NORMING CONSTANTS OF EMBEDDED BOUND STATES AND BOUNDED POSITON SOLUTIONS OF THE KORTEWEG-DE VRIES EQUATION

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Dedicated to the memory of Sergey Naboko, my teacher and friend

Abstract. In the context of the full line Schrodinger equation, we revisit the binary Darboux transformation (double commutation method) which inserts or removes any number of positive eigenvalues embedded into the absolutely continuous spectrum without altering the rest of scattering data. We then show that embedded eigenvalues produce an additional explicit term in the KdV solution. This term looks similar to multi-soliton solution and describes waves traveling in the direction opposite to solitons. It also resembles the known formula for (singular) multi-positon solutions but remains bounded, which answers in the affirmative Matveev's question about existence of bounded positons.

1. Introduction

We are concerned with the inverse scattering problem for the full line Schrodinger operator $\mathbb{L}_q = -\partial_x^2 + q(x)$ in the presence of embedded eigenvalues (i.e. positive eigenvalues in the continuous spectrum) and understanding how such eigenvalues affect solutions to the initial value problem for the Korteweg-de Vries (KdV) equation

$$\begin{aligned} &\partial_t u - 6u\partial_x u + \partial_x^3 u = 0, \quad -\infty < x < \infty,\ t \geq 0, \\ &u(x,0) = q(x). \end{aligned} \tag{1.1}$$

If $q(x) = O\left(|x|^{-2-\varepsilon}\right)$ as $x \to \pm\infty$ (short-range) then the classical inverse scattering transform (IST) yields essentially all the information about the solution one could ask for. However, if $q(x) = O\left(|x|^{-2}\right)$ then the classical IST is no longer well-defined in general as the standard scattering data no longer define the potential uniquely [1]. Note that if $q(x) = O\left(|x|^{-2-\varepsilon}\right)$ at $+\infty$ but quite arbitrary at $-\infty$ then a "right sided" IST still works[1] allowing to study KdV solutions with such initial data (see our recent [19] and the literature cited therein). As it was shown by

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[1]This is a strong manifestation of the unidirectional nature of KdV.

Naboko [26] slower than $q(x) = O\left(|x|^{-1}\right)$ may produce dense singular spectrum filling $(0,\infty)$ leaving any hope that a suitable IST can include such a situation. The main concern of our note is to develop the IST for those cases of *Wigner-von Neumann* type of initial data

$$q(x) = (A/x)\sin 2\omega x + O\left(x^{-2}\right), \quad |x| \to \infty, \tag{1.2}$$

that produce only finitely many *embedded bound states (*and no other positive singular spectrum). It is important that Wigner-von Neumann potentials are in L^2 and due to the seminal Bourgain's result [4] (1.1) remains well-posed.

In our recent work [28] we use L^2 well-posedness to treat a specific case of Wigner-von Neumann type of initial data that gives a hint for how IST may be extended shading some light on Vladimir Matveev's proposal [9]: "A very interesting unsolved problem is to study the large time behavior of the solutions to the KdV equation corresponding to the smooth initial data like $cx^{-1}\sin 2kx$, $c \in \mathbb{R}$", "The related inverse scattering problem is not yet solved and the study of the related large times evolution is a very challenging problem".

We recall that Wigner-von Neumann potentials were introduced as examples of quantum mechanical potentials that produce embedded eigenvalues (i.e. embedded into continuous spectrum). In the present paper, we concentrate on understanding the general effect of embedded eigenvalues on inverse scattering problem and KdV solutions. We show that to restore well-posedness of IST the classical scattering data need to be supplemented with embedded bound state data which are similar to that of negative bound states but come from a different type of singularity, embedded real poles of Jost solutions (also known as *resonances* or *spectral singularities*). The main new feature is an (explicit) extra term in the KdV solution that accounts for embedded eigenvalues and resembles the well-known multisoliton solution [23] (see also [29]). In the literature (see e.g. [25]) such solutions are commonly referred to as *positon* (since they correspond to positive eigenvalues) but only singular (double pole) positons are currently known. In fact, Matveev has repeatedly asked [25] if bounded (non-singular) positons exist. We offer an explicit construction of such solutions which should yield precise description of how positons interact with each other, as well as with solitons and the background. Our analysis is based on the *binary Darboux transformation* (see e.g. [20, 24]), also known as the *double commutation method* (see e.g. [8, 15]), but we rely on the new approach to it put forward in our recent [29] which is particularly well-suited to the IST setting. We refer the reader to Section 3 for more discussions, historical comments, and literature accounts.

We emphasize that we deal with a new type of coherent KdV structure associated with initial data that support zero transmission at positive energies[2]. Such a point gives rise to a spectral singularity which order determines main features of the KdV solution. In our recent paper [17] we show that if its order is less than 1/2 then, in fact, there are no interesting features to report on. In the context of Wigner-von Neumann initial data (1.2) it is the case when the ratio $\gamma := |A|/4\omega < 1/2$. In this paper we consider order 1 spectral singularities. Such singularities are generated, for example, by (1.2) with $\gamma = 1$. (Recall that such singularities are also referred to as resonances.) We are still far from the complete solution of Matveev's problem. But we now have a tool to turn an order one singularity into an embedded eigenvalue and

[2]At such points the reflection coefficient is unimodular (full reflection).

show that the new initial profile does generate a new distinct feature, a (bounded) positon. On the other hand, it is well-known that an embedded eigenvalue (bound state) is the result of a very complicated process of coherent reflections causing its instability (see e.g. [3]). For this reason there is unfortunately no easy (if any) way to tell initially a resonances from an embedded eigenvalue. However, under the KdV flow, over time, an embedded eigenvalue reveals itself (as a soliton does). The quantitative analysis of this phenomenon is very nontrivial and still work in progress.

Through the paper, we make the following notational agreement. The bar denotes the complex conjugate. Matrices (including rows and columns) are denoted by boldface letters. For instance, $\mathbf{x} = (x_n)$ is the row with entries x_n. Prime stands for the x-derivative and $W(f,g) = fg' - f'g$ is the Wronskian. We write $f(x) \sim g(x), x \to x_0$ (finite or infinite) if $f(x) - g(x) \to 0, x \to x_0$. The only function space we need is the standard $L^p(S)$ with $p = 1, 2$ with the convention $L^p := L^p(\mathbb{R})$, $L^p(\pm\infty) = L^p(a, \pm\infty)$ with any finite a. If $f(z)$ is analytic in some domain D of the complex plane, we call a boundary point z_0 an *embedded simple pole* if z_0 is a non-isolated singularity and $(z - z_0) f(z)$ tends to a finite limit $c \neq 0$ as $z \to z_0$ non-tangentially. We then denote $c = \operatorname{Res}_{z_0} f$. Continuity at a point means continuity in some neighborhood of the point. Finally, $\operatorname{Im} f(z_0) = (\operatorname{Im} f)(z_0)$ and the same agreement of course applies to the real part Re.

The paper is organized as follows. In Section 2 we fix our terminology and introduce our main ingredients. In Section 3 we state and prove the theorem on embedding eigenvalues into continuous spectrum and discuss how it addresses some open problems. In Section 4 we give our theorem on paring embedded bound states. In the final section 5 we work out an explicit example illustrating our main results.

2. Our framework and main ingredients

In this section we briefly review the necessary material and introduce our main ingredients. Let

$$\mathbb{L}_q = -\partial_x^2 + q(x) \tag{2.1}$$

denote the full line *Schrodinger operator* with a real potential $q(x)$. That is, we assume that $\mathbb{L}_q$ can be defined as a selfadjoint operator on L^2. We agree to retain the same notation $\mathbb{L}_q$ for a differential expression defined by (2.1). Occasionally we also consider half-line versions of $\mathbb{L}_q$. Through the rest of the paper we assume the following basic conditions:

Hypothesis 2.1. *q is a real locally integrable function on $\mathbb{R}$ subject to*

(1) *the operator $\mathbb{L}_q$ is semibounded below;*
(2) *the equation $\mathbb{L}_q u = k^2 u$ has a solution $\psi(x,k)$ subject for a.e.* $\operatorname{Im} k = 0$ *to*

$$\psi(x,k) \sim \mathrm{e}^{\mathrm{i}kx}, \psi'(x,k) \sim \mathrm{i}k\mathrm{e}^{\mathrm{i}kx},\ x \to +\infty. \text{ (right Jost solution)} \tag{2.2}$$

Hypothesis 2.1 covers a large class of step-type potentials, i.e. potentials decaying (but not necessarily short-range) at $+\infty$ but essentially arbitrary at $-\infty$. In our [18, 19] we develop the IST for the KdV equation assuming a short range decay at $+\infty$ in place of condition (2). (See also Subsections 2.1 and 2.2.)

2.1. **Weyl solution.**

Since some of the material of this subsection is not quite mainstream in the integrable systems community, for the reader's convenience we go over some basics of *Titchmarsh-Weyl theory.* We follow a modern exposition of this theory given in [31, Chapter 9] adapting it to our setting.

Definition 2.2 (Weyl solution)**.** *A real locally integrable potential* $q(x)$ *is said to be* Weyl limit point *at* $\pm\infty$ *if the* Schrödinger equation

$$\mathbb{L}_q u = -u'' + q(x)\, u = \lambda u, \quad x \in \mathbb{R}, \tag{2.3}$$

has a unique (up to a multiplicative constant) solution that is in $L^2(\pm\infty)$ *for each* $\lambda \in \mathbb{C}^+$*. Solutions* $\Psi_\pm(x,\lambda)$ *are called the right/left* Weyl solution *respectively.*

The concept of a Weyl solution is fundamental to the spectral theory of Schrodinger (Sturm-Liouville) operators in dimension one due to the fact that its uniqueness is equivalent to the selfadjointness of $\mathbb{L}_q$ on $L^2(a,\pm\infty)$ with a Dirichlet (or any other selfadjoint) boundary condition at $x = a \pm 0$, a is any finite number.

There is no criterion for the limit point case in terms of q but there are convenient sufficient conditions which are typically satisfied in realistic situations. For instance, if q is essentially bounded below,

$$\sup_{a\in\mathbb{R}} \int_a^{a+1} \max\{-q(x), 0\}\, \mathrm{d}x < \infty,$$

then it is in the limit point case at both $\pm\infty$. Thus, $\mathbb{L}_q$ with such q is selfadjoint on L^2. In fact, if the quadratic form $\langle \mathbb{L}_q f, f\rangle \geq c\,\|f\|^2$ with some finite c for any f from a dense subset of L^2 then $\mathbb{L}_q$ is selfadjoint and its spectrum $\operatorname{Spec}\mathbb{L}_q$ is bounded below by c. Hence $\mathbb{L}_q$ is also in the limit point case at both $\pm\infty$. Thus, the condition 1 of Hypothesis 2.1 implies that q is limit point at both $\pm\infty$. Also, if q obeys the condition 2 of Hypothesis 2.1 then the right Weyl solution $\Psi_+(x,\lambda)$ can be chosen to satisfy $\Psi_+\left(x,k^2\right) = \psi(x,k)$, where ψ is the right Jost solution (2.2). Note that $\Psi_+(x,\lambda)$ is a function of energy λ whereas $\psi(x,k)$ is a function of momentum k ($\lambda = k^2$).

In this connection we emphasize that the Weyl solution is a family of solutions different by a multiple $\alpha(\lambda)$. The logarithmic derivative though

$$m_\pm(\lambda,a) = \pm\frac{\Psi'_\pm(a\pm 0,\lambda)}{\Psi_\pm(a\pm 0,\lambda)}, \quad \lambda \in \mathbb{C}^+, \tag{2.4}$$

is clearly independent of the choice of $\Psi_\pm$, and is known as the right/left *Titchmarsh-Weyl m-function* (or just m-function for short).

It should be quite apparent that without loss of generality we can discuss only the right half-line case. Unless otherwise stated for the rest of the subsection we conveniently abbreviate

$$\Psi = \Psi_+, \quad m(\lambda) = m_+(\lambda, 0)\,.$$

The function $m(\lambda)$ is analytic mapping $\mathbb{C}^+$ to $\mathbb{C}^+$ (a Herglotz function) and hence admits the Herglotz representation

$$m(\lambda) = c + \int_{\mathbb{R}} \left(\frac{1}{s-\lambda} - \frac{s}{1+s^2}\right) \mathrm{d}\mu(s)\,, \quad c \in \mathbb{R},$$

with some positive measure μ subject to $\displaystyle\int_{\mathbb{R}} \frac{\mathrm{d}\mu(s)}{1+s^2} < \infty$. It is a fundamental fact of Titchmarsh-Weyl theory that μ coincides with the spectral measure of $\mathbb{L}_q^D$, the Schrodinger operator on $L^2(\mathbb{R}_+)$ with a Dirichlet boundary condition $u(+0) = 0$.

Note that E is an eigenvalue of $\mathbb{L}_q^D$ iff $m(E+\mathrm{i}\varepsilon)$ has a pole type singularity as $\varepsilon\to+0$.

The m-function m introduced by (3) is also known as Dirichlet or principal. However we will also need the Neumann m-function m_0 defined by

$$m_0(\lambda)=-\Psi(0,\lambda)/\Psi'(0,\lambda)=-1/m(\lambda). \tag{2.5}$$

It is a Heglotz function and its representing measure is the spectral measure of $\mathbb{L}_q^N$, the Schrodinger operator on $L^2(\mathbb{R}_+)$ with a Neumann boundary condition $u'(+0)=0$. If we normalize Ψ to satisfy

$$\Psi(x,\lambda)=c(x,\lambda)+m_0(\lambda)s(x,\lambda), \tag{2.6}$$

where $c(x,\lambda), s(x,\lambda)$ are solutions of $\mathbb{L}_q u=\lambda u$ on $\mathbb{R}_+$ satisfying

$$c(0,\lambda)=1, c'(0,\lambda)=0; s(0,\lambda)=0, s'(0,\lambda)=1,$$

then (see e.g. [31, Lemma 9.14]) for $\lambda\in\mathbb{C}^+$

$$\int_0^\infty|\Psi(x,\lambda)|^2\,\mathrm{d}x=\frac{\operatorname{Im}m_0(\lambda)}{\operatorname{Im}\lambda}. \tag{2.7}$$

We now have all ingredients to prove the following important statement.

Lemma 2.3. *Let $\mathbb{L}_q$ be selfadjoint on L^2 and $\Psi(x,\lambda)$ a right Weyl solution. If E is a real number such that:*

1. *$E>\inf\operatorname{Spec}\mathbb{L}_q$;*
2. *equation $\mathbb{L}_q u=Eu$ has a real solution $u_E(x)$ square integrable at $+\infty$;*
3. *$\lim_{\varepsilon\to+0}\Psi(x,E+\mathrm{i}\varepsilon)=:\Psi(x,E+\mathrm{i}0)$ exists and finite;*

then $u_E(x)$ and $\Psi(x,E+\mathrm{i}0)$ are linearly dependent.

Proof. Condition 1 implies that $u_E(x)$ has at least one zero (the Sturm comparison theorem). Without loss of generality we assume that it is 0. That is $u_E(0)=0$. Due to Condition 2, $u_E\in L^2(\mathbb{R}_+)$ and hence E is an eigenvalue of $\mathbb{L}_q^D$ on $L^2(\mathbb{R}_+)$. This means that the Dirichlet m-function $m(E+\mathrm{i}\varepsilon)$ has a pole type singularity as $\varepsilon\to+0$ and hence, due to (2.5), the Neumann m-function $m_0(E+\mathrm{i}\varepsilon)$ vanishes linearly as $\varepsilon\to+0$. Let Ψ_0 denote the Weyl solution subject to (2.6). It follows from (2.7) that

$$\int_0^\infty|\Psi_0(x,E+\mathrm{i}\varepsilon)|^2\,\mathrm{d}x=\frac{\operatorname{Im}m_0(E+\mathrm{i}\varepsilon)}{\varepsilon}.$$

Therefore, we must have

$$\int_0^\infty|\Psi(x,E+\mathrm{i}\varepsilon)|^2\,\mathrm{d}x\sim C>0,\quad \varepsilon\to+0. \tag{2.8}$$

But since $c(x,\lambda), s(x,\lambda)$ are entire functions in λ and $m_0(\lambda)$ has (nontangentional) boundary values a.e. on $\mathbb{R}$, it follows from (2.6) that boundary values of Ψ_0 are well-defined and

$$\begin{aligned}\Psi_0(x,E+\mathrm{i}0)&=c(x,E+\mathrm{i}0)+m_0(E+\mathrm{i}0)s(x,E+\mathrm{i}0)\\&=c(x,E).\end{aligned}$$

By the Fatou lemma we conclude that

$$\int_0^\infty|\Psi_0(x,E+\mathrm{i}0)|^2\,\mathrm{d}x=\int_0^\infty|c(x,E)|^2\,\mathrm{d}x\le C.$$

Thus $\Psi_0(x, E+\mathrm{i}0) \in L^2(\mathbb{R}_+)$. By the well-known (and easily verifiable) Wronskian identity:

$$W'(f_\lambda, f_\mu) = (\lambda - \mu) f_\lambda f_\mu, \tag{2.9}$$

where f_λ denotes a solution to $\mathbb{L}_q u = \lambda u$, one has

$$W(\Psi_0(x, E+\mathrm{i}\varepsilon), u_E(x)) = -\mathrm{i}\varepsilon \int_x^\infty \Psi_0(s, E+\mathrm{i}\varepsilon) u_E(s) \, \mathrm{d}s. \tag{2.10}$$

By taking in (2.10) $\varepsilon \to +0$, one immediately concludes from that that

$$W(\Psi_0(x, E+\mathrm{i}0), u_E(x)) = 0$$

if we show that the integral in (2.10) stays bounded. The latter follows from

$$\begin{aligned}
&\lim_{\varepsilon\to+0} \left| \int_x^\infty \Psi_0(s, E+\mathrm{i}\varepsilon) u_E(s) \, \mathrm{d}s \right|^2 \\
&\leq \lim_{\varepsilon\to+0} \int_x^\infty |\Psi_0(s, E+\mathrm{i}\varepsilon)|^2 \, \mathrm{d}s \cdot \int_x^\infty u_E(s)^2 \, \mathrm{d}s \\
&\leq \lim_{\varepsilon\to+0} \int_0^\infty |\Psi_0(s, E+\mathrm{i}\varepsilon)|^2 \, \mathrm{d}s \cdot \int_0^\infty u_E(s)^2 \, \mathrm{d}s \\
&< \infty.
\end{aligned}$$

It remains to notice that, as Weyl solutions, Ψ and Ψ_0 differ by a multiple $\alpha(\lambda)$. That is, $\Psi(x, \lambda) = \alpha(\lambda) \Psi_0(x, \lambda)$ for any x and hence

$$\alpha(\lambda) = \Psi(0, \lambda) / \Psi_0(0, \lambda) = \Psi(0, \lambda),$$

as by (2.6) $\Psi_0(0, \lambda) = 1$. Since, by Condition 3, $\Psi_0(x, E+\mathrm{i}0)$ is well-defined, so is $\alpha(E+\mathrm{i}0)$. Thus

$$W(\Psi(x, E+\mathrm{i}0), u_E(x)) = \alpha(E+\mathrm{i}0) W(\Psi_0(x, E+\mathrm{i}0), u_E(x)) = 0,$$

which concludes the proof. $\square$

In what follows E is a priori embedded into continuous spectrum and hence Condition 1 will be satisfied.

2.2. **Reflection coefficient** [18]**.** From now on, we assume Hypothesis 2.1 which lets us take the right Jost solution $\psi(x, k)$ defined by (2.2) as the right Weyl solution $\Psi_+(x, k^2)$ suitable for us. Namely, we set

$$\Psi_+(x, k^2) = \psi_+(x, k) = \psi(x, k).$$

We choose the left Weyl solution $\Psi_-(x, k^2)$, denote it by $\varphi(x, k)$, to satisfy

$$\varphi(x,k) = \overline{\psi(x,k)} + R(k)\psi(x,k), \quad (\textit{basic scattering relation}) \tag{2.11}$$

for a.e. real k with some $R(k)$ called the (right) *reflection coefficient*. Equation (2.11) is explained below. Thus

$$\Psi_-(x, k^2) = \varphi(x, k)$$

where φ is subject to (2.11).

Note that condition (2) of Hypothesis 2.1 assumes some decay at $+\infty$ and implies two important facts:

(1) As it immediately follows from (2.2),
$$W\left(\overline{\psi(x,k)},\psi\left(x,k\right)\right)=2\mathrm{i}k \tag{2.12}$$
and hence the pair $\{\psi,\overline{\psi}\}$ forms a fundamental set for (2.3). This means that (2.11) is nothing but an elementary fact saying that any solution is a linear combination of fundamental solutions.

(2) It follows form (2.11) that
$$R\left(k\right)=-\frac{W(\varphi\left(x,k\right),\overline{\psi}\left(x,k\right))}{W(\varphi\left(x,k\right),\psi\left(x,k\right))} \tag{2.13}$$
is well-defined for a.e. real k and $R\left(-k\right)=\overline{R\left(k\right)}$, $|R\left(k\right)|\leq 1$.

2.3. **Diagonal Green's function** [31]**.** If $q\in L^1\left(+\infty\right)$ then the Jost solution exists for any $k\neq 0$. Slower decay may give rise to real singularities of $\psi(x,k)$. The adequate object to deal with such singularities is the *diagonal Green's function* of $\mathbb{L}_q$ defined as
$$g\left(k^2,x\right)=\frac{\psi_+\left(x,k\right)\psi_-\left(x,k\right)}{W\left(\psi_+\left(x,k\right),\psi_-\left(x,k\right)\right)}=-\frac{\varphi\left(x,k\right)\psi\left(x,k\right)}{2\mathrm{i}k}, \tag{2.14}$$
the last equation being due to (2.12). The importance of g is due to

(1) it is analytic in k^2 from $\mathbb{C}^+$ to $\mathbb{C}^+$;
(2) its poles (necessarily real), both isolated and embedded, are eigenvalues of $\mathbb{L}_q$;
(3) the potential $q\left(x\right)$ can be found from
$$g\left(-\kappa^2,x\right)\sim 1-q\left(x\right)/2\kappa^2,\quad \kappa\to+\infty. \tag{2.15}$$

2.4. **Norming constants.** Recall that if (2.3) also has a left Jost solution $\psi_-\left(x,k\right)$ (i.e., subject to $\psi_-\left(x,k\right)\sim \mathrm{e}^{-\mathrm{i}kx}$) then $\varphi\left(x,k\right)=T\left(k\right)\psi_-\left(x,k\right)$ where $T\left(k\right)$ is called the *transmission coefficient*. It follows from (2.11) that $T\left(k\right)=2ik/W\left(\psi_-,\psi\right)$ meaning that $T\left(k\right)$ is meromorphic in $\mathbb{C}^+$ with simple poles (if any) $\{\mathrm{i}\kappa_n\}$, $\kappa_n>0$, and $k^2=-\kappa_n^2$ are the isolated poles of $g\left(k^2,x\right)$, i.e. negative bound states of $\mathbb{L}_q$. Since $R\left(k\right)$ in general is only defined on the real line, one needs to include pole information in the set of scattering data. It can be done via the relation
$$\operatorname*{Res}_{k=\mathrm{i}\kappa_n}\varphi\left(x,k\right)=\mathrm{i}c_n^2\psi\left(x,\mathrm{i}\kappa_n\right),\ (\textit{isolated pole condition}) \tag{2.16}$$
where positive c_n^2, called the (right) *norming constant* of bound state $-\kappa_n^2$, must be specified.

As was discussed, slower decay of q at $+\infty$ may give rise to *resonances* (also known as *spectral singularities*), i.e. real points $\pm\omega_n$ where $\psi\left(x,k\right)$, the other factor in (2.14), shows a blow up behavior. To the best of own knowledge only Wigner-von Neumann resonances are relatively well-understood [21]. In general $\psi\left(x,k\right)$ may blow up to any order. We however restrict our attention to the case $\psi\left(x,k\right)=O\left(\left(k-\omega_n\right)^{-1}\right)$, $k\to\omega_n$, i.e. ω_n is an embedded simple pole[3]. Since $g\left(k^2,x\right)$ may only have a simple embedded pole, $\varphi\left(x,\omega_n\right)$ is then well-defined. If $\varphi\left(x,\omega_n\right)\neq 0$ then ω_n^2 is an embedded bound state. As we show in [28], the

[3]The case of arbitrary order singularities is technically more difficult and is still work in progress.

reflection coefficient $R(k)$ alone can not tell if a resonance is a bound state or not. Therefore an extra condition is required. Using (2.16) as a pattern to follow, we set

$$\operatorname*{Res}_{k=\omega_n} \psi(x,k) = \frac{\mathrm{i}\alpha_n^2}{R(\omega_n)}\varphi(x,\omega_n) \ (\textit{embedded pole condition}) \tag{2.17}$$

with some $\alpha_n^2 > 0$ which we call the *norming constant* of embedded bound state ω_n^2. The reason for putting an extra $R(\omega_n)$ will be clear later. We shall see that (2.17) indeed works.

2.5. **Gauge transformation.** This is our last (but not least) ingredient.

Lemma 2.4 (on gauge transformation). *If $\varphi(x,k)$ and $\psi(x,k)$ are related by (2.11) then so are*

$$\begin{aligned}\widetilde{\varphi}(x,k) &= \varphi(x,k) + \sum_n a_n(x) W(\varphi(x,k), f_n(x,k)) \\ \widetilde{\psi}(x,k) &= \psi(x,k) + \sum_n a_n(x) W(\psi(x,k), f_n(x,k))\end{aligned} \tag{2.18}$$

for any real $a_n(x)$ and $f_n(x,k)$ real for real k.

The proof is by a direct consequence of the bi-linearity of the Wronskian and completely trivial. We will apply this lemma with a very specific choice of $f_n(x,k)$. The name 'gauge' (but not the transformation) is taken from the recent [2] where such transformations are crucially used in the context of matrix *Riemann-Hilbert problem* associated with the focusing NLS. We however learned about them from the recent [16] where it is used in a way similar to [2] but in the mKdV setting. Note that the form (2.18) is very different from those of [2, 16].

3. Inserting embedded eigenvalues

In this section we state, prove, and discuss the following

Theorem 3.1 (turning resonances into embedded eigenvalues). *Assume Hypothesis 2.1 and suppose that*

1. *(Resonance condition) for $\omega_n^2 > 0$, $1 \le n \le N < \infty$, $\mathbb{L}_q u = \omega_n^2 u$ has a unique (up to a scalar multiple) $L^2(-\infty)$ solution;*

2. *(Continuity condition) the (right) Jost solution $\psi(x,k)$ and the (right) reflection $R(k)$ coefficient are continuous at each $k = \omega_n$.*

Let

$$\mathbf{A} = (\alpha_n) = \begin{pmatrix} \alpha_1 & \alpha_2 & \dots & \alpha_N \end{pmatrix}$$

be a row-vector of arbitrary real nonzero numbers (norming constants) and[4]

$$\mathbf{\Phi}(x) := (\phi_n(x)), \quad \phi_n(x) := 2\operatorname{Re}\left[R(\omega_n)^{1/2}\psi(x,\omega_n)\right]. \tag{3.1}$$

Then

- *$\phi_n(x) \in L^2(-\infty)$ (hence $\mathbf{\Phi}(x) \in L^2(-\infty)$) and therefore*
- *the (square) matrix $\mathbf{G}_+(x)$ given by*

$$\mathbf{G}_+(x) := \mathbf{A}\left[\int_{-\infty}^{x} \mathbf{\Phi}(s)^T \mathbf{\Phi}(s)\,\mathrm{d}s\right]\mathbf{A}^T \ (\textit{the Gram matrix}) \tag{3.2}$$

is well-defined and (clearly) positive semi-definite;

[4] Where the root is chosen with a cut along $(-\infty, 0)$

- *the potential*
$$q_{+N}(x) = q(x) - 2\partial_x^2 \log\det\left(\mathbf{I} + \mathbf{G}_+(x)\right), \tag{3.3}$$
supports embedded bound states (eigenvalues) at ω_n^2 $(1 \le n \le N)$*;*
- *the associated (orthogonal in* L^2*) eigenfunctions* $(y_n(x))$ *can be (uniquely) found from the linear system*
$$\mathbf{y}\left(\mathbf{I} + \mathbf{G}_+(x)\right) = -\mathbf{A}^T \mathbf{\Phi}(x), \mathbf{y} := (y_n). \tag{3.4}$$

Before proceeding with the proof, note that the class of potentials satisfying the conditions of Theorem 3.1 is quite large. Indeed, as was discussed above, Hypothesis 2.1 requires only mild decay at $+\infty$ and general behavior at $-\infty$. Condition 1 is readily satisfied if on the left half line $q(x)$ behaves as a sum of N Wigner-von Neumann type potentials (1.2) with all γ's greater than 1/2. This is a classical fact known since at least the earlier 50s (see, e.g. [8]). Condition 2 is a bit more subtle. In section 5 we give specific examples with $\gamma = 1$ that produce analyticity (not just continuity) in condition 2. These examples and some considerations of [21] suggest a broad class of (long-range) potentials that guarantees condition 2 (work in progress).

Proof. We start with constructing a suitable pair $\varphi(x,k), \psi(x,k)$ of Weyl solutions for the original potential q at $\mp\infty$ respectively. The candidate for $\psi(x,k)$ is obvious, the right Jost solution. As in subsection 2.2 we define the Weyl solution at $-\infty$ by (2.11). It follows from (2.13) and (2.4) that for any x
$$|R(k)| = \left|\frac{m_-\left(k^2,x\right) + \overline{m_+\left(k^2,x\right)}}{m_-\left(k^2,x\right) + m_+\left(k^2,x\right)}\right|.$$
Using the same arguments as in the proof of Lemma 2.3, from condition 1 we conclude that for each $k^2 = \omega_n^2$ there is a point $x = a_n$ such that $m_-\left(k^2, a_n\right)$ has an embedded simple pole at ω_n^2. This immediately implies that $|R(\omega_n)| = 1$. In other words, a plane wave coming from $-\infty$ with energy ω_n^2 is completely reflected from q. Due to condition 2, it follows from (2.11) that
$$\begin{aligned} R(\omega_n)^{-1/2}\varphi(x,\omega_n) &= \overline{R(\omega_n)^{1/2}\psi(x,\omega_n)} + R(\omega_n)^{1/2}\psi(x,\omega_n) \\ &= 2\operatorname{Re} R(\omega_n)^{1/2}\psi(x,\omega_n), \end{aligned}$$
where the root is chosen with the argument in $(-\pi,\pi]$. Since $R(\omega_n)^{-1/2}\varphi(x,k)$ is a Weyl solution that has a finite boundary value at ω_n, by Lemma 2.3, from condition 1 we conclude that
$$\phi_n(x) = R(\omega_n)^{-1/2}\varphi(x,\omega_n) = 2\operatorname{Re} R(\omega_n)^{1/2}\psi(x,\omega_n) \tag{3.5}$$
is a real $L^2(-\infty)$ solution of $\mathbb{L}_q u = \omega_n^2 u$ and the first bullet item is proven. Since $\psi(x,\omega_n) \sim \mathrm{e}^{\mathrm{i}\omega_n x}$ at $+\infty$, (3.5) also yields
$$\phi_n(x) \sim 2\cos\left(\omega_n x + \frac{1}{2}\arg R(\omega_n)\right), \quad x \to +\infty. \tag{3.6}$$

We are ready now to present our candidates for a new pair $\varphi_{+N}(x,k)$, $\psi_{+N}(x,k)$ which is a suitable gauge transformation of $\varphi(x,k)$, $\psi(x,k)$. Taking in (2.18)
$$a_n(x) = \alpha_n y_n(x), f_n(x,k) = \frac{\phi_n(x)}{k^2 - \omega_n^2}$$

with some real (y_n) to be determined, we have

$$\varphi_{+N}(x,k)=\varphi(x,k)+\sum_{m=1}^{N}\alpha_m y_m(x)\frac{W(\varphi(x,k),\phi_m(x))}{k^2-\omega_m^2}, \tag{3.7}$$

$$\psi_{+N}(x,k)=\psi(x,k)+\sum_{m=1}^{N}\alpha_m y_m(x)\frac{W(\psi(x,k),\phi_m(x))}{k^2-\omega_m^2}. \tag{3.8}$$

Consider φ_{+N} first. Since $\varphi,\phi_n\in L^2(-\infty)$ (φ is a Weyl solution at $-\infty$), it follows from (2.9) that

$$\frac{W(\varphi(x,k),\phi_n(x))}{k^2-\omega_n^2}=\int_{-\infty}^{x}\varphi(s,k)\phi_n(s)\,\mathrm{d}s,\quad \operatorname{Im}k>0. \tag{3.9}$$

By Lemma 2.3 and (3.5) for any m,n one has

$$\left.\frac{W(\varphi(x,k),\phi_n(x))}{k^2-\omega_m^2}\right|_{k=\omega_n}=R(\omega_n)^{1/2}\int_{-\infty}^{x}\phi_n(s)\phi_m(s)\,\mathrm{d}s.$$

Thus φ_{+N} is continuous at each ω_n and it follows from (3.7) that

$$\varphi_{+N}(x,\omega_n)=R(\omega_n)^{1/2}\left\{\phi_n(x)+\sum_{m=1}^{N}\alpha_m y_m(x)\int_{-\infty}^{x}\phi_n(s)\phi_m(s)\,\mathrm{d}s\right\}. \tag{3.10}$$

Turn to ψ_{+N} now. One can see that it has an embedded simple pole at each ω_n. Let us compute its residue. Since ψ is Jost at $+\infty$, it follows from (2.11) that

$$W(\psi(x,k),\varphi(x,k))=W(\psi(x,k),\overline{\psi(x,k)})=-2\mathrm{i}k \tag{3.11}$$

and therefore by (3.5)

$$W(\psi(x,\omega_n),\phi_n(x))=R(\omega_n)^{-1/2}W(\psi(x,\omega_n),\varphi(x,\omega_n))=-2\mathrm{i}\omega_n R(\omega_n)^{-1/2}.$$

Thus, from (3.8) one obtains

$$\begin{aligned}\operatorname*{Res}_{k=\omega_n}\psi_{+N}(x,k)&=\alpha_n y_n(x)\frac{W(\psi(x,\omega_n),\phi_n(x))}{2\omega_n}\\&=-\mathrm{i}\alpha_n R(\omega_n)^{-1/2}y_n(x).\end{aligned} \tag{3.12}$$

We choose now (y_n) to satisfy our embedded pole condition (2.17):

$$\operatorname*{Res}_{k=\omega_n}\psi_{+N}(x,k)=\frac{\mathrm{i}\alpha_n^2}{R(\omega_n)}\varphi_{+N}(x,\omega_n). \tag{3.13}$$

Substituting (3.12) and (3.10) in (3.13) we have

$$\begin{aligned}&-\mathrm{i}\alpha_n R(\omega_n)^{-1/2}y_n\\&=\frac{\mathrm{i}\alpha_n^2}{R(\omega_n)}R(\omega_n)^{1/2}\left(\phi_n(x)+\sum_{m=1}^{N}\alpha_m y_m\int_{-\infty}^{x}\phi_m(s)\phi_n(s)\,\mathrm{d}s\right).\end{aligned}$$

$R(\omega_n)$ drops out[5] and we immediately arrive at the linear system

$$y_n(x)+\sum_{m=1}^{N}y_m(x)\int_{-\infty}^{x}\alpha_m\phi_m(s)\;\alpha_n\phi_n(s)\;\mathrm{d}s=-\alpha_n\phi_n(x) \tag{3.14}$$

[5]This was the reason for putting it in (2.17).

in y_n. In matrix form this system coincides with (3.4) which is nonsingular. Indeed,

$$\begin{aligned}\boldsymbol{G}_+(x) &= \left(\int_{-\infty}^{x} (\alpha_m\phi_m(s))(\alpha_n\phi_n(s))\,\mathrm{d}s\right) = \left(\alpha_m\left[\int_{-\infty}^{x}\phi_m(s)\,\phi_n(s)\,\mathrm{d}s\right]\alpha_n\right)\\ &= \mathbf{A}\left[\int_{-\infty}^{x}\mathbf{\Phi}(s)^T\,\mathbf{\Phi}(s)\,\mathrm{d}s\right]\mathbf{A}^T = \int_{-\infty}^{x}\left[\mathbf{\Phi}(s)\,\mathbf{A}^T\right]^T\left[\mathbf{\Phi}(s)\,\mathbf{A}^T\right]\,\mathrm{d}s.\end{aligned}$$

Therefore, $\mathbf{I}+\boldsymbol{G}_+(x)$ is positive definite and the system (3.14) has a unique solution (y_n) for any real α_n and x. Its main feature is that $y_n \in L^2(\mathbb{R})$. Indeed, since $\phi_n \in L^2(-\infty)$ we conclude $\|\boldsymbol{G}_+(x)\| = o(1)$, $x \to -\infty$, and $y_n(x) \sim -\alpha_n\phi_n(x) \in L^2(-\infty)$. To show that $y_n \in L^2(+\infty)$ we observe first that (3.6) implies that for each entry of $\boldsymbol{G}_+(x)$ we have $g_{nn}(x) = O(x)$, $g_{mn}(x) = O(1)$, $m \neq n$, as $x \to +\infty$. Therefore, $\left\|(\mathbf{I}+\boldsymbol{G}_+(x))^{-1}\right\| = O\left(x^{-1}\right)$, as $x \to +\infty$, and so $y_n(x) = O(1/x) \in L^2(+\infty)$.

Show now that $\varphi_{+N}(x,k) \in L^2(-\infty)$, $\psi_{+N}(x,k) \in L^2(+\infty)$ for $\operatorname{Im} k > 0$. Substituting (3.9) into (3.7) yields

$$\varphi_{+N}(x,k) = \varphi(x,k) + \sum_{n=1}^{N}\alpha_n y_n(x)\int_{-\infty}^{x}\varphi(s,k)\,\phi_n(s)\,\mathrm{d}s.$$

Since $\varphi(x,k)$ is (as a left Weyl solution) in $L^2(-\infty)$ for $\operatorname{Im} k > 0$, and (as is already proven) $y_n \in L^2$, and $\phi_n \in L^2(-\infty)$, one concludes that $\varphi_{+N}(x,k) \in L^2(-\infty)$ for $\operatorname{Im} k > 0$.

Turn to $\psi_{+N}(x,k)$. Since $\psi(x,k)$ is Jost at $+\infty$ and due to (3.6), one has $W(\psi(x,k),\phi_n(x)) = O(1)$, $x \to +\infty$, $\operatorname{Im} k \geq 0$. Therefore, (3.8) and (3.7) imply

$$\psi_{+N}(x,k) = \psi(x,k) + O(1/x)\,, \operatorname{Im} k \geq 0, x \to +\infty, \tag{3.15}$$

which proves that $\psi_{+N}(x,k)$ behaves like a Jost solution at $+\infty$ and hence $\psi_{+N}(x,k) \in L^2(+\infty)$ for $\operatorname{Im} k > 0$. By Lemma 2.4

$$\varphi_{+N}(x,k) = \overline{\psi_{+N}(x,k)} + R(k)\,\psi_{+N}(x,k)$$

holds for a.e. $\operatorname{Im} k = 0$, which together with (3.15) yields

$$\begin{aligned}W(\varphi_{+N}(x,k),\psi_{+N}(x,k)) &= W(\overline{\psi_{+N}(x,k)},\psi_{+N}(x,k)) \\ &= \lim_{x\to+\infty} W(\overline{\psi_{+N}(x,k)},\psi_{+N}(x,k)) = 2\mathrm{i}k.\end{aligned} \tag{3.16}$$

Assume for the time being that $\varphi_{+N}(x,k)\,,\psi_{+N}(x,k)$ also solve the Schrodinger equation with some potential $q_{+N}(x)$. Thus, we have constructed an ansatz $\varphi_{+N}(x,k)$, $\psi_{+N}(x,k)$ with desirable properties: $\varphi_{+N}(x,k)$ is a left Weyl solution and $\psi(x,k)$ is a right Weyl solution (i.e. for $\operatorname{Im} k > 0$ $\varphi_{+N}(x,k) \in L^2(-\infty)$, $\psi_{+N}(x,k) \in L^2(+\infty)$) and therefore (see e.g. [31])

$$\begin{aligned}g_{+N}\left(k^2,x\right) &= -\frac{\varphi_{+N}(x,k)\,\psi_{+N}(x,k)}{W(\varphi_{+N}(x,k)\,,\psi_{+N}(x,k))}\\ &= -\frac{\varphi_{+N}(x,k)\,\psi_{+N}(x,k)}{2\mathrm{i}k} \quad \text{(by (3.16))}.\end{aligned}$$

is the diagonal Green's function associated with $q_{+N}(x)$. Since by the construction $\psi_{+N}(x,k)$ has an embedded simple pole at each $k^2 = \omega_n^2$ (but φ_{+N} does not identically vanish there) we conclude that $g\left(k^2,x\right)$ also has embedded simple poles at $k^2 = \omega_n^2$ and thus all ω_n^2 are embedded eigenvalues of $q_{+N}(x)$ which is, in turn, can be computed from (2.15). There is a simpler alternative way to compute $q_{+N}(x)$ based on

$$\psi(x,k) \backsim \mathrm{e}^{\mathrm{i}kx}\left(1-\frac{1}{2\mathrm{i}k}\int_x^\infty q(s)\,\mathrm{d}s\right), \quad k\to\infty,\ \operatorname{Im}k\geq 0. \tag{3.17}$$

Since

$$W(\psi,\phi_n) \sim \mathrm{e}^{\mathrm{i}kx}\left(\phi_n' - \mathrm{i}k\phi_n\right), \quad k\to\infty,$$

we have: as $k\to\infty$

$$\begin{aligned}\mathrm{e}^{-\mathrm{i}kx}\left(\psi_{+N}-\psi\right)(x,k) &\sim -\sum_n \alpha_n y_n(x)\,\phi_n(x)\,\frac{\mathrm{i}k}{k^2-\omega_n^2}\\ &\sim \frac{1}{\mathrm{i}k}\sum_n y_n(x)\left(\alpha_n\phi_n(x)\right) = \frac{1}{\mathrm{i}k}\mathbf{y}(x)\,\boldsymbol{\phi}(x)^T\\ &= -\frac{1}{\mathrm{i}k}\boldsymbol{\phi}(x)\left(\mathbf{I}+\mathbf{G}_+(x)\right)^{-1}\boldsymbol{\phi}(x)^T,\end{aligned}$$

where $\mathbf{y} := (y_n)$, $\boldsymbol{\phi} := (\alpha_n\phi_n)$. But by Jacobi's formula on differentiation of determinants, we have (suppressing x)

$$\begin{aligned}\boldsymbol{\phi}\left(\mathbf{I}+\mathbf{G}_+\right)^{-1}\boldsymbol{\phi}^T &= \boldsymbol{\phi}\frac{\operatorname{adj}\left(\mathbf{I}+\mathbf{G}_+\right)}{\det\left(\mathbf{I}+\mathbf{G}_+\right)}\boldsymbol{\phi}^T\\ &= \sum_{m,n}\frac{\left(\operatorname{adj}\left(\mathbf{I}+\mathbf{G}_+\right)\right)_{mn}}{\det\left(\mathbf{I}+\mathbf{G}_+\right)}\phi_m\phi_n = \sum_{m,n}\frac{\left(\operatorname{adj}\left(\mathbf{I}+\mathbf{G}_+\right)\right)_{mn}}{\det\left(\mathbf{I}+\mathbf{G}_+\right)}g_{mn}'\\ &= \operatorname{tr}\left\{\left(\mathbf{I}+\mathbf{G}_+\right)'\frac{\operatorname{adj}\left(\mathbf{I}+\mathbf{G}_+\right)}{\det\left(\mathbf{I}+\mathbf{G}_+\right)}\right\} = \frac{\left(\det\left(\mathbf{I}+\mathbf{G}_+\right)\right)'}{\det\left(\mathbf{I}+\mathbf{G}_+\right)}\\ &= \left(\log\det\left(\mathbf{I}+\mathbf{G}_+\right)\right)',\end{aligned}$$

(where as before g_{mn} stands for the (m,n) entry of $\mathbf{G}_+$) and thus

$$\mathrm{e}^{-\mathrm{i}kx}\left(\psi_{+N}-\psi\right)(x,k) \sim -\frac{1}{\mathrm{i}k}\partial_x\log\det\left(\mathbf{I}+\mathbf{G}_+(x)\right).$$

By (3.17),

$$\mathrm{e}^{-\mathrm{i}kx}\left(\psi_{+N}-\psi\right)(x,k) \sim -\frac{1}{2\mathrm{i}k}\int_x^\infty \left(q_{+N}-q\right)(s)\,\mathrm{d}s, \quad k\to\infty,$$

and hence

$$q_{+N}(x) - q(x) = -2\partial_x^2\log\det\left(\mathbf{I}+\mathbf{G}_+(x)\right)$$

and (3.3) follows. By a direct verification (routinely performed for Darboux transformations), functions $\varphi_{+N}(x,k)$, $\psi_{+N}(x,k)$ indeed solve the Schrodinger equation with the potential q_{+N}. (See also the proof of Corollary 3.5).

As we have shown, $y_n(x)\in L^2(\mathbb{R})$ is, due to (3.12), proportional to $\operatorname{Res}_{k=\omega_n}\psi_{+N}$, which, in turn, solves $-u''+q_{+N}(x)\,u=\omega_n^2 u$ and we conclude that $y_n(x)$ is an eigenfunction of $\mathbb{L}_{q_{+N}}$. This concludes the proof. $\square$

Following the standard terminology [24], the transformation $(\varphi,\psi)\to(\varphi_{+N},\psi_{+N})$ constructed in the proof of Theorem 3.1 is directly related, as was mentioned in Introduction, to the binary Darboux transformation (double commutation method). As the very name (given by Deift [6] in 1978) suggests, the method rests on applying twice a commutation formula from operator theory. Note that basic formulas which the double commutation produces had been known to Gelfand and Levitan [10] already in 1951 in the context of their ground breaking study of the inverse spectral problem for Sturm-Liouville operators (although no commutation arguments were used). The full treatment of the double commutation method is given by Gesztesy et al [11]-[15] in the 1990s (see also the extensive literature cited therein). The double commutation method was introduced to study the effect of inserting/removing eigenvalues in spectral gaps on spectral properties of the underlying 1D Schrodinger operators while the binary Darboux transformation has been primarily a tool to produce explicit solutions. This is likely a reason why we could not find the literature where the two would be linked[6]. The double commutation method can also be applied to inserting/removing bound states into absolutely continuous spectra. In fact, in the half-line case it was first done (well before the term was coined) by Gelfand and Levitan [22, Section 6.6] and revisited in [8, Section 4] from the double commutation point of view. The formula derived in [8, Section 4] for the half-line case coincides with (3.3) for $N=1$ but no formula for $N>1$ is given. In [15] it is mentioned that the approach of [15] can yield such a formula in the full line case but to the best of our knowledge it has not been explicitly done. We emphasize however that our approach is unrelated to double commutation arguments and instead stems from the *Riemann-Hilbert problem* approach to the Darboux transformation put recently forward in [29]. The latter comes directly from inverse scattering and that is why it is much more suited for the IST (see Corollary 3.2 below).

Theorem 3.1 has some important corollaries.

Corollary 3.2. *Assume that $q(x)$ in Theorem 3.1 is short-range at $+\infty$[7] and has the scattering data $S(q)=\left\{R(k),\left(-\kappa_n^2,c_n^2\right)\right\}$. Then $S(q_{+N})=S(q)\cup\left\{\left(\omega_n^2,\alpha_n^2\right),1\le n\le N\right\}$ is the scattering data for q_{+N}.*

Proof. We only need to show that our binary Dabroux transformation preserves the discrete spectrum data $\left(-\kappa_n^2,c_n^2\right)$. To this end it suffices to show that

$$\operatorname*{Res}_{\mathrm{i}\kappa_n}\varphi_{+N}(x,k)=\mathrm{i}c_n^2\psi_{+N}(x,\mathrm{i}\kappa_n). \tag{3.18}$$

Indeed, since $\operatorname{Res}_{\mathrm{i}\kappa_n}\varphi(x,k)=ic_n^2\psi(x,\mathrm{i}\kappa_n)$ it immediately follows from (3.7) and (3.8) that

$$\begin{aligned}\operatorname*{Res}_{\mathrm{i}\kappa_n}\varphi_{+N}(x,k)&=\operatorname*{Res}_{\mathrm{i}\kappa_n}\varphi(x,k)+\sum_{m=1}^{N}\alpha_m y_m(x)\frac{W\left(\operatorname{Res}_{\mathrm{i}\kappa_n}\varphi(x,k),\phi_m(x)\right)}{k^2-\omega_m^2}\\&=\mathrm{i}c_n^2\left\{\psi(x,\mathrm{i}\kappa_n)+\sum_{m=1}^{N}\alpha_m y_m(x)\frac{W\left(\psi(x,\mathrm{i}\kappa_n),\phi_m(x)\right)}{k^2-\omega_m^2}\right\}\\&=\mathrm{i}c_n^2\psi_{+N}(x,\mathrm{i}\kappa_n).\end{aligned}$$

$\square$

Rowan Killip asked the author if embedded bound states require norming constants. Corollary 3.2 answers his question in the affirmative: $\left(\alpha_n^2\right)$ play the role of norming constants of embedded bound states.

[6] E.g. the book [20] pays much of attention to binary Darboux transformations but double commutation is not mentioned. The recent [30] briefly mentiones [20] and [15] but without discussing connections.

[7] That is $xq(x)\in L^1(+\infty)$.

Remark 3.3. *In particular, for one embedded eigenvalue* ω^2 *we have*

$$q_{+1}(x) = q(x) - 2\partial_x^2 \log\left(1 + \alpha^2 \int_{-\infty}^{x} \phi(s)^2 \, \mathrm{d}s\right), \tag{3.19}$$
$$\phi(s) = 2\,\mathrm{Re}\left[R(\omega)^{1/2}\,\psi(s,\omega)\right].$$

In this case $\|y\| = 1$. *To get to* $q_{+N}(x)$ *we can break our binary Darboux transformation into the chain of iterated transformations* $\psi_{+(n-1)}(x,k) \to \psi_{+n}(x,k)$,$1 \leq n \leq N$, *resulting in building* $q_{+N}(x)$ *by the simple recurrence formula*

$$\begin{aligned} q_{+n}(x) &= q_{+(n-1)}(x) \\ &\quad - 2\partial_x^2 \log\left(1 + 4\alpha_n^2 \int_{-\infty}^{x} \mathrm{Re}^2\left[R(\omega_n)^{1/2}\,\psi_{+(n-1)}(s,\omega_n)\right] \mathrm{d}s\right), \end{aligned} \tag{3.20}$$

each step being easy to control.

Remark 3.4. *It follows from (3.5) and (3.6) that* $q(x) - q_{+N}(x)$ *is continuous, in* $L^1(-\infty)$ *and* $O(1/x)$, $x \to +\infty$. *I.e., as expected* $q_{+N}(x)$ *is no longer short-range at* $+\infty$ *even if* $q(x)$ *is. More specifically, the discrepancy is*

$$q(x) - q_{+N}(x) \sim \sum_{n=1}^{N} \frac{A_n}{x} \sin(2\omega_n x + \delta_n), x \to +\infty, \tag{3.21}$$

with some A_n, δ_n. *Due to (3.20), it suffices to demonstrate (3.21) for* $N = 1$. *It follows from (3.6) that*

$$\begin{aligned} \tau(x) &:= 1 + 4\alpha^2 \int_{-\infty}^{x} \mathrm{Re}^2\left[R(\omega)^{1/2}\,\psi(s,\omega)\right] \mathrm{d}s \\ &= 1 + 4\alpha^2 \int_{-\infty}^{x} \phi(s)^2 \, \mathrm{d}s = O(x), \quad x \to +\infty, \end{aligned}$$

which, due to (3.19) and (3.6), implies that

$$\begin{aligned} q(x) - q_{+1}(x) &= 2\partial_x^2 \log \tau(x) = \tau''(x)/\tau(x) - \left[\tau'(x)/\tau(x)\right]^2 \\ &= 8\alpha^2 \phi(x)\,\phi'(x)/\tau(x) - \left[4\alpha^2 \phi(x)^2/\tau(x)\right]^2 \\ &\sim \frac{A}{x} \sin(2\omega x + \arg R(\omega)), \quad x \to +\infty, \end{aligned}$$

with some constant A. *These elementary arguments do not readily yield the coefficients in (3.21) though. As in the case of negative bound states (solitons) totally different arguments are needed to evaluate the coefficients (work in progress).*

Corollary 3.5 (bounded positons)**.** *Assume the conditions of Corollary 3.2. If* $q(x,t)$ *solves KdV with data* $S(q)$ *then*

$$q_{+N}(x,t) = q(x,t) - 2\partial_x^2 \log\det(\mathbf{I} + \mathbf{G}_+(x,t)), \tag{3.22}$$

where $\mathbf{G}_+(x,t)$ *is obtained from (3.2) by replacing* $\phi_n(x)$ *with*

$$\phi_n(x,t) = 2\,\mathrm{Re}\left[R(\omega_n)^{1/2}\,\mathrm{e}^{4\mathrm{i}\omega_n^3 t}\psi(x,t,\omega_n)\right],$$

solves KdV with data $S(q_{+N})$. *Moreover, embedded bound states* $\left(\omega_n^2\right)$ *are preserved under the KdV flow.*

Proof. Well-posedness of KdV under conditions of Corollary 3.2 is proven in [19], the time evolution of the scattering data $S(q)$ being the same as in the short-range case. For this reason, the main part of the proof goes along the same lines with that of Theorem 3.1. In particular, the embedded poles of $\psi_{+N}(x,t,k)$ by the very construction remain ω_n^2 and hence the time evolved diagonal Green's function has embedded poles at ω_n^2. One then concludes that embedded bound states (ω_n^2) are indeed preserved under the KdV flow. The only extra step required is to verify that $\psi_{+N}(x,t,k)$ solves the temporal part of the Lax pair equation. Such computations are performed in the literature for Darboux dressing. One can however check it independently. The simplest way to do it is, as always, to break our binary Darboux transformation into a chain of iterated transformations (3.20) that, adjusted for the time evolution, reads

$$\begin{aligned} q_{+n}(x,t) &= q_{+(n-1)}(x,t) \\ &\quad - 2\partial_x^2 \log\left(1+4\alpha_n^2\int_{-\infty}^{x} \mathrm{Re}^2\left[R(\omega_n)^{1/2}\,\mathrm{e}^{4\mathrm{i}\omega_n^3 t}\psi_{+(n-1)}(s,t,\omega_n)\right]\mathrm{d}s\right). \end{aligned}$$

□

In the KdV context, Matveev posed in [25] the following question: "The interesting question whether nonsingular positon solutions exists in the continuous integrable models remains open as yet." Corollary 3.5 answers his question in the affirmative (for one positon it was answer in our recent [28]). Matveev also conjectured that there may exist bounded positon solutions with a trivial scattering matrix (i.e. $R(k)=0$ and $T(k)=1$). Apparently Theorem 3.1 does not allow us to construct such solutions with a zero reflection coefficient.

Dmitry Pelinovsky asked the author "1) if the embedded eigenvalue disappears in the time evolution for $t>0$ and 2) if there is any impact of the embedded eigenvalues in the time evolution of KdV, e.g. propagation of an "embedded soliton" in the direction of linear dispersive waves?" One concludes from Corollary 3.5 that 1) the embedded eigenvalue does not disappear over time and 2) the effect of "embedded soliton" is manifested in the second log-derivative term of (3.22) which says that propagation of the ensemble of positons is determined by $4\omega_n^3 t+\omega_n x$ which is indeed in the direction of linear dispersive waves. Furthermore, we can show that there is a direct analog of (3.22) for (regular) solitons if we replace $\mathbf{G}_+(x,t)$ with the matrix

$$\left(c_m c_n \mathrm{e}^{8\left(\kappa_m^3+\kappa_n^3\right)t}\int_x^{\infty}\psi(s,t;\mathrm{i}\kappa_m)\,\psi(s,t;\mathrm{i}\kappa_n)\,\mathrm{d}s\right).$$

Here, as before, $(-\kappa_n^2)$ are negative bound states and (c_n^2) are associated norming constants. Thus both formulas are similar in nature and it is reasonable to expect that each soliton property has its positon counterpart. The main difference between the two is in-built in the profoundly different behavior of $\psi(x,t;\mathrm{i}\kappa_n)$ and $\psi(x,t,\omega_n)$: the former has finitely many zeros (n to be precise) while the latter has infinitely many zeros for any n.

Pelinovsky also asked "Does the "embedded solitons" disperse away in the time evolution?" Addressing this question amounts to understanding the behavior of $\psi(x,t,\omega_n)$ in the asymptotic regime around the "positon characteristic" $x=-12\omega_n^2 t$ as $t\to\infty$ (see our [28] for more detail). The main challenge is that $|R(\omega_n)|=1$ and the powerful nonlinear steepest descend method due to Deift-Zhou needs a serious modification, which to the best of our knowledge is only available in the case when

$|R(0)| = 1$ but less than 1 otherwise [7]. Note that in the NLS context and by totally different from [7] methods a treatment of the case $|R(\omega)| = 1$ was recently offered by Budylin [5]. A KdV adaptation of his techniques should yield the answer to the question if embedded solitons (bounded positons) will disperse away or not (i.e. present a KdV *breather*).

Remark 3.6. *Embedded bound states may not be created on a short-range background. Indeed we must have at least one real point $\omega \neq 0$ where $|R(\omega)| = 1$.*

Remark 3.7. *If q also has a Jost solution at $-\infty$ for a.e. $\operatorname{Im} k = 0$ then the transmission coefficient $T(k)$ is well-defined. It can be easily shown that*

$$T_{+N}(k) = T(k).$$

I.e., our binary Darboux transformation preserve both R and T. It follows from the conservation laws then that

$$\int_{-\infty}^{\infty} q_{+N}(x,t)\,\mathrm{d}x = \int_{-\infty}^{\infty} q(x,t)\,\mathrm{d}x,$$
$$\int_{-\infty}^{\infty} q_{+N}(x,t)^2\,\mathrm{d}x = \int_{-\infty}^{\infty} q(x,t)^2\,\mathrm{d}x.$$

4. Removing embedded bound states

In this section we show that we can as well remove (or rather pare) embedded bound states.

Theorem 4.1 (paring embedded eigenvalues)**.** *Assume Hypothesis 2.1. Let D be the set of embedded bound states of $\mathbb{L}_q$ and $D_0 = \{\omega_n^2, 1 \leq n \leq N < \infty\}$ be its subset such that ω_n^2 are simple and $R(k)$ defined by (2.13) and $(k-\omega_n)\psi(x,k)$ are functions continuous in $\operatorname{Im} k = 0$ at ω_n. If $\{\phi_n, 1 \leq n \leq N\}$ is an orthonormal set of real eigenfunction then the set of embedded eigenvalues of the potential*

$$q_{-N}(x) = q(x) - 2\partial_x^2 \log\det(\mathbf{I} - \mathbf{G}_-(x)),$$

where $\mathbf{G}_-$ is the Gram matrix defined by

$$\mathbf{G}_- := \left(\int_{-\infty}^{x} \phi_n(s)\,\phi_m(s)\,\mathrm{d}s\right),$$

coincides with $D \diagdown D_0$.

Proof. Our arguments go along the same lines with those in the proof of Theorem 3.1. Consider

$$\varphi_{-N}(x,k) := \varphi(x,k) + \sum_{n=1}^{N} y_n(x)\,\frac{W(\varphi(x,k),\phi_n(x))}{k^2-\omega_n^2},$$
$$\psi_{-N}(x,k) := \psi(x,k) + \sum_{n=1}^{N} y_n(x)\,\frac{W(\psi(x,k),\phi_n(x))}{k^2-\omega_n^2},$$

where φ, ψ are some Weyl solutions at $\mp\infty$ and y_n are real functions to be determined. By the Wronskian identity (2.9) ($\operatorname{Im} k > 0$)

$$\varphi_{-N}(x,k) := \varphi(x,k) + \sum_{n=1}^{N} y_n(x) \int_{-\infty}^{x} \varphi(s,k)\,\phi_n(s)\,\mathrm{d}s, \tag{4.1}$$

$$\psi_{-N}(x,k) := \psi(x,k) - \sum_{n=1}^{N} y_n(x) \int_x^\infty \psi(s,k)\,\phi_n(s)\,\mathrm{d}s. \tag{4.2}$$

As before, $\psi(x,k)$ is chosen to be a Jost solution at $+\infty$ and

$$\varphi(x,k) = \overline{\psi(x,k)} + R(k)\,\psi(x,k) \tag{4.3}$$

defines a Weyl solution at $-\infty$ for a.e. $\operatorname{Im} k = 0$. Since ω_n^2 is a bound state of $\mathbb{L}_q$ we conclude that the product $\varphi(x,k)\,\psi(x,k)$ has an embedded simple pole at ω_n. On the other hand, since $\psi(x,k)$ also has a embedded simple pole at ω_n, it follows from (4.3) and continuity that $\varphi(x,k+\mathrm{i}0)$ must be well defined at ω_n and different from zero. Since ω_n^2 is a simple eigenvalue, by Lemma 2.3 $\varphi(x,\omega_n+\mathrm{i}0)$ and ϕ_n are linearly dependent and thus $\varphi_{-N}(x,k)$ is well-defined at ω_n.

Turn to ψ_{-N}. From (4.2) one has

$$\operatorname*{Res}_{\omega_n} \psi_{-N}(x,k) := \psi_n(x) - \sum_{m=1}^{N} y_m(x) \int_x^\infty \psi_n(s)\,\phi_n(s)\,\mathrm{d}s,$$

where

$$\psi_n(x) := \operatorname*{Res}_{k=\omega_n} \psi(x,k),$$

is also an L^2 eigenfunction associated with ω_n^2. Since we want $\psi_{-N}(x,k)$ to be regular at ω_n, it follows that

$$\sum_{m=1}^{N} y_m(x) \int_x^\infty \psi_n(s)\,\phi_m(s)\,\mathrm{d}s = \psi_n(x).$$

Since ω_n^2 is a simple eigenvalue, ψ_n is proportional to ϕ_n and we arrive at the linear system

$$\sum_{m=1}^{N} y_m(x) \int_x^\infty \phi_m(s)\,\phi_n(s)\,\mathrm{d}s = \phi_n(x) \tag{4.4}$$

in (y_n). Its matrix

$$\left(\int_x^\infty \phi_m(s)\,\phi_n(s)\,\mathrm{d}s\right) = \mathbf{I} - \left(\int_{-\infty}^x \phi_m(s)\,\phi_n(s)\,\mathrm{d}s\right)$$

is Gram (in fact, positive definite) and hence the system has a unique solution for any finite x. Thus we have constructed two solutions $\varphi_{-N}(x,k)$, $\psi_{-N}(x,k)$ which are Weyl at $\mp\infty$ respectively and are regular at ω_n and hence so is the diagonal Green's function. Therefore, ω_n^2 is no longer an embedded bound state. □

Remark 4.2. *As is well-known, embedded bound states are unstable and may turn into resonances under an arbitrarily small perturbation* [3]*. Theorem 4.1 offers an explicit perturbation that purges only targeted embedded bound states.*

5. Explicit examples

In this section we work out an explicit example that clearly demonstrates how Theorems 3.1 and 4.1 apply shading, at the same time, some light on the nature of the conditions. We only consider the case of a single resonance ω. Without loss of generality, we can set $\omega = 1$. Our example is based on a construction from our [28]. Let

$$q_0(x) = -2\partial_x^2 \log \tau(x), \tag{5.1}$$

where (called the Hirota tau-function)

$$\tau(x) = 1 + 2\rho \int_0^{|x|} \sin^2 s \; \mathrm{d}s = 1 + \rho x - (\rho/2)\sin 2x \tag{5.2}$$

with some $\rho > 0$, and consider

$$q(x) = \begin{cases} q_0(x), & x < 0 \\ 0, & x \geq 0 \end{cases} . \tag{5.3}$$

One can easily see that $q(x)$ is continuous (but not continuously differentiable) and

$$q(x) \sim -\frac{4\sin 2x}{x}, \; x \to -\infty. \tag{5.4}$$

Thus, $q(x)$ is not short-range at $-\infty$ but in L^2 and it is certainly subject to Hypothesis 2.1. The main feature of $q(x)$ is that $\mathbb{L}_q$ admits explicit spectral and scattering theories. In particular, for the transmission T and right/left reflection R, L coefficients we have [28]

$$T(k) = \frac{P(k)}{P(k) + \mathrm{i}\rho}, \quad R(k) = \frac{-\mathrm{i}\rho}{P(k) + \mathrm{i}\rho} = L(k), \tag{5.5}$$

where $P(k) := k^3 - k$. The right Jost solution (recalling our agreement to drop $+$ sing) is apparently

$$\psi(x, k) = \mathrm{e}^{\mathrm{i}kx}, x \geq 0. \tag{5.6}$$

For the left Jost solution we have [28]

$$\psi_-(x, k) = \mathrm{e}^{-\mathrm{i}kx} - \left(\frac{\mathrm{e}^{-\mathrm{i}(k+1)x}}{k+1} - \frac{\mathrm{e}^{-\mathrm{i}(k-1)x}}{k-1} \right) \frac{\rho \sin x}{\tau(x)}, \quad x < 0, \tag{5.7}$$

where $\tau(x)$ is given by (5.2). Apparently, $\psi(x, k)$ and $R(k)$ are analytic at $k = 1$ and hence condition 2 of Theorem 3.1 is satisfied. Since $(k-1)\psi_-(x, k)$ is also a solution, we immediately conclude from (5.7) that

$$\varphi_0(x) = \frac{\sin x}{\tau(x)} = \frac{\sin x}{1 + 2\rho \int_0^{|x|} \sin^2 s \; \mathrm{d}s}, \qquad x < 0, \tag{5.8}$$

is clearly an $L^2(-\infty)$ solution and therefore condition 1 of Theorem 3.1 is also satisfied. Thus, Theorem 3.1 applies to our $q(x)$. We do not need to know $\varphi_0(x)$ for $x \geq 0$ yet (will be explicitly found later) but it is clear already that $+1$ is not a positive eigenvalue since a linear combination of plane waves $\mathrm{e}^{\pm\mathrm{i}x}$ is never in $L^2(+\infty)$. Thus $+1$ is a resonance of $\mathbb{L}_q$. This should also explain why we call condition 1 in Theorem 3.1 resonance.

Observe that $\varphi_0(0) = 0$ and hence $+1$ is a positive bound state of $\mathbb{L}_q^D$ on $L^2(\mathbb{R}_-)$ with a Dirichlet condition at 0.

Let us now apply Theorem 3.1 to our $q(x)$. Equation (3.19) reads

$$q_{+1}(x) = q(x) - 2\partial_x^2 \log\left(1 + \alpha^2 \int_{-\infty}^x \phi(s)^2 \, \mathrm{d}s\right), \tag{5.9}$$

where $\phi(s) = -\operatorname{Re}\left[R(1)^{1/2}\psi(s, 1)\right]$. Note that we chose minus sign for convenience. Evaluate

$$\phi(s) = -\lim \operatorname{Re}\left[R(k)^{1/2}\psi(s, k)\right], \quad k \to 1, \quad \operatorname{Im} k = 0.$$

It follows from (5.6) and (5.5) that for $s \geq 0$

$$\phi(s) = -\operatorname{Re}\left(\mathrm{i}e^{\mathrm{i}s}\right) = \sin s, \quad s \geq 0. \tag{5.10}$$

The case $s < 0$ needs some work as we do not know $\psi(s,k)$ on $\mathbb{R}_-$ yet. We compute it from the left basic scattering relation (cf. (2.11))

$$T(k)\,\psi(s,k) = \overline{\psi_-(s,k)} + L(k)\,\psi_-(s,k), \quad \operatorname{Im} k = 0.$$

It follows from (5.5) that $L(k) = T(k) - 1$ and hence

$$\begin{aligned}\psi(s,k) &= \frac{1}{T(k)}\left[\overline{\psi_-(s,k)} + (T(k)-1)\,\psi_-(s,k)\right] \\ &= \psi_-(s,k) + \frac{\overline{\psi_-(s,k)} - \psi_-(s,k)}{T(k)} \\ &= \psi_-(s,k) + \frac{P(k)+\mathrm{i}\rho}{P(k)}\left[\overline{\psi_-(s,k)} - \psi_-(s,k)\right].\end{aligned}$$

Thus

$$\psi(s,k) = \overline{\psi_-(s,k)} + \frac{2\rho}{P(k)}\operatorname{Im}\psi_-(s,k), \quad s < 0. \tag{5.11}$$

Observe that it is not clear why (5.11) is regular at $k = 1$ where $P(k)$ vanishes (but the general theory says that it is the case). It is an amusing exercise to demonstrate it directly. Since we only need the real part of it our computation will be easy:

$$\begin{aligned}&\operatorname{Re}\left[R(1)^{1/2}\,\psi(s,1)\right] \\ &= \lim_{k\to1}\operatorname{Re}\left[R(k)^{1/2}\,\overline{\psi_-(s,k)}\right] + 2\rho\lim_{k\to1}\frac{\operatorname{Re}R(k)^{1/2}}{P(k)}\lim_{k\to1}\operatorname{Im}\psi_-(s,1).\end{aligned} \tag{5.12}$$

Evaluate each of these limits separately. We start with the observation that as $k \to 1$

$$R(k) = -1 - \frac{\mathrm{i}}{\rho}P(k) \sim -1 - \frac{2\mathrm{i}}{\rho}(k-1),$$

and hence along the real line

$$\operatorname{Re}R(k)^{1/2} \sim \cos\left(\frac{\pi}{2} - \frac{k-1}{\rho}\right) = \sin\frac{k-1}{\rho}, \quad k \to 1.$$

We now immediately see that

$$\lim_{k\to1}\frac{\operatorname{Re}R(k)^{1/2}}{P(k)} = \frac{1}{\rho}. \tag{5.13}$$

It follows from (5.7) and (5.8) that for $s < 0$

$$\psi_-(s,k) = e^{-\mathrm{i}ks} - \rho\left(\frac{e^{-\mathrm{i}(k+1)s}}{k+1} - \frac{e^{-\mathrm{i}(k-1)s}}{k-1}\right)\varphi_0(s)$$

and we then have

$$\begin{aligned}\operatorname{Im}\psi_-(s,1) &= -\sin s - \frac{\rho}{2}(2s - \sin 2s)\,\varphi_0(s) \quad ((5.8)\text{ and }(5.2)) \\ &= -\sin s - \frac{\rho}{2}(2s - \sin 2s)\frac{\sin s}{1 - \rho s + (\rho/2)\sin 2s} \\ &= -\frac{\sin s}{1 - \rho s + (\rho/2)\sin 2s} = -\varphi_0(s), \quad s < 0.\end{aligned}$$

Thus $\operatorname{Im}\psi_-(s,1)$ is continuous at $k=1$ and

$$\operatorname{Im}\psi_-(s,1) = -\varphi_0(s), \quad s<0, \tag{5.14}$$

which also implies that for the first limit on the right hand side of (5.12) one must have

$$\lim_{k\to 1}\operatorname{Re}\left[R(k)^{1/2}\,\overline{\psi_-(s,k)}\right] = 0. \tag{5.15}$$

Substituting (5.13)-(5.15) into (5.12), we arrive at

$$\operatorname{Re}\left[R(1)^{1/2}\,\psi(s,1)\right] = -\varphi_0(s), \quad s<0. \tag{5.16}$$

Combining (5.10) with (5.16) we finally have

$$\phi(s) = -2\operatorname{Re}\left[R(1)^{1/2}\,\psi(s,1)\right] = 2\begin{cases}\varphi_0(s), & s<0\\ \sin s, & s\ge 0\end{cases}.$$

Thus, ϕ is a solution that square integrable at $-\infty$ and proportional to the sine function on $\mathbb{R}_+$. We are now able to find $q_{+1}(x)$ explicitly by (5.9). Indeed, for $x<0$

$$\begin{aligned} I(x) &:= \int_{-\infty}^{x}\phi(s)^2\,\mathrm{d}s = 4\int_{-\infty}^{x}\varphi_0(s)^2\,\mathrm{d}s \\ &= \int_{-\infty}^{x}\frac{4\sin^2 s\ \mathrm{d}s}{\left(1+2\rho\int_0^{-s}\sin^2 t\ \mathrm{d}t\right)} = -\frac{2}{\rho}\int_{-\infty}^{x}\frac{\mathrm{d}\tau(s)}{\tau(s)^2} = \frac{2}{\rho}\frac{1}{\tau(x)}. \end{aligned} \tag{5.17}$$

Note that, in particular,

$$\int_{-\infty}^{0}\phi(s)^2\,\mathrm{d}s = \frac{2}{\rho}.$$

For $x\ge 0$

$$\begin{aligned} I(x) &= \int_{-\infty}^{0}\phi(s)^2\,\mathrm{d}s + 4\int_0^x \sin^2 s\ \mathrm{d}s \\ &= \frac{2}{\rho}\left(1+2\rho\int_0^x\sin^2 s\ \mathrm{d}s\right) = \frac{2}{\rho}\tau(x). \end{aligned} \tag{5.18}$$

Substituting (5.17) and (5.18) into (5.9) yields

$$\begin{aligned} q_{+1}(x) &= q(x) - 2\partial_x^2\log\left(1+\alpha^2 I(x)\right) \\ &= q(x) - 2\partial_x^2\log\left(1+\frac{2\alpha^2}{\rho}\begin{cases}1/\tau(x), & x<0\\ \tau(x), & x\ge 0\end{cases}\right). \end{aligned}$$

This formula can be simplified nicely if we recall what our seed potential $q(x)$ is. Indeed, from (5.1)-(5.3) we have for $x<0$

$$\begin{aligned} q_{+1}(x) &= -2\partial_x^2\log\tau(x) - 2\partial_x^2\log\left(1+\frac{2\alpha^2}{\rho}1/\tau(x)\right) \\ &= -2\partial_x^2\log\left(1+\frac{\rho}{2\alpha^2}\tau(x)\right) \end{aligned}$$

and for $x>0$

$$\begin{aligned} q_{+1}(x) &= -2\partial_x^2\log\left(1+1/\tau(x)\right) \\ &= -2\partial_x^2\log\left(1+\frac{2\alpha^2}{\rho}\tau(x)\right), \end{aligned}$$

which can be conveniently put in one formula

$$q_{+1}(x) = -2\partial_x^2 \log\left(1+\left(\frac{\rho}{2\alpha^2}\right)^{\pm 1}\tau(x)\right), \quad \pm x > 0, \tag{5.19}$$
$$\tau(x) = 1 + 2\rho\int_0^{|x|} \sin^2 s \mathrm{d}s.$$

By Theorem 3.1, the Schrodinger operator with the potential given by (5.19) has an embedded eigenvalue $+1$.

There is a point in analyzing (5.19).

- One easily sees that

$$q_{+1}(x) \sim -4\,\frac{\sin 2x}{x}, \ |x| \to \infty.$$

  Thus, all q_{+1} share same large x asymptotics. Recall, that the seed potential q has this asymptotic behavior only at $-\infty$ and thus q_{+1} is long-range at $+\infty$ as well. This agrees, of course, with (3.21) with $A = -4$ and $\delta = 0$.
- By Corollary 3.2, the family of potentials given by (5.19) share the same scattering quantities (5.5) providing yet another example of the failure of the classical inverse scattering in the long-range setting. Recall, that in the short-range scattering $|R(k)| < 1$ for $k \neq 0$, which is clearly violated in our example as $R(\pm 1) = -1$.
- The function (5.19) is even if and only if $\rho = 2\alpha^2$. In this case,

$$q_{+1}^{\mathrm{Sym}}(x) = -2\partial_x^2 \log\left(1+\rho\int_0^{|x|}\sin^2 s\mathrm{d}s\right), \tag{5.20}$$

  which is the main example of an explicit Wigner-von Neumann type potential studied in [28] that has an embedded bound state $+1$. Note that there is no value of α that produces odd $q_{+1}(x)$.
- Turn now to the eigenfunction of $+1$. The system (3.4) simplifies to the single equation

$$\left(1+\alpha^2\int_{-\infty}^x \phi(s)^2\,\mathrm{d}s\right)y = \alpha\phi(x)$$

  for the eigenfunction y:

$$y(x) = \frac{\alpha\phi(x)}{1+\alpha^2\int_{-\infty}^x\phi(s)^2\,\mathrm{d}s}, \tag{5.21}$$

  which, as one can easily compute, has L^2 norm 1. It is worth noticing that as apposed to the right Jost solution $\psi(x,k)$ corresponding to the seed potential $q(x)$, by (3.8) the transformed Jost solution

$$\begin{aligned}&\psi_{+1}(x,k) \\ &= \mathrm{e}^{\mathrm{i}kx}\left\{1+\left(\frac{\mathrm{e}^{\mathrm{i}x}}{k+1}-\frac{\mathrm{e}^{-\mathrm{i}x}}{k-1}\right)\frac{\alpha^2\phi(x)}{1+\alpha^2\int_{-\infty}^x\phi(s)^2\,\mathrm{d}s}\right\}, \quad x \geq 0,\end{aligned} \tag{5.22}$$

indeed has a simple poles at $k=\pm 1$, as expected. It follows from (3.12) that[8]

$$\alpha=\left\|\operatorname*{Res}_{k=1}\psi_{+1}\left(\cdot,k\right)\right\|. \tag{5.23}$$

Recall that for the right norming constant of a negative bound state $-\kappa^2$ of a generic potential we have

$$c=\left\|\psi\left(\cdot,\mathrm{i}\kappa\right)\right\|^{-1}.$$

Comparing this with (5.23) suggests a new definition for a right norming constant of an embedded bound state (at least in the case of a single embedded bound state).

- Let us now briefly discuss how Theorem 4.1, removing embedded bound states, applies to our example. For simplicity, we consider $q_{+1}^{\mathrm{Sym}}\left(x\right)$ defined by (5.20) that has an embedded bound state $+1$. Check the conditions of Theorem 4.1. It follows from the general theory of Winger-von Neumann type potentials (see e.g. [8]) that $+1$ is necessarily simple eigenvalue. Indeed, for $k=1$ the Schrodinger equation has only one decaying solution (the other solution is increasing). It follows from (2.13) and (5.22) that $R\left(k\right)$ and $\left(k-1\right)\psi_{+1}\left(x,k\right)$ are both continuous (in fact, analytic) at $k=\pm 1$. Therefore, 4.1 applies to our $q_{+1}^{\mathrm{Sym}}\left(x\right)$. Performing computation similar to given above, one concludes that the transformed potential $q_{-1}\left(x\right)$ indeed coincides with $q\left(x\right)$ given by (5.3).
- Finally, we turn to the time evolution $q_{+1}\left(x,t\right)$ of $q_{+1}\left(x\right)$ under the KdV flow. Unfortunately, we no longer have an explicit formula and it is unreasonable to expect one[9]. Equation (3.22) in our case reads

$$q_{+1}\left(x,t\right)=q\left(x,t\right)-\partial_x^2\log\left(1+\alpha^2\int_{-\infty}^{x}\phi\left(s,t\right)^2\mathrm{d}s\right), \tag{5.24}$$
$$\phi\left(s,t\right)=2\operatorname{Im}\left[\mathrm{e}^{4\mathrm{i}t}\psi\left(s,t,1\right)\right].$$

Since $q\left(x\right)$ is supported on $\mathbb{R}_-$ and clearly bounded below, the results of our [18, 19, 27] apply and we have

$$q\left(x,t\right)=-\partial_x^2\log\det\left(I+\mathbb{H}\left(x,t\right)\right), \tag{5.25}$$

where $\mathbb{H}\left(x,t\right)$ is a trace class singular integral operator (in fact, Hankel) defined on the Hardy space H^2 of the upper half plane by

$$\mathbb{H}\left(x,t\right)f\left(k\right)=-\int_{\mathbb{R}}\frac{\Phi_{x,t}\left(s\right)f\left(s\right)}{s+k+\mathrm{i}0}\frac{\mathrm{d}s}{4\pi^2},\quad f\in H^2,$$

where the entire function $\Phi_{x,t}$ is given by

$$\Phi_{x,t}\left(s\right):=\int_{\operatorname{Im}z=b}\frac{R\left(z\right)\mathrm{e}^{\mathrm{i}\left(8z^3t+2zx\right)}}{z-s}\mathrm{d}z,\quad R\left(z\right)=\frac{-\mathrm{i}\rho}{z\left(z^2-1\right)+\mathrm{i}\rho}.$$

Here the line of integration $\operatorname{Im}z=b$ is chosen above the (only one) imaginary pole of $R\left(z\right)$. The determinant in (5.25) is infinite for $t>0$ and so

[8]Without loss of generality we can always assume that $\alpha>0$.

[9]Recall that for singular positons such a formula does exist [25]

(5.25) is only explicit at $t=0$, where it returns the initial profile (5.3). The right Jost solution for $q(x,t)$ can then be found by

$$\psi(x,t,k) = \mathrm{e}^{\mathrm{i}kx}\left\{1-(I+\mathbb{H}(x,t))^{-1}\,\mathbb{H}(x,t)\,1\right\},$$

where

$$\mathbb{H}(x,t)\,1 = -\int_{\mathbb{R}} \frac{\Phi_{x,t}(s)}{s+k+\mathrm{i}0}\frac{\mathrm{d}s}{4\pi^2},$$

which is well-defined even and in H^2 (though 1 is not in H^2). This step requires an inversion of the operator $I+\mathbb{H}(x,t)$, which does not come with an explicit formula. The KdV solution $q_{+1}(x,t)$ is then computed by (5.24). For $q_{+1}^{\mathrm{Sym}}(x)$ a different derivation of (5.24) is obtained by different means in our [28]. The first term in (5.24), is nothing but the classical *Dyson formula*. It looks exactly like the one in the short-range case but of course $q(x,0)=q(x)$ is not a short range potential at $-\infty$. Thus $q(x,t)$ comes from data with the missing embedded eigenvalue. On the other other hand, the second term in (5.24) takes into account the bound state $+1$. It resembles the (singular) positon solution

$$q_{\mathrm{pos}}(x,t) = -2\partial_x^2 \log\left\{1+x+12t-(1/2)\sin 2(x+4t)\right\}. \tag{5.26}$$

Such solutions seem to have appeared first in the late 70s earlier 80s but a systematic approach was developed a decade later by V. Matveev (see his 2002 survey [25]). Equation (5.26) readily yields basic properties of one-positon solutions considered in [25]. As a function of the spatial variable $q_{\mathrm{pos}}(x,t)$ has a double pole real singularity which oscillates in the 1/2 neighborhood of the moving point $x=-12t-1$, and for a fixed $t\geq 0$

$$q_{\mathrm{pos}}(x,t) \sim -4\frac{\sin 2(x+4t)}{x}, \quad x\to\pm\infty. \tag{5.27}$$

Observe that

$$q_{\mathrm{pos}}(x,0) = -2\partial_x^2\log(1+x-(1/2)\sin 2x)$$

coincides on $\mathbb{R}_+$ with our

$$q_{+1}^{\mathrm{Sym}}(x) = -2\partial_x^2\log(1+(\rho/2)\,x--(\rho/4)\sin 2x),$$

for $\rho=2$. But, of course, $q_{+1}^{\mathrm{Sym}}(x)$ is bounded on $\mathbb{R}_-$ while $q_{\mathrm{pos}}(x,0)$ is not. Note also that the positon is somewhat similar to the soliton given by

$$q_{\mathrm{sol}}(x,t) = -2\partial_x^2\log\cosh(x-4t) \tag{5.28}$$

but its double pole singularity moves in the opposite direction (i.e. to $-\infty$) three times as fast. We note that multi-positon as well as soliton-positon solutions have been studied in great detail (see [25] the references cited therein). We can also construct an explicit example of bounded multi-positon solutions to demonstrate Theorem 3.1 for any N. We hope to do this elsewhere.

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