NORMAL OPERATORS FOR MOMENTUM RAY TRANSFORMS, I: THE INVERSION FORMULA

SHUBHAM R. JATHAR, MANAS KAR, VENKATESWARAN P. KRISHNAN, AND VLADIMIR A. SHARAFUTDINOV

Abstract. The momentum ray transform I^k_m integrates a rank m symmetric tensor field f on $\mathbb{R}^n$ over lines with the weight t^k, $I^k_m f(x,\xi) = \int_{-\infty}^{\infty} t^k \langle f(x+t\xi), \xi^m\rangle\, \mathrm{d}t$. We compute the normal operator $N^k_m = (I^k_m)^* I^k_m$ and present an inversion formula recovering a rank m tensor field f from the data $(N^0_m f, \dots, N^m_m f)$.

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1. Introduction

Let $\langle\cdot,\cdot\rangle$ be the standard dot product on $\mathbb{R}^n$ and $|\cdot|$, the corresponding norm.

For Schwartz class functions, the ray transform (also called the X-ray transform) is defined by

$$If(x,\xi) = \int_{-\infty}^{\infty} f(x+t\xi)\,\mathrm{d}t \tag{1.1}$$

for all $(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n$ satisfying $|\xi| = 1$ and $\langle x, \xi\rangle = 0$. For a Schwartz class symmetric m-tensor field $f = (f_{i_1\dots i_m})$, the ray transform is defined by

$$\begin{aligned} I_m f(x,\xi) &= \int_{-\infty}^{\infty} f_{i_1\dots i_m}(x+t\xi)\xi^{i_1}\dots\xi^{i_m}\,\mathrm{d}t \\ &= \int_{-\infty}^{\infty} \langle f(x+t\xi), \xi^m\rangle\,\mathrm{d}t. \end{aligned} \tag{1.2}$$

We use the Einstein summation rule to sum from 1 to n over every repeated index in lower and upper positions in a monomial. In particular, when $m = 0$, the definition (1.2) coincides with (1.1) and when $m = 1$, (1.2) represents the ray transform of vector fields which is also called the Doppler transform.

In the case of $m = 0$, the ray transform If uniquely determines a function f and there is an explicit inversion formula. However, if $m \ge 1$, the ray transform I_m has a nontrivial kernel. In particular, $I(\sigma\nabla h) = 0$ whenever h is a smooth symmetric $(m-1)$-tensor field on $\mathbb{R}^n$ decaying at infinity, ∇ is the total covariant derivative, and σ denotes the symmetrization of a tensor. A symmetric m-tensor field f sufficiently fast decaying at infinity can be uniquely decomposed

$$f = f^s + \sigma\nabla h, \quad h(x) \to 0 \text{ as } |x| \to \infty$$

to the solenoidal (= divergence-free) part f^s and potential part $\sigma\nabla h$; see [Sha94, Theorem 2.6.2] and [PSU23, Theorem 6.4.7] for the detailed explanation in the Euclidean case as well as in the case of Riemannian manifolds. The solenoidal part of a symmetric m-tensor field f can be uniquely determined from $I_m f$ and there is an explicit inversion formula [Sha94, Theorem 2.12.2].

It is natural to ask: what additional information required along with $I_m f$ so that one could recover the entire tensor field f. This leads to the notion of the momentum ray transform I^k_m introduced in [Sha94, Section 2.17] by

$$\begin{aligned} I^k_m f(x,\xi) &= \int_{-\infty}^{\infty} t^k f_{i_1\dots i_m}(x+t\xi)\xi^{i_1}\dots\xi^{i_m}\,\mathrm{d}t \\ &= \int_{-\infty}^{\infty} t^k \langle f(x+t\xi),\xi^m\rangle\,\mathrm{d}t \quad (k=0,1,2\dots) \end{aligned}$$

for all $(x,\xi)\in\mathbb{R}^n\times\mathbb{R}^n$ satisfying $|\xi|=1$ and $\langle x,\xi\rangle=0$. In particular $I^0_m=I_m$.

We restrict ourselves by considering the data $(I^0_m f,\dots,I^m_m f)$ since, for $k>m$, the function $I^k_m f$ can be easily expressed through $I^0_m f,\dots,I^m_m f$.

A rank m symmetric tensor field f is uniquely determined by the data $(I^0_m f,\dots,I^m_m f)$. This was proved in [Sha94, Theorem 2.17.2]. Later this result was extended to a Helgason type support theorem for tensor fields on a simple real analytic Riemannian manifold [AM19]. An algorithm for recovering f from the data $(I^0_m f,\dots,I^m_m f)$ is presented in [KMSS19] as well as a stability estimate in (generalized) Sobolev norms. A range characterization for the operator $f\mapsto(I^0_m f,\dots,I^m_m f)$ on the Schwartz space was established in [KMSS20].

Let us introduce the normal operator $N^k_m=\left(I^k_m\right)^* I^k_m$, where $\left(I^k_m\right)^*$ is the L^2-adjoint of the momentum ray transform I^k_m. Since N^k_m is an averaging operator, the data $N^k_m f$ could represent a better measurement model rather than $I^k_m f$. We present an algorithm of recovering a rank m tensor field f from the data $(N^0_m f,\dots,N^m_m f)$. In terms of Fourier transforms $\widehat{f}$ and $(\widehat{N^0_m f},\dots,\widehat{N^m_m f})$, we derive the inversion formula

$$\widehat{f}(y)=|y|\sum_{k=0}^{m}P^k_m(\widehat{N^k_m f})$$

with some linear operators P^k_m on the space of rank m symmetric tensor fields. Given m, the operators P^k_m $(k=0,\dots,m)$ are calculated by explicit recurrent formulas; but the volume of calculations grows fast with m. We perform the calculations for $m=1,2,3$.

The ray transform has several important applications that include X-ray computer tomography (CT) in medical imaging when $m=0$. In the case of $m=1$, the ray transform is used in Doppler tomography to analyze vector fields. In cases where $m=2$ or $m=4$, the ray transform and its variants are applied to tomography problems in anisotropic media regarding the elasticity and Maxwell systems, see [Sha94] and [LS09, SW12]. Recently, the momentum ray transform has been adopted as a solution tool for the classical Calderón problem for the bi-Laplace model and other higher-order operators [BKS23, SS23, BK23]. The unique continuation principle for I_m and I^k_m is proved in [AKS22]. See also [IKS23] for a related work involving a fractional momentum operator.

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2. Basic definitions

First of all, mostly following [Sha94, Chapter 2], we introduce some notation and definitions concerning tensor algebra and analysis which will be used throughout the article.

2.1. **Tensor algebra over $\mathbb{R}^n$.** Let $T^m\mathbb{R}^n$ be the n^m-dimensional complex vector space of m-tensors on $\mathbb{R}^n$. In particular, $T^0\mathbb{R}^n = \mathbb{C}$ and $T^1\mathbb{R}^n = C^n$. We need complex tensors since we are going to use the Fourier transform. Assuming n to be fixed, the notation $T^m\mathbb{R}^n$ will be mostly abbreviated to T^m. For a fixed orthonormal basis $(e_1, \dots, e_n)$ of $\mathbb{R}^n$, by $u_{i_1\dots i_m} = u^{i_1\dots i_m} = u(e_{i_1}, \dots, e_{i_m})$ we denote *coordinates* (= *components*) of a tensor $u \in T^m$ with respect to the basis. There is no difference between covariant and contravariant tensors since we use orthonormal bases only. Given $u \in T^m$ and $v \in T^k$, *the tensor product* $u \otimes v \in T^{m+k}$ is defined by $(u \otimes v)_{i_1\dots i_{m+k}} = u_{i_1\dots i_m} v_{i_{m+1}\dots i_{m+k}}$. The standard dot product on $\mathbb{R}^n$ extends to T^m by $\langle u, v\rangle = u^{i_1\dots i_m}\overline{v_{i_1\dots i_m}}$. Throughout the article, the Einstein summation convention is used.

Let $S^m = S^m\mathbb{R}^n$ be the $\binom{n+m-1}{m}$-dimensional subspace of T^m consisting of symmetric tensors. *The partial symmetrization* $\sigma(i_1 \dots i_m) : T^{m+k} \to T^{m+k}$ in the indices $(i_1, \dots, i_m)$ is defined by

$$\sigma(i_1 \dots i_m) u_{i_1\dots i_m j_1\dots j_k} = \frac{1}{m!} \sum_{\pi\in\Pi_m} u_{i_{\pi(1)},\dots,i_{\pi(m)} j_1\dots j_k},$$

where the summation is performed over the group Π_m of all substitutions of the set $\{1, \dots, m\}$. In particular, $\sigma : T^m \to S^m$ is the symmetrization in all indices. Given $u \in S^m$ and $v \in S^k$, *the symmetric product* $uv \in S^{m+k}$ is defined by $uv = \sigma(u \otimes v)$. Being furnished with the symmetric product, $S^*\mathbb{R}^n = \bigoplus_{m=0}^{\infty} S^m\mathbb{R}^n$ becomes a commutative graded algebra that is called *the algebra of symmetric tensors over* $\mathbb{R}^n$. The algebra $S^*\mathbb{R}^n$ is canonically isomorphic to the algebra of polynomials on $\mathbb{R}^n$. Every statement on symmetric tensors can be translated to the langauge of polynomials, and vice versa.

Given $u \in S^m$, let $i_u : S^k \to S^{m+k}$ be the operator of symmetric multiplication by u and let $j_u : S^{m+k} \to S^k$ be the adjoint of i_u. These operators are written in coordinates as

$$\begin{aligned}(i_u v)_{i_1\dots i_{m+k}} &= \sigma(i_1 \dots i_{m+k})\, u_{i_1\dots i_m} v_{i_{m+1}\dots i_{m+k}}\\ (j_u v)_{i_1\dots i_k} &= v_{i_1\dots i_{m+k}} u^{i_{k+1}\dots i_{m+k}}.\end{aligned}$$

The tensor $j_u v$ will be also denoted by v/u. For the Kronecker tensor δ, the notations i_δ and j_δ will be abbreviated to i and j respectively.

2.2. **Tensor fields.** Recall that the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ is the topological vector space consisting of C^∞-smooth complex-valued functions on $\mathbb{R}^n$ fast decaying at infinity together with all derivatives, furnished with the standard topology. Let $\mathcal{S}(\mathbb{R}^n; S^m) = \mathcal{S}(\mathbb{R}^n) \otimes S^m$ be the topological vector space of smooth fast decaying symmetric m-tensor fields, defined on $\mathbb{R}^n$, whose components belong to the Schwartz space. In Cartesian coordinates, such a tensor field is written as $f = (f_{i_1\dots i_m})$ with coordinates (= components) $f_{i_1\dots i_m} = f^{i_1\dots i_m} \in \mathcal{S}(\mathbb{R}^n)$ symmetric in all indices. We again emphasize that there is no difference between covariant and contravariant coordinates since we use Cartesian coordinates only.

We use *the Fourier transform* $\mathcal{F}:\mathcal{S}(\mathbb{R}^n)\to\mathcal{S}(\mathbb{R}^n)$, $f\mapsto\widehat{f}$ in the form (hereafter i is the imaginary unit)

$$\mathcal{F}f(y)=\frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n}e^{-i\langle y,x\rangle}f(x)\,\mathrm{d}x.$$

The Fourier transform $\mathcal{F}:\mathcal{S}\left(\mathbb{R}^n;S^m\right)\to\mathcal{S}\left(\mathbb{R}^n;S^m\right)$, $f\mapsto\widehat{f}$ of symmetric tensor fields is defined component-wise: $\widehat{f}_{i_1\dots i_m}=\widehat{f_{i_1\dots i_m}}$.

Besides $\mathcal{S}\left(\mathbb{R}^n;S^m\right)$, we use some other spaces of tensor fields. In particular, $C^\infty\left(U;T^m\right)$ is the space of smooth m-tensor fields on an open set $U\subset\mathbb{R}^n$.See details in [Sha94, Section 2.1].

The L^2-product on $C_0^\infty\left(\mathbb{R}^n;T^m\right)$ is defined by

$$(f,g)_{L^2(\mathbb{R}^n;T^m)}=\int_{\mathbb{R}^n}\langle f(x),g(x)\rangle\,dx. \tag{2.1}$$

2.3. **Inner derivative and divergence.** The first order differential operator

$$d:C^\infty(\mathbb{R}^n;S^m)\to C^\infty(\mathbb{R}^n;S^{m+1})$$

defined by

$$(df)_{i_1\dots i_{m+1}}=\sigma(i_1\dots i_{m+1})\frac{\partial f_{i_1\dots i_m}}{\partial x^{i_{m+1}}}=\frac{1}{m+1}\Big(\frac{\partial f_{i_2\dots i_{m+1}}}{\partial x^{i_1}}+\dots+\frac{\partial f_{i_1\dots i_m}}{\partial x^{i_{m+1}}}\Big)$$

is called *the inner derivative.*

The divergence

$$\operatorname{div}:C^\infty(\mathbb{R}^n;S^{m+1})\to C^\infty(\mathbb{R}^n;S^m)$$

is defined by

$$(\operatorname{div}f)_{i_1\dots i_m}=\delta^{jk}\frac{\partial f_{i_1\dots i_m j}}{\partial x^k}.$$

The operators d and $-\operatorname{div}$ are formally adjoint to each other with respect to the L^2-product (2.1). The divergence is denoted by δ in [Sha94]. But we will always use the notation div since some our formulas involve the divergence and Kronecker tensor simultaneously.

2.4. **The space $\mathcal{S}(T\mathbb{S}^{n-1})$.** The Schwartz space $\mathcal{S}(E)$ is well defined for a smooth vector bundle $E\to M$ over a compact manifold with the help of a finite atlas and partition of unity subordinate to the atlas.

In particular, the Schwartz space $\mathcal{S}(T\mathbb{S}^{n-1})$ is well defined for the tangent bundle

$$T\mathbb{S}^{n-1}=\{(x,\xi)\in\mathbb{R}^n\times\mathbb{S}^{n-1}:\langle x,\xi\rangle=0\}\to\mathbb{S}^{n-1},\quad (x,\xi)\mapsto\xi$$

of the unit sphere $\mathbb{S}^{n-1}=\{x\in\mathbb{R}^n:|x|=1\}$.

The Fourier transform $\mathcal{F}:\mathcal{S}\left(T\mathbb{S}^{n-1}\right)\to\mathcal{S}\left(T\mathbb{S}^{n-1}\right),\varphi\mapsto\widehat{\varphi}$ is defined by

$$\mathcal{F}\varphi(y,\xi)=\frac{1}{(2\pi)^{(n-1)/2}}\int_{\xi^\perp}e^{-i\langle y,x\rangle}\varphi(x,\xi)\,\mathrm{d}x,$$

where $\mathrm{d}x$ is the $(n-1)$-dimensional Lebesgue measure on the hyperplane $\xi^\perp=\{x\in\mathbb{R}^n;\ \langle\xi,x\rangle=0\}$. Notice that it is the standard Fourier transform in the $(n-1)$-dimensional variable x while $\xi\in\mathbb{S}^{n-1}$ is considered as a parameter.

The L^2-product on $\mathcal{S}(T\mathbb{S}^{n-1})$ is defined by

$$(\varphi,\psi)_{L^2(T\mathbb{S}^{n-1})}=\int\limits_{\mathbb{S}^{n-1}}\int\limits_{\xi^\perp}\varphi(x,\xi)\overline{\psi(x,\xi)}\,\mathrm{d}x\,\mathrm{d}\xi, \tag{2.2}$$

where $\mathrm{d}\xi$ is the $(n-1)$-dimensional Euclidean volume form on the unit sphere $\mathbb{S}^{n-1}$.

2.5. **Ray transforms.** It is convenient to parameterize the family of oriented lines in $\mathbb{R}^n$ by points of the manifold $T\mathbb{S}^{n-1}$. Namely, a point $(x,\xi)\in T\mathbb{S}^{n-1}$ determines the line $\{x+t\xi : t\in\mathbb{R}\}$ through x in the direction ξ.

The ray transform

$$I_m : \mathcal{S}(\mathbb{R}^n;S^m)\to\mathcal{S}\left(T\mathbb{S}^{n-1}\right)$$

is the linear continuous operator defined by

$$I_m f(x,\xi)=\int_{\mathbb{R}} f_{i_1\dots i_m}(x+t\xi)\xi^{i_1}\dots\xi^{i_m}\,\mathrm{d}t=\int_{\mathbb{R}}\langle f(x+t\xi),\xi^m\rangle\,\mathrm{d}t.$$

The ray transform is related to the Fourier transform by the important formula [Sha94, formula 2.1.15].

$$\widehat{I_m f}(y,\xi)=(2\pi)^{1/2}\langle\widehat{f}(y),\xi^m\rangle\quad\left((y,\xi)\in T\mathbb{S}^{n-1}\right). \tag{2.3}$$

For $0\le k\le m$, *the momentum ray transform*

$$I^k_m : \mathcal{S}(\mathbb{R}^n;S^m)\to\mathcal{S}\left(T\mathbb{S}^{n-1}\right)$$

is the linear continuous operator defined by

$$I^k_m f(x,\xi)=\int_{\mathbb{R}} t^k f_{i_1\dots i_m}(x+t\xi)\xi^{i_1}\dots\xi^{i_m}\,\mathrm{d}t=\int_{\mathbb{R}} t^k\langle f(x+t\xi),\xi^m\rangle\,\mathrm{d}t. \tag{2.4}$$

The formula (2.3) is generalized as follows [KMSS19, formula (2.9)]:

$$\widehat{I^k_m f}(y,\xi)=(2\pi)^{1/2}i^k\left\langle d^k\widehat{f}(y),\xi^{m+k}\right\rangle\quad\left((y,\xi)\in T\mathbb{S}^{n-1}\right).$$

As we will see later, I^k_m should be considered together with lower degree operators $I^0_m,\dots,I^{k-1}_m$, i.e., the collection $(I^0_m f,\dots,I^k_m f)$ represents more convenient information about f than $I^k_m f$ alone.

2.6. **Normal operators.** The formal adjoint of the ray transform I_m with respect to L^2-products (2.1) and (2.2)

$$I^*_m : \mathcal{S}\left(T\mathbb{S}^{n-1}\right)\to C^\infty\left(\mathbb{R}^n;S^m\right)$$

is expressed by

$$\left(I^*_m\varphi\right)_{i_1\dots i_m}(x)=\int_{\mathbb{S}^{n-1}}\xi_{i_1}\dots\xi_{i_m}\varphi\big(x-\langle x,\xi\rangle\xi,\xi\big)\,\mathrm{d}\xi.$$

We emphasize that, for $\varphi\in\mathcal{S}(T\mathbb{S}^{n-1})$, the tensor field $I^*_m\varphi$ does not need fast decay at infinity.

Similarly, the formal L^2-adjoint of the momentum ray transform I^k_m

$$\left(I^k_m\right)^* : \mathcal{S}\left(T\mathbb{S}^{n-1}\right)\to C^\infty\left(\mathbb{R}^n;S^m\right)$$

is expressed by

$$\left((I^k_m)^*\varphi\right)_{i_1\dots i_m}(x)=\int_{\mathbb{S}^{n-1}}\langle x,\xi\rangle^k\xi_{i_1}\dots\xi_{i_m}\varphi\big(x-\langle x,\xi\rangle\xi,\xi\big)\,\mathrm{d}\xi. \tag{2.5}$$

Let

$$N_m=I^*_m I_m : \mathcal{S}\left(\mathbb{R}^n;S^m\right)\to C^\infty\left(\mathbb{R}^n;S^m\right)$$

be the normal operator for the ray transform I_m. Similarly, let

$$N^k_m=(I^k_m)^* I^k_m : \mathcal{S}\left(\mathbb{R}^n;S^m\right)\to C^\infty\left(\mathbb{R}^n;S^m\right)$$

be the normal operator for the momentum ray transform I^k_m.

Given $f \in \mathcal{S}(\mathbb{R}^n; S^m)$, the tensor field $N^k_m f$ does not grow too fast at infinity, i.e., the estimate

$$|N^k_m f(x)| \le C(1+|x|)^N$$

holds with some constants C and N. In particular, $N^k_m f$ can be considered as a tempered tensor field-distribution, i.e., $N^k_m f \in \mathcal{S}'(\mathbb{R}^n; S^m)$. Hence the Fourier transform $\widehat{N^k_m f} \in \mathcal{S}'(\mathbb{R}^n; S^m)$ is well defined at least in the distribution sense. We will show that, for $f \in \mathcal{S}(\mathbb{R}^n; S^m)$, the restriction of $\widehat{N^k_m f}$ to $\mathbb{R}^n \setminus \{0\}$ belongs to $C^\infty(\mathbb{R}^n \setminus \{0\}; S^m)$.

The operator N_m was computed in [Sha94, formula 2.11.3] where the notation μ^m was used instead of I^*_m. In Section 4, we will derive a similar formula for N^k_m. In [AKS22], a similar expression for the normal operator is considered to study the unique continuation principle for momentum ray transforms.

3. Main results

We start with inversion formulas for vector fields and for second rank symmetric tensor fields.

Theorem 3.1. *A vector field $f \in \mathcal{S}(\mathbb{R}^n; \mathbb{C}^n)$ $(n \ge 2)$ is recovered from the data $(N^0_1 f, N^1_1 f)$ by the inversion formula*

$$f(x) = \frac{2^{n/2-1}\Gamma\Big(\frac{n+1}{2}\Big)}{\sqrt{\pi}}(-\Delta)^{1/2}\Big[N^0_1 f - \frac{1}{n-1}\,d\,j_x\,N^0_1 f - \frac{1}{n-1}\,d\,\mathrm{div}\,N^1_1 f\Big]. \tag{3.1}$$

where Γ is Euler's Gamma function and the operator $(-\Delta)^{1/2}$ is defined with the help of the Fourier transform by $|y|\mathcal{F} = \mathcal{F}(-\Delta)^{1/2}$. The vector field in brackets belongs to the domain of $(-\Delta)^{1/2}$.

Theorem 3.2. *A tensor field $f \in \mathcal{S}(\mathbb{R}^n; S^2)$ is recovered from the data $(N^0_2 f, N^1_2 f, N^2_2 f)$ by the inversion formula*

(3.2)

$$\begin{aligned} f(x) = \frac{2^{n/2-1}\Gamma\Big(\frac{n+3}{2}\Big)}{\sqrt{\pi}}(-\Delta)^{1/2}\Big[&N^0_2 f - \frac{1}{n+1}\,ij\,N^0_2 f \\ &- \frac{2}{n+1}\,d\Big(j_x N^0_2 f + \mathrm{div}\,N^1_2 f\Big) \\ &+ \frac{1}{(n-1)(n+1)}\,d^2\Big(j^2_x\,N^0_2 f - 2\,j_x\,\mathrm{div}\,N^1_2 f + \frac{1}{2}\,\mathrm{div}^2\,N^2_2 f\Big)\Big]. \end{aligned}$$

The tensor field in brackets belongs to the domain of $(-\Delta)^{1/2}$.

We use the definition

$$(2l+1)!! = 1\cdot 3\cdots(2l+1), \quad (-1)!! = 1.$$

For tensor fields of arbitrary rank, our result is as follows.

Theorem 3.3. *Given integers $m \ge 0$ and $n \ge 2$, the Fourier transform of a tensor field $f \in \mathcal{S}(\mathbb{R}^n; S^m)$ is recovered from the data $(\widehat{N^0_m f}, j_y\widehat{N^0_m f}, \dots, j^m_y\widehat{N^m_m f})$ by the algorithm consisting of three steps.*

1. Compute tensor fields $F^{(m,k)} \in C^\infty(\mathbb{R}^n \setminus \{0\}; S^{m-k})$ $(0 \le k \le m)$ by

$$F^{(m,k)}(y) = \frac{c_{m,n}}{k!}\,j^k_y\,\widehat{N^k_m f}(y), \qquad c_{m,n} = \pi^{-1/2}\,(2m-1)!!\,2^{m+n/2-2}\,\Gamma\Big(\frac{2m+n-1}{2}\Big). \tag{3.3}$$

2. Compute tensor fields $H^{(m,k)} \in C^\infty(\mathbb{R}^n \setminus \{0\}; S^{m-k})$ $(0 \le k \le m)$ by

$$H^{(m,k)} = \frac{(2m-2k-1)!!(n+2m-2k-3)!!}{(2m-1)!!(n+2m-3)!!} \sum_{p=0}^{k} (-1)^p \binom{k}{p} \operatorname{div}^{k-p} F^{(m,p)}. \tag{3.4}$$

3. Recover $\widehat{f}$ by the formula

$$\begin{aligned} \widehat{f}(y) = \frac{|y|}{m!} \Big[& \frac{1}{(2m-1)!!} H^{(m,0)}(y) \\ & + \sum_{k=1}^{m} \frac{(-1)^k}{(2m-2k-1)!!} \binom{m}{k} \sum_{p=0}^{\min(k,m-k)} \frac{(-1)^p}{2^p} \binom{m-k}{p} i^p i_y^{k-p} j_y^p H^{(m,k)}(y) \Big]. \end{aligned} \tag{3.5}$$

Of course, (3.3)–(3.5) can be combined to give a formula that expresses the rank m symmetric tensor field $\widehat{f}$ through $(\widehat{N_m^0 f}, j_y \widehat{N_m^0 f}, \dots, j_y^m \widehat{N_m^m f})$ that does not involve $F^{(m,k)}$ and $H^{(m,k)}$. We will present the latter formula for $m = 1, 2, 3$ in the last section. The proof of Theorems 3.1 and 3.2 is also presented in the last section. Formulas (3.1) and (3.2) are obtained from (3.5) just by applying the inverse Fourier transform; nevertheless, some commutator formulas for the Fourier transform and operators participating in (3.5) should be used.

The rest of the article is mostly devoted to the proof of Theorem 3.3. Now, we discuss the scheme of the proof.

We introduce the tensor fields

$$A^{(m,k)} \in C^\infty(\mathbb{R}^n \setminus \{0\}; S^{2m-k}) \quad (0 \le k \le m) \tag{3.6}$$

by

$$A^{(m,k)}(y) = d^{2m-k} |y|^{2m-2k-1}. \tag{3.7}$$

These tensor fields play an important role in all our arguments.

In the next section, we compute the normal operators N_m^k $(0 \le k \le m)$ and prove that a tensor field $f \in \mathcal{S}(\mathbb{R}^n; S^m)$ satisfies $A^{(m,0)}/(d^k \widehat{f}) = F^{(m,k)}$, where $F^{(m,k)}$ is defined by (3.3). To avoid proliferation of the $\widehat{\cdot}$ symbol, we denote $g(y) = \widehat{f}(y)$ and write the latter equation in the form

$$A^{(m,0)}/(d^k g) = F^{(m,k)} \quad (0 \le k \le m). \tag{3.8}$$

Given the data $(F^{(m,0)}, \dots, F^{(m,m)})$, we consider (3.8) as a system of linear equations for the unknown tensor field g.

The first equation of the system (3.8)

$$A^{(m,0)}(y)/g(y) = F^{(m,0)}(y) \quad (y \in \mathbb{R}^n \setminus \{0\})$$

is a pure algebraic equation. More precisely, being written in coordinates, it constitutes a system of linear algebraic equations in the components of the tensor field $g(y)$ with coefficients depending on y. The system was considered in [Sha94, Theorem 2.12.1] where the tensor field $\varepsilon^m(y) = \frac{|y|}{((2m-1)!!)^2} d^{2m} |y|^{2m-1}$ was used instead of $A^{(m,0)}$. It allows to determine the *tangential part* of the tensor field g which corresponds to the solenoidal part of $f = \mathcal{F}^{-1} g$ (see [Sha94, Section 2.6] for the definition of the tangential part).

The second equation of the system (3.8), $A^{(m,1)}/(dg) = F^{(m,1)}$, constitutes a system of linear first order PDEs in components of the tensor field g, the third equation constitutes a system of linear second order PDEs, and so on.

At first sight, the following statement may seem incredible. The system (3.8) can be reduced to the purely algebraic system

$$A^{(m,k)}/g = H^{(m,k)} \quad (0 \le k \le m) \tag{3.9}$$

with right-hand side defined by (3.4). The reduction is presented in Section 5. The precise sense of the reduction is expressed by Proposition 5.3 below; see also the paragraph after Proposition 5.3.

Some consistency conditions should be imposed on right-hand side $H^{(m,k)}$ for solvability of the system (3.9). In the case of a general m, it is not easy to write down the consistency conditions explicitly. Fortunately, we do not need to know the consistency conditions; in our setting, the system (3.9) has a solution by Propositions 4.3 and 5.3 presented below. If the system (3.9) has a solution, then the solution is unique and is expressed by (3.5) with $\widehat{f} = g$. This fact is proved in Section 6.

4. Normal operator

We start with the proof of (2.5). For $f \in \mathcal{S}(\mathbb{R}^n; S^m)$ and $\varphi \in \mathcal{S}(T\mathbb{S}^{n-1})$,

$$\begin{aligned}(I^k_m f, \varphi)_{L^2(T\mathbb{S}^{n-1})} &= \int\limits_{\mathbb{S}^{n-1}} \int\limits_{\xi^\perp} (I^k_m f)(x,\xi)\, \overline{\varphi(x,\xi)}\, \mathrm{d}x\, \mathrm{d}\xi \\ &= \int\limits_{\mathbb{S}^{n-1}} \int\limits_{\xi^\perp} \int\limits_{-\infty}^{\infty} t^k \langle f(x'+t\xi), \xi^m\rangle\, \overline{\varphi(x',\xi)}\, \mathrm{d}t\, \mathrm{d}x'\, \mathrm{d}\xi.\end{aligned} \tag{4.1}$$

We transform the inner integral by the change $x = x' + t\xi$ of integration variables

$$\begin{aligned}\int\limits_{\xi^\perp} \int\limits_{-\infty}^{\infty} t^k \langle f(x'+t\xi), \xi^m\rangle\, \overline{\varphi(x',\xi)}\, \mathrm{d}t\, \mathrm{d}x' &= \int\limits_{\mathbb{R}^n} \langle x,\xi\rangle^k \langle f(x), \xi^m\rangle\, \overline{\varphi(x - \langle x,\xi\rangle\xi, \xi)}\, \mathrm{d}x \\ &= \int\limits_{\mathbb{R}^n} f^{i_1\dots i_m}(x)\, \xi_{i_1}\dots\xi_{i_m}\, \overline{\varphi(x - \langle x,\xi\rangle\xi, \xi)}\, \mathrm{d}x.\end{aligned}$$

Substituting this expression into (4.1), we obtain

$$\begin{aligned}(I^k_m f, \varphi)_{L^2(T\mathbb{S}^{n-1})} &= \int\limits_{\mathbb{R}^n} f^{i_1\dots i_m}(x)\, \overline{\int\limits_{\mathbb{S}^{n-1}} \xi_{i_1}\dots\xi_{i_m}\, \varphi(x - \langle x,\xi\rangle\xi, \xi)\, \mathrm{d}\xi}\, \mathrm{d}x \\ &= (f, (I^k_m)^*\varphi)_{L^2(\mathbb{R}^n;S^m)}.\end{aligned}$$

This proves (2.5).

Recall that $N^k_m = (I^k_m)^* I^k_m$ is the normal operator for the momentum ray transform.

Proposition 4.1. *Let $0 \le k \le m$ and $n \ge 2$. For a tensor field $f \in \mathcal{S}(\mathbb{R}^n; S^m)$,*

$$(N^k_m f)_{i_1\dots i_m}(x) = 2\sum_{l=0}^{k} \binom{k}{l} (x^{k+l} f)^{j_1\dots j_m p_1\dots p_{k+l}} * \frac{(x^{2m+k+l})_{i_1\dots i_m j_1\dots j_m p_1\dots p_{k+l}}}{|x|^{2m+2l+n-1}}, \tag{4.2}$$

where $*$ denotes the convolution.

The right-hand side of (4.2) needs the following comment. For $x \in \mathbb{R}^n$, according to our definition of the symmetric product, $x^{k+l} \in S^{k+l}$ with coordinates $(x^{k+l})^{p_1\dots p_{k+l}} = x^{p_1}\dots x^{p_{k+l}}$. Therefore, for $f \in S^m$,

$$(x^{k+l} f)^{j_1\dots j_m p_1\dots p_{k+l}} = \sigma(j_1\dots j_m p_1\dots p_{k+l})(x^{p_1}\dots x^{p_{k+l}} f^{j_1\dots j_m}).$$

Before proving Proposition 4.1, we observe that it implies some regularity of the tensor field $N_m^k f$. Indeed, the first factor $(x^{k+l}f)^{j_1\dots j_m p_1\dots p_{k+l}}$ on the right-hand side of (4.2) belongs to $\mathcal{S}(\mathbb{R}^n)$. The second factor is a function locally summable over $\mathbb{R}^n$ and bounded for $|x|\ge 1$. Hence the second factor can be considered as an element of the space $\mathcal{S}'(\mathbb{R}^n)$ of tempered distributions. As is well known [Vla79], for $u\in\mathcal{S}(\mathbb{R}^n)$ and $v\in\mathcal{S}'(\mathbb{R}^n)$, the convolution $u*v$ is defined and belongs to the space of smooth functions whose every derivative increases at most as a polynomial at infinity. In this case, the standard formula is valid: $\widehat{u*v}=\widehat{u}\widehat{v}$. Thus, we can state that

$$N_m^k:\mathcal{S}\left(\mathbb{R}^n;S^m\right)\to C^\infty\left(\mathbb{R}^n;S^m\right)$$

is a continuous operator.

To prove Proposition 4.1 we need the following

Lemma 4.2. *Let $k\ge 0$ be an integer, $0\ne a\in\mathbb{R}$, and $b\in\mathbb{R}$. Then*

$$\sum_{l=0}^{k}(-1)^l\binom{k}{l}\frac{(a^2+b)^{2k-l}}{a^{2k-2l}}=\sum_{l=0}^{k}\binom{k}{l}\frac{b^{k+l}}{a^{2l}}.$$

Proof. By the binomial formula,

$$\begin{aligned}\sum_{l=0}^{k}(-1)^l\binom{k}{l}\frac{(a^2+b)^{2k-l}}{a^{2k-2l}}&=\frac{1}{a^{2k}}\sum_{l=0}^{k}\binom{k}{l}(-a^2)^l(a^2+b)^{2k-l}\\&=\frac{1}{a^{2k}}\sum_{l=0}^{k}\binom{k}{l}(-a^2+b-b)^l(a^2+b)^{2k-l}\\&=\frac{(a^2+b)^k}{a^{2k}}\sum_{l=0}^{k}\binom{k}{l}(-a^2+b-b)^l(a^2+b)^{k-l}\\&=\frac{b^k\left(a^2+b\right)^k}{a^{2k}}=\sum_{l=0}^{k}\binom{k}{l}\frac{b^{k+l}}{a^{2l}}.\end{aligned}$$

□

Proof of Proposition 4.1. Using (2.4) and (2.5), we first compute

$$\begin{aligned}\left(N_m^k f\right)_{i_1\dots i_m}(x)&=\left(I_m^k\right)^*_{i_1\dots i_m}I_m^k f(x)\\&=\int_{\mathbb{S}^{n-1}}\langle x,\xi\rangle^k\xi_{i_1}\dots\xi_{i_m}(I_m^k f)(x-\langle x,\xi\rangle\xi,\xi)\,\mathrm{d}\xi\\&=\int_{\mathbb{S}^{n-1}}\int_{\mathbb{R}}t^k\langle x,\xi\rangle^k f^{j_1\dots j_m}(x-\langle x,\xi\rangle\xi+t\xi)\xi_{j_1}\dots\xi_{j_m}\xi_{i_1}\dots\xi_{i_m}\,\mathrm{d}t\mathrm{d}\xi\\&=2\int_{\mathbb{S}^{n-1}}\int_0^\infty t^k\langle x,\xi\rangle^k f^{j_1\dots j_m}(x-\langle x,\xi\rangle\xi+t\xi)(\xi^{2m})_{i_1\dots i_m j_1\dots j_m}\,\mathrm{d}t\mathrm{d}\xi.\end{aligned}$$

Replacing $t-\langle x,\xi\rangle$ by t in the last integral, we have

$$\begin{aligned}\left(N_m^k f\right)_{i_1\dots i_m}(x)&=2\int_{\mathbb{S}^{n-1}}\int_0^\infty\left(t+\langle x,\xi\rangle\right)^k\langle x,\xi\rangle^k f^{j_1\dots j_m}(x+t\xi)(\xi^{2m})_{i_1\dots i_m j_1\dots j_m}\,\mathrm{d}t\mathrm{d}\xi\\&=2\sum_{l=0}^{k}\binom{k}{l}\int_{\mathbb{S}^{n-1}}\int_0^\infty t^l\langle x,\xi\rangle^{2k-l}f^{j_1\dots j_m}(x+t\xi)(\xi^{2m})_{i_1\dots i_m j_1\dots j_m}\,\mathrm{d}t\mathrm{d}\xi.\end{aligned}$$

Changing integration variables by

$$x+t\xi=z,\quad t=|z-x|,\quad \xi=\frac{z-x}{|z-x|},\quad \mathrm{d}t\mathrm{d}\xi=|z-x|^{1-n}\mathrm{d}z,$$

we obtain

$$\big(N^k_m f\big)_{i_1\dots i_m}(x)=2\sum_{l=0}^{k}\binom{k}{l}\int_{\mathbb{R}^n}\langle x,z-x\rangle^{2k-l}\big((z-x)^{2m}\big)_{i_1\dots i_m j_1\dots j_m}\frac{f^{j_1\dots j_m}(z)}{|z-x|^{2m+2k-2l+n-1}}\,\mathrm{d}z.$$

Let us write this in the form

$$\begin{aligned}&\big(N^k_m f\big)_{i_1\dots i_m}(x)\\&=2\int_{\mathbb{R}^n}\Big[\sum_{l=0}^{k}(-1)^l\binom{k}{l}\frac{\langle x,x-z\rangle^{2k-l}}{|x-z|^{2k-2l}}\Big]\big((z-x)^{2m}\big)_{i_1\dots i_m j_1\dots j_m}\frac{f^{j_1\dots j_m}(z)}{|z-x|^{2m+n-1}}\,\mathrm{d}z.\end{aligned}$$

By Lemma 4.2 with $a=|x-z|$ and $b=\langle z,x-z\rangle$,

$$\begin{aligned}\sum_{l=0}^{k}(-1)^l\binom{k}{l}\frac{\langle x,x-z\rangle^{2k-l}}{|x-z|^{2k-2l}}&=\sum_{l=0}^{k}(-1)^l\binom{k}{l}\frac{\big(|x-z|^2+\langle z,x-z\rangle\big)^{2k-l}}{|x-z|^{2k-2l}}\\&=\sum_{l=0}^{k}\binom{k}{l}\frac{\langle z,x-z\rangle^{k+l}}{|x-z|^{2l}}=\sum_{l=0}^{k}\binom{k}{l}\frac{\langle z,x-z\rangle^{k+l}}{|z-x|^{2l}}.\end{aligned}$$

Substitute this expression into the previous formula

$$\big(N^k_m f\big)_{i_1\dots i_m}(x)=2\sum_{l=0}^{k}\binom{k}{l}\int_{\mathbb{R}^n}\langle z,x-z\rangle^{k+l}\big((x-z)^{2m}\big)_{i_1\dots i_m j_1\dots j_m}\frac{f^{j_1\dots j_m}(z)}{|x-z|^{2m+2l+n-1}}\,\mathrm{d}z.$$

Then we represent the first factor of the integrand as follows

$$\langle z,x-z\rangle^{k+l}=(z^{k+l})^{p_1\dots p_{k+l}}\big((x-z)^{k+l}\big)_{p_1\dots p_{k+l}}.$$

Substituting this expression into the previous formula, we write the result in the form

$$\begin{aligned}&\big(N^k_m f\big)_{i_1\dots i_m}(x)\\&=2\sum_{l=0}^{k}\binom{k}{l}\int_{\mathbb{R}^n}\big(z^{k+l}\otimes f(z)\big)^{p_1\dots p_{k+l}j_1\dots j_m}\frac{\big((x-z)^{2m+k+l}\big)_{i_1\dots i_m j_1\dots j_m p_1\dots p_{k+l}}}{|x-z|^{2m+2l+n-1}}\,\mathrm{d}z.\end{aligned}$$

We can replace $\big(z^{k+l}\otimes f(z)\big)^{p_1\dots p_{k+l}j_1\dots j_m}$ with $\big(z^{k+l}f(z)\big)^{j_1\dots j_m p_1\dots p_{k+l}}$ since the second factor in the integrand is symmetric in all indices. Hence

$$\begin{aligned}&\big(N^k_m f\big)_{i_1\dots i_m}(x)\\&=2\sum_{l=0}^{k}\binom{k}{l}\int_{\mathbb{R}^n}\big(z^{k+l}f(z)\big)^{j_1\dots j_m p_1\dots p_{k+l}}\frac{\big((x-z)^{2m+k+l}\big)_{i_1\dots i_m j_1\dots j_m p_1\dots p_{k+l}}}{|x-z|^{2m+2l+n-1}}\,\mathrm{d}z.\end{aligned}$$

Every integral on the right-hand side is the convolution of $\big(x^{k+l}f\big)^{j_1\dots j_m p_1\dots p_{k+l}}$ with $\frac{(x^{2m+k+l})_{i_1\dots i_m j_1\dots j_m p_1\dots p_{k+l}}}{|x|^{2m+2l+n-1}}$. We thus arrive at (4.2). □

We use the abbreviated notation $\partial_{i_1\dots i_k}=\frac{\partial^k}{\partial y^{i_1}\dots\partial y^{i_k}}$ for partial derivatives. Recall that indices can be written either in lower position or in upper position. In particular, $\partial^{i_1\dots i_k}=\partial_{i_1\dots i_k}$. Recall that $j_y:S^m\to S^{m-1}$ is the operator of contraction with y, see Subsection 2.1 where the operator j_u is defined. For $0 \le k \le m$, tensor fields $A^{(m,k)} \in C^\infty(\mathbb{R}^n \setminus \{0\}; S^{2m-k})$ are defined by (3.7) or in coordinates

$$A^{(m,k)}_{i_1\dots i_{2m-k}} = \partial_{i_1\dots i_{2m-k}} |y|^{2m-2k-1}.$$

Proposition 4.1 can be equivalently written in terms of Fourier transforms $\widehat{f}$ and $\widehat{N^k_m f}$.

Proposition 4.3. *Let $0 \le k \le m$ and $n \ge 2$. For $f \in \mathcal{S}(\mathbb{R}^n; S^m)$, the equation (3.8) holds with $g = \hat{f}$ and $F^{(m,k)}$ defined by (3.3).*

Proof. Applying the Fourier transform to the equality (4.2), we obtain

$$\widehat{N^k_m f}_{i_1\dots i_m} = 2\sum_{l=0}^{k} \binom{k}{l} \widehat{\left(x^{k+l} f\right)}^{j_1\dots j_m p_1\dots p_{k+l}} \mathcal{F}\Big[\frac{(x^{2m+k+l})_{i_1\dots i_m j_1\dots j_m p_1\dots p_{k+l}}}{|x|^{2m+2l+n-1}}\Big].$$

Using the standard properties of the Fourier transform [Hör83, Lemma 7.1.2]

$$\widehat{x_j f} = i\,\partial_j \widehat{f}, \quad \widehat{\partial_j f} = i\,y_j \widehat{f},$$

we transform our formula to the form

$$\widehat{N^k_m f}_{i_1\dots i_m} = 2(-1)^{m+k} \sum_{l=0}^{k} (-1)^l \binom{k}{l} \sigma(j_1 \dots j_m p_1 \dots p_{k+l}) \big(\partial^{p_1\dots p_{k+l}} \widehat{f}^{j_1\dots j_m}\big) \\ \times \partial_{i_1\dots i_m j_1\dots j_m p_1\dots p_{k+l}} \mathcal{F}\Big[|x|^{-2m-2l-n+1}\Big].$$

Here we can omit the symmetrization $\sigma(j_1 \dots j_m p_1 \dots p_{k+l})$ since the second factor $\partial_{i_1\dots i_m j_1\dots j_m p_1\dots p_{k+l}} \mathcal{F}\Big[|x|^{-2m-2l-n+1}\Big]$ is symmetric in all indices. Thus,

(4.3)
$$\widehat{N^k_m f}_{i_1\dots i_m} = 2(-1)^{m+k} \sum_{l=0}^{k} (-1)^l \binom{k}{l} \big(\partial^{p_1\dots p_{k+l}} \widehat{f}^{j_1\dots j_m}\big) \partial_{i_1\dots i_m j_1\dots j_m p_1\dots p_{k+l}} \mathcal{F}\Big[|x|^{-2m-2l-n+1}\Big].$$

Let $\mathcal{S}'(\mathbb{R}^n)$ be the space of tempered distributions. Recall that $\lambda \mapsto |x|^\lambda$ is the meromorphic $\mathcal{S}'(\mathbb{R}^n)$-valued function of $\lambda \in \mathbb{C}$ with simple poles at points $-n, -n-2, -n-4, \dots$. The Fourier transform of $|x|^\lambda$ is expressed by

$$\mathcal{F}[|x|^\lambda] = \frac{2^{\lambda+n/2}\Gamma\left(\frac{\lambda+n}{2}\right)}{\Gamma(-\lambda/2)} |y|^{-\lambda-n} \quad (\lambda, -\lambda-n \notin 2\mathbb{Z}^+),$$
$$\mathcal{F}[|x|^{2k}] = (2\pi)^{n/2}(-\Delta)^k \delta \quad (k \in \mathbb{Z}^+),$$

where δ is the Dirac function. In particular,

$$\mathcal{F}\Big[|x|^{-2m-2l-n+1}\Big] = \frac{\Gamma\left(\frac{1-2m-2l}{2}\right)}{2^{2m+2l+n/2-1}\Gamma\left(\frac{2m+2l+n-1}{2}\right)} |y|^{2m+2l-1}.$$

Substitute this value into (4.3)

(4.4)
$$\widehat{N^k_m f}_{i_1\dots i_m}(y) = \frac{(-1)^{m+k}}{2^{2m+n/2-2}} \sum_{l=0}^{k} \binom{k}{l} \frac{(-1)^l \Gamma\left(\frac{1-2m-2l}{2}\right)}{2^{2l}\Gamma\left(\frac{2m+2l+n-1}{2}\right)} \Big(\partial_{i_1\dots i_m j_1\dots j_m p_1\dots p_{k+l}} |y|^{2m+2l-1}\Big) \\ \times \partial^{p_1\dots p_{k+l}} \widehat{f}^{j_1\dots j_m}(y).$$

Let us contract the equation (4.4) with $y^{i_1}\dots y^{i_k}$, i.e., multiply the equation by $y^{i_1}\dots y^{i_k}$ and perform the summation over indices $i_1\dots i_k$

$$(4.5)\qquad \big(j^k_y\,\widehat{N^k_m f}\big)_{i_{k+1}\dots i_m}(y)=\frac{(-1)^{m+k}}{2^{2m+n/2-2}}\sum_{l=0}^{k}\binom{k}{l}\frac{(-1)^l\Gamma\Big(\frac{1-2m-2l}{2}\Big)}{2^{2l}\Gamma\Big(\frac{2m+2l+n-1}{2}\Big)}$$
$$\times\Big[y^{i_1}\dots y^{i_k}\partial_{i_1\dots i_k}\Big(\partial_{i_{k+1}\dots i_m j_1\dots j_m p_1\dots p_{k+l}}|y|^{2m+2l-1}\Big)\Big]\partial^{p_1\dots p_{k+l}}\widehat{f}^{j_1\dots j_m}(y).$$

On the right-hand side, all summands corresponding to $l>0$ are equal to zero. Indeed, $\partial_{i_{k+1}\dots i_m j_1\dots j_m p_1\dots p_{k+l}}|y|^{2m+2l-1}$ is the positively homogeneous function of degree $l-1$. By the Euler equation for homogeneous functions,

$$y^{i_1}\dots y^{i_k}\partial_{i_1\dots i_k}\Big(\partial_{i_{k+1}\dots i_m j_1\dots j_m p_1\dots p_{k+l}}|y|^{2m+2l-1}\Big)=\begin{cases}(-1)^k k!|y|^{2m-1} & \text{if } l=0,\\ 0 & \text{if } l>0.\end{cases}$$

The formula (4.5) becomes

(4.6)
$$\big(j^k_y\,\widehat{N^k_m f}\big)_{i_{k+1}\dots i_m}(y)=\frac{(-1)^m k!\,\Gamma\Big(\frac{1-2m}{2}\Big)}{2^{2m+n/2-2}\Gamma\Big(\frac{2m+n-1}{2}\Big)}\Big(\partial_{i_{k+1}\dots i_m j_1\dots j_m p_1\dots p_k}|y|^{2m-1}\Big)\partial^{p_1\dots p_k}\widehat{f}^{j_1\dots j_m}(y).$$

This is equivalent to (3.8). □

Lemma 4.4. *Let $0\le k\le m$ and $n\ge 2$. Then $j^{k+1}_y\,\widehat{N^k_m f}(y)=0$ for any tensor field $f\in\mathcal{S}(\mathbb{R}^n;S^m)$ and for any $y\in\mathbb{R}^n$.*

Proof. The statement trivially holds in the case of $k=m$. In the case of $k<m$ we apply the operator j_y to the equality (4.6)

$$\big(j^{k+1}_y\,\widehat{N^k_m f}\big)_{i_{k+2}\dots i_m}(y)=\frac{1}{C^k_m}\Big[\big(y^{i_{k+1}}\partial_{i_{k+1}}\big)\Big(\partial_{i_{k+2}\dots i_m j_1\dots j_m p_1\dots p_k}|y|^{2m-1}\Big)\Big]\partial^{p_1\dots p_k}\widehat{f}^{j_1\dots j_m}(y),$$

where $C^k_m=(-1)^m k!\Gamma\big(\frac{1-2m}{2}\big)/\Big(2^{2m+n/2-2}\Gamma\big(\frac{2m+n-1}{2}\big)\Big)$. The expression in brackets is equal to zero since $\partial_{i_{k+2}\dots i_m j_1\dots j_m p_1\dots p_k}|y|^{2m-1}$ is a positively homogeneous function of zero degree. □

From now on we can forget the momentum ray transform. The rest of the article is devoted to investigation of the system (3.8).

Lemma 4.4 implies that right-hand sides of equations (3.8) satisfy

$$(4.7)\qquad j_y\,F^{(m,k)}(y)=0\quad(0\le k\le m).$$

Thus, equalities (4.7) constitute necessary conditions for existence of a solution $g\in\mathcal{S}(\mathbb{R}^n;S^m)$ to the system (3.8). Most probably, equalities (4.7) are necessary and sufficient consistency conditions for the system (3.8), but this fact is not proved.

5. Reduction of the system (3.8) to an algebraic system

Tensor fields $A^{(m,k)}\in C^\infty(\mathbb{R}^n\setminus\{0\};S^{2m-k})$ $(0\le k\le m)$ are defined by (3.7). There exist two important relations between these tensor fields.

Lemma 5.1. *The following equalities are valid:*

$$(5.1)\qquad j_yA^{(m,k)}=-kA^{(m-1,k-1)}\quad(0\le k\le m),$$

$$(5.2)\qquad \operatorname{div}A^{(m,k)}=(2m-2k-1)(n+2m-2k-3)A^{(m,k+1)}\quad(0\le k\le m).$$

Proof. Applying the operator j_y to the equality (3.7), we have

$$j_y A^{(m,k)} = j_y d^{2m-k}|y|^{2m-2k-1}.$$

With the help of the operator $\langle y, \partial\rangle = y^j \frac{\partial}{\partial y^j}$, the latter formula can be written as

$$j_y A^{(m,k)} = \langle y, \partial\rangle\, d^{2m-k-1}|y|^{2m-2k-1}. \tag{5.3}$$

The tensor field $d^{2m-k-1}|y|^{2m-2k-1}$ is positively homogeneous of degree $-k$. By the Euler equation for homogeneous functions,

$$\langle y, \partial\rangle\, d^{2m-k-1}|y|^{2m-2k-1} = -k\, d^{2m-k-1}|y|^{2m-2k-1}.$$

Substituting this expression into (5.3), we obtain

$$j_y A^{(m,k)} = -k\, d^{2m-k-1}|y|^{2m-2k-1}.$$

By (3.6), the right-hand side of this formula is equal to $-kA^{(m-1,k-1)}$. This proves (5.1).

Let us write (3.7) in the coordinate form

$$A^{(m,k)}_{i_{k+1}\dots i_{2m}} = \partial_{i_{k+1}\dots i_{2m}}|y|^{2m-2k-1}.$$

Differentiate this equality

$$\frac{\partial A^{(m,k)}_{i_{k+1}\dots i_{2m}}}{\partial y^j} = \partial_{j i_{k+1}\dots i_{2m}}|y|^{2m-2k-1}.$$

From this

$$\begin{aligned}(\operatorname{div} A^{(m,k)})_{i_{k+2}\dots i_{2m}} &= \delta^{jl}\, \frac{\partial A^{(m,k)}_{l i_{k+2}\dots i_{2m}}}{\partial y^j} = \delta^{jl}\, \partial_{j l i_{k+2}\dots i_{2m}}|y|^{2m-2k-1}\\ &= \partial_{i_{k+2}\dots i_{2m}}(\Delta|y|^{2m-2k-1}).\end{aligned}$$

This can be written in the coordinate-free form

$$\operatorname{div} A^{(m,k)} = d^{2m-k-1}(\Delta|y|^{2m-2k-1}). \tag{5.4}$$

Using the obvious formula

$$\Delta|y|^{\alpha} = \alpha(\alpha+n-2)|y|^{\alpha-2},$$

we obtain

$$\Delta|y|^{2m-2k-1} = (2m-2k-1)(n+2m-2k-3)|y|^{2m-2k-3}.$$

Substituting this expression into (5.4), we have

$$\operatorname{div} A^{(m,k)} = (2m-2k-1)(n+2m-2k-3)\, d^{2m-k-1}|y|^{2m-2k-3}.$$

By (3.7),

$$d^{2m-k-1}|y|^{2m-2k-3} = A^{(m,k+1)}.$$

Two last formulas imply (5.2). □

From (5.2), one easily proves by induction on k

$$\operatorname{div}^k A^{(m,0)} = \frac{(2m-1)!!(n+2m-3)!!}{(2m-2k-1)!!(n+2m-2k-3)!!}\, A^{(m,k)} \quad (0 \le k \le m). \tag{5.5}$$

We reproduce the system (3.8)

$$A^{(m,0)}/(d^l g) = F^{(m,l)} \quad (l = 0, 1, \dots, m). \tag{5.6}$$

Here $F^{(m,l)} \in C^\infty(\mathbb{R}^n \setminus \{0\}; S^{m-l})$ $(0 \le l \le m)$ are arbitrary tensor fields belonging to the kernel of j_y.

Proposition 5.2. *If a tensor field $g \in C^\infty(\mathbb{R}^n \setminus \{0\}; S^m)$ satisfies* (5.6), *then*
(5.7)
$$(\operatorname{div}^k A^{(m,0)})/(d^l g) = (-1)^k \sum_{p=0}^{k} (-1)^p \binom{k}{p} \operatorname{div}^p F^{(m,k+l-p)} \quad (0 \le k \le m, 0 \le l \le m-k).$$

Proof. We prove (5.7) by induction on k. For $k = 0$, (5.7) coincides with (5.6). Assume (5.7) to be valid for some k. Apply the operator div to the equation (5.7)

$$\operatorname{div}\Big((\operatorname{div}^k A^{(m,0)})/(d^l g)\Big) = (-1)^k \sum_{p} (-1)^p \binom{k}{p} \operatorname{div}^{p+1} F^{(m,k+l-p)}. \tag{5.8}$$

We assume binomial coefficients to be defined for all integers k and p under the agreement:

$$\binom{k}{p} = 0 \quad \text{if either } k < 0 \text{ or } p < 0 \text{ or } k < p. \tag{5.9}$$

Due to the agreement, we can assume the summation to be performed over all integers p in (5.8) and formulas below.

The equality

$$\operatorname{div}(u/v) = (\operatorname{div} u)/v + u/(dv)$$

is valid for any two tensor fields. It is easily proved on the base of definitions of the operators d and div. With the help of this equality, we write (5.8) in the form

$$(\operatorname{div}^{k+1} A^{(m,0)})/(d^l g) + (\operatorname{div}^k A^{(m,0)})/(d^{l+1} g) = (-1)^k \sum_{p} (-1)^p \binom{k}{p} \operatorname{div}^{p+1} F^{(m,k+l-p)}.$$

By the induction hypothesis,

$$(\operatorname{div}^k A^{(m,0)})/(d^{l+1} g) = (-1)^k \sum_{p} (-1)^p \binom{k}{p} \operatorname{div}^{p} F^{(m,k+l-p+1)}.$$

Substituting this expression into the previous formula, we write the result in the form

$$(\operatorname{div}^{k+1} A^{(m,0)})/(d^l g) = (-1)^k \sum_{p} (-1)^p \binom{k}{p} \operatorname{div}^{p+1} F^{(m,k+l-p)} + (-1)^{k+1} \sum_{p} (-1)^p \binom{k}{p} \operatorname{div}^{p} F^{(m,k+l-p+1)}.$$

Changing the summation variable of the first sum as $p := p - 1$, we obtain

$$(\operatorname{div}^{k+1} A^{(m,0)})/(d^l g) = (-1)^{k+1} \sum_{p} (-1)^p \left[\binom{k}{p-1} + \binom{k}{p}\right] \operatorname{div}^{p} F^{(m,k+l-p+1)}.$$

By the Pascal triangle equality, $\binom{k}{p-1} + \binom{k}{p} = \binom{k+1}{p}$. Substituting this expression into the last formula, we arrive at (5.7) for $k := k + 1$. ☐

Setting $l = 0$ in (5.7), we obtain

$$(\operatorname{div}^k A^{(m,0)})/g = (-1)^k \sum_{p=0}^{k} (-1)^p \binom{k}{p} \operatorname{div}^p F^{(m,k-p)} \quad (0 \le k \le m).$$

Substituting the value (5.5) of $\operatorname{div}^k A^{(m,0)}$, we arrive at the equation

$$A^{(m,k)}/g = (-1)^k \frac{(2m-2k-1)!!(n+2m-2k-3)!!}{(2m-1)!!(n+2m-3)!!} \sum_{p=0}^{k} (-1)^p \binom{k}{p} \operatorname{div}^p F^{(m,k-p)}.$$

We have thus proved

Proposition 5.3. *If a tensor field $g \in C^\infty(\mathbb{R}^n \setminus \{0\}; S^m)$ satisfies* (3.8)*, then it also solves the system* (3.9) *with right-hand sides defined by* (3.4)*.*

We emphasize that (3.9) is a system of linear algebraic equations in coordinates of the unknown tensor field g. Of course the system (3.9) is not equivalent to (3.8). Proposition 5.3 states that (3.8) implies (3.9) but not vice versa. Nevertheless, we will see that g can be uniquely recovered from (3.9). In this sense the system (3.8) is reduced to the algebraic system (3.9).

6. Solution of the system (3.9)

The following statement completes the proof of Theorem 3.3.

Proposition 6.1. *If the system* (3.9) *is solvable, then the solution $g = \widehat{f}$ is unique and is expressed by the formula* (3.5)*.*

The proof of Proposition 6.1 is not easy. The main part of the proof is contained in the following two lemmas.

Lemma 6.2. *Given a tensor field $g \in C^\infty(\mathbb{R}^n \setminus \{0\}; S^{m+1})$, let us fix a Cartesian coordinate system on $\mathbb{R}^n$, fix a value of the index i_{m+1} and introduce the tensor field $\tilde g \in C^\infty(\mathbb{R}^n \setminus \{0\}; S^m)$ by*

$$\tilde g_{i_1\dots i_m} = g_{i_1\dots i_m i_{m+1}}. \tag{6.1}$$

Let us also introduce the vector field $\tilde\delta$ by

$$\tilde\delta_i = \delta_{i i_{m+1}}.$$

Then, for every $0 \le k \le m$,

$$\begin{aligned}(A^{(m,k)}/\tilde g)_{i_{k+1}\dots i_m} = \frac{1}{m+1}\Big[&\frac{1}{2m-2k+1}\,(A^{(m+1,k)}/g)_{i_{k+1}\dots i_{m+1}}\\ &- y_{i_{m+1}}\,(A^{(m+1,k+1)}/g)_{i_{k+1}\dots i_m} - (m-k)\,\Big(\tilde\delta\,(A^{(m,k)}/g)\Big)_{i_{k+1}\dots i_m}\Big].\end{aligned} \tag{6.2}$$

Proof. The identity

$$\partial_{i_1\dots i_p}(|y|^\alpha y_k) = y_k\,\partial_{i_1\dots i_p}|y|^\alpha + p\,\sigma(i_1\dots i_p)(\delta_{i_1k}\,\partial_{i_2\dots i_p}|y|^\alpha)$$

holds for any integer $p \ge 0$, any real α and any $1 \le k \le n$. It is easily proved by induction on p. With the help of this identity, we obtain

$$\begin{aligned}&\partial_{i_{k+1}\dots i_{m+1}j_1\dots j_{m+1}}|y|^{2m-2k+1} = (2m-2k+1)\partial_{i_{k+1}\dots i_m j_1\dots j_{m+1}}(|y|^{2m-2k-1}y_{i_{m+1}})\\ &= (2m-2k+1)\Big[y_{i_{m+1}}\,\partial_{i_{k+1}\dots i_m j_1\dots j_{m+1}}|y|^{2m-2k-1}\\ &+ (2m-k+1)\sigma(i_{k+1}\dots i_m j_1\dots j_{m+1})\Big(\delta_{i_{m+1}j_{m+1}}\,\partial_{i_{k+1}\dots i_m j_1\dots j_m}|y|^{2m-2k-1}\Big)\Big].\end{aligned}$$

Expanding the symmetrization $\sigma(i_{k+1}\dots i_m j_1\dots j_{m+1})$ with respect to the index j_{m+1} (see [Sha94, Lemma 2.4.1]), we write this in the form

$$\begin{aligned}\frac{1}{2m-2k+1}\,\partial_{i_{k+1}\dots i_{m+1}j_1\dots j_{m+1}}|y|^{2m-2k+1} &= y_{i_{m+1}}\,\partial_{i_{k+1}\dots i_m j_1\dots j_{m+1}}|y|^{2m-2k-1}\\ &+ \sigma(i_{k+1}\dots i_m j_1\dots j_m)\Big(\delta_{i_{m+1}j_{m+1}}\,\partial_{i_{k+1}\dots i_m j_1\dots j_m}|y|^{2m-2k-1}\\ &+ (2m-k)\delta_{i_{m+1}j_1}\,\partial_{i_{k+1}\dots i_m j_2\dots j_{m+1}}|y|^{2m-2k-1}\Big).\end{aligned}$$

Since the tensor $\delta_{i_{m+1}j_{m+1}}\,\partial_{i_{k+1}\dots i_m j_1\dots j_m}|y|^{2m-2k-1}$ is symmetric in $i_{k+1},\dots,i_m,j_1,\dots,j_m$, the formula can be written as follows:

$$
\begin{aligned}
&\frac{1}{2m-2k+1}\,\partial_{i_{k+1}\dots i_{m+1}j_1\dots j_{m+1}}|y|^{2m-2k+1}=y_{i_{m+1}}\,\partial_{i_{k+1}\dots i_m j_1\dots j_{m+1}}|y|^{2m-2k-1}\\
&+\delta_{i_{m+1}j_{m+1}}\,\partial_{i_{k+1}\dots i_m j_1\dots j_m}|y|^{2m-2k-1}\\
&+(2m-k)\sigma(i_{k+1}\dots i_m j_1\dots j_m)\Big(\delta_{i_{m+1}j_1}\,\partial_{i_{k+1}\dots i_m j_2\dots j_{m+1}}|y|^{2m-2k-1}\Big).
\end{aligned}
\tag{6.3}
$$

By (3.7),

$$
\begin{aligned}
\partial_{i_{k+1}\dots i_m j_1\dots j_{m+1}}|y|^{2m-2k-1}&=A^{(m+1,k+1)}_{i_{k+1}\dots i_m j_1\dots j_{m+1}},\\
\partial_{i_{k+1}\dots i_m j_1\dots j_m}|y|^{2m-2k-1}&=A^{(m,k)}_{i_{k+1}\dots i_m j_1\dots j_m},\\
\partial_{i_{k+1}\dots i_m j_2\dots j_{m+1}}|y|^{2m-2k-1}&=A^{(m,k)}_{i_{k+1}\dots i_m j_2\dots j_{m+1}}.
\end{aligned}
$$

Substitute these expressions into (6.3)

$$
\begin{aligned}
&\frac{1}{2m-2k+1}\,\partial_{i_{k+1}\dots i_{m+1}j_1\dots j_{m+1}}|y|^{2m-2k+1}=y_{i_{m+1}}\,A^{(m+1,k+1)}_{i_{k+1}\dots i_m j_1\dots j_{m+1}}\\
&+\delta_{i_{m+1}j_{m+1}}\,A^{(m,k)}_{i_{k+1}\dots i_m j_1\dots j_m}\\
&+(2m-k)\,\sigma(i_{k+1}\dots i_m j_1\dots j_m)\Big(\delta_{i_{m+1}j_1}\,A^{(m,k)}_{i_{k+1}\dots i_m j_2\dots j_{m+1}}\Big).
\end{aligned}
\tag{6.4}
$$

The equality

$$
\begin{aligned}
&\sigma(i_{k+1}\dots i_m j_1\dots j_m)\Big(\delta_{i_{m+1}j_1}\,A^{(m,k)}_{i_{k+1}\dots i_m j_2\dots j_{m+1}}\Big)\\
&=\frac{1}{2m-k}\sigma(i_{k+1}\dots i_m)\sigma(j_1\dots j_m)\Big[(m-k)\delta_{i_{m+1}i_m}\,A^{(m,k)}_{i_{k+1}\dots i_{m-1}j_1\dots j_{m+1}}\\
&\qquad\qquad+m\delta_{i_{m+1}j_1}\,A^{(m,k)}_{i_{k+1}\dots i_m j_2\dots j_{m+1}}\Big]
\end{aligned}
\tag{6.5}
$$

is easily proved on the base of the only fact: the tensor $A^{(m,k)}$ is symmetric. Formally speaking, the first term $(m-k)\delta_{i_{m+1}i_m}\,A^{(m,k)}_{i_{k+1}\dots i_{m-1}j_1\dots j_{m+1}}$ in brackets makes sense for $k\le m-2$ only. Nevertheless, the formula (6.5) holds for $k=m-1$ if we assume that $A^{(m,k)}_{i_m\dots i_{m-1}j_1\dots j_{m+1}}=A^{(m,k)}_{j_1\dots j_{m+1}}$. In the case of $k=m$, the first term in brackets is equal to zero because of the factor $(m-k)$. Thus, the formula (6.5) holds for all $0\le k\le m$.

With the help of (6.5), the formula (6.4) becomes

$$
\begin{aligned}
&\frac{1}{2m-2k+1}\,\partial_{i_{k+1}\dots i_{m+1}j_1\dots j_{m+1}}|y|^{2m-2k+1}=y_{i_{m+1}}\,A^{(m+1,k+1)}_{i_{k+1}\dots i_m j_1\dots j_{m+1}}\\
&+\delta_{i_{m+1}j_{m+1}}\,A^{(m,k)}_{i_{k+1}\dots i_m j_1\dots j_m}\\
&+(m-k)\,\sigma(i_{k+1}\dots i_m)\sigma(j_1\dots j_m)\Big(\delta_{i_{m+1}i_m}\,A^{(m,k)}_{i_{k+1}\dots i_{m-1}j_1\dots j_{m+1}}\Big)\\
&+m\,\sigma(i_{k+1}\dots i_m)\sigma(j_1\dots j_m)\Big(\delta_{i_{m+1}j_1}\,A^{(m,k)}_{i_{k+1}\dots i_m j_2\dots j_{m+1}}\Big).
\end{aligned}
$$

On the last line, the symmetrization $\sigma(i_{k+1}\dots i_m)$ can be deleted since the tensor $A^{(m,k)}_{i_{k+1}\dots i_m j_2\dots j_{m+1}}$ is symmetric in these indices. The formula simplifies to the following one:

$$
\begin{aligned}
\frac{1}{2m-2k+1}\,\partial_{i_{k+1}\dots i_{m+1}j_1\dots j_{m+1}}|y|^{2m-2k+1} &= y_{i_{m+1}}\,A^{(m+1,k+1)}_{i_{k+1}\dots i_m j_1\dots j_{m+1}}\\
&+\delta_{i_{m+1}j_{m+1}}\,A^{(m,k)}_{i_{k+1}\dots i_m j_1\dots j_m}\\
&+(m-k)\,\sigma(i_{k+1}\dots i_m)\sigma(j_1\dots j_m)\Big(\delta_{i_{m+1}i_m}\,A^{(m,k)}_{i_{k+1}\dots i_{m-1}j_1\dots j_{m+1}}\Big)\\
&+m\,\sigma(j_1\dots j_m)\Big(\delta_{i_{m+1}j_1}\,A^{(m,k)}_{i_{k+1}\dots i_m j_2\dots j_{m+1}}\Big).
\end{aligned}
\tag{6.6}
$$

Formulas (6.3) and (6.6) imply

$$
\begin{aligned}
\frac{1}{2m-2k+1}\,(A^{(m+1,k)}/g)_{i_{k+1}\dots i_{m+1}} &= \Big[y_{i_{m+1}}\,A^{(m+1,k+1)}_{i_{k+1}\dots i_m j_1\dots j_{m+1}}\\
&+\delta_{i_{m+1}j_{m+1}}\,A^{(m,k)}_{i_{k+1}\dots i_m j_1\dots j_m}\\
&+(m-k)\,\sigma(i_{k+1}\dots i_m)\sigma(j_1\dots j_m)\Big(\delta_{i_{m+1}i_m}\,A^{(m,k)}_{i_{k+1}\dots i_{m-1}j_1\dots j_{m+1}}\Big)\\
&+m\,\sigma(j_1\dots j_m)\Big(\delta_{i_{m+1}j_1}\,A^{(m,k)}_{i_{k+1}\dots i_m j_2\dots j_{m+1}}\Big)\Big]\,g^{j_1\dots j_{m+1}}.
\end{aligned}
$$

The symmetrization $\sigma(j_1\dots j_m)$ can be deleted after opening brackets since the tensor $g^{j_1\dots j_{m+1}}$ is symmetric in these indices. We thus obtain

$$
\begin{aligned}
\frac{1}{2m-2k+1}\,(A^{(m+1,k)}/g)_{i_{k+1}\dots i_{m+1}} &= y_{i_{m+1}}\,A^{(m+1,k+1)}_{i_{k+1}\dots i_m j_1\dots j_{m+1}}g^{j_1\dots j_{m+1}}\\
&+\delta_{i_{m+1}j_{m+1}}\,A^{(m,k)}_{i_{k+1}\dots i_m j_1\dots j_m}g^{j_1\dots j_{m+1}}\\
&+(m-k)\,\sigma(i_{k+1}\dots i_m)\Big(\delta_{i_{m+1}i_m}\,A^{(m,k)}_{i_{k+1}\dots i_{m-1}j_1\dots j_{m+1}}g^{j_1\dots j_{m+1}}\Big)\\
&+m\,\delta_{i_{m+1}j_1}\,A^{(m,k)}_{i_{k+1}\dots i_m j_2\dots j_{m+1}}g^{j_1\dots j_{m+1}}.
\end{aligned}
$$

Implementing the contraction with the Kronecker tensor in second and last lines, we obtain

$$
\begin{aligned}
\frac{1}{2m-2k+1}\,(A^{(m+1,k)}/g)_{i_{k+1}\dots i_{m+1}} &= y_{i_{m+1}}\,A^{(m+1,k+1)}_{i_{k+1}\dots i_m j_1\dots j_{m+1}}g^{j_1\dots j_{m+1}}\\
&+A^{(m,k)}_{i_{k+1}\dots i_m j_1\dots j_m}g^{j_1\dots j_m}_{i_{m+1}}\\
&+(m-k)\,\sigma(i_{k+1}\dots i_m)\Big(\delta_{i_{m+1}i_m}\,A^{(m,k)}_{i_{k+1}\dots i_{m-1}j_1\dots j_{m+1}}g^{j_1\dots j_{m+1}}\Big)\\
&+m\,A^{(m,k)}_{i_{k+1}\dots i_m j_1\dots j_m}g^{j_1\dots j_m}_{i_{m+1}}.
\end{aligned}
$$

In the last line, we have replaced the summation indices $j_2,\dots,j_{m+1}$ with $j_1,\dots,j_m$. We see now that second and last lines contain similar terms. Grouping this terms, we write the formula as follows:

$$
\begin{aligned}
\frac{1}{2m-2k+1}\,(A^{(m+1,k)}/g)_{i_{k+1}\dots i_{m+1}} &= y_{i_{m+1}}\,A^{(m+1,k+1)}_{i_{k+1}\dots i_m j_1\dots j_{m+1}}g^{j_1\dots j_{m+1}}\\
&+(m-k)\,\sigma(i_{k+1}\dots i_m)\Big(\delta_{i_{m+1}i_m}\,A^{(m,k)}_{i_{k+1}\dots i_{m-1}j_1\dots j_{m+1}}g^{j_1\dots j_{m+1}}\Big)\\
&+(m+1)\,A^{(m,k)}_{i_{k+1}\dots i_m j_1\dots j_m}g^{j_1\dots j_m}_{i_{m+1}}.
\end{aligned}
\tag{6.7}
$$

Recall that the value of the index i_{m+1} is fixed and $\tilde g^{j_1\dots j_m}=g^{j_1\dots j_m}_{i_{m+1}}$. The formula (6.7) can be written as

$$\begin{aligned}\frac{1}{2m-2k+1}\,(A^{(m+1,k)}/g)_{i_{k+1}\dots i_{m+1}}&=y_{i_{m+1}}\,(A^{(m+1,k+1)}/g)_{i_{k+1}\dots i_m}\\&+(m-k)\Big(\tilde\delta\,(A^{(m,k)}/g)\Big)_{i_{k+1}\dots i_m}+(m+1)\,(A^{(m,k)}/\tilde g)_{i_{k+1}\dots i_m}.\end{aligned}$$

This is equivalent to (6.2). ◻

Lemma 6.3. *For a tensor field $g\in C^\infty(\mathbb{R}^n\setminus\{0\};S^m)$, the following identity holds:*

$$A^{(m,0)}/g=m!(2m-1)!!\,|y|^{-1}g+\sum_{k=1}^{m}\sum_{p=0}^{\min(k,m-k)}\beta(m,k,p)\,i_y^{k-p}i^pj_y^p(A^{(m,k)}/g),\tag{6.8}$$

where the coefficients $\beta(m,k,p)$ are uniquely determined by the recurrent formulas

$$\begin{aligned}\tilde\beta(m{+}1,k,p)&=\frac{1}{2m-2k+1}\,\beta(m,k,p)-\frac{k-p}{k}\,\beta(m,k{-}1,p)\\&+\frac{m-k-p+2}{k}\,\beta(m,k{-}1,p{-}1)\end{aligned}\tag{6.9}$$

and

$$\beta(m+1,k,p)=\begin{cases}(2m{+}1)\big(\tilde\beta(m+1,k,p)+1\big) & \text{if } (k,p)=(1,0),\\(2m{+}1)\big(\tilde\beta(m+1,k,p)-m\big) & \text{if } (k,p)=(1,1),\\(2m{+}1)\tilde\beta(m+1,k,p) & \text{otherwise}\end{cases}\tag{6.10}$$

under the agreement

$$\beta(m,k,p)=0\quad\text{if either } k=0 \text{ or } k>m \text{ or } p<0 \text{ or } p>\min(k,m-k).\tag{6.11}$$

Proof. The proof is going by induction on m. For $m=0$, the sum on the right-hand side of (6.8) is absent and the formula holds since $A^{(m,0)}=|y|^{-1}$. Assume (6.8) to be valid for some $m\ge 0$ and let $g\in C^\infty(\mathbb{R}^n\setminus\{0\};S^{m+1})$. We fix a value of the index i_{m+1} and introduce the tensor field $\tilde g\in C^\infty(\mathbb{R}^n\setminus\{0\};S^m)$ by (6.1). By the induction hypothesis, the formula (6.8) holds for $\tilde g$. Let us write the formula in coordinates

$$\begin{aligned}A^{(m,0)}_{i_1\dots i_m j_1\dots j_m}g^{j_1\dots j_m}_{i_{m+1}}&=m!(2m-1)!!\,|y|^{-1}g_{i_1\dots i_{m+1}}\\&+\sigma(i_1\dots i_m)\sum_{k=1}^{m}\sum_{p=0}^{\min(k,m-k)}\beta(m,k,p)\,y^{l_1}\dots y^{l_p}\times\\&\times\delta_{i_1i_2}\dots\delta_{i_{2p-1}i_{2p}}y_{i_{2p+1}}\dots y_{i_{k+p}}A^{(m,k)}_{i_{k+p+1}\dots i_m l_1\dots l_p j_1\dots j_m}g^{j_1\dots j_m}_{i_{m+1}}.\end{aligned}\tag{6.12}$$

By Lemma 6.2,

$$\begin{aligned}&A^{(m,k)}_{i_{k+p+1}\dots i_m l_1\dots l_p j_1\dots j_m}g^{j_1\dots j_m}_{i_{m+1}}=(A^{(m,k)}/\tilde g)_{i_{k+p+1}\dots i_m l_1\dots l_p}\\&=\frac{1}{m+1}\Big[\frac{1}{2m-2k+1}\,(A^{(m+1,k)}/g)_{i_{k+p+1}\dots i_{m+1}l_1\dots l_p}-y_{i_{m+1}}\,(A^{(m+1,k+1)}/g)_{i_{k+p+1}\dots i_m l_1\dots l_p}\\&-(m-k)\Big(\tilde\delta\,(A^{(m,k)}/g)\Big)_{i_{k+p+1}\dots i_m l_1\dots l_p}\Big].\end{aligned}\tag{6.13}$$

In the case of $k=m$, the last term on the right-hand side of (6.13) is equal to zero. In the case of $k<m$, we transform the last term on the right-hand side of (6.13) with

the help of (5.1) as follows:

$$\text{(6.14)}\qquad \begin{aligned}\Big(\tilde\delta\,(A^{(m,k)}/g)\Big)_{i_{k+p+1}\dots i_m l_1\dots l_p} &= -\frac{1}{k+1}\Big(\tilde\delta(j_yA^{(m+1,k+1)}/g)\Big)_{i_{k+p+1}\dots i_m l_1\dots l_p}\\ = -\frac{1}{k+1}\sigma(i_{k+p+1}\dots i_m l_1\dots l_p)&\Big(\delta_{i_{m+1}i_{k+p+1}}(j_yA^{(m+1,k+1)}/g)_{i_{k+p+2}\dots i_m l_1\dots l_p}\Big).\end{aligned}$$

The equality

$$\begin{aligned}\sigma(i_{k+p+1}\dots i_m l_1\dots l_p)&\Big(\delta_{i_{m+1}i_{k+p+1}}(j_yA^{(m+1,k+1)}/g)_{i_{k+p+2}\dots i_m l_1\dots l_p}\Big)\\ &= \frac{m-k-p}{m-k}\,\sigma(i_{k+p+1}\dots i_m)\Big(\delta_{i_{m+1}i_{k+p+1}}(j_yA^{(m+1,k+1)}/g)_{i_{k+p+2}\dots i_m l_1\dots l_p}\Big)\\ &+\frac{p}{m-k}\,\sigma(l_1\dots l_p)\Big(\delta_{i_{m+1}l_1}(j_yA^{(m+1,k+1)}/g)_{i_{k+p+1}\dots i_m l_2\dots l_p}\Big)\end{aligned}$$

holds since $j_yA^{(m+1,k+1)}/g$ is a symmetric tensor. With the help of this, the formula (6.14) takes the form

$$\text{(6.15)}\qquad \begin{aligned}&\Big(\tilde\delta\,(A^{(m,k)}/g)\Big)_{i_{k+p+1}\dots i_m l_1\dots l_p}\\ &= -\frac{m-k-p}{(m-k)(k+1)}\,\sigma(i_{k+p+1}\dots i_m)\Big(\delta_{i_{m+1}i_{k+p+1}}(j_yA^{(m+1,k+1)}/g)_{i_{k+p+2}\dots i_m l_1\dots l_p}\Big)\\ &-\frac{p}{(m-k)(k+1)}\,\sigma(l_1\dots l_p)\Big(\delta_{i_{m+1}l_1}(j_yA^{(m+1,k+1)}/g)_{i_{k+p+1}\dots i_m l_2\dots l_p}\Big).\end{aligned}$$

Replacing the last term on the right-hand side of (6.13) with its value (6.15), we obtain (6.16)

$$\begin{aligned}&A^{(m,k)}_{i_{k+p+1}\dots i_m l_1\dots l_p j_1\dots j_m}g^{j_1\dots j_m}_{i_{m+1}} = (A^{(m,k)}/\tilde g)_{i_{k+p+1}\dots i_m l_1\dots l_p}\\ &= \frac{1}{m+1}\bigg[\frac{1}{2m-2k+1}\,(A^{(m+1,k)}/g)_{i_{k+p+1}\dots i_{m+1} l_1\dots l_p} - y_{i_{m+1}}\,(A^{(m+1,k+1)}/g)_{i_{k+p+1}\dots i_m l_1\dots l_p}\\ &\quad+\frac{m-k-p}{k+1}\,\sigma(i_{k+p+1}\dots i_m)\Big(\delta_{i_{m+1}i_{k+p+1}}(j_yA^{(m+1,k+1)}/g)_{i_{k+p+2}\dots i_m l_1\dots l_p}\Big)\\ &\quad+\frac{p}{k+1}\,\sigma(l_1\dots l_p)\Big(\delta_{i_{m+1}l_1}(j_yA^{(m+1,k+1)}/g)_{i_{k+p+1}\dots i_m l_2\dots l_p}\Big)\bigg].\end{aligned}$$

It is not quite obvious now that two last lines on the right-hand side of (6.16) are equal to zero in the case of $k = m$. Nevertheless, in the case of $k = m$ we are interested in (6.16) for $p = 0$ only, as is seen from (6.12). For $k = m$ and $p = 0$, the last two lines on the right-hand side of (6.16) are equal to zero.

We substitute the expression (6.16) into (6.12). After the substitution, the symmetrization $\sigma(i_{k+p+1}\dots i_m)$ can be omitted because of the presence of the "larger" symmetrization $\sigma(i_1\dots i_m)$. The symmetrization $\sigma(l_1\dots l_p)$ can be also omitted because of the presence of the factor $y^{l_1}\dots y^{l_p}$. We thus obtain

$$A^{(m,0)}_{i_1\dots i_m j_1\dots j_m} g^{j_1\dots j_m}_{i_{m+1}} = m!(2m-1)!!\,|y|^{-1} g_{i_1\dots i_{m+1}}$$
$$+\frac{1}{m+1}\,\sigma(i_1\dots i_m)\sum_{k=1}^{m}\sum_{p=0}^{\min(k,m-k)}\beta(m,k,p)\,\delta_{i_1i_2}\dots\delta_{i_{2p-1}i_{2p}}\,y_{i_{2p+1}}\dots y_{i_{k+p}}\,y^{l_1}\dots y^{l_p}\times$$
$$\times\Bigg[\frac{1}{2m-2k+1}\,(A^{(m+1,k)}/g)_{i_{k+p+1}\dots i_{m+1}l_1\dots l_p} - y_{i_{m+1}}\,(A^{(m+1,k+1)}/g)_{i_{k+p+1}\dots i_m l_1\dots l_p}$$
$$+\frac{m-k-p}{k+1}\,\delta_{i_{m+1}i_{k+p+1}}(j_y A^{(m+1,k+1)}/g)_{i_{k+p+2}\dots i_m l_1\dots l_p}$$
$$+\frac{p}{k+1}\,\delta_{i_{m+1}l_1}(j_y A^{(m+1,k+1)}/g)_{i_{k+p+1}\dots i_m l_2\dots l_p}\Bigg].$$

After pulling the factor $y^{l_1}\dots y^{l_p}$ inside brackets, this becomes

$$A^{(m,0)}_{i_1\dots i_m j_1\dots j_m} g^{j_1\dots j_m}_{i_{m+1}} = m!(2m-1)!!\,|y|^{-1} g_{i_1\dots i_{m+1}}$$
$$+\frac{1}{m+1}\,\sigma(i_1\dots i_m)\sum_{k=1}^{m}\sum_{p=0}^{\min(k,m-k)}\beta(m,k,p)\,\delta_{i_1i_2}\dots\delta_{i_{2p-1}i_{2p}}\,y_{i_{2p+1}}\dots y_{i_{k+p}}\times$$
$$\times\Bigg[\frac{1}{2m-2k+1}\,(j_y^p\,A^{(m+1,k)}/g)_{i_{k+p+1}\dots i_{m+1}} - y_{i_{m+1}}\,(j_y^p\,A^{(m+1,k+1)}/g)_{i_{k+p+1}\dots i_m}$$
$$+\frac{m-k-p}{k+1}\,\delta_{i_{m+1}i_{k+p+1}}(j_y^{p+1}\,A^{(m+1,k+1)}/g)_{i_{k+p+2}\dots i_m}$$
$$+\frac{p}{k+1}\,y_{i_{m+1}}(j_y^p\,A^{(m+1,k+1)}/g)_{i_{k+p+1}\dots i_m}\Bigg].$$

Observe that second and last terms in brackets differ by coefficients only. After grouping these terms, the formula becomes

$$A^{(m,0)}_{i_1\dots i_m j_1\dots j_m} g^{j_1\dots j_m}_{i_{m+1}} = m!(2m-1)!!\,|y|^{-1} g_{i_1\dots i_{m+1}}$$
$$+\frac{1}{m+1}\,\sigma(i_1\dots i_m)\sum_{k=1}^{m}\sum_{p=0}^{\min(k,m-k)}\beta(m,k,p)\,\delta_{i_1i_2}\dots\delta_{i_{2p-1}i_{2p}}\,y_{i_{2p+1}}\dots y_{i_{k+p}}\times$$
$$\times\Bigg[\frac{1}{2m-2k+1}\,(j_y^p\,A^{(m+1,k)}/g)_{i_{k+p+1}\dots i_{m+1}} - \frac{k-p+1}{k+1}\,y_{i_{m+1}}\,(j_y^p\,A^{(m+1,k+1)}/g)_{i_{k+p+1}\dots i_m}$$
$$+\frac{m-k-p}{k+1}\,\delta_{i_{m+1}i_{k+p+1}}(j_y^{p+1}\,A^{(m+1,k+1)}/g)_{i_{k+p+2}\dots i_m}\Bigg].$$

Next, we pull the factor $\delta_{i_{2p-1}i_{2p}}\,y_{i_{2p+1}}\dots y_{i_{k+p}}$ inside brackets

$$A^{(m,0)}_{i_1\dots i_m j_1\dots j_m} g^{j_1\dots j_m}_{i_{m+1}} = m!(2m-1)!!\,|y|^{-1} g_{i_1\dots i_{m+1}}$$
$$(6.17)\qquad +\frac{1}{m+1}\sum_{k=1}^{m}\sum_{p=0}^{\min(k,m-k)}\beta(m,k,p)\Bigg[\frac{1}{2m-2k+1}\Big(i_y^{k-p}i^p j_y^p(A^{(m+1,k)}/g)\Big)_{i_1\dots i_{m+1}}$$
$$-\frac{k-p+1}{k+1}\,y_{i_{m+1}}\Big(i_y^{k-p}i^p j_y^p(A^{(m+1,k+1)}/g)\Big)_{i_1\dots i_m}$$
$$+\frac{m-k-p}{k+1}\Big(i_{\tilde\delta}\,i_y^{k-p}i^p j_y^{p+1}(A^{(m+1,k+1)}/g)\Big)_{i_1\dots i_m}\Bigg].$$

Next, we apply Lemma 6.2 with $k=0$. More precisely, we reproduce the formula (6.7) from the proof of the lemma for $k=0$

$$\begin{aligned}(A^{(m+1,0)}/g)_{i_1\dots i_{m+1}} &= (m{+}1)(2m{+}1)\,A^{(m,0)}_{i_1\dots i_m j_1\dots j_m} g^{j_1\dots j_m}_{i_{m+1}}\\ &+ (2m{+}1)y_{i_{m+1}}\,A^{(m+1,1)}_{i_1\dots i_m j_1\dots j_{m+1}} g^{j_1\dots j_{m+1}}\\ &+ m(2m{+}1)\,\sigma(i_1\dots i_m)\Big(\delta_{i_{m+1}i_m}\,A^{(m,0)}_{i_1\dots i_{m-1}j_1\dots j_{m+1}} g^{j_1\dots j_{m+1}}\Big)\end{aligned}$$

.

This can be written in the form

$$\begin{aligned}(A^{(m+1,0)}/g)_{i_1\dots i_{m+1}} &= (m{+}1)(2m{+}1)\,A^{(m,0)}_{i_1\dots i_m j_1\dots j_m} g^{j_1\dots j_m}_{i_{m+1}}\\ &+ (2m{+}1)y_{i_{m+1}}\,(A^{(m+1,1)}/g)_{i_1\dots i_m} + m(2m{+}1)\Big(i_{\tilde\delta}(A^{(m,0)}/g)\Big)_{i_1\dots i_m}.\end{aligned}$$

By (5.1), $A^{(m,0)}=-j_yA^{(m+1,1)}$. Substituting this expression into the last line of the previous formula, we obtain

$$\begin{aligned}(A^{(m+1,0)}/g)_{i_1\dots i_{m+1}} &= (m{+}1)(2m{+}1)\,A^{(m,0)}_{i_1\dots i_m j_1\dots j_m} g^{j_1\dots j_m}_{i_{m+1}}\\ &+ (2m{+}1)y_{i_{m+1}}\,(A^{(m+1,1)}/g)_{i_1\dots i_m} - m(2m{+}1)\Big(i_{\tilde\delta}\,j_y(A^{(m+1,1)}/g)\Big)_{i_1\dots i_m}.\end{aligned}\tag{6.18}$$

Now, we replace the first term $A^{(m,0)}_{i_1\dots i_m j_1\dots j_m} g^{j_1\dots j_m}_{i_{m+1}}$ on the right-hand side of (6.18) with its expression (6.17)

$$\begin{aligned}\frac{1}{2m{+}1}(A^{(m+1,0)}/g)_{i_1\dots i_{m+1}} &= (m{+}1)!(2m-1)!!\,|y|^{-1}g_{i_1\dots i_{m+1}}\\ &+\sum_{k=1}^{m}\sum_{p=0}^{\min(k,m-k)}\beta(m,k,p)\Bigg[\frac{1}{2m{-}2k{+}1}\Big(i_y^{k-p}i^pj_y^p(A^{(m+1,k)}/g)\Big)_{i_1\dots i_{m+1}}\\ &\qquad -\frac{k{-}p{+}1}{k{+}1}\,y_{i_{m+1}}\Big(i_y^{k-p}i^pj_y^p(A^{(m+1,k+1)}/g)\Big)_{i_1\dots i_m}\\ &\qquad +\frac{m{-}k{-}p}{k{+}1}\Big(i_{\tilde\delta}\,i_y^{k-p}i^pj_y^{p+1}(A^{(m+1,k+1)}/g)\Big)_{i_1\dots i_m}\Bigg]\\ &+ y_{i_{m+1}}\,(A^{(m+1,1)}/g)_{i_1\dots i_m} - m\Big(i_{\tilde\delta}\,j_y(A^{(m+1,1)}/g)\Big)_{i_1\dots i_m}.\end{aligned}\tag{6.19}$$

From now on, we again let i_{m+1} be an arbitrary index. We apply the symmetrization $\sigma(i_1\dots i_{m+1})$ to the equation (6.19). The operator $i_{\tilde\delta}$ becomes i after the symmetrization and the result can be written in the coordinate-free form (recall that operators i_y and i commute)

(6.20)

$$\begin{aligned}\frac{1}{2m{+}1}\,A^{(m+1,0)}/g &= (m{+}1)!(2m-1)!!\,|y|^{-1}\,g\\ &+\sum_{k=1}^{m}\sum_{p=0}^{\min(k,m-k)}\beta(m,k,p)\Bigg[\frac{1}{2m{-}2k{+}1}\,i_y^{k-p}i^pj_y^p(A^{(m+1,k)}/g)\\ &\qquad -\frac{k{-}p{+}1}{k{+}1}\,i_y^{k-p+1}i^pj_y^p(A^{(m+1,k+1)}/g)+\frac{m{-}k{-}p}{k{+}1}\,i_y^{k-p}i^{p+1}j_y^{p+1}(A^{(m+1,k+1)}/g)\Bigg]\\ &+ i_y\,(A^{(m+1,1)}/g) - m\,ij_y\,(A^{(m+1,1)}/g).\end{aligned}$$

Let us write (6.20) in the form

$$
\begin{aligned}
&\frac{1}{2m+1}\,A^{(m+1,0)}/g = (m+1)!(2m-1)!!\,|y|^{-1}\,g\\
&+\sum_{k=1}^{m}\sum_{p=0}^{\min(k,m-k)}\frac{1}{2m-2k+1}\,\beta(m,k,p)\,i_y^{k-p}i^p j_y^p(A^{(m+1,k)}/g)\\
&-\sum_{k'=1}^{m}\sum_{p=0}^{\min(k',m-k')}\frac{k'-p+1}{k'+1}\,\beta(m,k',p)\,i_y^{k'-p+1}i^p j_y^p(A^{(m+1,k'+1)}/g)\\
&+\sum_{k'=1}^{m}\sum_{p'=0}^{\min(k',m-k')}\frac{m-k'-p'}{k'+1}\,\beta(m,k',p')\,i_y^{k'-p'}i^{p'+1}j_y^{p'+1}(A^{(m+1,k'+1)}/g)\\
&+i_y\,(A^{(m+1,1)}/g)-m\,ij_y\,(A^{(m+1,1)}/g).
\end{aligned}
$$

We change summation variables by $k'=k-1$ in the second sum and by $k'=k-1, p'=p-1$ in the third sum. The formula becomes

$$
\begin{aligned}
&\frac{1}{2m+1}\,A^{(m+1,0)}/g = (m+1)!(2m-1)!!\,|y|^{-1}\,g\\
&+\sum_{k=1}^{m}\sum_{p=0}^{\min(k,m-k)}\frac{1}{2m-2k+1}\,\beta(m,k,p)\,i_y^{k-p}i^p j_y^p(A^{(m+1,k)}/g)\\
&-\sum_{k=2}^{m+1}\sum_{p=0}^{\min(k-1,m-k+1)}\frac{k-p}{k}\,\beta(m,k-1,p)\,i_y^{k-p}i^p j_y^p(A^{(m+1,k)}/g)\\
&+\sum_{k=2}^{m+1}\sum_{p=1}^{\min(k-1,m-k+1)+1}\frac{m-k-p+2}{k}\,\beta(m,k-1,p-1)\,i_y^{k-p}i^p j_y^p(A^{(m+1,k)}/g)\\
&+i_y\,(A^{(m+1,1)}/g)-m\,ij_y\,(A^{(m+1,1)}/g).
\end{aligned}
\tag{6.21}
$$

We are going to equate summation limits in three sums on the right-hand side of (6.21) in order to unite the sums. Then we are going to involve two terms on the last line of (6.21) into the same sum. This needs some logical and arithmetic analysis.

In the first sum on the right-hand side of (6.21), the summation over k can be extended to $1 \le k \le m+1$ since $\beta(m,m+1,p)=0$ by the agreement (6.11). Let us demonstrate that the summation over p can be extended to $0 \le p \le \min(k, m-k+1)$. Indeed, $\min(k,m-k)=\min(k,m-k+1)$ if $k \le m-k$. If $k > m-k$, then there appears one extra term corresponding to $p=m-k+1$ in the first sum. But $\beta(m,k,m-k+1)=0$ by the agreement (6.11). Thus, summation limits of the first sum can be replaced with

$$
1 \le k \le m+1, \quad 0 \le p \le \min(k, m-k+1). \tag{6.22}
$$

In the second sum on the right-hand side of (6.21), the summation over k can be extended to $1 \le k \le m+1$ since $\beta(m,0,p)=0$ by the agreement (6.11). Let us demonstrate that the summation over p can be extended to $0 \le p \le \min(k, m-k+1)$. Indeed, $\min(k-1,m-k)=\min(k,m-k+1)$ if $k > m-k+1$. If $k \le m-k+1$, then there appears one extra term corresponding to $p=k$ in the second sum. But $\beta(m,k-1,k)=0$ by the agreement (6.11). Thus, summation limits of the second sum can be replaced with (6.22).

In the third sum on the right-hand side of (6.21), the summation over k can be extended to $1 \le k \le m+1$ since $\beta(m,0,p-1) = 0$ by the agreement (6.11). The lower summation limit over p can be replaced with zero since $\beta(m,k-1,-1) = 0$ by the agreement (6.11). Let us demonstrate that the upper summation limit over p can be replaced with $\min(k, m-k+1)$. Indeed, $\min(k-1, m-k+1)+1 = \min(k, m-k+1)$ if either $2k < m+2$ or $2k > m+2$. The only critical case is $2k = m+2$ when $\min(k-1, m-k+1)+1 = m-k+2$ and $\min(k, m-k+1) = m-k+1$. We are going to loose the term corresponding to $p = m-k+2$ after the replacement. But this term is equal to zero due to the presence of the factor $\frac{m-k-p+2}{k}$.

Thus, summation limits can be replaced with (6.22) in all sums on the right-hand side of (6.21). After the replacement, we unite three sums and write (6.21) in the form

$$
\begin{aligned}
A^{(m+1,0)}/g &= (m+1)!(2m+1)!!\,|y|^{-1}\,g \\
&+ (2m+1)\sum_{k=1}^{m+1}\sum_{p=0}^{\min(k,m-k+1)} \tilde\beta(m+1,k,p)\, i_y^{k-p} i^p j_y^p (A^{(m+1,k)}/g) \\
&+ (2m+1)\, i_y\,(A^{(m+1,1)}/g) - m(2m+1)\, i j_y\,(A^{(m+1,1)}/g),
\end{aligned}
\tag{6.23}
$$

where $\tilde\beta(m+1,k,p)$ is defined by (6.9).

Finally, we have to include two terms on the last line of (6.23) into the sum. The term $i_y\,(A^{(m+1,1)}/g)$ corresponds to $(k,p) = (1,0)$ and the term $i j_y\,(A^{(m+1,1)}/g)$ corresponds to $(k,p) = (1,1)$. Therefore we define $\beta(m+1,k,p)$ by (6.10). The formula (6.23) becomes now

$$
\begin{aligned}
A^{(m+1,0)}/g &= (m+1)!(2m+1)!!\,|y|^{-1}\,g \\
&\quad + \sum_{k=1}^{m+1}\sum_{p=0}^{\min(k,m-k+1)} \beta(m+1,k,p)\, i_y^{k-p} i^p j_y^p (A^{(m+1,k)}/g).
\end{aligned}
$$

This coincides with (6.8) for $m := m+1$. □

Proof of Proposition 6.1. Coefficients $\beta(m,k,p)$ are determined by pretty complicated recurrent formulas (6.9)–(6.11). Nevertheless, the coefficients can be expressed by the explicit formula

$$
\beta(m,k,p) = \begin{cases} (-1)^{k+p+1}\dfrac{(2m-1)!!}{(2m-2k-1)!!}\,2^{-p}\binom{m}{k}\binom{m-k}{p} & \text{for } k > 0, \\ 0 \quad \text{for } k \le 0. \end{cases}
\tag{6.24}
$$

Indeed, being defined by (6.9), $\beta(m,k,p)$ satisfy (6.11) under the agreement (5.9). Formulas (6.9)–(6.10) can be equivalently written in the form

$$
\beta(m+1,1,0) = (2m+1)\Big(\frac{1}{2m-1}\,\beta(m,1,0) + 1\Big),
\tag{6.25}
$$

$$
\beta(m+1,1,1) = (2m+1)\Big(\frac{1}{2m-1}\,\beta(m,1,1) - m\Big),
\tag{6.26}
$$

$$
\begin{aligned}
\beta(m+1,k,p) &= (2m+1)\Big[\frac{1}{2m-2k+1}\,\beta(m,k,p) - \frac{k-p}{k}\,\beta(m,k-1,p) \\
&+ \frac{m-k-p+2}{k}\,\beta(m,k-1,p-1)\Big] \quad \big((k,p) \ne (1,0), (k,p) \ne (1,1)\big).
\end{aligned}
\tag{6.27}
$$

Unlike (6.9)–(6.10), formulas (6.25)–(6.27) do not involve $\tilde\beta(m+1,k,p)$. One easily proves that equations (6.25)–(6.27) are satisfied by values (6.24). We express g from (6.8).

Substituting the value (6.24) of $\beta(m,k,p)$ into the expression, we arrive at the formula (3.5) with $\hat f = g$. Substituting the value (6.24) into (6.8) we arrive at the formula (3.5) with $\widehat{f} = g$. This completes the proof of Proposition 6.1 as well as of Theorem 3.3. $\square$

7. Proof of Theorems 3.1 and 3.2

7.1. Vector fields. In the case of $m=1$, the formula (3.5) looks as follows:

$$\widehat{f} = |y|\big(H^{(1,0)} - i_y H^{(1,1)}\big). \tag{7.1}$$

By (3.4),

$$H^{(1,0)} = F^{(1,0)}, \quad H^{(1,1)} = \frac{1}{n-1}\Big(\operatorname{div} F^{(1,0)} - F^{(1,1)}\Big)$$

and by (3.3),

$$F^{(1,0)} = c_{1,n}\,\widehat{N_1^0 f}, \quad F^{(1,1)} = c_{1,n}\,j_y\widehat{N_1^1 f},$$

where

$$c_{1,n} = -\frac{2^{n/2}\Gamma\big(\frac{n+1}{2}\big)}{\Gamma\big(-\frac12\big)} = \frac{2^{n/2-1}\Gamma\big(\frac{n+1}{2}\big)}{\sqrt{\pi}}.$$

From this

$$H^{(1,0)} = c_{1,n}\,\widehat{N_1^0 f}, \quad H^{(1,1)} = \frac{c_{1,n}}{n-1}\Big(\operatorname{div}\widehat{N_1^0 f} - j_y\widehat{N_1^1 f}\Big).$$

Substitute these expressions into (7.1)

$$\widehat{f} = c_{1,n}\,|y|\Big(\widehat{N_1^0 f} - \frac{1}{n-1}\,i_y \operatorname{div}\widehat{N_1^0 f} + \frac{1}{n-1}\,i_y j_y\,\widehat{N_1^1 f}\Big). \tag{7.2}$$

We apply the inverse Fourier transform to the formula (7.2) and use the commutator formulas

$$\mathcal{F}^{-1}|y| = (-\Delta)^{1/2}\mathcal{F}^{-1}, \quad \mathcal{F}^{-1}i_y j_y = -d\operatorname{div}\mathcal{F}^{-1}, \quad \mathcal{F}^{-1}i_y\operatorname{div} = d\,j_x\,\mathcal{F}^{-1}. \tag{7.3}$$

In this way we obtain

$$f = c_{1,n}(-\Delta)^{1/2}\Big(N_1^0 f - \frac{1}{n-1}\,d\,j_x\,N_1^0 f - \frac{1}{n-1}\,d\operatorname{div} N_1^1 f\Big).$$

This completes the proof of Theorem 3.1.

7.2. Second rank tensor fields. In the case of $m=2$, the formula (3.5) looks as follows:

$$\widehat{f} = \frac16\,|y|\Big(H^{(2,0)} - 6\,i_y H^{(2,1)} + 3\,ij_y H^{(2,1)} + 3\,i_y^2 H^{(2,2)}\Big). \tag{7.4}$$

By (3.4),

$$H^{(2,0)} = F^{(2,0)}, \quad H^{(2,1)} = \frac{1}{3(n+1)}\Big(\operatorname{div} F^{(2,0)} - F^{(2,1)}\Big),$$
$$H^{(2,2)} = \frac{1}{3(n-1)(n+1)}\Big(\operatorname{div}^2 F^{(2,0)} - 2\operatorname{div} F^{(2,1)} + F^{(2,2)}\Big)$$

and by (3.3),

$$F^{(2,0)} = c_{2,n}\,\widehat{N_2^0 f}, \quad F^{(2,1)} = c_{2,n}\,j_y\widehat{N_2^1 f}, \quad F^{(2,2)} = \frac12\,c_{2,n}\,j_y^2\widehat{N_2^2 f},$$

where

$$c_{2,n} = 3\,\frac{2^{n/2}\Gamma\big(\frac{n+3}{2}\big)}{\sqrt{\pi}}. \tag{7.5}$$

From this

$$H^{(2,0)} = c_{2,n}\,\widehat{N_2^0 f}, \quad H^{(2,1)} = \frac{c_{2,n}}{3(n+1)}\Big(\operatorname{div}\widehat{N_2^0 f} - j_y\widehat{N_2^1 f}\Big),$$
$$H^{(2,2)} = \frac{c_{2,n}}{3(n-1)(n+1)}\Big(\operatorname{div}^2\widehat{N_2^0 f} - 2\operatorname{div} j_y\,\widehat{N_2^1 f} + \frac{1}{2}\,j_y^2\widehat{N_2^2 f}\Big).$$

Substitute these expressions into (7.4)

$$\begin{aligned}\widehat{f} = \frac{c_{2,n}}{6}\,|y|\Big[\widehat{N_2^0 f} &- \frac{2}{n+1}\,i_y\Big(\operatorname{div}\widehat{N_2^0 f} - j_y\widehat{N_2^1 f}\Big)\\ &+ \frac{1}{n+1}\,i\Big(j_y\operatorname{div}\widehat{N_2^0 f} - j_y^2\widehat{N_2^1 f}\Big)\\ &+ \frac{1}{(n-1)(n+1)}\,i_y^2\Big(\operatorname{div}^2\widehat{N_2^0 f} - 2\operatorname{div} j_y\,\widehat{N_2^1 f} + \frac{1}{2}\,j_y^2\widehat{N_2^2 f}\Big)\Big].\end{aligned}$$

By Lemma 4.4, $j_y^2\widehat{N_2^1 f} = 0$, and the latter formula is simplified to the following one:

$$\begin{aligned}\widehat{f} = \frac{c_{2,n}}{6}\,|y|\Big[\widehat{N_2^0 f} &+ \frac{1}{n+1}\,ij_y\operatorname{div}\widehat{N_2^0 f}\\ &- \frac{2}{n+1}\,i_y\Big(\operatorname{div}\widehat{N_2^0 f} - j_y\widehat{N_2^1 f}\Big)\\ &+ \frac{1}{(n-1)(n+1)}\,i_y^2\Big(\operatorname{div}^2\widehat{N_2^0 f} - 2\operatorname{div} j_y\,\widehat{N_2^1 f} + \frac{1}{2}\,j_y^2\widehat{N_2^2 f}\Big)\Big].\end{aligned} \tag{7.6}$$

The second term on the right-hand side of (7.6) can be simplified. Indeed, the commutator formula

$$j_y\operatorname{div} = \operatorname{div} j_y - j \tag{7.7}$$

is proved by an easy calculation in coordinates. By this formula,

$$j_y\operatorname{div}\widehat{N_2^0 f} = \operatorname{div} j_y\widehat{N_2^0 f} - j\widehat{N_2^0 f}.$$

By Lemma 4.4, $j_y\widehat{N_2^0 f} = 0$, and the latter formula gives $j_y\operatorname{div}\widehat{N_2^0 f} = -j\widehat{N_2^0 f}$. Substitute this value into (7.6)

$$\begin{aligned}\widehat{f} = \frac{c_{2,n}}{6}\,|y|\Big[\widehat{N_2^0 f} &- \frac{1}{n+1}\,ij\,\widehat{N_2^0 f}\\ &- \frac{2}{n+1}\Big(i_y\operatorname{div}\widehat{N_2^0 f} - i_y\,j_y\widehat{N_2^1 f}\Big)\\ &+ \frac{1}{(n-1)(n+1)}\Big(i_y^2\operatorname{div}^2\widehat{N_2^0 f} - 2\,i_y^2\operatorname{div} j_y\,\widehat{N_2^1 f} + \frac{1}{2}\,i_y^2\,j_y^2\widehat{N_2^2 f}\Big)\Big].\end{aligned} \tag{7.8}$$

We apply the inverse Fourier transform to the formula (7.8) and use the commutator formulas (7.3) as well as

$$\mathcal{F}^{-1}i_y^2 = -d^2\mathcal{F}^{-1}, \quad \mathcal{F}^{-1}\operatorname{div}^2 = -j_x^2\mathcal{F}^{-1}, \quad \mathcal{F}^{-1}j_y d = \operatorname{div} i_x\mathcal{F}^{-1}, \quad \mathcal{F}^{-1}\operatorname{div} j_y = j_x\operatorname{div}\mathcal{F}^{-1}.$$

In this way we obtain

$$f(x) = \frac{c_{2,n}}{6}\,(-\Delta)^{1/2}\Big[N_2^0 f - \frac{1}{n+1}\,ij\,N_2^0 f - \frac{2}{n+1}\,d\Big(j_x\,N_2^0 f + \operatorname{div} N_2^1 f\Big) + \frac{1}{(n-1)(n+1)}\,d^2\Big(j_x^2 N_2^0 f - 2\,j_x \operatorname{div} N_2^1 f + \frac{1}{2}\operatorname{div}^2 N_2^2 f\Big)\Big].$$

Substituting the value (7.5) of $c_{2,n}$, we arrive at (3.2). This completes the proof of Theorem 3.2.

7.3. **Third rank tensor fields.** In the case of $m = 3$, the formula (3.5) looks as follows:

$$\widehat{f} = \frac{|y|}{90}\Big(H^{(3,0)} - 15\,i_y H^{(3,1)} + 15\,ij_y H^{(3,1)} + 45\,i_y^2 H^{(3,2)} - \frac{45}{2}\,ii_y j_y H^{(3,2)} - 15\,i_y^3 H^{(3,3)}\Big). \tag{7.9}$$

By (3.4),

$$H^{(3,0)} = F^{(3,0)},\quad H^{(3,1)} = \frac{1}{5(n+3)}\Big(\operatorname{div} F^{(3,0)} - F^{(3,1)}\Big),$$

$$H^{(3,2)} = \frac{1}{15(n+1)(n+3)}\Big(\operatorname{div}^2 F^{(3,0)} - 2\operatorname{div} F^{(3,1)} + F^{(3,2)}\Big),$$

$$H^{(3,3)} = \frac{1}{15(n-1)(n+1)(n+3)}\Big(\operatorname{div}^3 F^{(3,0)} - 3\operatorname{div}^2 F^{(3,1)} + 3\operatorname{div} F^{(3,2)} - F^{(3,3)}\Big),$$

and by (3.3),

$$F^{(3,0)} = c_{3,n}\,\widehat{N_3^0 f},\quad F^{(3,1)} = c_{3,n}\,j_y\widehat{N_3^1 f},\quad F^{(3,2)} = \frac{1}{2}\,c_{3,n}\,j_y^2\widehat{N_3^2 f},\quad F^{(3,3)} = \frac{1}{6}\,c_{3,n}\,j_y^3\widehat{N_3^2 f},$$

where

$$c_{3,n} = 15\,\pi^{-1/2}\,2^{n/2+1}\,\Gamma\Big(\frac{n+5}{2}\Big). \tag{7.10}$$

From this

$$H^{(3,0)} = c_{3,n}\,\widehat{N_3^0 f},\quad H^{(3,1)} = \frac{c_{3,n}}{5(n+3)}\Big(\operatorname{div}\widehat{N_3^0 f} - j_y\widehat{N_3^1 f}\Big),$$

$$H^{(3,2)} = \frac{c_{3,n}}{15(n+1)(n+3)}\Big(\operatorname{div}^2\widehat{N_3^0 f} - 2\operatorname{div} j_y\,\widehat{N_3^1 f} + \frac{1}{2}\,j_y^2\widehat{N_3^2 f}\Big),$$

$$H^{(3,3)} = \frac{c_{3,n}}{15(n-1)(n+1)(n+3)}\Big(\operatorname{div}^3\widehat{N_3^0 f} - 3\operatorname{div}^2 j_y\,\widehat{N_3^1 f} + \frac{3}{2}\operatorname{div} j_y^2\widehat{N_3^2 f} - \frac{1}{6}\,j_y^3\widehat{N_3^3 f}\Big).$$

Substitute these expressions into (7.9)

$$\widehat{f} = \frac{c_{3,n}}{90}\,|y|\Big[\widehat{N_3^0 f} - \frac{3}{n+3}\,i_y\Big(\operatorname{div}\widehat{N_3^0 f} - j_y\widehat{N_3^1 f}\Big) + \frac{3}{n+3}\,i\Big(j_y\operatorname{div}\widehat{N_3^0 f} - j_y^2\widehat{N_3^1 f}\Big) + \frac{3}{(n+1)(n+3)}\,i_y^2\Big(\operatorname{div}^2\widehat{N_3^0 f} - 2\operatorname{div} j_y\,\widehat{N_3^1 f} + \frac{1}{2}\,j_y^2\widehat{N_3^2 f}\Big) - \frac{3}{2(n+1)(n+3)}\,ii_y\Big(j_y\operatorname{div}^2\widehat{N_3^0 f} - 2\,j_y\operatorname{div} j_y\,\widehat{N_3^1 f} + \frac{1}{2}\,j_y^3\widehat{N_3^2 f}\Big) - \frac{1}{(n-1)(n+1)(n+3)}\,i_y^3\Big(\operatorname{div}^3\widehat{N_3^0 f} - 3\operatorname{div}^2 j_y\widehat{N_3^1 f} + \frac{3}{2}\operatorname{div} j_y^2\widehat{N_3^2 f} - \frac{1}{6}\,j_y^3\widehat{N_3^3 f}\Big)\Big].$$

By Lemma 4.4, $j_y^2\widehat{N_3^1f}=0$ and $j_y^3\widehat{N_3^2f}=0$. The previous formula is simplified to the following one:

(7.11)

$$\begin{aligned}\widehat f=\frac{c_{3,n}}{90}\,|y|\Big[&\widehat{N_3^0f}+\frac{3}{n+3}\,ij_y\operatorname{div}\widehat{N_3^0f}-\frac{3}{n+3}\,i_y\Big(\operatorname{div}\widehat{N_3^0f}-j_y\widehat{N_3^1f}\Big)\\&+\frac{3}{(n{+}1)(n{+}3)}\,i_y^2\Big(\operatorname{div}^2\widehat{N_3^0f}-2\operatorname{div}j_y\,\widehat{N_3^1f}+\frac12\,j_y^2\widehat{N_3^2f}\Big)\\&-\frac{3}{2(n{+}1)(n{+}3)}\,ii_y\Big(j_y\operatorname{div}^2\widehat{N_3^0f}-2\,j_y\operatorname{div}j_y\,\widehat{N_3^1f}\Big)\\&-\frac{1}{(n{-}1)(n{+}1)(n{+}3)}\,i_y^3\Big(\operatorname{div}^3\widehat{N_3^0f}-3\operatorname{div}^2j_y\widehat{N_3^1f}+\frac32\operatorname{div}j_y^2\widehat{N_3^2f}-\frac16\,j_y^3\widehat{N_3^3f}\Big)\Big].\end{aligned}$$

At least three terms on the right-hand side of (7.11) can be simplified. Indeed, using Lemma 4.4 and the commutator formula (7.7), we transform

$$j_y\operatorname{div}\widehat{N_3^0f}=\operatorname{div}j_y\,\widehat{N_3^0f}-j\,\widehat{N_3^0f}=-j\,\widehat{N_3^0f}. \tag{7.12}$$

Quite similarly,

(7.13)

$$j_y\operatorname{div}j_y\,\widehat{N_3^1f}=(\operatorname{div}j_y-j)j_y\,\widehat{N_3^1f}=\operatorname{div}j_y^2\,\widehat{N_3^1f}-jj_y\,\widehat{N_3^1f}=-jj_y\,\widehat{N_3^1f}=-j_yj\,\widehat{N_3^1f}.$$

We have used that operators j and j_y commute. Let us also transform the term containing $j_y\operatorname{div}^2$

$$\begin{aligned}j_y\operatorname{div}^2\widehat{N_3^0f}&=(j_y\operatorname{div})\operatorname{div}\widehat{N_3^0f}=(\operatorname{div}j_y-j)\operatorname{div}\widehat{N_3^0f}\\&=\operatorname{div}(j_y\operatorname{div})\widehat{N_3^0f}-j\operatorname{div}\widehat{N_3^0f}=\operatorname{div}(\operatorname{div}j_y-j)\widehat{N_3^0f}-j\operatorname{div}\widehat{N_3^0f}\\&=\operatorname{div}^2j_y\,\widehat{N_3^0f}-\operatorname{div}j\,\widehat{N_3^0f}-j\operatorname{div}\widehat{N_3^0f}=-\operatorname{div}j\,\widehat{N_3^0f}-j\operatorname{div}\widehat{N_3^0f}.\end{aligned}$$

Using that j and div commute [DS10], we get

$$j_y\operatorname{div}^2\widehat{N_3^0f}=-2\,j\operatorname{div}\widehat{N_3^0f}. \tag{7.14}$$

Substituting expressions (7.12)–(7.14) and the value (7.10) of the constant $c_{3,n}$ into (7.11), we obtain the inversion formula recovering the Fourier transform of a tensor field $f\in\mathcal S(\mathbb R^n;S^3)$ $(n\ge2)$ through the data $(\widehat{N_3^0f},j_y\widehat{N_3^1f},j_y^2\widehat{N_3^2f},j_y^3\widehat{N_3^3f})$

$$\begin{aligned}\widehat f=\frac{2^{n/2}\,\Gamma\big(\frac{n+5}{2}\big)}{3\sqrt\pi}\,|y|\Big[&\widehat{N_3^0f}-\frac{3}{n+3}\,ij\,\widehat{N_3^0f}\\&-\frac{3}{(n{+}1)(n{+}3)}\,i_y\Big((n{+}1)\operatorname{div}\widehat{N_3^0f}-\,ij\operatorname{div}\widehat{N_3^0f}-(n{+}1)\,j_y\widehat{N_3^1f}+\,ij\,j_y\widehat{N_3^1f}\Big)\\&+\frac{3}{(n{+}1)(n{+}3)}\,i_y^2\Big(\operatorname{div}^2\widehat{N_3^0f}-2\operatorname{div}j_y\widehat{N_3^1f}+\frac12\,j_y^2\widehat{N_3^2f}\Big)\\&-\frac{1}{(n{-}1)(n{+}1)(n{+}3)}\,i_y^3\Big(\operatorname{div}^3\widehat{N_3^0f}-3\operatorname{div}^2j_y\widehat{N_3^1f}+\frac32\operatorname{div}j_y^2\widehat{N_3^2f}-\frac16\,j_y^3\widehat{N_3^3f}\Big)\Big].\end{aligned}$$

The same approach can be used for deriving the inversion formula for $m=4,5,\dots$ The length of the formula grows with m as well as the volume of calculations.

References

[AKS22] Divyansh Agrawal, Venkateswaran P. Krishnan, and Suman Kumar Sahoo. Unique continuation results for certain generalized ray transforms of symmetric tensor fields. *J. Geom. Anal.*, 32(10):Paper No. 245, 27, 2022.

[AM19] Anuj Abhishek and Rohit Kumar Mishra. Support theorems and an injectivity result for integral moments of a symmetric m-tensor field. *J. Fourier Anal. Appl.*, 25(4):1487–1512, 2019.

[BK23] Sombuddha Bhattacharyya and Pranav Kumar. Local data inverse problem for the polyharmonic operator with anisotropic perturbations. *arXiv preprint arXiv:2307.10608*, 2023.

[BKS23] Sombuddha Bhattacharyya, Venkateswaran P. Krishnan, and Suman K. Sahoo. Momentum ray transforms and a partial data inverse problem for a polyharmonic operator. *SIAM J. Math. Anal.*, 55(4):4000–4038, 2023.

[DS10] N. S. Dairbekov and V. A. Sharafutdinov. Conformal Killing symmetric tensor fields on Riemannian manifolds. *Mat. Tr.*, 13(1):85–145, 2010.

[Hör83] Lars Hörmander. *The Analysis of Linear Partial Differential Operators. I Distribution theory and Fourier analysis*, volume 256 of *Grundlehren der mathematischen Wissenschaften.* Springer-Verlag, Berlin, 1983.

[IKS23] Joonas Ilmavirta, Pu-Zhao Kow, and Suman Kumar Sahoo. Unique continuation for the momentum ray transform. *arXiv preprint arXiv:2304.00327*, 2023.

[KMSS19] Venkateswaran P. Krishnan, Ramesh Manna, Suman Kumar Sahoo, and Vladimir A. Sharafutdinov. Momentum ray transforms. *Inverse Probl. Imaging*, 13(3):679–701, 2019.

[KMSS20] Venkateswaran P. Krishnan, Ramesh Manna, Suman Kumar Sahoo, and Vladimir A. Sharafutdinov. Momentum ray transforms, II: range characterization in the Schwartz space. *Inverse Problems*, 36(4):045009, 33, 2020.

[LS09] William Lionheart and Vladimir Sharafutdinov. Reconstruction algorithm for the linearized polarization tomography problem with incomplete data. In *Imaging microstructures*, volume 494 of *Contemp. Math.*, pages 137–159. Amer. Math. Soc., Providence, RI, 2009.

[PSU23] Gabriel P. Paternain, Mikko Salo, and Gunther Uhlmann. *Geometric inverse problems—with emphasis on two dimensions*, volume 204 of *Cambridge Studies in Advanced Mathematics.* Cambridge University Press, Cambridge, 2023. With a foreword by András Vasy.

[Sha94] V. A. Sharafutdinov. *Integral geometry of tensor fields.* Inverse and Ill-posed Problems Series. VSP, Utrecht, 1994.

[SS23] Suman Kumar Sahoo and Mikko Salo. The linearized Calderón problem for polyharmonic operators. *J. Differential Equations*, 360:407–451, 2023.

[SW12] Vladimir Sharafutdinov and Jenn-Nan Wang. Tomography of small residual stresses. *Inverse Problems*, 28(6):065017, 17, 2012.

[Vla79] V. S. Vladimirov. *Generalized functions in mathematical physics.* Mir Publishers, Moscow, 1979.

Indian Institute of Science Education and Research (IISER) Bhopal, India
Email address: `shubham18@iiserb.ac.in`

Indian Institute of Science Education and Research (IISER) Bhopal, India
Email address: `manas@iiserb.ac.in`

Centre for Applicable Mathematics, Tata Institute of Fundamental Research, India
Email address: `vkrishnan@tifrbng.res.in`

Sobolev Institute of Mathematics, 4 Koptyug Av., 630090, Novosibirsk, Russia
Email address: `sharaf@math.nsc.ru`