A CONTINUOUS ANALOG OF THE BINARY DARBOUX TRANSFORMATION FOR THE KORTEWEG-DE VRIES EQUATION

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Abstract. In the KdV context we put forward a continuous version of the binary Darboux transformation (aka the double commutation method). Our approach is based on the Riemann-Hilbert problem and yields a new explicit formula for perturbation of the negative spectrum of a wide class of step-type potentials without changing the rest of the scattering data. This extends the previously known formulas for inserting/removing finitely many bound states to arbitrary sets of negative spectrum of arbitrary nature. In the KdV context our method offers same benefits as the classical binary Darboux transformation does.

1. Introduction

As the title suggests, we are concerned with the *binary Darboux transformation* in the context of the *Korteweg-de Vries equation* (KdV). The literature on the Darboux transformation goes back to the nineteenth century and is immensely extensive and diverse. We only review some of what is directly related to our paper and where the interested reader may find further references. The very term appears to be introduced in 1985 by Babich-Matveev-Salle [3] in the context of the Toda lattice and then extended to the Kadomtsev-Petviashvili equation (see the influential 1991 book [45] by Matveev and Salle). The name owes to the fact that the *single Darboux transformation* (also know as Crum, elementary, or standard) is applied twice: to the associated AKNS system and its conjugate. We also refer the interested reader to Ling et al [40] and the extensive literature cited therein and to Cieslinski [4] where the binary Darboux transformation is revisited from a different point of view. Note that the binary Darboux transformation was originally introduced to generate explicit solutions to integrable systems by algebraic means (c.f. [31, 45]) and the *inverse scattering transform* (IST) was not directly used (while our approach is based on the IST).

It is interesting to note that what in the Darboux transformation community is referred to as the binary Darboux transformation is, in fact, also known in spectral theory of Sturm-Liouville operators as the *double commutation method* introduced in 1951 by Gelfand and Levitan in their seminal paper [19] in the context of their

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ground breaking study of the inverse spectral problem for Sturm-Liouville operators. The name seems however to be cast by Deift [8] in 1978 as the method rests on applying twice a commutation formula from operator theory. Note that Gelfand and Levitan did not use commutation arguments but relied on *transformation operator* techniques (see also the book [38, Section 6.6] by Levitan). The full treatment of the double commutation method is given by Gesztesy et al [20]-[25] in the 1990s (see also the extensive literature cited therein). The double commutation method was introduced to study the effect of inserting/removing eigenvalues in spectral gaps on spectral properties of the underlying 1D Schrodinger operators while the binary Darboux transformation, as we have mentioned, has been primarily a tool to produce explicit solutions. This is likely a reason why we could not find the literature where the two would be explicitly linked[1].

The main feature of the Darboux transformation (both, single and binary) is that it allows us to add or remove finitely many eigenvalues of the underlying system without altering the rest of the spectrum, which offers a powerful tool to study completely integrable systems. We are concerned with altering certain types of continuous spectrum too. More specifically, in the context of the KdV equation we introduce a broad class of the initial profiles, referred below to as *step-type* (see Definition 3.4), that admits an extension of the binary Darboux transformation allowing us to perturb (in particular, add or remove) the negative spectrum in nearly unrestricted way without affecting the rest of the scattering data. This class includes, as a very particular case, initial profiles approaching different constant values at $\pm\infty$ that recently drew renewed interest (see e.g. the recent Egorova et al [12] and Girotti et al [27] and the extensive literature cited therein). We start out from the Riemann-Hilbert version of the binary Darboux transformation put forward in our recent [50] and show that our construction is not really restricted to isolated negative eigenvalues and can be readily extended to the negative spectrum of arbitrary nature that step-type potentials can produce. As is well-known, Darboux transformations are particularly convenient in studying soliton propagation over various backgrounds and it is reasonable to expect that our approach can be used to the same effect for whole intervals of continuous negative spectrum. This becomes particularly relevant in the light of the recent spike of interest to soliton gases (see section 4).

The paper is organized as follows. In Section 2 we outline our notational agreements. In Section 3 we go over some basics necessary to fix our notation and terminology as well as some of our previous results needed in what follows. Section 4 is devoted to the statement and discussions of our main result, Theorem 4.1. In Section 5 we rewrite the classical formulation of the Riemann-Hilbert problem approach to the KdV equation in the form convenient for our generalizations. In Section 6 we state our Riemann-Hilbert problem with a jump matrix which entries are distributions. In Section 7 we state and prove a Riemann-Hilbert version of the continuous binary Darboux transformation. Theorem 7.1 proven therein is essentially equivalent to Theorem 4.1 but we hope it may present an independent interest, especially to the reader who prefers the Riemann-Hilbert problem framework. Section 8 is auxiliary and devoted to discretization of our main integral

[1] E.g. the book [31] pays much of attention to binary Darboux transformations but double commutation is not mentioned. The recent [52] briefly mentiones [31] and [25] but without discussing connections.

operator which our limiting arguments are based on. Section 9 is another auxiliary section where a convenient formula is derived for the proof of the main theorem. In Section 10 we finally prove Theorem 4.1 which essentially amounts to combining the ingredients prepared in the previous sections. In Section 11 we consider explicit examples. The first one is a new derivation of the well-known formula for pure soliton solutions and the second one is an explicit construction of reflectionless step-type potential. Appendix is devoted to some more auxiliary statements.

2. Notation

Through the paper, we make the following notational agreement. The bar denotes the complex conjugate. Prime $'$ means the derivative (perhaps generalized) in the main variable (typically spatial or in spectral parameter). The temporal variable t appears only as a parameter and is frequently suppressed. $W\{f,g\} = fg' - f'g$ is the Wronskian with obvious interpretation if one of f is a vector.

Given a non-negative finite Borel measure μ on the real line, $L^2(\mathrm{d}\mu)$ is the real Hilbert space with the inner product $\langle f, g\rangle = \int f(x)\, g(x) \mathrm{d}\mu(x)$, where the integral is taken over the support $\operatorname{Supp}\mu$. In particular, $\mathrm{d}\mu = \mathrm{d}x$, the Lebesgue measure on the whole real line, then we conveniently abbreviate $L^2(\mathrm{d}x) = L^2$. We apply the same agreement to other Lebesgue spaces L^p. Thus, we conveniently write

$$\int \cdot \,\mathrm{d}\mu = \int_{\operatorname{Supp}\mu} \cdot \,\mathrm{d}\mu, \int_{-\infty}^{\infty} = \int .$$

If a function $f(x)$ is defined on a set larger than $\operatorname{Supp}\mu$ we write $f \in L^p(\mathrm{d}\mu)$ if its restriction to $\operatorname{Supp}\mu$ is in $L^p(\mathrm{d}\mu)$. We adopt the following notation common in scattering theory:

$$L^1\left((1+|x|)^N \mathrm{d}x\right) = L^1_N, \quad N \geq 1.$$

χ_S is the characteristic function of a set S. In particular, $\chi_\pm = \chi_{\mathbb{R}_\pm}$ for the Heaviside functions. We write

$$f \in L^p(\pm\infty) \text{ if } f\chi_{(a,\pm\infty)} \in L^p \text{ for any finite } a.$$

The same convention is used for $L^p_N(\pm\infty)$. If S is not a subset of $\operatorname{Supp}\mu$ we write $\mu(S) = \mu(S \cap \operatorname{Supp}\mu)$. We will refer to $\mathrm{d}\mu(x)/\mathrm{d}x$ as generalized density/derivative or distributional derivative. As always, it is defined by

$$\int_I f(x) \frac{\mathrm{d}\mu(x)}{\mathrm{d}x} \mathrm{d}x = \int_I f(x)\, \mathrm{d}\mu(x)$$

for any continuous function $f(x)$ and finite interval I. In particular, $\delta_a(x) = \mathrm{d}\chi_{(a,+\infty)}(x)/\mathrm{d}x$ is the Dirac delta-function supported at a. We call a measure μ symmetric if $\mathrm{d}\mu(x) = \mathrm{d}\mu(|x|)$.

Square matrices are denoted by boldface capital letters, except for the identity matrix I. We use boldface lowercase letters for vectorial (row or column) quantities. Except for $\mathbb{R}$, $\mathbb{C}$, blackboard bold letters denote operators. Less common notations and conventions will be introduced later.

3. Brief Review of the IST Method

We are concerned with the Cauchy problem for the KdV equation

$$\partial_t u - 6u' + u''' = 0, \quad x, t \in \mathbb{R}, \tag{3.1}$$

$$u(x,0) = q(x), \tag{3.2}$$

the first nonlinear evolution PDE solved in 1967 by Gardner, Greene, Kruskal, and Miura [11] by the inverse scattering transform (IST). For the reader's convenience and to fix our notation we review the necessary material following [1, 43, 47]. The IST method consists, as the standard Fourier transform method, of three steps:

Step 1. (direct transform)

$$q(x) \longrightarrow S_q,$$

where S_q is a new set of variables which turns (3.1) into a simple first order linear ODE for $S_q(t)$ with the initial condition $S_q(0) = S_q$.

Step 2. (time evolution)

$$S_q \longrightarrow S_q(t).$$

Step 3. (inverse transform)

$$S_q(t) \longrightarrow q(x,t).$$

3.1. **Classical short-range IST.** Step 1. Suppose that the initial condition q for (3.2) is real and rapidly decaying. This means that the solution $q(x,t)$ to (3.1)-(3.2) is subject to the decay condition

$$\int_{-\infty}^{\infty} (1+|x|)\,|q(x,t)|\,\mathrm{d}x < \infty, \quad t \geq 0, \quad \text{(short-range)}. \tag{3.3}$$

Associate with $q(x)$ the full line *Schrödinger operator* $\mathbb{L}_q = -\partial_x^2 + q(x)$. (For simplicity we retain the same notation for the differential operation.) Its spectrum $\operatorname{Spec}\mathbb{L}_q$ consists of a two fold absolutely continuous part filling $[0,\infty)$ and finitely many negative simple eigenvalues (bound states) $\left(-\kappa_n^2\right)_{n=1}^N$. An important feature of the short-range q is that it supports two (right/left) *Right/left Jost solutions* $\psi_\pm(x,k)$. I.e. solutions of the *Schrödinger equation*

$$\mathbb{L}_q u = -u'' + q(x)\,u = k^2 u, \tag{3.4}$$

having plane wave asymptotics at infinity: $\psi_\pm(x,k) \sim \mathrm{e}^{\pm\mathrm{i}kx}, x \to \pm\infty$. Note that $\psi_\pm(x,k)$ are determined by $q(x)$ on $(\pm\infty, x)$ respectively. These solutions are analytic for $\operatorname{Im} k > 0$, continuous down to the real line where $\psi_\pm(x,-k) = \overline{\psi_\pm(x,k)}$. The pair $\{\psi_+, \overline{\psi_+}\}$ forms a fundamental set for (3.4) and hence

$$T(k)\psi_-(x,k) = \overline{\psi_+(x,k)} + R(k)\psi_+(x,k),\ k \in \mathbb{R}, \tag{3.5}$$

with some T and R called the transmission and (right) reflection coefficients respectively. While totally elementary, (3.5), called the *right basic scattering identity,* serves as a foundation for inverse scattering theory. It immediately follows from (3.5) that

$$T(k) = \frac{2\mathrm{i}k}{W\{\psi_-(x,k), \psi_+(x,k)\}}, \tag{3.6}$$

which means that $T(k)$ can be analytically continued into the upper half plane with simple poles at $\mathrm{i}\kappa_n$ where $\psi_-(x,k)$, $\psi_+(x,k)$ are linearly dependent. Moreover, $\psi_-(x,i\kappa_n)$, $\psi_+(x,i\kappa_n)$ are real and decay exponentially at both $\pm\infty$ and hence $-\kappa_n^2$ is a (negative) bound state (eigenvalue) of $\mathbb{L}_q$. The number

$$c_n = \left(\int \psi_+(x,\mathrm{i}\kappa_n)^2 \mathrm{d}x\right)^{-1/2} \tag{3.7}$$

is called the *norming constant* of the bound state $-\kappa_n^2$. The reflection coefficient $R(k)$ is continuous but need not analytically extend outside the real line. It obeys $R(-k) = \overline{R(k)}$, $|R(k)| \leq 1$, $|R(k)| = 1$ only at $k = 0$.

The main feature of the short-range case is that $R(k)$ (continuous component) and $\{(\kappa_n, c_n), 1 \leq n \leq N\}$ (discrete component) determine the potential q uniquely. It will be convenient for our purposes to write the discrete component as the (discrete) measure

$$\mathrm{d}\rho(k) = \sum_{n=1}^{N} c_n^2 \delta_{\kappa_n}(k)\, \mathrm{d}k, \tag{3.8}$$

which alone carries over all necessary information about the discrete spectrum. We can now introduce the (right) *scattering data*

$$S_q := \{R, \mathrm{d}\rho\}.$$

We emphasize that S_q determines q uniquely (i.e. S_q is indeed data) in general only for short-range q's. (See e.g. our recent [51] for explicit counterexamples.)

Step 2. The main reason why the IST works is that the (necessarily unique) solution $q(x,t)$ to the problem (3.1)-(3.3) gives rise to the "time evolved" Schrödinger operator $\mathbb{L}_{q(\cdot,t)}$ for which

$$R(k,t) = R(k)\,\mathrm{e}^{8\mathrm{i}k^3 t}, \kappa_n(t) = \kappa_n, c_n(t) = \mathrm{e}^{4\kappa_n^3 t} c_n.$$

Since $q(x,t)$ is short-range for $t > 0$

$$S_q(t) = \left\{R(k)\,\mathrm{e}^{8\mathrm{i}k^3 t}, \mathrm{e}^{8k^3 t}\mathrm{d}\rho(k) : k \geq 0\right\}$$

is ("time evolved") scattering data for $q(x,t)$.

Step 3. Solve the inverse scattering problem for $S_q(t)$ by any applicable method.

Note that this scheme runs for the left scattering data equally well. Another remarkable feature of the classical short-range IST is that potentials subject to (3.3) can be characterized in term of the scattering data (known as *Marchenko's characterization* [43]).

3.2. **One sided IST.** The IST was extended in [32] and [54] to step-like initial data q, i.e. q's approaching two different values as $x \to \pm\infty$ rapidly enough (aka an initial hydraulic jump or bore wave) or and then to q's approaching periodic function on one end and a constant on another in [33]. The main difference from the short-range case is that the measure ρ in the scattering data gains an absolutely continuous component and the right and left ISTs are different. In fact, [32] uses the left IST and [54] does the right IST to prove that the absolutely continuous component of ρ gives rise to an infinite sequence of asymptotic solitons twice as high as the initial hydraulic jump. The latter is the main feature of such initial profiles. Another interesting feature of step-like initial data is appearance of rarefaction waves studied recently in [12] and [27].

In [29] we extended IST to initial data that approach zero at $+\infty$ fast enough but are essentially arbitrary at $-\infty$. As opposed to the step-like data studied in [32, 54] such potentials support in general only the right inverse scattering. The approach is based on the notion of a Weyl solution. Recall that a real-valued locally integrable potential q is said to be Weyl limit point at $\pm\infty$ if the equation $\mathbb{L}_q u = k^2 u$ has a unique (up to a multiplicative constant) solution (called Weyl) $\Psi_\pm(\cdot, k^2)$ that is square integrable at $\pm\infty$ for each $k^2 \in \mathbb{C}^+$. Note that as apposed to Jost solutions, Weyl solutions exist under much more general conditions on q's and no decay of any kind is required. As is well-known, if $q(x)$ is limit point at both $\pm\infty$ then $\mathbb{L}_q$ is selfadjoint on L^2.

It is convenient to give a special name to potentials that we shall deal with through the rest of the paper.

Definition 3.1 (Step-type potentials)**.** *We call a (real) potential $q(x)$* step-type *if*

(1) $q(x) \in L_1^1(+\infty)$ *(short-range at $+\infty$);*
(2) $\operatorname{Spec} \mathbb{L}_q > -\infty$ *(essential boundedness below).*

The (right-sided) scattering theorem for such potentials is studied in our [28, Section 7] where the interested reader can finds the details. Choose the Weyl solution $\varphi(x,k)$ (note our variable k not k^2) at $-\infty$ to satisfy

$$\varphi(x,k) = \overline{\psi(x,k)} + R(k)\psi(x,k), \text{ a.e. } \operatorname{Im} k = 0, \tag{3.9}$$

with some coefficient $R(k)$ (c.f. (3.5)), which can be called the (right) reflection coefficient. Note that in the short-range case

$$\varphi(x,k) = T(k)\,\psi_-(x,k). \tag{3.10}$$

Proposition 3.2 (On reflection coefficient)**.** *The (right) reflection coefficient R of a step-type potential $q(x)$ is well defined, symmetric $R(-k) = \overline{R(k)}$, and $|R(k)| \leq 1$ a.e. Moreover, if $\Delta \subseteq \operatorname{Spec}(\mathbb{L}_q)$ is the minimal support of the two fold a.c. spectrum of $\mathbb{L}_q$ then $|R(k)| < 1$ for a.e. real k such that $k^2 \in \Delta$ and $|R(k)| = 1$ otherwise.*

This statement describes the positive spectrum of $\mathbb{L}_q$ only. The negative spectrum is described in

Proposition 3.3 (On norming measure)**.** *If $q(x)$ is step-type then on the imaginary line φ and ψ are related by*[2]

$$\operatorname{Im} \varphi(x, \mathrm{i}k + 0)\,\mathrm{d}k = \pi\psi(x, \mathrm{i}k)\,\mathrm{d}\rho(k), \quad k \geq 0, \tag{3.11}$$

for some non-negative finite measure ρ. Moreover,

$$\operatorname{Supp} \rho = \left\{k \geq 0 : -k^2 \in \operatorname{Spec}(\mathbb{L}_q)\right\}.$$

Proposition 3.2 is proven in [28, Proposition 7.10]. Proposition 3.3 follows from Proposition 7.12 of [28, Section 7] (but it shall also become transparent from considerations below). Note that $\psi(x,k)$ is analytic for $\operatorname{Im} k > 0$ and $\varphi(x,t)$ is analytic for $\operatorname{Im} k > 0$ away from $\mathrm{i}\operatorname{Supp}\rho$. If q is short-range then $\mathrm{d}\rho$ in (3.11) coincides with (3.8) (see below). For this reason we call ρ the *norming measure*. Note that the important in the short-range scattering representation (3.10) defining the transmission coefficient $T(k)$ is lost for a generic step-type potential. It can of course be introduced, as frequently done in the literature on bore-type q's discussed above, by replacing the left Jost solution in (3.10) with another Weyl solution which asymptotic in $x \to -\infty$ is know for a.e. real k. But the transmission coefficient introduced this way will depend on such choice. We refer the interested reader to [26] for a general framework on scattering theory with different spatial asymptotics at $\pm\infty$. For our purposes we do not need the transmission coefficient but it may become necessary to study asymptotic behavior of KdV solutions as $x \to -\infty$.

[2]Here and below, as always, $\operatorname{Im} f(x + \mathrm{i}0)\,\mathrm{d}x = w^* - \lim \operatorname{Im} f(x + \mathrm{i}y)\,\mathrm{d}x,\ y \to +0.$

If $q(x)$ is a pure step function, i.e. $q(x) = -h^2,\ x < 0,\ q(x) = 0,\ x \geq 0$ then $\operatorname{Spec}(\mathbb{L}_q) = (-h^2, \infty)$ and purely absolute continuous, $(-h^2, 0)$ and $(0, \infty)$ being its simple and two fold components respectively. Moreover

$$R(k) = -\left(\frac{h}{\sqrt{k^2} + \sqrt{k^2 + h^2}}\right)^2, \quad \mathrm{d}\rho(k) = \frac{2k}{\pi h^2}\sqrt{h^2 - k^2}\mathrm{d}k. \tag{3.12}$$

Let us now look at what happens to Step 1- Step 3 (the "time evolved" picture). Using the classical short-range IST as a pattern to follow, introduce

$$S_q = \{R(k), \mathrm{d}\rho(k) : k \geq 0\}$$

for Step 1. The problem though is that no analog of Marchenko's characterization is known to date and, while S_q can be formed by solving right sided scattering problem we cannot claim that there is only one q that corresponds to S_q. But we move on to Step 2 and form

$$S_q(t) = \left\{R(k)\,\mathrm{e}^{8\mathrm{i}k^3 t}, \mathrm{e}^{8k^3 t}\mathrm{d}\rho(k) : k \geq 0\right\}.$$

This step is formal as there are no general well-posedness results for the KdV equation with general step-type initial data. Thus we don't know what $S_q(t)$ actually represents and cannot go over to step 3. These problems can however be detoured by understanding a KdV solution as a suitable limit. The following definition is convenient for this. It is also quite natural from the physical point of view.

Definition 3.4 (Step-type KdV solutions)**.** *We call $q(x,t)$ a (right)* step-type KdV solution *with scattering data $S_q = \{R(k), \mathrm{d}\rho(k) : k \geq 0\}$ if for $t \geq 0$*

(1) *$q(x,t)$ is step-type (in the sense of Definition 3.1);*
(2) *there is a sequence $S_n = \{R_n, \mathrm{d}\rho_n\}$ such that each set S_n is the scattering data for some short-range (real) potential $q_n(x)$ and in the weak* topology*

$$\begin{array}{ccc} S_n & \to & S_q \\ q_n(x,t) & \to & q(x,t) \end{array}, \text{ (double convergence)}, \tag{3.13}$$

where $q_n(x,t)$ is the KdV solution with initial data $q_n(x)$.

The comments below should clarify the nature of this concept and why it could be convenient.

1. We emphasize again that no well-posedness for step-type KdV solutions is available in general. The main issue is that it is not clear what Banach space such solutions can be included to even speak about well-posedness. For bore-type initial conditions the best known well-posedness result in given in the recent [37] (see also [39]). These papers also suggest that well-posedness for (3.1)-(3.2) for general step-type q's may be out of reach.

2. In [5] the authors use certain IST constructions to give examples of nonuniqueness of the Cauchy problem for KdV. One example gives a nontrivial C^∞ solution $q(x,t)$ in a domain $\{(x,t) : 0 < t < H(x)\}$ for a positive nondecreasing function H, such that $q(x,t)$ vanishes to all orders as $t \to 0$. This solution decays rapidly as $x \to +\infty$, but cannot be "well behaved" as x moves left. Further analysis is required to tell if such $q(x,t)$ may in fact be a step-type potential[3] or not but these disturbing examples suggest that condition 2 may indeed be needed to single out a solution that we think of as physically relevant.

[3]Explicit IST constructions given in [2] produce double pole singularity moving left and thus such solutions fail condition 1 of Definition 3.4.

3. In a more restrictive form, Definition 3.1 appears first in our [49] (where it is referred to as natural). The problem (3.1)-(3.2) with $q(x)$ subject to

$$\operatorname{Spec}\mathbb{L}_q > -\infty, \tag{3.14}$$

$$q(x) \in L^1_N(+\infty) < \infty, \quad N \geq 5/2, \tag{3.15}$$

is studied in our [28, 29]. It is shown that the sequence $q_n(x,t)$ of KdV solutions corresponding to cut-off approximations $q_n(x) = q(x)\chi_{(-n,\infty)}(x)$ of initial data $q(x)$ converges to a classical solution $q(x,t)$ of (3.1) (that is three times continuously differentiable in x and once in t) uniformly on compacts in $\mathbb{R}\times\mathbb{R}_+$. (That is, convergence $q_n(x,t) \to q(x,t)$ in (3.13) is actually much better than weak*.) The sequence of scattering data S_n, however, converges in the weak* topology and no better. Recall that, as is well-known [9], even in the short-range case cut-off approximations lead to the sequence $R_n(k)$ that fails to converge uniformly to $R(k)$ at $k = 0$. Thus under conditions (3.14) and (3.15) the double convergence (3.13) works as follows:

$$\begin{aligned} q_n(x) \to q(x) \text{ a.e. } &\Rightarrow S_n \to S \text{ star-weakly} \\ &\Rightarrow q_n(x,t) \to q(x,t) \text{ locally uniformly for } t > 0. \end{aligned} \tag{3.16}$$

4. Since the KdV flow is isospectral (already assumed in the Lax pair formulation of the KdV equation), condition (3.14) holds for $t > 0$. Condition (3.15) is however not time invariant. Technics of [29] can be readily used to tell how the rate of decay at $+\infty$ drops under the KdV flow (in fact, to tell how the KdV trades decay at $+\infty$ for gain in smoothness (work in progress)). [5] suggests that if $q(x) \in L^1_N(+\infty)$, then $q(x,t) \in L^1_{N-5/4}(+\infty) \in L^1_1(+\infty)$ and hence $q(x,t) \in L^1_1(+\infty)$ when $N \geq 5/2$.

5. The chain (3.16) is not always convenient. The one

$$S_n \to S \text{ a.e. } \Rightarrow q_n(x,t) \to q(x,t) \text{ a.e. for } t \geq 0 \tag{3.17}$$

may, as our [51] suggests, work better than (3.16). Below we have to implement both (3.16) and (3.17) in (3.13). This should explain the meaning of condition 2.

6. The weak* convergence of $q_n(x,t) \to q(x,t)$ in (3.13) is assumed to be on the save side and may be upgraded. The real upgrade though would of course be indicating a Banach space where it takes place. The latter does not appear possible unless we impose strong assumptions on the behavior of $q(x)$ as $x \to -\infty$. E.g. [29] if

$$\sum_{m=-\infty}^{\infty}\left(\int_m^{m+1}|q(x)|\,dx\right)^2 < \infty, \tag{3.18}$$

then $q_n(x,t) \to q(x,t)$ holds at least in the Sobolev space $H^{-1}(\mathbb{R})$. The latter is a direct consequence of well-posedness of the KdV in H^{-1} recently proven in [35]. (In fact, any $H^{-\varepsilon}$ with $0 < \varepsilon \leq 1$ will do.)

7. Condition (3.14) is satisfied if

$$\operatorname*{Sup}_{|I|=1}\int_I \max(-q(x), 0)\,\mathrm{d}x < \infty. \tag{3.19}$$

The latter covers a large class of initial profiles without any assumption on a pattern of behavior at $-\infty$. Examples include white-noise restricted to the left half-line. Note in this connection, that it was recently proven in [34] that the white noise is invariant on the whole line.

4. Main theorem

In this section we present our main statement and offer relevant discussions.

Theorem 4.1 (Pertubration of negative spectrum)**.** *Let $q(x,t)$ be a step-type KdV solution (in the sense of Definition 3.4) with the scattering data*

$$S_q = \{R(k), \mathrm{d}\rho(k) : k \geq 0\}, \tag{4.1}$$

$\psi(x,t,k)$ its right Jost solution, and

$$K(k/\mathrm{i}, s; x, t) := \int_x^\infty \psi(z,t;k)\,\psi(z,t;\mathrm{i}s)\,\mathrm{d}z, \quad \operatorname{Im} k \geq 0, s \geq 0. \tag{4.2}$$

Then for any finite signed measure σ supported on a compact set of $[0,\infty)$ satisfying the conditions

$$\int |\mathrm{d}\sigma(k)|/k < \infty, \quad \mathrm{d}\rho + \mathrm{d}\sigma \geq 0, \tag{4.3}$$

the Fredholm integral equation

$$y(\alpha) + \int K(\alpha, s; x, t)\, y(s)\, \mathrm{d}\sigma_t(s) = \psi(x,t;\mathrm{i}\alpha), \ \alpha \in \operatorname{Supp}\sigma, \tag{4.4}$$

$$\mathrm{d}\sigma_t(s) := \mathrm{e}^{8s^3 t}\mathrm{d}\sigma(s),$$

has a unique solution $y(s;x,t)$ in $L^2(\mathrm{d}\sigma)$ and

$$\psi_\sigma(x,t;k) = \psi(x,t;k) - \int K(k/\mathrm{i}, s; x, t)\, y(s,x,t)\, \mathrm{d}\sigma_t(s), \tag{4.5}$$

is the right Jost solution corresponding to the potential

$$\begin{aligned} q_\sigma(x,t) &= q(x,t) \\ &+ 2\left[\int \psi_\sigma(x,t;\mathrm{i}s)\,\psi(x,t;\mathrm{i}s)\,\mathrm{d}\sigma_t(s)\right]^2 + 4\int \psi_\sigma(x,t;\mathrm{i}s)\,\psi'(x,t;\mathrm{i}s)\,\mathrm{d}\sigma_t(s), \end{aligned} \tag{4.6}$$

which is a step-type KdV solution with the scattering data

$$S_{q_\sigma} = \{R(k), \mathrm{d}\rho(k) + \mathrm{d}\sigma(k) : k \geq 0\}.$$

Moreover,

(1) *For $s \in \operatorname{Supp}\sigma$*
$$\psi_\sigma(x,t,\mathrm{i}s) = y(s,x,t); \tag{4.7}$$
(2) *$q_\sigma(x,t)$ is as smooth as $q(x,t)$ (i.e. $q(x,t) \in C^{(n)} \Leftrightarrow q_\sigma(x,t) \in C^{(n)}$);*
(3) *If $0 \notin \operatorname{Supp}\sigma$ then $q_\sigma(x,t) - q(x,t)$ decays exponentially as $x \to +\infty$ for every fixed $t > 0$;*
(4) *If κ_0 is a pure point of σ with a positive weight then $-\kappa_0^2$ is a (negative) bound state of $\mathbb{L}_{q_\sigma}$ which is embedded if $\kappa_0 \in \operatorname{Supp}\rho$;*
(5) *The binary Darboux transformation is invertible in the following sense*
$$(\psi_\sigma)_{-\sigma} = \psi, \quad (q_\sigma)_{-\sigma} = q.$$

Some comments:

1. Theorem 4.1 is an extension of Theorem 1 of our recent [50] where all considerations are conducted in the short-range setting. It was observed in [50] (Remark 3) that only the right Jost solution ψ explicitly appears and the left Jost solution does not explicitly appear in any formulas. This led us to conjecture that q may be quite general at $-\infty$. Theorem 4.1 answers this conjecture in the affirmative. The relaxation of the decay assumption at $-\infty$ results in the appearance of rich negative

spectrum described by a measure ρ of arbitrary nature (in the short-range case it is discrete with finitely many pure points). To remain in the short-range setting we could consider in [50] perturbations σ supported only at finitely many points. In Theorem 4.1 we no longer have to assume this and moreover σ can be continuous. For this reason Theorem 4.1 can also be named the *continuous binary Darboux transformation* or the continuous analog of the double commutation method [25]. Note that the formula (4.6) appears to be new even in this case when σ is discrete with finitely many points (c.f. [25, 50]). Furthermore, binary Darboux transformations are typically written differently for adding and removing eigenvalues and thus Theorem 4.1 combines the two.

2. The main feature of the transformation (4.6) is that it allows us to modify the negative spectrum in a nearly unrestricted way while leaving the reflection coefficient unchanged.

3. Since $\psi(\cdot, \mathrm{i}s)$ is real, if $\mathrm{d}\sigma \geq 0$ then the kernel $K(\alpha, s; x, t)$ is positive definite and (4.4) is automatically uniquely solvable. However, if $\mathrm{d}\sigma < 0$ then this is no longer true in general. A counterexample is easily produced by $R = 0$, $\rho = 0$, and $\mathrm{d}\sigma(k) = -c^2\delta(k-\kappa)\mathrm{d}k, \kappa > 0$. Indeed a simple formal computation gives then

$$q_\sigma(x,t) = -2\partial_x^2 \log\left(1 - c^2 \mathrm{e}^{8\kappa^3 t - 2\kappa x}\right),$$

which is a singular KdV solution (has moving real double pole). Thus the second condition (4.3) cannot be dropped. In other words, removing an inexistent eigenvalue produces a singular solution. Note that this way our approach offers a setting for generating singular solutions on nonzero backgrounds. It would be interesting to compare our method to Wronskian considerations commonly used in the this context (see, e.g. the influential [42]).

4. The first condition (4.3) guarantees that 0 is not an eigenvalue of q_σ. If 0 were an eigenvalue then q_σ would not be short-range at $+\infty$. We believe it can be relaxed to read that σ is a Carleson measure (see, e.g. for the definition [36]) but cannot be completely removed.

5. Equations (4.5) and (4.7) give an explicit formula for reconstruction of the Jost solution $\psi_\sigma(x,t;k)$ for any $\operatorname{Im} k \geq 0$ via its values on $\operatorname{Supp}\sigma$. Note that there is no such formula for a generic analytic function.

6. Step-type potentials admit embedded discrete negative spectrum. Note that there are no positive embedded bound states if q is short-range at $+\infty$.

7. The formula (4.6) can also be written as

$$q_\sigma(x,t) = q(x,t) - 2\partial_x^2 \log\det\left[I + \mathbb{K}(x,t)\right],$$

where $\mathbb{K}(x,t)$ is a trace class integral operator acting by the formula

$$\mathbb{K}(x,t) f = \int \left[\int_x^\infty \psi(z,t;\mathrm{i}\alpha)\,\psi(z,t;\mathrm{i}s)\,\mathrm{d}z\right] f(s)\,\mathrm{d}\sigma_t(s).$$

8. As we have mentioned, it is proven in [32, 54] that a short-range perturbation of a pure step function, i.e. $q(x) = -h^2,\ x < 0,\ q(x) = 0,\ x \geq 0$ (hydraulic jump) gives rise to an infinite sequence of asymptotic solitons of height $-2h^2$ (twice as high as the initial hydraulic jump). Theorem 4.1 suggests that this effect is a much more general phenomenon and the fastest soliton always propagates with the asymptotic velocity $2h^2$ where $h^2 = -\inf \operatorname{Spec} \mathbb{L}_q$. Computing asymptotic phases is the most difficult part. (Work in progress).

9. As one of our referees drew our attention to, if $q = 0$ our (4.6) produces a notion of generalized reflectionless potentials that is reminiscent of the construction due to Lundina [41] and Marchenko [44]. More specifically, it is proven in [44] that if the integral equation

$$\begin{aligned} &\mathrm{e}^{-4\kappa^3 t+\kappa x}\left\{a(\kappa)\,y(\kappa)-\frac{1}{2\kappa}\left[\int\frac{y(s)-y(\kappa)}{s-\kappa}\mathrm{d}\sigma(s)-1\right]\right\}\\ &=\mathrm{e}^{4\kappa^3 t-\kappa x}\left\{[a(\kappa)-1]\,y(-\kappa)-\frac{1}{2\kappa}\left[\int\frac{y(s)-y(-\kappa)}{s+\kappa}\mathrm{d}\sigma(s)-1\right]\right\} \end{aligned} \tag{4.8}$$

is uniquely solvable for $y(\kappa, x, t)$, then

$$q(x,t) = -2\partial_x \int y(\kappa, x, t)\,\mathrm{d}\sigma(\kappa) \tag{4.9}$$

satisfies the KdV equation with data $q(x, 0) = q(x)$, a and σ being related to $q(x)$. It is not obvious how (4.8) and (4.9) are related to (4.4) and (12.1), respectively, but it would certainly be an interesting question to ask (especially because an open question related to the measure σ is stated in [44]). Note that, as apposed to our (4.4), solubility of (4.8) is not on the surface. Finally, we also mention that our methods are very different from [44], where smoothness of $q(x)$ is essential, while it is not in our construction.

10. And last but not least, we discuss the relevance of Theorem 4.1 to soliton gases. Back in 1971, Zakharov [55] pioneered a statistical description of multisoliton solutions (*rarefied soliton gas*) which became a big deal in this millennium after the introduction of *integrable turbulence* and general framework for random solutions of integrable PDEs in his influential [57]. This phenomenon was observed in shallow water wind waves in Currituck Sound, NC [7] and was experimentally reproduced in a water tank [48] and optical fibers, drawing even greater interest in a number of research groups (see e.g. [6, 10, 14, 16, 27, 46]) with different approaches. *Dense soliton gas and condensate*, particular important from the physical view point, can be modelled as a closure of pure soliton solutions (c.f. [10, 13, 21, 17, 18]). We mention only [10] where the *Zakharov-Manakov dressing method* [56] was used to yield *primitive potentials*, which are *one-gap* but neither periodic nor decaying. Such solutions are parametrized by *dressing functions* r_1, r_2 and essentially only $r_2 = 0$ has been studied rigorously [27] via RHP technics. For $r_2 \neq 0$ the only case of $r_1 = r_2$ was just considered in [46] (*elliptic one-gap potential* if $r_1 = r_2 = 1$) but the general case is still out of reach. Note that the dressing method isn't quite IST and cannot solve a Cauchy problem [44]. While seemingly unrelated, Theorem 4.1 may put many KdV soliton gas considerations in the context of the IST for the Cauchy problem for the KdV equation and provide a rigorous framework to study soliton gases. In fact, in the soliton gas community they actually study statistical quantities (*density of states*, *effective velocity*, *collision rate*, etc.) of our left step-type KdV solutions from Theorem 4.1 with $q(x, t) = 0$ (zero background) and specific a.c. $\mathrm{d}\sigma \geq 0$ supported on $\left[-b^2, -a^2\right]$ with $a > 0$. Inclusion of $q(x, t) \neq 0$ (nonzero backgrounds) and $a = 0$ (small solitons) into the picture are good open problems. Another open problem comes from some numerics suggesting that "injection" of a soliton into *soliton condensates* may locally in time and space "evaporate" the latter but this effect is not described mathematically. We are yet to look into these questions but at least Theorem 4.1 eases our concern about rather formal

realization of limiting (scaling) arguments quite common in the physical literature on the subject.

5. Classical meromorphic vector Riemann-Hilbert problem

In this section we review the standard meromorphic vector Riemann-Hilbert problem that arises from the classical inverse scattering formalism for the KdV equation following [30]. It will be the starting point in our search for a suitable formulation of the corresponding Riemann-Hilbert problem for arbitrary step-type potentials. Through this section we denote

$$R(k,t)=R(k)\mathrm{e}^{8\mathrm{i}k^3t}, c_n(t)=c_n\mathrm{e}^{4\kappa_n^3t}.$$

Meromorphic Riemann-Hilbert Problem (MRHP): Let $S_q=\{R,(\kappa_n,c_n)\}$ be the scattering data of a short-range potential and let

$$\mathbf{J}(k,t):=\begin{pmatrix}1-|R(k,t)|^2 & -\overline{R(k,t)}\\ R(k,t) & 1\end{pmatrix}\quad (\textit{jump matrix}),$$

and t real parameter. Find a row function $\mathbf{v}=\begin{pmatrix}\varphi & , & \psi\end{pmatrix}$ meromorphic in $\operatorname{Im}k\neq 0$ with simple poles $(\pm\mathrm{i}\kappa_n)$, such that:

(1) *Symmetry condition*:

$$\overline{\mathbf{v}(\overline{k})}=\mathbf{v}(-k)=\mathbf{v}(k)\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\quad \operatorname{Im}k\neq 0. \tag{5.1}$$

(2) *Jump condition*: The boundary values $\mathbf{v}(k\pm\mathrm{i}0)$ are related by

$$\mathbf{v}(k+\mathrm{i}0)=\mathbf{v}(k-\mathrm{i}0)\mathbf{J}(k,t),\quad \operatorname{Im}k=0. \tag{5.2}$$

(3) *Pole conditions*:

$$\operatorname*{Res}_{\pm\mathrm{i}\kappa_n}\mathbf{v}=\mathrm{i}c_n^2(t)\begin{cases}\left.\mathbf{v}\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\right|_{\mathrm{i}\kappa_n}\\ \left.-\mathbf{v}\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\right|_{-\mathrm{i}\kappa_n}\end{cases},\quad 1\leq n\leq N. \tag{5.3}$$

(4) *Asymptotic condition*[4]: For real x

$$\mathbf{v}(k)\sim\begin{pmatrix}\mathrm{e}^{-\mathrm{i}kx} & , & \mathrm{e}^{\mathrm{i}kx}\end{pmatrix},\quad k\to\infty. \tag{5.4}$$

As is well-known,

$$\mathbf{v}(k)=\begin{pmatrix}\varphi(k) & , & \psi(k)\end{pmatrix}, \tag{5.5}$$

where

$$\varphi(k)=T(k)\psi_-(x,t;k),\quad \psi(k)=\psi_+(x,t;k).$$

solves MRHP and the potential $q(x,t)$ is recovered from its second component. It is important that this component has no poles.

Note that the pole conditions (5.3) can be written in the scalar form

$$\operatorname*{Res}_{\mathrm{i}\kappa_n}\varphi(k)=\mathrm{i}c_n^2(t)\psi(\mathrm{i}\kappa_n),\ \operatorname*{Res}_{-\mathrm{i}\kappa_n}\varphi(k)=-\mathrm{i}c_n^2(t)\psi(\mathrm{i}\kappa_n), \tag{5.6}$$

[4]It is more common to set $\mathbf{v}(k)\sim\begin{pmatrix}1 & , & 1\end{pmatrix}, k\to\infty$, to remove oscillations but then the jump matrix gains an undesirable dependence on x.

which we now represent in an integral form. Due to symmetry it is enough to consider only the first equation in (5.6). Let I be an open interval in $\mathbb{R}_+$, then one can easily see that

$$\begin{aligned} \int_I \operatorname{Im}\varphi(\mathrm{i}s-0)\,\mathrm{d}s &= \lim_{\varepsilon\to+0}\int_I \operatorname{Im}\varphi(\mathrm{i}s-\varepsilon)\,\mathrm{d}s \\ &= \lim_{\varepsilon\to+0}\int_I \frac{\varphi(\mathrm{i}s-\varepsilon)-\overline{\varphi(\mathrm{i}s-\varepsilon)}}{2\mathrm{i}}\mathrm{d}s \\ &= -\lim_{\varepsilon\to+0}\int_I \frac{\varphi(\mathrm{i}s-\varepsilon)-\varphi(\mathrm{i}s+\varepsilon)}{2}\mathrm{i}\mathrm{d}s \\ &= \lim_{\varepsilon\to+0}\frac{1}{2}\int_{C_\varepsilon(I)}\varphi(z)\,\mathrm{d}z = \mathrm{i}\pi\sum_{\kappa_n\subset I}\operatorname*{Res}_{\mathrm{i}\kappa_n}\varphi. \end{aligned} \tag{5.7}$$

Here $C_\varepsilon(I)$ is a contour in $\mathbb{C}^+$ enclosing I and shrinking to I as $\varepsilon\to 0$. It follows now from this and (5.6) that

$$\begin{aligned} \int_I \operatorname{Im}\varphi(\mathrm{i}s-0)\,\mathrm{d}s &= -\sum_{\kappa_n\subset I}\pi c_n^2\psi(\mathrm{i}\kappa_n) \\ &= -\pi\int_I \psi(\mathrm{i}s)\,\mathrm{d}\rho(s), \end{aligned} \tag{5.8}$$

where

$$\mathrm{d}\rho(s) := \sum_n c_n^2(t)\,\delta_{\kappa_n}(s)\,\mathrm{d}s, \tag{5.9}$$

and δ_{κ_n} is the Dirac delta function supported at κ_n. Since I is arbitrary, it follows from (5.7) that

$$\operatorname{Im}\varphi(\mathrm{i}s-0)\,\mathrm{d}s = -\pi\psi(\mathrm{i}s)\,\mathrm{d}\rho(s)$$

and, since $\operatorname{Im}\psi(\mathrm{i}s-0)=0$ (ψ is analytic away from $\mathbb{R}$), we can rewrite now the pole conditions (5.3) as

$$\begin{aligned} \operatorname{Im}\mathbf{v}(\mathrm{i}s-0) &= \begin{cases} \mathbf{v}(\mathrm{i}s+0)\begin{pmatrix} 0 & 0 \\ -\pi\chi_+(s)\,\mathrm{d}\rho(|s|)/\mathrm{d}s & 0\end{pmatrix}, & s>0 \\ -\mathbf{v}(\mathrm{i}s+0)\begin{pmatrix} 0 & -\pi\chi_-(s)\,\mathrm{d}\rho(|s|)/\mathrm{d}s \\ 0 & 0\end{pmatrix}, & s<0 \end{cases} \\ &= \mathbf{v}(\mathrm{i}s+0)\begin{pmatrix} 0 & -2\mathrm{i}\pi\chi_-(s)\,\mathrm{d}\rho(|s|)/\mathrm{d}s \\ -2\mathrm{i}\pi\chi_+(s)\,\mathrm{d}\rho(|s|)/\mathrm{d}s & 0\end{pmatrix}, \end{aligned} \tag{5.10}$$

where the derivatives are understood in the sense of distributions and, as always, $\chi_\pm$ is the characteristic function of $\mathbb{R}_\pm$.

Since $2\mathrm{i}\operatorname{Im}\mathbf{v}(\mathrm{i}s-0) = \mathbf{v}(\mathrm{i}s-0)-\overline{\mathbf{v}(\mathrm{i}s-0)} = \mathbf{v}(\mathrm{i}s-0)-\mathbf{v}(\mathrm{i}s+0)$, (5.10) yields

$$\mathbf{v}(\mathrm{i}s-0) = \mathbf{v}(\mathrm{i}s+0)\begin{pmatrix} 1 & -2\mathrm{i}\pi\chi_-(s)\,\delta(s) \\ -2\mathrm{i}\pi\chi_+(s)\,\delta(s) & 1\end{pmatrix}, \tag{5.11}$$

where

$$\delta(s) := \operatorname{sgn}(s)\sum_n c_n^2(t)\,\delta_{\kappa_n}(|s|). \tag{5.12}$$

Observe that we have reformulated the pole conditions (5.3) as a jump condition (5.11) across the imaginary line, the negative spectrum data being encoded in (5.12). Note that the jump matrix in (5.11) is not a continuous function but a distribution. The main advantage of (5.11) over (5.3) is that it readily yields a generalization to an arbitrary negative spectrum once we allow δ to be the distributional derivative of an arbitrary positive finite measure. This will be done in Section 7.

6. Quadrant-analytic vector Riemann-Hilbert problem

In this section we introduce a jump matrix with singular entries that plays the central role in our considerations.

Let Σ be a contour consisting of three lines: the real line $\mathbb{R}$ oriented from left to right, the part of the imaginary line $\mathrm{i}\mathbb{R}_+$ in the upper half-plane oriented upwards, and the part of the imaginary line $-\mathrm{i}\mathbb{R}_-$ in the lower half-plane oriented downwards. Apparently, Σ divides the complex plane into quadrants. Given a function $f(k)$ analytic on $\mathbb{C}\setminus\Sigma$, call such functions *quadrant analytic*[5], we denote by $f_\pm(k)$ nontangentional boundary values of f from the positive/negative $(\pm)$ side of Σ. Here the positive/negative side is the one that lies to the left/right from Σ as we traverse the contour in the direction of orientation.

Let $R(k)$ be as in Proposition 3.2

$$R(k,t)=R(k)\mathrm{e}^{8\mathrm{i}k^3t},$$

$\rho(k)$ a nonnegative finite measure on $\mathbb{R}_+$ and $\delta(k,t)=\mathrm{e}^{4k^3t}\mathrm{d}\rho(k)/\mathrm{d}k$ (generalized density).

Quadrant analytic vector Riemann-Hilbert problem (QARHP) Let $\boldsymbol{J}(k,t)$ be a 2×2 matrix-valued function defined on Σ as follows

$$\mathbf{J}(k,t)=\begin{cases}\mathbf{J}_R(k,t), & \operatorname{Im}k=0\\ \mathbf{J}_\rho(k,t), & \operatorname{Re}k=0\end{cases}, \tag{6.1}$$

where

$$\mathbf{J}_R(k,t):=\begin{pmatrix}1-|R(k,t)|^2 & -\overline{R(k,t)}\\ R(k,t) & 1\end{pmatrix}, k\in\mathbb{R},$$

$$\mathbf{J}_\rho(k,t):=\begin{pmatrix}1 & -2\mathrm{i}\pi\chi_-(s)\,\delta(s,t)\\ -2\mathrm{i}\pi\chi_+(s)\,\delta(s,t) & 1\end{pmatrix},\ k=\mathrm{i}s,\ s\in\mathbb{R}.$$

Find a row function $\mathbf{v}=\begin{pmatrix}\varphi & , & \psi\end{pmatrix}$ analytic in each quadrant such that:

(1) *Symmetry conditions:*

$$\overline{\mathbf{v}(\overline{k})}=\mathbf{v}(-k)=\mathbf{v}(k)\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}. \tag{6.2}$$

(2) *Jump condition:* at least in the sense of distributions

$$\mathbf{v}_+(k)=\mathbf{v}_-(k)\mathbf{J}(k,t),\ \text{for } k\in\Sigma. \tag{6.3}$$

Remark 6.1. (1) *We stated our QARHP in abstract terms but it, of course, comes from the Riemann-Hilbert formulation of the IST for the KdV equation. Thus, each component of* $\mathbf{v}=\begin{pmatrix}\varphi & , & \psi\end{pmatrix}$ *solves the Schrodinger equation (and hence depends on the spatial variable x) and the jump matrix* $\mathbf{J}(k,t)$ *depends on t (time) as it takes into account the time evolution of scattering data. Therefore,* $\mathbf{v}(k)$ *everywhere below depends on (x,t) as parameters (and consequently many other quantities) but we agree to suppress this dependence when it causes no confusion.*

[5]In fact, f may be analytic on some sets of the imaginary line. In particular, may be analytic on both $\mathbb{C}^\pm$.

(2) *The asymptotic condition (5.4) imposes* [30] *the following condition on* $\mathbf{J}$

$$\mathbf{J}(-k,t)=\begin{pmatrix}0&1\\1&0\end{pmatrix}\mathbf{J}(k,t)^{-1}\begin{pmatrix}0&1\\1&0\end{pmatrix},$$

which our $\mathbf{J}$ *clearly obeys.*

(3) *The asymptotic condition (5.4) is clearly missing which results in non-uniqueness. Indeed, a scalar multiple of a solution satisfies conditions (6.2) and (6.3). In our setting it is more convenient to restore uniqueness by imposing a slightly different from (5.4) condition.*

The following proposition is easy but crucial to our considerations. Since the variables (x,t) appear in the QARHP as parameters we leave them out.

Proposition 6.2 (Gauge transformation)**.** *If* $\mathbf{v}(k)$ *is a solution of QARHP and* μ *is any finite (signed) symmetric measure, then*

$$\widetilde{\mathbf{v}}(k)=\mathbf{v}(k)+\int\frac{W\{\mathbf{v}(k,\cdot),\psi(\cdot,\mathrm{i}s)\}}{k^2+s^2}\mathrm{d}\mu(s),\ \ k\notin\Sigma, \tag{6.4}$$

satisfies the symmetry condition (6.2) and

$$\widetilde{\mathbf{v}}_+(k)=\widetilde{\mathbf{v}}_-(k)\mathbf{J}_R(k,t)\ \ \operatorname{Im}k=0 \text{ (jump across the real line),}$$

but not across the imaginary line.

Proof. Check the symmetry conditions (6.2):

$$\begin{aligned}\overline{\widetilde{\mathbf{v}}(\overline{k})}&=\overline{\mathbf{v}(\overline{k})}+\int\frac{W\left\{\overline{\mathbf{v}(\overline{k})},\psi\right\}}{k^2+s^2}\mathrm{d}\mu\\&=\mathbf{v}(-k)+\int\frac{W\{\mathbf{v}(-k),\psi\}}{k^2+s^2}\mathrm{d}\mu=\widetilde{\mathbf{v}}(-k)\\&=\mathbf{v}(k)\begin{pmatrix}0&1\\1&0\end{pmatrix}+\int\frac{W\left\{\mathbf{v}(k)\begin{pmatrix}0&1\\1&0\end{pmatrix},\psi\right\}}{k^2+s^2}\mathrm{d}\mu\\&=\widetilde{\mathbf{v}}(k)\begin{pmatrix}0&1\\1&0\end{pmatrix}\end{aligned}$$

and both symmetry conditions follow. Check the jump across condition (6.3) across the real line. Suppressing the variables, we have

$$\begin{aligned}\widetilde{\mathbf{v}}_+&=\mathbf{v}_++\int\frac{W\{\mathbf{v}_+,\psi\}}{k^2+s^2}\mathrm{d}\mu=\mathbf{v}_-(k)\,\mathbf{J}_R+\int\frac{W\{\mathbf{v}_-\mathbf{J}_R,\psi\}}{k^2+s^2}\mathrm{d}\mu\\&=\left(\mathbf{v}_-+\int\frac{W\{\mathbf{v}_-,\psi\}}{k^2+s^2}\mathrm{d}\mu\right)\mathbf{J}_R=\widetilde{\mathbf{v}}_-\mathbf{J}_R\end{aligned}$$

and the jump condition follows. □

Proposition 6.2 says that the transformation $\mathbf{v}_+\to\widetilde{\mathbf{v}}_+$ preserves the jump condition across the real line but not across the imaginary line. By choosing μ we will be able to modify the jump matrix $\mathbf{J}_\rho$ in (6.1) in nearly unrestricted way (without altering the reflection coefficient).

7. Continuous binary Darboux transformation

In this section we state and prove a Riemann-Hilbert version of the continuous binary Darboux transformation. It is of course directly related to Theorem 4.1 but we hope it deserves special attention. Through this section all statements and proofs admit time dependent situation but since t appears as just a parameter we drop it from the list of variables emphasizing that the material of this section need not be considered in the KdV context.

Theorem 7.1 (Continuous Darboux transformation)**.** *Let $q(x)$ be a step-type potential, $\psi(x,k)$ a right Jost solution, and for all real x*

$$\mathbf{v}(x,k) = \begin{pmatrix} \varphi(x,k) & , & \psi(x,k) \end{pmatrix}$$

solve the QARHP with the jump matrix $(\mathbf{J}_R, \mathbf{J}_\rho)$ given by (6.1). Let σ be a (signed) finite measure supported on $\mathbb{R}_+$ such that the Fredholm integral equation

$$y(\alpha,x) + \int K(\alpha,s,x)\, y(s,x)\, \mathrm{d}\sigma(s) = \psi(x,\mathrm{i}\alpha), \quad \alpha \in \operatorname{Supp}\sigma, \tag{7.1}$$

with the kernel

$$K(\alpha,s,x) = \int_x^\infty \psi(z,\mathrm{i}\alpha)\,\psi(z,\mathrm{i}s)\,\mathrm{d}z$$

has a unique solution in $L^2(\mathrm{d}\sigma)$. Then

$$\begin{aligned}\widetilde{\mathbf{v}}(x,k) &= \begin{pmatrix} \widetilde{\varphi}(x,k) & , & \widetilde{\psi}(x,k) \end{pmatrix} \\ &= \mathbf{v}(x,k) + \int W\{\mathbf{v}(x,k), \psi(x,\mathrm{i}s)\}\frac{y(s,x)\,\mathrm{d}\sigma(s)}{k^2+s^2}\end{aligned} \tag{7.2}$$

solves the QARHP with the jump matrix $(\mathbf{J}_R, \mathbf{J}_{\rho+\sigma})$. Moreover $\widetilde{\psi}(x,k)$ is the right Jost solution corresponding to the potential

$$\begin{aligned}\widetilde{q}(x,t) &= q(x,t) \\ &+ 2\left[\int \widetilde{\psi}(x;\mathrm{i}s)\,\psi(x,\mathrm{i}s)\,\mathrm{d}\sigma(s)\right]^2 + 4\int \widetilde{\psi}(x,\mathrm{i}s)\,\psi'(x,\mathrm{i}s)\,\mathrm{d}\sigma(s),\end{aligned} \tag{7.3}$$

and

$$\widetilde{\psi}(x,\mathrm{i}\alpha) = y(\alpha,x), \quad \alpha \in \operatorname{Supp}\sigma. \tag{7.4}$$

Note that the requirement that ψ is a Jost solution plays the role of the normalization condition missing in the QARHP. We also observe that since the Jost solution is real for $\operatorname{Re} k = 0$, the right hand side of (7.3) is also real. And finally it is worth mentioning that Condition (6.1) implies that on the real line $\varphi(x,k)$ and $\psi(x,k)$ are related by the basic scattering identity (3.9) and thus $\varphi(x,k)$ is unique for a given the latter $\psi(x,k)$.

Proof. Consider the gauge transform (6.4). By Proposition 6.2 the reflection coefficient R is then preserved and it remains to find a measure $\mathrm{d}\mu(s,x)$ in (6.4) that produces the desirable jump matrix across the imaginary line. Due to the symmetry condition (6.2) we can assume that $\operatorname{Im} k > 0$. For the time being we suppress the dependence on x whenever it leads to no confusion.

Rewriting (6.4) component-wise we have

$$\widetilde{\varphi}(k)=\varphi(k)+\int\frac{W\{\varphi(k),\psi(\mathrm{i}s)\}}{k^2+s^2}\mathrm{d}\mu(s), \tag{7.5}$$

$$\widetilde{\psi}(k)=\psi(k)+\int\frac{W\{\psi(k),\psi(\mathrm{i}s)\}}{k^2+s^2}\mathrm{d}\mu(s). \tag{7.6}$$

Since the new pair $\begin{pmatrix}\widetilde{\varphi} & , & \widetilde{\psi}\end{pmatrix}$ must satisfy the jump condition across $\mathrm{i}\mathbb{R}$, we have

$$\begin{aligned}&\begin{pmatrix}\widetilde{\varphi}(\mathrm{i}s-0) & , & \widetilde{\psi}(\mathrm{i}s-0)\end{pmatrix}\\ &=\begin{pmatrix}\widetilde{\varphi}(\mathrm{i}s+0) & , & \widetilde{\psi}(\mathrm{i}s+0)\end{pmatrix}\begin{pmatrix}1 & -2\mathrm{i}\pi\chi_-(s)\widetilde{\delta}(s)\\ -2\mathrm{i}\pi\chi_+(s)\widetilde{\delta}(s) & 1\end{pmatrix},\end{aligned}$$

where $\widetilde{\delta}(s)$ is the density of the perturbed measure $\widetilde{\rho}=\rho+\sigma$, we conclude that $\widetilde{\psi}(k)$ is analytic for $\operatorname{Im}k>0$ and

$$\operatorname{Im}\widetilde{\psi}(\mathrm{i}\alpha-0)=0,\quad \operatorname{Im}\widetilde{\varphi}(\mathrm{i}\alpha-0)=-\pi\widetilde{\delta}(\alpha)\widetilde{\psi}(\mathrm{i}\alpha).$$

It follows from (7.5) that

$$\begin{aligned}&\operatorname{Im}\widetilde{\varphi}(\mathrm{i}\alpha-0)-\operatorname{Im}\varphi(\mathrm{i}\alpha-0) \qquad (7.7)\\ &=\lim_{\varepsilon\to0}\operatorname{Im}\int\frac{W\{\varphi(\mathrm{i}\alpha-\varepsilon),\psi(\mathrm{i}s)\}}{(\mathrm{i}\alpha-\varepsilon)^2+s^2}\mathrm{d}\mu(s)\\ &=\lim_{\varepsilon\to0}\int W\{\operatorname{Im}\varphi(\mathrm{i}\alpha-\varepsilon),\psi(\mathrm{i}s)\}\operatorname{Re}\frac{1}{(\mathrm{i}\alpha-\varepsilon)^2+s^2}\mathrm{d}\mu(s)\\ &+\lim_{\varepsilon\to0}\int W\{\operatorname{Re}\varphi(\mathrm{i}\alpha-\varepsilon),\psi(\mathrm{i}s)\}\operatorname{Im}\frac{1}{(\mathrm{i}\alpha-\varepsilon)^2+s^2}\mathrm{d}\mu(s)\\ &=I_1+I_2.\end{aligned}$$

It follows from the jump condition for (φ,ψ) that

$$\operatorname{Im}\varphi(\mathrm{i}\alpha-\varepsilon)=-\pi\delta(\alpha)\psi(\mathrm{i}\alpha). \tag{7.8}$$

Therefore, for I_1 we have

$$I_1=-\pi\delta(\alpha)\lim_{\varepsilon\to0}\int W\{\psi(\mathrm{i}\alpha),\psi(\mathrm{i}s)\}\operatorname{Re}\frac{1}{(\mathrm{i}\alpha-\varepsilon)^2+s^2}\mathrm{d}\mu(s).$$

Recall the following Wronskian identity: if f_λ is a solution to the Schrödinger equation $-f''+q(x)f=\lambda^2 f$ then

$$W'\{f_\lambda,f_\nu\}=\left(\lambda^2-\nu^2\right)f_\lambda f_\nu \text{ for any } \lambda,\nu. \tag{7.9}$$

Observe that if f_λ, f_ν decay sufficiently fast at $+\infty$, then (7.9) implies

$$\frac{W\{f_\lambda,f_\nu\}}{\lambda^2-\nu^2}=-\int_x^\infty f_\lambda(s)f_\nu(s)\,\mathrm{d}s. \tag{7.10}$$

Since due to (7.10)

$$\begin{aligned}W\{\psi(\mathrm{i}\alpha),\psi(\mathrm{i}s)\}&=-\left(s^2-\alpha^2\right)\int_x^\infty\psi(z,\mathrm{i}\alpha)\psi(z,\mathrm{i}s)\,\mathrm{d}z \qquad (7.11)\\ &=-\left(s^2-\alpha^2\right)K(\alpha,s),\end{aligned}$$

where we have denoted (suppressing x as before)

$$K(\alpha, s) = K(\alpha, s; x) = \int_x^\infty \psi(z, \mathrm{i}\alpha)\,\psi(z, \mathrm{i}s)\,\mathrm{d}z,$$

the last equation can be continued

$$I_1 = \pi\delta(\alpha)\lim_{\varepsilon\to 0}\int \left(s^2-\alpha^2\right)\operatorname{Re}\frac{1}{(\mathrm{i}\alpha-\varepsilon)^2+s^2}K(\alpha, s)\,\mathrm{d}\mu(s).$$

Observe that

$$\begin{aligned}\operatorname{Re}\frac{1}{(\mathrm{i}\alpha-\varepsilon)^2+s^2} &= \frac{1}{2s}\operatorname{Re}\left(\frac{1}{s-\alpha-\mathrm{i}\varepsilon}+\frac{1}{s+\alpha+\mathrm{i}\varepsilon}\right)\\ &= \frac{1}{2s}\left[\frac{s-\alpha}{(s-\alpha)^2+\varepsilon^2}+\frac{s+\alpha}{(s+\alpha)^2+\varepsilon^2}\right],\end{aligned}$$

and hence

$$\begin{aligned}&\left(s^2-\alpha^2\right)\operatorname{Re}\frac{1}{(\mathrm{i}\alpha-\varepsilon)^2+s^2}\\ &= \frac{1}{2s}\left[\frac{(s-\alpha)^2(s+\alpha)}{(s-\alpha)^2+\varepsilon^2}+\frac{(s+\alpha)^2(s-\alpha)}{(s+\alpha)^2+\varepsilon^2}\right]\\ &\to \frac{1}{2s}\left[(s+\alpha)+(s-\alpha)\right] = 1, \quad \varepsilon\to 0 \quad \text{uniformly.}\end{aligned}$$

Therefore,

$$I_1 = \pi\delta(\alpha)\int K(\alpha, s)\,\mathrm{d}\mu(s). \tag{7.12}$$

Turn to I_2. We have

$$\begin{aligned}\operatorname{Im}\frac{1}{(\mathrm{i}\alpha-\varepsilon)^2+s^2} &= \frac{1}{2s}\operatorname{Im}\left(\frac{1}{s-\alpha-\mathrm{i}\varepsilon}+\frac{1}{s+\alpha+\mathrm{i}\varepsilon}\right)\\ &= \frac{1}{2s}\left[\frac{\varepsilon}{(s-\alpha)^2+\varepsilon^2}-\frac{\varepsilon}{(s+\alpha)^2+\varepsilon^2}\right]\\ &= \frac{\pi}{2s}P_{\alpha+\mathrm{i}\varepsilon}(s)-\frac{1}{2s}\frac{\varepsilon}{(s+\alpha)^2+\varepsilon^2},\end{aligned}$$

where

$$P_{x+\mathrm{i}y}(t) = \frac{1}{\pi}\frac{y}{(t-x)^2+y^2}$$

is the the Poisson kernel. Recall the classical fact (see, e.g. [36])

$$\int \mathrm{d}x g(x)\lim_{y\to 0}\int P_{x+\mathrm{i}y}(t)\,\mathrm{d}\mu(t) = \int g(x)\,\mathrm{d}\mu(x)$$

or

$$\mathrm{d}\mu_y(x) = \left(\int P_{x+\mathrm{i}y}(t)\,\mathrm{d}\mu(t)\right)\mathrm{d}x \to \mathrm{d}\mu(x), y\to 0, \tag{7.13}$$

in the weak* topology. The second term goes to zero uniformly as $\varepsilon \to 0$ and we get

$$\begin{aligned} I_2 &= \lim_{\varepsilon \to 0} \int W\left\{\operatorname{Re}\varphi\left(\mathrm{i}\alpha - \varepsilon\right), \psi\left(\mathrm{i}s\right)\right\} \operatorname{Im} \frac{1}{\left(\mathrm{i}\alpha - \varepsilon\right)^2 + s^2} \mathrm{d}\mu\left(s\right) \\ &= \lim_{\varepsilon \to 0} \int W\left\{\operatorname{Re}\varphi\left(\mathrm{i}\alpha - 0\right), \psi\left(\mathrm{i}s\right)\right\} \frac{\pi}{2s} P_{\alpha + \mathrm{i}\varepsilon}\left(s\right) \mathrm{d}\mu\left(s\right) \\ &= \pi \int \frac{W\left\{\operatorname{Re}\varphi\left(\mathrm{i}\alpha - 0\right), \psi\left(\mathrm{i}\alpha\right)\right\}}{2\alpha} \mathrm{d}\mu\left(\alpha\right). \end{aligned}$$

Here we have used (7.13) to pass to the limit.

It follows from (6.3) that

$$\varphi\left(k + \mathrm{i}0\right) = \overline{\psi\left(k + \mathrm{i}0\right)} + R\left(k\right) \psi\left(k + \mathrm{i}0\right)$$

and hence, since ψ is the right Jost solution,

$$W\left\{\varphi\left(k + \mathrm{i}0\right), \psi\left(k + \mathrm{i}0\right)\right\} = W\left\{\overline{\psi\left(k + \mathrm{i}0\right)}, \psi\left(k + \mathrm{i}0\right)\right\} = 2\mathrm{i}k.$$

Thus $W\left\{\varphi\left(k\right), \psi\left(k\right)\right\} = 2\mathrm{i}k$ also for $\operatorname{Im} k > 0$ and in particular

$$\begin{aligned} &W\left\{\operatorname{Re}\varphi\left(\mathrm{i}\alpha - 0\right), \psi\left(\mathrm{i}\alpha\right)\right\} \\ &= \operatorname{Re} W\left\{\varphi\left(\mathrm{i}\alpha - 0\right), \psi\left(\mathrm{i}\alpha\right)\right\} = -2\alpha. \end{aligned}$$

It follow then that in the sense of distributions

$$I_2 = -\pi \mu'\left(\alpha\right). \tag{7.14}$$

Substituting (7.12) and (7.14) into (7.7) yields

$$\begin{aligned} \operatorname{Im} \widetilde{\varphi}\left(\mathrm{i}\alpha - 0\right) &= \operatorname{Im} \varphi\left(\mathrm{i}\alpha - 0\right) + I_1 + I_2 \\ &= -\delta\left(\alpha\right) \psi\left(\mathrm{i}\alpha\right) + \pi \delta\left(\alpha\right) \int K\left(\alpha, s\right) \mathrm{d}\mu\left(s\right) - \pi \mu'\left(\alpha\right) \\ &= -\pi \widetilde{\delta}\left(\alpha\right) \widetilde{\psi}\left(\mathrm{i}\alpha\right). \end{aligned}$$

Here we have taken (7.8) into account. It follows that

$$\delta\left(\alpha\right) \left[\psi\left(\mathrm{i}\alpha\right) - \int K\left(\alpha, s\right) \mathrm{d}\mu\left(s\right)\right] + \mu'\left(\alpha\right) = \widetilde{\delta}\left(\alpha\right) \widetilde{\psi}\left(\mathrm{i}\alpha\right). \tag{7.15}$$

Evaluate now $\widetilde{\psi}\left(\mathrm{i}\alpha\right)$. From (7.6) and (7.11) we see that

$$\begin{aligned} \widetilde{\psi}\left(k\right) &= \psi\left(k\right) + \int \frac{W\left\{\psi\left(k\right), \psi\left(\mathrm{i}s\right)\right\}}{k^2 + s^2} \mathrm{d}\mu\left(s\right) \\ &= \psi\left(k\right) - \int K\left(\alpha, s\right) \mathrm{d}\mu\left(s\right) \end{aligned}$$

and hence

$$\widetilde{\psi}\left(\mathrm{i}\alpha\right) = \psi\left(\mathrm{i}\alpha\right) - \int K\left(\alpha, s\right) \mathrm{d}\mu\left(s\right). \tag{7.16}$$

Substituting this into (7.15) one immediately obtains

$$\begin{aligned} &\delta\left(\alpha\right) \left[\psi\left(\mathrm{i}\alpha\right) - \int K\left(\alpha, s\right) \mathrm{d}\mu\left(s\right)\right] + \mu'\left(\alpha\right) \\ &= \widetilde{\delta}\left(\alpha\right) \left\{\psi\left(\mathrm{i}\alpha\right) - \int K\left(\alpha, s\right) \mathrm{d}\mu\left(s\right)\right\}, \end{aligned}$$

which, recalling that $\sigma = \widetilde{\rho} - \rho$, can be rearranged as

$$\sigma'(\alpha)\left\{\psi(\mathrm{i}\alpha) - \int K(\alpha, s)\,\mathrm{d}\mu(s)\right\} = \mu'(\alpha),$$

or in terms of measures

$$\left\{\psi(\mathrm{i}\alpha) - \int K(\alpha, s)\,\mathrm{d}\mu(s)\right\}\mathrm{d}\sigma(s) = \mathrm{d}\mu(\alpha),$$

which is an integral equation on the measure μ. Take $\mathrm{d}\mu$ to be absolutely continuous with respect to the measure $\mathrm{d}\sigma$ and let $y = \mathrm{d}\mu/\mathrm{d}\sigma$ be the Radon-Nikodym derivative. By the Radon-Nikodym theorem then

$$\psi(\mathrm{i}\alpha) - \int K(\alpha, s)\,y(s)\,\mathrm{d}\sigma(s) = y(\alpha)$$

and we finally have

$$y(\alpha, x) + \int K(\alpha, s, x)\,y(s, x)\,\mathrm{d}\sigma(s) = \psi(x, \mathrm{i}\alpha).$$

Thus we arrive at the Fredholm integral equation

$$y + \mathbb{K}y = \psi, \tag{7.17}$$

where $\mathbb{K}$ is the integral operator with the kernel $K(\alpha, s; x)$ with respect to the measure σ.

It remains to show (7.4). This immediately follows from (7.16) and (7.17). Indeed,

$$\begin{aligned}\widetilde{\psi}(\mathrm{i}\alpha) &= \psi(\mathrm{i}\alpha) - \int K(\alpha, s)\,\mathrm{d}\mu(s)\\ &= \psi(\mathrm{i}\alpha) - \int K(\alpha, s)\,y(s)\,\mathrm{d}\sigma(s)\\ &= \psi - \mathbb{K}y = y.\end{aligned}$$

The proof that $\widetilde{\psi}$ is a right Jost solution and the representation (7.3) for $\widetilde{q}$ will be given in Section 10. $\square$

8. An integral operator and its discretization

In this section we study the trace class integral operator $\mathbb{K}$ arising in the previous section and approximate it with a sequence of finite matrices. It is convenient to do so in independent terms. Let μ be a non-negative finite Borel measure on the real line and $L^2(\mathrm{d}\mu)$ the real Hilbert space with the inner product $\langle f, g\rangle = \int f(x)\,g(x)\mathrm{d}\mu(x)$. Let $g(x, s)$ be a real continuous function for $s \in S$, where S is an interval (finite or infinite) such that

$$|||g|||^2 := \int_S \int g(x, s)^2\,\mathrm{d}\mu(x)\,\mathrm{d}s < \infty. \tag{8.1}$$

Define a family of rank one operators $\mathbb{G}(s)$ on $L^2(\mathrm{d}\mu)$ by

$$(\mathbb{G}(s)\,f)(x) = \langle f, g(\cdot, s)\rangle\,g(x, s).$$

Clearly, $\mathbb{G}$ is positive and ($||\cdot||_2$ stands for the Hilbert-Schmidt norm)

$$\operatorname{tr}\mathbb{G}(s) = ||\mathbb{G}(s)||_2 = ||g(\cdot, s)||^2 = \int g(x, s)^2\,\mathrm{d}\mu(x). \tag{8.2}$$

Consider an operator defined by

$$\mathbb{K}=\int_S \mathbb{G}(s)\,\mathrm{d}s.$$

It is an integral operator

$$(\mathbb{K}f)(x)=\int K(x,y)\,f(y)\,\mathrm{d}\mu(y)$$

on $L^2(\mathrm{d}\mu)$ with the kernel

$$K(x,y)=\int_S g(x,s)\,g(y,s)\,\mathrm{d}s.$$

It follows from (8.1) and (8.2) that

$$\|\mathbb{K}\|_2\le\int_S\|\mathbb{G}(s)\|_2\,\mathrm{d}s=\int_S\int g(x,s)^2\,\mathrm{d}\mu(x)\,\mathrm{d}s=|||g|||^2<\infty$$

and hence the operator $\mathbb{K}$ is Hilbert-Schmidt on $L^2(\mathrm{d}\mu)$, positive, it is also trace class, and

$$\operatorname{tr}\mathbb{K}=\int_S\int g(x,s)^2\,\mathrm{d}\mu(x)\,\mathrm{d}s=|||g|||^2.$$

Discretize $\mathbb{K}$ as follows. Let I be a finite interval containing $\operatorname{Supp}\mu$ and $(I_n)_{n=1}^N$ be a finite partition of I. In each I_n pick up an interior point x_n that is also in $\operatorname{Supp}\mu$. Take a piece-wise constant approximation g_N of g

$$g_N(x,s)=\sum_{n=1}^N g(x_n,s)\,\chi_{I_n}(x)$$

and consider the integral operator

$$(\mathbb{K}_N f)(x)=\int K_N(x,y)\,f(y)\,\mathrm{d}\mu(y)$$

on $L^2(\mathrm{d}\mu)$ with the kernel

$$K_N(x,y)=\int_S g_N(x,s)\,g_N(y,s)\,\mathrm{d}s.$$

Proposition 8.1. *If $|||g-g_N|||\to 0, N\to\infty$, then $\mathbb{K}_N\to\mathbb{K}$ in the trace norm.*

The proof is given in the Appendix. We show that $\mathbb{K}_N$ can be realized as the $N\times N$ matrix

$$\boldsymbol{K}_N=\left(c_n^2\int_S g_m(s)\,g_n(s)\,\mathrm{d}s\right)_{1\le m,n\le N},$$

where $g_n(s):=g(x_n,s)$ and $c_n:=\mu(I_n)^{1/2}$, as follows. Identify a simple function

$$f_N(x)=\sum_{n=1}^N f_n\chi_{I_n}(x)$$

with an N column $(f_n):=\boldsymbol{f}_N$. We show that

$$\mathbb{K}_N f_N|_{I_n}=(\boldsymbol{K}_N\boldsymbol{f}_N)_n\,,$$

where the subscript n denotes the $n$the component of a column. Indeed,

$$\begin{aligned}(\mathbb{K}_N f_N)(x) &= \int K_N(x,y)\, f_N(y)\,\mathrm{d}\mu(y)\\ &= \int \left[\int_S g_N(x,s)\, g_N(y,s)\,\mathrm{d}s\right] f_N(y)\,\mathrm{d}\mu(y)\\ &= \int_S g_N(x,s)\left[\int g_N(y,s)\, f_N(y)\,\mathrm{d}\mu(y)\right]\mathrm{d}s\\ &= \int_S g_N(x,s)\left[\sum_{m=1}^{N} g_N(x_m,s)\, f_m \mu(I_m)\right]\mathrm{d}s\\ &= \int_S g_N(x,s)\left[\sum_{m=1}^{N} c_m^2 g_m(s)\, f_m\right]\mathrm{d}s\\ &= \sum_{m=1}^{N} c_m^2 \left[\int_S g_N(x,s)\, g_m(s)\,\mathrm{d}s\right] f_m.\end{aligned}$$

It follows that for the restriction to I_n we have

$$\begin{aligned}\mathbb{K}_N f_N|_{I_n} &= \sum_{m=1}^{N} c_m^2 \left[\int_S g_n(s)\, g_m(s)\,\mathrm{d}s\right] f_m\\ &= (\boldsymbol{K}_N \boldsymbol{f}_N)_n\end{aligned}$$

as desired.

Thus we have shown that $(I+\mathbb{K}_N)^{-1}$ is well-defined on $L^2(\mathrm{d}\mu)$ and the integral equation

$$(I+\mathbb{K}_N)\, f = g_N$$

has a unique $L^2(\mathrm{d}\mu)$ solution

$$f_N = (I+\mathbb{K}_N)^{-1} g_N,$$

which is due to Proposition 8.1 converges in $L^2(\mathrm{d}\mu)$ to some f.

9. Abstract log-determinant formula

In this section we derive a log-determinant formula needed in the proof of Theorem 4.1. It is convenient to do so in independent terms.

Proposition 9.1. *Let $\mathbb{A}(x)$ be a self-adjoint trace class operator-valued function on a real Hilbert space $\mathfrak{H}$ such that*

$$\partial_x \mathbb{A}(x) = -\langle \cdot, a(x)\rangle\, a(x),\ a \in \mathfrak{H}$$

is rank-one. If $I+\mathbb{A}(x)$ is nonsingular then

$$\begin{aligned}\partial_x^2 \log\det[I+\mathbb{A}(x)] = &-\left\langle a(x), [I+\mathbb{A}(x)]^{-1} a(x)\right\rangle^2 \qquad (9.1)\\ &- 2\left\langle \partial_x a(x), [I+\mathbb{A}(x)]^{-1} a(x)\right\rangle.\end{aligned}$$

Proof. We suppress dependence on x. We base our proof on the followings well-known formulas (see e.g. [53])

$$\log\det(I+\mathbb{A}) = \operatorname{tr}\log(I+\mathbb{A}), \qquad (9.2)$$

$$\partial \operatorname{tr}\log(I+\mathbb{A}) = \operatorname{tr}(I+\mathbb{A})^{-1}\partial\mathbb{A}, \qquad (9.3)$$

$$\operatorname{tr}\langle\cdot,f\rangle g=\langle g,f\rangle. \tag{9.4}$$

$$\partial(I+\mathbb{A})^{-1}=-(I+\mathbb{A})^{-1}\partial\mathbb{A}(I+\mathbb{A})^{-1}. \tag{9.5}$$

By (9.2) and (9.3) we have

$$\begin{aligned}\partial^2\log\det(I+\mathbb{A})&=\partial\operatorname{tr}(I+\mathbb{A})^{-1}\partial\mathbb{A}\\&=\operatorname{tr}(I+\mathbb{A})^{-1}\partial\mathbb{A}.\end{aligned}$$

But $(I+\mathbb{A})^{-1}\partial\mathbb{A}$ is a rank one operator and hence by (9.4)

$$\begin{aligned}\operatorname{tr}(I+\mathbb{A})^{-1}\partial\mathbb{A}&=-\operatorname{tr}\langle\cdot,a\rangle(I+\mathbb{A})^{-1}a\\&=-\left\langle(I+\mathbb{A})^{-1}a,a\right\rangle.\end{aligned}$$

Differentiating this equation one more time, by (9.5) we have

$$\begin{aligned}\partial\operatorname{tr}(I+\mathbb{A})^{-1}\partial\mathbb{A}=-\partial\left\langle(I+\mathbb{A})^{-1}a,a\right\rangle&=-\left\langle\partial(I+\mathbb{A})^{-1}a,a\right\rangle\\&-\left\langle(I+\mathbb{A})^{-1}\partial a,a\right\rangle-\left\langle(I+\mathbb{A})^{-1}a,\partial a\right\rangle\\&=\left\langle(I+\mathbb{A})^{-1}\partial\mathbb{A}(I+\mathbb{A})^{-1},a\right\rangle-2\left\langle(I+\mathbb{A})^{-1}a,\partial a\right\rangle\\&=\left\langle\partial\mathbb{A}(I+\mathbb{A})^{-1},(I+\mathbb{A})^{-1}a\right\rangle-2\left\langle(I+\mathbb{A})^{-1}a,\partial a\right\rangle.\end{aligned} \tag{9.6}$$

But $\partial\mathbb{A}(I+\mathbb{A})^{-1}$ is a rank one operator and therefore

$$\partial\mathbb{A}(I+\mathbb{A})^{-1}\cdot=-\left\langle\cdot,(I+\mathbb{A})^{-1}a\right\rangle a.$$

Thus, we have

$$\begin{aligned}\left\langle\partial\mathbb{A}(I+\mathbb{A})^{-1}a,(I+\mathbb{A})^{-1}a\right\rangle&=-\left\langle a,(I+\mathbb{A})^{-1}a\right\rangle\left\langle a,(I+\mathbb{A})^{-1}a\right\rangle\\&=-\left\langle a,(I+\mathbb{A})^{-1}a\right\rangle^2.\end{aligned}$$

Substituting this into (9.6) finally yields (9.1). □

10. Proof of the main theorem

The proof of Theorem 4.1 amounts to combining the ingredients prepared above and is based in part on limiting arguments. We start out from the following statement.

Proposition 10.1 (Adding/removing bound states). *Let $q(x,t)$ be a step-type KdV solution with the scattering data $S_q=\{R,\mathrm{d}\rho\}$ and $\psi(x,t,k)$ the right Jost solution. Fix the discrete measure*

$$\mathrm{d}\sigma(k)=\sum_{n=1}^{N}c_n^2\delta_{\kappa_n}(k)\,\mathrm{d}k,\quad N<\infty,$$

with some positive c_n^2, κ_n, and introduce the $N\times N$ matrix function $\mathbf{K}(x,t)$ with entries

$$K_{mn}(x,t)=c_n^2e^{8\kappa_n^3t}\int_x^\infty\psi(s,t,\mathrm{i}\kappa_m)\,\psi(s,t,\mathrm{i}\kappa_n)\,\mathrm{d}s.$$

Suppose that $d\rho$ is discrete with a finitely many pure points then

(1) *the matrix* $I+\mathbf{K}(x,t)$ *is nonsingular,*
$$q_{\sigma}(x,t)=q(x,t)-2\partial_x^2\log\det\{I+\mathbf{K}(x,t)\} \tag{10.1}$$
is also a step-type KdV solution with the scattering data $\{R,\mathrm{d}\rho+\mathrm{d}\sigma\}$*, and*
$$\psi_{\sigma}(x,t,k)=\psi(x,t,k) \tag{10.2}$$
$$-\sum_{n=1}^{N}c_n^2e^{8\kappa_n^3t}y_n(x,t)\int_x^{\infty}\psi(s,t,k)\,\psi(s,t,\mathrm{i}\kappa_n)\,\mathrm{d}s$$
is the associated right Jost solution, where the column $\mathbf{y}=(y_n)$ *is given by*
$$\mathbf{y}(x,t)=\{I+\mathbf{K}(x,t)\}^{-1}\psi(x,t),\quad \psi(x,t):=(\psi(x,t,\mathrm{i}\kappa_n));$$
(2) *If* $d\sigma\leq d\rho$*, then* $I-\mathbf{K}(x,t)$ *is nonsingular,*
$$q_{-\sigma}(x,t)=q(x,t)-2\partial_x^2\log\det\{I-\mathbf{K}(x,t)\} \tag{10.3}$$
is also a step-type KdV solution with the scattering data $\{R,\mathrm{d}\rho-\mathrm{d}\sigma\}$*, and*
$$\psi_{-\sigma}(x,t,k)=\psi(x,t,k) \tag{10.4}$$
$$+\sum_{n=1}^{N}c_n^2e^{8\kappa_n^3t}y_n(x,t)\int_x^{\infty}\psi(s,t,k)\,\psi(s,t,\mathrm{i}\kappa_n)\,\mathrm{d}s$$
is the associated right Jost solution, where the column $\mathbf{y}=(y_n)$ *is given by*
$$\mathbf{y}(x,t)=\{I-\mathbf{K}(x,t)\}^{-1}\psi(x,t),\quad \psi(x,t):=(\psi(x,t,\mathrm{i}\kappa_n));$$
(3) *The binary Darboux transform is invertible in the following sense*
$$(q_{\sigma})_{-\sigma}=q,\quad (\psi_{\sigma})_{-\sigma}=\psi.$$

This statement is not new. Part 1 follows from [24, Theorem 4.1] where it is proven in the most general spectral situation and for arbitrary Sturm-Liouville operators but not in the IST context. Part 2 (removing eigenvalues) and 3 are not explicitly addressed therein but it can of course be done along the same lines readily suggested in [24]. For short-range q's both parts are proven in our [50] by completely different methods and in the IST context. The main difference in the approaches is that in [24] is obtained by adding one eigenvalue at a time while in [50] all eigenvalues are added simultaneously. The formulation of Proposition 10.1 is, however, new.

Note that Proposition 10.1 can actually be independently derived from [50, Theorem 3.1] by the following limiting arguments. Consider the sequence $q_n=q\chi_{(-n,\infty)}$. As is proven in [28], $S_{q_n}\to S_q$ weakly and the corresponding sequence $q_n(x,t)$ converges point-wise to some $q(x,t)$ for every $t>0$. An explicit formula for $q(x,t)$ via S_q is given in [28] in terms of Hankel operators. The corresponding sequence $\psi_n(k;x,t)$ converges uniformly to $\psi(x,t;k)$ on compact in $\operatorname{Im}k>0$ (this is a general classical fact that holds even for Weyl solutions). Part 3 for q_n follows directly from Marchenko's characterization of the inverse scattering problem for short-range potentials [43]. Indeed, his characterization of the scattering data $S_q=\{R,\mathrm{d}\rho\}$ imposes no other condition on $\mathrm{d}\rho$ but $N<\infty$. The passage to the limit as $n\to\infty$ should then be in order.

Observe that Proposition 10.1 is a particular case of Theorem 4.1. However an independent proof of Theorem 4.1 involves cumbersome technicalities which can be avoided by taking limits in already known results.

We now proceed to the proof of Theorem 4.1. As before, since (x,t) appear as parameters, we may drop them from the list of variable whenever it is convenient and leads to no confusion. We recall that prime stands for the derivative in x (no derivatives in other variable appear). It is sufficiently to show that Theorem 4.1 holds for Case 1: $S_q=\{R,0\}$, $\mathrm{d}\sigma\geq 0$ and Case 2: $S_q=\{R,\mathrm{d}\rho\}$, $-\mathrm{d}\sigma\leq 0$ where $\mathrm{d}\sigma$ is a restriction of the (non-negative) measure $\mathrm{d}\rho$, as the general case is a combination of the two. We concentrate on the proof for Case 1 since Case 2 relies on the very same arguments.

Case 1 (adding negative spectrum). Discretize the measure $\mathrm{d}\rho$ in (4.1) as is done in Section 8. More specifically, take

$$\mathrm{d}\rho_N\left(k\right)=\sum_{n=1}^{N}\rho\left(I_n\right)\delta_{\kappa_n}\left(k\right)\mathrm{d}k,\quad N<\infty,$$

where the partition (I_n) is chosen the same way as in Section 8. Proposition 10.1, part 1, then applies with $S_q=\{R,0\}$, $\mathrm{d}\sigma=\mathrm{d}\rho_N\geq 0$, and $\mathbf{K}=\mathbf{K}_N$, where the entries are given by

$$\left(\mathbf{K}_N\right)_{mn}=\rho\left(I_n\right)e^{8\kappa_n^3t}\int_x^\infty\psi\left(s,t,\mathrm{i}\kappa_m\right)\psi\left(s,t,\mathrm{i}\kappa_n\right)\mathrm{d}s.$$

Then

$$q_{\rho_N}\left(x,t\right)=q\left(x,t\right)-2\partial_x^2\log\det\left\{I+\mathbf{K}_N\left(x,t\right)\right\}$$

is a step-type KdV solution. By Proposition 9.1 we have

$$\begin{aligned}q_{\rho_N}\left(x,t\right)=q\left(x,t\right)&\\&+2\left[\int\psi_{\rho_N}\left(x,t;\mathrm{i}s\right)\psi\left(x,t;\mathrm{i}s\right)\mathrm{d}\rho_N\left(s\right)\right]^2\\&+4\int\psi_{\rho_N}\left(x,t;\mathrm{i}s\right)\psi'\left(x,t;\mathrm{i}s\right)\mathrm{d}\rho_N\left(s\right),\end{aligned}\tag{10.5}$$

where

$$\psi_{\rho_N}\left(x,t,k\right)=\psi\left(x,t,k\right)-\sum_{n=1}^{N}\rho\left(I_n\right)e^{8\kappa_n^3t}y_n\left(x,t\right)\int_x^\infty\psi\left(s,t,k\right)\psi\left(s,t,\mathrm{i}\kappa_n\right)\mathrm{d}s,\tag{10.6}$$

and

$$\left(y_n\right)=\left(I+\mathbf{K}_N\right)^{-1}\boldsymbol{\psi},\quad\boldsymbol{\psi}:=\left(\psi\left(\cdot,\mathrm{i}\kappa_n\right)\right).$$

Note that the integrals in (10.5) are actually finite sums.

By Proposition 8.1 with $g=\psi$, $S=(x,\infty)$ we can pass in (10.5) and (10.6) to the limit as $N\to\infty$. Thus we have both, the weak convergence of $\{R,\mathrm{d}\rho_N\}$ to $\{R,\mathrm{d}\rho\}$ and point-wise convergence of $q_{\rho_N}\left(x,t\right)$ to

$$\begin{aligned}q_\rho\left(x,t\right)=q\left(x,t\right)&\\+2\left[\int\psi_\rho\left(x,t;\mathrm{i}s\right)\psi\left(x,t;\mathrm{i}s\right)\mathrm{d}\rho_t\left(s\right)\right]^2&+4\int\psi_\rho\left(x,t;\mathrm{i}s\right)\psi'\left(x,t;\mathrm{i}s\right)\mathrm{d}\rho_t\left(s\right).\end{aligned}\tag{10.7}$$

By Definition 3.4, $q_\rho\left(x,t\right)$ is a step-type KdV solution, which proves (4.5) and (4.6) for $S_q=\{R,0\}$ and $\mathrm{d}\sigma=\mathrm{d}\rho\geq 0$ if we show that $q_\rho\left(x,t\right)$ is short-range at $+\infty$

for each $t \geq 0$. This will be done later. The solubility of the Fredholm integral equation (4.4), which we rewrite in the form

$$y + \mathbb{K}y = \psi, \tag{10.8}$$

where $\mathbb{K}$ is the integral operator on $L^2(\mathrm{d}\rho)$ with the kernel

$$K(\alpha, s; x, t) := \int_x^\infty \psi(z, t; \mathrm{i}\alpha)\, \psi(z, t; \mathrm{i}s)\, \mathrm{d}z,$$

is obvious as $\mathbb{K}$ is clearly positive (see Section 8) and hence $I + \mathbb{K}$ is positive definite. Since the Hilbert-Schmidt norm $\|\mathbb{K}(x,t)\|_2$ and the L^2 norm $\|\psi(x,t;\cdot)\|$ are small for large x and any fixed $t \geq 0$, we immediately see from (10.8) that $\|y(\cdot, x, t)\|$ is also small as $x \to \infty$ and (4.5) readily yields

$$\psi_\rho(x, t; k) = \psi(x, t; k)\,[1 + o(1)] \to 0, \quad x \to \infty, \quad \operatorname{Im} k \geq 0. \tag{10.9}$$

If we now show that ψ_ρ solves the the Schrödinger equation, it will be its right Jost solution. We rely on the following general fact (directly verifiable): if the Wronskian identity (7.9) holds for two functions $f_\lambda(x)$, $f_\nu(x)$ then $f_\lambda''(x)/f_\lambda(x) + \lambda^2$ is independent of λ and hence is equal to some $q(x)$. Thus, f_λ solves the Schrödinger equation $-f'' + q(x) f = \lambda^2 f$.

By Proposition 10.1 ψ_{ρ_N} is a right Jost solution and hence is subject to (7.9). But as we have shown, ψ_{ρ_N} converges in $L^2(\mathrm{d}\rho)$ to some ψ_ρ and hence there is a subsequence convergent almost everywhere. It follows that we can pass in (7.9) to the limit and therefore ψ_ρ is a solution to the Schrodinger equation[6]. By Theorem 7.1, (7.4), we can claim that $\psi_\rho(x, t; \mathrm{i}s) = y(s, x, t)$, $s \in \operatorname{Supp}\rho$, which also proves property (1). We are ready to show now that

$$\begin{aligned} Q(x) &:= q_\rho(x,t) - q(x,t) \\ &= 2\left[\int \psi_\rho(x,t;\mathrm{i}s)\,\psi(x,t;\mathrm{i}s)\,\mathrm{d}\rho_t(s)\right]^2 + 4\int \psi_\rho(x,t;\mathrm{i}s)\,\psi'(x,t;\mathrm{i}s)\,\mathrm{d}\rho_t(s) \end{aligned}$$

is in $L_1^1(+\infty)$. This will conclude the proof that q_ρ is indeed a step-type KdV solution. But due to (10.9) $Q \in L_1^1(+\infty)$ if

$$Q_0(x) := 2\left[\int \psi(x,t;\mathrm{i}s)\,\psi(x,t;\mathrm{i}s)\,\mathrm{d}\rho_t(s)\right]^2 + 4\int \psi(x,t;\mathrm{i}s)\,\psi'(x,t;\mathrm{i}s)\,\mathrm{d}\rho_t(s)$$

is in $L_1^1(+\infty)$. The latter immediately follows from Lemma 12.2 as each term of Q_0 is subject to its conditions.

It remains to prove the properties. (1) has already been proven. (2) clearly holds as each term in $Q(x)$ is at least absolutely continuous. The same applies to the derivatives if q is differentiable sufficient number of times. (3) is also obvious since both $\psi_\rho(x,t;\mathrm{i}s)$, $\psi(x,t;\mathrm{i}s)$ decay exponentially as $x \to \infty$ for every $s > 0$. Thus if the sets $\operatorname{Supp}\rho$ and $\{0\}$ are separated then each term in $Q(x)$ decay exponentially. To show (5) we recall the following general representation of the diagonal Green's function

$$G\left(k^2, x\right) = \frac{f_+(x,k)\, f_-(x,k)}{W\{f_+(x,k), f_-(x,k)\}}, \tag{10.10}$$

[6] Note it can also be shown by a direct but rather involved inspection.

where $f_\pm$ are Weyl solutions at $\pm\infty$ respectively. Take $f_\pm$ as in Theorem 7.1: $f_+ = \psi$, $f_- = \varphi$. Then (10.10) reads

$$G\left(k^2, x\right) = -\frac{\varphi(x,k)\,\psi(x,k)}{2\mathrm{i}k}. \tag{10.11}$$

As is well-known, $G(\lambda, x)$ is a Herglotz function for each x, that is an analytic function from $\operatorname{Im}\lambda > 0$ into itself. Such functions may have singularities only on $\operatorname{Im}\lambda = 0$ of at most simple pole type. In the context of Schrodinger operators, a pole type singularity may only occur at a bound state. By Theorem 7.1 all pole type singularities on the right hand side of (10.11) come from singularities of φ, which, in turn, coincide with pure points of ρ. This proves (4) and Case 1 is proven now.

Case 2 (removing negative spectrum). Suppose that $\{R, \mathrm{d}\rho\}$ are scattering data for a step-type KdV solution and $\mathrm{d}\sigma = \mathrm{d}\rho|_\Delta$, where $\Delta \subseteq \operatorname{Supp}\rho$. Let $\mathrm{d}\sigma_N$ be a discretization of $\mathrm{d}\sigma$ in Case 1. By Proposition 10.1 we construct a step-type KdV solution $q_{-\sigma_N}$ by (10.3) and the associated Jost solution $\psi_{-\sigma_N}$ by (10.4). As in Case 1, we can pass to the limit in Proposition 10.1 as $\mathrm{d}\sigma_N \to \mathrm{d}\sigma$, the equations

$$(\psi_\sigma)_{-\sigma} = \psi, \quad (q_\sigma)_{-\sigma} = q$$

being preserved in the limit. This completes our proof of Theorem 4.1.

We are now in the position to complete the proof of Theorem 7.1. It is just enough to notice that $\widetilde{\psi}$ and $\widetilde{q}$ in Theorem 7.1 appear as ψ_σ and q_σ in Theorem 4.1.

Remark 10.2. *One of the referees asked if the process of adding negative spectrum is commutative. I.e. adding σ_1 and then σ_2 produces the same result as adding σ_2 and then σ_1. We are positive that this is indeed the case. Take in (10.1) two one point measures* $\mathrm{d}\sigma_n(s) = c_n^2\delta(s-\kappa_n)\,\mathrm{d}s$, $\kappa_n > 0$, $n = 1, 2$. *We have two matrices* $\mathbf{K}_{12}$ *and* $\mathbf{K}_{12}$

$$\mathbf{K}_{12} = \int_x^\infty \begin{pmatrix} c_1^2\psi(s,\mathrm{i}\kappa_1)^2 & c_1c_2\psi(s,\mathrm{i}\kappa_1)\,\psi(s,\mathrm{i}\kappa_2) \\ c_1c_2\psi(s,\mathrm{i}\kappa_1)\,\psi(s,\mathrm{i}\kappa_2) & c_2^2\psi(s,\mathrm{i}\kappa_2)^2 \end{pmatrix} \mathrm{d}s,$$

(adding σ_1 and then σ_2)

$$\mathbf{K}_{21} = \int_x^\infty \begin{pmatrix} c_2^2\psi(s,\mathrm{i}\kappa_2)^2 & c_1c_2\psi(s,\mathrm{i}\kappa_1)\,\psi(s,\mathrm{i}\kappa_2) \\ c_1c_2\psi(s,\mathrm{i}\kappa_1)\,\psi(s,t,\mathrm{i}\kappa_2) & c_1^2\psi(s,t,\mathrm{i}\kappa_1)^2 \end{pmatrix} \mathrm{d}s.$$

(adding σ_2 and then σ_1)

Since

$$\begin{aligned} &\begin{pmatrix} c_2^2\psi(s,\mathrm{i}\kappa_2)^2 & c_1c_2\psi(s,\mathrm{i}\kappa_1)\,\psi(s,\mathrm{i}\kappa_2) \\ c_1c_2\psi(s,\mathrm{i}\kappa_1)\,\psi(s,t,\mathrm{i}\kappa_2) & c_1^2\psi(s,t,\mathrm{i}\kappa_1)^2 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c_1^2\psi(s,\mathrm{i}\kappa_1)^2 & c_1c_2\psi(s,\mathrm{i}\kappa_1)\,\psi(s,\mathrm{i}\kappa_2) \\ c_1c_2\psi(s,\mathrm{i}\kappa_1)\,\psi(s,t,\mathrm{i}\kappa_2) & c_2^2\psi(s,\mathrm{i}\kappa_2)^2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \end{aligned}$$

we conclude that

$$\mathbf{K}_{21} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{K}_{12} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Hence $\det(I + \mathbf{K}_{21}) = \det(I + \mathbf{K}_{12})$ *and therefore*

$$\begin{aligned} q_{\sigma_1\sigma_2}(x,t) &= q(x,t) - 2\partial_x^2 \log\det\{I + \mathbf{K}_{12}(x,t)\} \\ &= q(x,t) - 2\partial_x^2 \log\det\{I + \mathbf{K}_{21}(x,t)\} = q_{\sigma_2\sigma_1}(x,t). \end{aligned}$$

This simple computation can be easily extended to two general discrete measures σ_1, σ_2 *and then we apply our density argument to go over to arbitrary measures.*

11. Examples

In this section we offer two examples. The first one is a new derivation of the KdV solution and the other one is an explicit construction of a step-type potential (KdV solution) which has the same norming measure ρ as the pure step potential (3.12) but zero reflection coefficient.

11.1. **Classical pure soliton solution.** We show that Theorem 4.1 immediately recovers the well-known classical formula for pure N soliton solution [43]

$$q(x,t) = -2\partial_x^2 \log\det\left(\delta_{mn} + c_n^2 \frac{\mathrm{e}^{-(\kappa_m+\kappa_n)x+8\left(\kappa_m^3+\kappa_n^3\right)t}}{\kappa_m+\kappa_n}\right). \tag{11.1}$$

Indeed, take in Theorem 4.1 $S_q = 0$ (the zero background) and a discrete measure ρ given by

$$\mathrm{d}\rho(k,t) = \sum_{n=1}^{N} c_n(t)^2 \,\delta_{\kappa_n}(k)\,\mathrm{d}k, \quad c_n(t) = c_n \mathrm{e}^{4\kappa_n^3 t}$$

In this case $\psi(x,k) = \exp(\mathrm{i}kx)$ and the Fredholm integral equation (4.4) becomes a linear system,

$$y_m + \sum_{n=1}^{N} K_{mn}(x,t)\, y_n = \mathrm{e}^{-\kappa_m x},$$

where

$$K_{mn}(x,t) = c_n^2 \mathrm{e}^{8\kappa_n^3 t} \frac{e^{-(\kappa_m+\kappa_n)x}}{\kappa_m+\kappa_n}.$$

Furthermore,

$$\begin{aligned} \psi_\rho(x,t;\kappa_m) &= \mathrm{e}^{-\kappa_m x} - \sum_{n=1}^{N} c_n^2 \mathrm{e}^{8\kappa_n^3 t} \frac{\mathrm{e}^{-(\kappa_m+\kappa_n)x}}{\kappa_m+\kappa_n} y_n(x,t) \\ &= \mathrm{e}^{-\kappa_m x}\left[1 - \sum_{n=1}^{N} c_n^2 \mathrm{e}^{8\kappa_n^3 t - \kappa_n x} \frac{y_n(x,t)}{\kappa_m+\kappa_n}\right] \end{aligned} \tag{11.2}$$

and thus for the KdV solution we have

$$\begin{aligned} q_\rho(x,t) = 2&\left[\sum_{n=1}^{N} c_n^2 \mathrm{e}^{8\kappa_n^3 t-\kappa_n x}\psi_\rho(x,t;\kappa_n)\right]^2 \\ &- 4\sum_{n=1}^{N} \kappa_n c_n^2 \mathrm{e}^{8\kappa_n^3 t-\kappa_n x}\psi_\rho(x,t;\kappa_n). \end{aligned} \tag{11.3}$$

The formula (11.3) is a new derivation of (11.1). Due to (10.1), (11.3) is equivalent to (11.1). We note that the representation (11.1) is not as convenient for asymptotic analysis for large x and t as the methods based on the Riemann-Hilbert problem (see e.g. [30]). We are hopeful that for similar reasons (11.3) could be a suitable starting point to do asymptotic analysis that could work for arbitrary enough measures.

We conclude this section with a simple computation showing how (11.2) implies the famous one soliton solution. Taking in (11.2) $\rho(k) = c^2\delta(k - \mathrm{i}\kappa)$

$$\psi_\rho(x, t; \mathrm{i}\kappa) = e^{-\kappa x}\frac{2\kappa}{2\kappa + c^2\exp(8\kappa^3 t - 2\kappa x)}$$

and substituting it in (11.3) implies

$$\begin{aligned} q_\rho(x,t) &= 2\left[\frac{2c^2\exp\left(8\kappa^3 t - 2\kappa x\right)}{2\kappa + 2c^2\exp\left(8\kappa^3 t - 2\kappa x\right)}\right]^2 - 4\kappa\frac{2c^2\exp\left(8\kappa^3 t - 2\kappa x\right)}{2\kappa + 2c^2\exp\left(8\kappa^3 t - 2\kappa x\right)} \\ &= -2\kappa^2\operatorname{sech}^2\left(4\kappa^3 t - \kappa x + \log\frac{c}{\sqrt{2\kappa}}\right), \end{aligned}$$

which is the one soliton solution, as expected.

11.2. **Reflectionless step-type potential.** As was discussed in section 4, such potentials naturally appear in the study of soliton gases. A pure step potential (3.12) serves as a model of soliton condensate but apparently it is not reflectionless. In this subsection we offer a construction that produces a step-type potential that has the same norming measure as a pure step potential but reflectionless. To this end, take in Theorem 4.1 $S_q = 0$ (the zero background) and the measure ρ with density $\mathrm{d}\rho(k)/\mathrm{d}k = 2k\sqrt{1-k^2}$ supported on $[0,1]$. Recall that our ρ is known as the *Wigner semicircle distribution.* Apparently,

$$\int_0^1 \mathrm{d}\rho(k)/k < \infty, \quad \mathrm{d}\rho \geq 0,$$

and Theorem 4.1 applies. In this case $\psi(x,k) = \exp(\mathrm{i}kx)$ and

$$K(k, \mathrm{i}s, x) = \frac{\mathrm{e}^{(\mathrm{i}k - s)x}}{s - \mathrm{i}k},$$

are independent of t. Then the Fredholm integral equation (4.4) turns into

$$Y(\alpha) + \int_0^1 2s\sqrt{1-s^2}\frac{\mathrm{e}^{8s^3 t - 2sx}}{s+\alpha}Y(s)\,\mathrm{d}s = 1, \quad \alpha \in [0,1], \tag{11.4}$$

where $Y(s) = y(s)\mathrm{e}^{sx}$. Eq (11.4) has a unique solution $Y(\alpha; x, t)$ and for the potential we have

$$\begin{aligned} q_\rho(x,t) = 8&\left[\int_0^1 s\sqrt{1-s^2}e^{-2sx}Y(s;x,t)\,\mathrm{d}s\right]^2 \\ &- 8\int_0^1 s^2\sqrt{1-s^2}e^{-2sx}Y(s;x,t)\,\mathrm{d}s, \end{aligned} \tag{11.5}$$

which is a reflectionless step-type KdV solution with the scattering data

$$S_{q_\rho} = \{0, \mathrm{d}\rho\}.$$

By Theorem 4.1 $q_\rho(x,t)$ decays at $+\infty$ sufficiently fast but it cannot be readily seen how it behaves at $-\infty$. In fact $q_\rho \to -1$ as $x \to -\infty$. To show this one needs to perform a standard transformation of the QARHP in Theorem 7.1 (see e.g. [27]). We will come back to this elsewhere.

12. Appendix

12.1. Proof of Proposition 8.1.

Lemma 12.1. *Let* $a_1(x,t)$ *and* $a_2(x,t)$ *be real continuos functions and*

$$A(x,y)=\int_S a(x,s)\,a(y,s)\,\mathrm{d}s,\ B(x,y)=\int_S b(x,s)\,b(y,s)\,\mathrm{d}s.$$

Then for the Hilbert-Schmidt norm of the integral operators

$$(\mathbb{A}f)(x)=\int A(x,y,s)\,f(y)\,\mathrm{d}\mu(y)$$

we have

$$\begin{aligned}||\mathbb{A}-\mathbb{B}||_2 &= |||\mathbb{A}-\mathbb{B}||| \\ &\le (|||a|||+|||b|||)\,|||a-b|||\,.\end{aligned}$$

Proof. Rewrite

$$\begin{aligned}&\int_S [a(x,s)\,a(y,s)-b(x,s)\,b(y,s)]\,\mathrm{d}s \\ &=\int_S [a(x,s)-b(x,s)]\,a(y,s)\,\mathrm{d}s+\int_S [a(y,s)-b(y,s)]\,b(x,s)\,\mathrm{d}s \\ &=A_1(x,y)+A_2(x,y)\,.\end{aligned}$$

Consider the $L^2(\mathrm{d}\mu\times\mathrm{d}\mu)$ norm of each A_1, A_2:

$$\begin{aligned}&\int A_1(x,y)^2\,\mathrm{d}\mu(x)\,\mathrm{d}\mu(y) \\ &\le \int\int\left\{\int_S |a(x,s)-b(x,s)|\,a(y,s)\,\mathrm{d}s\right\}^2 \mathrm{d}\mu(x)\,\mathrm{d}\mu(y) \\ &\le \int\int\left[\int_S |a(x,s)-b(x,s)|^2\,\mathrm{d}s\right]\left\{\int_S a(y,s)^2\,\mathrm{d}s\right\}\mathrm{d}\mu(x)\,\mathrm{d}\mu(y) \\ &=\int\left[\int_S |a(x,s)-b(x,s)|^2\,\mathrm{d}s\right]\mathrm{d}\mu(x)\int\left\{\int_S a(y,s)^2\,\mathrm{d}s\right\}\mathrm{d}\mu(y) \\ &=|||a-b|||\cdot|||a|||\,.\end{aligned}$$

Similarly,

$$\int A_2(x,y)^2\,\mathrm{d}\mu(x)\,\mathrm{d}\mu(y)\le |||a-b|||\cdot|||b|||\,.$$

□

12.2. Auxiliary estimates for Jost solutions.

Let $m(x,k)=\mathrm{e}^{-\mathrm{i}kx}f(x,k)$, where $f(x,k)$ is the right Jost solution. Then [9, page 130] $m(x,k)$ is uniformly bounded in $\operatorname{Im}k\ge 0$ for every real x and for $x\ge 0$

$$|m(x,k)-1|\le \mathrm{const}\cdot\frac{\int_x^\infty (1+|s|)\,|q(s)|\,\mathrm{d}s}{1+|k|}, \tag{12.1}$$

$$|m'(x,k)|\le \mathrm{const}\cdot\frac{\int_x^\infty |q(s)|\,\mathrm{d}s}{1+|k|}, \tag{12.2}$$

uniformly for $\operatorname{Im}k\ge 0$.

Lemma 12.2. *Let $q \in L_1^1(+\infty)$ and $f(x, \mathrm{i}s) = e^{-sx} m(x, \mathrm{i}s)$, $s \geq 0$, where m is subject to (12.1) and (12.2). Suppose that a finite measure σ supported on a finite subset of $\mathbb{R}_+$ satisfies*

$$\int \mathrm{d}\sigma(s)/s < \infty.$$

Then

$$F(x) := \int f(x, \mathrm{i}s)^2 \,\mathrm{d}\sigma(s) \in L^1(+\infty)$$

and $F(x)^2$, $F'(x)$ are both in $L_1^1(+\infty)$.

Proof. Without loss of generality we may assume that $\mathrm{d}\sigma \geq 0$. Observe that (12.1) and (12.2) imply $m(x,k) = 1 + o(1)$, $m'(x,k) = o(1/x)$ as $x \to \infty$ uniformly in $\operatorname{Im} k \geq 0$. Therefore

$$F(x) = \int e^{-2sx} m(x, \mathrm{i}s)^2 \,\mathrm{d}\sigma(s) = F_0(x) \cdot (1 + o(1)), \quad x \to \infty, \tag{12.3}$$

$$\begin{aligned} F'(x) &= -2 \int \mathrm{e}^{-2sx} m(x, \mathrm{i}s)^2 \, s\mathrm{d}\sigma(s) + 2 \int \mathrm{e}^{-2sx} m(x, \mathrm{i}s)\, m'(x, \mathrm{i}s) \,\mathrm{d}\sigma(s) \\ &= F_0'(x) \cdot (1 + o(1)) + F_0(x) \cdot o(1/x), \quad x \to \infty, \end{aligned} \tag{12.4}$$

where

$$F_0(x) := \int \mathrm{e}^{-2sx} \mathrm{d}\sigma(s).$$

We show first that $F_0 \in L^1(+\infty)$. For any finite a we have

$$\begin{aligned} \int_a^\infty F_0(x)\,\mathrm{d}x &= \int \left[\int_a^\infty \mathrm{e}^{-2sx} \mathrm{d}x\right] \mathrm{d}\sigma(s) \\ &= \int \mathrm{e}^{-2as} \frac{\mathrm{d}\sigma(s)}{2s} \leq \max_{s \in \operatorname{Supp}\sigma} \mathrm{e}^{-2as} \cdot \int \frac{\mathrm{d}\sigma(s)}{2s} < \infty. \end{aligned}$$

Since $\operatorname{Supp}\sigma \subset \mathbb{R}_+$ and σ is finite, it immediately follows that (1) $F_0 \in L^1(+\infty)$ and (2) in the computations below we can set for simplicity $a = 0$. Due to (12.3), we conclude that $F \in L^1(+\infty)$. Let us show that $F^2 \in L_1^1(+\infty)$. Indeed, due to (12.3) again, it is enough to show that $F_0^2 \in L_1^1(+\infty)$:

$$\int_0^\infty x F_0^2(x)\,\mathrm{d}x \leq \max_{x \geq 0} [x F_0(x)] \int_0^\infty F_0(x)\,\mathrm{d}x < \infty,$$

as F_0 is monotonic function from $L^1(+\infty)$.

We now show that

$$F_0'(x) = -2 \int \mathrm{e}^{-2sx} s\mathrm{d}\sigma(s) \in L_1^1(+\infty).$$

Indeed, integrating by parts, one has

$$\begin{aligned} \int_0^\infty x\,|F_0'(x)|\,\mathrm{d}x &= 2 \int_0^\infty \left[\int \mathrm{e}^{-2sx} s\mathrm{d}\sigma(s)\right] x\mathrm{d}x \\ &= 2 \int \left[\int_0^\infty \mathrm{e}^{-2sx} x\mathrm{d}x\right] s\mathrm{d}\sigma(s) \\ &= 2 \int \left(\frac{1}{2s}\right)^2 s\mathrm{d}\sigma(s) = \int \frac{\mathrm{d}\sigma(s)}{2s} < \infty. \end{aligned}$$

Therefore, it follows from (12.4) that $F_0' \in L_1^1(+\infty)$. ∎

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