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A nonlinear observer for bilinear systems in block form

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ABSTRACT

This paper presents the design of a high-gain nonlinear observer for bilinear systems in the block form. We prove the uniform exponential stability of the observer error by finding a concrete exponential bound. In particular, we extend the result of O. I. Goncharov on the exponential stability of high-gain bilinear observers into observers of block forms. Our work results in a generalization of that work. We also study and simulate two examples of observers for bioreactor systems in two and three dimensions. Finally, the results show the effectiveness of the proposed approach.

1. Introduction

Among the different nonlinear observers' approaches, the high-gain observers (HGO) are among the most important in the control community. A high-gain observer is a dynamical system with a corrective term involving the observer gain factor, $\theta > 0$, chosen big enough to guarantee the convergence of observation error. Usually, this parameter appears by positive exponents on different lines or subspaces of the state space, where its highest power is the dimension of the system. In this case, the gain factor θ enters into a numerical estimation problem for the system when a relatively high value is under demand.

There can be a classification of high-gain systems into two branches. The first is the class of uniformly observable systems, i.e., high-gain observers where the system is observable from any point. The second class is non-uniform observer systems, in which the input is required to satisfy excitation conditions to ensure observability. An example of the latter kind is considered in [15], in which the authors study a class of Multi-Input/Multi-Output observers where the gain factor varies with time.

This paper follows certain hierarchies of high-gain observer systems to study the uniform observation problem for an HGO in block matrix form when the blocks are all the exact sizes. First, we explain a high-gain hierarchy for linear systems of special block forms. Our proof is based on a determinantal formula for matrices with blocks of the same size when the different blocks commute. Next, we employ some results in [28] to formulate our contribution. In addition, we show a similar result for regular nonlinear systems in triangular form.

1.1. Related work

Recently some solutions have been made toward the stability of nonlinear systems, [1–3,6–8,11,14,34,37,39–41,54]. Bilinear systems are a particular case of nonlinear systems, prevalent for modeling special systems such as biological ones. In some of these bilinear systems, various observation problems are comparatively complicated. Among these is the uniform observation problem for extended bilinear systems in block form. The problem here is to find a uniform estimate for the state vector.

The uniform observability question for bilinear systems is of intense interest from a practical point of view. For instance, different criteria for uniform observability have been studied in [5,22,25,27,29–32,56,57,59,60]. When the problem is solvable, we say the system is uniformly observable. One way to attain this is to add sufficient constraints to the system. One can put the constraints on the input $u = u(t)$ like to be sufficiently smooth or even linear, bounded or having known estimates, to obtain the uniform observation property. The possibility of reconstructing the state vector of a bilinear system in the general case depends on the input. One can impose additional constraints on the system's structure to ensure uniform observability. In this regard, a criterion for the uniform observability of scalar systems was originally obtained in the paper [59], where an observer based on the differentiation of the inputs was suggested.

This paper is motivated by the two works [28], and [15] on solving the observer design problem for specific bilinear systems in

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dimension n. Ref. [15] studies the error dynamics of the extended high-gain bilinear system in block form in the presence of perturbation noise. The authors prove the existence of a bound on the high-gain observer's error dynamics. However, the bound is complicated, and their proof is very long. On the other hand, in reference [28], the author proves the uniform and exponential stability of the bilinear system in the scalar variables, i.e., for a SISO system. The motivation to generalize the result of [28] to the block matrices is to consider MIMO systems, a natural generalization of SISO systems. Therefore, a significant effort in this paper is to prove a high-gain hierarchy for a bilinear MIMO system in a block form when the size of the blocks is the same.

1.1.1. Contribution

On the one hand, the contribution is in the generalization of the method presented in reference [28] in which Goncharov addresses the uniform observation problem for the bilinear system in a block form when the blocks have the same size. On the other hand, in reference [15], a form of boundedness stability for observers of systems in the form $\dot{x} = F(u,x)x + \phi(u,x)$ has been proved. Then, we prove a stronger statement with more abstract methods. Also, we demonstrate a nonlinear observer's uniform exponential stability by finding a more concrete exponential bound (by a different approach). Moreover, our method proves the uniform boundedness of the error, a stronger result than the one in [15]. Finally, we use the feedback linearization of observer systems [2,8,13,43], to prove a similar uniform exponential stability theorem in nonlinear observer systems. Besides, to investigate our method's behavior, we apply the observer to two versions of a bioreactor model; the bioreactor explains a bacteria growth model and population distribution [10]. Simulations are presented for SISO and MISO bioreactor systems.

This observer has many possible applications, especially in control schemes such as output feedback or active disturbance rejection control [20,46]. Observing the complete or even an augmented state is required in those scenarios. Examples of real systems of that kind are mechatronic systems [38], UAVs control, [16,19], visual SLAM [12], aerospace control [17], micromechanical devices [18,21], and flexible systems [44,45].

1.2. Content

The remainder of this paper is organized as follows. Section 2 presents observer design for systems in bilinear form. Section 3 demonstrates the exponential stability of the proposed observer for bilinear systems of the form,

$$\dot{x} = Ax + (Bu)x, \qquad y = Cx.$$

Then, in Section 4 results are extended for bilinear systems in the form,

$$\dot{x} = F(u,x)x + \phi(u,x), \qquad y = Cx.$$

We present two cases of study in Section 5: SISO and MISO versions of the bioreactor system, where the proposed observer is applied and simulated. Finally, some conclusions and future work are given in Section 6.

2. Problem setting

In [28], O. I. Goncharov poses the problem of uniform observation for a scalar bilinear system:

$$\dot{x} = Ax + u(Bx), \qquad y = Cx \tag{1}$$

where $x \in \mathbb{R}^n,\ u \in \mathbb{R}^1,\ A, B \in \mathbb{R}^{n\times n},\ C \in \mathbb{R}^{1\times n}$, are constant matrices. One may assume without loss of generality that A, B and C are in canonical controllability forms [37], that is,

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ a_1 & a_2 & a_3 & \dots & a_n \end{pmatrix},$$

$$B = \begin{pmatrix} b_{11} & 0 & 0 & \cdots & 0 \\ b_{21} & b_{22} & 0 & \cdots & 0 \\ b_{31} & b_{32} & b_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & b_{n3} & \cdots & b_{nn} \end{pmatrix}, \quad C = [1\ 0\ 0\ \cdots\ 0]. \tag{2}$$

An observer for system (1) is

$$\dot{\hat{x}} = A\hat{x} + uB\hat{x} + K(y - C\hat{x}), \qquad \hat{x}, y \in \mathbb{R}^n. \tag{3}$$

where $K = [k_1, \dots, k_n]^T$ is a column block matrix of feedback coefficients, and $u(t)$ is bounded [28]. Error $e = x - \hat{x}$ satisfies

$$\dot{e} = A_0 e + uBe \tag{4}$$

where

$$A_0 = A - KC = \begin{pmatrix} -k_1 & 1 & 0 & \cdots & 0 \\ -k_2 & 0 & 1 & \cdots & 0 \\ -k_3 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -k_{n-1} & 0 & 0 & \cdots & 1 \\ -k_n + a_1 & a_2 & a_3 & \cdots & a_n \end{pmatrix} \tag{5}$$

One uses the standard hierarchy of high-gain observers such that the eigenvalues of the matrix A_0 can be proportional to a high-gain factor [see [28] lemma 1]. The hierarchy states that if the feedback coefficients k_i are big enough, then the eigenvalues of the matrix A_0 are all real and negative (A is Hurwitz). Also, the contribution of the coefficients a_i in the characteristic polynomial of A can be neglected; see [22,57]. Goncharov proves that the error $e(t)$ is exponentially bounded, that is, there exist constants $M, a_0 > 0$ and a polynomial $P(\theta)$ such that,

$$\parallel e \parallel \leqslant P(\theta)\exp((-(\theta - a_0)t)), \qquad (\theta > M). \tag{6}$$

where θ is called *the high-gain factor*. We generalize this result for similar systems where the matrices A, B, and C are in block forms with blocks of the same size. We state that in the following problem statement.

2.1. Problem statement

This section presents the problem statement related to the design of observers, divided into two cases: a linear version and a nonlinear version of a dynamic system. Let us consider first the case of the linear one.

Problem 1 (Linear case)**.** Design an observer for the MIMO bilinear system of the form,

$$\dot{x} = Ax + (Bu)x, \qquad y = Cx, \tag{7}$$

where $x = [x_1,\ x_2,\ \dots,\ x_q]$, with $x_i \in \mathbb{R}^r, u \in \mathbb{R}^r$, and $A,\ B \in \mathbb{R}^{qr\times qr}$ and $qr = n$. Notice that the system above has the same shape as (2) but with entries given by block matrices of size $r \times r$. Besides, $u = \mathrm{diag}[u_1 I_r,\ \dots,\ u_q I_r]$, $C = [I_r,\ 0,\ \dots,\ 0]$ hold.

In addition, we consider the multi-input multi-output (MIMO) nonlinear system, for which one has to design an observer. The problem is as follows.

Problem 2 (Nonlinear case)**.** Consider the multi-input multi-output (MIMO) nonlinear system,

$$\dot{x} = F(u,x)x + \phi(u,x), \qquad y = Cx, \tag{8}$$

where $x \in \mathbb{R}^{qr}$, and

$$F(u,x) = \begin{pmatrix} 0 & F_1 & 0 & \cdots & 0 \\ 0 & 0 & F_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & F_{q-1} \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \qquad \phi(u,x) = \begin{bmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_q \end{bmatrix},$$

where $F_k = F_k(u, x_1, \ldots, x_q)$, $\phi_k = \phi_k(u, x_1, \ldots, x_k)$, are vector functions and $u \in \mathbb{R}^r$, $x_i \in \mathbb{R}^r$ and $C = [I_r \quad 0 \ldots \; 0]$. The problem is to find an observer for the aforementioned bilinear MIMO system.

Remark 1. It is known that under some regularity conditions, a vast majority of control systems can be brought into the block form (7) and (8) by some change of coordinates, [1,15,28,37]. Thus, the motivation for using block matrices is to consider MIMO systems (which generalize SISO systems) when the blocks are the same size.

3. Solution to problem 1: a high-gain observer for bilinear systems of the form $\dot{x} = Ax + (Bu)x$

In this section, we present a solution to the observation problem 1 for a bilinear system of the form,

$$\dot{x} = Ax + (Bu)x, \quad y = Cx \tag{9}$$

where

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_q \end{bmatrix}, \; x_i \in \mathbb{R}^r, \; qr = n, \qquad A = \begin{pmatrix} 0 & I_r & 0 & \cdots & 0 \\ 0 & 0 & I_r & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & I_r \\ A_1 & A_2 & A_3 & \cdots & A_q \end{pmatrix}, \; A_k \in \mathbb{R}^{r\times r}$$

$$A_iA_j = A_jA_i, \qquad i,j = 1, \ldots, q \tag{10}$$

$$B = \begin{pmatrix} B_{11} & 0 & 0 & \cdots & 0 \\ B_{21} & B_{22} & 0 & \cdots & 0 \\ B_{31} & B_{32} & B_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ B_{q1} & B_{q2} & B_{q3} & \cdots & B_{qq} \end{pmatrix}, \; B_{ij} \in \mathbb{R}^{r\times r}.$$

With that aim, we consider the following assumptions.

Assumption 1. The matrices A_i pairwise commute, i.e., $A_iA_j = A_jA_i$, $i,j = 1, \ldots, q$, and $C = [I_r, \; 0, \; \ldots \; 0]$.

Assumption 2. The control input is structured as $u = \mathrm{diag}[u_0 I_r, u_1 I_r, \ldots, u_{q-1} I_r]$, where $u_i \in \mathbb{R}$ are assumed to be bounded.

Considering assumptions 1 and 2 the proposed observer for (9) is given by

$$\dot{\hat{x}} = A\hat{x} + (Bu)\hat{x} + K(y - C\hat{x}), \qquad \hat{x}, y \in \mathbb{R}^n \tag{11}$$

where $K = [K_1 I_r, \; \ldots \; K_q I_r]^T$ is a column block matrix of feedback coefficients. The error $e = x - \hat{x}$ has dynamics given by,

$$\dot{e} = A_0 e + (Bu)e \tag{12}$$

where

$$A_0 = A - KC = \begin{pmatrix} -K_1 I_r & I_r & 0 & \cdots & 0 \\ -K_2 I_r & 0 & I_r & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -K_{q-1} I_r & 0 & 0 & \cdots & I_r \\ -K_q I_r + A_1 & A_2 & A_3 & \cdots & A_q \end{pmatrix}. \tag{13}$$

The matrix A_0 can be decomposed as,

$$A_0 = \bar{A} + \hat{A} \tag{14}$$

with

$$\bar{A} = \begin{pmatrix} -K_1 I_r & I_r & 0 & \cdots & 0 \\ -K_2 I_r & 0 & I_r & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -K_{q-1} I_r & 0 & 0 & \cdots & I_r \\ -K_q I_r & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad \hat{A} = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A_1 & A_2 & A_3 & \cdots & A_q \end{pmatrix}. \tag{15}$$

In the following, we present and prove the theorem that is the spirit of the main result. It generalizes Lemma 1 in the work of O. I. Goncharov [28]. The theorem states that the feedback coefficients K_i above can be chosen high enough such that the eigenvalues of A_0, (denoted by $\mathrm{Sp}(A_0)$) are all negative, distinct, and proportional to the high-gain factor θ [25,59]. Besides, the effect of the matrix $\hat{A}$ on the sign of the eigenvalues of A_0 can be neglected.

Theorem 3.1. *Assume we start from a set of n real numbers,*

$$\mathrm{Sp}(\bar{A}) = \{\bar{\alpha}_1, \ldots, \bar{\alpha}_n\}, \;\; \bar{\alpha}_i \neq \bar{\alpha}_j, \;\; i \neq j \tag{16}$$

and consider the feedback coefficients as

$$K_i(\theta) = \overline{K_i}\theta^i + o(i) \tag{17}$$

where θ is the gain factor and the coefficients $\overline{K_i}$ are defined via

$$\bar{\varrho}(s) := \prod_{i=1}^{n}(s - \bar{\alpha}_i) = |s^q I_r + \overline{K_1} s^{q-1} I_r + \ldots + \overline{K_q} I_r|; \tag{18}$$

and $o(i)$ are the terms of degree strictly less than i in θ. Then, the characteristic polynomial of $A_0 = A - KC$ in (13) corresponding to the error observer (12) is of the form

$$\varrho(s) = |s^q I_r + K_1(\theta)s^{q-1} I_r + \ldots + K_q(\theta) I_r| \tag{19}$$

where $K_i(\theta)$ was defined above.

Before presenting the proof of Theorem 3.1, we present the following proposition required for the proof.

Proposition 3.1 [42,53]**.** *If*

$$A = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1q} \\ A_{21} & A_{22} & \cdots & A_{2q} \\ \vdots & \vdots & \ddots & \vdots \\ A_{q1} & A_{q2} & \cdots & A_{qq} \end{pmatrix} \tag{20}$$

is a $q \times q$ block matrix of $A_{ij} \in \mathbb{R}^{r\times r}$ such that $A_{ij}A_{kl} = A_{kl}A_{ij}$ for all entries, then[1]

$$|A| = \left| \sum_{\sigma \in S_q} A_{1\sigma(1)} \ldots A_{q\sigma(q)} \right|, \tag{21}$$

where S_q is the symmetric group of permutations on $\{1, \ldots, q\}$.

Remark 2. The above formula has been proved by induction and the axiomatic definition of determinant functions in the different references. It can also be proved using the method of Schur complements in matrix analysis, see for instance, references [48–52,55,61].

We are ready to present the proof of Theorem 3.1.

[1] We denote the sum inside the above determinant as $D(A) = \sum_{\sigma \in S_q} A_{1,\sigma(1)} \ldots A_{q,\sigma(q)}$, associated to the block matrix A in (20), where S_q is the symmetric group on q elements.

Proof of Theorem 3.1.. Let us consider the matrix,

$$sI - A_0 = \begin{pmatrix} (K_1+s)I_r & -I_r & \cdots & 0 & 0 \\ K_2 I_r & sI_r & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ K_{q-1}I_r & 0 & \cdots & sI_r & -I_r \\ K_q I_r - A_1 & -A_2 & \cdots & -A_{q-1} & sI_r - A_q \end{pmatrix}. \quad (22)$$

Expanding and grouping the block entries as the Proposition 3.1, similar to the argument in [28], we obtain,

$$|sI - A_0| = \left| sD(N_1(1)) + \sum_{k=1}^{q-1} (-1)^{k-1} D(N_k(s)) + (-1)^q A_1 (-1)^{q-1} D(N_q)(s) \right| \quad (23)$$

where

$$N_p = \begin{pmatrix} sI_r & -I_r & 0 & \cdots & 0 \\ 0 & sI_r & -I_r & \cdots & 0 \\ 0 & 0 & sI_r & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -A_{p+1} & -A_{p+2} & -A_{p+3} & \cdots & sI_r - A_q \end{pmatrix} \quad (24)$$

is the $(n-k)$ minor in the lowest left corner. We note that the number of the blocks producing the exponent of s in $D(N_k(s))$ reduces as k grows. Therefore, we have

$$D(N_k(s)) = s^{q-k} I_r + o(s^{q-k}). \quad (25)$$

Replacing (25) in (23) we obtain, $|sI - A_0| = \left| s^q I_r + \Sigma_k (-1)^{q-k} (s^{n-k} I_r + o(s^{n-k}) \right|$. The proof of the Theorem is complete. □

Corollary 3.1.1. *Assume the setting of Theorem 3.1 holds. Then in the decomposition of A_0 as (14) (on the matter relevant to stability), the effect of the second factor, i.e., matrices A_j, $j = 1, \ldots, q$ in (15) on $\mathrm{Sp}(A_0)$, can be neglected subject to the condition that the high-gain factor θ is big enough. In this case,*

$$\mathrm{Sp}(A_0) = \{\theta\alpha_1(\theta),\ \theta\alpha_2(\theta),\ \ldots\ \theta\alpha_n(\theta)\} \quad (26)$$

such that $\lim_{\theta\to\infty} \alpha_i(\theta) = \bar{\alpha}_i$.

Proof. The proof is mainly that of Goncharov in the scalar case (Lemma 1 in Ref. [28]), plus the extended determinant formula used in Theorem 3.1. The elements $\theta\alpha_1(\theta),\ \theta\alpha_2(\theta),\ \ldots\ \theta\alpha_n(\theta)$ are the roots of the characteristic polynomial of A_0. Therefore, the characteristic polynomial of A_0 is as follows,

$$\varrho(s) = \prod_{i=1}^{n} (s - \theta\alpha_i) = \theta^n \prod_{i=1}^{n} \left(\frac{s}{\theta} - \alpha_i \right) = \theta^n \left| \bar{s}^q I_r + \frac{K_1(\theta)}{\theta} \bar{s}^{q-1} I_r + \ldots \right|, \quad (27)$$

where we have used the simple relation $\bar{s} = s/\theta$. Then, we have $\lim_{\theta\to\infty} \frac{K_i(\theta)}{\theta^i} = \overline{K_i}$ for all i. In other words $\frac{\varrho(s)}{\theta^n} \to \bar{\varrho}(s)$. It follows that the eigenvalues satisfy $\lim \alpha_i(\theta) \to \overline{\alpha}_i$ as $\theta \to +\infty$. □

Remark 3. From the previous discussion, one can conclude that with the above choice of gain factors K_i of the feedback system, the spectrum of A_0 tends to $-\infty$ and is proportional by a gain factor θ. Thus, in the remainder, we always assume that the condition above on feedback coefficients is satisfied.

Now, we are ready to study the stability of the observer's error dynamics (12). For that, first, notice that a general solution for a bilinear system,

$$\dot{x} = Ax + (Bu)x \quad (28)$$

is given by

$$x(t) = \exp(At)x_0 + \int_0^t u(s) \exp(A(t-s)) Bx(s) ds. \quad (29)$$

A similar formula holds for the error dynamics. If θ is high enough, the eigenvalues of A_0 satisfy $\lambda_i \le -\theta,\ 1 \le i \le n$. Therefore,

$$\| \exp(At) \| \le P(\theta) \exp(-\theta t) \quad (30)$$

where P is a polynomial that tends to ∞ with θ, [28]. By Lagrange interpolation for matrix functions ([23] Chapter 5) one can find an expression,

$$\exp(At) = \sum_{l=0}^{n-1} A^l P_l(t) \quad (31)$$

for specific polynomials P_l. Therefore, we must analyze the matrix $\exp(At)B$. We do that in the following.

Theorem 3.2. *Assume the hierarchy of Theorem 3.1 with $\bar{\alpha}_i < -1$. Then, the error dynamics in (12) associated to the observer (11) is,*

$$\dot{e} = A_0 e + (Bu)e \quad (32)$$

and its solution is exponentially stable, i.e. there exists constants $M, a_0 > 0$ and a polynomial $P(\theta)$ such that,

$$\| e(t) \| \leqslant P(\theta) \exp(-(\theta - a_0)t), \qquad (\theta > M). \quad (33)$$

Our strategy to prove Theorem 3.2 is to extend Lemmas 1–4 in reference [28] to block matrices of the form in (10). For that, we introduce the following definition.

Definition 1. Let $A(\theta) = [a_{ij}(\theta)]$ and $B(\theta) = [b_{ij}(\theta)]$ be two $n \times m$ matrices with polynomial entries in θ. If there exist non-negative constants K and M such that the entries of these matrices satisfy:

$$a_{ij}(\theta) \le K b_{ij}(\theta), \qquad i = 1, \ldots, n,\ \ j = 1, \ldots, m. \qquad \forall \theta \ge M. \quad (34)$$

then we can define the following relation,

$$A(\theta) \lesssim B(\theta). \quad (35)$$

Such a condition essentially means that $\deg a_{ij}(\theta) \le \deg b_{ij}(\theta)$ pairwise for all i and j.

The following properties hold [see Definition 1 in [28] page 1602]:

(1) the relation $\lesssim$ is transitive and reflexive,
(2) if $A(\theta)$ and $B(\theta)$ are in block form then $A(\theta) \lesssim B(\theta)$ iff the blocks of the matrices satisfy the same relation,
(3) if $A(\theta) \lesssim B(\theta)$ and $C(\theta) \lesssim D(\theta)$, then $A(\theta) + C(\theta) \lesssim B(\theta) + D(\theta)$,
(4) if $A(\theta) \lesssim B(\theta)$ and $c \in \mathbb{R}$, then $cA(\theta) \lesssim B(\theta)$,
(5) if $A(\theta) \lesssim B(\theta)$ and $C(\theta) \lesssim D(\theta)$, then $AC(\theta) \lesssim BD(\theta)$.

The proofs of the properties 1–5 are given in reference [28]. We employ the relation $\lesssim$ and the properties 1–5 in the following two lemmas.

Lemma 3.1. *Let the matrix A_0 has the form given in (14), and let the feedback coefficients $K_i(\theta)$ be chosen from Theorem 3.1, then*

$$A_0^p \lesssim \begin{pmatrix} \theta^p I_r & \theta^{p-1} I_r & \cdots & \theta^{p-n+1} I_r \\ \theta^{p+1} I_r & \theta^p I_r & \cdots & \theta^{p-n+2} I_r \\ \vdots & \vdots & \ddots & \vdots \\ \theta^{p+n-1} I_r & \theta^{p+n-2} I_r & \cdots & \theta^p I_r \end{pmatrix} =: D^p. \quad (36)$$

Moreover we have

$$A_0^pB \lesssim \begin{pmatrix} \theta^p I_r & \theta^{p-1}I_r & \cdots & \theta^{p-n+1}I_r \\ \theta^{p+1}I_r & \theta^p I_r & \cdots & \theta^{p-n+2}I_r \\ \vdots & \vdots & \ddots & \vdots \\ \theta^{p+n-1}I_r & \theta^{p+n-2}I_r & \cdots & \theta^p I_r \end{pmatrix} \begin{pmatrix} I_r & 0 & \cdots & 0 \\ I_r & I_r & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ I_r & I_r & \cdots & I_r \end{pmatrix} \lesssim D^p. \tag{37}$$

Proof of Lemma 3.1.. To prove this lemma, we use the properties given above. The proof of the first inequality is by induction on p. The case $p=1$ is trivial from the general form of the matrix A_0 in (13). For the inductive step, one uses the property (5) to pass from $A_0^{p-1} \lesssim D^{p-1}$ to $A_0^p \lesssim D^p$. According to property (2) above, one compares the matrices in block form in the same way as the scalar case. To prove the second inequality, we use the first inequality and again with property (5), and we have

$$A_0^pB \lesssim D^p \begin{pmatrix} I_r & 0 & \cdots & 0 \\ I_r & I_r & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ I_r & I_r & \cdots & I_r \end{pmatrix} = \begin{pmatrix} \theta^p I_r & \theta^{p-1}I_r & \cdots & \theta^{p-n+1}I_r \\ \theta^{p+1}I_r & \theta^p I_r & \cdots & \theta^{p-n+2}I_r \\ \vdots & \vdots & \ddots & \vdots \\ \theta^{p+n-1}I_r & \theta^{p+n-2}I_r & \cdots & \theta^p I_r \end{pmatrix} \begin{pmatrix} I_r & 0 & \cdots & 0 \\ I_r & I_r & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ I_r & I_r & \cdots & I_r \end{pmatrix} \lesssim D^p. \tag{38}$$

The proof is now complete. □

The inequalities in Lemma 3.1 can be used to construct estimates for the exponential $\exp(A_0t)$.

Lemma 3.2. *Assume the matrices A_0 and B are given as in (13) and (10), respectively, and let the feedback coefficients $K_i(\theta)$ be chosen in accordance with Theorem 3.1, then*

$$A_0^pB \lesssim \begin{pmatrix} I_r & \theta^{-1}I_r & \cdots & \theta^{p-n+1}I_r \\ \theta^{+1}I_r & I_r & \cdots & \theta^{-n+2}I_r \\ \vdots & \vdots & \ddots & \vdots \\ \theta^{+n-1}I_r & \theta^{+n-2}I_r & \cdots & \theta^p I_r \end{pmatrix} \exp(-\theta t). \tag{39}$$

Proof. (sketch) The proof is based on lemma 4 of reference [28]. We apply the Lagrange interpolation formula in matrix form to the matrix A_0 (one notes that the eigenvalues are distinct) [please see reference [23] page 108, and the proof of lemma 4 in [28]]. By the Lagrange interpolation formula [see [28] page 1604], it follows that,[2]

$$\begin{aligned} \exp(A_0t) &= \sum_l \frac{(A_0-\theta\lambda_1(\theta)I)\ldots[l]\ldots(A_0-\theta\lambda_n(\theta)I)}{(\theta\lambda_k(\theta)-\theta\lambda_1(\theta))\ldots[l]\ldots(\theta\lambda_k(\theta)-\theta\lambda_n(\theta))} \\ &\quad \exp(\theta\lambda_l(\theta)t) \\ &= \sum_l \exp(\theta\lambda_l(\theta)t)\sum_p G_{lp}\frac{{A_0}^pB}{\theta^p}(\theta) \end{aligned} \tag{40}$$

where the functions $G_{lp}(\theta)$ are the same as in the proof of lemma 4 in [28], page 1604. They satisfy $G_{lp}(\theta) \to G_{lp}(\overline{\lambda}_1, \ldots, \overline{\lambda}_n)$ as $\theta \to +\infty$. Using this formula with the inequalities in Lemma 3.1, it follows that,

$$\begin{aligned} \exp(A_0)B &= \sum_l \exp(\theta\lambda_l(\theta)t)\sum_p G_{lp}\frac{{A_0}^pB}{\theta^p}(\theta) \\ &\lesssim \sum_l \exp(\theta\lambda_l(\theta)t)\sum_p G\frac{D^p}{\theta^p}(\theta) \\ &\lesssim \begin{pmatrix} I_r & \theta^{-1}I_r & \cdots & \theta^{p-n+1}I_r \\ \theta^{+1}I_r & I_r & \cdots & \theta^{-n+2}I_r \\ \vdots & \vdots & \ddots & \vdots \\ \theta^{+n-1}I_r & \theta^{+n-2}I_r & \cdots & \theta^p I_r \end{pmatrix} \exp(\theta t). \end{aligned} \tag{41}$$

The factor G is a constant such that $|G_{lp}(\theta)| < G$ for large θ. □

We are ready to present the proof of Theorem 3.2.

Proof of Theorem 3.2.. We sketch the method of Theorem 1 in [28] for block matrices. We use the following change of coordinates:

$$e = (e_1, \ldots, e_q) = (\epsilon_1, \theta\epsilon_2, \ldots, \theta^{q-1}\epsilon_q). \tag{42}$$

Set $\Delta_\theta = \text{diag}[I_r, \theta I_r, \ldots, \theta^{q-1}I_r]$. By Lemma 3.2 one has the inequality,

$$\Delta_{1/\theta}\exp(A_0(t-\tau))B\Delta_\theta \lesssim$$

$$\begin{aligned} &\Delta_{1/\theta}\exp(A_0(t-\tau))B\Delta_\theta \lesssim \\ &\Delta_{1/\theta}\begin{pmatrix} I_r & \theta^{-1}I_r & \cdots & \theta^{p-n+1}I_r \\ \theta^{+1}I_r & I_r & \cdots & \theta^{-n+2}I_r \\ \vdots & \vdots & \ddots & \vdots \\ \theta^{+n-1}I_r & \theta^{+n-2}I_r & \cdots & \theta^p I_r \end{pmatrix}\Delta_\theta\exp(-\theta(t-\tau)) \\ &= \begin{pmatrix} I_r & I_r & \cdots & I_r \\ I_r & I_r & \cdots & I_r \\ \vdots & \vdots & \ddots & \vdots \\ I_r & I_r & \cdots & I_r \end{pmatrix}\exp(-\theta(t-\tau)). \end{aligned} \tag{43}$$

Thus, from definition of $\lesssim$ we have $\Delta_{1/\theta}\exp(A_0(t-\tau))B\Delta_\theta \leq K\exp(-\theta(t-\tau))$ in the extended form. Following the ideas from the theorem above, the proof follows as the scalar case presented in [28]. We obtain the inequality

$$\exp(\theta t) \parallel \epsilon(t) \parallel \leq P(\theta) \parallel e_0 \parallel + u_0K\int_0^t \exp(\theta\tau) \parallel \epsilon(t) \parallel d\tau. \tag{44}$$

Thus by Gronwall-Bellman lemma, we get,

$$\begin{aligned} \parallel \epsilon(t) \parallel &\leqslant P(\theta)\exp(-\theta t) \parallel e_0 \parallel \\ &\quad + u_0K\exp(-(\theta-u_0K)t)\int_0^t P(\theta) \parallel e_0 \parallel \exp(-u_0K\tau)d\tau, \end{aligned} \tag{45}$$

in the block form, where u_0 and K are constants independent of θ, and P is a polynomial. Notice that for a fixed $\theta > M$, when $t \to \infty$, the exponential terms dominate the rest of the terms, and the RHS of the above inequality converges rapidly to zero. □

4. Solution to problem 2: a high-gain observer for bilinear systems in the form $\dot{\boldsymbol{x}} = \boldsymbol{F}(\boldsymbol{u}, \boldsymbol{x})\boldsymbol{x} + \boldsymbol{\phi}(\boldsymbol{u}, \boldsymbol{x})$

In this section, we consider the following nonlinear multi-input multi-output (MIMO) system,

$$\dot{x} = F(u, x)x + \phi(u, x), \qquad y = Cx \tag{46}$$

[2] Here, the symbol $[l]$ indicates that the lth element in a product or a sum is omitted.

where

$$x=\begin{bmatrix}x_1\\x_2\\\vdots\\x_{q-1}\\x_q\end{bmatrix},\ \text{with } \mathrm{x_i}\in\mathbb{R}^{\mathrm{r}},\qquad \mathrm{u}\in\mathbb{R},$$

$$F=\begin{pmatrix}0&F_1&0&\cdots&0\\0&0&F_2&\cdots&0\\\vdots&\vdots&\vdots&\ddots&\vdots\\0&0&0&\cdots&F_{q-1}\\0&0&0&\cdots&0\end{pmatrix},\tag{47}$$

$$\phi(u,x)=\begin{bmatrix}\phi_1\\\phi_2\\\vdots\\\phi_{q-1}\\\phi_q\end{bmatrix},\ \text{where } \mathrm{qr}=\mathrm{n},$$

and $F_k=F_k(u,x_1,\ldots,x_q)$, $\phi_k=\phi_k(u,x_1,\ldots,x_k)$ and $C=[I_r\ \ 0\ \ldots\ 0]$. Thus, the matrix F consists of blocks, each one with size $r\times r$. More generally, we discuss the case $y=h(u,x)$. The system has been studied in [1,14,15] in slightly different setups. Denote

$$f(u,x)=F(u,x)x+\phi(u,x),\tag{48}$$

and introduce new input variables

$$u_0=u,\qquad \dot u_i=u_{i+1}.\tag{49}$$

Let us consider the change of variables,

$$z_1=\psi_1(u,x)=h(u_0,x),\quad z_i=\psi_i(u,x)=L_f\psi_{i-1}.\tag{50}$$

We shall assume that the following regularity condition is satisfied for system (46).

Assumption 3 ([1] page 7)**.** The canonical flag of system (46) is uniform, i.e. the family of n distributions

$$D_i(u):\ x\longmapsto \ker\left[\frac{\partial\psi_i}{\partial x}(x,u)\right]\tag{51}$$

have dimension $n-i$ regardless of u.

According to Theorem 1.3 in [1], the above change of variables will transform the Eq. (46) into

$$\dot z=\begin{bmatrix}z_2\\z_3\\\vdots\\z_q\\L_f\psi_{q-1}\end{bmatrix}+\begin{bmatrix}b_1(u_0,z_1)\\b_2(u_0,u_1,z_1,z_2)\\\vdots\\b_{q-1}(u_0,\cdots,u_{q-2},z_1,\cdots,z_{q-1})\\b_q(u_0,\cdots,u_{q-1},z_1,\cdots,z_q)\end{bmatrix}.\tag{52}$$

Besides, Assumption 3 implies that the exact linearization problem is solvable for system (46) or the same for (52) [see [1], theorem 1.3]. However, there is a little difference. In our setup, the variables are vector variables of size r; this does not affect the formulas in the coordinate changes and is still valid in the vector variable form. Therefore, the proof of Theorem 1.3 in [1] also carries over to this case without any changes.

Now, we are ready to present our second main result. We propose an observer of system (46) of the form:

$$\begin{aligned}&\dot{\hat x}=F(u,\hat x)\hat x+\phi(u,\hat x)-\theta\Delta_\theta P(t)C(\hat x-y)\\&\dot\epsilon=\theta(-\epsilon+\theta E^T\epsilon+C^T(C\hat x-y)),\qquad \epsilon(0)=0\end{aligned}\tag{53}$$

where $\Delta_\theta=\mathrm{diag}[I_r,\ \theta^{-1}I_r,\ \ldots\ \theta^{-q+1}I_r]$, and $P(t)$ is a solution to the Ricatti equation [15],

$$\dot P(t)=\theta\big[P(t)+F(u,\hat x)P(t)+P(t)F(u,\hat x)^T-P(t)C^TCP(t)\big],\tag{54}$$

where $P(0)=P(0)^T>0$, and

$$\epsilon=\begin{bmatrix}\epsilon_1\\\epsilon_2\\\vdots\\\epsilon_{q-1}\\\epsilon_q\end{bmatrix}\ \text{with } \epsilon_i\in\mathbb{R}^r,\qquad E=\begin{pmatrix}0&I_r&0&\cdots&0\\0&0&I_r&\cdots&0\\\vdots&\vdots&\vdots&\ddots&\vdots\\0&0&0&\cdots&I_r\\0&0&0&\cdots&0\end{pmatrix}.\tag{55}$$

The scalar θ is the gain factor. The stability of this observer is demonstrated in the following theorem.

Theorem 4.1. *Let $e=x-\hat x$ be the error of the observer (53). Then, there exist constants $a_0, M>0$ and a polynomial $C(\theta)$ such that,*

$$\parallel e\parallel\leqslant C(\theta)\exp(-(\theta-a_0)t),\qquad (\theta>M).\tag{56}$$

Proof. As mentioned above, Assumption 3 and Theorem 1.3 in [1] imply that the exact linearization problem for system (46) or (52) is solvable. Because the triangular coordination appears, the local linearization of system (46) is conjugated to a bilinear system of the form (9). According to the standard method of change of variables in feedback control (see [37] pages 142–143, see also [1] chapter 1), the vector functions b_j in (52) are linear in u_j, $j\geq 1$, and also the variables are sorted in a triangular form. The only variable in the u-part, which possibly appears nonlinear, is u_0 [see for example [1] Section 1]. On the other hand, if we consider the canonical form presented in theorem 1.3 in reference [1], where the variables z_j may have higher dimensions, then we can solve the last coordinate of the equation for the control function u to make the whole system linear. Thus, the feedback canonical linear form of (46) is in the form (9). The change of coordinates in the systems above also applies to the corresponding observer. Using the form (46) we may assume the system is already given as $\dot x=Ax+\phi(u,x)$. In this case the error can be written as $\dot e=A_0e+\theta(\hat\phi(u,\hat x)-\phi(u,x))$, where A_0 is as in (13), [see for instance reference [1] page 14]. When the exact linearization problem is solvable for (46), the error associated with the observer (53) after linearization of the system can be written as $\dot e=A_0e+(Bu)e$, i.e., in the form that was used in Theorem 3.2. Thus, one can first change coordinates so that the Eq. (52) becomes linear, i.e., in the form (9). Then in the new coordinates, the observer and the error get the desired form. Thus, the observer for system (46) transforms to the observer (11) under the same change of coordinates. It follows that the error dynamics of observers (11) and (53) are also conjugate by the change of coordinates. This proves that the estimates in Theorem 3.2 are satisfied in the nonlinear case. Therefore, our theorem is a consequence of Theorem 3.2 in the linear case. □

Remark 4. Theorem 4.1 provides a uniform exponential boundedness property for the error of the observer (53) independent to x when the high-gain parameter is large enough. Our theorem is stronger than the result in [15] in the case the blocks appearing in (8) are of the same size. The bound in [15] depends on other system parameters. Besides, our method of proof is more intuitive and shorter.

Remark 5. Various examples of system (46) with a different setup have been considered in references [1–3,6–8,11,14,34,39–41,54], where asymptotic stability of the observer error has been demonstrated. The Eq. (46) in its general form (where the blocks may

Table 1
The system's parameters of Example 5.1 used in the simulation experiments.

$a_1 = 1$	$a_2 = 1$	$a_3 = 1$	$a_4 = 0.1$	$\theta = 1$

Table 2
The system's parameters for the MISO system of Example 5.2 used in the simulation experiments.

$a_1 = 3$	$a_2 = 2$	$a_3 = 3$	$a_4 = 100$	$a_5 = 12$
$a_6 = 9.82$	$a_7 = 0.5$	$a_8 = 0.1$	$\theta = 3.5$	

have different sizes) is related to differential systems on *Siegel upper half spaces* and flag varieties. Moreover, their stability analysis produces significant interactions between dynamical systems and algebraic geometry [9].

5. A study case: the bioreactor bilinear system

In this section, two bioreactors' models are investigated to estimate their states utilizing the observer presented in previous sections. The bioreactor system is a model of bacteria growth or measurement of population density that can be modeled in various dimensions, with different inputs and outputs, [10]. In our case, we study two bioreactor models of dimensions two and three, the first with a single input and the latter with multiple inputs. Both are bilinear systems. These systems classify as non-minimal systems since their zero dynamics show no convergence; [see [40] Ch. 6]. The bioreactor models of the following two examples are taken from [26] and [15], respectively; however, our analysis is different.

Example 5.1 (Bioreactor as a SISO system)**.** We consider the following system [26],

$$\begin{aligned}\dot{x}_1 &= \frac{a_1x_1x_2}{a_2x_1+x_2} + ux_1\\ \dot{x}_2 &= -\frac{a_3a_1x_1x_2}{a_2x_1+x_2} - ux_2 + ua_4\\ y &= h(x) = x_1\end{aligned} \tag{57}$$

where $x = (x_1, x_2) \in \mathbb{R}^2,\ u \in \mathbb{R},\ a_1, a_2, a_3 \in \mathbb{R}^+$. Set

$$\mu(x) = \frac{a_1x_1x_2}{a_2x_1+x_2},\ g(x) = \begin{bmatrix} x_1 \\ -x_2+a_4\end{bmatrix},\ f(x) = \begin{bmatrix}\mu(x)\\ -a_3\mu(x)\end{bmatrix}. \tag{58}$$

We can write system (57) in the form of (46) as follows,

$$\dot{x} = \begin{pmatrix} 0 & \mu(x)/x_2 \\ 0 & 0\end{pmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} + \begin{bmatrix} ux_1 \\ -a_3\mu(x) + u(-x_2+a_4)\end{bmatrix}. \tag{59}$$

The observer for this system is in the general form of (53) and described by,

$$\begin{aligned}\dot{\hat{x}} &= \begin{pmatrix} 0 & \mu(\hat{x})/\hat{x}_2 \\ 0 & 0\end{pmatrix}\begin{bmatrix}\hat{x}_1\\ \hat{x}_2\end{bmatrix} + \begin{bmatrix} u\hat{x}_1 \\ -a_3\mu(\hat{x}) + u(-\hat{x}_2+a_4)\end{bmatrix}\\ &\quad + \theta\Delta_\theta(C\hat{x}-y)\end{aligned} \tag{60}$$

where $C = [1\ \ 0]$ and $\Delta_\theta = [1\ \theta^{-1}]$, [26]. It can be written as

$$\dot{\hat{x}} = \begin{bmatrix}\mu(\hat{x})\\ -a_3\mu(\hat{x})\end{bmatrix} + u\begin{bmatrix}\hat{x}_1 \\ -\hat{x}_2 + a_4\end{bmatrix} + \theta\Delta_\theta(C\hat{x}-y). \tag{61}$$

Table 1 shows the system's parameters used in the simulation. The system's initial conditions are $x_1(0) = 0.9$, $x_2(0) = 6$, while the observer's initial conditions are $\hat{x}_1(0) = 2$, $\hat{x}_2(0) = 0.1$. We applied the control input given by

$$u = \begin{cases} 0.08 & 0 \le t \le 10\\ 0.02 & 10 < t \le 20 \\ 0.08 & 20 < t\end{cases} \tag{62}$$

As we are dealing with the estimation problem and not a control problem, we first try the control input in (62) as a function of time with finite and not differentiable points to investigate the observer convergence. Then, the results of the simulations are depicted in Fig. 1. Notice how the observer states exponentially converge to the actual system states, as shown in Fig. 1.

We also simulate the above system with the same initial conditions and parameters but with the output feedback control,

$$u = 10\hat{x}_1. \tag{63}$$

The results are depicted in Fig. 2.

Moreover, Fig. 3 depicts the observer response under the effect of noisy output $y = x_1 + n(t)$, where $n(t)$ is white noise. Such an output is used to compute the control

$$u = 10y = 10(x_1 + n(t)) \tag{64}$$

with the same structure as in (63). Notice that the observer converges to the actual states despite the noise in the output signal and control.

Example 5.2 (Bioreactor as a MISO system)**.** Next, consider the multi-input single-output bioreactor model studied in [15]

$$\begin{aligned}\dot{x}_1 &= a_1x_1^2x_2 - a_1x_1x_2u_1\\ \dot{x}_2 &= \bar{\mu}(x)\nu(x)x_2 - x_2u_2\\ \dot{x}_3 &= -a_2\bar{\mu}(x)\nu(x)x_2 - (x_3 - a_3)u_2\\ y_1 &= x_1\end{aligned} \tag{65}$$

where

$$\bar{\mu}(x_1, x_2) = a_4\frac{x_1x_3}{(a_5+x_1)(a_6+x_3)},\quad \nu(x_1,x_2) = a_7\frac{x_2}{a_8+x_1}. \tag{66}$$

The coefficients a_i are kinetic parameters that, for the case of the present simulation, take the values shown in Table 2. We set $\kappa(x) = \bar{\mu}(x)\nu(x)$,

$$f(x) = \begin{bmatrix} a_1x_1^2x_2 \\ x_2\kappa(x) \\ -a_2\kappa(x)x_2\end{bmatrix},\quad g_1(x) = \begin{bmatrix} -a_1x_1x_2 \\ 0 \\ 0\end{bmatrix},$$
$$g_2(x) = \begin{bmatrix} 0 \\ -x_2 \\ -(x_2 - a_3)\end{bmatrix}, \tag{67}$$

and then (65) finds the standard form

$$\dot{x} = f(x) + g_1(x)u_1 + g_2(x)u_2. \tag{68}$$

One can write this system in the form of (46) as follows,

$$\dot{x} = \begin{pmatrix} 0 & a_1x_1^2 & 0 \\ 0 & 0 & \frac{x_2}{x_3}\kappa(x) \\ 0 & 0 & 0\end{pmatrix}x(t) + \begin{bmatrix} a_1x_1x_2u_1 \\ -x_2u_2 \\ a_2\kappa(x)x_2 - (x_2 - a_3)u_2\end{bmatrix}. \tag{69}$$

Notice that the system is already in the form (52). An observer for system (65) is

$$\begin{aligned}\dot{\hat{x}} &= \begin{pmatrix} 0 & a_1\hat{x}_1^2 & 0 \\ 0 & 0 & \frac{\hat{x}_2}{\hat{x}_3}\kappa(\hat{x}) \\ 0 & 0 & 0\end{pmatrix}\hat{x}(t) + \begin{bmatrix} a_1\hat{x}_1\hat{x}_2u_1 \\ -\hat{x}_2u_2 \\ a_2\kappa(\hat{x})\hat{x}_2 - (\hat{x}_2 - a_3)u_2\end{bmatrix}\\ &\quad - \theta\Delta_\theta^{-1}P(t)C^T(C\begin{bmatrix}\hat{x}_1\\ \hat{x}_2\\ \hat{x}_3\end{bmatrix} - y(t))\end{aligned} \tag{70}$$

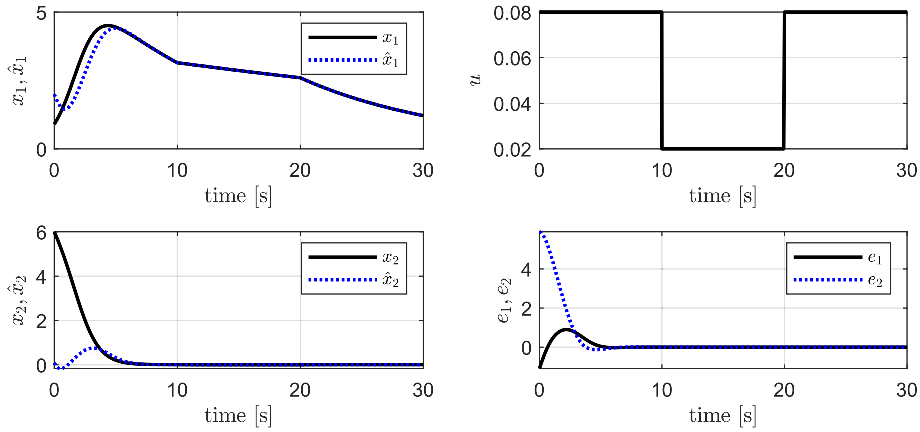

Fig. 1. This figure shows the trajectory convergence of Example 5.1 under control (62), where we show the system states, observer states, control, and observer errors.

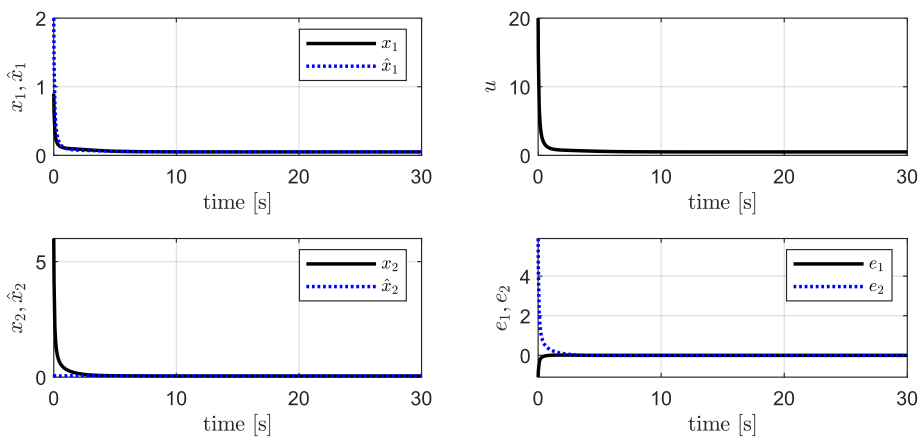

Fig. 2. This figure shows the trajectory convergence of Example 5.1 under output feedback control (63). It is shown the system and observer states, the output feedback control, and the observer errors.

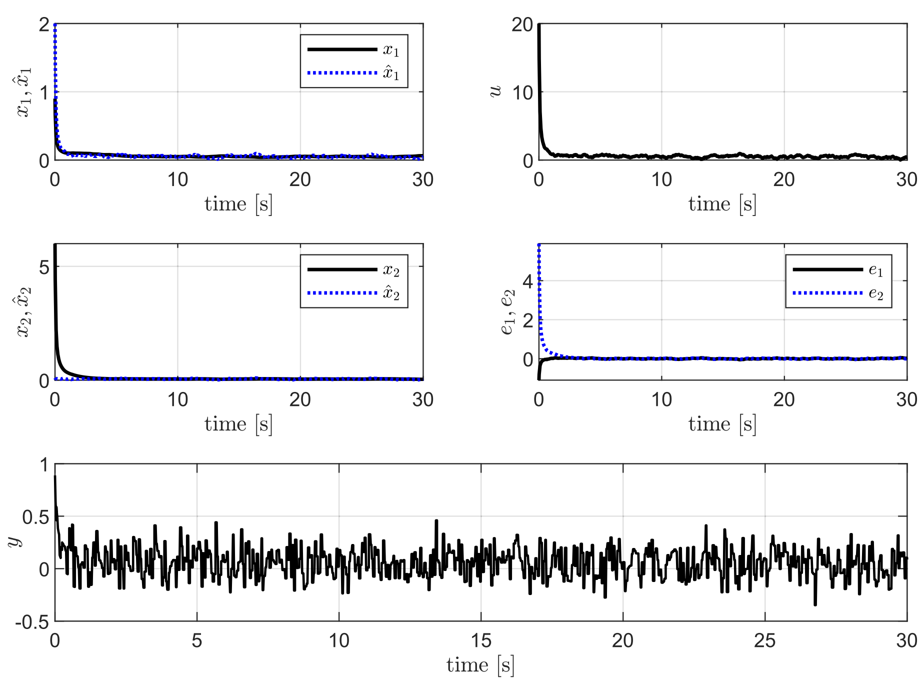

Fig. 3. This figure shows the trajectory convergence of Example 5.1 for the case of noisy output $y = x_1 + n(t)$, where $n(t)$ is random noise under control (64). Notice that the observer errors converge to zero with some noise in their response. This is normal since we are taking the noisy signal y for feedback in control.

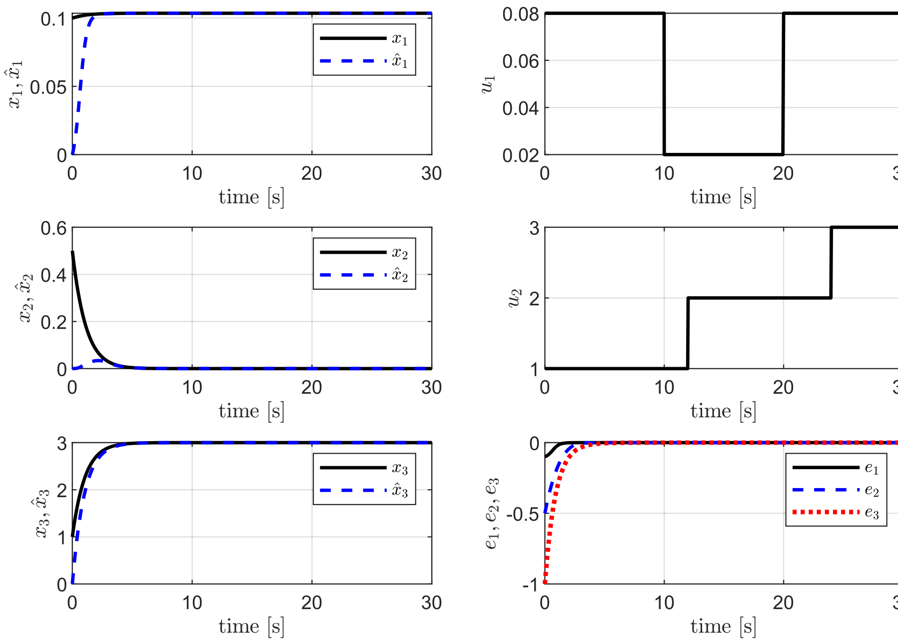

Fig. 4. This figure shows the original states and their estimates, the control inputs given by (72), and the observer error convergence to zero for the Example 5.2.

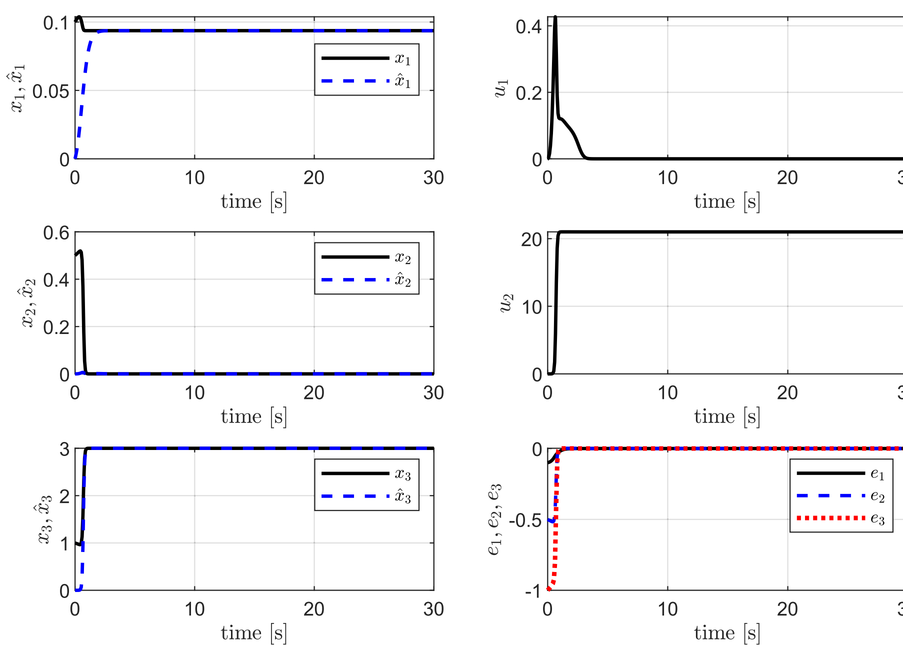

Fig. 5. This figure shows the original states and their estimates under the effect of an output feedback control given in (73) for the Example 5.2. Besides, it is shown the observer error convergence to zero.

where $C = \begin{bmatrix}1 & 0 & 0\end{bmatrix}$, $\Delta_\theta = \mathrm{diag}[1 \quad \theta^{-1} \quad \theta^{-2}]$, and $P(t)$ is a symmetric positive matrix given by the solution of the Ricatti Eq. (54):

$$\dot{P}(t) = \theta\left(P(t) + \begin{pmatrix}0 & a_1x_1^2 & 0\\ 0 & 0 & \frac{x_2}{x_3}\kappa(x)\\ 0 & 0 & 0\end{pmatrix} P(t) + \right.$$
$$\left. + P(t)\begin{pmatrix}0 & a_1x_1^2 & 0\\ 0 & 0 & \frac{x_2}{x_3}\kappa(x)\\ 0 & 0 & 0\end{pmatrix}^T - P(t)C^TCP(t)\right), \tag{71}$$

where $P(t) > 0$, [15]. As we mentioned, the error dynamics is conjugated to a system of the form (32) where the inequality (33) can be proved.

For the simulation, the parameters of this MISO bioreactor model are depicted in Table 2. The system's initial conditions are $x_1(0) = 0.1$, $x_2(0) = 0.5$, $x_3(0) = 1$; while the observer's initial conditions are $\hat{x}_1(0) = 0$, $\hat{x}_2(0) = 0$, $\hat{x}_3(0) = 0$; and $P(0) = 0.1I_{3\times3}$ for the Riccati differential equation. We have considered the control inputs given by,

$$u_1 = \begin{cases}0.08, & 0 \le t \le 10\\ 0.02, & 10 < t \le 20,\\ 0.08, & 20 < t\end{cases} \quad u_2 = \begin{cases}1, & 0 \le t \le 12\\ 2, & 12 < t \le 24\\ 3, & 24 < t\end{cases} \tag{72}$$

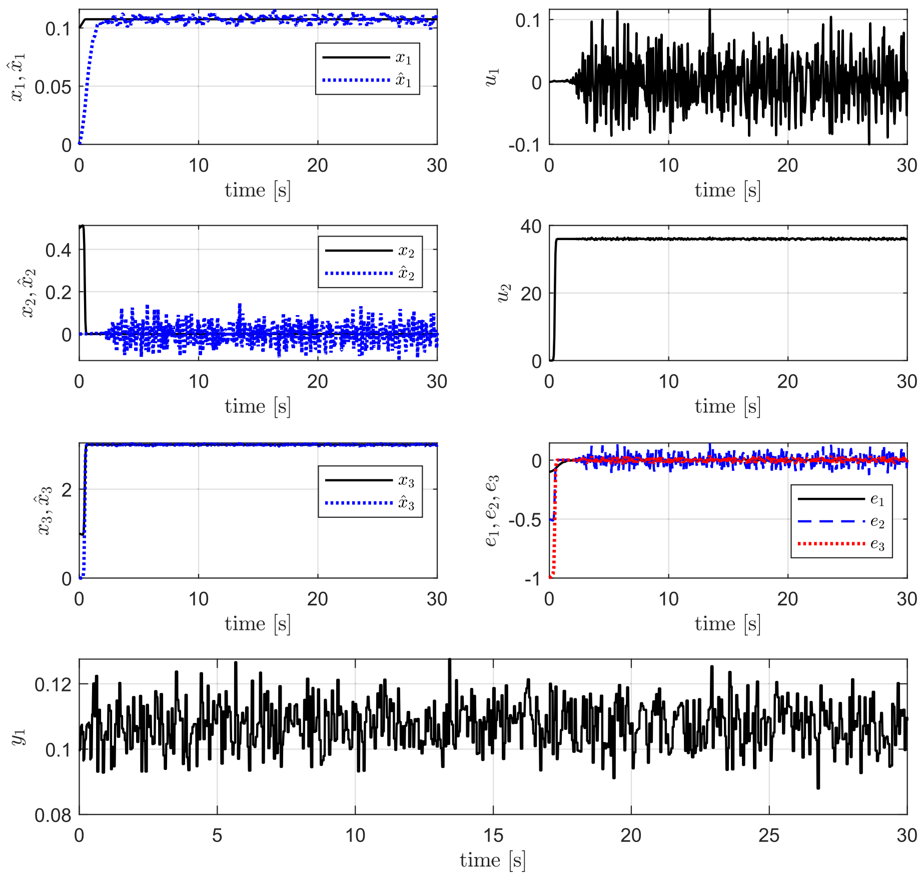

Fig. 6. This figure shows the original states and their estimates under the effect of the output feedback control given in (73) with $\kappa_1 = 0.8$, and $\kappa_2 = 12$ for the Example 5.2. Besides, the observer error converges to zero despite the noisy output given by $y_1 = x_1 + n(t)$.

Since the paper's contribution is in the observer design and not in control, we have chosen the above control inputs with finite discontinuous points to highlight the observer response despite such conditions. Notice that the above control inputs maintain the system states bounded, but they are not designed to achieve particular stability. Control inputs (u_1, u_2), observer states, actual system states, and observer's errors are shown in Fig. 4; notice how the convergence of all the observer states to the actual states is achieved.

Additionally, we conducted a simulation with the output feedback control,

$$u_1 = \kappa_1 \hat{x}_2 \quad u_2 = \kappa_2 \hat{x}_3. \tag{73}$$

where $\kappa_1 = 70$, $\kappa_2 = 7$. All the system parameters and initial conditions are the same as in the previous simulation scenario. The results are depicted in Fig. 5.

Finally, we conducted a simulation case with the output system $y_1 = x_1 + n(t)$, where $n(t)$ is white noise. Due to the noise, we have tuned the control gains to achieve a good closed-loop system performance. The control gains are now $\kappa_1 = 0.8$, and $\kappa_2 = 12$. The results are depicted in Fig. 6. Although the observer error converges to zero, such a convergence is achieved with certain noise because the observer is using the signal $y_1 = x_1 + n(t)$ in its error term producing white noise in the estimated states. Such noise can be reduced or even eliminated with proper filters. However, this is out of the scope of the article.

Remark 6. The reader can consult references [4,24,33,35,36,47,58] for more example of bioreactor models. The bioreactor model is one of the examples of non-minimal bilinear systems.

6. Conclusion

The exponential stability of bilinear systems in block form has been proved. Notice that a similar statement has been proved for the extended observer in the nonlinear case under a regularity condition on the system canonical flag. The latter provides a uniform exponential bound for the error dynamics, more robust than the result of [15]. Finally, two bilinear systems are presented as an application, for which two corresponding observers are proposed and simulated. This shows the effectiveness of the proposed approach.

A natural challenge is to extend the result of the paper to an arbitrary extended block form in the presence of disturbances. Besides, we intend to apply this observer in mechanical underactuated systems, such as UAVs, to design output feedback controllers.

Author contributions statement

M. Reza-Rahmati and G. Flores reviewed the manuscript and contributed equally to this paper.

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Declaration of Competing Interest

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