Soliton shielding of the focusing nonlinear Schrödinger equation

Marco Bertola,[1,2,3] Tamara Grava,[1,4,3] and Giuseppe Orsatti[1,3]

[1]*SISSA, via Bonomea 265, 34136, Trieste, Italy*

[2]*Concordia University, 1455 av. de Maisonneuve W., Montréal Canada*

[3]*INFN, Sezione di Trieste*

[4]*School of Mathematics, University of Bristol, Fry Building, Bristol, BS8 1UG, UK*

We first consider a deterministic gas of N solitons for the Focusing Nonlinear Schrödinger (FNLS) equation in the limit $N \to \infty$ with a point spectrum chosen to interpolate a given spectral soliton density over a bounded domain of the complex spectral plane. We show that when the domain is a disk and the soliton density is an analytic function, then the corresponding deterministic soliton gas surprisingly yields the one-soliton solution with point spectrum the center of the disk. We call this effect *soliton shielding*. We show that this behaviour is robust and survives also for a *stochastic* soliton gas: indeed, when the N soliton spectrum is chosen as random variables either uniformly distributed on the circle, or chosen according to the statistics of the eigenvalues of the Ginibre random matrix the phenomenon of soliton shielding persists in the limit $N \to \infty$. When the domain is an ellipse, the soliton shielding reduces the spectral data to the soliton density concentrating between the foci of the ellipse. The physical solution is asymptotically step-like oscillatory, namely, the initial profile is a periodic elliptic function in the negative x–direction while it vanishes exponentially fast in the opposite direction.

Introduction. The wave propagation in a variety of physical systems is well described by dispersive integrable nonlinear wave equations. Integrability implies the existence of nonlinear modes that interact elastically and are called *solitons*. The inverse scattering, also called nonlinear Fourier transform, is the tool to analyze how a general given wave packet can be viewed as a nonlinear superposition of solitons. Recently several investigations, both on the mathematical and the physical side, have been carried out in which a very large number of solitons is considered. Coherent nonlinear superposition of many solitons occurs when one tries to optimally correlate the parameters of many nonlinear modes in order to produce a "macroscopic" wave profile that behaves more like a single broad wave packet than a combination of many smaller objects. This typically occurs in small dispersion limit or semiclassical limits [1–3]. Incoherent/random nonlinear superpositions of solitons are more closely related to the notion of a soliton gas in an infinite statistical ensemble of interacting solitons that was first introduced by Zakharov [4] for the Korteweg de Vries (KdV) equation. Further generalization were later derived for KdV (see e.g.[5]) and for the Focusing Nonlinear Schrödinger equation (FNLS) in [6, 7]. Connection between statistical properties of a soliton gas and generalized hydrodynamic has been recently established in [8–10]. Statistical properties of solutions of large set of random solitons have been numerically investigated in [11],[12],[13],[14],[15],[6]. Experimental realizations of behaviour of large sets of solitons are obtained in [16] and [17]. In this note, following the lines of [18, 19], we consider a soliton gas that originates from the limit $N \to \infty$ of the N-soliton solution of the FNLS equation

$$i\psi_t + \frac{1}{2}\psi_{xx} + |\psi|^2\psi = 0. \tag{1}$$

We consider both the cases in which the N-soliton spectra is chosen in a deterministic and random way.

Let us recall the one–soliton solution, given by

$$\psi(x,t) = 2b\,\mathrm{sech}[2b(x+2at-x_0)]\mathrm{e}^{-2i[ax+(a^2-b^2)t+\frac{\phi_0}{2}]}, \tag{2}$$

where x_0 is the initial peak position of the solition, ϕ_0 is the initial phase, $2b$ is the modulus of the wave maximal amplitude and $-2a$ is the soliton velocity. The general N soliton solution can be obtained from the Zakharov-Shabat [20] linear spectral problem, reformulated as a *Riemann-Hilbert Problem* (RHP) for a 2×2 matrix $Y^N(z;x,t)$ with the following data [21]: the discrete spectrum $S := \{z_0; \dots; z_{N-1}; \bar{z}_0; \dots; \bar{z}_{N-1}\}$, $z_j \in \mathbb{C}^+$ the upper half space, and its norming constants $\{c_0, \dots, c_{N-1}\}$ with $c_j \in \mathbb{C}$. Here and below $\bar{z}$ stands for the complex conjugate of z.

The matrix $Y^N(z;x,t)$ is analytic for $z \in \mathbb{C}\backslash S$ and has *simple* poles in S with the residue condition

$$\begin{aligned}
\mathrm{Res}_{z=z_j} Y^N(z) &= \lim_{z\to z_j} Y^N(z)\begin{pmatrix} 0 & 0 \\ c_j\mathrm{e}^{2\theta(z,x,t)} & 0\end{pmatrix} \\
\mathrm{Res}_{z=\bar{z}_j} Y^N(z) &= \lim_{z\to \bar{z}_j} Y^N(z)\begin{pmatrix} 0 & -\bar{c}_j\mathrm{e}^{-2\theta(z,x,t)} \\ 0 & 0\end{pmatrix} \\
Y^N(z) &= \mathbb{I} + \mathcal{O}\left(\frac{1}{z}\right), \quad \text{as } z \to \infty,
\end{aligned} \tag{3}$$

where $\theta(z,x,t) = i(z^2t + zx)$ and $\mathbb{I}$ is the identity matrix. The equations (3) uniquely determine $Y^N(z;x,t)$ as a rational matrix function of z in the form

$$Y^N(z;x,t) = \mathbb{I} + \sum_{j=0}^{N-1}\frac{\begin{pmatrix} f_j(x,t) & 0 \\ g_j(x,t) & 0\end{pmatrix}}{z-z_j} + \sum_{j=0}^{N-1}\frac{\begin{pmatrix} 0 & -\overline{g_j(x,t)} \\ 0 & \overline{f_j(x,t)}\end{pmatrix}}{z-\overline{z_j}}, \tag{4}$$

where the coefficients $f_j(x,t)$ and $g_j(x,t)$ are determined from a linear system by imposing the residue conditions in (3). The solution of the FNLS equation is recovered from $Y^N(z;x,t)$ by the relation

$$\psi_N(x,t) = 2i \lim_{z\to\infty} z(Y^N(z;x,t))_{12} \tag{5}$$

which gives the N-soliton solution in the form $\psi_N(x,t) = -2i\sum_{j=0}^{N-1}\overline{g_j(x,t)}$. In the case of one soliton solution, we have that the point spectrum $z_0 = a+ib$ determines the speed and amplitude of the soliton (2) and the coefficient c_0 determines the position $x_0 = \frac{\ln(|c_0|)}{2b}$ of the soliton peak and the phase $\phi_0 = \frac{\pi}{2} + \arg(c_0)$ of the soliton.

The FNLS equation can have a soliton of order N when the matrix function $Y^N(z)$ has a pole of order N. Such a solution can be viewed as N soliton solution where the simple poles coalesce to a pole of order N. The limit as $N\to\infty$ of such solution has been studied in [22, 23] where it has been shown that its near field structure is described by the Painlevé III equation. An analogous asymptotical study has been performed for breathers in [24].

In this letter we consider the case when the norming constants $\{c_j\}_{j=0}^{N-1}$, scale as $1/N$ as the number N of simple poles (i.e. the number of solitons) tends to infinity. On the physical side, scaling the norming constants to be small means that the individual solitons are centered at positions that are logarithmically large in N, so that in the finite part of the (x,t) plane only the tails of the solitons add up. The resulting gas of solitons is a condensate in the terminology of [6].

Differently from [18], [19] where the infinite set of solitons is obtained by letting the soliton spectra accumulate on lines of the spectral complex plane, here we consider the case in which soliton spectra accumulate on one or more simply connected bounded domains $\mathcal{D}$ of the complex upper plane $\mathbb{C}^+$ and their complex conjugate $\overline{\mathcal{D}}$. We let the number of solitons goes to infinity in such a way that their point spectrum z_j ($\bar{z}_j$) fills **uniformly** the domain $\mathcal{D}$. The corresponding norming constants c_j are interpolated by a smooth function $\beta(z,\bar{z})$, namely

$$c_j = \frac{\mathcal{A}}{\pi N}\beta(z_j,\bar{z}_j), \tag{6}$$

where $\mathcal{A}$ is the area of the domain $\mathcal{D}$ and N is the total number of solitons.

The remarkable emerging feature is that as $N\to\infty$, for certain types of domains and densities, we have a "soliton shielding", namely, the gas behaves as a *finite* number of solitons. This happens for example if the distribution function is $\beta(z,\bar{z}) = \bar{z}^{n-1}r(z)$ with $r(z)$ an analytic function in $\mathcal{D}$, and the domain is described by $\mathcal{D} := \{z\in\mathbb{C}\text{ s.t. } |(z-d_0)^n - d_1| < \rho\}$, $n\in\mathbb{N}$, with $d_0\in\mathbb{C}^+$, $|d_1|$ and $\rho>0$ sufficiently small so that $\mathcal{D}\in\mathbb{C}^+$. Then the deterministic soliton gas is equivalent to a n-soliton solution. In the case $n=1$, the domain $\mathcal{D}$ is a disk centered at $\lambda_0 = d_0 + d_1$ and the infinite number of solitons superimpose nonlinearly in their tails to produce a single soliton solution with point spectrum λ_0 and norming constant equal to $\rho^2 r(\lambda_0)$. We are going to see that this behaviour persists also when the N soliton spectrum is a random variable distributed according to the Ginibre ensemble [25] or the uniform distribution on the disk.

When the domain $\mathcal{D}$ is an ellipse we show that such deterministic soliton gas is a step-like periodic elliptic wave at $x=-\infty$ and rapidly decreasing at $x=+\infty$ as in [18].

Deterministic soliton gas. In order to obtain the limit of the N-soliton solution as $N\to\infty$, we impose that the norming constants c_j scale as $1/N$. Then we use a transformation that removes the singularities of Y^N. Indeed let γ_+ be a closed anticlockwise oriented contour that encircles all the poles in the upper half space and D_{γ_+} the finite domain with boundary γ_+ and similarly we define $\gamma_- = -\overline{\gamma_+}$ and D_{γ_-} encircles all the poles in the lower half space.

One ends up with the RHP for the matrix function $\widetilde{Y}^N(z;x,t)$ analytic in $\mathbb{C}\backslash\{\gamma_+\cup\gamma_-\}$, subject to the conditions

$$\begin{aligned} &\widetilde{Y}^N_+(z,x,t) = \widetilde{Y}_-^N(z,x,t)\tilde{J}_N(z,x,t), \quad z\in\gamma_+\cup\gamma_- \\ &\widetilde{Y}^N(z;x,t) = \mathbb{I} + \mathcal{O}\left(\frac{1}{z}\right), \text{ as } z\to\infty, \end{aligned} \tag{7}$$

where the subscripted $Y_\pm$ denote the left/right boundary values along the oriented contour and

$$\tilde{J}_N(z,x,t) = \begin{cases} \begin{pmatrix} 1 & 0 \\ -\sum\limits_{j=0}^{N-1}\frac{c_j e^{2\theta(z_j,x,t)}}{z-z_j} & 1 \end{pmatrix}, & z\in\mathbb{C}_+ \\ \begin{pmatrix} 1 & \sum\limits_{j=0}^{N-1}\frac{\bar{c}_j e^{-2\theta(\bar{z}_j,x,t)}}{z-\bar{z}_j} \\ 0 & 1 \end{pmatrix}, & z\in\mathbb{C}_- . \end{cases} \tag{8}$$

We call the matrix $\tilde{J}_N(z,x,t)$ the *jump matrix*. The solution $\widetilde{Y}^N(z,x,t)$ is obtained from $Y^N(z,x,t)$ by the relation $\widetilde{Y}^N(z,x,t) = Y(z,x,t)$ for z in $\mathbb{C}\backslash\{D_{\gamma_+}\cup D_{\gamma_-}\}$ and $\widetilde{Y}^N(z,x,t) = Y(z,x,t)\tilde{J}_N(z,x,t)$ for $z\in D_{\gamma_+}\cup D_{\gamma_-}$. In this case the coefficients f_j and g_j in (4) are recovered by imposing $\widetilde{Y}^N(z,x,t)$ to be analytic at z_j and $\overline{z_j}$ for $j=0,\dots,N-1$.

Let $\mathcal{D}$ be a domain so that the closure of $\mathcal{D}$ is strictly contained in the domain D_{γ_+} bounded by γ_+ and the closure of $\overline{\mathcal{D}}$ is completely contained in the domain D_{γ_-} bounded by γ_-. We let the number of solitons goes to infinity in such a way that their point spectrum z_j ($\bar{z}_j$) fills **uniformly** the domain $\mathcal{D}$ contained in γ_+ and we choose the norming constants c_j as in (6) so that

$$\sum_{j=0}^{N-1}\frac{c_j}{(z-z_j)} = \sum_{j=0}^{N-1}\frac{\mathcal{A}}{\pi N}\frac{\beta(z_j,\bar{z}_j)}{z-z_j} \underset{N\to\infty}{\longrightarrow} \iint_{\mathcal{D}}\frac{\beta(w,\bar{w})}{z-w}\frac{d^2w}{\pi},$$

where the infinitesimal area measure is $d^2w = (d\overline{w}\wedge$

$\mathrm{d}w)/(2i)$. Consequently the RH-problem (7) becomes

$$\widetilde{Y}^{\infty}_{+}(z,x,t)=\widetilde{Y}^{\infty}_{-}(z,x,t)\tilde{J}_{\infty}(z,x,t),\quad \tilde{J}_{\infty}(z,x,t)=\begin{pmatrix} 1 & \iint\limits_{\overline{\mathcal{D}}}\frac{\mathrm{e}^{-2\theta(w,x,t)}\beta^{*}(w,\bar{w})\mathrm{d}^2w}{\pi(z-w)}\chi_{\gamma_-} \\ \iint\limits_{\mathcal{D}}\frac{\mathrm{e}^{2\theta(w,x,t)}\beta(w,\bar{w})\mathrm{d}^2w}{\pi(w-z)}\chi_{\gamma_+} & 1\end{pmatrix}$$
$$\widetilde{Y}^{\infty}(z;x,t)=\mathbb{I}+\mathcal{O}\left(\frac{1}{z}\right),\quad \text{as } z\to\infty, \tag{9}$$

with $\beta^{*}(w,\bar{w})=\overline{\beta(\bar{w},w)}$. The limiting FNLS solution is given by

$$\psi_{\infty}(x,t)=2i\lim_{z\to\infty}z(\widetilde{Y}^{\infty}(z;x,t))_{12}. \tag{10}$$

For a general bounded domain $\mathcal{D}$ and smooth function $\beta(z,\bar{z})$, the class of solutions of FNSL obtained from (9) and (10) is unexplored. In the case $\beta(z,\bar{z})=n\bar{z}^{n-1}r(z)$, with $r(z)$ analytic in $\mathcal{D}$, we can apply Green theorem for $z\notin\mathcal{D}$ and obtain

$$\iint\limits_{\mathcal{D}}\frac{\mathrm{e}^{2\theta(w,x,t)}\beta(w,\bar{w})\mathrm{d}^2w}{\pi(z-w)}=\int\limits_{\partial\mathcal{D}}\frac{r(w)\overline{w}^{n}\mathrm{e}^{2\theta(w;x,t)}}{z-w}\frac{\mathrm{d}w}{2\pi i}, \tag{11}$$

and similarly for the integral over $\overline{\mathcal{D}}$.

For sufficiently smooth simply connected domains $\mathcal{D}$ the boundary $\partial\mathcal{D}$ can be described by the so–called Schwarz function $S(z)$ [26] of the domain $\mathcal{D}$ through the equation

$$\bar{z}=S(z).$$

The Schwarz function admits analytic extension to a maximal domain $\mathcal{D}^0\subset\mathcal{D}$. For example, for quadrature domains, $\mathcal{D}^0$ is just $\mathcal{D}$ minus a finite collection of points [26]. The simplest such quadrature domain is the disk, which is one of our examples below. For other classes of domains we have that $\mathcal{D}\setminus\mathcal{D}^0$ may consist of a *mother-body*, i.e., a collection of smooth arcs [27]. An example of this is the ellipse, which will be our second example.

Shielding of soliton gas for quadrature domains. We start by considering the class of domains

$$\mathcal{D}:=\left\{z\in\mathbb{C}\text{ s.t. }\left|(z-d_0)^m-d_1\right|<\rho\right\},\quad m\in\mathbb{N}, \tag{12}$$

with $d_0\in\mathbb{C}^+$ and $|d_1|,\rho>0$ sufficiently small so that $\mathcal{D}\in\mathbb{C}^+$. When $m=1$ such domain coincides with the disk $\mathbb{D}_\rho(\lambda_0)$ of radius $\rho>0$ centred at $\lambda_0=d_0+d_1$. When $m>1$ the domain $\mathcal{D}$ has a m–fold symmetry about d_0 and is simply connected if $|d_1|\le\rho$, and otherwise it has m connected components [28]. The boundary of $\mathcal{D}$ is described by

$$\bar{z}=S(z),\quad S(z)=\overline{d_0}+\left(\bar{d}_1+\frac{\rho^2}{(z-d_0)^m-d_1}\right)^{\frac{1}{m}}. \tag{13}$$

The n-soliton solution. This solution is obtained from (11) by choosing $m=n$ in (13). We then substitute $\overline{w}=S(w)$ in the contour integral (11) and use the residue theorem at the n poles given by the solution $\{\lambda_0,\dots,\lambda_{n-1}\}$ of the equation $(z-d_0)^n=d_1$. Then

$$\int\limits_{\partial\mathcal{D}}\frac{\overline{w}^{n}\,r(w)\mathrm{e}^{2\theta(w;x,t)}}{z-w}\frac{\mathrm{d}w}{2\pi i}=\int\limits_{\partial\mathcal{D}}S(w)^n r(w)\frac{\mathrm{e}^{2\theta(w;x,t)}}{z-w}\frac{\mathrm{d}w}{2\pi i}$$
$$=\rho^2\sum_{j=0}^{n-1}\frac{r(\lambda_j)}{\prod_{k\neq j}(\lambda_j-\lambda_k)}\frac{\mathrm{e}^{2\theta(\lambda_j;x,t)}}{z-\lambda_j},\quad z\notin\mathcal{D},$$

which gives, up to a sign the entry 21 of the jump matrix (8). Namely the solution $\psi_\infty(x,t)$ in (10) coincides with the n soliton solution $\psi_n(x,t)$ in (5) with spectrum $\{\lambda_0,\dots,\lambda_{n-1}\}$ and corresponding norming constants $c_j=\rho^2r(\lambda_j)/\prod_{k\neq j}(\lambda_j-\lambda_k)$ for $j=0,\dots,n-1$.

One soliton solution. In particular, in the case $n=m=1$ and $\mathcal{D}=\mathbb{D}_\rho(\lambda_0)$ the disk centred at $\lambda_0=d_0+d_1$ of radius ρ, we obtain exactly the RH-problem (7) for $N=1$ and $c_0=\rho^2r(\lambda_0)$. Namely we recover the one soliton solution (2) of the FNLS (1) equation with $\lambda_0=d_0+d_1=a+ib$, with peak position x_0 and phase shift ϕ_0 given respectively by

$$x_0:=\frac{\log(|\rho^2r(\lambda_0)|)-\log(2b)}{2b},\ \ \phi_0:=\arg(r(\lambda_0))-\frac{\pi}{2}. \tag{14}$$

We observe that the radius ρ of the disk and the value of the function $r(z)$ at λ_0 contribute to the phase shift of the soliton but not to its amplitude or velocity, which are uniquely determined by the center of the disk λ_0.

Soliton solution of order n. By considering $m=1$, namely, the disk $\mathbb{D}_\rho(\lambda_0)$ and $\beta(z)=n(\bar{z}-\bar{\lambda}_0)^{n-1}r(z)$ for $n>1$ one obtains the soliton solution of order n. This degenerate solution and the limit $n\to\infty$ has been extensively analyzed in [23].

Remark. In Figure 1 we plot the resulting "effective" soliton using an approximation of the uniform measure on the unit disk by means of N Fekete points, namely the set of N points described by the vector $\boldsymbol{w}=(w_0,\dots,w_{N-1})$ that minimizes the energy

$$E(\boldsymbol{w})=-2\sum_{0\le j<k\le N-1}\log|w_j-w_k|+\frac{N}{2}\sum_{j=0}^{N-1}|w_j|^2, \tag{15}$$

(suitably translated/rescaled) over all possible configurations. Then the uniform measure on the disk $D_\rho(\lambda_0)$ is obtained by the rescaling $z_j=\rho(w_j-\lambda_0)$. The train of solitons on the left (albeit slowly) will move towards $-\infty$ as $\mathcal{O}(\log N)$.

Elliptic Domain. We now consider the case in which $\mathcal{D}$ coincide with an elliptic domain $\mathcal{E}$ and with $\beta(z,\bar{z})=r(z)$ analytic. For the sake of simplicity, we assume that the focal points $i\alpha_1$ and $i\alpha_2$ of the ellipse lie on the imaginary

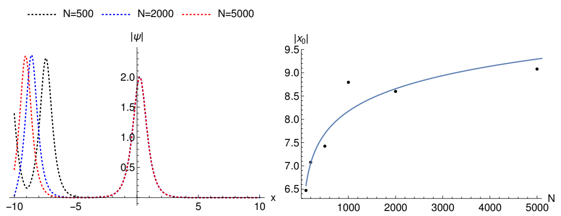

FIG. 1: On the left, the plot of the gas that approximates the area measure using N Fekete points for $N = 500, 2000, 5000$, all centred in a disk of ray $1/10$ and center $\lambda_0 = i$ (with $\beta(z) = \pi/\rho^2$), and the emerging limiting (one-soliton) solution $\psi^{\infty}(x,t)$ centred at $x = 0.226$. On the right, a fit (with a curve of the form $q + p\log(N)$) of the distance between the peak of the limiting soliton solution $\psi^{\infty}(x,t)$ and the first peak of the remaining part of the solution that is going to infinity as $N \to \infty$.

axis and $\alpha_2 > \alpha_1 > 0$. The equation of the ellipse is $\sqrt{x^2 + (y-\alpha_1)^2} + \sqrt{x^2 + (y-\alpha_2)^2} = 2\rho > 0$, where ρ is chosen sufficiently small so that $\mathcal{E}$ lies in the upper half space. We choose $\beta(z) = r(z)$ to be analytic in $\mathcal{E}$. In this case in equations (11) $n = 1$ and one has to consider the Schwarz function of the ellipse, namely $\bar{z} = S(z)$:

$$S(z) = \left(1 - \frac{2\rho^2}{c^2}\right)(z - iy_0) + 2\frac{\rho}{c^2}\sqrt{\rho^2 - c^2}R(z) - iy_0, \tag{16}$$

where $R(z) := \sqrt{(z - i\alpha_1)(z - i\alpha_2)}$, $y_0 = \frac{\alpha_1+\alpha_2}{2}$ and $c = \frac{\alpha_2-\alpha_1}{2}$. The function $S(z)$ is analytic in $\mathbb{C}$ away from the segment $\mathcal{I} := [i\alpha_1, i\alpha_2]$, with boundary values $S_\pm(z)$. For $z \notin \mathcal{E} \cup \overline{\mathcal{E}}$, the integral along the boundary $\partial\mathcal{E}$ ($\partial\overline{\mathcal{E}}$) of the ellipse in (11) can be deformed to a line integral on the segment $\mathcal{I} = [i\alpha_1, i\alpha_2]$ ($\overline{\mathcal{I}} := [-i\alpha_2, -i\alpha_1]$), namely

$$\int_{\partial\mathcal{E}} \frac{r(w)\overline{w}\mathrm{e}^{2\theta(w;x,t)}}{z-w}\frac{\mathrm{d}w}{2\pi i} = \int_{\mathcal{I}} \frac{r(w)\delta S(w)\mathrm{e}^{2\theta(w;x,t)}}{z-w}\frac{\mathrm{d}w}{2\pi i},$$

where $\delta S(z) = S_+(z) - S_-(z)$. Next we define

$$\Gamma(z) := \begin{cases} \widetilde{Y}^{\infty}(z), & z \in \mathbb{C}\backslash\{D_{\gamma_+} \cup \mathcal{D}_{\gamma_-}\} \\ \widetilde{Y}^{\infty}(z)J(z), & z \in D_{\gamma_+} \cup \mathcal{D}_{\gamma_-} \end{cases} \tag{17}$$

where $J(z) =$

$$\begin{pmatrix} 1 & \displaystyle\int_{\overline{\mathcal{I}}} \frac{r^*(w)\delta S^*(w)\mathrm{e}^{-2\theta(w;x,t)}}{w-z}\frac{\mathrm{d}w}{2\pi i}\chi_{\mathcal{D}_{\gamma_-}} \\ \displaystyle\int_{\mathcal{I}} \frac{r(w)\delta S(w)\mathrm{e}^{2\theta(w;x,t)}}{z-w}\frac{\mathrm{d}w}{2\pi i}\chi_{\mathcal{D}_{\gamma_+}} & 1 \end{pmatrix}.$$

In this way $\Gamma(z)$ does not have a jump on $\gamma_+ \cup \gamma_-$. Since $J(z)$ has a jump in $\mathcal{I} \cup \overline{\mathcal{I}}$ it follows that $\Gamma(z)$ is analytic in $\mathbb{C}\backslash\{\mathcal{I} \cup \overline{\mathcal{I}}\}$ with jump conditions

$$\begin{aligned} \Gamma_+(z) &= \Gamma_-(z)\mathrm{e}^{\theta(z;x,t)\sigma_3}G(z)\mathrm{e}^{-\theta(z;x,t)\sigma_3} \\ G(z) &= \begin{pmatrix} 1 & \chi_{\overline{\mathcal{I}}}\delta S^*(z)r^*(z) \\ -\chi_{\mathcal{I}}\delta S(z)r(z) & 1 \end{pmatrix}, \end{aligned} \tag{18}$$

and $\Gamma(z) = \mathbb{I} + O(\frac{1}{z})$, as $z \to \infty$. We can find the same RHP (18) also when we study the problem (7) with an infinite number of spectral points uniformly distributed along the segments $\mathcal{I} \cup \overline{\mathcal{I}}$.

For $t = 0$, the initial datum $\psi_0(x)$ associated to the solution of the RHP (18) turns out to be step-like oscillatory. Indeed from the steepest descent method following the lines [18] as $x \to -\infty$ we derive [29] the elliptic function

$$\psi_0(x) = i(\alpha_2 + \alpha_1)\mathrm{dn}\left[(\alpha_2 + \alpha_1)(x - x_0); m\right] + \mathcal{O}(x^{-1}), \tag{19}$$

where $\mathrm{dn}(z;m)$ is the Jacobi elliptic function of modulus $m = \frac{4\alpha_2\alpha_1}{(\alpha_2+\alpha_1)^2}$ and x_0 is a constant which depends on $\delta S(z)$, $\beta(z)$ and the geometry of the problem. For $x \to +\infty$ the initial datum goes to zero exponentially fast. When $\alpha_1 \to \alpha_2$ the ellipse degenerates to the circle and one recovers the one soliton solution.

Random soliton gas with Ginibre and uniform statistics

Let us now introduce randomness in the system by choosing the points $z_j = \rho(w_j - \lambda_0)\chi_{\mathbb{C}^+}$ with $(w_0, \ldots, w_{N-1}) \in \mathbb{C}^N$ distributed according to the probability density (*Ginibre ensemble*)

$$\mu_N = \frac{1}{Z_N}e^{-E(w_0,\ldots,w_{N-1})}d^2w_0 \ldots d^2w_{N-1}, \tag{20}$$

where Z_N is the normalizing constant and $E(w_0, \ldots, w_{N-1})$ is the energy defined in (15). In the limit $N \to \infty$ the random points $\{w_0, \ldots, w_{N-1}\}$ fill uniformly the unit disk centered at zero (see e.g. [25]). For any smooth function $h : \mathbb{C} \to \mathbb{C}$, let us consider the random variable $X_h^N := \sum_{j=1}^N h(w_j)$. It is known [30] that

$$\frac{1}{N}\mathbb{E}[X_h] \underset{N\to\infty}{\longrightarrow} \int_{|w|\le 1} h(w)\mathrm{d}^2w, \tag{21}$$

where $\mathbb{E}$ is the expectation with respect to the probability measure μ_N. Actually more is true [27] [30]: the limit of the random variable $X_h - \mathbb{E}[X_h]$ converges to a normal random variable $\mathcal{N}(0,\sigma)$ centred at zero and with finite variance σ^2 depending on h.

From the above arguments it is expected that the jump of the RH-problem (9), in probability, satisfies

$$\mathbb{P}\left(\left|\sum_{j=0}^{N-1}\frac{\mathcal{A}}{N}\frac{\beta(z_j,\bar{z}_j)}{z - z_j} - \iint_{\mathcal{D}}\frac{\beta(w,\bar{w})}{z-w}\mathrm{d}^2w\right| > \epsilon\right) = \mathcal{O}(\frac{1}{N}),$$

for $z \notin \mathcal{D}$. Using small norm arguments on the RH-problem [31], one may argue that the random N soliton solution $\psi_N(x, t, z_0, \ldots, z_{N-1})$ converges as $N \to \infty$ in probability to the one-soliton solution $\psi_\infty(x,t)$. Similar arguments can be used also when the soliton spectrum is sampled according to the uniform distribution on the unit disk. The complete mathematical proof would require

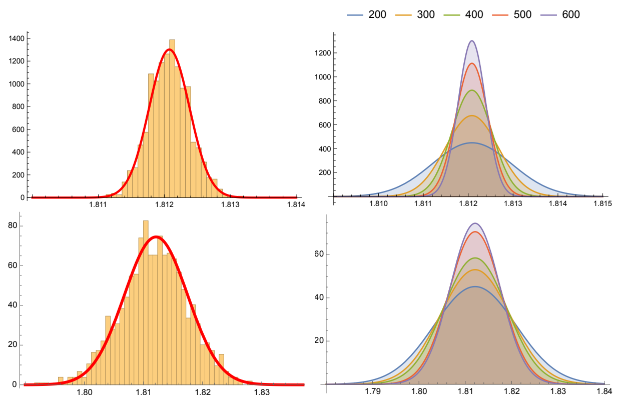

FIG. 2: Left: the Gaussian fitting of the fluctuations of the $N = 600$ soliton solution $\psi_N(0,0)$ with respect to the limiting solution $\psi^\infty(0,0) \simeq 1.812$ and 1000 trials. The point spectrum is sampled in the disk $\mathbb{D}_{1/10}(i)$ according to the Ginibre ensemble (top) and the uniform distribution (bottom). On the right figure the corresponding Gaussian fitting for $N = 200, 300, 400, 500, 600$. The Gaussian distribution is centred at $\psi^\infty(0,0)$ and the error σ scales numerically as $0.178/N$ (Ginibre) and $0.129/N^{\frac{1}{2}}$ (uniform distribution). The scaling does not depend on the point $x = 0, t = 0$ chosen to make the statistics.

a more elaborated argument, which is postponed to a subsequent publication. From numerical simulations, the fluctuations of $\psi_N(x,t,z_0,\dots,z_{N-1})$ around the limiting value $\psi_\infty(x,t)$ are Gaussian with error that decreases at the rate $\mathcal{O}(N^{-1})$, when the random points $\{z_0,\dots,z_{N-1}\}$ are sampled from the Ginibre ensemble while the rate is $\mathcal{O}(N^{-1/2})$ for the uniform distribution on the disk, see Figure 2.

Conclusions. We have considered a gas of N solitons solution of the FNLS equation in the limit $N \to \infty$. The soliton spectrum $\{z_j\}_{j=0}^{N-1}$ is chosen at first as the discretization of the uniform measure of a compact domain $\mathcal{D}$ of the complex upper half space and the norming constants $\{c_j\}_{j=0}^{N-1}$ are interpolated by a smooth function $\beta(z,\overline{z})$, namely $c_j = \frac{\mathcal{A}}{\pi N}\beta(z_j,\overline{z}_j)$ where $\mathcal{A}$ is the area measure of the domain $\mathcal{D}$. We then showed that when the domain $\mathcal{D}$ is a disk and the soliton density $\beta(z,\overline{z})$ is an analytic function, then the corresponding N-soliton solution condensates in the limit $N \to \infty$ and fixed (x,t), to the one-soliton solution with point spectrum coinciding with the center of the disk. We call this surprising effect *soliton shielding* because the interaction of infinite solitons reduces out to a one-soliton solution. Our result is robust and persists also when the soliton spectrum is a random variable sampled according to the Ginibre ensemble or the uniform distribution on the disk.

The determination of the N-soliton solution in the double scaling limit $N \to \infty$ and $x \to \infty$ in such a way that $x \simeq \log N$ remains a challenging open problem.

For other choices of domains $\mathcal{D}$ or density $\beta(z,\overline{z})$ we obtained a n-soliton solution or a one-soliton solution of order n. When the domain $\mathcal{D}$ is an ellipse, we showed that the spectral measure concentrates on lines connecting the foci of the ellipse and the soliton gas initial datum is asymptotically step-like oscillatory.

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