Whitham modulation theory for the defocusing nonlinear Schrödinger equation in two and three spatial dimensions

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Abstract. The Whitham modulation equations for the defocusing nonlinear Schrödinger (NLS) equation in two, three and higher spatial dimensions are derived using a two-phase ansatz for the periodic traveling wave solutions and by period-averaging the conservation laws of the NLS equation. The resulting Whitham modulation equations are written in vector form, which allows one to show that they preserve the rotational invariance of the NLS equation, as well as the invariance with respect to scaling and Galilean transformations, and to immediately generalize the calculations from two spatial dimensions to three. The transformation to Riemann-type variables is described in detail; the harmonic and soliton limits of the Whitham modulation equations are explicitly written down; and the reduction of the Whitham equations to those for the radial NLS equation is explicitly carried out. Finally, the extension of the theory to higher spatial dimensions is briefly outlined. The multidimensional NLS-Whitham equations obtained here may be used to study large amplitude wavetrains in a variety of applications including nonlinear photonics and matter waves.

30 January 2023

1. Introduction

The nonlinear Schrödinger (NLS) equation in one, two and three spatial dimensions is a ubiquitous model in nonlinear science. One reason is its universality as a model for the evolution of weakly nonlinear dispersive wave trains [9, 18, 53]. The NLS equation arises as the governing equation in a broad variety of physical contexts, ranging from water waves to optics, acoustics, Bose-Einstein condensates and beyond [6, 36, 39, 44]. As a result, enormous attention has been devoted over the last half century to the study of its solutions. It is also the case that in many physical situations, dispersive effects are much weaker than nonlinear ones and these scenarios, which can be formulated as small dispersion limits of the governing equations, give rise to a variety of interesting physical phenomena [25]. In particular, the small dispersion limits often lead to the formation of dispersive shock waves, a coherent, slowly modulated and expanding train of nonlinear oscillations.

A powerful tool in the study of small dispersion limits is Whitham modulation theory (also simply called Whitham theory) [56, 57]. Whitham theory is an asymptotic framework within which one can derive the Whitham modulation equations or Whitham equations for brevity. The Whitham equations are a system of first-order, quasi-linear partial differential equations (PDEs) that govern the evolution of the periodic traveling wave solutions of the original PDE over spatial and temporal scales that are larger than the traveling wave solution's wavelength and period, respectively. Whitham theory does not require integrability of the original PDE, and therefore it can also be applied to non-integrable PDEs. Thanks to Whitham theory and, when applicable, the inverse scattering transform (IST), much is known about small dispersion limits for (1+1)-dimensional nonlinear wave equations (e.g., see [13, 21, 25, 30, 38, 45] and references therein). On the other hand, small dispersion limits for (2+1)-dimensional systems have been much less studied and (3+1)-dimensional systems apparently have not been studied at all. Recently, the Whitham modulation equations for the Kadomtsev-Petviashvili (KP) and two-dimensional Benjamin-Ono equations and, more generally, a class of (2+1)-dimensional equations of KP type were derived [2, 3, 1]. The properties of the resulting KP-Whitham equations were then studied in [14] and the soliton limit of these equations was used in [48, 50, 49] to study the time evolution of a variety of piecewise-constant initial conditions in the modulation equations and, in the process, characterize the resulting dynamics of the solutions of the KP equation. Recently, the Whitham equations for the radial NLS equation [4] and those for focusing and defocusing two-dimensional nonlinear Schrödinger (2DNLS) equations [5] were also derived using a multiple scales approach.

The goal of this work is to derive and study the Whitham modulation equations for the defocusing multi-dimensional nonlinear Schrödinger equation, which we write in the semiclassical scaling as

$$i\varepsilon\psi_t + \varepsilon^2\nabla^2\psi - 2|\psi|^2\psi = 0 \tag{1.1}$$

for a complex-valued field $\psi(\mathbf{x}, t)$, where $\mathbf{x} = (x_1, \dots, x_N)^T$ and $\nabla^2\psi = \psi_{x_1x_1} + \cdots + \psi_{x_Nx_N}$ is the spatial Laplacian, and subscripts x_j and t denote partial differentiation throughout. Equation (1.1) arises as a governing equation in water waves [6], optics [44], plasmas [36], Bose-Einstein condensates [39], magnetic materials [59] and beyond. The small parameter $0 < \varepsilon \ll 1$ quantifies the relative strength of dispersive effects compared to nonlinear ones and sets a spatial and temporal scale for oscillatory solutions. In the (1+1)-dimensional case, the Whitham modulation equations have been shown to provide quantitative predictions for experiments in ultracold quantum fluids [34, 35] and nonlinear optics [55, 58, 10, 8].

While the Whitham equations for the two-dimensional version of (1.1) (hereafter referred to as the 2DNLS equation) were obtained in [5], this work differs from [5] in several important respects. First, our derivation employs a two-phase ansatz for the periodic solutions of the 2DNLS equation, which has several practical advantages. For one thing, it immediately yields a second conservation of waves equation in vector form that was missed in [5]. It is well known that several methods can be used to derive the Whitham equations: averaged conservation laws, averaged Lagrangian, and multiple scales perturbation theory. Our derivation employs averaged conservation laws which are directly tied to the physical symmetries of the NLS equation, rather than secularity conditions as used in [5]. Moreover, the ability to take advantage of the second conservation of waves equation also simplifies the calculations. In contrast, one of the secularity conditions obtained in [5] is equivalent to the averaged energy equation, which is more complicated and requires more significant manipulation than the second conservation of waves equation. Our approach dramatically simplifies the calculations and enables us to carry out the whole derivation in vector form. Consequently, the resulting NLS-Whitham equations are obtained in a simpler way, which lays the groundwork for generalizations to other NLS-type equations and higher dimensions.

In this work, we also show how our approach allows one to easily generalize the derivation of the Whitham equations to the NLS equation in an arbitrary number of spatial dimensions. We primarily concentrate on the two and three dimensional cases, though some of our results apply to an arbitrary number of spatial dimensions. This generalization to higher dimensions is particularly relevant because the NLS equation in three spatial dimensions is the zero-potential version of the Gross-Pitaevski equation, and is therefore of fundamental importance in describing the dynamics of Bose-Einstein condensates [39], so we expect our results to be directly applicable in that context.

We use our representation of the NLS-Whitham equations to identify several symmetries and reductions of the Whitham equations. For example, we verify that the Whitham equations preserve the invariance of the (N+1)-dimensional NLS equation with respect to scaling and Galilean transformations, and we take advantage of the vector formulation of the modulation equations, which we use to show that they preserve the rotation symmetry of the multidimensional NLS equation. We also explicitly write down both the harmonic and soliton limits of the Whitham equations in a mathematically convenient set of independent variables (which we refer to as Riemann-type variables) and in physical variables. We identify the self-consistent reduction of the 2DNLS-Whitham equations to the Whitham equations for the radial NLS equation.

The outline of this work is as follows. In section 2 we write the NLS equation in hydrodynamic form, write down its conservation laws, and obtain a representation for the periodic solutions. In section 3 we average the conservation laws to obtain the Whitham equations in physical variables. In section 4 we begin to study the reductions of the Whitham equations in physical variables, including one-dimensional reductions as well as the harmonic and soliton limits. In section 5 we discuss two different transformations to Riemann-type variables. In section 6 we derive further symmetries and reductions of the Whitham equations, including the reduction to the Whitham equations of the radial NLS equation and the harmonic and soliton limits of the Whitham equations in Riemann-type variables. In section 7 we present the generalization of the results to the NLS equation in three spatial dimensions, and in section 8 we end this work with a discussion of the results and some final remarks. The details of various calculations are relegated to the Appendix.

2. Hydrodynamic form, conservation laws and periodic solutions of the NLS equation

2.1. Madelung form of the NLS equation and its conservation laws

We begin by writing down the first few conservation laws of the NLS equation (1.1) in an arbitrary number of dimensions. It is convenient to introduce the Madelung transformation

$$\psi(\mathbf{x},t) = \sqrt{\rho(\mathbf{x},t)}\,e^{i\Phi(\mathbf{x},t)}\,, \tag{2.1a}$$

$$\mathbf{u}(\mathbf{x},t) = \varepsilon\boldsymbol{\nabla}\Phi(\mathbf{x},t)\,. \tag{2.1b}$$

where $\mathbf{u} = (u_1,\dots,u_N)^T$, $\mathbf{x} = (x_1,\dots,x_N)^T$ and $\boldsymbol{\nabla} = (\partial_{x_1},\dots,\partial_{x_N})^T$. Substituting (2.1) into the NLS equation (1.1), separating into real and imaginary parts, and differentiating the real part with respect to each of the spatial variables yields the following dispersive hydrodynamic system of PDEs:

$$\rho_t + 2\boldsymbol{\nabla}\cdot(\rho\mathbf{u}) = 0\,, \tag{2.2a}$$

$$\mathbf{u}_t + 2(\mathbf{u}\cdot\boldsymbol{\nabla})\,\mathbf{u} + 2\boldsymbol{\nabla}\rho - \tfrac{1}{4}\varepsilon^2\boldsymbol{\nabla}\Big(\nabla^2\ln\rho + \frac{1}{\rho}\nabla^2\rho\Big) = 0\,. \tag{2.2b}$$

The conservation laws for (1.1) for the mass E, momentum $\mathbf{P}$ and energy H in integrated form are:

$$\frac{dE}{dt} = 0\,,\qquad \frac{d\mathbf{P}}{dt} = 0\,,\qquad \frac{dH}{dt} = 0\,, \tag{2.3a}$$

where

$$E = \int_{\mathbb{R}^N} |\psi|^2\,(\mathrm{d}\mathbf{x})\,,\quad \mathbf{P} = \tfrac{i}{2}\varepsilon\int_{\mathbb{R}^N}(\psi\boldsymbol{\nabla}\psi^* - \psi^*\boldsymbol{\nabla}\psi)\,(\mathrm{d}\mathbf{x})\,,\quad H = \int_{\mathbb{R}^N}\big(\varepsilon^2\|\boldsymbol{\nabla}\psi\|^2 + |\psi|^4\big)\,(\mathrm{d}\mathbf{x})\,, \tag{2.3b}$$

$\|\mathbf{v}\|^2 = |v_1|^2 + \cdots + |v_N|^2$ is the Euclidean vector norm and $(\mathrm{d}\mathbf{x}) = \mathrm{d}x_1\cdots\mathrm{d}x_N$ is the volume element in $\mathbb{R}^N$. These conservation laws correspond, via Noether's theorem, to the invariance of the NLS equation (1.1) with respect to phase rotations, space and time translations, respectively [53]. In differential form, and in terms of the Madelung variables, these conservation laws become [37]

$$\rho_t + 2\boldsymbol{\nabla}\cdot(\rho\mathbf{u}) = 0\,, \tag{2.4a}$$

$$(\rho\mathbf{u})_t + 2\boldsymbol{\nabla}\cdot(\rho\,\mathbf{u}\otimes\mathbf{u}) + \boldsymbol{\nabla}(\rho^2) = \tfrac{1}{2}\varepsilon^2\Big(\boldsymbol{\nabla}(\nabla^2\rho) - \boldsymbol{\nabla}\cdot\Big(\frac{1}{\rho}\nabla\rho\otimes\nabla\rho\Big)\Big)\,, \tag{2.4b}$$

$$h_t + 2\boldsymbol{\nabla}\cdot\big((h+\rho^2)\mathbf{u}\big) = \varepsilon^2\boldsymbol{\nabla}\cdot\Big(\mathbf{u}\nabla^2\rho - \frac{1}{\rho}(\boldsymbol{\nabla}\cdot\rho\mathbf{u})\nabla\rho\Big)\,, \tag{2.4c}$$

where $\otimes$ denotes the dyadic [namely, $\mathbf{v}\otimes\mathbf{w} = \mathbf{v}\mathbf{w}^T$, so that $(\mathbf{v}\otimes\mathbf{w})_{i,j} = v_i w_j$] and the mass density, momentum density and energy density of (1.1) are, respectively

$$\rho = |\psi|^2\,,\quad \rho\mathbf{u} = i2\varepsilon(\psi\boldsymbol{\nabla}\psi^* - \psi^*\boldsymbol{\nabla}\psi)\,,\qquad h = \varepsilon^2\|\boldsymbol{\nabla}\psi\|^2 + |\psi|^4 = \rho\|\mathbf{u}\|^2 + \rho^2 + \frac{\varepsilon^2}{4\rho}\|\boldsymbol{\nabla}\rho\|^2\,. \tag{2.5}$$

The first two of the conservation laws (2.4) are equivalent to the real and imaginary parts of NLS equation in hydrodynamic form (2.2), but only up to an extra differentiation, an issue that we will return to later.

2.2. Periodic solutions of the NLS equation via a two-phase ansatz

The Whitham modulation equations govern the slow dynamics of the parameters of the periodic solutions of the PDE of interest. Next, we therefore write down the periodic solutions of the hydrodynamic system (2.2) in arbitrary dimensions. We begin by looking for solutions in the form of the following two-phase ansatz:

$$\rho(\mathbf{x},t)=\rho(Z)\,,\qquad \Phi(\mathbf{x},t)=\phi(Z)+S\,, \tag{2.6a}$$

where $\rho(Z)$ and $\phi(Z)$ are periodic function of Z with period one, and the "fast phases" Z and S are

$$Z(\mathbf{x},t)=(\mathbf{k}\cdot\mathbf{x}-\omega t)/\varepsilon\,,\qquad S(\mathbf{x},t)=(\mathbf{v}\cdot\mathbf{x}-\mu t)/\varepsilon\,. \tag{2.6b}$$

where $\mathbf{k}=(k_1,\dots,k_N)^T$ and $\mathbf{v}=(v_1,\dots,v_N)^T$. The reason for using a two-phase ansatz is the fact that the solution $\psi(\mathbf{x},t)$ of the NLS equation (1.1) is complex-valued, unlike that of the Korteweg-deVries (KdV) equation (of which the KP equation mentioned earlier is a two-dimensional generalization), which is real-valued. Therefore, a one-phase ansatz (e.g., as in [5]) leads only to a subclass of all periodic solutions, and one would need to apply a Galilean boost a posteriori in order to capture the most general family of periodic solutions of the NLS equation. Two-phase ansatzes are standard when deriving the Whitham equations using Lagrangian averaging (e.g., see [57]); the novelty here is that such a two-phase ansatz is combined with the use of averaged conservation laws. A key benefit of this approach is the immediate deduction of an additional conservation law compared to [5].

In light of (2.6), the definition (2.1) yields

$$\mathbf{u}(Z)=\mathbf{k}\phi'(Z)+\mathbf{v}, \tag{2.7}$$

using primes to denote derivatives with respect to Z for brevity. The fact that $\phi(Z)$ is periodic implies

$$\bar{\mathbf{u}}=\mathbf{v}, \tag{2.8}$$

where throughout this work the overbar will denote the integral of a quantity with respect to Z over the unit period. Moreover, the definition (2.1) implies the irrotationality condition

$$\boldsymbol{\nabla}\wedge\mathbf{u}=0\,. \tag{2.9}$$

Hereafter, $\mathbf{v}\wedge\mathbf{w}$ is the N-dimensional wedge product, which in two and three spatial dimensions can be replaced by the standard cross product [17, 29]. We substitute the two-phase ansatz (2.6) into the hydrodynamic equations (2.2*a*) and (2.2*b*) and collect the leading-order terms, obtaining:

$$-\omega\rho'+2\mathbf{k}\cdot(\rho\mathbf{u})'=0\,, \tag{2.10a}$$

$$-\omega\mathbf{u}'+2(\mathbf{k}\cdot\mathbf{u})\,\mathbf{u}'+2\mathbf{k}\rho'-\tfrac14\Big((\ln\rho)''+\frac{\rho''}{\rho}\Big)'\|\mathbf{k}\|^2\mathbf{k}=\mathbf{0}\,. \tag{2.10b}$$

Integrating (2.10*a*) and using (2.7) yields

$$\phi'(Z)=\frac{1}{\|\mathbf{k}\|}\Big(U+\frac{J}{\rho}-\hat{\mathbf{k}}\cdot\bar{\mathbf{u}}\Big), \tag{2.11}$$

where $U=\omega/(2\|\mathbf{k}\|)$ is the phase speed, $\hat{\mathbf{k}}=\mathbf{k}/\|\mathbf{k}\|$, and the integration constant J will be determined later. Using (2.11), we can rewrite (2.7) as:

$$\mathbf{u}(Z)=\Big(\frac{J}{\rho}+U\Big)\hat{\mathbf{k}}+\bar{\mathbf{u}}_\perp\,, \tag{2.12}$$

where $\mathbf{u}_\perp = \mathbf{u} - (\hat{\mathbf{k}}\cdot\mathbf{u})\,\hat{\mathbf{k}}$. Importantly, the requirement that $\phi(Z)$ is periodic implies that $\phi'(Z)$ must have zero mean. Taking the inner product of (2.12) with $\hat{\mathbf{k}}$ and averaging the result over the wave period yields a relation between $\bar{\mathbf{u}}$ and U, and therefore determines $\omega = 2\|\mathbf{k}\|U$:

$$U = \hat{\mathbf{k}}\cdot\bar{\mathbf{u}} - J\,\overline{\rho^{-1}}\,. \tag{2.13}$$

Next, substituting (2.11) into (2.10b) and simplifying yields two ODEs for ρ. Note that the two ODEs are consistent thanks to the constraint (2.9), which becomes, to leading order,

$$\mathbf{k}\wedge\mathbf{u}' = 0\,. \tag{2.14}$$

Integrating the resulting ODE for ρ one obtains [see Appendix A.2 for details]

$$(\rho')^2 = P_3(\rho)\,, \tag{2.15}$$

with

$$P_3(\rho) = \frac{4}{\|\mathbf{k}\|^2}(\rho-\lambda_1)(\rho-\lambda_2)(\rho-\lambda_3)\,, \tag{2.16}$$

whose solution is

$$\rho(Z) = A + 4m\|\mathbf{k}\|^2 K_m^2\,\mathrm{sn}^2(2K_m z|m)\,, \tag{2.17}$$

where A is a free parameter, and with

$$J^2 = A\big(A + 4\|\mathbf{k}\|^2K_m^2\big)\big(A + 4m\|\mathbf{k}\|^2K_m^2\big)\,, \tag{2.18}$$

The roots $\lambda_1,\ldots,\lambda_3$ are related to the coefficients in the solution (2.17) as

$$\lambda_1 = A, \qquad \lambda_2 = A + 4mK_m^2\|\mathbf{k}\|^2, \qquad \lambda_3 = A + 4K_m^2\|\mathbf{k}\|^2. \tag{2.19}$$

Conversely, when $\lambda_1,\lambda_2,\lambda_3$ are known, A, $\|\mathbf{k}\|$ and m are given by

$$A = \lambda_1 \qquad \|\mathbf{k}\|^2 = (\lambda_3-\lambda_1)/4K_m^2\,, \qquad m = (\lambda_2-\lambda_1)/(\lambda_3-\lambda_1)\,. \tag{2.20}$$

The amplitude of the periodic oscillations of the density is $\lambda_2-\lambda_1$. The requirements $\rho\geqslant 0$, $\|\mathbf{k}\|\geqslant 0$ and $0\leqslant m\leqslant 1$ immediately yield the constraints $A\geqslant 0$ as well as

$$0\leqslant\lambda_1\leqslant\lambda_2\leqslant\lambda_3\,. \tag{2.21}$$

The symmetric polynomials $e_1,\ldots,e_3$ defined by the roots $\lambda_1,\ldots,\lambda_3$ will also be useful later:

$$e_1 = \lambda_1+\lambda_2+\lambda_3\,, \qquad e_2 = \lambda_1\lambda_2+\lambda_2\lambda_3+\lambda_3\lambda_1\,, \qquad e_3 = \lambda_1\lambda_2\lambda_3 = J^2\,. \tag{2.22}$$

Note that (2.22) only determines J up to a sign, i.e., $J = \sigma\sqrt{\lambda_1\lambda_2\lambda_3}$, with $\sigma=\pm1$. Both sign choices lead to valid solutions of the NLS equation (1.1). Some care is deserved when determining the value of σ in the presence of modulations of the periodic solutions, as discussed in section 3.2.

The leading-order periodic solution of the hydrodynamic system (2.2) defined by (2.11) and (2.17) contains the following independent parameters: A, m, $\mathbf{k}$, $\bar{\mathbf{u}}$ and μ. However, recall that, to derive the hydrodynamic equation (2.2b) from the NLS equation (1.1), one differentiates the real part with respect to the spatial variables. Imposing that the solution of the dispersive hydrodynamic system (2.2) also solves the NLS equation (by substituting into the undifferentiated imaginary part of the NLS equation (1.1)) yields a constraint that determines μ in terms of the other constants. Deriving this relation directly from the above expressions is a bit cumbersome, but seeking a periodic solution of (1.1) without writing it in hydrodynamic form [cf. Appendix A.1], one obtains

$$\mu = 4(1+m)\|\mathbf{k}\|^2K_m^2 + 3A + \|\bar{\mathbf{u}}\| - \big(J\overline{\rho^{-1}}\big)^2\,. \tag{2.23}$$

One can now verify that adding this relation to the above solution of the hydrodynamic system does indeed yield a solution of the NLS equation (1.1). Alternatively, one can obtain (2.23) using the undifferentiated version of (2.10b); see Appendix A.2. Thus, the periodic solutions of the NLS equation (1.1) in N spatial dimensions contain $2N+2$ scalar independent parameters: A, m, $\mathbf{k}$ and $\mathbf{v}=\bar{\mathbf{u}}$, as one would expect based on the invariances of the PDE (cf. [53]).

2.3. Harmonic and soliton limits of the periodic solutions

Recall that the harmonic ($m = 0$) and soliton ($m = 1$) limits of the Whitham equations for the one-dimensional NLS (1DNLS) equation have special significance [25]. The same will be true for the multi-dimensional NLS equation. It is therefore useful to evaluate the corresponding limits of the above periodic solutions.

In the limit $m \to 0$ (i.e., $\lambda_2 \to \lambda_1^+$), the solution (2.1) reduces to a plane wave. Indeed, in this limit, we have

$$\rho(Z) = A, \qquad B = 0, \qquad \mu = 2A + \|\bar{\mathbf{u}}\|^2, \qquad J^2 = A^2(\pi^2\|\mathbf{k}\|^2 + A), \tag{2.24}$$

and

$$\psi(\mathbf{x}, t) = \sqrt{A}\,\mathrm{e}^{i(\bar{\mathbf{u}}\cdot\mathbf{x} - (\|\bar{\mathbf{u}}\|^2 + 2A)t)}. \tag{2.25}$$

Therefore, the only independent parameters in this case are A and $\bar{\mathbf{u}}$.

In the opposite limit ($m \to 1$, i.e., $\lambda_2 \to \lambda_3^-$), the solution (2.1) reduces to the soliton solution of the NLS equation. Indeed, in this limit, (2.17) and (2.20) yield

$$\rho(Z) = \lambda_1 + (\lambda_3 - \lambda_1)\tanh^2\big[\sqrt{\lambda_3 - \lambda_1}\,(\hat{\mathbf{k}}\cdot\mathbf{x} - \omega t/\|\mathbf{k}\|)\big], \tag{2.26a}$$

$$B = \lambda_3 - \lambda_1, \quad J^2 = \lambda_1\lambda_3^2, \quad U = \hat{\mathbf{k}}\cdot\bar{\mathbf{u}} - \sigma\sqrt{\lambda_1}, \quad \mu = 2\lambda_3 + \|\bar{\mathbf{u}}\|^2. \tag{2.26b}$$

Note that $\|\mathbf{k}\| \to 0$ as $m \to 1$, but $K_m \to \infty$ in such a way that their product remains finite: $\|\mathbf{k}\|K_m \to \sqrt{\lambda_3 - \lambda_1}/2$. Using (2.11) we then obtain

$$\phi(Z) = \arctan\Big[\sqrt{\lambda_3 - \lambda_1}\,\tanh\Big(\sqrt{\lambda_3 - \lambda_1}\,(\hat{\mathbf{k}}\cdot\mathbf{x} - \omega t/\|\mathbf{k}\|)\Big)/\sqrt{\lambda_1}\Big], \tag{2.27}$$

implying

$$e^{i\phi + iS} = e^{iS}\Big[\sqrt{\lambda_1} + i\sqrt{\lambda_3 - \lambda_1}\,\tanh\Big(\sqrt{\lambda_3 - \lambda_1}\,(\hat{\mathbf{k}}\cdot\mathbf{x} - \omega t/\|\mathbf{k}\|)\Big)\Big]\Big/\sqrt{\rho(Z)}, \tag{2.28}$$

with $S = \bar{\mathbf{u}}\cdot\mathbf{x} - \mu t$ as before. Putting everything together, we obtain

$$\psi(\mathbf{x}, t) = A_o\mathrm{e}^{-2iA_o^2 t}\mathrm{e}^{i(\bar{\mathbf{u}}\cdot\mathbf{x} - \|\bar{\mathbf{u}}\|^2 t)}\{\cos\theta + i\sin\theta\tanh[A_o\sin\theta\,[\hat{\mathbf{k}}\cdot\mathbf{x} - 2(\hat{\mathbf{k}}\cdot\bar{\mathbf{u}} - A_o\cos\theta)t]]\}, \tag{2.29}$$

with $\bar{\mathbf{u}}$ as in (2.6), $A_o = \sqrt{\lambda_3}$ and $\theta = \arctan\big[\sqrt{(\lambda_3 - \lambda_1)/\lambda_1}\big]$. The independent parameters of the solution in this case are λ_1, λ_3 (or equivalently A_o or θ), $\hat{\mathbf{k}}$ and $\bar{\mathbf{u}}$. One can further reduce (2.29) to the more familiar form of the dark soliton solutions of the defocusing NLS equation by choosing $\bar{\mathbf{u}} = \mathbf{0}$.

3. Derivation of the NLS-Whitham equation by averaged conservation laws

We are now ready to study slow modulations of the periodic solutions described above and derive the Whitham modulation equations that govern them.

3.1. Nonlinear modulations and averaged conservation laws

We begin by introducing the following multiple scales ansatz for the solution of the NLS equation (1.1):

$$\rho(\mathbf{x}, t) = \rho(Z, \mathbf{X}, T), \qquad \Phi(\mathbf{x}, t) = \phi(Z, \mathbf{X}, T) + S, \tag{3.1}$$

where $\mathbf{X} = \mathbf{x}$ and $T = t$, with ρ and ϕ periodic in Z with period one and

$$\nabla Z = \frac{\mathbf{k}(\mathbf{X}, T)}{\varepsilon}, \qquad Z_t = -\frac{\omega(\mathbf{X}, T)}{\varepsilon}, \tag{3.2a}$$

$$\nabla S = \frac{\mathbf{v}(\mathbf{X}, T)}{\varepsilon}, \qquad S_t = -\frac{\mu(\mathbf{X}, T)}{\varepsilon}, \tag{3.2b}$$

where, as per the results of section 2.2, $\mathbf{v} = \bar{\mathbf{u}}$. The above multiple scales ansatz implies

$$\nabla_{\mathbf{x}} \mapsto \frac{\mathbf{k}}{\varepsilon}\partial_Z + \frac{\mathbf{v}}{\varepsilon}\partial_S + \nabla_{\mathbf{X}}, \qquad \partial_t \mapsto -\frac{\omega}{\varepsilon}\partial_Z - \frac{\mu}{\varepsilon}\partial_S + \partial_T, \tag{3.3}$$

Substituting (3.1) into (1.1), to leading order we recover the periodic solution (2.1), but where all $2N+2$ parameters A, m, $\mathbf{k}$ and $\bar{\mathbf{u}}$ are now slowly varying functions of $\mathbf{X}$ and t. We then seek modulation equations to determine the space-time dependence of these parameters. To avoid complicating the notation unncessarily, below we will write derivatives in $\mathbf{X}$ and T as derivatives in $\mathbf{x}$ and t. Equations (3.2) immediately yield the equations of conservation of waves:

$$\mathbf{k}_t + \nabla\omega = \mathbf{0}, \tag{3.4a}$$

$$\nabla \wedge \mathbf{k} = 0, \tag{3.4b}$$

$$\bar{\mathbf{u}}_t + \nabla\mu = \mathbf{0}, \tag{3.4c}$$

$$\nabla \wedge \bar{\mathbf{u}} = 0. \tag{3.4d}$$

Of course only N equations among (3.4*a*) and (3.4*b*) are independent, and similarly for (3.4*c*) and (3.4*d*). Equations (3.4*a*) and (3.4*c*) form the first two vectorial Whitham modulation equations, whereas (3.4*b*) and (3.4*d*) are compatibility constraints.

Next, we obtain the remaining Whitham modulation equations by averaging the conservation laws (2.4) over the fast variable Z. Using (3.3) to replace all spatial and temporal derivatives in (2.2) and (2.4), expanding all terms in powers of ε, and averaging, we obtain at order $\mathcal{O}(\varepsilon^0)$

$$(\bar{\rho})_t + 2\nabla\cdot(\overline{\rho\mathbf{u}}) = 0, \tag{3.4e}$$

$$(\overline{\rho\mathbf{u}})_t + 2\nabla\cdot(\overline{\rho\mathbf{u}\otimes\mathbf{u}}) + \nabla(\overline{\rho^2}) + \nabla\left(\overline{\frac{(\rho')^2}{2\rho}}\,\mathbf{k}\otimes\mathbf{k}\right) = \mathbf{0}, \tag{3.4f}$$

$$\bar{h}_t + \nabla\cdot\left(2\overline{h\mathbf{u}} + 2\overline{\rho^2\mathbf{u}} + \left(\mathbf{k}\cdot\overline{\frac{\rho'}{\rho}(\rho\mathbf{u})'}\right)\mathbf{k} - \|\mathbf{k}\|^2\overline{\rho''\mathbf{u}}\right) = 0, \tag{3.4g}$$

where $\bar{h}$ denotes the averaged energy density:

$$\bar{h} = \overline{\rho\|\mathbf{u}\|^2} + \overline{\rho^2} + \frac{1}{4}\|\mathbf{k}\|^2\overline{(\rho')^2/\rho}. \tag{3.5}$$

Together with (3.4*a*) and (3.4*c*), equations (3.4*e*)–(3.4*g*) are $3N+2$ scalar PDEs for the $2N+2$ dependent variables A, m, $\mathbf{k}$ and $\mathbf{v} = \bar{\mathbf{u}}$ subject to the $2N$ spatial constraints (3.4*b*), (3.4*d*), and are the desired Whitham modulation equations in physical variables in any number of spatial dimensions. Of course, not all of these equations are independent. We will see later that choosing different subsets of equations still leads to equivalent results, and in the end the number of independent modulation equations is $2N+2$. At the same time, however, we emphasize the simplicity and directness of this approach compared to [5] in deriving the Whitham equations in multiple spatial dimensions.

3.2. Modified form of the modulation equations

In preparation for further simplification of the above system of Whitham equations, it is convenient to express the periodic solutions in terms of the roots λ_1, λ_2, λ_3, thereby replacing A, m and $\|\mathbf{k}\|^2$ as dependent variables. Explicitly, (2.12) and (2.17) become:

$$\rho(Z) = \lambda_1 + (\lambda_2 - \lambda_1)\,\mathrm{sn}^2(2K_m z|m), \tag{3.6a}$$

$$\mathbf{u}(Z) = \mathbf{U} + \frac{J}{\rho(Z)}\hat{\mathbf{k}}, \tag{3.6b}$$

with

$$\mathbf{U} = \bar{\mathbf{u}} - J\overline{\rho^{-1}}\hat{\mathbf{k}}, \tag{3.6c}$$

which also implies

$$\omega = 2\mathbf{k}\cdot\mathbf{U}, \qquad \mu = \lambda_1+\lambda_2+\lambda_3+\|\mathbf{U}\|^2+2UJ\overline{\rho^{-1}}, \tag{3.6d}$$

with $\hat{\mathbf{k}} = \mathbf{k}/\|\mathbf{k}\|$ as before and J, A, $\|\mathbf{k}\|$ and m given in terms of $\lambda_1,\ldots,\lambda_3$ by (2.18) and (2.20). In turn, using (3.6), we can write the Whitham modulation equations (3.4) as

$$\mathbf{k}_t + 2\boldsymbol{\nabla}(\mathbf{k}\cdot\mathbf{U}) = \mathbf{0}, \tag{3.7a}$$

$$\boldsymbol{\nabla}\wedge\mathbf{k} = 0, \tag{3.7b}$$

$$\big(\mathbf{U}+J\overline{\rho^{-1}}\,\hat{\mathbf{k}}\big)_t + \boldsymbol{\nabla}\big(e_1+\|\mathbf{U}\|^2+2J\overline{\rho^{-1}}\,\mathbf{U}\cdot\hat{\mathbf{k}}\big) = \mathbf{0}, \tag{3.7c}$$

$$\boldsymbol{\nabla}\wedge\big(\mathbf{U}+J\overline{\rho^{-1}}\,\hat{\mathbf{k}}\big) = 0, \tag{3.7d}$$

$$\bar\rho_t + 2\nabla\cdot\big(J\hat{\mathbf{k}}+\bar\rho\mathbf{U}\big) = 0, \tag{3.7e}$$

$$(J\hat{\mathbf{k}}+\overline{\rho}\mathbf{U})_t + \boldsymbol{\nabla}(\overline{\rho^2}) + \boldsymbol{\nabla}\Big[\Big(2\bar\rho\mathbf{U}+2J\hat{\mathbf{k}}\Big)\otimes\mathbf{U} + 2J\mathbf{U}\otimes\hat{\mathbf{k}} + \frac{2}{3}\big(2e_2-e_1\bar\rho\big)\hat{\mathbf{k}}\otimes\hat{\mathbf{k}}\Big] = \mathbf{0}. \tag{3.7f}$$

$$\bar h_t + \boldsymbol{\nabla}\cdot\left[2J(2\bar\rho+\overline{\|\mathbf{u}\|^2})\hat{\mathbf{k}} + 2(\overline{\rho^2}+\bar h)\mathbf{U} + \left(\mathbf{U}\cdot\hat{\mathbf{k}}\left(\overline{\frac{(\rho')^2}{\rho}}\right) - \frac{J}{2}\left(\overline{\frac{(\rho')^2}{\rho^2}}\right)\right)\mathbf{k}\right] = 0. \tag{3.7g}$$

See Appendix A.2 for details on how to obtain (3.7*f*). The next step is the evaluation of the elliptic integrals in (3.7). To this end, we have [46]

$$\overline{\rho} = \int_0^1 \rho(Z)\,\mathrm{d}z = \lambda_3 - (\lambda_3-\lambda_1)\frac{E_m}{K_m}, \tag{3.8a}$$

$$\overline{\rho^{-1}} = \int_0^1 \rho^{-1}(Z)\,\mathrm{d}z = \frac{1}{\lambda_1 K_m}\,\Pi\Big(1-\frac{\lambda_2}{\lambda_1}\Big|m\Big), \tag{3.8b}$$

where $K_m = K(m)$, $E_m = E(m)$ and $\Pi(\cdot|m)$ are the complete elliptic integrals of the first, second and third kind respectively. We also note, for convenience, that

$$\bar{\mathbf{u}} = \int_0^1 \mathbf{u}(Z)\,\mathrm{d}z = \mathbf{U} + \sigma\frac{\sqrt{\lambda_2\lambda_3}}{\sqrt{\lambda_1}K_m}\Pi\Big(1-\frac{\lambda_2}{\lambda_1}\Big|m\Big)\hat{\mathbf{k}}, \tag{3.9a}$$

$$\overline{\rho\mathbf{u}} = \int_0^1 \rho(Z)\mathbf{u}(Z)\,\mathrm{d}z = \bar\rho\mathbf{U} + J\hat{\mathbf{k}} = \Big(\lambda_3-(\lambda_3-\lambda_1)\frac{E_m}{K_m}\Big)\mathbf{U} + \sigma\sqrt{\lambda_1\lambda_2\lambda_3}\,\hat{\mathbf{k}}. \tag{3.9b}$$

We reiterate that not all of the equations (3.7) are independent. For example, one can obtain (3.7*d*) using (3.7*c*) and (3.7*e*). This is relevant because it allows us to work with the most convenient subset of equations among all the PDEs in (3.7), as long as the compatibility constraints (3.7*b*) and (3.7*d*) are satisfied. To this end, recall that $\bar h$ is given by (3.5), and

$$\overline{\|\mathbf{u}\|^2} = \|\mathbf{U}\|^2 + 2J\mathbf{U}\cdot\hat{\mathbf{k}}\,\overline{\rho^{-1}} + J^2\overline{\rho^{-2}}, \tag{3.10a}$$

$$\overline{\rho\|\mathbf{u}\|^2} = J^2\overline{\rho^{-1}} + 2J\mathbf{U}\cdot\hat{\mathbf{k}} + \bar\rho\|\mathbf{U}\|^2. \tag{3.10b}$$

Moreover, the terms $(\rho')^2/\rho$ and $(\rho')^2/\rho^2$, which appear in (3.7*g*), can be computed using (2.15). On the other hand, the averaged energy conservation law (3.7*g*) is the most complicated among all of the equations (3.7). In section 7 we will show that, thanks to the use of the two-phase ansatz and the resulting second conservation of waves equations (3.7*c*) and (3.7*d*), one can avoid having to deal with the averaged energy equation (3.7*g*), which greatly simplifies the transformation to Riemann-type variables.

We also point out that the sign of J, as determined by the initial conditions for the system through the value of σ—see the discussion after (2.22)—affects $\bar{\mathbf{u}}$ via (3.9*a*) and $\overline{\rho\mathbf{u}}$ via (3.9*b*). Therefore, when considering modulations of the periodic solutions, the value of σ depends on $\mathbf{x}$ and t, and its value must be chosen in such a way to ensure smoothness of $\overline{\rho\mathbf{u}}$. In particular, a sign change of J occurs when the solution hits a vacuum point, i.e., $\lambda_1 = 0$. At such a point, $\bar{\mathbf{u}}$ is singular but $\overline{\rho\mathbf{u}}$ is not. See [33] for additional discussion.

4. Symmetries and reductions of the NLS-Whitham system in physical variables

We now present several reductions of the Whitham modulation system (3.7) in physical variables in arbitrary number of spatial dimensions. Further symmetries and reductions in the two-dimensional case will be discusssed in section 6 after we introduce Riemann-type variables in section 5.

4.1. Unidirectional reductions of the modulation equations

We begin by showing that the NLS-Whitham equations (3.7) reduce to the 1DNLS-Whitham equations (i.e., the Whitham equations for the 1DNLS equation) when $k_2 = \cdots = k_N = \nu_2 = \cdots = \nu_N = 0$ and all quantities are independent of $x_2, \ldots, x_N$. In this case, we have:

$$\|\mathbf{k}\|^2 = k_1^2\,, \qquad u_1(Z) = \frac{J}{\rho} + U\,, \qquad \omega = 2kU\,, \qquad U = \bar{u}_1 - J\overline{\rho^{-1}}\,, \qquad u_2(Z) = 0 \tag{4.1a}$$

The Whitham equations (3.7*b*) and (3.7*d*) and the second components of (3.7*a*), (3.7*c*), and (3.7*e*) are satisfied trivially, while the rest simplify to:

$$k_t + 2(kU)_x = 0\,, \tag{4.2a}$$

$$\left(U + J\overline{\rho^{-1}}\right)_t + \left(e_1 + 2JU\overline{\rho^{-1}} + U^2\right)_x = 0\,, \tag{4.2b}$$

$$(\bar{\rho})_t + 2(U\bar{\rho} + J)_x = 0\,, \tag{4.2c}$$

$$(U\bar{\rho} + J)_t + \left(\overline{\rho^2} + 2U^2\bar{\rho} + 2J^2\overline{\rho^{-1}} + \frac{k^2}{2}\overline{\frac{(\rho')^2}{\rho}} + 4UJ\right)_x = 0\,, \tag{4.2d}$$

$$\left(\overline{\rho^2} + U^2\bar{\rho} + J^2\overline{\rho^{-1}} + \frac{k^2}{4}\overline{\frac{(\rho')^2}{\rho}} + 2JU\right)_t + \left(\frac{3Uk^2}{2}\overline{\left(\frac{(\rho')^2}{\rho}\right)} - \frac{k^2J}{2}\overline{\frac{(\rho')^2}{\rho^2}} + 4U\overline{\rho^2} + (4J + 2U^3)\bar{\rho} + 6JU^2 + 6J^2U\overline{\rho^{-1}} + 2J^3\overline{\rho^{-2}}\right)_x = 0\,, \tag{4.2e}$$

with $x = x_1$. The system (4.2) coincides with the modulation equations for the 1DNLS equation [34] (cf. (4.41) and (4.42) in [34]) upon trivial rescalings resulting from the different normalization of the NLS equation in [34] compared to (1.1). Note that (4.2) comprise five PDEs for the four solution parameters $A, m, k \& U$ (or equivalently $\lambda_1, \lambda_2, \lambda_3, U$). Once again, one can verify that the modulation equation obtained from (4.2*e*) is consistent with those obtained from the first four PDEs above.

The above scenario is not the only one in which the Whitham modulation system (3.7) reduces to that of the 1DNLS equation. Next we consider so-called "rotated" one-dimensional reductions where the rotated coordinate frame is determined by R, an $N \times N$ orthogonal matrix. We introduce the rotated vector $\mathbf{w}^\sharp = \mathsf{R}\mathbf{w}$ for any vector $\mathbf{w}$. Then, the rotated one-dimensional reduction is obtained through the requirement that $\mathbf{k}$ and $\bar{\mathbf{u}}$ (or equivalently $\hat{\mathbf{k}}$ and $\mathbf{U}$) be parallel and that both depend only on t and the first component of $\mathbf{x}^\sharp$. We choose R so that $\hat{\mathbf{k}}^\sharp = (1, 0, \ldots, 0)^T$, i.e., $k_2^\sharp = \ldots = k_N^\sharp = 0$, which also implies $U_2^\sharp = \ldots = U_N^\sharp = 0$. Since the Whitham modulation equations (3.7) are invariant under rotations of the coordinate axes (see below), we recover the one-dimensional reduction (4.2) when all quantities are independent of $x_2^\sharp, \ldots, x_N^\sharp$ in the rotated coordinate frame, i.e., with x and all modulation variables in (4.2) replaced by their rotations $x_1^\sharp$, etc.

4.2. Invariances of the modulation equations

The Whitham modulation equations (3.7) are manifestly invariant under translations of the spatial and temporal coordinates. Next we show that the Whitham system (3.7) preserves the invariance of the NLS equation under rotations of the Cartesian coordinates. Namely, if $\mathbf{x} \mapsto \mathbf{x}^\sharp = \mathsf{R}\mathbf{x}$, where R is an arbitrary constant rotation matrix, (3.7) remain unchanged upon $\mathbf{U} \mapsto \mathbf{U}^\sharp = \mathsf{R}\mathbf{U}$ and $\mathbf{k} \mapsto \mathbf{k}^\sharp = \mathsf{R}\mathbf{k}$. One can verify that this is indeed the case using the following identities:

$$\mathsf{R}\nabla_{\mathbf{x}} = \nabla_{\mathbf{x}^\sharp}\,, \quad \mathbf{U}\cdot\mathbf{k} = \mathbf{U}^\sharp\cdot\mathbf{k}^\sharp\,, \quad \|\mathbf{U}\| = \|\mathbf{U}^\sharp\|\,, \tag{4.3a}$$

$$\nabla_{\mathbf{x}}\cdot(\alpha\mathbf{k}) = \nabla_{\mathbf{x}^\sharp}\cdot(\alpha\mathbf{k}^\sharp)\,, \quad \nabla_{\mathbf{x}}\cdot(\alpha\mathbf{U}) = \nabla_{\mathbf{x}^\sharp}\cdot(\alpha\mathbf{U}^\sharp)\,, \tag{4.3b}$$

$$\mathsf{R}\nabla_{\mathbf{x}}\cdot(\alpha\mathbf{U}\otimes\mathbf{U}) = \nabla_{\mathbf{x}^\sharp}\cdot(\alpha\mathbf{U}^\sharp\otimes\mathbf{U}^\sharp)\,, \quad \mathsf{R}\nabla_{\mathbf{x}}\cdot(\alpha\mathbf{k}\otimes\mathbf{k}) = \nabla_{\mathbf{x}^\sharp}\cdot(\alpha\mathbf{k}^\sharp\otimes\mathbf{k}^\sharp)\,, \tag{4.3c}$$

$$\mathsf{R}\nabla_{\mathbf{x}}\cdot(\alpha\mathbf{k}\otimes\mathbf{U}) = \nabla_{\mathbf{x}^\sharp}\cdot(\alpha\mathbf{k}^\sharp\otimes\mathbf{U}^\sharp)\,. \tag{4.3d}$$

where α is an arbitrary real number.

Next we show that the Whitham system (3.7) also preserves the invariance of the NLS equation with respect to scaling and spatial reflections and Galilean transformations. Recall that, if $q(\mathbf{x},t)$ is any solution of the NLS equation, so are $q^\sharp(\mathbf{x},t) = \alpha q(\alpha\mathbf{x},\alpha^2 t)$, $q^\sharp(\mathbf{x},t) = q(-\mathbf{x},t)$ and $q^\sharp(\mathbf{x},t) = q(\mathbf{x}-2\mathbf{w}t,t)e^{i(\mathbf{w}\cdot\mathbf{x}-\|\mathbf{w}\|^2 t)}$ where all transformation parameters are real-valued. We next show that the modulation equations (3.7) are invariant under each of these transformations. Specifically, letting $q^\sharp(\mathbf{x},t) = [\rho^\sharp(\mathbf{x},t)]^{1/2}\,e^{i\phi^\sharp(\mathbf{x},t)}$, we have, for the scaling symmetry,

$$\rho^\sharp(\mathbf{x},t) = \alpha^2\rho(\alpha\mathbf{x},\alpha^2 t)\,, \qquad \phi^\sharp(\mathbf{x},t) = \phi(\alpha\mathbf{x},\alpha^2 t)\,, \tag{4.4}$$

and the dependent variables of the Whitham equations become

$$\lambda_j^\sharp(\mathbf{x},t) = \alpha^2\lambda_j(\alpha\mathbf{x},\alpha^2 t)\,,\; j=1,2,3\,, \tag{4.5a}$$

$$\mathbf{k}^\sharp(\mathbf{x},t) = \alpha\mathbf{k}(\alpha\mathbf{x},\alpha^2 t)\,, \quad \mathbf{U}^\sharp(\mathbf{x},t) = \alpha\mathbf{U}(\alpha\mathbf{x},\alpha^2 t)\,, \quad J^\sharp(\mathbf{x},t) = \alpha^3 J(\alpha\mathbf{x},\alpha^2 t)\,. \tag{4.5b}$$

Using (4.4) and (4.5), one can show that the Whitham modulation equations (3.7) remain unchanged. Similarly, it can be seen that spatial reflections leave the modulation equations invariant upon the following transformation of the dependent variables:

$$\rho^\sharp(\mathbf{x},t) = \rho(-\mathbf{x},t)\,, \quad \lambda_j^\sharp(\mathbf{x},t) = \lambda_j(-\mathbf{x},t)\,,\; j=1,2,3\,, \quad \mathbf{k}^\sharp(\mathbf{x},t) = -\mathbf{k}(-\mathbf{x},t)\,, \tag{4.6a}$$

$$\mathbf{U}^\sharp(\mathbf{x},t) = -\mathbf{U}(-\mathbf{x},t)\,, \quad J^\sharp(\mathbf{x},t) = J(-\mathbf{x},t)\,. \tag{4.6b}$$

Finally, with regards to Galilean transformations, writing $q^\sharp(\mathbf{x},t) = \sqrt{\rho^\sharp(\mathbf{x},t)}\,e^{i\phi^\sharp(\mathbf{x},t)}$ implies

$$\rho^\sharp(\mathbf{x},t) = \rho(\mathbf{x}-2\mathbf{w}t,t)\,, \qquad \phi^\sharp(\mathbf{x},t) = \phi(\mathbf{x}-2\mathbf{w}t,t) + \mathbf{w}\cdot\mathbf{x} - \|\mathbf{w}\|^2 t\,. \tag{4.7}$$

The dependent variables in the modulation equations (3.7) become

$$\lambda_j^\sharp(\mathbf{x},t) = \lambda_j(\mathbf{x}-2\mathbf{w}t,t)\,,\; j=1,2,3\,, \tag{4.8a}$$

$$\mathbf{k}^\sharp(\mathbf{x},t) = \mathbf{k}(\mathbf{x}-2\mathbf{w}t,t)\,, \quad \mathbf{U}^\sharp(\mathbf{x},t) = \mathbf{U}(\mathbf{x}-2\mathbf{w}t,t)+\mathbf{w}\,, \quad J^\sharp(\mathbf{x},t) = J(\mathbf{x}-2\mathbf{w}t,t)\,. \tag{4.8b}$$

Using (4.8), one can verify that the modulation equations (3.7) remain invariant under the above Galilean transformation. The Riemann-type variables, which will be introduced in section 5, change as follows under the above transformations:

$$r_j^\sharp(\mathbf{x},t) = \alpha r_j(\alpha\mathbf{x},\alpha^2 t), \quad r_j^\sharp(\mathbf{x},t) = r_j(-\mathbf{x},t), \quad r_j^\sharp(\mathbf{x},t) = r_j(\mathbf{x}-2\mathbf{w}t,t) + \mathbf{w}\cdot\hat{\mathbf{k}}/2, \quad j=1,2,3,4. \tag{4.9}$$

4.3. Harmonic and soliton limits of the modulation equations in physical variables

The harmonic and soliton limits of the Whitham equations for the KdV and 1DNLS equations have proven to be quite useful to study various nonlinear dynamical scenarios of practical interest [20, 42, 52]. The same is true for the harmonic and soliton limits of the KP-Whitham equations [14, 48, 50, 49]. We therefore expect that the same will also be true for the harmonic and soliton limit of the NLS equation in multiple spatial dimensions.

Like with the periodic solution, the harmonic limit of the Whitham equations is the limit $m \to 0$, corresponding to $\lambda_2 \to \lambda_1^+$. Recall that in this limit the solution becomes a plane wave. The integrals in (3.8) simplify considerably:

$$\overline{\rho} = \lambda_1\,, \quad \overline{\rho^2} = \lambda_1^2\,, \quad \overline{\Big(\frac{(\rho')^2}{\rho}\Big)} = 0\,, \quad \overline{\rho^{-1}} = 1/\lambda_1\,, \quad \overline{\rho^{-2}} = 1/\lambda_1^2\,, \qquad J = \sigma\lambda_1\sqrt{\lambda_3}\,. \tag{4.10}$$

Then, the linear dispersion relation is

$$\omega = 2\|\mathbf{k}\|\left(\hat{\mathbf{k}}\cdot\bar{\mathbf{u}} - \sigma\sqrt{\pi^2\|\mathbf{k}\|^2 + \bar{\rho}^2}\right), \tag{4.11}$$

the averaged energy limits to $\bar{h} = \bar{\rho}\|\bar{\mathbf{u}}\|^2 + \bar{\rho}^2$ and the Whitham equations (3.7) reduce to:

$$\mathbf{k}_t + \boldsymbol{\nabla}\omega = \mathbf{0}\,, \tag{4.12a}$$
$$\bar{\mathbf{u}}_t + \boldsymbol{\nabla}\big(2\bar{\rho} + \|\bar{\mathbf{u}}\|^2\big) = \mathbf{0}\,, \qquad \boldsymbol{\nabla}\times\bar{\mathbf{u}} = 0\,, \tag{4.12b}$$
$$\bar{\rho}_t + 2\boldsymbol{\nabla}\cdot(\bar{\rho}\bar{\mathbf{u}}) = 0\,, \tag{4.12c}$$
$$(\bar{\rho}\bar{\mathbf{u}})_t + \boldsymbol{\nabla}(\bar{\rho}^2) + 2\boldsymbol{\nabla}\cdot\big(\bar{\rho}\bar{\mathbf{u}}\otimes\bar{\mathbf{u}}\big) = \mathbf{0}\,, \tag{4.12d}$$
$$\bar{h}_t + \nabla\cdot\big(2(\bar{h} + \bar{\rho}^2)\bar{\mathbf{u}}\big) = 0\,. \tag{4.12e}$$

Again, not all of these equations are independent. For example, one can derive (4.12*d*) using (4.12*b*) and (4.12*c*). Also, note that the variable $\mathbf{k}$ is immaterial, since its value does not affect the solution, and (4.12) is decoupled from the other PDEs. Thus, equations (4.12*b*) and (4.12*c*), which are equivalent to the shallow water equations, are by themselves a closed system of evolution PDEs for the parameters of the plane wave solution, $\bar{\rho}$ and $\bar{\mathbf{u}}$. Nonetheless, (4.12*a*) describes the evolution of a harmonic wave propagating on top of the mean flow.

Finally, we discuss the soliton limit of the Whitham modulation system (3.7), obtained for $m \to 1$ corresponding to $\lambda_2 \to \lambda_3$. In this limit, the integrals in (3.8) become:

$$\overline{\rho} = \lambda_3\,, \quad \overline{\rho^2} = \lambda_3^2\,, \quad \|\mathbf{k}\|^2\overline{\Big(\frac{(\rho')^2}{\rho}\Big)} = 0\,, \quad \overline{\rho^{-1}} = 1/\lambda_3\,, \quad \overline{\rho^{-2}} = 1/\lambda_3^2\,. \tag{4.13}$$

Then (3.7*a*) and (3.7*b*) are trivially satisfied, and the rest simplify to:

$$\bar{\mathbf{u}}_t + \nabla\big(2\bar{\rho} + \|\bar{\mathbf{u}}\|^2\big) = \mathbf{0}\,, \tag{4.14a}$$
$$\bar{\rho}_t + 2\nabla\cdot(\bar{\rho}\bar{\mathbf{u}}) = 0\,, \tag{4.14b}$$
$$(\bar{\rho}\bar{\mathbf{u}})_t + \nabla(\bar{\rho}^2) + 2\nabla\cdot\big(\bar{\mathbf{u}}\otimes\bar{\rho}\bar{\mathbf{u}}\big) = \mathbf{0}\,, \tag{4.14c}$$
$$\bar{h}_t + \nabla\cdot\big(2(\bar{h} + \bar{\rho}^2)\bar{\mathbf{u}}\big) = 0\,. \tag{4.14d}$$

Note that, similar to before, we can derive equation (4.14*c*) and (4.14*d*) from (4.14*a*) and (4.14*b*). Therefore, we have a system of $N+2$ PDEs for the dependent variables $\bar{\mathbf{u}}$ and $\bar{\rho} = \lambda_2$. But in this case, we are missing PDEs for λ_1 and $\hat{\mathbf{k}}$ that define the soliton amplitude and its propagation direction, which are needed to completely determine the soliton solution. This deficiency is also present in the one-dimensional case. The one-dimensional case is simpler, however, because, in that case, $\mathbf{k}$ is a one-component vector, and therefore $\hat{\mathbf{k}} = \pm 1$, constant. The soliton limit is singular so care must be taken in its calculation. In any case, both in the one-dimensional and higher-dimensional situation, the problem is eliminated by the transformation to Riemann-type variables, as we will see later.

5. 2DNLS-Whitham equations in Riemann-type variables

In this section and the next one we temporarily restrict our attention to the two-dimensional case and perform suitable changes of dependent variables to simplify the form of the 2DNLS-Whitham equations.

When $N = 2$, the modulation system (3.7) consists of eight PDEs for six dependent variables in the independent variables $\mathbf{x} = (x, y)^T$ and t, plus the two scalar constraints (3.7*b*) and (3.7*d*).

We will use the four scalar conservation of waves equations (3.7*a*) and (3.7*c*) together with the averaged conservation of mass (3.7*e*) and one of the components of the conservation of momentum equations (3.7*f*), neglecting the compatibility conditions (3.7*b*) & (3.7*d*) as well as the conservation of energy (3.7*g*). Importantly, however, the resulting Whitham equations are equivalent to those obtained by working with a different set of averaged equations [5].

As in the one-dimensional case, the transformation involves two steps. The first step is the change of dependent variables from $(A, k_1, k_2, m, \bar{u}_1, \bar{u}_2)$ to $\mathbf{Y} = (\sqrt{\lambda_1}, \sqrt{\lambda_2}, \sqrt{\lambda_3}, U_1, U_2, q)$, with

$$q = k_2/k_1 = \tan\varphi, \tag{5.1}$$

similar to [2], where $\varphi = \arctan(k_2/k_1)$ [not to be confused with the fast phase $\phi(Z)$ that was used in sections 2 and 3] identifies the direction of the periodic wave's fronts:

$$\hat{\mathbf{k}} = (\cos\varphi, \sin\varphi)^T. \tag{5.2}$$

The second step of the transformation is then defined by the map from $\lambda_1, \lambda_2, \lambda_3$ and U_1 to the "Riemann-type" variables $\dot{r}_1, \dot{r}_2, \dot{r}_3, \dot{r}_4$ via the transformation

$$U_1 = \tfrac{1}{2}\cos\varphi\,(\dot{r}_1 + \dot{r}_2 + \dot{r}_3 + \dot{r}_4), \tag{5.3a}$$

$$\lambda_1 = \tfrac{1}{4}(\dot{r}_1 - \dot{r}_2 - \dot{r}_3 + \dot{r}_4)^2, \quad \lambda_2 = \tfrac{1}{4}(\dot{r}_1 - \dot{r}_2 + \dot{r}_3 - \dot{r}_4)^2, \quad \lambda_3 = \tfrac{1}{4}(\dot{r}_1 + \dot{r}_2 - \dot{r}_3 - \dot{r}_4)^2. \tag{5.3b}$$

The variables $\dot{r}_1, \ldots, \dot{r}_4$ are one possible two-dimensional generalization of the Riemann invariants of the Whitham equations for the 1DNLS equation. Note that in this work the overdot does not denote differentiation with respect to time.

Recall that the existence of Riemann invariants for $(1+1)$-dimensional hydrodynamic-type systems is intimately tied to the integrability properties of the modulation equations. Using the one-dimensional Riemann invariants as dependent variables in higher-dimensional systems diagonalizes their one-dimensional reductions, and makes the equations more advantageous for analysis (e.g., see [25]). We will show below that, for both the two-dimensional and three-dimensional cases, a suitable generalization of the one-dimensional Riemann invariants allows one to write the modulation equations in a concise and convenient form.

In terms of $\dot{r}_1, \ldots, \dot{r}_4$, the periodic solution (2.17) becomes

$$\rho(Z) = \tfrac{1}{4}(\dot{r}_1 - \dot{r}_2 - \dot{r}_3 + \dot{r}_4)^2 + (\dot{r}_2 - \dot{r}_1)(\dot{r}_4 - \dot{r}_3)\,\mathrm{sn}^2(2K_m Z|m), \tag{5.4a}$$

$$m = \frac{(\dot{r}_2 - \dot{r}_1)(\dot{r}_4 - \dot{r}_3)}{(\dot{r}_3 - \dot{r}_1)(\dot{r}_4 - \dot{r}_2)}. \tag{5.4b}$$

Moreover, $\dot{\mathbf{R}} = (\dot{r}_1, \dot{r}_2, \dot{r}_3, \dot{r}_4, U_\perp, q)^T$ satisfies the hydrodynamic system

$$\dot{\mathbf{R}}_t + \mathsf{M}_1 \dot{\mathbf{R}}_x + \mathsf{M}_2 \dot{\mathbf{R}}_y = 0. \tag{5.5}$$

The matrices M_1 and M_2 are rather complicated, and we therefore omit them for brevity. When $k_2 = U_\perp = 0$, however, the last two equations in (5.5) are trivially satisfied, and the first four reduce to the Whitham equations for the 1DNLS equation in Riemann invariant (diagonal) form [28, 47]:

$$\frac{\partial \dot{\mathbf{r}}}{\partial t} + \mathsf{V}\frac{\partial \dot{\mathbf{r}}}{\partial x} = 0, \tag{5.6}$$

with

$$\dot{\mathbf{r}} = (\dot{r}_1, \ldots, \dot{r}_4)^T, \qquad \dot{\mathsf{V}} = \mathrm{diag}(\dot{\mathbf{V}}), \qquad \dot{\mathbf{V}} = (\dot{V}_1, \ldots, \dot{V}_4)^T, \tag{5.7}$$

$$\dot{V}_1 = 2V_o + \frac{2(\dot{r}_2 - \dot{r}_1)(\dot{r}_4 - \dot{r}_1)K_m}{(\dot{r}_4 - \dot{r}_2)E_m - (\dot{r}_4 - \dot{r}_1)K_m}, \quad \dot{V}_2 = 2V_o + \frac{2(\dot{r}_2 - \dot{r}_1)(\dot{r}_3 - \dot{r}_2)K_m}{(\dot{r}_3 - \dot{r}_2)K_m - (\dot{r}_3 - \dot{r}_1)E_m}, \tag{5.8a}$$

$$\dot{V}_3 = 2V_o + \frac{2(\dot{r}_3 - \dot{r}_2)(\dot{r}_4 - \dot{r}_3)K_m}{(\dot{r}_4 - \dot{r}_2)E_m - (\dot{r}_3 - \dot{r}_2)K_m}, \quad \dot{V}_4 = 2V_o + \frac{2(\dot{r}_4 - \dot{r}_1)(\dot{r}_4 - \dot{r}_3)K_m}{(\dot{r}_4 - \dot{r}_1)K_m - (\dot{r}_3 - \dot{r}_1)E_m}, \tag{5.8b}$$

with $V_o = U_1$.

The Whitham modulation system (5.5) can be further simplified by introducing a modified set of Riemann-type variables:

$$r_j = \cos\varphi\, \dot r_j\,, \quad j = 1,\dots,4\,, \tag{5.9a}$$

with $q = \tan\varphi$ as before. Moreover, the curl-free constraint (3.4d) yields [see section 7 for details]

$$p = \sec\varphi\, U_\perp\,, \tag{5.9b}$$

where the perpendicular component of $\mathbf{U}$ is defined by

$$U_\perp = \mathbf{U}\cdot\hat{\mathbf{k}}_\perp\,, \qquad \hat{\mathbf{k}}_\perp = (-\sin\varphi, \cos\varphi)^T\,. \tag{5.10}$$

The Whitham modulation equations (5.5) then reduce to the following form:

$$\frac{\partial \mathbf{R}}{\partial t} + \mathsf{A}\,\frac{\partial \mathbf{R}}{\partial x} + \mathsf{B}\,\frac{\partial \mathbf{R}}{\partial y} = 0\,, \tag{5.11}$$

where $\mathbf{R} = (r_1,\dots,r_4,q,p)^T$,

$$\mathsf{A} = \begin{pmatrix} A_{4\times4} & A_{4\times2} \\ A_{2\times4} & A_{2\times2} \end{pmatrix}, \qquad \mathsf{B} = \begin{pmatrix} B_{4\times4} & B_{4\times2} \\ B_{2\times4} & B_{2\times2} \end{pmatrix}, \tag{5.12a}$$

with $g = 1+q^2$ as in [5] and

$$A_{4\times4} = \mathsf{V} - q^2 U_1 \mathbb{1}_4 + q^2(\mathbf{1}\otimes\mathbf{r} + \mathbf{r}\otimes\mathbf{1})\,, \quad A_{2\times2} = 2\begin{pmatrix} (1-q^2)U_1 & -q^2 \\ q^2(2U_1^2 - s_2) & gU_1 \end{pmatrix}, \tag{5.12b}$$

$$A_{4\times2} = -q\big(\, 2U_1\mathbf{r} - \mathsf{V}\mathbf{r} + \mathbf{a}/g\;,\; U_1\mathbf{1} - (2(q^2-1)\mathbf{r} + \mathbf{V})/g\,\big)\,, \quad A_{2\times4} = gq\big(-\mathbf{1}\;,\; 2(U_1\mathbf{1} - \mathbf{r})\,\big)^T, \tag{5.12c}$$

$$B_{4\times4} = q(\mathsf{V} + U_1\mathbb{1}_4) + 2p\,\mathsf{I}_4 - q(\mathbf{1}\otimes\mathbf{r} + \mathbf{r}\otimes\mathbf{1})\,, \quad B_{2\times2} = 2\begin{pmatrix} p + 2qU_1 & q \\ -q(2U_1^2 - s_2) & p \end{pmatrix}, \tag{5.12d}$$

$$B_{4\times2} = 1/(2g)\big(2\mathbf{a}\;,\; (1-q^2)(4\mathbf{r} - \mathbf{V})\big)\,, \qquad B_{2\times4} = -A_{2\times4}/q\,, \tag{5.12e}$$

with

$$\mathbf{r} = (r_1,\dots,r_4)^T, \tag{5.12f}$$

$$\mathsf{V} = \mathrm{diag}(\mathbf{V}), \quad \mathbf{V} = (V_1,\dots,V_4)^T, \tag{5.12g}$$

$$\mathbf{a} = \tfrac{1}{3}\big[4U_1(1-3q^2)\mathbf{r} - 2U_1\mathbf{V} - (1+3q^2)((s_2 - 2U_1^2)\mathbf{1} - \mathsf{V}\mathbf{r})\big]\,, \tag{5.12h}$$

where $U_1 = (r_1+r_2+r_3+r_4)/2$, $V_1,\dots,V_4$ are as in (5.8) but with $(r_1,\dots,r_4)$ instead of $(\dot r_1,\dots,\dot r_4)$, $\mathbf{1} = (1,\dots,1)^T$, I_n is the $n\times n$ identity matrix, $\mathbb{1}_n$ denotes the $n\times n$ matrix with all entries equal to one, and

$$s_n = r_1^n + r_2^n + r_3^n + r_4^n\,. \tag{5.13}$$

In component form, the Whitham modulation equations (5.11) are [5]

$$\frac{\partial r_j}{\partial t} + V_j\frac{\partial r_j}{\partial x} + (qV_j + 2p)\frac{\partial r_j}{\partial y} + h_j = 0, \quad j = 1,2,3,4, \tag{5.14a}$$

$$\frac{\partial q}{\partial t} + 2\big(gU_1 + pq\big)\frac{\partial q}{\partial x} + 2\frac{D}{Dy}\big[gU_1 + pq\big] = 0\,, \tag{5.14b}$$

$$\frac{\partial p}{\partial t} + 2gU_1\frac{\partial p}{\partial x} + 2p\frac{\partial p}{\partial y} + \frac{D}{Dy}\big[g(s_2 - 2U_1^2)\big] = 0\,, \tag{5.14c}$$

where

$$h_j = 2q(U_1 - r_j)\frac{DU_1}{Dy} - \frac{1}{2}q\frac{Ds_2}{Dy} + q(V_j - 2U_1)\left(r_j\frac{\partial q}{\partial x} + \frac{1}{2}\frac{\partial p}{\partial x}\right) + \frac{a_j}{g}\frac{Dq}{Dy} - \frac{1-q^2}{2g}(V_j - 4r_j)\frac{Dp}{Dy} \tag{5.15a}$$

and D_y is the "convective" derivative as in [2]:

$$\frac{D}{Dy} = \frac{\partial}{\partial y} - q\frac{\partial}{\partial x}\,. \tag{5.15b}$$

The steps to obtain (5.14) are just a special case of the ones needed to simplify the Whitham equations for the three-dimensional NLS equation, which will be discussed in section 7. All the calculations in section 7 can be trivially reduced to the two-dimentional case by simply taking $(q_1,q_2) = (q,0)$ and $(p_1,p_2) = (p,0)$ there. Therefore we omit the details here for brevity.

Note the necessary compatibility condition for equations (5.14) in which the initial data is subject to the curl-free constraints $\nabla\times\bar{\mathbf{u}} = \nabla\times\mathbf{k} = 0$, similarly to the KP equation [2, 5]. In section 7 we will show how these constraints can be written out explicitly in terms of the Riemann-type variables.

6. Further symmetries and reductions of the 2DNLS-Whitham equations

Both of the sets of Riemann-type variables $\dot{\mathbf{R}}$ and $\mathbf{R}$ introduced in section 5 are useful to study further symmetries of the 2DNLS-Whitham system.

6.1. Reduction to the Whitham equations for the radial NLS equation

The Whitham equations for the 2DNLS equation admit a self-consistent reduction to the Whitham equations for the radial NLS equation, which were recently derived [4]. To show this, we first perform a change of independent variables from the Cartesian coordinates x and y to the polar coordinates

$$R = \sqrt{x^2+y^2}\,, \qquad \theta = \arctan(y/x)\,. \tag{6.1}$$

Using the definition of the convective derivative D_y in (5.15*b*), we find

$$\frac{Df}{Dy} = \frac{(y-qx)}{R}\frac{\partial f}{\partial R} + \frac{(x+qy)}{R^2}\frac{\partial f}{\partial\theta}\,. \tag{6.2}$$

Equations (5.14*b*) and (5.14*c*) in polar coordinates then become, respectively,

$$q_t + g\sum_{i=1}^{4}\Big[(\sin\theta - q\cos\theta)(r_i)_R + \frac{(\cos\theta+q\sin\theta)}{R}(r_i)_\theta\Big] + 2q(\sin\theta-q\cos\theta)p_R + \frac{2q}{R}(\cos\theta+q\sin\theta)p_\theta$$
$$+\big[2U_1(1-q^2)\cos\theta + 2(p+2qU_1)\sin\theta\big]q_R + \big[2(p+2qU_1)\cos\theta - 2U_1(1-q^2)\sin\theta\big]\frac{q_\theta}{R} = 0\,, \tag{6.3a}$$

$$p_t + 2g\sum_{i=1}^{4}(r_i-U_1)\Big[(\sin\theta-q\cos\theta)(r_i)_r + (\cos\theta+q\sin\theta)\frac{(r_i)_\theta}{r}\Big] + 2(gU_1\cos\theta + p\sin\theta)p_R$$
$$+2(p\cos\theta - gU_1\sin\theta)\frac{p_\theta}{R} + 2q(s_2-2U_1^2)\Big[(\sin\theta-q\cos\theta)q_R + (\cos\theta+q\sin\theta)\frac{q_\theta}{R}\Big] = 0\,. \tag{6.3b}$$

We then look for a reduction of (6.3) and the remaining four Whitham equations (5.14*a*) in which $q = \tan\theta = y/x$. With this assumption, (6.3) simplify considerably. We also seek solutions in which the Riemann-type variables $\dot{r}_1,\dots,\dot{r}_4$ are independent of the angular coordinate θ. Recall that the variables $r_1,\dots,r_4$ appearing in (6.3) are related to $\dot{r}_1,\dots,\dot{r}_4$ by (5.9*a*). Thus

$$\frac{\partial r_i}{\partial R} = \frac{1}{\sqrt{g}}\frac{\partial\dot{r}_i}{\partial R} - \frac{qr_i}{g^{3/2}}\frac{\partial q}{\partial R}\,, \qquad \frac{\partial r_i}{\partial\theta} = -\frac{q\dot{r}_i}{g^{3/2}}\frac{\partial q}{\partial\theta}\,. \tag{6.4}$$

Substituting the above expression into (6.3*a*) and (6.3*b*) yields, respectively,

$$p_\theta + \cot\theta\, p = 0\,, \tag{6.5a}$$
$$p_t + 2(U_1\sec\theta + p\sin\theta)p_R + 2(p\cos\theta - \tan\theta\sec\theta U_1)\,p_\theta/R = 0\,. \tag{6.5b}$$

Equation (6.5a) yields $p(R,\theta,t) = C(R,t)\csc\theta$, with $C(R,t)$ to be determined. Then, substituting this expression into (6.5b) yields $C_t + 2(U_1\sec\theta + C)\,C_R - 2(C\cot^2\theta - U_1\sec\theta)\,C/R = 0$, whose only self-consistent solution is $C = 0$, implying $p(R,\theta,t) = 0$.

Now we turn our attention to the reduction of the first four Whitham modulation equations, namely (5.14a). Tedious but straightforward calculations show that, when written in the polar coordinates (6.1), and using $q = \tan\theta$ and $p = 0$ as well as (6.4), the four modulation equations (5.14a) become exactly the Whitham equations for the radial NLS equation derived in [4]:

$$\frac{\partial\dot{\mathbf{r}}}{\partial t} + \dot{\mathsf{V}}\,\frac{\partial\dot{\mathbf{r}}}{\partial R} + \frac{\dot{\mathbf{b}}}{R} = 0\,, \tag{6.6}$$

with $\dot{\mathbf{r}} = (\dot r_1,\dots,\dot r_4)^T$ and $\dot{\mathsf{V}} = \mathrm{diag}(\dot{\mathbf{V}})$ as in section 5, with $\dot{\mathbf{b}} = (\dot b_1,\dots,\dot b_4)^T$,

$$\dot b_1 = 2V_o^2 - \tfrac{1}{3}(\dot r_2 + \dot r_3 + \dot r_4)V_1 - \tfrac{1}{3}[(\dot r_2 + \dot r_3)^2 + (\dot r_3 + \dot r_4)^2 + (\dot r_2 + \dot r_4)^2]\,, \tag{6.7a}$$

$$\dot b_2 = 2V_o^2 - \tfrac{1}{3}(\dot r_1 + \dot r_3 + \dot r_4)V_2 - \tfrac{1}{3}[(\dot r_1 + \dot r_3)^2 + (\dot r_3 + \dot r_4)^2 + (\dot r_1 + \dot r_4)^2]\,, \tag{6.7b}$$

$$\dot b_3 = 2V_o^2 - \tfrac{1}{3}(\dot r_1 + \dot r_2 + \dot r_4)V_3 - \tfrac{1}{3}[(\dot r_1 + \dot r_2)^2 + (\dot r_2 + \dot r_4)^2 + (\dot r_1 + \dot r_4)^2]\,, \tag{6.7c}$$

$$\dot b_4 = 2V_o^2 - \tfrac{1}{3}(\dot r_1 + \dot r_2 + \dot r_3)V_4 - \tfrac{1}{3}[(\dot r_1 + \dot r_2)^2 + (\dot r_2 + \dot r_3)^2 + (\dot r_3 + \dot r_1)^2]\,, \tag{6.7d}$$

and $V_o = \frac{1}{2}(\dot r_1 + \dot r_2 + \dot r_3 + \dot r_4)$ as before. In terms of the physical variables, the assumption $q = \tan\theta$ implies that the wavefronts are oriented radially, and the requirement $p = 0$ means that the mean flow has no transversal component either, which are both conditions that are consistent with a radially symmetric reduction.

6.2. Harmonic limit and soliton limit of the 2DNLS-Whitham equations in Riemann-type variables

In section 4 we studied the harmonic limit and the soliton limit of the modulation equations in physical variables, and we saw that the singular soliton limit yields fewer equations than are needed to describe the parameters of the soliton solutions of the 2DNLS equation. We next study the corresponding limits of the Whitham modulation equations in Riemann-type variables, and we show how the transformation to Riemann-type variables eliminates this problem and yields a closed system of equations.

The harmonic limit ($m \to 0$) corresponds to either $r_2 \to r_1^+$ or $r_3 \to r_4^-$. In the former case, the PDE (5.14a) with $j = 1$ and the one with $j = 2$ coincide, as needed for the limit to be a self-consistent reduction, and the Whitham modulation system (5.11) then becomes

$$\mathbf{R}_t + \mathsf{A}_{o.1}\mathbf{R}_x + \mathsf{B}_{o.1}\mathbf{R}_y = 0\,, \tag{6.8a}$$

with $\mathbf{R} = (r_1, r_3, r_4, q, p)^T$. The matrices $\mathsf{A}_{o.1}$ and $\mathsf{B}_{o.1}$ are simply the matrices A and B from section 5 with $r_2 = r_1$ and the second row and column omitted. Moreover, the Riemann speeds reduce to

$$V_1 = V_2 = 4r_1 - \frac{(r_3 - r_4)^2}{2r_1 - r_3 - r_4}\,, \qquad V_3 = 3r_3 + r_4\,, \qquad V_4 = r_3 + 3r_4\,, \tag{6.8b}$$

while $h_1,\dots,h_4$ are still given by (5.15a) with $r_2 = r_1$. In the latter case (i.e., $r_3 \to r_4^-$), the PDE (5.14a) with $j = 3$ and the one with $j = 4$ coincide, and the Whitham modulation system (5.11) then becomes

$$\mathbf{R}_t + \mathsf{A}_{o.2}\mathbf{R}_x + \mathsf{B}_{o.2}\mathbf{R}_y = 0\,, \tag{6.3a}$$

with $\mathbf{R} = (r_1, r_2, r_3, q, p)^T$. The matrices $\mathsf{A}_{o.2}$ and $\mathsf{B}_{o.2}$ are just the matrices A and B from section 5 with $r_4 = r_3$ and the fourth row and column omitted. The Riemann speeds reduce to

$$V_1 = 3r_1 + r_2\,, \qquad V_2 = r_1 + 3r_2\,, \qquad V_4 = V_3 = 4r_3 + \frac{(r_1 - r_2)^2}{r_1 + r_2 - 2r_3}\,, \tag{6.3b}$$

with $h_1,\dots,h_4$ now given by (5.15a) with $r_4 = r_3$. In both cases, it is straightforward to verify that, once the transformation to Riemann-type variables is inverted and the modulation equations are written back in terms of the physical variables, one recovers the system (4.12).

The soliton limit ($m \to 1$) corresponds to $r_3 \to r_2^+$. In this case, the PDEs (5.14a) with $j = 3$ and the one with $j = 2$ coincide, and the remaining equations become

$$\mathbf{R}_t + \mathsf{A}_1 \mathbf{R}_x + \mathsf{B}_1 \mathbf{R}_y = 0\,, \tag{6.3a}$$

with $\mathbf{R} = (r_1, r_2, r_4, q, p)^T$. The matrices A_1 and B_1 are A and B from section 5 with $r_3 = r_2$ and the fourth row and column omitted. The Riemann speeds reduce to

$$V_1 = 3r_1 + r_4\,, \qquad V_2 = V_3 = r_1 + 2r_2 + r_4 \qquad V_4 = r_1 + 3r_4\,, \tag{6.3b}$$

where $h_1, \ldots, h_4$ are still given by (5.15a) with $r_3 = r_2$. As in the harmonic limit, it is straightforward to verify that, once the transformation to Riemann-type variables is inverted and the modulation equations are written back in terms of the physical variables, one recovers the system (4.14). In this case, however, the equations in Riemann-type variables also allow us to obtain the two previously missing modulation equations, which determine the evolution of $\hat{\mathbf{k}}$ and the soliton amplitude $a = \lambda_3 - \lambda_1$. One of these equations is immediate, since (5.14b) directly determines $q = \tan\varphi$ and therefore $\hat{\mathbf{k}}$. As for the amplitude equation, note that (5.3b) yields $\lambda_3 - \lambda_1 = \sec^2\varphi\,(r_4 - r_2)(r_3 - r_1)$. Therefore, the modulation equation for r_1, $r_2 = r_3$, r_4, and q determine the evolution of the soliton amplitude and direction.

7. Whitham modulation equations for the NLS equation in three spatial dimensions

We now show how, thanks to the rotation-invariant form of all equations in sections 2 and 3, the results of section 5 are easily generalized to the NLS equation in three spatial dimensions.

7.1. Set-up and resulting 3DNLS-Whitham system

The Madelung transformation (2.1) yields the same hydrodynamic system of PDEs (2.2) as well as the mass, momentum and energy conservation laws (2.4) in differential form, now with $\mathbf{u} = (u_1, u_2, u_3)^T$, $\mathbf{x} = (x, y, z)^T$ and $\boldsymbol{\nabla} = (\partial_x, \partial_y, \partial_z)^T$. The two-phase ansatz (2.6) is also the same, now with $\mathbf{k} = (k_1, k_2, k_3)^T$ and $\mathbf{v} = (v_1, v_2, v_3)^T$, and the curl-free condition (2.9) is now $\nabla \times \mathbf{u} = 0$. The only difference is the number of independent parameters in the periodic solutions: eight in three spatial dimensions as opposed to six in two spatial dimensions. The whole derivation in section 3 also remains the same, including the averaged conservation laws (3.4) and the Whitham modulation equations (3.7), again the only difference being the number of equations, which in three dimensions is eleven evolutionary equations.

The first point at which the derivation for the three-dimensional case diverges from the two-dimensional one is the transformation to Riemann-type variables. Compared to [5], the process here is made much easier by the availability of the second conservation of waves equation (3.4c), which allows us to bypass the averaged conservation of energy, which, in turn, greatly simplifies the calculations even in the presence of a third spatial dimension. We begin with the natural generalization of the parametrization (5.2) for $\hat{\mathbf{k}}$, namely:

$$\hat{\mathbf{k}} = (\cos\varphi, \sin\varphi\cos\alpha, \sin\varphi\sin\alpha)^T\,. \tag{7.1a}$$

$$q_1 = k_2/k_1 = \tan\varphi\,\cos\alpha\,, \qquad q_2 = k_3/k_1 = \tan\varphi\,\sin\alpha\,, \tag{7.1b}$$

$$g = 1 + q_1^2 + q_2^2 = 1/k_1^2 = \sec^2\varphi\,. \tag{7.1c}$$

The leading-order part (2.14) of the curl-free condition (2.9) now consists of three equations. The first two of them are $k_1 u_2' = k_2 u_1'$ and $k_1 u_3' = k_3 u_1'$, which, when integrated, yield

$$\mathbf{u}_\flat(Z) = (u_2(Z), u_3(Z))^T = u_1(Z)\,\mathbf{q} + \mathbf{p}\,, \tag{7.2}$$

with $\mathbf{q} = (q_1, q_2)^T$, $\mathbf{p} = (p_1, p_2)^T$, and p_1, p_2 are additional modulation variables depending on the slow variables $\mathbf{x}$ and t that appear due to integration in Z. For any three-component vector $\mathbf{w} = (w_1, w_2, w_3)^T$ we introduce the "flat" notation $\mathbf{w}_\flat = (w_2, w_3)^T$, which we use extensively, to denote the two-component vector comprised of the second and third components of the vector $\mathbf{w}$. The third equation, $k_2 u_3' = k_3 u_2'$ is automatically satisfied. Also, averaging (7.2), we obtain the two additional relations:

$$\bar{\mathbf{u}}_\flat = \bar{u}_1 \mathbf{q} + \mathbf{p}\,, \tag{7.3a}$$

$$\mathbf{U}_\flat = U_1 \mathbf{q} + \mathbf{p}\,. \tag{7.3b}$$

Similarly, the first component of (3.6*b*) yields

$$u_1(Z) = U_1 + Jk_1/(\|\mathbf{k}\|\rho(Z))\,, \tag{7.3c}$$

together with

$$\omega = 2k_1(gU_1 + \mathbf{q}\cdot\mathbf{p})\,. \tag{7.3d}$$

Finally, we define the Riemann-type variables $r_1,\dots,r_4$ via the same transformation as in section 5, namely:

$$U_1 = \tfrac{1}{2}(r_1 + r_2 + r_3 + r_4), \tag{7.4a}$$

$$\lambda_1 = \tfrac{1}{4}\,g(r_1 - r_2 - r_3 + r_4)^2, \quad \lambda_2 = \tfrac{1}{4}\,g(r_1 - r_2 + r_3 - r_4)^2, \quad \lambda_3 = \tfrac{1}{4}\,g(r_1 + r_2 - r_3 - r_4)^2. \tag{7.4b}$$

Then, in sections 7.2, 7.3 and 7.4 below, we show that the Whitham modulation equations (3.7) yield the eight-component system of equations

$$\frac{\partial \mathbf{r}}{\partial t} + \mathsf{V}\frac{\partial \mathbf{r}}{\partial x} + (\mathbf{q}\otimes\mathsf{V} + 2\mathbf{p}\otimes\mathsf{I}_4)\cdot\boldsymbol{\nabla}_\flat \mathbf{r} + \mathbf{h}(\mathbf{r},\mathbf{q},\mathbf{p}) = 0, \tag{7.5a}$$

$$\frac{\partial \mathbf{q}}{\partial t} + 2(U_1 + \mathbf{q}\cdot\mathbf{U}_\flat)\frac{\partial \mathbf{q}}{\partial x} + 2\mathbf{D}_\flat(U_1 + \mathbf{q}\cdot\mathbf{U}_\flat) = 0\,, \tag{7.5b}$$

$$\frac{\partial \mathbf{p}}{\partial t} + 2(U_1 + \mathbf{q}\cdot\mathbf{U}_\flat)\frac{\partial \mathbf{p}}{\partial x} + \mathbf{D}_\flat(g(\tilde{e}_1 - U_1^2) + \|\mathbf{p}\|^2) = 0\,. \tag{7.5c}$$

Here, as before, $\mathbf{r} = (r_1,\dots,r_4)^T$, $\mathsf{V} = \mathrm{diag}(\mathbf{V})$ with $\mathbf{V} = (V_1,\dots,V_4)^T$ as in (5.12*g*), and the dot product in (7.5*a*) operates on the two-component vectors to its left and its right. That is, in component form, for each $j = 1,\dots,4$ the third term in (7.5*a*) is the dot product between $\mathbf{q}\,V_j + 2\mathbf{p}$ and $\boldsymbol{\nabla}_\flat r_j$. Additionally, (7.5*b*) and (7.5*c*) contain the three-dimensional generalization of the convective derivative of [2] and section 5, namely:

$$\mathbf{D}_\flat = (D_y, D_z)^T = \boldsymbol{\nabla}_\flat - \mathbf{q}\,\partial_x\,, \tag{7.6a}$$

where $\boldsymbol{\nabla}_\flat = (\partial_y, \partial_z)^T$ and

$$\frac{D}{Dy} = \frac{\partial}{\partial y} - q_1\frac{\partial}{\partial x}\,, \qquad \frac{D}{Dz} = \frac{\partial}{\partial z} - q_2\frac{\partial}{\partial x}\,. \tag{7.6b}$$

The term $\mathbf{h}(\mathbf{r},\mathbf{q},\mathbf{p}) = (h_1,\dots,h_4)^T$ in (7.5*a*) is given by

$$\begin{aligned} h_j = {}& 2(U_1 - r_j)\mathbf{q}\cdot\mathbf{D}_\flat U_1 - \tfrac{1}{2}\mathbf{q}\cdot\mathbf{D}_\flat s_2 + (V_j - 2U_1)\,\mathbf{q}\cdot\left(r_j\frac{\partial \mathbf{q}}{\partial x} + \frac{1}{2}\frac{\partial \mathbf{p}}{\partial x}\right) - \tfrac{1}{4}(V_j - 4r_j)\,\mathbf{D}_\flat\cdot\mathbf{p} \\ & + a_j\mathbf{D}_\flat\cdot\mathbf{q} + (b_j/g)\,\mathrm{tr}[(\mathbf{q}\otimes\mathbf{q})(\mathbf{D}_\flat\otimes\mathbf{q})] + ((V_j - 4r_j)/g)\,\mathrm{tr}[(\mathbf{q}\otimes\mathbf{q})(\mathbf{D}_\flat\otimes\mathbf{p})], \end{aligned} \tag{7.7a}$$

with

$$a_j = \tfrac{1}{3}[2(2r_j - V_j)U_1 - s_2 + 2U_1^2 + V_j r_j], \quad b_j = r_j(V_j - 4U_1) - s_2 + 2U_1^2 + a_j, \tag{7.7b}$$

for $j = 1,\dots,4$. The s_n are as in (5.13), and $\tilde{e}_1 = g(\lambda_1 + \lambda_2 + \lambda_3)$ in (7.5*c*), is similar to (2.22). Equations (7.5*a*), (7.5*b*), (7.5*c*) and (7.7*a*) should be compared to (5.14*a*), (5.14*b*), (5.14*c*) and (5.15*a*) in the two-dimensional case. Note that, while $\mathbf{h}_4(\mathbf{r},\mathbf{q},\mathbf{p})$ might give the impression of a forcing term in (7.5*a*), that is not the case in reality, as (7.7*a*) shows that $\mathbf{h}_4(\mathbf{r},\mathbf{q},\mathbf{p})$ is in fact a homogenous first-order differential polynomial in $\mathbf{r}$, $\mathbf{q}$ and $\mathbf{p}$, and therefore (7.5) is indeed a system of PDEs of hydrodynamic type like its one-dimensional and two-dimensional counterparts.

Similarly to the two-dimensional case, the 3DNLS-Whitham modulation equations (7.5) are subject to the compatibility conditions $\nabla\times\bar{\mathbf{u}}(\mathbf{x},0)=\mathbf{0}$ and $\nabla\times\mathbf{k}(\mathbf{x},0)=\mathbf{0}$ at $t=0$. In Appendix A.3 we show that, in terms of the dependent variables defined above, these constraints become, respectively,

$$k_1\mathbf{q}_x=\mathbf{D}_\flat k_1\,,\qquad k_1\mathbf{p}_x=2((\boldsymbol{\nabla}_{\mathbf{r}}k_1)^T\mathsf{R}_4)\cdot\mathbf{D}_\flat\mathbf{r}-2U_1\mathbf{D}_\flat k_1\,,\tag{7.8}$$

where $\boldsymbol{\nabla}_{\mathbf{r}}=(\partial_{r_1},\dots,\partial_{r_4})^T$, $\mathsf{R}_4=\mathrm{diag}(r_1,\dots,r_4)$, and the dot product operates on the four-component vectors to its left and its right. Equations (7.8) are conditions that must be satisfied by the initial conditions for (7.5) in order for its solutions to represent modulations of actual one-phase solutions of the NLS equation (1.1).

7.2. Derivation of the 3DNLS-Whitham system: Equations for the auxiliary variables

To derive the evolution equation (7.5*b*) for $\mathbf{q}$, we split the first conservation of waves equation (3.4*a*) and rewrite it using the convective derivatives D_y and D_z defined in (7.6*b*), to obtain

$$k_{1,t}+\omega_x=0\,,\qquad\mathbf{q}_t+(\mathbf{D}_\flat\omega)/k_1=0\,,\qquad\mathbf{q}_x=(\mathbf{D}_\flat k_1)/k_1\,,\tag{7.9}$$

with ω as in (7.3*d*). The first of equations (7.9) will be used later to derive (7.5*a*). Substituting the third equation in (7.9) into the second one and using (7.3*d*) yields the desired evolution equation (7.5*b*). Note also that the third equation in (7.9) is precisely the first of the constraints (7.8).

Next, to derive the evolution equation (7.5*c*) for $\mathbf{p}$, we start with the constraint (3.4*d*) for the second conservation of waves equation. Using (7.3), (3.4*d*) yields

$$(\bar{u}_1\mathbf{q}+\mathbf{p})_x=\boldsymbol{\nabla}_\flat\bar{u}_1\,.\tag{7.10}$$

Averaging (7.3*c*) over the unit period, we can rewrite the above as

$$\partial_x\big[(U_1+J\overline{\rho^{-1}}/g^{1/2})\mathbf{q}+\mathbf{p}\big]=\boldsymbol{\nabla}_\flat\big(U_1+J\overline{\rho^{-1}}/g^{1/2}\big)\,,\tag{7.11}$$

and simplifying further we obtain

$$\mathbf{p}_x=\mathbf{D}_\flat U_1+\mathbf{D}_\flat\big(J\overline{\rho^{-1}}/g^{1/2}\big)-\big(U_1+J\overline{\rho^{-1}}/g^{1/2}\big)(\mathbf{D}_\flat k_1)/k_1\,.\tag{7.12}$$

Now we use the second conservation of waves (3.4*c*), written in the form of (3.7*c*). From the second and third components, together with the above relations, we have

$$\partial_t\big((U_1+J\overline{\rho^{-1}}/g^{1/2})\mathbf{q}+\mathbf{p}\big)+\boldsymbol{\nabla}_\flat\big(g(U_1^2+\tilde{e}_1)+\|\mathbf{p}\|^2+2U_1\mathbf{q}\cdot\mathbf{p}+2(gU_1+\mathbf{q}\cdot\mathbf{p})J\overline{\rho^{-1}}/g^{1/2}\big)=0\,.\tag{7.13}$$

Simplifying yields

$$\begin{aligned}\mathbf{p}_t+\mathbf{D}_\flat\big(g(U_1^2+\tilde{e}_1)+\|\mathbf{p}\|^2+2U_1\mathbf{p}\cdot\mathbf{q}\big)-2U_1\,\mathbf{D}_\flat(gU_1+\mathbf{p}\cdot\mathbf{q})\\+2(gU_1+\mathbf{p}\cdot\mathbf{q})\big(\mathbf{D}_\flat\big(J\overline{\rho^{-1}}/g^{1/2}\big)-\big(U_1+J\overline{\rho^{-1}}/g^{1/2}\big)(\mathbf{D}_\flat k_1)/k_1\big)=0\,.\end{aligned}\tag{7.14}$$

Using (7.12) yields the desired equation (7.5*c*).

7.3. Derivation of the 3DNLS-Whitham system: Convective derivatives

It remains to derive the four equations in (7.5*a*) for the Riemann-type variables $r_1,\dots,r_4$. To this end, we use the two conservation of waves equations (3.7*a*) and (3.7*c*) (as well as the compatibility conditions (3.7*b*) and (3.7*d*)) along with the averaged conservation of mass and momentum equations (3.7*e*), (3.7*f*). The process comprises three main steps.

The first step is the further simplification of the averaged conservation laws. Note that

$$k_1^2=g(\lambda_3-\lambda_1)/4K_m^2\,.\tag{7.15}$$

For convenience we also introduce the quantity $\mathbf{M} = \overline{\rho\mathbf{u}}/g = (M_1, M_2, M_3)^T$, with

$$M_1 := \overline{\rho u_1}/g = U_1\overline{\rho}/g + \tilde{J}, \qquad \mathbf{M}_\flat := M_1\mathbf{q} + (\overline{\rho}/g)\,\mathbf{p} = (\overline{\rho}/g)\mathbf{U}_\flat + \tilde{J}\mathbf{q}, \tag{7.16}$$

and $\tilde{J} = \hat{k}_1 J/g$. Then, using (7.3$d$) one can rewrite the modulation equations (3.7a) and (3.7e) as follows:

$$k_{1,t} + 2[k_1(U_1 + \mathbf{q}\cdot\mathbf{U}_\flat)] = 0, \tag{7.17}$$

$$(\overline{\rho})_t + 2(gM_1)_x + 2\mathbf{\nabla}_\flat\cdot(g\,\mathbf{M}_\flat) = 0, \tag{7.18}$$

while the first component of the second conservation of waves equation (3.7c) becomes

$$(U_1 + J\overline{\rho^{-1}}/g^{1/2})_t + \big[g\big(\tilde{e}_1 + U_1^2 + 2J\overline{\rho^{-1}}/g^{1/2}U_1\big) + \|\mathbf{p}\|^2 + 2(U_1 + J\overline{\rho^{-1}}/g^{1/2})\mathbf{p}\cdot\mathbf{q}\big]_x = 0. \tag{7.19}$$

Moreover, using the equation (A.19b) we can write the averaged momentum equation (3.7f) in component form as

$$(gM_1)_t + \big[g\big(2U_1(M_1+\tilde{J}) + \tilde{e}_2 - \overline{\rho}^2/g^2\big) + \overline{\rho}^2\big]_x + \mathbf{\nabla}_\flat\cdot\big[2g(M_1+\tilde{J})\mathbf{U}_\flat - 2g\tilde{J}\mathbf{p} + g(\tilde{e}_2 - \overline{\rho}^2/g^2)\mathbf{q}\big] = 0, \tag{7.20a}$$

$$(g\mathbf{M}_\flat)_t + \Big[g\big(2(M_1+\tilde{J})\mathbf{U}_\flat + (\tilde{e}_2 - \overline{\rho}^2/g^2)\mathbf{q} - 2\tilde{J}\mathbf{p}\big)\Big]_x + \mathbf{\nabla}_\flat\overline{\rho}^2) + \mathbf{\nabla}_\flat\cdot\Big[2g\mathbf{M}_\flat\otimes\mathbf{U}_\flat + 2g\tilde{J}\mathbf{U}_\flat\otimes\mathbf{q} + g(\tilde{e}_2 - \overline{\rho}^2/g^2)\,\mathbf{q}\otimes\mathbf{q}\Big] = \mathbf{0}. \tag{7.20b}$$

Next, we perform a second, intermediate step to write the Whitham modulation equations in terms of convective derivatives. First, we derive some identities that will be useful later. Equation (3.7b) and the definition of $\mathbf{q}$ in (7.1b) yield

$$\mathbf{q}_x = \frac{1}{k_1}\mathbf{D}_\flat k_1, \quad \mathbf{q}_y = \frac{1}{k_1}\mathbf{D}_\flat(k_1q_1), \quad \mathbf{q}_z = \frac{1}{k_1}\mathbf{D}_\flat(k_1q_2). \tag{7.21}$$

Moreover, in Appendix A.3, we show that these relations also yield the two constraints

$$D_y q_2 = D_z q_1, \tag{7.22a}$$

$$D_y p_2 = D_z p_1, \tag{7.22b}$$

which will prove to be useful. We then define the additional convective derivatives

$$D_x = \frac{\partial}{\partial x} + \mathbf{q}\cdot\mathbf{\nabla}_\flat, \qquad D_t = \frac{\partial}{\partial t} + 2U_1\frac{\partial}{\partial x} + 2\mathbf{U}_\flat\cdot\mathbf{\nabla}_\flat. \tag{7.23}$$

Now we rewrite the evolution equations for $\mathbf{q}$ using these convective derivatives. Specifically, in Appendix A.3 we show that (7.17) and (7.19) yield, respectively,

$$\frac{D_t k_1}{k_1} + 2D_xU_1 + W_1 = 0, \tag{7.24}$$

$$D_t(U_1 + J\overline{\rho^{-1}}/g^{1/2}) + (U_1 - J\overline{\rho^{-1}}/g^{1/2})\frac{D_tk_1}{k_1} + D_x(\tilde{e}_1 + U_1^2) + 2W_2 = 0, \tag{7.25}$$

where

$$gW_1 = \mathbf{q}\cdot[D_t\mathbf{q} + 2U_1\,D_x\mathbf{q} + 2D_x\mathbf{p}], \tag{7.26a}$$

$$gW_2 = \mathbf{q}\cdot\big[U_1\,D_t\mathbf{q} + s_2D_x\mathbf{q} + \frac{1}{2}D_t\mathbf{p} + U_1\,D_x\mathbf{p}\big]. \tag{7.26b}$$

Moreover, in Appendix A.3 we also show that the conservation of mass equation (7.18) and conservation of momentum equation (7.20a) yield, respectively,

$$g\Big[D_t(\overline{\rho}/g) - (\overline{\rho}/g)\frac{D_tk_1}{k_1} + 2D_x\tilde{J}\Big] + (\overline{\rho}/g)(\mathbf{q}\cdot D_t\mathbf{q}) + 6\tilde{J}\mathbf{q}\cdot D_x\mathbf{q} + 2M_1\big(g\mathbf{\nabla}_\flat\cdot\mathbf{q} - \mathbf{q}\cdot D_x\mathbf{q}\big) + 2(\overline{\rho}/g)\big(g\mathbf{\nabla}_\flat\cdot\mathbf{p} - \mathbf{q}\cdot D_x\mathbf{p}\big) = 0, \tag{7.27}$$

$$g\Big[(\overline{\rho}/g)D_tU_1 + D_t\tilde{J} - 2\tilde{J}\frac{D_tk_1}{k_1} + D_x\tilde{e}_2\Big] + \mathbf{q}\cdot[M_1D_t\mathbf{q} + (\overline{\rho}/g)D_t\mathbf{p} + 4\tilde{e}_2D_x\mathbf{q}] + (\tilde{e}_2 - (\overline{\rho}^2/g^2) + 2U_1\tilde{J})\big(g\mathbf{\nabla}_\flat\cdot\mathbf{q} - \mathbf{q}\cdot D_x\mathbf{q}\big) + 2\tilde{J}\big(g\mathbf{\nabla}_\flat\cdot\mathbf{p} - \mathbf{q}\cdot D_x\mathbf{p}\big) = 0. \tag{7.28}$$

Equations (7.24), (7.25), (7.27) and (7.28) comprise the four modified modulation equations written in terms of the variables U_1, λ_1, λ_2 and λ_3 and the convective derivatives D_x and D_t.

7.4. Derivation of the 3DNLS-Whitham system: Equations for Riemann-type variables

The third and final step in the derivation of (7.5*a*) is to express the modulation equations in terms of $r_1,\ldots,r_4$. Recall the transformation (7.4) to the Riemann-type variables. Note that the arrangement of indices in (7.4) is dictated by the requirement that the constraint (2.21) be satisfied when $r_1 \leqslant r_2 \leqslant r_3 \leqslant r_4$, since

$$\lambda_2 - \lambda_1 = g(r_4 - r_3)(r_2 - r_1)\,, \tag{7.29a}$$

$$\lambda_3 - \lambda_1 = g(r_4 - r_2)(r_3 - r_1)\,, \tag{7.29b}$$

$$\lambda_3 - \lambda_2 = g(r_4 - r_1)(r_3 - r_2)\,. \tag{7.29c}$$

In Appendix A.3, using the above definitions, we show that (7.24), (7.25), (7.27) and (7.28) yield, respectively,

$$(\boldsymbol{\nabla}_{\mathbf{r}}[\log k_1])^T D_t\mathbf{r} + D_x s_1 + W_1 \qquad = 0\,, \tag{7.30a}$$

$$2(\boldsymbol{\nabla}_{\mathbf{r}}[\log k_1])^T \mathsf{R}_4\, D_t\mathbf{r} + D_x s_2 + 2W_2 = 0\,, \tag{7.30b}$$

$$3(\boldsymbol{\nabla}_{\mathbf{r}}[\log k_1])^T \mathsf{R}_4^2 D_t\mathbf{r} + D_x s_3 + 3W_3 = 0\,, \tag{7.30c}$$

$$4(\boldsymbol{\nabla}_{\mathbf{r}}[\log k_1])^T \mathsf{R}_4^3\, D_t\mathbf{r} + D_x s_4 + 4W_4 = 0\,, \tag{7.30d}$$

where $\mathbf{r} = (r_1,\ldots,r_4)^T$, $\boldsymbol{\nabla}_{\mathbf{r}} = (\partial_{r_1},\ldots,\partial_{r_4})^T$ and $\mathsf{R}_4 = \mathrm{diag}(r_1,\ldots,r_4)$ as before, with W_1 and W_2 as in (7.26*a*) and (7.26*b*), and

$$\begin{aligned} gW_3 &= \tfrac{1}{4}(s_2 - 2U_1^2)gW_1 + U_1 gW_2 + \tfrac{1}{2}\mathbf{q}\cdot[(\overline{\rho}/g)D_t\mathbf{q} + 6\tilde{J}D_x\mathbf{q}] \\ &\quad + (U_1\bar{\rho}/g + \tilde{J})\big(g(\boldsymbol{\nabla}_\flat\cdot\mathbf{q}) - \mathbf{q}\cdot D_x\mathbf{q}\big) + (\overline{\rho}/g)\big(g(\boldsymbol{\nabla}_\flat\cdot\mathbf{p}) - \mathbf{q}\cdot D_x\mathbf{p}\big)\,, \end{aligned} \tag{7.31a}$$

$$\begin{aligned} gW_4 &= \tfrac{1}{8}(6\tilde{J} - U_1 s_2 + 2U_1^3)gW_1 + \tfrac{1}{4}(s_2 - 4U_1^2)gW_2 + \tfrac{3}{2}U_1 gW_3 \\ &\quad + \tfrac{1}{4}\big[\mathbf{q}\cdot(M_1 D_t\mathbf{q} + (\overline{\rho}/g)D_t\mathbf{p} + 4\tilde{e}_2 D_x\mathbf{q}) \\ &\quad + (\tilde{e}_2 - (\overline{\rho}^2/g^2) + 2U_1\tilde{J})\big(g(\boldsymbol{\nabla}_\flat\cdot\mathbf{q}) - \mathbf{q}\cdot D_x\mathbf{q}\big) + 2\tilde{J}\big(g(\boldsymbol{\nabla}_\flat\cdot\mathbf{p}) - \mathbf{q}\cdot D_x\mathbf{p}\big)\big]\,. \end{aligned} \tag{7.31b}$$

Importantly, note that, even though the second conservation of waves equation (7.25) contains the third complete elliptic integral $\Pi(\cdot,m)$ via $\overline{\rho^{-1}}$ [cf. (3.8*b*)], the third elliptic integral does not appear in the resulting modulation equation (7.30*a*). Note that $\Pi(\cdot,m)$ is also contained in the conservation of energy equation. Next, one can collect the four equations (7.30) and rewrite them in matrix form as

$$\mathsf{M}(\mathbf{r})\left(\nabla_{\mathbf{r}}[\log k_1]\cdot D_t\mathbf{r} + D_x\mathbf{r}\right) + \mathbf{W} = \mathbf{0}\,, \tag{7.32}$$

where $\mathbf{W} = (W_1,\cdots,W_4)^T$ and $\mathsf{M}(\mathbf{r})$ is the Vandermonde matrix

$$\mathsf{M}(\mathbf{r}) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ r_1 & r_2 & r_3 & r_4 \\ r_1^2 & r_2^2 & r_3^2 & r_4^2 \\ r_1^3 & r_2^3 & r_3^3 & r_4^3 \end{pmatrix}. \tag{7.33}$$

Multiplying (7.32) by $\mathsf{M}^{-1}(\mathbf{r})$, we then finally obtain (7.5*a*), with

$$h_j = \frac{(-1)^{j+1}\Delta_{ilm}}{|\Delta|\,(\partial k/\partial r_j)/k}[r_i r_l r_m W_1 - (r_i r_l + r_l r_m + r_m r_i)W_2 + (r_i + r_l + r_m)W_3 - W_4], \quad j = 1,\ldots,4\,, \tag{7.34}$$

where $j \neq i, j \neq l, j \neq m$, $i < l < m$, summation of repeated indices is implied, and

$$|\Delta| = \prod_{j>l}^{4}(r_j - r_l)\,, \qquad \Delta_{ilm} = (r_i - r_l)(r_l - r_m)(r_m - r_i)\,. \tag{7.35}$$

Finally, using equations (A.22*a*) and (A.22*b*), one can simplify $h_1,\ldots,h_4$ in (7.34) to obtain (7.7*a*).

8. Discussion and perspectives

In summary, we derived the Whitham modulation equations for the defocusing NLS equation in two, three and higher spatial dimensions using a two-phase ansatz and the averaged conservation laws of the NLS equation written in coordinate-free vector form, and we elucidated various symmetries and reductions of the resulting equations, including the reduction to the Whitham equations of the radial NLS equation as well as the harmonic and soliton limits. We point out that, long after this work was completed, we learned that modulation equations for multi-dimensional equations of NLS type were written down in physical variables using a general framework in [7], and the modulation equations were used to study the stability of the plane wave solutions. On the other hand, no transformation to Riemann-type variables was carried out in [7].

We reiterate that the use of a two-phase ansatz in this work (as opposed to a one-phase ansatz as in [5]) greatly simplifies the derivation, since it results in a second conservation of waves equation that allows us to avoid using the conservation of energy equation, which is much more complicated in comparison. Moreover, the advantage of using a two-phase ansatz increases with the number of spatial dimensions. This is because the number of modulation equations needed is $2N+2$. Therefore, if one tried to derive the modulation equations in three spatial dimensions with a one-phase ansatz, one would need to use additional conservation laws for the NLS equation. This would not only lead to a much more complicated derivation, but one would quickly exhaust the number of available conservation laws, since the NLS equation in more than one spatial dimensions is not completely integrable, and therefore does not have hidden symmetries resulting in an infinite number of conservation laws.

In contrast, the results of section 7 can be generalized in a straightforward way to obtain the Whitham modulation equations in simplified form in an arbitrary number of spatial dimensions. The system of modulation equations (7.5) is already written in vectorial, dimension-independent form, with the only caveat that, with N spatial dimensions, $\mathbf{q}$ and $\mathbf{p}$ have $N-1$ components. Moreover, all the steps of the derivation in section 7 are written in a way that generalizes to any number of spatial dimensions. Indeed, one can introduce spherical coordinates in N spatial dimensions by generalizing (7.1*a*) as $\hat{k}_1 = \cos\varphi_1$, $\hat{k}_2 = \sin\varphi_1\cos\varphi_2$, $\hat{k}_3 = \sin\varphi_1\sin\varphi_2\cos\varphi_3$, etc., up to $\hat{k}_{N-1} = \sin\varphi_1\cdots\sin\varphi_{N-2}\cos\varphi_{N-1}$ and $\hat{k}_N = \sin\varphi_1\cdots\sin\varphi_{N-2}\sin\varphi_{N-1}$. Then, one introduces $q_1,\dots,q_N$ via the generalization of (7.1*b*), namely, $q_1 = k_2/k_1$, $q_2 = k_3/k_1$, etc., up to $q_{N-1} = k_N/k_1$, as well as $p_1,\dots,p_N$ via the natural generalization of (7.2). In this way, one obtains the generalization of (7.1*c*) as $g = 1+q_1^2+\cdots+q_N^2 = \sec\varphi_1$, and all the calculations and equations in section 7 remain valid as long as one also redefines the operators $\boldsymbol{\nabla}_\flat$ and $\mathbf{D}_\flat$ accordingly.

We point out that, even though we have not done so explicitly in this work, it would be straightforward to obtain the reduction to the Whitham equations for the radial NLS equation from the 3DNLS-Whitham equations derived in section 7 using spherical coordinates. It would also be straightforward to write down explicitly the harmonic and soliton limits in three spatial dimensions, as well as all of these corresponding limits in higher dimensions.

We should comment on the importance of the constraints $\boldsymbol{\nabla}\wedge\mathbf{k} = 0$ and $\boldsymbol{\nabla}\wedge\bar{\mathbf{u}} = 0$. On one hand, these constraints play a key role in the derivation. On the other hand, the final Whitham equations [e.g., (7.5*a*), (7.5*b*) and (7.5*c*)] do not automatically ensure that these constraints are satisfied, only that their time derivative is (similar to [2]). Because of this, the compatibility between solutions of the Whitham system and modulated one-phase solutions of the NLS equation is not guaranteed a priori, and, similar to [2], one must give initial conditions that are compatible with the one-phase assumption. These constraints are also likely to be related to the integrability properties of the system, as discussed below.

We emphasize that this work is foundational and that, similar to [2], the results presented here open up a number of interesting problems, which are expected to lead to several further advances in the near future. Specifically, we next mention and briefly discuss some of these possible of avenues for further research.

One direction for future work is the derivation of the Whitham equations for the focusing NLS equation in three spatial dimensions. We expect that this will be straightforward. Indeed, the Whitham equations in the two-dimensional focusing case were already written in [5] (although not in rotation-invariant form). Once the derivation of the one-phase solutions of the NLS equation is done in dimension-invariant form, as was the case in section 2.2, the rest of the machinery presented in this work will carry over to the case of the focusing case in three and higher dimensions without significant changes. Of course, as in the one-dimensional case, the resulting Whitham equations will be elliptic (i.e., the characteristic velocities will be complex), and therefore require suitable interpretation of initial value problems; see [15, 23, 32, 33, 34, 40] as well as [13, 25] and references therein. The Whitham equations for the NLS equation in one spatial dimension have also proved to be useful in some situations, even in the focusing case [11, 16, 24], so one can expect that those in two and three spatial dimensions will be useful as well.

Another important direction for future work is a study to determine whether the Whitham modulation system derived here, or any of its reductions, are completely integrable. A notion of integrability for multidimensional systems was put forth in [26, 27], based on the existence of infinitely many N-component reductions. Of course, the NLS equation in more than one spatial dimension is not integrable, and therefore one would have no reason to expect that the corresponding NLS-Whitham systems are. Still, the reductions to one-dimensional NLS-Whitham equations are indeed integrable, and therefore it is a natural question whether there are other integrable reductions. In this regard, we should point out that, even for the KP equation (which is integrable), the original Whitham system derived in [2] appears not to be integrable, but its harmonic and soliton limits are [14]. Moreover, so are various less-trivial one-dimensional reductions beyond the obvious reduction to the Whitham system for the KdV equation, once one properly takes into account the analogue of the compatibility conditions (3.4*a*) and (3.4*b*) [12].

Yet another interesting problem for future work is the issue of whether one can establish a precise relation between the 2DNLS-Whitham system and the KP-Whitham system. It is well known that the 1DNLS-Whitham system admits a reduction to the KdV-Whitham system [32]. It is also well known that the 2DNLS equation admits a reduction to the KP equation [41]. A natural question is therefore whether the 2DNLS-Whitham system admits a reduction to the KP-Whitham system. It is straightforward to see that, if one considers the same reduction as in [32], the PDEs for $r_1,\dots,r_4$ in the 2DNLS-Whitham system naturally reduce to those for $r_1,\dots,r_3$ in the KP-Whitham system. The PDE for q also reduces to the corresponding equation in the KP-Whitham system, since it just comes from the second component of the conservation of waves equation in both systems. The open question, however, is how one can obtain a PDE for p that does not contain a time derivative, as prescribed in the KP-Whitham system.

Finally, and most importantly from a practical point of view, an obvious opportunity for future work will be the use of the modulation equations derived here to characterize the dynamical behavior in physically significant scenarios. One important application is to the description of dispersive shock waves (DSWs) [32, 13]. Some of the earliest experiments on DSWs in nonlinear optics and Bose-Einstein condensates (BECs)—where the defocusing NLS equation is an excellent model—involved inherently multidimensional nonlinear wave propagation [22, 51, 34, 55]. One intriguing feature, observed in both BEC and optics [34, 55], is the coherent propagation of multidimensional DSWs with stable ring/spherical and elliptical/ellipsoidal patterns. These observations are at odds with the known transverse instability of planar cnoidal wave solutions of (1.1) [54]. Further analysis of the 2D and 3DNLS-Whitham modulation equations may provide some analytical insight in this. Moreover, BECs are three-dimensional, so the (3+1)-dimensional modulation equations derived here are needed to describe large amplitude matter waves. Three-dimensional effects have been shown to be decisive in some BEC DSW experiments [19, 43].

Various applications of the Whitham equations for the focusing and defocusing NLS equations in one spatial dimension were already mentioned above. We should also note that, while the full modulation system composed of equations (7.5*a*), (7.5*b*) and (7.5*c*) might appear complicated, even its reductions can be useful in this regard. For example, of particular interest from an applicative point of view are the harmonic and soliton limits. In the one-dimensional case, soliton modulation theory and its applications were studied for the KdV equation in [42] and for the defocusing NLS equation in [52], while the harmonic limit of the Whitham equations for the KdV equation was studied in [20]. Similarly, the harmonic and soliton limits of the Whitham equations for the KP equation, which were derived and analyzed in [2, 14] have found concrete applications in [48, 50, 49]. These reductions analytically describe the evolution of a soliton or linear waves in the presence of the slowly varying mean field $\bar{\rho}$, $\bar{\mathbf{u}}$. Obtaining these modulation equations using multiple scales and a soliton ansatz is quite tedious and, to our knowledge, has apparently only been carried out for the KdV equation in [31]. We believe that, like with Whitham equations for the KP equation [2], the modulation equations derived in this work will prove to be an effective tool to study several physically significant problems. The soliton limit should prove to be particularly important in this respect, similar to the KP equation [48, 49, 50].

We hope that the results of this work and the present discussion will provide a stimulus for several further studies on these and related problems.

Acknowledgments

We thank Alexandr Chernyavskiy and Dmitri Kireyev for many useful discussions on topics related to this work. This research was partially supported by the National Science Foundation under grant numbers DMS-1816934 and DMS-2009487.

Appendix

A.1. Direct derivation of the periodic solutions of the NLS equation

Here we derive the periodic solutions of the NLS equation in an arbitrary number of dimensions directly, without using the hydrodynamic system. We start with the one-phase ansatz

$$\psi(\mathbf{x},t) = \sqrt{\rho(z/\varepsilon)}\,\mathrm{e}^{i\Phi(z/\varepsilon)}, \tag{A.1}$$

where, as before, the "fast variable" is $Z = \mathbf{k}\cdot\mathbf{x} - \omega t$. Substituting (A.1) into (1.1) and separating into real and imaginary parts yields respectively:

$$(\sqrt{\rho})'' - \sqrt{\rho}(\Phi')^2 + \frac{\omega}{\|\mathbf{k}\|^2}\sqrt{\rho}\,\Phi' - \frac{2}{\|\mathbf{k}\|^2}\rho^{3/2} = 0, \tag{A.2a}$$

$$\sqrt{\rho}\,\Phi'' + \Big(2\Phi' - \frac{\omega}{\|\mathbf{k}\|^2}\Big)(\sqrt{\rho})' = 0, \tag{A.2b}$$

where, for brevity in this section, we denote $a = \|\mathbf{k}\|^2$. Integrating (A.2b) yields Φ' up to an integration constant J

$$\Phi' = \frac{J}{\|\mathbf{k}\|\,\rho} + \frac{\omega}{2a}. \tag{A.3}$$

Substituting the phase relation (A.3), the real part (A.2a) reduces to:

$$(\sqrt{\rho})'' - \frac{J^2}{a\,\rho^{3/2}} + \Big(\frac{\omega}{2a}\Big)^2\sqrt{\rho} - \frac{2}{a}\rho^{3/2} = 0. \tag{A.4}$$

Multiplying by $2(\sqrt{\rho})'$ and integrating with respect to Z and letting $f = \rho$ yields:

$$(f')^2 = \frac{4}{a}f^3 - 4\Big(\frac{\omega}{2a}\Big)^2 f^2 + 4c_1 f - \frac{4J^2}{a}. \tag{A.5}$$

By substituting $f(Z) = A + By^2(Z)$, we get the following ODE for y:

$$(y')^2 = \frac{1}{B^2}\Big[\frac{A^3}{a} - A^2\Big(\frac{\omega}{2a}\Big)^2 + Ac_1 - \frac{J^2}{a}\Big]\frac{1}{y^2} + \frac{1}{B}\Big[\frac{3A^2}{a} - 2A\Big(\frac{\omega}{2a}\Big)^2 + c_1\Big] + \Big[\frac{3A}{a} - \Big(\frac{\omega}{2a}\Big)^2\Big]y^2 + \frac{B}{a}y^4. \tag{A.6}$$

Now recall that the Jacobian elliptic sine $y(Z) = \mathrm{sn}(cZ|m)$ solves the ODE $(y'/c)^2 = (1-y^2)(1-my^2)$. By requiring that (A.6) matches the ODE for the elliptic sine, one then obtains (2.17), with $B = 4m\|\mathbf{k}\|^2 K_m^2$ as before, and with

$$J^2 = 4aK_m^2 A\left(1+\frac{A}{4K_m^2 a}\right)(A+4mK_m^2 a)\,, \tag{A.7a}$$

$$\left(\frac{\omega}{2a}\right)^2 = 4K_m^2(1+m)+\frac{3A}{a}\,,\quad c_1 = \frac{1}{a}\left[\left(4mK_m^2 a+A\right)(4K_m^2 a+2A)+A\left(4K_m^2 a+A\right)\right]. \tag{A.7b}$$

Similar to section 2.2, we write the ODE (A.5) as $(f')^2 = P_3(f)$, where

$$P_3(f) = \frac{4}{a}\left[f^3 - a\left(\frac{\omega}{2a}\right)^2 f^2 + c_1\,a\,f - J^2\right] = \frac{4}{a}(f-\lambda_1)(f-\lambda_2)(f-\lambda_3)\,, \tag{A.8}$$

with $\lambda_1,\ldots,\lambda_3$ given by (2.19). Note that the requirements $a \geqslant 0$ and $0 \leqslant m \leqslant 1$ again immediately imply (2.21). The symmetric polynomials defined by λ_1, λ_2, and λ_3 are related to the above constants as

$$e_1 = \lambda_1+\lambda_2+\lambda_3 = \omega^2/4a\,,\quad e_2 = \lambda_1\lambda_2+\lambda_2\lambda_3+\lambda_3\lambda_1 = c_1\,a\,,\quad e_3 = \lambda_1\lambda_2\lambda_3 = J^2. \tag{A.9}$$

which also allow one to recover A, a, and m when λ_1, λ_2, and λ_3 are known. The above solution contains $N+2$ independent parameters: A, m and $\mathbf{k}$ [since J and ω are determined by (A.7a), (A.7b)]. Next we employ the Galilean invariance of the NLS equation to apply a Galilean boost and thereby obtain the more general family of solutions

$$\tilde{\psi}(\mathbf{x},t) = \psi(\mathbf{x}-2\mathbf{v}t,t)\,\mathrm{e}^{i(\mathbf{v}\cdot\mathbf{x}-\|\mathbf{v}\|^2 t)/\varepsilon} = \sqrt{\rho(\tilde{z}/\varepsilon)}\,\mathrm{e}^{i\tilde{\Phi}(\tilde{z}/\varepsilon,\mathbf{x},t)}\,, \tag{A.10a}$$

where $\tilde{z} = \mathbf{k}\cdot\mathbf{x}-\tilde{\omega}t$, with $\tilde{\omega} = \omega+2\mathbf{k}\cdot\mathbf{v}$, and where

$$\tilde{\Phi}(\tilde{z}/\varepsilon,\mathbf{x},t) = \Phi(\tilde{z}/\varepsilon)+(\mathbf{v}\cdot\mathbf{x}-\|\mathbf{v}\|^2 t)/\varepsilon\,. \tag{A.10b}$$

The transformation adds the N new independent parameters $v_1,\ldots,v_N$. Therefore, the periodic solution of the NLS equation (1.1) in N spatial dimensions contains $2N+2$ independent real parameters: A, m, $\mathbf{k}$ and $\mathbf{v}$, as expected.

A.2. Calculation of the solution amplitude and second frequency and simplification of certain terms

Here we give a few additional details on the calculation of the fluid density. Starting from (2.10b), using (2.12) and simplifying the resulting ODE, one has

$$a\rho''' + a\rho'\frac{(\rho')^2-2\rho\rho''}{\rho^2} - 4\rho\rho' + \frac{4J^2\rho'}{\rho^2} = 0\,, \tag{A.11}$$

where $a = \|\mathbf{k}\|^2$ for brevity. Integrating w.r.t. Z yields

$$a\rho'' - a\frac{(\rho')^2}{\rho} - 2\rho^2 + 2c_1 - \frac{4J^2}{\rho} = 0\,, \tag{A.12}$$

where c_1 is an arbitrary integration constant. Multiplying (A.12) by $2\rho'/\rho^2$ and integrating with respect to Z again yields

$$a(\rho')^2 = 4\rho^3 - 4c_2\rho^2 + 4c_1\rho - 4J^2\,, \tag{A.13}$$

with c_2 another arbitrary integration constant. Letting $\rho(Z) = A + By^2(Z)$ yields the following ODE:

$$(y')^2 = \frac{1}{B^2 a}(A^3 - A^2 c_2 + Ac_1 - J^2)\frac{1}{y^2} + \frac{1}{Ba}(3A^2 - 2Ac_2 + c_1) + \left(\frac{3A}{a} - \frac{c_2}{a}\right)y^2 + \frac{B}{a}y^4. \tag{A.14}$$

Now recall that the Jacobi elliptic sine $y(Z) = \mathrm{sn}(cz|m)$ solves the ODE $(y'/c)^2 = (1-y^2)(1-my^2)$. By requiring that (A.14) matches the ODE for the elliptic sine, one obtains (2.17), with the coefficients as in (A.7a).

Next we obtain (2.23), which determines the frequency μ of the second phase. As mentioned in section 2, to this end one can use the undifferentiated version of (2.10*b*) [obtained from the real part of (1.1) using (2.1) and (2.6)], which is

$$-\omega\phi' + 2(\mathbf{k}\cdot\bar{\mathbf{u}})\phi' + a(\phi')^2 + 2\rho - \mu + \|\bar{\mathbf{u}}\|^2 - \frac{a}{4}\Big((\ln\rho)'' + \frac{\rho''}{\rho}\Big) = 0. \tag{A.15}$$

Differentiating (A.15) w.r.t. x and y and collecting leading-order terms yields (2.10*b*). However, (A.15) allows us to determine μ in a more straightforward manner. Indeed, substituting (2.11) into equation (A.15) and simplifying yields,

$$2a\rho'' - a\frac{(\rho')^2}{\rho} - 8\rho^2 + C\rho - \frac{4J^2}{\rho} = 0, \tag{A.16a}$$

where

$$C = 4\mu - 4\big(\|\bar{u}\|^2 - (J\overline{\rho^{-1}}/g^{1/2})^2\big) \tag{A.16b}$$

Multiplying (A.16*a*) by ρ'/ρ and integrating with respect to Z yields

$$a(\rho')^2 = 4\rho^3 - C\rho^2 + 4c_3\rho - 4J^2, \tag{A.17}$$

with an arbitrary integration constant c_3. Comparing the coefficients in (A.13) and (A.17) we have $C = 4c_2$ [as well as $c_1 = c_3$], which, when inserted in (A.16*b*), finally yields (2.23) for μ.

Finally, we provide further details on how to simplify the modulation equations (3.4) and in particular on how to obtain (3.7*f*). The averaged conservation of momentum equation (3.4*f*), when written in terms of $\lambda_1,\ldots,\lambda_3$ and $\mathbf{U}$, is

$$(J\hat{\mathbf{k}} + \bar{\rho}\mathbf{U})_t + \nabla(\overline{\rho^2}) + \nabla\cdot\Big[2\bar{\rho}\mathbf{U}\otimes\mathbf{U} + 2J(\hat{\mathbf{k}}\otimes\mathbf{U} + \mathbf{U}\otimes\hat{\mathbf{k}}) + \Big(\Big(\overline{\frac{(\rho')^2}{2\rho}}\Big) + 2\frac{J^2\overline{\rho^{-1}}}{\|\mathbf{k}\|^2}\Big)\mathbf{k}\Big)\otimes\mathbf{k}\Big] = \mathbf{0}. \tag{A.18}$$

Notice that averages containing ρ_z can be evaluated by recalling that $\rho(Z)$ satisfies the ODE (2.15). Differentiating (2.15) and using the definition of the symmetric polynomials yields

$$\frac{\|\mathbf{k}\|^2}{2}\rho'' = 3\rho^2 - 2e_1\rho + e_2, \tag{A.19a}$$

and averaging over the fast variable Z gives

$$\overline{\rho^2} = \frac{2e_1\bar{\rho} - e_2}{3}. \tag{A.19b}$$

Reordering the ODE (2.15) gives us

$$\frac{\|\mathbf{k}\|^2\rho_2^2 + 4J^2}{\rho} = 4\rho^2 - 4e_1\rho + 4e_2. \tag{A.19c}$$

Averaging again over the fast variable Z and using (A.19*b*) yields

$$\frac{\|\mathbf{k}\|^2}{4}\Big(\Big(\overline{\frac{(\rho')^2}{\rho}}\Big) + \frac{4J^2}{\|\mathbf{k}\|^2}\overline{\rho^{-1}}\Big) = \frac{1}{3}(2e_2 - e_1\bar{\rho}). \tag{A.19d}$$

Finally, using (A.19*d*), equation (A.18) yields (3.7*f*).

A.3. Detailed steps in the derivation of the 3DNLS-Whitham system

We begin by expressing the modulation equations in terms of convective derivatives. Using (7.21) one can see that

$$D_y(q_2) = q_{2,y} - q_1 q_{2,x} = \frac{D_z(k_1 q_1)}{k_1} - q_1 \frac{D_z k_1}{k_1} = D_z(q_1)\,, \tag{A.20}$$

which proves (7.22*a*). Moreover, using (3.4*d*) and the fact that $\mathbf{p} = \bar{\mathbf{u}}_\flat - \bar{u}_1 \mathbf{q}$, one has

$$\begin{aligned} D_y(p_2) &= (\bar{u}_3 - q_2 \bar{u}_1)_y - q_1(\bar{u}_3 - q_2 \bar{u}_1)_x = \bar{u}_{3,y} - q_1 \bar{u}_{3,x} - q_2 \bar{u}_{1,y} + q_1 q_2 \bar{u}_{1,x} - \bar{u}_1 D_y(q_2)\,, \\ &= \bar{u}_{2,z} - q_1 \bar{u}_{1,z} - q_2 \bar{u}_{2,x} + q_1 q_2 \bar{u}_{1,x} - \bar{u}_1 D_z(q_1) = D_z(p_1)\,. \end{aligned} \tag{A.21a}$$

which yields (7.22*b*).

Using the identity (7.22*a*) and straightforward algebra, we can rewrite (7.5*b*) and (7.5*c*) as

$$D_t \mathbf{q} + 2g \mathbf{D}_\flat U_1 + 2q_1(U_1 \mathbf{D}_\flat q_1 + \mathbf{D}_\flat p_1) + 2q_2(U_1 \mathbf{D}_\flat q_2 + \mathbf{D}_\flat p_2) = 0\,, \tag{A.22a}$$

$$D_t \mathbf{p} - 2q_1 U_1 D_y \mathbf{p} - 2q_2 U_1 D_z \mathbf{p} + \mathbf{D}_\flat(g(\tilde{e}_1 - U_1^2)) = 0\,, \tag{A.22b}$$

where D_t is as in (7.23).

Next, we express the first conservation of waves equation in convective derivative form. Recalling equations (7.17) and using (7.21) one can obtain the following,

$$\frac{D_t k_1}{k_1} + 2(U_1)_x + 2\mathbf{q} \cdot (\mathbf{U}_\flat)_x = 0\,. \tag{A.23}$$

Simplifying further we have (7.24), with D_x as in (7.23) and with

$$W_1 = U_1(\|\mathbf{q}\|^2)_x - 2\mathbf{q} \cdot \mathbf{D}_\flat U_1 + 2\mathbf{q} \cdot \mathbf{p}_x\,. \tag{A.24}$$

Moreover, using (A.22*a*) one can simplify W_1 further and obtain (7.26*a*).

Next, it can be easily seen that (7.19) becomes

$$\begin{aligned} &D_t(U_1 + J\overline{\rho^{-1}}/g^{1/2}) + g(\tilde{e}_1)_x + 2gJ\overline{\rho^{-1}}/g^{1/2}(U_1)_x + 2U_1\|\mathbf{q}\|^2(J\overline{\rho^{-1}}/g^{1/2} + U_1)_x \\ &\quad -2(U_1\mathbf{q} + \mathbf{p}) \cdot \boldsymbol{\nabla}_\flat(J\overline{\rho^{-1}}/g^{1/2} + U_1) + 2\mathbf{p} \cdot \big((J\overline{\rho^{-1}}/g^{1/2} + U_1)\mathbf{q}\big)_x \\ &\quad +(\tilde{e}_1 + U_1^2 + 2J\overline{\rho^{-1}}/g^{1/2}U_1)\|\mathbf{q}\|_x^2 + 2(\mathbf{p} + U_1\mathbf{q} + J\overline{\rho^{-1}}/g^{1/2}\mathbf{q}) \cdot \mathbf{p}_x = 0\,. \end{aligned} \tag{A.25}$$

As a direct consequence of equation (3.7*d*) we obtain

$$\mathbf{p}_x = \boldsymbol{\nabla}_\flat(U_1 + J\overline{\rho^{-1}}/g^{1/2}) - \big((U_1 + J\overline{\rho^{-1}}/g^{1/2})\mathbf{q}\big)_x\,. \tag{A.26}$$

Using the above relation for $\mathbf{p}_x$ and eliminating $(U_1)_x$ with the help of (A.23), one can obtain the simplified second wave conservation equation (7.25) in terms of convective derivatives, where

$$\begin{aligned} W_2 = \tfrac{1}{2}\big[&(\tilde{e}_1 + U_1^2)\|\mathbf{q}\|_x + 2U_1\mathbf{q} \cdot \big(2(\mathbf{U}_\flat)_x - \boldsymbol{\nabla}_\flat U_1 + J\overline{\rho^{-1}}/g^{1/2}\mathbf{q}_x - \mathbf{D}_\flat S\big) \\ &-\mathbf{q} \cdot \mathbf{D}_\flat(\tilde{e}_1 + U_1^2) - (U_1^2\|\mathbf{q}\|^2)_x\big]. \end{aligned} \tag{A.27}$$

Equation (A.26) yields

$$J\overline{\rho^{-1}}/g^{1/2}\mathbf{q}_x - \mathbf{D}_\flat\big(J\overline{\rho^{-1}}/g^{1/2}\big) = \mathbf{D}_\flat U_1 - \mathbf{p}_x - U_1\mathbf{q}_x\,. \tag{A.28}$$

Using the above relation along with equations (A.22*a*), (A.22*b*), (7.22*b*), (7.22*a*) and some tedious but straightforward algebra, one obtains (7.26*b*).

To express the averaged mass equation (7.18) in terms of convective derivatives, first we replace $\mathbf{M}$ using (7.16) and obtain:

$$D_t(\overline{\rho}) + 2\overline{\rho}\big((U_1)_x + \boldsymbol{\nabla}_\flat \cdot \mathbf{U}_\flat\big) + 2\big((g\tilde{J})_x + \boldsymbol{\nabla}_\flat \cdot (g\tilde{J}\mathbf{q})\big) = 0\,, \tag{A.29}$$

Simplifying further we obtain

$$D_t(\overline{\rho})+2\overline{\rho}D_xU_1+2D_x(g\tilde{J})+2gM_1(\nabla_\flat\cdot\mathbf{q})+2\overline{\rho}(\nabla_\flat\cdot\mathbf{p})=0\,. \tag{A.30}$$

To rewrite (A.30) in simpler form, we consider the combination (A.30) − $g(\overline{\rho}/g)$(7.24), which yields (7.27).

Finally, we consider the first component of the averaged momentum equation (7.20*a*), using a similar approach as before one can rewrite it as follows:

$$\begin{aligned}&D_t(gM_1)+2g(M_1+\tilde{J})D_xU_1+2U_1D_x(g\tilde{J})+D_x(g\tilde{e}_2)-\mathbf{q}\cdot\mathbf{D}_\flat(g(\overline{\rho}^2/g^2))\\&\quad+2(\overline{\rho}^2/g^2)g(\mathbf{q}\cdot\mathbf{q}_x)+g\big(\tilde{e}_2-(\overline{\rho}^2/g^2)+2U_1(M_1+\tilde{J})\big)\nabla_\flat\cdot\mathbf{q}+2M_1g\nabla_\flat\cdot\mathbf{p}=0\,.\end{aligned} \tag{A.31}$$

Next, taking the combination (A.31) − U_1(A.30) yields

$$\begin{aligned}&g(\overline{\rho}/g)D_tU_1+D_t(g\tilde{J})+4g\tilde{J}D_xU_1+D_x(g\tilde{e}_2)-\mathbf{q}\cdot\mathbf{D}_\flat(g(\overline{\rho}^2/g^2))\\&\quad+2g(\overline{\rho}^2/g^2)(\mathbf{q}\cdot\mathbf{q}_x)+g(\tilde{e}_2-(\overline{\rho}^2/g^2)-2U_1\tilde{J})+2g\tilde{J}\nabla_\flat\cdot\mathbf{p}=0\,.\end{aligned} \tag{A.32}$$

To simplify this equation more we consider the combination (7.20*b*)−(7.20*a*)$\mathbf{q}$−(7.18)$\mathbf{p}$ and obtain the following vector equation:

$$\begin{aligned}&M_1D_t\mathbf{q}+(\overline{\rho}/g)D_t\mathbf{p}+(2U_1\tilde{J}+\tilde{e}_2)D_x\mathbf{q}+2\tilde{J}D_x\mathbf{p}\\&\quad-(\overline{\rho}^2/g^2)D_x\mathbf{q}+g\mathbf{D}_\flat((\overline{\rho}^2/g^2))+2(\overline{\rho}^2/g^2)\mathbf{D}_\flat g=0\,.\end{aligned} \tag{A.33}$$

Finally, we consider the combination (A.32)−$2g\tilde{J}$(7.24)+$\mathbf{q}\cdot$(A.33). Using (7.22*a*) and after extensive simplifications, we obtain (7.28).

Next we show that, using the transformation to Riemann-type variables (7.4) (7.24), (7.25), (7.27) and (7.28) yield (7.30). Note first that, using (7.4), we have the following identities:

$$U_1=\tfrac{1}{2}s_1\,, \tag{A.34a}$$
$$\tilde{e}_1=s_2-\tfrac{1}{4}s_1^2\,, \tag{A.34b}$$
$$\tilde{e}_2=s_4+\tfrac{1}{16}[-16s_3s_1-4s_2^2+8s_1^2s_2-s_1^4]\,, \tag{A.34c}$$
$$\tilde{J}=\tfrac{1}{3}s_3-\tfrac{1}{24}s_1(6s_2-s_1^2)\,, \tag{A.34d}$$

[where again the s_n are as in (5.13)], which allow us to express $\tilde{e}_1,\dots,\tilde{e}_3$ in terms of the Riemann invariants via $s_1,\dots,s_4$. The identity (A.34*d*) is especially important, since it allows us to eliminate square roots from the modulation equations. Recall that (2.22) only determines J^2, and $J=\sigma(\lambda_1\lambda_2\lambda_3)^{1/2}$, with $\sigma=\pm1$. On the other hand, (7.4) yields $\sigma\lambda_1^{1/2}=\frac{1}{2}\sqrt{g}\,(r_1-r_2-r_3+r_4)$, where the sign σ here is needed because but one needs $\lambda_3\geqslant\lambda_2\geqslant\lambda_1\geqslant0$ (cf. section 2.2), but $r_1-r_2-r_3+r_4$ can be either positive or negative depending on the relative magnitude of $r_1,\dots,r_4$. (In contrast, no ambiguity arises for $\lambda_2^{1/2}$ and $\lambda_3^{1/2}$ when $r_1,\dots,r_4$ are well-ordered.) One can verify that, with this choices, the sign of both the left-hand side and right-hand side of (A.34*d*) equal σ. The following formulae are also useful:

$$k_1=\frac{\sqrt{(r_4-r_2)(r_3-r_1)}}{2K_m}\,, \tag{A.35a}$$

$$\begin{aligned}\left(\frac{\partial k_1}{\partial r_1},\dots,\frac{\partial k_1}{\partial r_4}\right)^T=\frac{\sqrt{(r_4-r_2)(r_3-r_1)}}{4K_m^2}\bigg(&\frac{(r_1-r_4)K_m+(r_4-r_2)E_m}{(r_2-r_1)(r_4-r_1)},\\&\frac{(r_3-r_2)K_m+(r_1-r_3)E_m}{(r_2-r_1)(r_3-r_2)},\frac{(r_3-r_2)K_m+(r_2-r_4)E_m}{(r_3-r_2)(r_3-r_4)},\frac{(r_4-r_1)K_m+(r_1-r_3)E_m}{(r_4-r_1)(r_4-r_3)}\bigg)^T,\end{aligned} \tag{A.35b}$$

We are now ready to present the final steps of the derivation. We begin by deriving (7.24), which is the simplest of the four equations. In this case we simply need to express D_tk_1 in terms of the Riemann invariants, i.e.,

$$D_tk=\sum_{j=1}^4\frac{\partial k_1}{\partial r_j}D_tr_j\,, \tag{A.36}$$

which immediately yields (7.30a), with W_1 as in (7.26a). Next, equation (7.25) simplifies due to the identity

$$D_t(U_1 + J\overline{\rho^{-1}}/g^{1/2}) + (U_1 - J\overline{\rho^{-1}}/g^{1/2})\,\frac{D_t k_1}{k_1} = \frac{2}{k_1}\sum_{j=1}^{4} r_j \frac{\partial k_1}{\partial r_j} D_t r_j\,, \tag{A.37}$$

and takes the form of (7.30b), with W_2 as in (7.26b). Next, taking the combination (7.27)/2 + $gU_1/2\times$(7.25) + $g(s_2 - 2U_1^2)/4\times$(7.24) and using identities (A.34) and (A.37), yields (7.30c), where W_3 is as in (7.31a). Finally, considering the linear combination (7.28)/2 + $3U_1/2\times$(7.27) $g(s_2 + 2U_1^2)/4\times$(7.25) + $g(3\tilde{J} + U_1 s_2 - 2U_1^3)/2\times$(7.24), and using identities (A.34), (A.37) again, and after some tedious algebra, one finds (7.30d), with W_4 as in (7.31b).

Our last task is to show that the compatibility relations $\nabla\times\mathbf{k} = \nabla\times\bar{\mathbf{u}} = \mathbf{0}$, when written in terms of the Riemann-type variables $\mathbf{r} = (r_1,\ldots,r_4)^T$ as well as $\mathbf{q}$ and $\mathbf{p}$, yield (7.8). To this end, we first use the definition of $\mathbf{q}$ as in (7.1b) along with the compatibility condition $\nabla\times\mathbf{k} = 0$. It can be easily seen that

$$k_1\mathbf{q}_x = (\mathbf{k}_\flat)_x - k_{1,x}\,\mathbf{q} = \nabla_\flat k_1 - k_{1,x}\,\mathbf{q} = \mathbf{D}_\flat k_1 \tag{A.38}$$

(cf. the third equation in (7.9)), which yields the first half of (7.8). Next, using the compatibility condition $\nabla\times\bar{\mathbf{u}} = 0$ with the definition of $\bar{u}_1$ as in (7.3c) one can derive (7.12), namely,

$$\mathbf{p}_x = \mathbf{D}_\flat\big(U_1 + J\overline{\rho^{-1}}/g^{1/2}\big) + \big(U_1 - J\overline{\rho^{-1}}/g^{1/2}\big)\frac{\mathbf{D}_\flat k_1}{k_1} - 2U_1\frac{\mathbf{D}_\flat k_1}{k_1}\,. \tag{A.39}$$

Using the identity (A.37), one then obtains the second half of (7.8).

References

[1] M. J. Ablowitz, G. Biondini and I. Rumanov, "Whitham modulation theory for (2+1)-dimensional equations of Kadomtsev-Petviashvili type", *J. Phys. A* **51**, 215501 (2018)

[2] M. J. Ablowitz, G. Biondini and Q. Wang, "Whitham modulation theory for the Kadomtsev-Petviashvili equation", *Proc. Roy. Soc. A* **473**, 20160695 (2017)

[3] M. J. Ablowitz, G. Biondini and Q. Wang, "Whitham modulation theory for the two-dimensional Benjamin-Ono equation", *Phys. Rev. E* **96**, 032225 (2017)

[4] M J Ablowitz, J Cole and I Rumanov, "On the Whitham system for the radial nonlinear Schrödinger equation", Stud. Appl. Math. **142**, 269–313 (2019)

[5] M J Ablowitz, J Cole and I Rumanov, "On the Whitham system for the (2+1)-dimensional nonlinear Schrödinger equation", arXiv:2109.09719 [nlin.ps]

[6] M. J. Ablowitz and H. Segur, *Solitons and the inverse scattering transform* (SIAM, Philadelphia, 1981)

[7] C. Audiard and L. M. Rodrigues, "About plane periodic waves of the nonlinear Schrödinger equations", Bull. Soc. Math. France **97**, 111 (2022)

[8] A. Bendahmane, G. Xu, M. Conforti, A. Kudlinski, A. Mussot, and S. Trillo, "The Piston Riemann Problem in a Photon Superfluid", Nat. Commun. **13**, 1 (2022)

[9] D. J. Benney and A. C. Newell, "The propagation of nonlinear wave envelopes," J. Math. Phys. (Stud. Appl. Math.) **46**, 133 (1967)

[10] T. Bienaimé, M. Isoard, Q. Fontaine, A. Bramati, A. M. Kamchatnov, Q. Glorieux, and N. Pavloff, "Quantitative Analysis of Shock Wave Dynamics in a Fluid of Light", Phys. Rev. Lett. **126**, 183901 (2021)

[11] G. Biondini, "Riemann problems and dispersive shocks in self-focusing media", Phys. Rev. E **98**, 052220 (2018)

[12] G. Biondini, A. J. Bivolcic and M. A. Hoefer, "One-dimensional reductions of the Whitham equations for the Kadomtsev-Petviashvili equation", in preparation

[13] G. Biondini, G. A. El, M. A. Hoefer and P. D. Miller, "Dispersive hydrodynamics: Preface", Physica D **333**, 1–5 (2016)

[14] G. Biondini, M. A. Hoefer, and A. Moro, "Integrability, exact reductions and special solutions of the KP-Whitham equations," Nonlinearity **33**, 4114–4132 (2020)

[15] G. Biondini and Y. Kodama, "On the Whitham equations for the defocusing nonlinear Schrödinger equation with step initial data", J. Nonlin. Sci. **16**, 435–481 (2006)

[16] G. Biondini and D. Mantzavinos, "Universal nature of the nonlinear stage of modulational instability", Phys. Rev. Lett. **116**, 043902 (2016)

[17] R. L. Bryant, S. S. Chern, R. B. Gardner, H. L. Goldschmidt and P. A. Griffiths, *Exterior differential systems* (Springer-Verlag, 1991)

[18] F. Calogero and W. Eckhaus, "Nonlinear evolution-equations, rescalings, model PDEs and their integrability", Inv. Probl. **3**, 229 (1987)

[19] J. J. Chang, P. Engels, and M. A. Hoefer, "Formation of Dispersive Shock Waves by Merging and Splitting Bose-Einstein Condensates", Phys. Rev. Lett. **101**, 170404 (2008)

[20] T. Congy, G. A. El, and M. A. Hoefer, "Interaction of linear modulated waves with unsteady dispersive hydrodynamic states with application to shallow water waves", J. Fluid Mech. **875**, 1145–1174 (2019)

[21] T. Congy, G. El, and G. Roberti, "Soliton gas in bidirectional dispersive hydrodynamics", Phys. Rev. E **103**, 042201 (2021)

[22] Z. Dutton, M. Budde, C. Slowe, and L. V. Hau, "Observation of Quantum Shock Waves Created with Ultra-Compressed Slow Light Pulses in a Bose-Einstein Condensate", Science **293**, 663 (2001)

[23] G. A. El, V. V. Geogjaev, A. V. Gurevich and A. L. Krylov, "Decay of an initial discontinuity in the defocusing NLS hydrodynamics", Phys. D **87**, 186–192, (1995)

[24] G.A. El, A.V. Gurevich, V.V. Khodorovskii and A.L. Krylov, "Modulational instability and formation of a nonlinear oscillatory structure in a focusing medium", Phys. Lett. A, **177**, 357–361 (1993)

[25] G. A. El and M. A. Hoefer, "Dispersive shock waves and modulation theory", Phys. D **333**, 11–65 (2016)

[26] E. V. Ferapontov and K. R. Khusnutdinova, "On the integrability of (2+1)-dimensional quasilinear systems", *Comm. Math. Phys.* **248**, 187–206 (2004)

[27] E. V. Ferapontov and K. R. Khusnutdinova, "The Haantjes tensor and double waves for multi-dimensional systems of hydrodynamic type: a necessary condition for integrability", *Proc. R. Soc. Lond. A* **462**, 1197–1219 (2006)

[28] M. G. Forest and J.-E. Lee, "Geometry and modulation theory for periodic nonlinear Schrödinger equation", in *Oscillation theory, computation and methods of compensated compactness*, C. Dafermos, J. L. Ericksen, D. Kinderlehrer, and M. Slemrod, Eds. (Springer, New York, 1986)

[29] T. Frankel, *The geometry of physics* (Cambridge, 1997)

[30] R. Gong and D.-S. Wang, "Formation of the undular bores in shallow water generalized Kaup-Boussinesq model", Physica D **439**, 133398 (2022)

[31] R. Grimshaw, "Slowly Varying Solitary Waves. I. Korteweg-de Vries Equation", Proc. R. Soc. Lond. A **368**, 359 (1979)

[32] A. V. Gurevich and A. L. Krylov "Dissipationless shock waves in media with positive dispersion", Sov. Phys. JETP **65**, 944 (1987)

[33] M. A. Hoefer and M. J. Ablowitz, "Interactions of dispersive shock waves", Phys. D **236**, 44–64 (2007)

[34] M. A. Hoefer, M. J. Ablowitz, I. Coddington, E. A. Cornell, P. Engels, and V. Schweikhard, "Dispersive and classical shock waves in Bose-Einstein condensates and gas dynamics," Phys. Rev. A **74**, 023623 (2006)

[35] M. A. Hoefer, P. Engels, and J. J. Chang, "Matter-Wave Interference in Bose-Einstein Condensates: A Dispersive Hydrodynamic Perspective", Physica D **238**, 1311 (2009)

[36] E. Infeld and G. Rowlands, *Nonlinear waves, solitons and chaos* (Cambridge University Press, 2000)

[37] S. Jin, C. D. Levermore, and D. W. McLaughlin, "The semiclassical limit of the defocusing NLS hierarchy", Commun. Pure Appl. Math. **52**, 613 (1999)

[38] A. M. Kamchatnov, *Nonlinear periodic waves and their modulations* (World Scientific, 2000)

[39] P. Kevrekidis, D. Frantzeskakis, and R. Carretero-Gonzalez, *Emergent nonlinear phenomena in Bose-Einstein condensates: Theory and experiment* (Springer, Berlin, 2009)

[40] Y. Kodama, "The Whitham equations for optical communications: mathematical theory of NRZ", SIAM J. Appl. Math. **59**, 2162–2192 (1999)

[41] E. A. Kuznetsov and S. K. Turitsyn "Instability and collapse of solitons in media with a defocusing nonlinearity", Sov. Phys. JETP **67**, 1583 (1988)

[42] M. D. Maiden, D. V. Anderson, N. A. Franco, G. A. El, and M. A. Hoefer, "Solitonic dispersive hydrodynamics: Theory and observation", Phys. Rev. Lett. **120**, 144101 (2018)

[43] M. E. Mossman, M. A. Hoefer, K. Julien, P. G. Kevrekidis, and P. Engels, "Dissipative Shock Waves Generated by a Quantum-Mechanical Piston", Nature Comm. **9**, 4665 (2018)

[44] A. C. Newell and J. V. Moloney, *Nonlinear optics* (Addison-Wesley, 1992)

[45] L. T. K. Nguyen and N. F. Smyth, "Dispersive shock waves for the Boussinesq Benjamin-Ono equation", Stud. Appl. Math. **147**, 32–59 (2021)

[46] F. W. Olver, D. W. Lozier, R. F. Boisvert and C. W. Clark, *NIST handbook of mathematical functions* (Cambridge University Press, New York, 2010)

[47] M. V. Pavlov, "Nonlinear Schrödinger equation and the Bogolyubov-Whitham method of averaging", Theor. Math. Phys. **71**, 584–588 (1987)

[48] S Ryskamp, M A Hoefer and G Biondini, "Oblique interactions between solitons and mean flow in the Kadomtsev-Petviashvili equation", *Nonlinearity* **34**, 3583 (2021)

[49] S Ryskamp, M A Hoefer and G Biondini, "Modulation theory for soliton resonance and Mach reflection", *Roy. Soc. Proc. A* **478**, 20210823 (2022)

[50] S Ryskamp, M Maiden, G Biondini and M A Hoefer, "Evolution of truncated and bent wave solitons: the Mach expansion problem", *J. Fluid Mech.* **909**, A24 (2021)

[51] T. P. Simula, P. Engels, I. Coddington, V. Schweikhard, E. A. Cornell, and R. J. Ballagh, "Observations on Sound Propagation in Rapidly Rotating Bose-Einstein Condensates", Phys. Rev. Lett. **94**, 080404 (2005)

[52] P. Sprenger, M. A. Hoefer, and G. A. El, "Hydrodynamic optical soliton tunneling", Phys. Rev. E **97**, 032218 (2018)

[53] C. Sulem and P. L. Sulem, *The nonlinear Schrödinger equation: Self-focusing and wave collapse* (Springer, 1999)

[54] R. J. Thelwell, J. D. Carter, and B. Deconinck, "Instabilities of One-Dimensional Stationary Solutions of the Cubic Nonlinear Schrödinger Equation", J. Phys. A **39**, 73 (2006)

[55] W. Wan, S. Jia, and J. W. Fleischer, “Dispersive Superfluid-like Shock Waves in Nonlinear Optics”, Nat. Phys. **3**, 46 (2007)
[56] G. B. Whitham, “Non-linear dispersive waves”, *Proc. Roy. Soc. Ser. A* **283**, 238–261 (1965)
[57] G. B. Whitham, *Linear and nonlinear waves* (Wiley, 1974)
[58] G. Xu, M. Conforti, A. Kudlinski, A. Mussot, and S. Trillo, “Dispersive dam-break flow of a photon fluid”, Phys. Rev. Lett. **118**, 254101 (2017)
[59] A. K. Zvezdin and A. F. Popkov, “Contribution to the nonlinear theory of magnetostatic spin waves”, Sov. Phys. JETP **57**, 350(1983)