the Borel plane by determining their (positive) alien derivatives. The singularities on the positive real lines are interplayed such a way that the TBA result agrees with the median resummation. We supported our calculations with the explicit examples of the bosonic integrable models, in particular of the $O(N)$ non-linear sigma models.

In the statistical physical applications the systems are not relativistically but Galiean-invariant. This implies that the source terms and the moments has to be changed from $\cosh n\theta$ to θ^j, see [11, 10] for details in constructing the analogue of our basis and for describing observables at the perturbative level. Our formulas can be related to those by differentiating wrt. n and putting n to zero. However, here we go beyond the perturbative level, and construct the full non-perturbative trans-series. We think that by making the appropriate differentiation the non-perturbative parts of the non-relativistic moments can also be extracted based on our formulae.

We thus hope that our generic solution of the TBA equation will be useful and find applications both in the statistical and particle physics.

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