Half-time range description for the free space wave operator and the spherical means transform

Abstract. The forward problem arising in several hybrid imaging modalities can be modeled by the Cauchy problem for the free space wave equation. Solution to this problems describes propagation of a pressure wave, generated by a source supported inside unit sphere S. The data g represent the time-dependent values of the pressure on the observation surface S. Finding initial pressure f from the known values of g consitutes the inverse problem. The latter is also frequently formulated in terms of the spherical means of f with centers on S.

Here we consider a problem of range description of the wave operator mapping f into g. Such a problem was considered before, with data g known on time interval at least $[0, 2]$ (assuming the unit speed of sound). Range conditions were also found in terms of spherical means, with radii of integration spheres lying in the range $[0, 2]$. However, such data are redundant. We present necessary and sufficient conditions for function g to be in the range of the wave operator, for g given on a half-time interval $[0, 1]$. This also implies range conditions on spherical means measured for the radii in the range $[0, 1]$.

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1. Introduction

In the thermo- and photoacoustic tomography (TAT/PAT) [18, 19, 29] the forward problem can be modeled (in the simplest case) by the following Cauchy problem for the whole space wave equation:

$$u_{tt}(t,x) = \Delta u(t,x), \quad (t,x) \in (0,\infty) \times \mathbb{R}^d, \tag{1}$$

$$u(0,x) = f(x), \qquad u_t(0,x) = 0. \tag{2}$$

Solution $u(t,x)$ of this problem represents the excess pressure in the acoustic wave. This wave is caused by an instantaneous thermoelastic expansion at $t = 0$ leading to the initial condition $u(0,x) = f(x)$. Function $f(x)$ is assumed to be supported within the open unit ball $B \subset \mathbb{R}^d$ centered at the origin, and is extended by zero to $\mathbb{R}^d$. The measurements in TAT/PAT are the values $g(t,\theta)$ of the pressure registered during the time interval $t \in (0,T]$ on the unit sphere $\theta \in S := \mathbb{S}^{d-1}$:

$$g(t,\theta) := u(t,\theta), \quad (t,\theta) \in Z_T, \tag{3}$$

where $Z_T := (0,T] \times S$ is the cylinder supporting the data. The data $g(t,\theta)$ can be viewed as an action of **the observation operator** $\mathcal{A}$ on function f :

$$\mathcal{A} : f(x) \in C_0^\infty(B) \mapsto g(t,\theta), \quad (t,\theta) \in Z_T. \tag{4}$$

The inverse source problem of TAT/PAT consists of finding f from data $g = \mathcal{A}f$. It has been studied extensively in the last two decades (see [20–23] and references therein). In particular, explicit inversion formulas have been found [9, 10, 24, 32] allowing one to invert the operator $\mathcal{A}$ and reconstruct f from g under the assumption $T \geq 2$. In this case, $T = 2$ represents the time sufficient for all characteristics originated within B to reach the boundary ∂B of B. Thus, the most studied case is $T \geq 2$. However, shorter acquisition times are of practical and theoretical importance. We concentrate here on the case $T = 1$, which is known to be sufficient, since by that time at least one of two characteristics originating at any point in B reaches the boundary ∂B. In addition, explicit inversion formulas have been found [4, 7] for reconstructing f from "half-time" data g known on the cylinder $Z_1 := (0,1] \times S$. It is well known [17] that the data $g(t,\theta)$ obtained by solving the problem (1)-(3) can be expressed in terms of the spherical means $M(r,\theta)$ of f with centers $\theta \in S$:

$$M(r,\theta) := \int\limits_S f(\theta + r\tau) d\tau, \quad (r,\theta) \in \mathbb{R}^+ \times S, \tag{5}$$

where $d\tau$ is the standard area element on the unit sphere. Indeed, let us consider the causal free-space Green's function for the wave equation (1), that we will denote by $\mathbf{G}(t,x)$. It is well known (see, e.g., [30]) that this function can be expressed through a dimension-dependent radial function $G_d(t,r)$ as follows:

$$\begin{gathered} \mathbf{G}(t,x) = G_d(t,|x|), \quad G_2(t,r) = \tfrac{\chi_+(t-r)}{2\pi\sqrt{t^2-r^2}}, \quad G_3(t,r) = \tfrac{\delta(t-r)}{4\pi r}, \\ \text{and for } d > 3, \\ G_{d+2}(t,r) = -\tfrac{1}{2\pi r}\tfrac{\partial}{\partial r} G_d(t,r), \end{gathered} \tag{6}$$

where $\delta(t)$ is the Dirac's delta function, and $\chi_+(s)$ is the Heaviside function, equal to 1 when $s > 0$ and equal to 0 otherwise. Now data $g(t,\theta)$ can be expressed through the spherical means as follows:

$$g(t,\theta) = \frac{\partial}{\partial t} \int\limits_0^t M(r,\theta) G_d(t,r) r^{d-1} dr. \tag{7}$$

Equation (7) is easily solvable with respect to $M(r,\theta)$, so that the knowledge of $g(t,\theta)$ on Z_T implies the knowledge of $M(r,\theta)$ on Z_T and *vice versa* (see, e.g. [1, 17]). As a consequence of this, historically, most of the results in this area have been obtained in terms of the spherical means M, rather than in terms of the wave data g.

We are interested here in the range of the operator $\mathcal{A}$ (see (4)). Namely, we study the necessary and sufficient conditions for a function $g \in C_0^\infty(Z_T)$ to lie in the range of $\mathcal{A}$. Such range descriptions were obtained in [1–3, 11] in terms of spherical means $M(r,\theta)$ defined on $C_0^\infty([0,2] \times S)$, which corresponds to the case of "full time" observation, i.e., $T = 2$.

However, these results are not optimal, in the sense that they use "too much" data. Indeed, as we have already mentioned, it is known that a stable solution of the inverse source problem of TAT/PAT is possible if data $g(t,\theta)$ are given on a twice shorter cylinder Z_1 (see, e.g., [4,7]). To put it differently, this implies that the data on the time interval $[1,\infty)$ are uniquely determined by the data on interval $(0,1]$. In terms of applications to TAT/PAT, this means that the data acquisition time can be halved [5]. This makes the measurement system cheaper, and potentially improves the quality of the image, since, for a variety of reasons, the acoustic wave quickly deteriorates in time, and thus "late" data get degraded. We refer to such data as "half-time" data, as is done in the title of this paper.

In the present paper, we find the range description of the operator

$$\mathcal{A} : C_0^\infty(B) \to C_0^\infty(Z_1),$$

i.e. we are working with the data given on time interval $(0,1]$. Formulation of the problem and the main result are given in Section 2, some preliminary considerations are made in Section 3 after which the main results are described in Sections 4 and 6. The proofs are presented in Section 7.

2. Formulation of the problem

We consider operator

$$\begin{aligned} &\mathcal{A} : C_0^\infty(B) \to C_0^\infty(Z_1), \\ &\mathcal{A} : f \mapsto g(t,\theta), \quad (t,\theta) \in Z_1, \end{aligned} \tag{8}$$

where $g(t,\theta)$ is the trace of the solution of the Cauchy problem (1)-(2) on Z_1 (see (3)). Our aim is to find necessary and sufficient condition for a function $b(t,\theta) \in C_0^\infty(Z_1)$ to be in the range of $\mathcal{A}$, i.e., to be represented as $b = \mathcal{A}f$ for some $f \in C_0^\infty(B)$. We somewhat abuse the notations here, since functions from $C_0^\infty(Z_1)$ and their derivatives are not required to vanish at $t = 1$. They vanish only near $t = 0$.

3. Some preliminary constructions

Our approach is based on studying the Radon transform of a solution to a certain auxiliary initial/boundary value problem (IBVP) for the wave equation in the exterior of the closed ball $\overline{B}$. We start by defining and analyzing the tools we need.

3.1. The Radon transform and its range

Consider a compactly supported continuous function $q(x)$ on $\mathbb{R}^d$. For a unit direction vector $\omega \in S$ and $p \in \mathbb{R}$, the values of the Radon transform $\mathcal{R}q(\omega,p)$ of q are given by the following formula (see, e.g., [27]):

$$\begin{aligned} &[\mathcal{R}q](\omega,p) := \int\limits_{\mathbb{R}^d} q(x)\delta(x\cdot\omega - p)dx \\ &= \int\limits_{\Pi(\omega,p)} q(x)dx \text{ for } (\omega,p) \in S\times\mathbb{R}. \end{aligned} \tag{9}$$

Here $\delta(\cdot)$ is the Dirac's delta function and $\Pi(\omega,p)$ is the plane defined by the equation

$$x\cdot\omega = p. \tag{10}$$

The following description of the range of the Radon transform $\mathcal{R}$ of functions from $C_0^\infty(B)$ is well known [12–16, 20, 26, 27]:

Theorem 1. *A function $F(\omega,p)$ defined on $S\times(-1,1)$ can be represented as the Radon transform of a function $f\in C_0^\infty(B)$, if and only if the following conditions are satisfied:*

(i) Symmetry condition: $F(\omega,p)=F(-\omega,-p)$,

(ii) Smoothness and support condition: $F(\omega,p)\in C_0^\infty(S\times(-1,1))$,

(iii) Moment conditions: for any $n=0,1,2,...$, the moment $M_n(\omega)$

$$M_n(\omega):=\int\limits_{-1}^{1}F(\omega,p)p^n dp \tag{11}$$

is the restriction from $\mathbb{R}^d$ to S of a homogeneous polynomial of degree n in ω.

We will use the real-valued spherical harmonics $Y_l^{\mathbf{m}}(\omega)$ on the unit $(d-1)$-dimensional sphere S, where $l=0,1,2,\dots$ and $|\mathbf{m}|\le l$ (see, for instance, [28]). They form an orthonormal basis in $L_2(S)$:

$$\int\limits_S Y_l^{\mathbf{m}}(\omega)Y_{l'}^{\mathbf{m}'}(\omega)d\omega=1 \text{ if } l=l', \mathbf{m}=\mathbf{m}', \text{ and zero otherwise.} \tag{12}$$

Using these, the moment conditions (3) of the theorem can be equivalently re-written as

$$\begin{gathered}\int\limits_S M_n(\omega)Y_l^{\mathbf{m}}(\omega)d\omega=\int\limits_S\left[\int\limits_{-1}^{1}F(\omega,p)p^n dp\right]Y_l^{\mathbf{m}}(\omega)d\omega=0\\ \text{for } n=0,1,2,...,l>n,\ \ |\mathbf{m}|\le l.\end{gathered} \tag{13}$$

3.1.1. Exterior Radon transform We will also need the **exterior (with respect to $\overline{B}$) Radon transform $\mathcal{R}^E$**. For a continuous function $h(x)$ defined and compactly supported in the exterior B^c of B, $\mathcal{R}^E$ is defined by the formula similar to (9) but with $|p|\ge 1$:

$$\begin{gathered}\left[\mathcal{R}^E h\right](\omega,p):=\int\limits_{\mathbb{R}^d}h(x)\delta(x\cdot\omega-p)dx\\ =\int\limits_{\Pi(\omega,p)}h(x)dx,\ \text{for } (\omega,p)\in S\times(\mathbb{R}\setminus(-1,1)).\end{gathered} \tag{14}$$

Remark 2. *If function h were defined in the whole $\mathbb{R}^d$, not just outside of the "hole" B, transform $\mathcal{R}^E$ would be just a restriction of the full Radon transform of h.*

Our goal is to obtain range description for the wave operator, similar to the above range description for the Radon transform $\mathcal{R}$. We will achieve this by relating the Radon transform to a solution of a certain IBVP (initial/boundary value problem) for the wave equation in the exterior $B^c:=\mathbb{R}^d\setminus\overline{B}$ of the ball $\overline{B}$. This technique was previously used in [8] to reduce the problem of inverting operator $\mathcal{A}$ to inverting the classical Radon transform.

3.2. Exterior initial/boundary value problem

We will denote by $F(\omega, p)$ the Radon transform of the initial function $f(x) \in C_0^\infty(B)$:

$$F(\omega, p) := [\mathcal{R}f]\,(\omega, p). \tag{15}$$

Further, given a function $b \in C_0^\infty(Z_1)$, consider the solution $v^{(b)}(t, x)$ to the following exterior problem in $(0, \infty) \times \mathbb{R}^d \backslash \bar{B}$:

$$\begin{cases} \frac{\partial^2 v^{(b)}(t,x)}{\partial t^2} - \Delta v^{(b)}(t, x) = 0, & (t, x) \in (0, 1] \times B^c, \\ v^{(b)}(0, x) = 0, \quad v_t^{(b)}(0, x) = 0, & x \in B^c, \\ v^{(b)}(t, \theta) = b(t, \theta). & (t, \theta) \in Z_1. \end{cases} \tag{16}$$

The domain $(0, 1) \times B^c$ where the problem is solved, and the support of the solution $v^{(b)}(t, x)$ are shown in Figure 1.

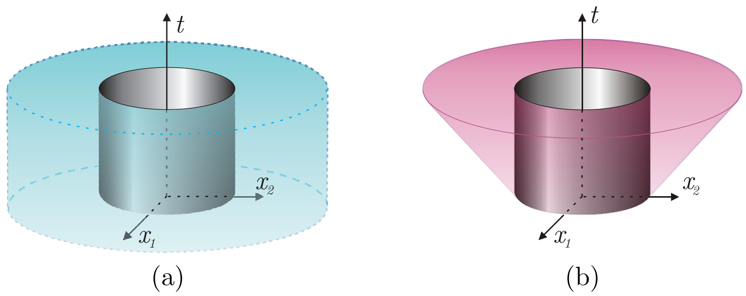

Figure 1. Geometry of the exterior problem:
(a) Domain $(0, 1) \times B^c$ where the problem is solved;
(b) Support of the solution $v^{(b)}(t, x)$.

Note that the solution u of the problem (1), (2) also solves the exterior problem (16), if $b(t, \theta)$ is equal to $g(t, \theta)$. Therefore, if $b = g$, the solution $v^{(g)}$ of the exterior problem (16) coincides with u in $(0, 1] \times B^c$. Let us now consider for each fixed t the Radon projections $[\mathcal{R}u]\,(t, \omega, p)$ (defined by (9)) of u. It is well known (e.g., see [17]) that, for a fixed ω, such projections satisfy the 1D wave equation:

$$\frac{\partial^2}{\partial t^2}\,[\mathcal{R}u]\,(t, \omega, p) = \frac{\partial^2}{\partial p^2}\,[\mathcal{R}u]\,(t, \omega, p), \qquad (t, p) \in (0, 1) \times \mathbb{R}. \tag{17}$$

Due to (2), $\mathcal{R}u$ satisfies the following initial conditions:

$$[\mathcal{R}u]\,(0, \omega, p) = F(\omega, p), \quad \frac{\partial}{\partial t}\,[\mathcal{R}u]\,(0, \omega, p) = 0, \qquad p \in \mathbb{R}. \tag{18}$$

According to d'Alembert's formula, one can express $\mathcal{R}u$ on $(0, 1) \times \mathbb{R}$ as follows:

$$\begin{gathered} [\mathcal{R}u](t, \omega, p) = \tfrac{1}{2}\,[F(\omega, p + t) + F(\omega, p - t)] \\ \text{for } (t, p) \in (0, 1) \times \mathbb{R}. \end{gathered} \tag{19}$$

Since $f(x)$ is compactly supported in B, its Radon projections $F(\omega, p)$ are supported inside the region $\{(\omega, p) \,:\, p \in [-1, 1]\}$. Therefore, for $t = 1$ we get

$$[\mathcal{R}u](1, \omega, p) = \begin{cases} \frac{1}{2}F(\omega, p - 1), & p \in [1, 2], \\ \frac{1}{2}F(\omega, p + 1), & p \in [-2, -1] \end{cases}, \tag{20}$$

or

$$F(\omega,p)=\begin{cases}2[\mathcal{R}u](1,\omega,p+1), & p\in[0,1]\\ 2[\mathcal{R}u](1,\omega,p-1), & p\in[-1,0]\end{cases}.$$

Equivalently, if $v^{(g)}$ is the solution of the exterior problem (16) with $b=g=\mathcal{A}f$, the Radon transform F of f can be expressed as follows:

$$F(\omega,p)=\begin{cases}2[\mathcal{R}v^{(g)}](1,\omega,p+1), & p\in[0,1]\\ 2[\mathcal{R}v^{(g)}](1,\omega,p-1), & p\in[-1,0]\end{cases}. \tag{21}$$

We thus have proven the following theorem (that first appeared in [8]):

Theorem 3. *The Radon transform $F(\omega,p)$ of the initial function $f(x)$, and thus the function itself, can be recovered by using (21) from the solution v of the exterior problem (16).*

4. The implicit range characterization theorem

The considerations of Section 3 lead to the following characterization of the range of observation operator $\mathcal{A}$ in terms of solution of the exterior problem:

Theorem 4. *Suppose that function g is defined on Z_1, $v^{(g)}$ is the solution to the exterior problem (16), and $F(\omega,p)$ is defined by (21). Then function g belongs to the range of the operator $\mathcal{A}$ (8) (i.e.,$g=\mathcal{A}f$ for some $f\in C_0^\infty(B)$)* ***if and only if*** *the following four conditions are satisfied:*

(i) Smoothness of g: $g\in C_0^\infty(Z_1)$.

(ii) Symmetry of F: $F(\omega,p)=F(-\omega,-p)$.

(iii) Smoothness of F: $F(\omega,p)\in C_0^\infty(S\times(-1,1))$.

(iv) Moment conditions for F: $\int\limits_S\left[\int\limits_{-1}^{1}F(\omega,p)p^n dp\right]Y_l^{\mathbf{m}}(\omega)d\omega=0$, for $n=0,1,2,...,\ l>n,\ \ |\mathbf{m}|\le l$.

Proof. **Necessity of conditions (1)-(4)**. Smoothness and support of g (condition (1)) follow from the well known properties of solutions of the initial value problem for the wave equation in $\mathbb{R}^d$ with initial conditions in $C_0^\infty(\mathbb{R}^d)$. On the other hand, function $F(\omega,p)$ defined by (21) is the Radon transform of $f(x)\in C_0^\infty(B)$ and, therefore, satisfies conditions (1)-(3) of Theorem 1, thus proving statements(2)-(4).

Sufficiency of conditions (1)-(4). If conditions (2)-(4) are satisfied, then according to Theorem 1 there exists function $f\in C_0^\infty(B)$ such that $F(\omega,p)=\mathcal{R}f$. One can then solve the Cauchy problem (1), (2) thus obtaining solution $u(t,x)$ in $(0,1]\times\mathbb{R}^d$. The Radon transform $\mathcal{R}u$ of such a solution is given by formula (19), for each fixed $t\in(0,1]$ and each $\omega\in S$. This implies that $[\mathcal{R}^E u](t,\omega,p)=[\mathcal{R}^E v^{(g)}](t,\omega,p)$ for all $t\in(0,1]$, $\omega\in S$, and $p\in[1,2]$. Since $\mathcal{R}^E$ is uniquely invertible, $u(t,x)=v^{(g)}(t,x)$ for $t\in(0,1]$, $x\in B^c$, and, in particular, $u(t,x)=v^{(g)}(t,x)=g(t,x)$ for all $x\in S$, $t=(0,1]$, implying $g=\mathcal{A}f$. $\square$

Remark 5.

(i) *When considering sufficiency of condition (3) in the above Theorem, one can notice that $F(\omega, p)$ defined by (21) is automatically a C_0^∞-function on $(-1, 0] \times S$ and on $[0, 1) \times S$, due to the smoothness of $v^{(g)}$. Only the continuity of all the derivatives of $F(\omega, p)$ at $p = 0$ needs to be verified. This condition is non-trivial, since function F is constructed of two "halves" (see equation (21)). This condition is essential, since without it one cannot guarantee smoothness of the candidate function $f(x)$ at $x = 0$.*

(ii) *This range description is implicit, since it requires solving the auxiliary exterior IBVP with the given data $g(x, t)$. The remaining part of this paper is dedicated to expressing the range conditions in terms of the Fourier coefficients of the candidate function $b(t, \theta)$.*

5. A more explicit range characterization

As mentioned above, Theorem 4 provides a rather implicit range description, which requires, first, solving the exterior IBVP and then finding its Radon projections. The spherical symmetry of our domain $\mathbb{R}^d \setminus \overline{B}$ will allow us to use standard separation of variables technique to solve the exterior problem and obtain the desired range conditions in terms of Fourier coefficients of $b(t, \theta)$. This requires some preliminary work.

5.1. Range conditions for the Radon transform in terms of expansion in spherical harmonics

In order to avoid the redundancy in the Radon transform defined on $S \times [-1, 1]$, we will work with the Radon transform restricted to $S \times [0, 1]$. Let us first formulate the necessary and sufficient conditions for a function $F(\omega, p)$ defined on $S \times [0, 1]$ to be in the range of such a restricted Radon transform, i.e.

$$F(\omega, p) = [\mathcal{R}q]\,(\omega, p),\ \ \omega \in S,\ \ p \in [0, 1], \tag{22}$$

for some $q \in C_0^\infty(B)$.

Let us start with necessary conditions. Assume $F = \mathcal{R}q$. An obvious necessary condition following from Theorem 1 is

$$F(\omega, p) \in C_0^\infty(S \times [0, 1)). \tag{23}$$

We need to express remaining conditions in terms of the Fourier coefficients $F_l^{\mathbf{m}}$ of F. Define

$$F_l^{\mathbf{m}}(p) := \int\limits_S F(\omega, p) Y_l^{\mathbf{m}}(\omega) d\omega = 0, \quad p \in [0, 1], \quad l = 0, 1, 2, ..., \quad |\mathbf{m}| \le l, \tag{24}$$

so that

$$F(\omega, p) = \sum_{l,\mathbf{m}} F_l^{\mathbf{m}}(p) Y_l^{\mathbf{m}}(\omega), \quad p \in [0, 1].$$

Note that for any $\omega \in S$,

$$[\mathcal{R}q](\omega, -p) = [\mathcal{R}q](-\omega, p) = F(\omega, -p) \text{ for } p \in [0, 1]. \tag{25}$$

On the other hand,

$$F(\omega,-p)=\sum_{l,\mathbf{m}} F_l^{\mathbf{m}}(p)Y_l^{\mathbf{m}}(-\omega)=\sum_{l,\mathbf{m}}(-1)^l F_l^{\mathbf{m}}(p)Y_l^{\mathbf{m}}(\omega), \quad p\in[0,1].$$

For any fixed $\omega\in S$, the Radon transform $[\mathcal{R}q](\omega,p)$ is an infinitely differentiable function of p on the interval $[-1,1]$. Therefore, functions ${}^*F_l^{\mathbf{m}}(p)$ defined as

$$^*F_l^{\mathbf{m}}(p):=\left\{\begin{array}{cc} F_l^{\mathbf{m}}(p), & p\in[0,1] \\ (-1)^l F_l^{\mathbf{m}}(-p), & p\in[-1,0) \end{array}\right. \tag{26}$$

must be infinitely differentiable on $[-1,1]$, and, in particular, at $p=0$.

Note that ${}^*F_l^{\mathbf{m}}(p)$ are odd functions for odd l, and even functions for even $l=0,2,4,\ldots$. Smoothness of these functions implies that for even l, all odd order right derivatives of $F_l^{\mathbf{m}}(p)$ at $p=0$ should vanish. For odd l, right derivatives of even orders (including zero) of $F_l^{\mathbf{m}}(p)$ must vanish at $p=0$:

$$\frac{\partial^n}{\partial p^n}F_l^{\mathbf{m}}(0)=0,\ l \text{ is even},\ |\mathbf{m}|\le l,\ n=1,3,5,..., \tag{27}$$

$$\frac{\partial^n}{\partial p^n}F_l^{\mathbf{m}}(0)=0,\ l \text{ is odd},\ \ |\mathbf{m}|\le l,\ n=0,2,4,\ldots . \tag{28}$$

The moment conditions (13) can be expressed in terms of coefficients ${}^*F_l^{\mathbf{m}}$ as follows:

$$\int_{-1}^{1} {}^*F_l^{\mathbf{m}}(p)p^n\ dp=0,\ n=0,1,2,...,,\ \text{if } l>n,\ \text{even } |\mathbf{m}|\le l, \tag{29}$$

or, in terms of $F_l^{\mathbf{m}}$

$$\int_{0}^{1} F_l^{\mathbf{m}}(p)p^n\ dp=0,\ n=0,1,2,...,,\ \text{if } l>n,\ l+n \text{ even},\ |\mathbf{m}|\le l. \tag{30}$$

Conditions (23), (27), (28) and (30) are necessary for a function $F(\omega,p)$ defined on $S\times[0,1]$ to be in the range of the Radon transform restricted to $S\times[0,1]$. These conditions are also sufficient, as proven in the theorem below.

Theorem 6. *Suppose function $F(\omega,p)$ defined on $S\times[0,1]$ satisfies conditions (23), (27), (28) and (30) with coefficients $F_l^{\mathbf{m}}(p)$ defined by equation (24). Then, there exists a function $q\in C_0^\infty(B)$ such that representation (22) holds.*

Proof. Let us define function ${}^*F(\omega,p)$ as follows

$$^*F(\omega,p)=\left\{\begin{array}{cc} F(\omega,p), & p\in[0,1] \\ F(-\omega,-p), & p\in[-1,0) \end{array}\right.,\ \omega\in S.$$

The Fourier coefficients of ${}^*F(\omega,p)$ are given by equation (26). Therefore, ${}^*F(\omega,p)$ satisfies the moment conditions (13) and the symmetry condition ${}^*F(\omega,p)=F(-\omega,-p)$. Moreover, ${}^*F(\omega,p)$ is a C^∞ function on each of the two halves $S\times(0,1]$ and $S\times[-1,0)$ of the cylinder $S\times[-1,1]$, i.e.

$$^*F(\omega,p)\in C_0^\infty(S\times[0,1)) \text{ and } {}^*F(\omega,p)\in C_0^\infty(S\times(-1,0]). \tag{31}$$

Due to conditions (27) and (28), each of the coefficients ${}^*F_l^{\mathbf{m}}(p)$ is a $C_0^\infty((-1,1))$ function. Thus, for a fixed ω, ${}^*F(\omega,p)$ is $C_0^\infty((-1,1))$ function in p. Moreover, since all the mixed derivatives of $F(\omega,p)$ are well defined at $p=0$ for each value of ω, the limits

$$\lim_{p\to 0^\pm} {}^*D^k F(\omega,p)$$

of any derivative $D^k F$ are well defined and coincide with $D^k F(\omega,0)$. It follows that ${}^*F(\omega,p)$ is $C_0^\infty(S\times(-1,1))$, and, due to Theorem 1, ${}^*F(\omega,p)$ is in the range of the Radon transform. In other words, there exists function $q\in C_0^\infty(B)$ such that ${}^*F(\omega,p)=[\mathcal{R}q](\omega,p)$. Therefore, $F(\omega,p)=[\mathcal{R}q](\omega,p)$ for $\omega\in S$ and $p\in[0,1]$. $\square$

5.2. Solving the exterior problem using separation of variables

5.2.1. Fourier transform of $b(t,\theta)$ in time

According to the well known Borel extension theorem (see, e.g. [31]), function $b(t,\theta)$ can be extended (non-uniquely) to $b\in C_0^\infty(0,T)$ for any $T>1$. Let us fix such an extension. (Even though such an extension is arbitrary, we will prove later on that the values of the extension for $t>1$ do not enter the final formulas).

Further, we extend $b(t,\theta)$ by zero to all of $\mathbb{R}$ in t. Abusing the notation, we will still call the extended function $b(t,\theta)$.

Such an extension allows us to define the Fourier transform of b with respect to the time variable t:

$$\hat{b}(\lambda,\theta) := [\mathcal{F}b](\lambda,\theta) := \int_{\mathbb{R}} b(t,\theta)e^{i\lambda t}dt,$$

$$b(t,\theta) = [\mathcal{F}^{-1}\hat{b}](t,\theta) := \frac{1}{2\pi}\int_{\mathbb{R}} \hat{b}(\lambda,\theta)e^{-i\lambda t}d\lambda.$$

We proceed by expanding $b(t,\theta)$ and $\hat{b}(\lambda,\theta)$ into spherical harmonics:

$$b(t,\theta) = \sum_{l,\mathbf{m}} b_l^{\mathbf{m}}(t)Y_l^{\mathbf{m}}(\theta), \qquad b_l^{\mathbf{m}}(t) = \int_S b(t,\theta)Y_l^{\mathbf{m}}(\theta)d\theta, \tag{32}$$

$$\hat{b}(\lambda,\theta) = \sum_{l,\mathbf{m}} \hat{b}_l^{\mathbf{m}}(\lambda)Y_l^{\mathbf{m}}(\theta), \qquad \hat{b}_l^{\mathbf{m}}(\lambda) = \int_S \hat{b}(\lambda,\theta)Y_l^{\mathbf{m}}(\theta)d\theta. \tag{33}$$

5.2.2. Solution obtained by expansion in spherical harmonics Our goal is to find the conditions for a candidate function $b(t,\theta)$ to be in the range of the wave operator $\mathcal{A}$ in terms of the spherical harmonic expansion coefficients of b, rather than in terms of $v^b(t,x)$. To do so, we need to be able to express the exterior Radon transform of the solution $v^{(b)}$ to the exterior problem (16), in terms of coefficients $b_l^{\mathbf{m}}(t)$ of the spherical harmonic expansion of $b(t,\theta)$ (see equation (32)).

This will require the use of certain special functions. The **spherical Hankel function** $h_l^d(\lambda)$ is defined through the standard **Hankel function** $H^{(1)}_{l+d/2-1}(\lambda)$ of order $l + d/2 - 1$ by the following formulas:

$$h_l^d(\lambda) = \frac{H^{(1)}_{l+d/2-1}(\lambda)}{\lambda^{d/2-1}}, l = 0, 1, 2, ..., \lambda \geq 0,$$

$$h_l^d(-\lambda) = \overline{h_l^d(\lambda)}, l = 0, 1, 2, ..., \lambda > 0$$

(see [28] for details.)

The desired connection between $\mathcal{R}^E v^{(b)}$ and $b_l^{\mathbf{m}}$ is given by the following theorem.

Theorem 7. *Let $v^{(b)}$ be the solution to the exterior problem (16) with boundary condition $b(t, \theta) \in C_0(Z_1)$, and $[\mathcal{R}^E v^{(b)}](t, \omega, p)$ be the exterior Radon transform of $v^{(b)}$. Then $\mathcal{R}^E v^{(b)}$ can be expressed as the following series*

$$[\mathcal{R}^E v^{(b)}](t, \omega, p) = \sum_{l,\mathbf{m}} R_l^{\mathbf{m}}(t - p) Y_l^{\mathbf{m}}(\omega), \tag{34}$$

with functions $R_l^{\mathbf{m}}$ expressed through the Fourier coefficients $\hat{b}_l^{\mathbf{m}}(\lambda)$ (see equation (33)) as follows:

$$R_l^{\mathbf{m}}(t) = -2^{\frac{d}{2}} \pi^{\frac{d}{2}-1} i^l \left[\mathcal{F}^{-1} \left(\frac{\hat{b}_l^{\mathbf{m}}(\lambda)}{\lambda^{d-1} h_l^d(\lambda)} \right) \right](t). \tag{35}$$

Alternatively, functions $R_l^{\mathbf{m}}$ can be represented as convolutions

$$R_l^{\mathbf{m}}(t) = (b_l^{\mathbf{m}} * K_l)(t) = \int_0^{1+t} b_l^{\mathbf{m}}(s) K_l(t - s) ds, \tag{36}$$

with convolution kernels $K_l(t)$ defined through the inverse Fourier transform as follows

$$K_l(t) = -\left[\mathcal{F}^{-1} \left(\frac{2^{\frac{d}{2}} \pi^{\frac{d}{2}-1} i^l}{\lambda^{d-1} h_l^d(\lambda)} \right) \right](t). \tag{37}$$

We postpone the lengthy proof of this Theorem till Section 7.

Remark 8. *The above theorem, by solving explicitly the exterior problem, relates the Radon transform of $v^{(b)}$ directly to the Fourier coefficients of b. Below we use these formulas at the value of $t = 1$ and values of p lying in the interval $[1, 2]$. Then, integration in the equation (36) requires only the values of $b_l^{\mathbf{m}}(s)$ in the interval $s \in [0, 1]$. Thus, our choice of the arbitrary smooth extension of $b(t, \theta)$ to times $t > 1$ does not affect functions $R_l^{\mathbf{m}}(t)$ that are being computed.*

6. Range theorem in terms of spherical harmonics expansion

Theorem 9. *A function $b(t, \theta) \in C_0^\infty(Z_1)$ is in the range of $\mathcal{A}$, i.e. can be expressed as $b = \mathcal{A}f$ if and only if the following two sets of conditions are satisfied:*

(i) ***moment conditions***

$$\int_0^1 R_l^{\mathbf{m}}(p) p^n \, dp = 0, \;\; n = 0, 1, 2, ...,, \;\; \text{if } l > n, \;\; l{+}n \text{ is even}, \;\; |\mathbf{m}| \leq l.$$

(ii) ***"smoothness" conditions*** *at* $p = 0$

$$\frac{\partial^n}{\partial p^n} R_l^{\mathbf{m}}(0) = 0,\ l \textit{ is even},\ |\mathbf{m}| \le l,\ n = 1, 3, 5, ...,$$
$$\frac{\partial^n}{\partial p^n} R_l^{\mathbf{m}}(0) = 0,\ l \textit{ is odd},\ \ |\mathbf{m}| \le l,\ n = 0, 2, 4, ...$$

where functions $R_l^{\mathbf{m}}(t)$*,* $l = 0, 1, 2, ...,$ $|\mathbf{m}| \le l$*, are defined by (34).*

Moreover, if conditions (1) and (2) are satisfied, the Radon projections $F(\omega, p)$ *of* $f(x)$ *are given explicitly by the formulas*

$$F(\omega, p) = \sum_{l,\mathbf{m}} R_l^{\mathbf{m}}(p) Y_l^{\mathbf{m}}(\omega),\ F(\omega, -p) = F(-\omega, p),\ p \in [0, 1],\ \omega \in S,$$

and so f *can be reconstructed from* F *by inverting the Radon transform.*

Proof. Define function $F(\omega, p)$ for $p \in [0, 1]$, $\omega \in S$, as follows

$$F(\omega, p) = [\mathcal{R}^E v^{(b)}](1, \omega, p + 1) = \sum_{l,\mathbf{m}} R_l^{\mathbf{m}}(p) Y_l^{\mathbf{m}}(\omega).$$

The coefficients of the spherical harmonic expansion of $F(\omega, p)$ are

$$F_l^{\mathbf{m}}(p) = \int_S F(\omega, p) Y_l^{\mathbf{m}}(\omega) d\omega = R_l^{\mathbf{m}}(p),$$

with functions $R_l^{\mathbf{m}}$ defined by equations (36) and (37). Proposition 6 yields the necessary and sufficient conditions on $F(\omega, p)$ and $F_l^{\mathbf{m}}(p)$ for b being in the range of $\mathcal{A}$. These conditions coincide with the moment and smoothness conditions of the theorem. This completes the proof. □

7. Proof of Theorem 7

We need to prepare several preliminary results before Theorem 7 can be proven.

First, let us use separation of variables to obtain explicit expression for the solution $v^{(b)}(t, x)$ to the exterior problem (16). We will also need the time derivative $v_t^{(b)}(t, x)$ of $v^{(b)}(t, x)$, and the expression for the normal derivative $\frac{\partial}{\partial n} v^{(b)}(t, \theta)$, where $\theta \in S$.

Proposition 10. *For a given condition* $b(t, \theta)$*, solution* $v^{(b)}(t, x)$ *to the exterior problem (16) and its derivatives are given by the following formulas:*

$$v^{(b)}(t, x) = \frac{1}{2\pi} \int_{\mathbb{R}} \left(\sum_{l,\mathbf{m}} \hat{b}_l^{\mathbf{m}}(\lambda) \frac{h_l^d(\lambda|x|)}{h_l^d(\lambda)} Y_l^{\mathbf{m}}(\hat{x}) \right) e^{-i\lambda t} d\lambda, \tag{38}$$

$$v_t^{(b)}(t, x) = -\frac{1}{2\pi} \int_{\mathbb{R}} \left(i\lambda \sum_{l,\mathbf{m}} \hat{b}_l^{\mathbf{m}}(\lambda) \frac{h_l^d(\lambda|x|)}{h_l^d(\lambda)} Y_l^{\mathbf{m}}(\hat{x}) \right) e^{-i\lambda t} d\lambda, \tag{39}$$

$$\frac{\partial}{\partial n} v^{(b)}(t, \theta) = \frac{1}{2\pi} \int_{\mathbb{R}} \left(\lambda \sum_{l,\mathbf{m}} \hat{b}_l^{\mathbf{m}}(\lambda) \frac{(h_l^d(\lambda))'}{h_l^d(\lambda)} Y_l^{\mathbf{m}}(\theta) \right) e^{-i\lambda t} d\lambda. \tag{40}$$

Proof. We observe that the following combinations of functions

$$\frac{h_l^d(\lambda|x|)}{h_l^d(\lambda)} e^{-i\lambda t} Y_l^{\mathbf{m}}(\hat{x}), \qquad \hat{x} = x/|x|,$$

are outgoing solutions of Helmholz equation. Note that functions $h_l^d(\lambda)$ in the denominator do not have zeros for real values of λ (see, e.g. [28]). With x restricted to S, these functions are equal to $e^{-i\lambda t} Y_l^{\mathbf{m}}(x/|x|)$. Thus, they form a complete orthonormal system on $\mathbb{R} \times S$. Therefore, the solution $v^{(b)}$ for the exterior problem corresponding to the boundary values $b(t,\theta)$ can be written in the form (38). In Appendix A we provide an alternative, more detailed proof of this formula.

Due to the smoothness of the function $b(t,\theta)$ the series (38) converges fast (faster than any negative power of $|\mathbf{m}|$). This justifies the term-wise differentiation of the series, resulting in the formulas (39) and (40). □

Our intention is to compute the exterior Radon transform of $v^{(b)}(t,x)$ given by formula (38). A natural idea would be to try to integrate equation (38) at each frequency λ term-by-term. However, at dimensions d higher than 2, integrals over planes of functions $h_l^d(\lambda|x|) Y_l^{\mathbf{m}}(\hat{x})$ diverge, due to the slow decrease of $h_l^d(\lambda|x|)$ at infinity. Thus, we need the following workaround that expresses integrals of $v_t^{(b)}(t,x)$ over planes through integrals over sphere S. Since the sphere is bounded, the question of convergence does not arise.

Proposition 11. *The exterior Radon transform $\mathcal{R}^E v_t^{(b)}$ of the time derivative $v_t^{(b)}$ of the exterior solution $v^{(b)}(t,x)$ can be expressed through integrals over sphere S as follows:*

$$\begin{aligned} &2[\mathcal{R}^E v_t^{(b)}](t,\omega,p) = \\ &\int\limits_S \frac{\partial}{\partial n} v^{(b)}(\omega\cdot\theta - p + t, \theta) d\theta - \int\limits_S v_t^{(b)}(\omega\cdot\theta - p + t,\theta)(\theta\cdot\omega) d\theta. \end{aligned} \tag{41}$$

Proof. The well known Kirchhoff representation (see, e.g., [25] and references therein) allows us to express $v^{(b)}$, using Green's function $\mathbf{G}(t,x)$ given by equation (6), through the boundary values of $v^{(b)}$ and $\frac{\partial}{\partial n} v^{(b)}$ as follows;

$$\begin{aligned} &v^{(b)}(t,x) \\ &= \int\limits_S \int\limits_0^t \left(\frac{\partial}{\partial n} v^{(b)}(\tau,\theta) \mathbf{G}(t-\tau, x-\theta) - v^{(b)}(\tau,\theta) \frac{\partial}{\partial n(\theta)} \mathbf{G}(t-\tau,x-\theta) \right) d\tau d\theta. \end{aligned}$$

Correspondingly, the time derivative $v_t^{(b)}(t,x)$ can be represented by the following formula:

$$\begin{aligned} &v_t^{(b)}(t,x) \\ &= \int\limits_S \int\limits_0^t \left(\frac{\partial}{\partial n} v^{(b)}(\tau,\theta) \mathbf{G}_t(t-\tau, x-\theta) - v^{(b)}(\tau,\theta) \frac{\partial}{\partial n(\theta)} \mathbf{G}_t(t-\tau,x-\theta) \right) d\tau d\theta, \end{aligned}$$

where $\mathbf{G}_t$ denotes the time derivative of $\mathbf{G}$. Let us evaluate the exterior Radon transform $[\mathcal{R}^E v_t^{(b)}](t,\omega,p)$ of $v_t^{(b)}$, for $p \geq 1$. For each fixed t we obtain

$$\begin{aligned}
&[\mathcal{R}^E v_t^{(b)}](t,\omega,p) \\
&= \int\limits_{\Pi(\omega,p)} \left[\int\limits_S \int\limits_0^t \left(\frac{\partial}{\partial n} v^{(b)}(\tau,\theta) \mathbf{G}_t(t-\tau, x-\theta) \right.\right. \\
&\qquad\qquad \left.\left. - v^{(b)} \frac{\partial}{\partial n(\theta)} \mathbf{G}_t(t-\tau, x-\theta) \right) d\tau d\theta \right] dx \\
&= \int\limits_S \int\limits_0^t \frac{\partial}{\partial n} v^{(b)}(\tau,\theta) \left[\int\limits_{\Pi(\omega,p)} \mathbf{G}_t(t-\tau, x-\theta) dx \right] d\tau d\theta \\
&- \int\limits_S \int\limits_0^t v^{(b)}(\tau,\theta) \left[\theta \cdot \nabla_\theta \int\limits_{\Pi(\omega,p)} \mathbf{G}_t(t-\tau, x-\theta) dx \right] d\tau d\theta,
\end{aligned}$$

where reversing the order of integration is justified since for any fixed t function $v^{(b)}(t,x)$ is finitely supported, and where we equate the normal to the unit sphere S at the point θ, with θ. We now observe that, for $t > 0$,

$$2 \int\limits_{\Pi(\omega,p)} \mathbf{G}_t(t, x-y) dx = \delta(\omega \cdot y - p - t) + \delta(\omega \cdot y - p + t), \tag{42}$$

$$2\nabla_y \int\limits_{\Pi(\omega,p)} \mathbf{G}_t(t, x-y) dx = \omega \left[\delta'(\omega \cdot y - p - t) + \delta'(\omega \cdot y - p + t) \right]. \tag{43}$$

In the Appendix B we provide a proof for the known equality (42). Equation (43) is obtained by taking a gradient of equation (42).

Each of the equations (42), (43) represents two singular waves propagating in the opposite directions $\pm\omega$ as t increases. Now with a substitution $t \to t - \tau$, for $t > \tau$ we obtain

$$2 \int\limits_{\Pi(\omega,p)} \mathbf{G}_t(t-\tau, x-y) dx = \delta(\omega \cdot y - p - (t-\tau)) + \delta(\omega \cdot y - p + t - \tau),$$

$$2\nabla_y \int\limits_{\Pi(\omega,p)} \mathbf{G}_t(t-\tau, x-y) dx = \omega \left[\delta'(\omega \cdot y - p - (t-\tau)) + \delta'(\omega \cdot y - p + t - \tau) \right].$$

Since we consider values of $p \geq 1$ and $\tau \leq t$, only one of the terms in the above formulas will contribute to the integral over sphere S:

$$
\begin{aligned}
&2[\mathcal{R}^E v_t^{(b)}](t,\omega,p) \\
&= \int\limits_S \int\limits_0^t \frac{\partial}{\partial n} v^{(b)}(\tau,\theta)\delta(\omega\cdot\theta - p + t - \tau)d\tau d\theta \\
&- \int\limits_S \int\limits_0^t v^{(b)}(t,\theta)(\theta\cdot\omega)\delta'(\omega\cdot\theta - p + t - \tau)d\tau d\theta \\
&= \int\limits_S \frac{\partial}{\partial n} v^{(b)}(\omega\cdot\theta - p + t,\theta)d\theta - \int\limits_S v_t^{(b)}(\omega\cdot\theta - p + t,\theta)(\theta\cdot\omega)d\theta. \qquad (44)
\end{aligned}
$$

□

Next, we formulate and prove the following proposition.

Proposition 12. *Let $v^{(b)}$ be the solution to the exterior problem (16) with boundary condition $b(t,\theta) \in C_0(Z_1)$, and $[\mathcal{R}^E v^{(b)}](t,\omega,p)$ be the exterior Radon transform of $v^{(b)}$. Then $\mathcal{R}^E v^{(b)}$ can be expressed as the series (34), with functions $R_l^{\mathbf{m}}$ expressed through the Fourier coefficients $\hat{b}_l^{\mathbf{m}}(\lambda)$ as shown in equation (35). Moreover, functions $R_l^{\mathbf{m}}(t)$ can be represented as convolutions*

$$
R_l^{\mathbf{m}}(t) = (b_l^{\mathbf{m}} * K_l)(t) = \int\limits_{\mathbb{R}} b_l^{\mathbf{m}}(s)K_l(t-s)ds, \qquad (45)
$$

with kernels $K_l(t)$ defined by equation (37); these functions are infinitely differentiable for all values of t.

Proof. Let us utilize the expression for $\mathcal{R}^E v_t^{(b)}$ given by equation (44), and substitute into it expressions (39) and (40) for $v_t^{(b)}$ and $\frac{\partial}{\partial n}v^{(b)}$. Then the second integral on the last line of (44) can be transformed as follows:

$$
\begin{aligned}
&- \int\limits_S v_t^{(b)}(\omega\cdot y - p + t,\theta)(\theta\cdot\omega)d\theta \\
&= \frac{1}{2\pi}\int\limits_S \left[\int\limits_{\mathbb{R}} \left\{ \sum_{l,\mathbf{m}} \hat{b}_l^{\mathbf{m}}(\lambda) Y_l^{\mathbf{m}}(\theta) \right\} i\lambda e^{-i\lambda\omega\cdot\theta}(\theta\cdot\omega)e^{-i\lambda(t-p)}d\lambda \right] d\theta \\
&= -\frac{1}{2\pi}\int\limits_S \left[\int\limits_{\mathbb{R}} \left\{ \sum_{l,\mathbf{m}} \hat{b}_l^{\mathbf{m}}(\lambda) Y_l^{\mathbf{m}}(\theta) \right\} \lambda \left\{ \frac{d}{d\lambda} e^{-i\lambda\omega\cdot\theta} \right\} e^{-i\lambda(t-p)}d\lambda \right] d\theta \\
&= -\frac{1}{2\pi}\int\limits_{\mathbb{R}} \lambda e^{-i\lambda(t-p)} \left[\sum_{l,\mathbf{m}} \hat{b}_l^{\mathbf{m}}(\lambda) \frac{d}{d\lambda} \left\{ \int\limits_S Y_l^{\mathbf{m}}(\theta) e^{-i\lambda\omega\cdot\theta} d\theta \right\} \right] d\lambda \qquad (46)
\end{aligned}
$$

In order to simplify the expression in the curly braces, we use Jacobi-Anger expansion formula (Lemma 9.10.2 in [6]) which in our notation has the following form:

$$
\int\limits_S e^{-2\pi i t(\hat{x}\cdot\hat{y})} Y_l^{\mathbf{m}}(\hat{x})d\hat{x} = 2\pi i^l Y_l^{\mathbf{m}}(\hat{y}) \frac{J_{l+\frac{d}{2}-1}(2\pi t)}{t^{\frac{d}{2}-1}},
$$

or, with substitution $2\pi t = \lambda$, $\hat{y} = \omega$, $\hat{x} = \theta$,

$$\int\limits_S e^{-i\lambda\omega\cdot\theta} Y_l^{\mathbf{m}}(\theta) d\theta = (2\pi)^{\frac{d}{2}} i^l Y_l^{\mathbf{m}}(\omega) j_l^d(\lambda). \tag{47}$$

where

$$j_l^d(\lambda) = \operatorname{Re} h_l^d(\lambda) = \frac{J_{l+\frac{d}{2}-1}(\lambda)}{\lambda^{\frac{d}{2}-1}}, \qquad l = 0, 1, 2, ...$$

Now, using (47), equation (46) takes the following form:

$$\begin{aligned} &-\int\limits_S v_t^{(b)}(\omega\cdot y - p + t, \theta)(\theta\cdot\omega) d\theta \\ &= -(2\pi)^{\frac{d}{2}-1} \int\limits_{\mathbb{R}} \left[\sum_{l,\mathbf{m}} \hat{b}_l^{\mathbf{m}}(\lambda) i^l Y_l^{\mathbf{m}}(\omega)(j_l^d(\lambda))' \right] \lambda e^{-i\lambda(t-p)} d\lambda. \end{aligned} \tag{48}$$

A similar technique can be used to simplify the first term in (44). Indeed, taking into account formula (40) we obtain

$$\begin{aligned} &\int\limits_S \frac{\partial}{\partial n} v^{(b)}(\omega\cdot\theta - p + t, \theta) d\theta \\ &= \frac{1}{2\pi} \int\limits_S \left[\int\limits_{\mathbb{R}} \left\{ \sum_{l,\mathbf{m}} \hat{b}_l^{\mathbf{m}}(\lambda) \frac{\lambda (h_l^d(\lambda))'}{h_l^d(\lambda)} Y_l^{\mathbf{m}}(\theta) e^{-i\lambda\omega\cdot\theta} \right\} e^{-i\lambda(t-p)} d\lambda \right] d\theta \\ &= \frac{1}{2\pi} \int\limits_{\mathbb{R}} \lambda \left(\sum_{l,\mathbf{m}} \hat{b}_l^{\mathbf{m}}(\lambda) \frac{(h_l^d(\lambda))'}{h_l^d(\lambda)} \left[\int\limits_S e^{-i\lambda\omega\cdot\theta} Y_l^{\mathbf{m}}(\theta) d\theta \right] \right) e^{-i\lambda(t-p)} d\lambda. \end{aligned}$$

Again, using the Jacobi-Anger formula to compute the term in the brackets,

$$\begin{aligned} &\int\limits_S \frac{\partial}{\partial n} v^{(b)}(\omega\cdot\theta - p + t, \theta) d\theta \\ &= (2\pi)^{\frac{d}{2}-1} \int\limits_{\mathbb{R}} \lambda \left(\sum_{l,\mathbf{m}} i^l \hat{b}_l^{\mathbf{m}}(\lambda) \frac{\left(h_l^d\right)'(\lambda)}{h_l^d(\lambda)} Y_l^{\mathbf{m}}(\omega) j_l^d(\lambda) \right) e^{-i\lambda(t-p)} d\lambda. \end{aligned} \tag{49}$$

Now we can combine expressions (49) and (48) for the terms appearing in (44):

$$\begin{aligned} 2[\mathcal{R}^E v_t^{(b)}](t,\omega,p) &= (2\pi)^{\frac{d}{2}-1} \int\limits_{\mathbb{R}} \lambda \left(\sum_{l,\mathbf{m}} i^l \hat{b}_l^{\mathbf{m}}(\lambda) \frac{\left(h_l^d\right)'(\lambda)}{h_l^d(\lambda)} Y_l^{\mathbf{m}}(\omega) j_l^d(\lambda) \right) e^{-i\lambda(t-p)} d\lambda \\ &\quad - (2\pi)^{\frac{d}{2}-1} \int\limits_{\mathbb{R}} \lambda \left(\sum_{l,\mathbf{m}} i^l \hat{b}_l^{\mathbf{m}}(\lambda) Y_l^{\mathbf{m}}(\omega) (j_l^d)'(\lambda) \right) e^{-i\lambda(t-p)} d\lambda \\ = (2\pi)^{\frac{d}{2}-1} \int\limits_{\mathbb{R}} \lambda &\left(\sum_{l,\mathbf{m}} i^l \hat{b}_l^{\mathbf{m}}(\lambda) \frac{\left(h_l^d\right)'(\lambda) j_l^d(\lambda) - h_l^d(\lambda)(j_l^d(\lambda))'}{h_l^d(\lambda)} Y_l^{\mathbf{m}}(\omega) j_l^d(\lambda) \right) e^{-i\lambda(t-p)} d\lambda \end{aligned}$$

The following well-known identity for the Wronskian of functions J_μ and $H_\mu^{(1)}$ (formula 10.5.3 in [28])

$$\mathcal{W}(J_\mu, H_\mu^{(1)})(\lambda) = 2i/(\pi\lambda)$$

yields

$$\left(h_l^d(\lambda)\right)' j_l^d(\lambda) - h_l^d(\lambda)(j_l^d(\lambda))' = \frac{1}{\lambda^{d-2}}\mathcal{W}\left(J_{l+d/2-1}, H^{(1)}_{l+d/2-1}\right)(\lambda) = \frac{2i}{\pi\lambda^{d-1}},$$

which, in turn, leads to a simplification of the expression for $\mathcal{R}^E v_t^{(b)}$:

$$\begin{aligned}
[\mathcal{R}^E v_t^{(b)}](t,\omega,p) &= (2\pi)^{\frac{d}{2}-1}\frac{i}{\pi}\int_{\mathbb{R}}\left(\sum_{l,\mathbf{m}} i^l \frac{\hat{b}_l^{\mathbf{m}}(\lambda)}{\lambda^{d-2}h_l^d(\lambda)}Y_l^{\mathbf{m}}(\omega)\right)e^{-i\lambda(t-p)}d\lambda \\
&= 2^{\frac{d}{2}-1}\pi^{\frac{d}{2}-2}\sum_{l,\mathbf{m}}\left(i^{l+1}\int_{\mathbb{R}}\frac{\hat{b}_l^{\mathbf{m}}(\lambda)}{\lambda^{d-2}h_l^d(\lambda)}e^{-i\lambda(t-p)}d\lambda\right)Y_l^{\mathbf{m}}(\omega).
\end{aligned}$$

Since $v^{(b)}(0,x)=0$, we have $[\mathcal{R}^E v^{(b)}](0,\omega,p)=0$. Now the Radon transform $\mathcal{R}^E v^{(b)}$ of $v^{(b)}$ can be found by integration in time t:

$$\begin{aligned}
[\mathcal{R}^E v^{(b)}](t,\omega,p) &= \int_0^t [\mathcal{R}^E v_\tau^{(b)}](\tau,\omega,p)d\tau \\
&= 2^{\frac{d}{2}-1}\pi^{\frac{d}{2}-2}\int_0^t\left[\sum_{l,\mathbf{m}}\left(i^{l+1}\int_{\mathbb{R}}\frac{\hat{b}_l^{\mathbf{m}}(\lambda)}{\lambda^{d-2}h_l^d(\lambda)}e^{-i\lambda\tau}e^{i\lambda p}d\lambda\right)Y_l^{\mathbf{m}}(\omega)\right]d\tau \\
&= -2^{\frac{d}{2}-1}\pi^{\frac{d}{2}-2}\sum_{l,\mathbf{m}}\left(i^{l}\int_{\mathbb{R}}\frac{\hat{b}_l^{\mathbf{m}}(\lambda)}{\lambda^{d-1}h_l^d(\lambda)}e^{-i\lambda(t-p)}d\lambda\right)Y_l^{\mathbf{m}}(\omega),
\end{aligned}$$

or

$$[\mathcal{R}^E v^{(b)}](t,\omega,p) = \sum_{l,\mathbf{m}} R_l^{\mathbf{m}}(t-p)Y_l^{\mathbf{m}}(\omega) \text{ with} \tag{50}$$

$$\begin{aligned}
R_l^{\mathbf{m}}(t) &= -2^{\frac{d}{2}-1}\pi^{\frac{d}{2}-2}i^l\int_{\mathbb{R}}\frac{\hat{b}_l^{\mathbf{m}}(\lambda)}{\lambda^{d-1}h_l^d(\lambda)}e^{-i\lambda t}d\lambda \\
&= -2^{\frac{d}{2}}\pi^{\frac{d}{2}-1}i^l\left[\mathcal{F}^{-1}\left(\frac{\hat{b}_l^{\mathbf{m}}(\lambda)}{\lambda^{d-1}h_l^d(\lambda)}\right)\right](t).
\end{aligned} \tag{51}$$

Equations (50) and (51) coincide with equations (34) and (35) announced in Theorem (7) and in the statement of this proposition. It follows that functions $R_l^{\mathbf{m}}(t)$ can be represented as convolutions (45). Moreover, since we have extended function $b(t,\theta)$ to a $C_0^\infty(\mathbb{R}\times S)$ function, coefficients $b_l^{\mathbf{m}}(t)$ are infinitely smooth. Therefore, functions $R_l^{\mathbf{m}}(t)$ are also $C_0^\infty(\mathbb{R})$ functions, as convolutions of distributions with C_0^∞ functions. $\square$

Remark 13.

(i) *The appearance of convolutions in (45) is not surprising, since the problem we consider is invariant with respect to the shift in time.*

(ii) *We will only need to know the values of $R_l^{\mathbf{m}}(t)$ on the interval $[-1,0]$, and the values of all the derivatives of $R_l^{\mathbf{m}}(t)$ at $t=0$. In what follows we will show that distributions $K_l(t)$ have finite support, and as a result, integration in (37) is performed over finite intervals, such that the values of the smooth extension of $b(t,\theta)$ to $t>1$ are, in fact, not used at all.*

Proposition 14. *Convolution kernels $K_l(t)$ given by equation (37) vanish for $t<-1$ in the sense of distributions, and thus integration interval in (45) can be reduced as follows:*

$$R_l^{\mathbf{m}}(\tau) = \int_0^{1+\tau} b_l^{\mathbf{m}}(s) K_l(\tau - s) ds. \tag{52}$$

Moreover, in order to compute values of $R_l^{\mathbf{m}}(t)$ in the interval $t\in[-1,0]$ using the above formula, functions $b_l^{\mathbf{m}}(t)$ need to be known only on the interval $t\in[0,1]$.

Proof. Consider an arbitrary $C_0^\infty(\mathbb{R}\times S)$ function $b(t,\theta)$, finitely supported in t on the interval $(0,T)$ for some $T>1$, and function $v^{(b)}(t,x)$ that is a solution to the exterior problem in $\mathbb{R}\times B^c$, vanishing at $t=-\infty$:

$$\begin{cases} \frac{\partial^2 v^{(b)}(t,x)}{\partial t^2} - \Delta v^{(b)}(t,x) = 0, & (t,x)\in\mathbb{R}\times B^c, \\ v^{(b)}(t,x) = 0, \quad v_t^{(b)}(t,x) = 0, & (t,x)\in(-\infty,0)\times B^c, \\ v^{(b)}(t,\theta) = b(t,\theta). & (t,\theta)\in\mathbb{R}\times S. \end{cases}$$

Due to the finite speed of sound, solution $v^{(b)}(t,x)$ of this problem is identically zero outside of the interior of the cone $|x|\le t+1$. Therefore, for the exterior Radon transform of $v^{(b)}$ we have

$$[\mathcal{R}^E v^{(b)}](t,\omega,p) = 0 \text{ for } p-t>1, \quad \omega\in S.$$

In terms of Fourier coefficients of $\mathcal{R}^E$ (see equation (50)), this implies

$$R_l^{\mathbf{m}}(t-p) = 0 \text{ for } t-p<-1, \quad l=0,1,2,3,.., \quad |\mathbf{m}|\le l,$$

or, equivalently $R_l^{\mathbf{m}}(t)=0$ for $t<-1$. Values of the Fourier coefficients $R_l^{\mathbf{m}}(t)$ can be expressed through the Fourier coefficients $b_l^{\mathbf{m}}(t)$ of $b(t,\theta)$ by convolutions (45). Since $b(t,\theta)$ vanishes for $t<0$, coefficients $b_l^{\mathbf{m}}(t)$ also vanish for these values of t. Therefore, equation (45) can be re-written as follows

$$R_l^{\mathbf{m}}(t) = \int_0^{\infty} b_l^{\mathbf{m}}(s) K_l(t-s) ds. \tag{53}$$

Since function $b(t,\theta)$ is arbitrary, so are coefficients $b_l^{\mathbf{m}}(t)$ (subject to conditions of being smooth and supported on $(0,T)$). Then, for a fixed $l,\mathbf{m}$, vanishing of $R_l^{\mathbf{m}}(t)$ for $t<-1$ implies vanishing of the integral (53) for $t<-1$, for an arbitrary test function $b_l^{\mathbf{m}}(t)$. It follows that for any $t<-1$, distribution $K_l(t-s)$ vanishes for all $s\ge 0$. In other words,

$$K_l(\tau) = 0 \text{ for } \tau<-1, \quad l=0,1,2,3,.., \quad |\mathbf{m}|\le l,$$

as we wanted to show. This implies that the integration interval in (53) can be further reduced, yielding equation (52). Clearly, in order to compute values of $R_l^{\mathbf{m}}(t)$ in the interval $t \in [-1, 0]$ using formula (52) functions $b_l^{\mathbf{m}}(t)$ need to be known only on the interval $t \in [0, 1]$.This completes the proof of the proposition. □

Finally, the proof of Theorem 7 results by combining the statements of Propositions 12 and 14.

8. Remarks and conclusions

- In order to solve the exterior problem in terms of expansion in spherical harmonics, we smoothly extended the $C_0^\infty(Z_1)$ candidate function $b(t, \theta)$ to $C_0^\infty(\mathbb{R} \times S)$. Such an extension has allowed us to obtain the expression (34) for the exterior Radon transform of the solution $v^{(b)}(t, x)$, with functions $R_l^{\mathbf{m}}(t)$ computed as convolutions (36). Such a smooth extension can be constructed rather arbitrarily, using the Borel's lemma. However, as proven in Proposition 14, computation of $R_l^{\mathbf{m}}(t)$ in the interval $t \in [-1, 0]$ needed for Theorem 9 requires the knowledge of the values of $b(t, \theta)$ only on $C_0^\infty(Z_1)$. This eliminates any effect of the freedom of choice of the smooth extension of b.
- The results of this paper are somewhat related to the results obtained in [11] and [3], where in odd dimensions certain symmetries (with respect to $t = 1$) were observed in the range of the wave operator and the spherical means operator. The presence of such symmetries serves as a necessary and sufficient condition for a function to be in the range of the corresponding operator. Results of our paper, however, pertain to functions defined on the half-time interval $t \in [0, 1]$. They hold in both even and odd dimensions.
- We have formulated our range conditions for the wave operator. However, related results can be obtained for the spherical means operator, by utilizing the connection between the two problems, expressed through equations (5), (7).

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Appendix A

In this Appendix we provide a more detailed proof for the formula (38). The latter formula gives a series solution to the exterior problem (16). Solution $v^{(b)}$ to this exterior problem can be expressed using the Kichhoff's representation as a combination of the single- and double- layer potentials as follows:

$$v^{(b)}(t,x) = \int\limits_S \int\limits_0^t \left[\frac{\partial v^{(b)}(\tau,\theta)}{\partial n(\theta)} \mathbf{G}(t-\tau, x-\theta) - b(\tau,\theta) \frac{\partial}{\partial n(\theta)} \mathbf{G}(t-\tau, x-\theta) \right] d\tau d\theta$$

for $x \in B^c$ and $t > 0$. Since $\mathbf{G}(t,x)$ vanishes for $t < 0$, the above formula still holds if we $v^{(b)}(t,x)$ by 0 to $t \le 0$, and make the upper integration limit (in τ) infinite:

$$v^{(b)}(t,x) = \int\limits_S \int\limits_{\mathbb{R}} \left[\frac{\partial v^{(b)}(\tau,\theta)}{\partial n(\theta)} \mathbf{G}(t-\tau, x-\theta) - b(\tau,\theta) \frac{\partial}{\partial n(\theta)} \mathbf{G}(t-\tau, x-\theta) \right] d\tau d\theta$$

where, with abuse of notation, we keep the same notation for the function $v^{(b)}(t,x)$. The convolution represented by the inner integral can be expressed using the Fourier transform. Namely, introduce the following notations:

$$\begin{aligned} \hat{G}(\lambda,x) \quad &: = [\mathcal{F}G](\lambda,\theta) := \int\limits_{\mathbb{R}} G(t,x) e^{i\lambda t} dt, \\ \frac{\partial \hat{v}^{(b)}(\lambda,\theta)}{\partial n(\theta)} &: = \left[\mathcal{F} \frac{\partial v^{(b)}}{\partial n} \right] (\lambda,\theta) := \int\limits_{\mathbb{R}} \frac{\partial v^{(b)}(t,\theta)}{\partial n(\theta)} e^{i\lambda t} dt. \end{aligned}$$

Then

$$v^{(b)}(t,x) = \int\limits_S \int\limits_{\mathbb{R}} \left[\frac{\partial \hat{v}^{(b)}(\lambda,\theta)}{\partial n(\theta)} \hat{G}(\lambda, x-\theta) - \hat{b}(\lambda,\theta) \frac{\partial}{\partial n(\theta)} \hat{G}(\lambda, x-\theta) \right] e^{-i\lambda t} d\lambda d\theta$$

Previously, we extended $\hat{b}(\lambda,\theta)$ into spherical harmonics (see equation (33)):

$$\hat{b}(\lambda,\theta) = \sum_{l,\mathbf{m}} \hat{b}_l^{\mathbf{m}}(\lambda) Y_l^{\mathbf{m}}(\theta).$$

Let us also expand functions $\frac{\partial \hat{v}^{(b)}(\lambda,\theta)}{\partial n(\theta)}$ into spherical harmonics:

$$\frac{\partial \hat{v}^{(b)}(\lambda,\theta)}{\partial n(\theta)} = \sum_{l,\mathbf{m}} d_l^{\mathbf{m}}(\lambda) Y_l^{\mathbf{m}}(\theta), \qquad d_l^{\mathbf{m}}(\lambda) = \int\limits_S \frac{\partial \hat{v}^{(b)}(\lambda,\theta)}{\partial n(\theta)} Y_l^{\mathbf{m}}(\theta) d\theta.$$

We obtain

$$\begin{aligned} &v^{(b)}(t,x) \\ &= \frac{1}{2\pi} \int\limits_S \int\limits_{\mathbb{R}} \left[\left(\sum_{l,\mathbf{m}} \hat{d}_l^{\mathbf{m}}(\lambda) Y_l^{\mathbf{m}}(\theta) \right) \hat{G}(\lambda, x-\theta) \right. \\ &- \left. \left(\sum_{l,\mathbf{m}} \hat{b}_l^{\mathbf{m}}(\lambda) Y_l^{\mathbf{m}}(\theta) \right) \frac{\partial}{\partial n(\theta)} \hat{G}(\lambda, x-\theta) \right] e^{-i\lambda t} d\lambda d\theta \\ &= \frac{1}{2\pi} \int\limits_{\mathbb{R}} \left(\sum_{l,\mathbf{m}} \hat{d}_l^{\mathbf{m}}(\lambda) \left[\int\limits_S Y_l^{\mathbf{m}}(\theta) \hat{G}(\lambda, x-\theta) d\theta \right] \right) e^{-i\lambda t} d\lambda \\ &- \frac{1}{2\pi} \int\limits_{\mathbb{R}} \left(\sum_{l,\mathbf{m}} \hat{b}_l^{\mathbf{m}}(\lambda) \left[\int\limits_S Y_l^{\mathbf{m}}(\theta) \frac{\partial}{\partial n(\theta)} \hat{G}(\lambda, x-\theta) d\theta \right] \right) e^{-i\lambda t} d\lambda. \quad (54) \end{aligned}$$

It is well known that the Fourier transform $\hat{G}(\lambda, x)$ of the fundamental solution $\mathbf{G}(t, x)$ is, in fact, the fundamental solution of the Helmholtz equation, and that it can be expressed through the Hankel function $h_0^d(\lambda|x|)$:

$$\hat{G}(\lambda, x) = c(d) h_0^d(\lambda|x|),$$

where $c(d)$ is a known constant depending on dimension d. With this in mind, one sees that the integrals

$$\int\limits_S Y_l^{\mathbf{m}}(\theta)\hat{G}(\lambda, x-\theta)d\theta \text{ and } \int\limits_S Y_l^{\mathbf{m}}(\theta)\frac{\partial}{\partial n(\theta)}\hat{G}(\lambda, x-\theta)d\theta$$

represent, respectively, a single-layer and a double-layer potentials for the Helmholtz equation, with densities $Y_l^{\mathbf{m}}(\theta)$ supported on the unit sphere S. It is known (see, e.g., Appendix A in [24]) that the single layer potential has the following representation:

$$\int\limits_S Y_l^{\mathbf{m}}(\theta)\hat{G}(\lambda, x-\theta)d\theta = {}^{(s)}c_l^{\mathbf{m}}(\lambda)Y_l^{\mathbf{m}}(\hat{x})h_l^d(\lambda|x|), \qquad x = \hat{x}|x|, \qquad |x| \geq 1, \quad (55)$$

where ${}^{(s)}c_l^{\mathbf{m}}(\lambda)$ are certain known factors. Using a similar reasoning as in [24], for double layer potentials one obtains

$$\int\limits_S Y_l^{\mathbf{m}}(\theta)\frac{\partial}{\partial n(\theta)}\hat{G}(\lambda, x-\theta)d\theta = {}^{(d)}c_l^{\mathbf{m}}(\lambda)Y_l^{\mathbf{m}}(\hat{x})h_l^d(\lambda|x|), \qquad x = \hat{x}|x|, \qquad |x| \geq 1, \quad (56)$$

where ${}^{(d)}c_l^{\mathbf{m}}(\lambda)$ are also known factors.

By substituting (55) and (56) into (54) one sees that the solution $v^{(b)}(t, x)$ can be expressed as the following series

$$v^{(b)}(t, x) = \frac{1}{2\pi}\int\limits_{\mathbb{R}} \left[\sum_{l,\mathbf{m}} \hat{\psi}_l^{\mathbf{m}}(\lambda)h_l^d(\lambda|x|)Y_l^{\mathbf{m}}(\hat{x})\right] e^{-i\lambda t}dt, \quad (57)$$

where coefficients $\hat{\psi}_l^{\mathbf{m}}(\lambda)$ need to be found (since the normal derivative $\frac{\partial v^{(b)}}{\partial n}$ is not known *a priori*). These coefficients can be determined by utilizing the boundary condition

$$v^{(b)}(t, \theta) = b(t, \theta) = \frac{1}{2\pi}\int\limits_{\mathbb{R}} \hat{b}(\lambda, \theta)e^{-i\lambda t}dt = \frac{1}{2\pi}\int\limits_{\mathbb{R}} \left[\sum_{l,\mathbf{m}} \hat{b}_l^{\mathbf{m}}(\lambda)Y_l^{\mathbf{m}}(\theta)\right] e^{-i\lambda t}dt, \quad (58)$$

where we used equations (32) and (33).

Further, by substituting θ for x in (57) we can see that

$$v^{(b)}(t, \theta) = \frac{1}{2\pi}\int\limits_{\mathbb{R}} \left[\sum_{l,\mathbf{m}} \hat{\psi}_l^{\mathbf{m}}(\lambda)h_l^d(\lambda)Y_l^{\mathbf{m}}(\theta)\right] e^{-i\lambda t}dt. \quad (59)$$

Now, by comparing equations (58) and (59) and using the orthogonality of the spherical harmonics one can find coefficients $\hat{\psi}_l^{\mathbf{m}}(\lambda)$:

$$\hat{\psi}_l^{\mathbf{m}}(\lambda)h_l^d(\lambda) = \hat{b}_l^{\mathbf{m}}(\lambda) \text{ or } \hat{\psi}_l^{\mathbf{m}}(\lambda) = \frac{\hat{b}_l^{\mathbf{m}}(\lambda)}{h_l^d(\lambda)}. \quad (60)$$

Note that functions $h_l^d(\lambda)$ do not have zeroes for real values of λ, thus justifying the division in the above formula.

Finally, substitution of (60) into (57) results in

$$v^{(b)}(t,x) = \frac{1}{2\pi}\int\limits_{\mathbb{R}} \left[\sum_{l,\mathbf{m}} \hat{b}_l^{\mathbf{m}}(\lambda)\frac{h_l^d(\lambda|x|)}{h_l^d(\lambda)}Y_l^{\mathbf{m}}(\hat{x})\right] e^{-i\lambda t}dt, \tag{61}$$

which coincides with formula (38). Note that, due to the smoothness of the function $b(t,\theta)$ the series (61) converges fast (faster than any negative power of $|\mathbf{m}|$). This justifies the interchanges of order of integrations and summations in the above formulas.

Appendix B

We would like to prove equation (42). Equivalently, we need to show that, for $t > 0$,

$$\int\limits_{\Pi(\omega,p)} \mathbf{G}_t(t,x-y)dx = \frac{1}{2}\delta(\omega\cdot y - p - t) + \frac{1}{2}\delta(\omega\cdot y - p + t). \tag{62}$$

For a fixed value of $t > 0$, let us evaluate the action of the distributions contained in (62) on a test function $\varphi(x) \in C_0^\infty(\mathbb{R}^d)$:

$$\left\langle \varphi, \int\limits_{\Pi(\omega,p)} \mathbf{G}_t(t,x-y)dx \right\rangle = \int\limits_{\mathbb{R}^d} \varphi(y)\left[\int\limits_{\Pi(\omega,p)} \mathbf{G}_t(t,x-y)dx\right]dy = \int\limits_{\Pi(\omega,p)} s(t,x)dx,$$

where we interchanged the order of integrations and end where $s(t,x)$ is defined as follows

$$s(t,x) := \int\limits_{\mathbb{R}^d} \varphi(y)\mathbf{G}_t(t,x-y)dy.$$

Since the fundamental solution $\mathbf{G}_t(t,x)$ depends only on $|x|$ (and t), function $s(t,x)$ is a solution to the following Cauchy problem in the whole space:

$$\begin{aligned} s_{tt}(t,x) &= \Delta s(t,x), \quad (t,x) \in (0,\infty)\times\mathbb{R}^d, \\ s(0,x) &= \varphi(x), \qquad s_t(0,x) = 0. \end{aligned}$$

Consider now the Radon transform $S(t,\omega,p)$ of $s(t,x)$:

$$S(t,\omega,p) = \int\limits_{\Pi(\omega,p)} s(t,x)dx.$$

As was explained previously, for a fixed ω, function $S(t,\omega,p)$ is the solution of the wave equation in t and p given by the Kirchhoff formula

$$S(t,\omega,p) = \frac{1}{2}\left(S_0(p-t) + S_0(p+t)\right)$$

with

$$S_0(\omega,p) = S(0,\omega,p) = \int\limits_{\Pi(\omega,p)} s(0,y)dy = \int\limits_{\Pi(\omega,p)} \varphi(y)dy.$$

Therefore,

$$
\begin{aligned}
S(t,\omega,p) = & \left\langle \varphi, \int\limits_{\Pi(\omega,p)} \mathbf{G}_t(t,x-y)dx \right\rangle \\
&= \frac{1}{2}\left(\int\limits_{\Pi(\omega,p+t)} \varphi(y)dy + \int\limits_{\Pi(\omega,p-t)} \varphi(y)dy \right) \\
&= \frac{1}{2}\left(\int\limits_{\mathbb{R}^d} \varphi(x)[\delta(\omega\cdot y-p-t)+\delta(\omega\cdot y-p+t)]dy \right),
\end{aligned}
$$

which proves equation (62) and, thus, equation (42).

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