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ON BRUNOVSKY NUMBERS AND OBSERVABILITY AND CONTROLLABILITY INDICES IN NONLINEAR MIMO SYSTEMS*

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Abstract. When the exact linearization problem is solvable for nonlinear multi-input multi-output systems, it is possible to conduct the linearization in two standard ways. The first way employs a sequence of integrable distributions defined by the vector fields involved in the system. The second way uses the output functions and the codistributions defined as the kernels of codistributions. In both cases, one ends up with a change of coordinates, which transforms the system to canonical block forms, where the sizes of the blocks are invariants of the system. One can associate two sets of indices, canonical invariants of the system, called *Brunovsky indices*. This work compares these two sets of invariants obtained in a system of dimension n. The indices are classically called the Brunovsky controllability indices and Brunovsky observability indices. We prove that the two sets of invariants give transpose partitions of n. That is, if $(h_1 \geq \cdots \geq h_N)$ are the controllability indices and $(h'_1 \geq \cdots \geq h'_M)$ are observability indices of the same nonlinear system, then the two partitions $n = h_1 + h_2 + \cdots + h_N = h'_1 + \cdots + h'_{M-1} + h'_M$ are transposed to each other. In other words, the sizes of blocks that appear in the above two canonical forms are not only in general identical, but they may also have a different number of blocks. Therefore, they generally determine two different partitions of the dimension of the system. In addition, we present several conditions that characterize the Brunovsky canonical forms, in both the controllable and observable cases, and prove their mutual equivalence. We also discuss a relevant duality between the flag varieties parametrizing the Brunovsky systems of the same type, i.e., with the same Brunovsky numbers. The duality says that the parabolic algebras associated to the two flag varieties parametrizing systems with the given Brunovsky indices $(h_1, \dots, h_N)$ and $(h'_1, \dots, h'_M)$ are dual parabolic algebras.

Key words. Brunovsky canonical form, Brunovsky numbers, observability indices, controllability indices, feedback linearization

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1. Introduction.

Assuming the exact linearization problem is solvable for a multi-input multi-output (MIMO) system, there are two canonical ways to linearize it. One uses only the integrability condition on a sequence of distributions defined by the vector fields involved in the differential equation. At the same time, the second employs differential forms defined by the system's outputs [25]; see also [6, 4, 24, 31, 32, 33, 34, 36, 38, 40]. One can arrange the coordinates in each method to obtain a unique canonical form. However, the two canonical block forms obtained at the end are not identical. Although both are linearizations of the same systems, the block forms that appear in the linearizations may have different sizes. Nevertheless, the two linear forms of the MIMO system are canonical, and the blocks' sizes (see (1.9) and (1.10) below) appearing, in that case, are called *Brunovsky indices.* The two sets of Brunovsky canonical forms of a linearizable MIMO system are classically known in the theoretical literature. Each Brunovsky form determines a set of natural numbers that

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partition the system's dimensions. However, to our knowledge, nobody has compared the two sets of invariants.

The problem of comparing the controllability and observability of Brunovsky indices is interesting even in the simplest case of a linear system. Consider the linear system

$$\dot{x} = Ax + Bu, \qquad y = Cx, \tag{1.1}$$

where $x \in \mathbb{R}^n,\ A \in M_{n\times n},\ B \in M_{n\times m},\ u \in \mathbb{R}^m$, and $C = \begin{bmatrix} c_1 & c_2 & \dots & c_m & 0\dots \end{bmatrix}$. Define the distributions

$$\mathfrak{g}_k = \text{Image}[B \quad AB \quad A^2B \quad \dots \quad A^kB] \tag{1.2}$$

as the span of the *columns* of all the matrices in the bracket. Then, we set $h_j = \text{rank}(\mathfrak{g}_{j-1}) - \text{rank}(\mathfrak{g}_{j-2})$. One can also define the codistributions

$$\Omega_k = \left\langle\ C,\ CA,\ \dots,\ \ CA^{k-1}\ \right\rangle_{\mathbb{R}} \tag{1.3}$$

as the span of the *rows* of all matrices. Then, we set $h'_k = \text{rank } \Omega_k$. The collection of numbers $\{h_i\}$ are called Brunovsky controllability indices, while the numbers $\{h'_j\}$ are called Brunovsky observability indices; see [25]. Let us assume the ranks of the distributions $\mathfrak{g}_k$ and Ω_k fill out the total dimension of the system, i.e., n for large k. In this case, both collections give a partition of n, the dimension of the system. Examples show that the above two sets of indices are quite different and nontrivially related; see Example 4.9. The controllability and observability Brunovsky indices can also be defined for nonlinear systems (see section 1.1). We wish to be able to compare these two sets of indices when the exact linearization problem is solvable for the system. The significance of the relationship between the two sets of indices appearing in the above two feedback problems encouraged us to write this work.

Certain geometric conditions characterize the exact linearization property for a control system [25]. In [27], two sets of conditions, namely **Conditions A** and **B** (see subsection 2.2), are presented, which characterize when a control system is equivalent to a Brunovsky canonical form. In this text, we present the third set of conditions, namely **Conditions C**, that characterize the Brunovsky observability form. We prove that the above sets of conditions are mutually equivalent. We also generalize existing results in nonlinear systems, which appear in our contributions.

1.1. Setting and context. We shall consider a nonlinear system as follows:

$$\dot{x} = f(x) + \sum_{j=1}^{r} u_j g_j(x), \qquad x \in \mathbb{R}^n, \tag{1.4}$$

where $f, g_1, \dots, g_r$ are smooth vector fields on an open subset of $\mathbb{R}^n$ and $u_j : \mathbb{R}^n \to \mathbb{R}$ are input control functions. This text assumes u_j is smooth on an open subset of $\mathbb{R}^n$. However, these functions can generally be considered more general (see [25], for instance). The classical control provides a systematic approach to determine when the system (1.4) is equivalent to a linear form [25, 27]; see subsection 2.2. In this case, one says that the exact linearization problem is solvable (or the system is controllable). Then the exact linearization problem provides the following form of system (1.4):

$$\dot{z}_1^{(j)} = z_2^{(j)},\ \ \dot{z}_2^{(j)} = z_3^{(j)},\ \ \dots,\ \dot{z}_{n_j}^{(j)} = v^{(j)}, \qquad j = 1, \dots, N, \quad N \geq 1, \tag{1.5}$$

where $z_i^{(j)}$ are the new coordinates of the system for some natural numbers $n_1, \ldots, n_h$. The latter form plays a crucial role in controls. For example, when the system (1.4) can be written in the form (1.5), we say it is controllable, i.e., there exist a change of coordinates and also a choice of the input functions that transforms it into a particular block diagonal linear form. The system's total dimension is divided into several smaller loops by possibly permuting the new coordinates. After obtaining the form (1.5), one defines new vector coordinates

$$\begin{aligned} y_1 &:= (z_1^{(1)}, \ldots, z_1^{(h)}), \\ \dot{y}_j &= y_{j+1}, \qquad j = 1, 2, \ldots, \end{aligned} \tag{1.6}$$

by grouping the variables. Differentiating with respect to t provides a sequence of blocks with the same entries as in (1.5) but possibly permuting the coordinates. As a result, the system can be written in a simple block form. The latter form (1.6) of the system (1.4) is canonical, i.e., it is unique. In particular, the block matrices that appear in the canonical form (1.6) are invariants of (1.4) called *the Brunovsky controllability indices.* We denote them by $(h_1, \ldots, h_N)$. We have $n = h_1 + \cdots + h_N$.

The system (1.4) can also be studied as a MIMO system with outputs

$$\begin{aligned} \dot{x} &= f(x) + \sum_{j=1}^{r} u_j g_j(x), \qquad & x \in \mathbb{R}^n, \\ y &= \lambda(x), & y \in \mathbb{R}^r, \end{aligned} \tag{1.7}$$

and considering the same assumption as in (1.4) for the functions f and u_j, where the functions $\lambda_j : \mathbb{R}^n \to \mathbb{R}$ are the system's outputs. The exact linearization problem for (1.7) is also referred to as *the observability problem* (see [25, Chap. 5]). In general, one can define a certain set of indices called relative degrees associated to (1.7) (see subsection 2.3 and Chapter 5 of [25]). In the case that the observability problem is solvable for (1.7) the relative degrees determine a form of (1.7) in the shape of (1.5). Again by grouping the variables as (1.6) we reach a canonical form, called the Brunovsky canonical form of (1.7), and the sizes of the blocks determine a second set of indices $(h'_1, \ldots, h'_M)$ which are also invariants of the MIMO system (1.7). However, this canonical form and the associated indices are in general different from the former canonical form for the control system (1.4) and the controllability indices.

On the other hand, notice that Brunovsky indices always appear in nonincreasing order. If $\underline{h} = (h_1, \ldots, h_n)$ and $\underline{h}' = (h'_1, \ldots, h'_n)$ are, respectively, the Brunovsky column and row indices of a linearizable control system, then there are positive integers N and M, such that $h_1 \geq \cdots \geq h_N > 0 = h_{N+1} = \cdots = h_n = 0$, $h'_1 \geq \cdots \geq h'_M > 0 = h'_{M+1} = \cdots = h'_n = 0$ and, as mentioned above, $h_1 + \cdots + h_n = h'_1 + \cdots + h'_n = n$.

This article aims to compare the observability indices with the controllable ones. Both sets of Brunovsky indices are partitions of the dimension n. Moreover, we show that these two partitions of n are transposed to each other. The result can be regarded as an effort to understand the above-mentioned invariants.

1.2. Contribution. To summarize and briefly explain the paper's contributions, let us consider system (1.4) with the setup explained above and three fundamental conditions mentioned next that involve the contributions. The vector fields f and g_j and their brackets determine a sequence of integrable distributions $\mathfrak{g}_0 = \langle g_1, \ldots, g_r \rangle_{\mathbb{R}}$, $\mathfrak{g}_k = \langle [f + \mathfrak{g}_0, \mathfrak{g}_{k-1}] \rangle_{\mathbb{R}}$, $k \geq 1$, in an inductive way whose dimensions are constant on an open subset of the ambient space (see subsection 2.2 for a precise definition). Moreover, $\dim \mathfrak{g}_N = n$ for some N. The integrability conditions on the distributions $\mathfrak{g}_k$ provide a standard method to determine certain conditions under which the system (1.4) is transformable into a Brunovsky canonical form. Such conditions are as follows.

Conditions A:

(A1) The submodules $\mathfrak{g}_k$ are closed under bracket operation of vector fields.

(A2) The numbers $b_j(x) := \dim \mathfrak{g}_j(x)$ are constant, i.e., they are independent of x.

(A3) $b_N(x) = n$ for some N, where N is the smallest such number.

According to [27], the set of these conditions can be equivalently replaced by the following **Conditions B**.

Conditions B:

(B1) There are smooth functions a_{ij} such that for each $i \leq j$

$$[ad^i(f)g_r, ad^j(f)g_s] = \sum_{1\leq i\leq r} \sum_{l\leq j} a_{il} ad^l(f) g_i. \tag{1.8}$$

(B2) The numbers $\dim \operatorname{span}\{ad^j(f)g_i(x) \mid j \leq k\} = b'_k(x)$ are constant (independent of x).

(B3) $\dim \operatorname{span}\{ad^j(f)g_i(x) \mid j \leq n-1\} = n$.

By conditions (A2) and (B2) we have assumed that the numbers $b_j(x)$ and $b'_j(x)$ are constant and independent of a special point x. Thus, we denote them as b_j and b'_j in the following. We refer the interested reader to [25] for a complete discussion of the properties of the numbers $b_j(x)$ and $b'_j(x)$.

The linearization problem of (1.7) can also be approached by defining the sequence of codistributions $\Omega_0 = \langle d\lambda_1, \ldots, d\lambda_r\rangle_{\mathbb{R}}$, $\Omega_k = \langle \Omega_{k-1} + \sum_j L_{g_j}\Omega_{k-1} + L_f\Omega_{k-1}\rangle_{\mathbb{R}}$. We present a new and third set of conditions, namely **Conditions C**.

Conditions C:

(C1) The codistributions Ω_j have constant dimensions, $j = 1, 2, \ldots, M$.

(C2) Ω_M has dimension n, where the number M is the smallest number with this property.

(C3) For each j the codistribution Ω_j is invariant under f, g_l for all $l \leq r$ (i.e., $L_f\Omega_j,\ L_{g_l}\Omega_j \subset \Omega_j$).

That determines whether the system (1.7) can be transformed into a Brunovsky canonical form (see section 2.3 on how these conditions can be checked).

The set of **Conditions C** employs the sequence of codistributions defined by the output function λ. In other words, **Conditions C** describe the observability indices analogous to **Conditions A, B**, which describe the controllability indices. We show that the three sets of **Conditions A, B,** and **C** are mutually equivalent. The theorem and its proof are our first contribution (Theorem 4.1 below). It also provides a method of how the controllability and observability indices are related. It follows that the MIMO system (1.7) can be linearized in two canonical ways.

Based on the above observation, assume the sets of **Conditions A, B, C** are satisfied. In particular, the two systems (1.4) and (1.7) are completely linearizable. There are two standard approaches to linearizing the two systems above. In the first case, there exist a change of coordinates and a choice of the input control functions u_j such that the differential equation (1.4) gets transformed into the Brunovsky canonical controllable form,

$$\dot{x} = Ax + Bu, \qquad A = \begin{pmatrix} 0 & E_1 & \dots & 0 & 0 \\ 0 & 0 & E_2 \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & & \\ 0 & 0 & \dots & 0 & E_{N-1} \\ 0 & 0 & \dots. & 0 & 0 \end{pmatrix}_{n\times n}, \ B = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \dots \\ E_N \end{pmatrix}_{n\times m}, \tag{1.9}$$
$$u = \begin{bmatrix} u_1 & u_2 & \dots & u_r \end{bmatrix}^T,$$

where $E_j = [I_{h_j} \ \ 0]^T$, $j = 1, 2, \dots, N$. In the second case, the system (1.7) is equivalent to the canonical form,

$$\dot{x} = A'x + B'u, \qquad A' = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 \\ E_2' & 0 & \dots & 0 & 0 \\ 0 & E_3' & \dots & 0 & 0 \\ \dots & \dots & \dots & & \\ 0 & 0 & \dots & E_M' & 0 \end{pmatrix}, \ B' = \begin{pmatrix} E_1' \\ 0 \\ 0 \\ \dots \\ 0 \end{pmatrix}, \tag{1.10}$$
$$y = Cx, \qquad C = [E_1' \ \ 0 \ \ 0 \ \ \dots \ 0], \qquad u = \begin{bmatrix} u_1 & u_2 & \dots & u_r \end{bmatrix}^T,$$

where $E_j' = [I_{h_i'} \ \ 0]$, $j = 1, 2, \dots, M$. As explained, we obtain two Brunovsky canonical forms of the same system. The two set of indices $(h_1, \dots, h_N)$ and $(h_1', \dots, h_M')$ are invariants of the systems (1.4) and (1.7). The observability indices of the system (1.7), due to the choice of outputs $\lambda_1, \dots, \lambda_r$, give a transpose partition of n to the case of Brunovsky controllability indices. In other words, the two partitions

$$n = h_1 + h_2 + \dots + h_N = h_1' + \dots + h_{M-1}' + h_M' \tag{1.11}$$

are transpose to each other; see Definition 4.3.

The paper has the following main contributions. The first involves the proof of the equivalence of the set of **Conditions C** with the sets of **Conditions A, B** mentioned above. This appears in Theorem 4.1 in section 4. The second result is the proof of (1.11), which appears in Theorem 4.4. We also check this result with an example. In addition, another contribution of this paper is the transpose duality between the Brunovsky indices of (1.4), i.e., (1.11). It has direct applications in studying the geometric properties of (1.4) and especially its symmetries.

We present a series of examples for the aforementioned duality (1.11) arising in a pencil of linear controllable (resp., observable) equations over the projective line,

$$S(\lambda) = \begin{bmatrix} A & B \\ C & 0 \end{bmatrix} - \lambda \begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix}. \tag{1.12}$$

Following [14], a change of coordinates exists that transforms the total pencil in a Kronecker canonical form; see Example 4.10. However, some extra blocks appear according to the existence of Jordan blocks at the generalized eigenvalues. We shall call the triple (A, B, C) a Kronecker triple and the pencil (1.12) a Kronecker system. A series of invariant characteristic indices can be associated with a Kronecker triple. A complete set of indices structurally associated with the pencil (1.12) is presented in [14]. When $\lambda \in \mathbb{C}$ is generic (see Example 4.10), these indices fall in our above classification controllability and observability indices. In this case, our result Theorem 4.4 applies to compare the corresponding indices. Also, we discuss that the duality of

(1.11) *can be mutually applied to the most generic locus on the pencil.* It follows that the transpose duality (1.11) holds for a family of linear equations in the most generic cases.

We define several aspects for the moduli (classifying) space $\mathcal{D}$ of Kronecker systems with the same Kronecker indices; see [10, 14]. These are flag manifolds with stratified boundaries. The Lie group $G = GL_n\mathbb{C} \times GL_m\mathbb{C} \times GL_p\mathbb{C}$ acts transitively on the domain $\mathcal{D}$ identifying this moduli space as conjugacy classes of the stabilizer of a fixed point. We define two new notions, namely the Kronecker (Brunovsky) domain and Kronecker (Brunovsky) group, associated with a feedback system and its observer; see section 5. The moduli of the controllable pair (A, B) and the observable pair (A, C) can be identified as open subsets of $\mathcal{D}$. Our result expresses that an involutive map on $\mathcal{D}$ transforms these two subsets into one another. Thus, we obtain a duality between the corresponding flag varieties, parametrizing all linear pencils of the same Kronecker (Brunovsky) numbers.

1.3. Comparison with recent literature. The most natural question on a nonlinear control system is when a local (or global) change of coordinates exists that carries the given nonlinear system into a linear one. Krener [30] showed the importance of the Lie algebra of vector fields associated with the system in studying such a question and answered this problem. Jakubczyk and Respondek [27] and Isidori and Ruberti [26] found necessary and sufficient conditions, together with a constructive procedure, for linearization of the input-output response. A survey of linearization problems can be found in [39]; see [11]. In [8] conducted by Brunovsky, the indices were first identified as a complete set of invariants of the orbits of controllability pairs under state-coordinate, input-coordinate, and feedback transformations. Brunovsky calls such indices the *controllability indices.* Rosenbrock and MacFarlane [41] and Kalman [28] show the connection between controllability indices and Kronecker indices of a singular matrix pencil. In [35], a comparative study of the different definitions of controllability indices has been done based on the idea of minimal bases of polynomial modules. That paper discusses the connection between degree-preserving module isomorphisms and minimal bases of rational vector spaces. The authors show that the problem of finding such indices can be reduced to determining degrees of minimal basis vectors of a special submodule. In addition, they establish the equality of controllability indices and minimal indices. The result of [35] mainly involves defining controllability indices in several ways and their compatibility. In [20], the authors develop functional techniques applicable to studying the classes of singular observers. They study the geometric properties of observability subspaces. In [29], an approach via the truncations of the Hankel matrix is presented, which defines the controllability indices and observability ones.

The transpose duality we prove in this article is precise and concrete in relation to Brunovsky indices. Although these indices can be defined using different methods, the specific relation between the controllable and observer indices has yet to be investigated in the above works. The insight toward comparing the indices on the two sides remains to be determined. A different approach is taken in [2] based on the realization of a set $L = (L_1, \ldots, L_n)$ of $m \times n$ real or complex matrices which are realizable by systems of minimal order (see the reference for the exact definition). The authors associate to L its partial Kronecker column indices and its partial Kronecker row indices. The sum of the numbers in any of these two sequences is the same. Moreover, the sum equals the order of the minimal realizations of L. The authors indicate the existence of a duality between the row and column indices; however, our result in the current paper is still new compared to the result and method in [2]. This text compares the two ways of linearization that apply to a MIMO system. The main result of this paper concerns a dimensional duality related to the classification of MIMO control systems. With that aim, we analyze a transpose duality that affects Brunovsky indices associated with linearizable MIMO systems and is useful in studying their geometric properties.

We shall consider an example of linear pencils of differential equations with observer; cf. [14]. We present a series of examples appearing from a pencil of linear equations. The setup may be compared with the texts [14, 12, 13, 42]. Then, the duality (1.11) can be stated for the Brunovsky numbers of the linear system on the most generic locus of the pencil. The pencil of linear controllable (resp., observable) systems leads to studying the classifying spaces of Kronecker systems with similar Kronecker indices. In section 5 we discuss and show that the above duality (1.11) is related to a duality on the most generic locus of the classifying spaces. Then, we provide a duality between the corresponding flag varieties parametrizing systems with the same Brunovsky numbers.

1.4. Structure of the paper. The remainder of this paper is given as follows. In subsection 2.2, we define controllability indices for a feedback linearizable system. Subsection 2.3 defines the dual concept of observability indices for the same system with specific outputs. We had stated the proofs when the exact result did not exist in the literature. The problem statement is presented in section 3. The contributions are given in section 4. Specifically, Theorem 4.1 is the new contribution, and Proposition 4.2 has a partial contribution. In section 4, we express and prove the main result stated in Theorem 4.1 (see also Theorem 4.4). We also give some complete examples. In section 5, we provide several new definitions and results related to our main result. We define the Kronecker groups and domains and prove a duality theorem on equivalent linear pencils' classifying (moduli) space. Finally, in section 6 we give some final remarks and conclusions.

2. Preliminaries. Throughout the text, the vector fields f, g_j and the input functions are assumed to be defined over $\mathbb{R}^n$. So, let us assume that our system is defined on an open subset of $\mathbb{R}^n$, the intersection of the domains where all these functions are well defined.

2.1. Notations. This article considers all the differential systems over the real numbers denoted $\mathbb{R}$. We use German letters such as $\mathfrak{g}$ for the Lie algebra of vector fields on a real manifold. Letters f, g, and X denote vector fields. The Lie algebra and the vector fields are defined over $\mathbb{R}$. Capital letters such as A, B, and C are used for matrices. The notation $L_f h$ denotes the covariant derivative of the function h along the flows of the vector field f. All manifolds are assumed to be open subsets of $\mathbb{R}^n$ for some n.

2.2. Controllability indices. We shall use the language of distributions on a differentiable manifold to analyze the system in question. A distribution is identified by a set of vector fields on a smooth manifold, denoted by $\mathfrak{g} = \langle X_1, \ldots, X_r \rangle_{\mathbb{R}}$, where the angles mean the local span of the vector fields in $\mathbb{R}^n$. We say the distribution is smooth if the vector fields X_j defining it are smooth vector fields. The dimension of a distribution at a point x is the dimension of the span of the vectors $X_1(x), \ldots, X_r(x)$, regarded as vectors in $\mathbb{R}^n$. The point x is called a nonsingular point for the distribution $\mathfrak{g}$ if in a neighborhood of x, the dimension is constant and equal to r, the number of vector fields. The operations on the vector fields on a differentiable manifold naturally extend over the distributions; see [25]. An important notion is that of involutiveness of a distribution, meaning that the bracket of any two vector fields in the distribution is still a member of the distribution. We sometimes refer to this notion by saying the distribution is closed under a bracket. By Frobenius theorem, this is equivalent to the distribution being integrable, meaning that the system of differential equations defined by the vector fields in the distribution is locally solvable by integral foliations; see [25].

Consider the nonlinear control system of the form (1.4). One applies three kinds of coordinate changes to the above system:

(1) The first is the usual change of coordinates, i.e., the diffeomorphisms

$$\phi : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0), \tag{2.1}$$

which transforms the differential equation (1.4) by $f \longmapsto D\phi(f)$, $g_j \longmapsto D\phi(g_j)$ $(j \geq 1)$.

(2) The second is coordinate changes in the input space, $\mathbb{R}^r$, which are nonlinear in x. That is, transformations of the form

$$g_j \longmapsto \sum_l a_{jl} g_l, \qquad j = 1, 2, \ldots, r, \tag{2.2}$$

where the matrix $A(x) = (a_{ij}(x))$ is smooth.

(3) In control systems, one studies the system's linearization of (1.4) under a specific choice of the functions u_j, called feedback control. Thus, a third kind of transformation is applicable:

$$f \longmapsto f + \sum_j \beta_j g_j, \qquad \beta_j : \mathbb{R}^n \to \mathbb{R},\ \beta_j(0) = 0. \tag{2.3}$$

We denote the group generated by the above three transformations by G. We say the system (1.4) is G-linearizable or G-equivalent to a controllable system if a sequence of transformations in G transforms it to a bilinear system.

On the other hand, let us denote the Lie algebra of smooth vector fields on $\mathbb{R}^n$ by $\mathfrak{X}^\infty(\mathbb{R}^n)$. Define the submodules of $\mathfrak{X}^\infty(\mathbb{R}^n)$,

$$\mathfrak{g}_0 = \langle g_1, \ldots, g_r \rangle_{\mathbb{R}}, \qquad \mathfrak{g}_k = \langle [f + \mathfrak{g}_0, \mathfrak{g}_{k-1}] \rangle_{\mathbb{R}}, \tag{2.4}$$

generated over $C^\infty(\mathbb{R}^n)$ associated to the system (1.4), and let

$$b_j(x) = \dim \mathfrak{g}_j(x), \tag{2.5}$$

where we use the language of distributions on manifolds. In particular, we denote $\mathfrak{g}_k(x) = \langle X(x), X \in \mathfrak{g}_k \rangle_{\mathbb{R}} \subset T_x\mathbb{R}^n$. The submodules $\mathfrak{g}_k$, $k \geq 0$, are invariant under the action of the group of transformation G, or sometimes called G-invariant.

Conditions A:

(A1) The submodules $\mathfrak{g}_k$ are closed under bracket operation of vector fields.
(A2) The numbers $b_j(x)$ are constant.
(A3) $b_N(x) = n$ for some N, where N is the smallest such number.

Because the Lie bracketing is invariant under the G-action, the above conditions are independent of the aforementioned coordinate changes. **Conditions A** are invariant under the group of transformations generated by the elements in (1)–(3) mentioned earlier in this subsection. According to [27], these conditions can be equivalently replaced by the following **Conditions B**.

Conditions B:

(B1) There are smooth functions a_{ij} such that for each $i \leq j$

$$[ad^i(f)g_r, ad^j(f)g_s] = \sum_{1\leq i\leq r}\sum_{l\leq j} a_{il} ad^l(f)g_i. \tag{2.6}$$

(B2) The numbers $\dim\operatorname{span}\{ad^j(f)g_i(x) \mid j \leq k\} = b'_k(x)$ are constant.

(B3) $\dim\operatorname{span}\{ad^j(f)g_i(x) \mid j \leq n-1\} = n$.

The number N is the smallest natural number such that $b_N(x) = n$. Its existence is guaranteed by condition (A3). One easily sees that $b_N(x) \geq b_{N-1}(x) \geq \cdots \geq b_0$, and similarly $b'_N(x) \geq b'_{N-1}(x) \geq \cdots \geq b'_0$. By condition (B1) we have $b_j(x) = b'_j(x)$. We may also define the invariants

$$h_0(x) = b_0(x), \qquad h_j(x) = b_j(x) - b_{j-1}(x) = b'_j(x) - b'_{j-1}(x). \tag{2.7}$$

It is easy to see that $h_1(x) \geq h_2(x) \geq \cdots$ (cf. [27]). The set of **Conditions A** or its equivalent set B are closely related to the linearization problem for (1.4) in control systems. In this regard, we have the following important definition.

DEFINITION 2.1 (Brunovsky canonical form [27]). *Assume that the numbers $b_j(x)$ are independent of x (as they is the same for $b'_j(x)$ or $h'_j(x)$). Write the coordinate $x = (x_0, \ldots, x_N)$ such that $\dim x_j = h_j$. A system of the form (1.4) such that*

$$f = \begin{bmatrix} 0 \\ \cdots \\ 0 \\ \tilde{x}_1 \\ \tilde{x}_2 \\ \cdots \\ \tilde{x}_{N-1} \end{bmatrix},\ g_1 = \begin{bmatrix} 1 \\ 0 \\ \cdots \\ 0 \end{bmatrix},\ g_2 = \begin{bmatrix} 0 \\ 1 \\ \cdots \\ 0 \end{bmatrix},\ \ldots,\ g_{b_0} = \begin{bmatrix} 0 \\ 0 \\ \cdots \\ 1 \\ \cdots \\ 0 \end{bmatrix},\ g_j = \begin{bmatrix} 0 \\ 0 \\ \cdots \\ 0 \end{bmatrix},\ j > b_0, \tag{2.8}$$

where the 1 in g_{b_0} is in the b_0 place, is called a system in Brunovsky canonical form.

The following theorem relates the sets of **Conditions A** and **B** to the exact linearization problem of the system (1.4) and the Brunovsky canonical form.

THEOREM 2.2 (see [27]). *The following conditions are equivalent locally near $0 \in \mathbb{R}^n$.*

- *The system (1.4) is G-equivalent to a Brunovsky canonical form system.*
- *The set of **Conditions A** is satisfied.*
- *The set of **Conditions B** is satisfied.*
- *The system (1.4) is G-linearizable to a controllable system.*

Proof. The proof of the above theorem is based on the following. If a sequence of distributions (defined over $\mathbb{R}$)

$$\mathfrak{g}_0 \subset \mathfrak{g}_1 \subset \cdots \subset \mathfrak{g}_N \tag{2.9}$$

on a manifold M of dimension n have constant dimensions $b_0 \leq b_1 \leq \cdots \leq b_N$, then there exists a coordinate system $(x_0, \ldots, x_n)$ on M such that the integral manifolds of $\mathfrak{g}_j$ are of the form

$$x_j = C_j, \qquad j = h_j + 1, \ldots, n, \quad C_j \text{ constant.} \tag{2.10}$$

□

DEFINITION 2.3 (Brunovsky controllability indices [27, 9, 22]). *The numbers* h_j *defined in* (2.7) *(or the same in Definition* 2.1*) are called Brunovsky controllability indices of the system* (1.4).

The controllability indices characterize a unique canonical block form, which is equivalent to (1.4). These indices are invariants of (1.4). We express this in the following proposition.

PROPOSITION 2.4 (see [27, 25, 9, 22, 5, 15, 30, 7]). *Assume any one of the sets of* ***Conditions*** **A** *or* **B** *is satisfied for the control system* (1.4). *Then, there exist a change of coordinates and a choice of the input control functions* u_j *such that the differential equation* (1.4) *gets transformed to the Brunovsky canonical controllable form,*

(2.11)
$$\dot{x} = Ax + Bu, \qquad A = \begin{pmatrix} 0 & E_1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & E_2 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & E_{N-1} & 0 \\ 0 & 0 & 0 & \cdots & 0 & E_N \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}_{n\times n}, \quad B = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \cdots \\ E_N \end{pmatrix}_{n\times m},$$
$$u = \begin{bmatrix} u_1 & u_2 & \ldots & u_r \end{bmatrix}^T,$$

where $E_j = [I_{h_j} \quad 0]^T,\ j = 1, 2, \ldots, N$. *The sizes of the block matrices* E_j *are the Brunovsky controllability indices.*

Proof. See the references cited in the proposition. ∎

Remark 2.5. To obtain the Brunovsky indices, it is not sufficient that the system just is linearized. It is also necessary that the system is in the Brunovsky canonical form. For example, the simple system

$$\dot{x}_1 = a_1 x_1 + b_1 u_1 \quad \ldots \quad \dot{x}_n = a_n x_n + b_n u_n \tag{2.12}$$

is not in controller canonical form. The above sort of feedback linearization is different from the one in ordinary differentiable dynamics (see Theorem 2.2). After the linearization, the system can be solved by algebraic equations for the functions u_j in such a way that it finds the form (1.5). The Brunovsky form will then be obtained by exchanging the order of variables so that the block forms will appear correctly.

Example 2.6 (linear system [25]). When the functions $f(x)$ and $g_j(x)$ in (1.4) are already given by linear matrices, the aforementioned distributions can be easily determined. Consider the system

$$\dot{x} = Ax + Bu, \qquad x \in \mathbb{R}^n,\ A \in M_{n\times n},\ B \in M_{n\times m},\ u \in \mathbb{R}^m. \tag{2.13}$$

The distributions $\mathfrak{g}_k$ get the form

$$\mathfrak{g}_k = \text{Image}[B \quad AB \quad A^2B \quad \ldots \quad A^kB] \tag{2.14}$$

meaning the span of the *columns* of all the matrices in the bracket. The matrix

$$\mathfrak{g}_n(A, B) = [B \quad AB \quad A^2B \quad \ldots \quad A^{n-1}B] \in \mathbb{C}^{n\times nm} \tag{2.15}$$

is called the controllability matrix of the pair (A,B). Thus, we have

$$h_1 = \operatorname{rank}(B), \qquad h_j = \operatorname{rank}(\mathfrak{g}_{j-1}) - \operatorname{rank}(\mathfrak{g}_{j-2}), \tag{2.16}$$

where $\mathfrak{g}_j$ is given by (2.14). The aforementioned canonical form of Brunovsky is used in order to classify the linear system (2.13) (see [9]).

DEFINITION 2.7 (see [9]). *Two pairs of matrices (A,B) and (C,D) in the form of* (2.13) *are block equivalent if there exist matrices $R_{n\times n}, S_{m\times m}, T_{m\times n}$ such that*

$$[C \quad D] = R[A \quad B]\begin{pmatrix} R^{-1} & 0 \\ T & S \end{pmatrix}, \tag{2.17}$$

where R,S are invertible.

Remark 2.8 (see [9]). Assume that A,B are constant matrices and that the system (2.13) is completely controllable, i.e.,

$$\operatorname{rank}(B, AB, \ldots, A^n B) = n. \tag{2.18}$$

We translate the preceding definition into some feedback control statements to formulate it more precisely. By adding linear feedback to (A,B), we mean that in (2.13), we substitute

$$u = Tx + v, \tag{2.19}$$

where T is an $m \times n$ constant matrix. As a result of this transformation, we obtain a system (C'', D''), with

$$C'' = A + BT, \qquad D'' = B. \tag{2.20}$$

We say (C'', D'') behaves like (C', D') if there are nonsingular matrices R and S of type $n\times n, m\times m$, respectively, such that

$$C' = R^{-1}A''R, \qquad D' = R^{-1}BS. \tag{2.21}$$

Summarizing, the definition asks whether for given systems $(A,B),(C',D')$ there are matrices $R_{m\times n}$, $T_{m\times n}$, $S_{m\times m}$ with R,S being nonsingular, such that

$$C' = R^{-1}(A+BT)R, \qquad D' = R^{-1}BS. \tag{2.22}$$

If the answer is positive, we shall say that (A,B) and (C',D') are feedback (or briefly, F-) equivalent. A straightforward computation shows that F-equivalence is an equivalence relation, i.e., it is symmetric, reflexive, and transitive.

The following proposition relates Definition (2.7) to the Brunovsky numbers.

PROPOSITION 2.9 (see [43, 37, 10, 22, 23, 3, 19]). *Two matrices are block-equivalent if and only if they have the same (controllability) Brunovsky numbers.*

Proof. See the references cited in the proposition. ∎

By considering Definition 2.7 and Proposition 2.9, we understand that the Brunovsky numbers are invariants of the bilinear control systems. In the following, we present another way to interpret the Brunovsky numbers for analytic matrices in one variable. The analysis of this section can also be expressed in terms of global analytic block similarity of a pair of matrices $(A(z), B(z))$ for bilinear systems. In case (1.4) could be defined by the matrices $A(z)$ and $B(z)$, then we have the following format of Theorem 2.2.

THEOREM 2.10 (see [22]). *Let $A : X \to \mathbb{C}^n$, $B : X \to \mathbb{C}^{n\times n}$ be analytic matrices, where X is connected and open in $\mathbb{C}$. Then the pair $(A(z), B(z))$ is (globally) block equivalent to a pair in Brunovsky canonical form*

$$[G(z) \quad J(z)] := R(z)[A(z) \quad B(z)] \begin{pmatrix} R(z)^{-1} & 0 \\ T(z) & S(z) \end{pmatrix}, \tag{2.23}$$

where P, Q are invertible if and only if the following holds:

- *The Brunovsky numbers of $(A(z), B(z))$, namely $h_1(z) \geq h_2(z) \geq \cdots \geq h_N(z)$, are independent of $z \in X$.*
- *The size and the numbers of the Jordan blocks in the Jordan part of the Brunovsky form of $(A(z), B(z))$ are independent of z.*
- *There are s different analytic eigenfunctions $\alpha_i : X \to \mathbb{C}$, where s is the number of eigenvalues of A having eigenvector in $\ker(B)$, denoted by*

  $$\sigma(A(z), B(z)) = \{\alpha_1, \ldots, \alpha_s\}, \tag{2.24}$$

  such that if

  $$Z = \{z \in X | \alpha_i(x) = \alpha_j(x)\}, \tag{2.25}$$

  then the sum of generalized eigenspace

  $$N(z) = \lim_{x \to z} (R_{\lambda_1} + \cdots + R_{\lambda_s}) \tag{2.26}$$

  is direct and $\dim N(z) = n - \sum h_i$.

Remark 2.11. Theorem 2.10 is stated for a pair of matrices whose entries are analytic functions defined on a complex plane domain. However, it is possible to modify the arguments in the proofs in [22] to work for functions of several complex variables. We have mentioned the theorem as an alternative way or approach to the Brunovsky canonical form and controllability indices. One may consider the Brunovsky canonical form as just associated with the pairs of matrices $(A(z), B(z))$ where the group of symmetries is defined as in the beginning of this section (the same as in Remark 2.8). We will not include the details of this theorem in this paper and refer the interested reader to [22] and the references therein.

2.3. Observability indices. The language of codistributions may also describe the integrability conditions for a collection of vector fields on a manifold. This notion is dual to that of distributions. This problem originally goes back to different formulations of the Frobenius theorem on a smooth manifold's integrability of differential systems. By definition, a codistribution is identified by the linear span of 1-forms on (an open subset of) a manifold, denoted $\mathfrak{g}' = \langle w_1, \ldots, w_{n-r} \rangle_{\mathbb{R}}$, ($\langle .,. \rangle$ means the span of covectors). The codistribution is called smooth if the 1-forms in a generator are smooth differential forms. The annihilators $\mathfrak{g}'^{\perp}$ of a codistribution $\mathfrak{g}'$ at a point x are the vectors in the tangent space to x that are killed by all the elements of $\mathfrak{g}'$. The dual notion of codistribution can state the Frobenius theorem's integrability criteria. We say a system of differential equations given by a distribution $\mathfrak{g}$ of dimension r is integrable if there can be found $(n - r)$ smooth functions $\lambda_1, \ldots, \lambda_{n-r}$ such that $\mathfrak{g}' := \langle d\lambda_1, \ldots, d\lambda_{n-r} \rangle_{\mathbb{R}} = \mathfrak{g}^{\perp}$ (see Chapter 1 of [25] for details on the terminology).

On the other hand, *the Brunovsky observability indices* are closely related to the notion of relative degrees for the MIMO observer systems. Thus we first recall the definition of relative degrees in this case. Relative degrees can generally be defined for any observer system with an arbitrary algebraic observer. The Brunovsky indices refer to the choice of specific observer equations so that the exact linearization problem is solvable for (1.7). We define the observability indices through a maximal set of codistributions defined by the system (1.7) itself. The two notions of relative degrees and observability indices are closely related. Now, let us consider the MIMO system (1.7). To this system, one classically assigns relative degrees as follows.

Definition 2.12 (relative degrees [25, Chap. 5]). *We say the system* (1.7) *has relative degrees* $(a_1,\dots,a_M)$ *at a point* x *if the following two conditions hold:*

- $L_{g_i}L_f^j\lambda_l = 0$ *for* $j < a_l - 1$ *in a neighborhood of* x.
- *The matrix*

$$D(x) = \begin{pmatrix} L_{g_1}L_f^{a_1-1}\lambda_1(x) & \cdots & L_{g_r}L_f^{a_1-1}\lambda_1(x) \\ L_{g_1}L_f^{a_2-1}\lambda_2(x) & \cdots & L_{g_r}L_f^{a_2-1}\lambda_2(x) \\ \vdots & \ddots & \vdots \\ L_{g_1}L_f^{a_r-1}\lambda_r(x) & \cdots & L_{g_r}L_f^{a_r-1}\lambda_r(x) \end{pmatrix} \tag{2.27}$$

is nonsingular at x.

Relative degrees play a crucial role in the analysis of the system (1.7). The following theorem provides a canonical way to compute relative degrees.

Theorem 2.13 (see [25]). *Assume that system* (1.7) *has relative degrees* $(a_1,\dots, a_M)$. *Then* $a = a_1 + a_2 + \cdots + a_M \le n$. *Define the functions*

$$\phi_j^i(x) = L_f^{j-1}h_i(x). \tag{2.28}$$

Then, it is always possible to find $n-a$ *functions* $(\phi_{a+1},\dots,\phi_n)$ *such that the change of coordinates*

$$\begin{aligned} &\phi : \mathbb{R}^n \longrightarrow \mathbb{R}^n, \\ &\phi = (\phi_1^1(x),\dots,\phi_{a_1}^1,\dots,\phi_{a+1},\dots,\phi_n) \end{aligned} \tag{2.29}$$

transforms (1.7) *to the system of the form*

$$\begin{aligned} \dot{\phi}_j^{a_l} &= \phi_{j+1}^i, \qquad j < a_i - 1,\ l = 1,2,\dots,r, \\ \dot{\phi}_{a_l-1}^{a_l} &= L_f^{a_l}\lambda_l + \sum_{s=1}^{r} L_{g_s}L_f^{a_l-1}\lambda_l u_s, \\ \dot{\phi}_j &= Z(x) = L_f\phi_j, \qquad j > n-a. \end{aligned} \tag{2.30}$$

If the distribution $\mathfrak{g}_0 = \langle g_1,\dots,g_r\rangle$ *is closed under bracket operation, then it is possible to choose* ϕ_j, $j > n-a$, *such that*

$$L_{g_i}\phi_j = 0, \qquad j > n-a,\ i = 1,\dots,r. \tag{2.31}$$

Proof. See [25, Chap. 5]. ∎

We consider the case when the inequality in Theorem 2.13 is an equality, i.e., when $a_1 + a_2 + \cdots + a_M = n$. In this case, one says that the exact linearization problem is solvable for the system (1.7). We have the following theorem.

THEOREM 2.14 (Theorem 5.2.3 in [25]). *There exist outputs* $\lambda_1, \dots, \lambda_r$ *such that*

$$a_1 + a_2 + \cdots + a_r = n \tag{2.32}$$

if and only if the following conditions are satisfied:

- *The distributions* $\mathfrak{g}_j$ *have constant dimensions,* $j = 1, 2, N-1$.
- $\mathfrak{g}_N$ *has dimension* n.
- $\mathfrak{g}_k$ *are closed under bracket.*

Proof. See [25, Thm. 5.2.3]. ∎

The set of conditions in Theorem 2.14 is the same as the set of **Conditions A** (thus, also **Conditions B**). We have to say that the indices $a_1, \dots, a_r$ are invariants of the system (1.7) when the linearization problem is solvable. However, these invariants may not be canonical in this form. In order to make the index set $\{a_i \mid 1 \le i \le r\}$ canonical, one needs to group the variables to obtain the Brunovsky form. We explain this as follows (see also the proof of Theorem 4.1). Assume we are in the situation of Theorem 2.14, i.e., the exact linearization problem is solvable for (1.7). Define the codistributions,

$$\Omega_0 = \langle d\lambda_1, \dots, d\lambda_r \rangle_{\mathbb{R}}, \qquad \Omega_k = \left\langle \Omega_{k-1} + \sum_j L_{g_j} \Omega_{k-1} + L_f \Omega_{k-1} \right\rangle_{\mathbb{R}}, \tag{2.33}$$

and denote

$$h'_0 = \operatorname{rank}(\Omega_0), \qquad h'_k = \operatorname{rank}(\Omega_k) - a'_{k-1} \tag{2.34}$$

where we have $h'_1 + h'_2 + \cdots + h'_M = \dim \Omega_M = n$. Also, we have

$$\Omega_0^\perp \supset \Omega_1^\perp \supset \cdots \supset \Omega_M^\perp. \tag{2.35}$$

In the notation at the beginning of this section, we have set $\mathfrak{g}'_{M-k} = \Omega_k^\perp$ (we use both of the notations $\mathfrak{g}'_{M-k}$ and $\Omega_k^\perp$ in the following). We have the simple relation $\operatorname{rank}(\Omega_k) + \operatorname{rank}(\Omega_k)^\perp = n$. Theorem 2.14 is the base of the following definition.

DEFINITION 2.15 (Brunovsky observability indices). *Assume that the exact observation problem is solvable for the system* (1.7)*. The indices defined by*

$$h'_1, h'_2, \dots, h'_M, \qquad h'_i = \operatorname{rank}(\Omega_i) - \operatorname{rank}(\Omega_{i-1}), \tag{2.36}$$

are called observability indices of the system (1.7) *(*$\Omega_{-1} = 0$*). We have*

$$h'_1 + h'_2 + \cdots + h'_M = n. \tag{2.37}$$

The observability indices characterize a unique block canonical form of (1.7), which plays a crucial role in control systems; see Remark 2.8. It also follows that these indices are invariants of (1.7). We mention this in the following proposition.

PROPOSITION 2.16 (see [27, 25, 9, 22, 5, 15, 30, 7]). *Assume the set of conditions in Theorem* 2.14 *is satisfied (**Conditions** **A** or **B**). Then, the system* (1.7) *is equivalent to the Brunovsky canonical observable form*

$$\begin{gathered} \dot{x} = A'x + B'u, \qquad A' = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 \\ E'_2 & 0 & \dots & 0 & 0 \\ 0 & E'_3 & \dots & 0 & 0 \\ \dots & \dots & \dots & & \\ 0 & 0 & \dots & E'_M & 0 \end{pmatrix}, \; B' = \begin{pmatrix} E'_1 \\ 0 \\ 0 \\ \dots \\ 0 \end{pmatrix}, \\ y = C.x, \qquad C = [E'_1 \;\; 0 \;\; 0 \;\; \dots \;\; 0], \qquad u = \begin{bmatrix} u_1 & u_2 & \dots & u_r \end{bmatrix}^T, \end{gathered} \tag{2.38}$$

where $E'_j = [I_{h'_i} \quad 0]$, $j = 1, 2, \ldots, M$. *The sizes of the block matrices* E'_j *are the Brunovsky observability indices.*

Proof. Please refer to [27, 25, 9, 22, 5, 15, 30, 7]. ∎

By Theorems 2.13 and 2.14, there exists a change of coordinates that will transform the system (1.7) to the form explained by (2.30) while (2.32) holds. In this case, the observability problem for (1.7) is solvable. Say also (1.7) is observable. We obtain a set of indices that fulfill the total dimension n. We also call this case a maximal case. According to Theorem 2.13, the desired change of variable, in this case, is given by

$$\psi = \bigoplus_i \begin{bmatrix} \lambda_i \\ L_f \lambda_i \\ \cdots \\ L_f^{a_i - 1} \lambda_i \end{bmatrix} : \mathbb{R}^n \to \mathbb{R}^n. \tag{2.39}$$

Example 2.17 (linear case [25]). Example 2.6 has an analogous version for MIMO systems. In this case, we work with codistributions and their kernels. Consider the linear system

$$\begin{aligned} &\dot{x} = Ax + Bu, && x \in \mathbb{R}^n,\ A \in M_{n\times n},\ B \in M_{n\times m},\ u \in \mathbb{R}^m, \\ &y = Cx, && C = \begin{pmatrix} c_1 & c_2 & \ldots & c_m & 0\ldots \end{pmatrix},\ c_i \in \mathbb{R}^1. \end{aligned} \tag{2.40}$$

In this case, the codistributions defined by (2.33) find the following form:

$$\Omega_k = \left\langle\ C,\ CA,\ \ldots\ CA^{k-1}\ \right\rangle_{\mathbb{R}}, \tag{2.41}$$

where the angles mean the subspace of the span of the *rows* of all matrices. Consequently

$$\Omega_0^{\perp} = \ker(C), \qquad \Omega_k^{\perp} = \ker \begin{bmatrix} C \\ CA \\ \cdots\cdot \\ CA^{k-1} \end{bmatrix}; \tag{2.42}$$

therefore we have

$$h'_k = \operatorname{rank} \begin{bmatrix} C \\ CA \\ \cdots\cdot \\ CA^{k-1} \end{bmatrix}. \tag{2.43}$$

If one defines $Z = \bigcap_i \ker(CA^i)$, the maximal integral submanifolds of Z are of the form $z_0 + Z$. Again, similar to the controllability canonical form, the canonical observability form can be used to classify MIMO linear systems.

3. Problem statement. To the nonlinear control system (1.4) (generally multi-input system), one can associate a set of indices that divide n as a partition and provide a canonical form of the exact linearization of (1.4). There are a change of coordinates and a choice of control functions u_j such that the above differential equation gets transformed into a special block form called Brunovsky canonical controllable form. The block matrices' size provides invariants of the controllable system (1.4) called the Brunovsky controllability indices. We denote them by $(h_1, \ldots, h_M)$ (see subsection 2.2). These indices are defined by certain invariants of distributions constructed from the vector fields f and g_j and the bracket operations. In [27], two equivalent sets of conditions, namely **Conditions A** and **Conditions B**, have been settled that characterize the Brunovsky controllability indices (see subsection 2.2). The controllability indices $(h_1, h_2, \ldots, h_N)$ give a partition of n, i.e.,

$$n = h_1 + \cdots + h_N. \tag{3.1}$$

Consider the system (1.7) with the same conditions above. In an alternative method, by using the outputs, λ_j, one can define another set of indices that are defined somehow in a dual manner to the previous procedure. Again, the block's size of matrices appears to provide invariants of the system (1.7); they are called the Brunovsky observability indices, denoted by $(h'_1, \ldots, h'_M)$ (see subsection 2.3). The observability indices $(h'_1, \ldots, h'_M)$ also give a partition of n, that is,

$$n = h'_1 + \cdots + h'_M. \tag{3.2}$$

Observability indices give a canonical form of the relative degrees when the variables are appropriately grouped. The main problem is as follows. In the same way that the sets of **Conditions A** and **B** describe the Brunovsky controllability indices, one can present similar dual **Conditions C** that describe Brunovsky observability indices.

Problem 3.1. Prove that the three sets of **Conditions A**, **Conditions B**, and **Conditions C** are mutually equivalent to each other.

The above problem seems to be a natural step in comparing the controllability and observability indices. We also make the following second question.

Problem 3.2. How are the two set of indices $(h_1, \ldots, h_N)$ and $(h'_1, \ldots, h'_M)$ related? In other words, how are the system's controllability indices of the system (1.4) related to the observability indices of the system (1.7)? Do we have the equality $M = N$? Provide a geometric insight for this comparison.

We answer both problems in the next section. These appear as Theorems 4.1 and 4.4. In Example 4.9, we show the two canonical forms of linearizing (1.7) are not identical, and the two sets of indices are not generally the same; nor do we have $M = N$, i.e., the number of the blocks in the above two linearization processes are not equal. Some geometric motivations of these two problems are described in section 5.

4. Main results. According to the explanation given in sections 2 and 3, there are two different ways of linearizing a system given by (1.4) or a system in the form of (1.7), when they are exactly linearizable. The nontrivial expectation is that these two linearizations may not be the same, and the two-block forms we mentioned in the previous sections can be different. Naturally, one finds that the two canonical sets of indices we encountered are unequal. In general, the two sets of indices are far from being identical, although they both give a partition of n; they may not even have an equal number of elements. This section investigates the relationship between two sets of indices defined in subsections 2.2 and 2.3. Because the controllability and observability indices given in Definition 2.3 and Definition 2.15, respectively, characterize a particular form of exact linearity of the same systems of differential equations, it is natural to expect a simple relation between them. We note that the ways these two series of numbers are defined use dual concepts of distributions versus codistributions on $\mathbb{R}^n$. Thus, the connections between the invariants above should also be related to duality. Below we state another set of conditions that can be compared with the sets of **Conditions A** and **B** given above and also in [27]. This also appears to be one of our contributions to the literature.

Conditions C:

(C1) The codistributions Ω_j have constant dimensions, $j=1,2,\dots,M$.

(C2) Ω_M has dimension n, where the number M is the smallest number with this property.

(C3) For each j the codistribution Ω_j is invariant under f, g_l for all $l \le r$, i.e., $L_f\Omega_j,\ L_{g_l}\Omega_j \subset \Omega_j$.

Recall that a codistribution Ω is invariant under a vector field f if $L_f w \in \Omega$ for all forms $w \in \Omega$. The notion is dual to one of the distributions so that we call a distribution $\mathfrak{g}$ invariant under a vector field f if $[f,X] \in \mathfrak{g}$ for all $X \in \mathfrak{g}$. One can think of the set of **Conditions C** as dual to the sets **A, B**. The way that the different items in **Conditions C** can be checked was explained in subsection 2.3. The following theorem explains the relation between the **Conditions A, B** and **Conditions C**. It serves as part of our main result.

THEOREM 4.1. *The conditions in Theorem* 2.14 *can be equivalently replaced by the set of* ***Conditions*** **C**. *Each of the three sets of* ***Conditions*** **A, B, C** *is equivalent to the other.*

Proof. The action of the codistributions Ω_j on the distributions $\mathfrak{g}_i$ can be presented in an $n \times n$-matrix consisting of different blocks where the (ij)-block is of the form

$$\begin{bmatrix} d\lambda_i \\ dL_f\lambda_i \\ \dots \\ dL_f^{a_i-1}\lambda_i \end{bmatrix} \begin{pmatrix} g_j & ad(f)g_j & \dots & ad(f)^{a_i-1}g_j \end{pmatrix} = \begin{pmatrix} 0 & \dots & L_{ad(f)^{a_i-1}g_j}\lambda_i \\ 0 & \dots & * \\ & \dots & * \\ L_{g_j}L_f^{a_i-1}\lambda_i & * & * \end{pmatrix}, \tag{4.1}$$

where the last matrix is nonsingular. This can be done by arranging a basis for a suitable choice of λ_i. On the other hand, as in Lemma 4.1.2 of [25], the following two sets of conditions (identities) are equivalent (the same proof as in [25]):

(a) $L_g\psi_i(x) = L_gL_f\psi_i(x) = \dots = L_gL_f^k\psi_i(x) = 0$.

(b) $L_g\psi_i(x) = L_{ad(f)g}\psi_i(x) = \dots = L_{ad(f)^kg}\psi_i(x) = 0$

for each i, with

$$\psi_i = \begin{bmatrix} \lambda_i \\ L_f\lambda_i \\ \dots \\ L_f^{a_i-1}\lambda_i \end{bmatrix}, \qquad g = [g_1,\dots,g_r]. \tag{4.2}$$

Now, let's prove **Conditions A** and **C** are equivalent. We show that such conditions are involutive and equivalent. If **Conditions C** hold, one looks for suitable ψ_j's where the list of equations (a) in the above holds. Then the involutive condition for Ω_j reads as the list of equations in (a), where the next term after the step k does not vanish; i.e., $L_gL_f^{k+1}\psi_i(x) \ne 0$ (see [25, Lem. 4.1.2, p. 141]). On the other hand, if **Conditions A** hold, then one looks for ϕ_i in the second line of equations (b). However, the coordinates and their directional derivatives are different. The involutive property or invariant under f and g is read as equations (b), with the one after the last stage not vanishing, i.e., $L_{ad(f)^{k+1}g}\psi_i(x) \neq 0$. These two sets of equations characterize the observability and controllability indices, respectively. Because the two sets of equations in (a) and (b) above are equivalent, the above two criteria of involutivity in **Conditions A** and **C** are also equivalent.

The equivalence of constant dimensions in **Conditions A** (or **B**) and **C** can be checked easily. Notice that the distribution has a constant dimension if its annihilating codistribution (denoted by the upper perpendicular symbol above) has a constant dimension. Therefore, the comparison of the other two conditions is trivial. One notes that the items in **Conditions C** are G-invariant. Therefore, one may check the criteria by its G-equivalent linearized form. The block multiplication in the identity (4.1) gets the following form in the linear case,

$$\begin{bmatrix} c_i \\ c_iA \\ \dots \\ c_iA^{a_i-1} \end{bmatrix} \begin{pmatrix} B_j & AB_j & \dots & A^{a_i-1}B_j \end{pmatrix} = \begin{pmatrix} 0 & \dots & c_iA^{a_i-1}B_j \\ 0 & \dots & * \\ & \dots & * \\ c_iA^{a_i-1}B_j & * & * \end{pmatrix}, \tag{4.3}$$

in which one can verify the claim of equivalence of conditions (A2) and (C1) on linear spaces. Also, the conditions for having a constant dimension of distributions and codistributions can be seen concretely in the linear case. ∎

The equivalence of the set of **Conditions C** and those in **Conditions A, B** does not imply that the two series of indices, i.e., the Brunovsky controllability and observability indices, coincide. Nevertheless, the relation between the sets of indices appears to be our paper's main result. Therefore, we first state the following proposition.

PROPOSITION 4.2. *In the setup of Theorems* 2.13 *and* 2.14 *assume that* $\lambda_1, \dots, \lambda_r$ *is a set of outputs such that* (2.32) *holds. Then, for any other choice of these functions* $\tilde{\lambda}_1, \dots, \tilde{\lambda}_r$, *the associated relative degrees* $\tilde{a}_1, \dots, \tilde{a}_M$ *satisfy*

$$\tilde{a}_i \leq a_i, \qquad i = 1, 2, \dots, M. \tag{4.4}$$

Proof. The claim of the proposition is a MIMO system analogue of Theorem 4.8.2 in [25], where a similar statement is proved for a single-input single-output (SISO) system. Because MIMO systems can be considered as several blocks of SISO systems, Proposition 4.2 results as an application of Theorem 4.8.2 in [25] to several blocks of coordinates. The claim can also be understood from the block matrices illustration in (4.1). The relation (4.1) implies that the length of the first column matrix cannot exceed the length of the second-row matrix, which is independent of λ_i and depends on g_j (see also Theorem 4.8.2 in [25]). ∎

Proposition 4.2 is a generalization of Theorem 4.8.2 in [25] to a MIMO system. For our next result, we recall the following definition.

DEFINITION 4.3 (transpose partition). *The pairs of partitions for a single number, whose Ferrers diagrams (Young tableaux) transform into each other when reflected about the line* $y = -x$, *with the coordinates of the upper left dot taken as* $(0, 0)$, *are called conjugate (or transpose) partitions. In this case, the corresponding Young tableaux are transposed to each other in their rows and columns. An example is illustrated in the following picture:*

(4.5)

a	d	f
b	e	g
c		

$$3+3+1=3+2+2$$

a	b	c
d	e	
f	g	

The operation is called transposition.

Another example is

(4.6)

a	d	f	h	i
b	e	g		
c				

$$5+3+1=3+2+2+1+1$$

a	b	c
d	e	
f	g	
h		
i		

The following theorem explains the relation between Brunovsky controllability indices and the Brunovsky observability indices introduced in subsections 2.2 and 2.3.

THEOREM 4.4. *Assume the exact linearization problem is solvable for the system* (1.4) *and the system* (1.7) *due to a choice of outputs* $\lambda_1, \ldots, \lambda_r$. *The observability indices of* (1.7) *give a transpose partition of* n *concerning the Brunovsky controllability indices for* (1.4). *In other words, the two partitions*

$$n = h_1 + h_2 + \cdots + h_N = h'_1 + \cdots + h'_{M-1} + h'_M \tag{4.7}$$

are transposed to each other.

In terms of the corresponding Young tableaux, we get a picture like this:

(4.8) N rows

M rows,

where the jth row of the left-hand tableau has h_j boxes and the jth row of the right-hand tableau has h'_j boxes. The number of columns in each tableau equals the number of rows in the other by transposition. The above phenomenon for the SISO systems is not an exciting example; however, it gives exciting examples in MIMO systems.

Proof of Theorem 4.4. We first consider the Brunovsky observability indices. According to Theorem 2.13 the change of coordinates

$$\phi = \bigoplus_i \begin{bmatrix} \lambda_i \\ L_f \lambda_i \\ \cdots \\ L_f^{a_i - 1} \lambda_i \end{bmatrix} : \mathbb{R}^n \to \mathbb{R}^n \tag{4.9}$$

transforms system (1.7) into

$$\dot{x} = A''x + B''u, \quad A'' = \begin{pmatrix} A_1 & 0 & \dots & 0 & 0 \\ 0 & A_2 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & & \\ 0 & 0 & \dots & 0 & A_M \end{pmatrix}, \; u = \begin{bmatrix} u_1 & u_2 & \dots & u_r \end{bmatrix}^T, \tag{4.10}$$
$$y = C''x,$$

where the matrices A_j (of size $a_j \times a_j$) are in Brunovsky canonical form in a SISO system. We may put $x = (x_1, \dots, x_M)$, where x_j has dimension a_j. We can group the coordinates' content so that in the first group, set all the first coordinates of $x_1, \dots, x_M$ and in the second group all the second coordinates. If the sizes of A_j were different, we would fill the coordinate gaps with zeros. Because of their canonical forms of A_j, differentiating the first block gives the second block, and so on. This permutation of coordinates gives the Brunovsky canonical form (2.38)

$$\dot{x} = A'x + B'u, \qquad A' = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 \\ E'_2 & 0 & \dots & 0 & 0 \\ 0 & E'_3 & \dots & 0 & 0 \\ \dots & \dots & \dots & & \\ 0 & 0 & \dots & E'_M & 0 \end{pmatrix}, \; B' = \begin{pmatrix} E'_1 \\ 0 \\ 0 \\ \dots \\ 0 \end{pmatrix}, \tag{4.11}$$
$$y = Cx, \qquad C = [E'_1 \;\; 0 \;\; 0 \;\; \dots \; 0], \qquad u = \begin{bmatrix} u_1 & u_2 & \dots & u_r \end{bmatrix}^T.$$

Therefore, the observability indices can be obtained as

$$h'_k = \sharp\{a_j \geq k, \; j \geq 0\}, \tag{4.12}$$

where $\sharp$ means *number of elements,* considering our new arrangement. Equation (4.1) and the set of equations after that show that another way exists to arrange the coordinates; that is, we use the set of equations (b) in the proof of Theorem 4.1. The equation $L_{ad(f)^j g}\psi(x) = D\psi(ad(f)^j g)$ indicates that in these coordinates, we differentiate first in the direction of flows of g_j for different j. On the other hand, if we consider the set of **Conditions B**, we find that a_j also defines the controllability indices. The theorem's claim follows from the fact that (4.12) defines the transpose or conjugate partition to the one given by a_j. This, for instance, can be seen by noting that the transpose partition is the reflection of the Young diagram under the line $y = -x$. The identity (4.1) implies that $n = h_1 + h_2 + \cdots + h_N = h'_1 + \cdots + h'_{M-1} + h'_M$. ∎

Remark 4.5. Some explanations are required about Theorem 4.4. The first is regarding the Lie group G of symmetries of (1.4) generated by the transformations (2.1), (2.2), and (2.3) presented in subsection 2.2. The Lie group G has certain additional features rather than symmetries of an ordinary differential equation. In fact, according to the definition, it considers additional symmetries over the feedback control functions u_j. As we saw in the linear case, Remark 2.8, the equivalence classes of control systems and their observers have more complicated behavior due to the presence of input functions. This fact reflects the nontriviality of the result in Theorem 4.4. The second point is that the transpose duality in (4.7) is handy in the study of geometric properties of the system (1.4) and its observers. It directly implies dimensional dualities on the group of symmetries, namely G. Also, a MIMO system has two groups of symmetries that are dimensionally relevant according to (4.7), which recalls further works.

COROLLARY 4.6. *The observability (resp., controllability) indices of an n-dimensional SISO system are* $(1,1,\ldots,1)$ *(resp.,* (n)*). The aforementioned duality takes the simple form* $1+1+\cdots+1=n$ *in this case.*

Remark 4.7. The point to compare the Brunovsky indices in the controllable and observable cases is essential to clarify the relation (4.7). The identity (4.7) shows a transposition duality between these two sets of indices. Despite its simplicity, we claim it is not addressed in the literature.

Remark 4.8. Because the vectors

$$\mathfrak{d}_i = \begin{bmatrix} d\lambda_i \\ dL_f\lambda_i \\ \ldots \\ dL_f^{a_i-1}\lambda_i \end{bmatrix}, \qquad i=1,\ldots,r, \tag{4.13}$$

are independent (cf. [25, Lem. 5.1.1]), one has

$$\dim\ker\mathfrak{d}_i = n-h_1-\cdots-\widehat{h_i}-\cdots-h_r, \tag{4.14}$$

where the hat $(\widehat{\cdot})$ means deletion.

Next, we give a simple example of checking the relation (4.7) and also the computation of Brunovsky indices.

Example 4.9 (see [25, Example 5.2.6]). We consider the following differential equation near $0\in\mathbb{R}^5$:

$$\dot{x} = \begin{bmatrix} x_2+x_2^2 \\ x_3-x_1x_4+x_4x_5 \\ x_2x_4+x_1x_5-x_5^2 \\ x_5 \\ x_2^2 \end{bmatrix} + u_1\begin{bmatrix} 0 \\ 0 \\ \cos(x_1-x_2) \\ 0 \\ 0 \end{bmatrix} + u_2\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}. \tag{4.15}$$

Then, we define the following distributions on $\mathbb{R}^5$:

$$\begin{aligned} \mathfrak{g}_0 &= \langle g_1, g_2\rangle, \\ \mathfrak{g}_1 &= \langle g_1, g_2, ad_f g_1, ad_f g_2\rangle, \\ \mathfrak{g}_2 &= \left\langle g_1, g_2, ad_f g_1, ad_f^2 g_1, ad_f g_2, ad_f^2 g_2\right\rangle. \end{aligned} \tag{4.16}$$

We calculate the following:

$$ad_f g_1 = \begin{bmatrix} 0 \\ -\cos(x_1-x_5) \\ -x_2\sin(x_1-x_5) \\ 0 \\ 0 \end{bmatrix}, \quad ad_f g_2 = \begin{bmatrix} 0 \\ -1 \\ -(x_1-x_5) \\ -1 \\ 0. \end{bmatrix}. \tag{4.17}$$

One can quickly check the following relations:

$$\begin{aligned} [g_1, ad_f g_1] = [g_2, ad_f g_1] = [g_2, ad_f g_1] = [g_1, ad_f g_2] = 0, \\ [ad_f g_1, ad_f g_2] = \tan(x_1-x_5)g_1(x). \end{aligned} \tag{4.18}$$

Using the above relations, each distributor's $\mathfrak{g}_j$ are closed under brackets and have constant ranks 2, 4, and 5 near $0\in\mathbb{R}^5$. Therefore the Brunovsky controllability indices for this system are $h_1=2, h_2=4-2=2, h_3=5-4=1$.

On the other hand, $\dim \mathfrak{g}_1^\perp = 1$. We may easily find a function λ_1 such that $\langle d\lambda_1 \rangle = \mathfrak{g}_1^\perp$. A trivial check shows that $y_1 = \lambda_1(x) = x_1 - x_5$ is such a function. Now, $\dim \mathfrak{g}_0^\perp = 3$. Thus, we have

$$d\lambda_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & -1 \end{bmatrix}, \qquad dL_f\lambda_1 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \end{bmatrix}. \tag{4.19}$$

We may also easily guess a function λ_2 such that $\langle d\lambda_1,\ dL_f\lambda_1, d\lambda_2 \rangle = \mathfrak{g}_0^\perp$. A simple choice is $\lambda_2(x) = x_2$, and therefore, methods in control can be used to find the functions λ_1, λ_2. Thus, we have

$$\begin{aligned} &L_{g_1}\lambda_1 = L_{g_2}\lambda_1 = L_{g_1}L_f\lambda_1 = L_{g_2}L_f\lambda_1 = 0, \\ &L_{g_1}\lambda_2 = L_{g_2}\lambda_2 = 0, \end{aligned} \tag{4.20}$$

and the matrix

$$\begin{pmatrix} L_{g_1}L_f^2\lambda_1 & L_{g_2}L_f^2\lambda_1 \\ L_{g_1}L_f\lambda_1 & L_{g_1}L_f\lambda_1 \end{pmatrix} \tag{4.21}$$

is nonsingular. The Brunovsky observability indices are $h_1' = 3, h_2' = 2$. Thus, these numbers are maximal, i.e., they fulfill the total dimension $n = 5$. Now one sees that the two partitions of the system dimension by Brunovsky indices obtained by the above two ways are transposed to each other:

$$\begin{aligned} &h = (h_1 = 2, h_2 = 2, h_3 = 1), \\ &h' = (h_1' = 3, h_2' = 2), \\ &5 = 2 + 2 + 1 = 3 + 2. \end{aligned} \tag{4.22}$$

We can also see this in the Young diagrams

(4.23)

a	b
d	e
f	

$$2 + 2 + 1 = 3 + 2$$

a	d	f
b	e	

Next, we present another example where Theorem 4.4 can be applied. The following example concerns a pencil of linear differential equations with their observers. We apply Theorem 4.4 for the Brunovsky indices over the most generic locus (see below for the definition).

Example 4.10 (linear pencil). This example is an application of Theorem 4.4. We work over the field $\mathbb{C}$ of complex numbers for simplicity. The invariants of the feedback systems can be studied through a parametrized family of differential equations, parametrized by a variable $\lambda \in \mathbb{C}$. The family can be made into a pencil by adding a point as infinity to the line $\mathbb{C}$. We explain this in the linear case. Consider a linear finite-dimensional system of differential equations:

$$\begin{aligned} &\dot{x}(t) = Ax(t) + Bu(t), \\ &y(t) = Cx(t), \end{aligned} \tag{4.24}$$

where $A \in M_{n\times n}(\mathbb{C}), B \in M_{n\times m}(\mathbb{C}), C \in M_{p\times n}(\mathbb{C})$. We associate a system pencil with (4.24) as follows:

$$S(\lambda) = \mathcal{A} - \lambda\mathcal{H} := \begin{bmatrix} A & B \\ C & 0 \end{bmatrix} - \lambda \begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix}, \tag{4.25}$$

where $\mathcal{A}$ and $\mathcal{H}$ are of size $(n+p)\times(n+m)$. This provides a more general setup than Definition 2.1. According to [16, 14] there exist invertible matrices U and V such that the system $U(\mathcal{A}-\lambda\mathcal{H})V^{-1}$ can be written as

$$S(\lambda)\equiv R\bigoplus L\bigoplus J(\lambda_1)\bigoplus\cdots\bigoplus J(\lambda_q)\bigoplus N, \tag{4.26}$$

called Kronecker canonical form (KCF) where

$$R=\bigoplus_{k=1}^{h_0} R_{i_k},\quad L=\bigoplus_{k=1}^{h_0'} L_{j_k},\quad J(\lambda_i)=\bigoplus_{k=1}^{l_i} J_{m_k}(\lambda_i),\quad N=\bigoplus_{k=1}^{l_\infty} N_{n_k}. \tag{4.27}$$

The blocks $J_{m_k}(\lambda_i)$ are Jordan blocks associated with each distinct finite eigenvalue λ_i, and the blocks N_{n_k} are Jordan blocks associated with the eigenvalue at infinity. These two types of blocks constitute the regular part of a matrix pencil and are defined by

$$J_{m_k}(\lambda_i)=\begin{bmatrix}\lambda_i-\lambda & 1 & & \\ & \cdots & & \\ & & \lambda_i-\lambda & 1\\ & & & \lambda_i-\lambda\end{bmatrix},\qquad N_{n_k}=\begin{bmatrix}1 & -\lambda & \\ & \cdots & \\ & 1 & -\lambda\end{bmatrix}. \tag{4.28}$$

If $m\neq p$ or $\det(\mathcal{A}-\lambda\mathcal{H})\equiv 0$ for all $\lambda\in\mathbb{C}$, then the matrix pencil also includes a singular part which consists of the h_0 right singular blocks L_{i_k} of size $i_k\times(i_{k+1})$ and the l left singular blocks $L_{j_k}^T$ of size $(j_k+1)\times j_k$:

$$L_{i_k}=\begin{bmatrix}-\lambda & 1 & \\ & \cdots & \\ & -\lambda & 1\end{bmatrix},\qquad L_{j_k}^T=\begin{bmatrix}-\lambda & & \\ 1 & \cdots & \\ & \cdots & -\lambda\\ & & 1\end{bmatrix}, \tag{4.29}$$

where the L_0 and L_0^T blocks are of size 0×1 and 1×0, respectively, and each of them contributes with a column or row of zeros. Two matrix pencils are equivalent if they have the same KCF (4.26), that is, they have the same R, L, J, and N blocks [14, 12, 13, 42].

The decomposition (4.26) can be related to the Brunovsky forms as follows. Given a controllability pair (A,B) with the pencil S_{c} defined as above, there exists a feedback equivalent or block similar matrix pair (A_1,B_1) in BCF, such that

$$P\begin{bmatrix}A-\lambda I_n & B\end{bmatrix}\begin{bmatrix}P^{-1} & 0\\ R & Q^{-1}\end{bmatrix}=\begin{bmatrix}A_1-\lambda I_n & B_1\end{bmatrix}=\begin{bmatrix}A_\epsilon & 0 & B_\epsilon\\ 0 & A_\lambda & 0\end{bmatrix}. \tag{4.30}$$

The transformation matrices $P\in M_{n\times n}(\mathbb{C})$ and $Q\in M_{m\times m}(\mathbb{C})$ are nonsingular, and $R\in M_{m\times n}(\mathbb{C})$. Moreover, the matrix pair (A_ϵ,B_ϵ) is controllable and corresponds to the R-blocks in the KCF of $S_{\mathrm{c}}(\lambda)$. If rank $S_{\mathrm{c}}(\lambda)<n$ for some $\lambda\in\mathbb{C}$, then (A,B) is uncontrollable and there exists a regular pencil A_μ whose eigenvalues correspond to the uncontrollable eigenvalues. The dual form of the Brunovsky canonical form for the observability pair (A,C) is

$$\begin{bmatrix}P & S\\ 0 & T\end{bmatrix}\begin{bmatrix}A-\lambda I_n\\ C\end{bmatrix}P^{-1}=\begin{bmatrix}A_2-\lambda I_n\\ C_2\end{bmatrix}=\begin{bmatrix}A_\eta & 0\\ 0 & A_\lambda\\ C_\eta & 0\end{bmatrix}. \tag{4.31}$$

The transformation matrices $P \in M_{n\times n}(\mathbb{C})$ and $T \in M_{p\times p}(\mathbb{C})$ are nonsingular and $S \in M_{n\times p}(\mathbb{C})$. The matrix pair (A_η, C_η) is observable and corresponds to the L-blocks. If rank $S_{\mathrm{o}}(\lambda) < n$ for some $\lambda \in \mathbb{C}$, then (A, C) is unobservable and there exists a regular pencil A_λ whose eigenvalues correspond to the unobservable eigenvalues [14, 42].

In general, the corresponding invariants of the system pencils $S(\lambda)$, $S_{\mathrm{c}}(\lambda)$, and $S_{\mathrm{o}}(\lambda)$, are different. For example, the system pencil $S_c(\lambda)$ of a completely controllable system associated with the pair (A, B) can only have L-blocks in its KCF. In contrast, $S(\lambda)$ may have both types of singular invariants (blocks) and eigenvalues in its KCF.

The most generic cases of the matrix pairs correspond to completely controllable and observable systems, while the most degenerate cases correspond to systems with n uncontrollable and n unobservable multiple modes, respectively. The generic property can be defined according to the points in moduli space of pairs (A, B) or (A, C). In general, the moduli **spaces** $\mathcal{M}$ of the controllability (or observability) pairs are stratified manifolds [14]. This stratification appears according to the existence of Jordan blocks and their elementary divisors.

In the following, we will avoid the degeneracy locus and concentrate on where no degeneracy exists; see [14]. We call a subset in this moduli space generic if it contains an open dense subset of positive measure. The above generic condition has been analyzed in Theorem 5 of [42]. According to [42], when $m, p \geq 1$, the controllability indices of system (4.24) are given by

$$\begin{aligned} &h_1 = h_2 = \cdots = h_\nu = [n/m]^+, \\ &h_{\nu+1} = h_{\nu+1} = \cdots = h_m = [n/m]^-, \qquad \nu = n(\bmod m). \end{aligned} \tag{4.32}$$

Similarly, when $n, p \geq 1$, the observability indices are given by

$$\begin{aligned} &h'_1 = h'_2 = \cdots = h'_\mu = [n/p]^+, \\ &h'_{\mu+1} = h'_{\mu+1} = \cdots = h'_p = [n/p]^-, \qquad \mu = n(\bmod p). \end{aligned} \tag{4.33}$$

Thus, Theorem 4.4 allows us to compare the structure integer-partitions. When $m, p \geq 1$, in the most generic cases, we have a conjugacy relation between (4.32) and (4.33) as

$$n = h_1 + h_2 + \cdots + h_\nu + h_{\nu+1} + \cdots + h_m = h'_1 + h'_2 + \cdots + h'_\mu + h'_{\mu+1} + \cdots + h'_p, \tag{4.34}$$

where h_i and h'_j are defined by (4.32) and (4.33), respectively.

The most degenerate controllability pair has m times $R = 0$ blocks and n Jordan blocks of size 1×1 corresponding to an eigenvalue of multiplicity n. Similarly, the most degenerate observability pair has p times $L = 0$ blocks and n number of 1×1 Jordan blocks.

5. Moduli of feedback equivalent systems. As we have mentioned before, the Brunovsky indices of a feedback control system are crucial in the classification problem of such systems. A relevant concept related to the classification of feedback control systems is the classifying (moduli) space of systems with the same invariants (Brunovsky indices). This concept is relatively important in the literature [10, 14, 12]. In this section, we relate the result of Theorem 4.4 to a similar question on the classifying space $\mathcal{D}$ of feedback equivalent systems with a fixed set of indices. We briefly define these spaces and explain their basic properties.

We keep the notation of Example 4.10, and we identify the linear system (4.24) by the triple of matrices (A, B, C). These matrices can also be a linear transformation between the corresponding linear spaces.

Definition 5.1. *Let* (A', B', C') *be the associated triple to a system of type* (4.24). *We call* (A', B', C') *a subsystem of* (A, B, C) *if the transformations* A', B', *and* C' *are restrictions of* A, B, *and* C, *respectively, so that the corresponding KCF via* (4.26), *for the triple* (A', B', C'), *is obtained from that of* (A, B, C) *by restriction of the components in the KCF* (4.26) *of* (A, B, C).

More specifically, if we regard the system (4.24) as a feedback system defined on $\mathbb{C}^n \times \mathbb{C}^m \times \mathbb{C}^p$, a subsystem of (4.24) corresponding to (A', B', C') is a feedback linear differential equation of the same type where the flows (solutions) of the subsystem are restrictions of the flows of (4.24). Thus, the corresponding subspaces where the subsystem is defined must be chosen to be invariant for the dynamics of (A', B', C'); we also call (A, B, C) a Kronecker triple. Let's make the following general definition.

Definition 5.2. *Given a Kronecker triple* (A, B, C), *the classifying space* $\mathcal{D}$ *of Kronecker triples of type* (A, B, C) *is the set of all pairs* (A', B', C') *which are block equivalent to* (A, B, C); *i.e., all elements of* $\mathcal{D}$ *have the same KCF form* (4.26).

The group $G = GL_n\mathbb{C} \times GL_m\mathbb{C} \times GL_p\mathbb{C}$ acts on $\mathcal{D}$ with finitely many orbits. Let H be the stabilizer group of (A, B, C) in G. It is a normal subgroup of G, and the domain $\mathcal{D}$ equals the set of conjugacy classes of H in G, where

$$\mathcal{D} = \{\text{set of conjugacy classes } \ g^{-1}Hg \ \text{ of } H \text{ in } G\}. \tag{5.1}$$

Considering a triple (A, B, C) together with all its subsystems, we can introduce the following notions.

Definition 5.3. *We associate the following two notions to the triple* (A, B, C):

1. *Define the Kronecker group* K *of the Kronecker triple* (A, B, C) *as the subgroup of* G *with the property that* K*-stable triples* (A', B', C') *are exactly the subsystems of* (A, B, C).
2. *Define the Kronecker domain* D *of the Kronecker triple* (A, B, C) *to be its orbit in* $\mathcal{D}$ *under the action of the subgroup* K.

The group K is, in general, smaller than G. The definition of the group K does not depend on the choice of the base point $(A, B, C) \in \mathcal{D}$ *up to conjugacy by an element in* G. Also, the domain D does not depend on the choice of base point $(A, B, C) \in \mathcal{D}$ up to an isomorphism. The definitions above are specialized to the cases of controllability pairs and observability pairs separately.

Given the controllability pair of matrices (A, B), we define the classifying space $\mathcal{D}'$ to be the set of all pairs (A', B') which are block equivalent to (A, B). Similarly, for an observability pair (A, C), we define the associated classifying space $\mathcal{D}''$ as the set of all pairs (A', C') which are block equivalent to (A, C). The group $G' = GL_n\mathbb{C} \times GL_m\mathbb{C}$ acts on the space $\mathcal{D}'$ by (4.30), and similarly $G'' = GL_n\mathbb{C} \times GL_p\mathbb{C}$ acts transitively on $\mathcal{D}''$ by (4.31) with finitely many orbits. Letting H' and H'' be the respective stabilizers of the above actions, then one has $\mathcal{D}'$ and $\mathcal{D}''$ equal to the set of conjugacy classes of H' and H'' in the corresponding action, where

$$\begin{aligned} \mathcal{D}' &= \{ \text{ set of conjugacy classes } \ g^{-1}H'g \ \text{ of } \ H' \ \text{ in } G' \ \}, \\ \mathcal{D}'' &= \{ \text{ set of conjugacy classes } \ g^{-1}H''g \ \text{ of } \ H'' \ \text{ in } G'' \ \}. \end{aligned} \tag{5.2}$$

The notions of Brunovsky group K' (resp., K'') and Brunovsky domain D' (resp., D'') can be similarly defined as in Definition 5.3. The groups K, K', and K'' are algebraic, that is, the set of each group's points can be defined by algebraic equations over Euclidean spaces. The aforementioned moduli spaces are manifolds with boundaries that accept stratifications; see [14, 10, 42, 18, 17].

The dimension of the orbit of a Kronecker triple $(A,B,C)\in D$ under the action of the algebraic group K can be measured by the rank of the tangent bundle $T_{(A,B,C)}D$. The duality mentioned in Theorem 4.4 shall influence a duality between the corresponding classifying spaces. This duality can be easily investigated in terms of the structure theory of the associated Lie algebras. We stress the duality on the open subsets $D'_\circ \subset D'$ and $D''_\circ \subset D''$ of the locus for the most generic points. Let us now introduce the following general definition.

DEFINITION 5.4 (see [1]). *Let G be a complex semisimple Lie group. Fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}=Lie(G)$, let R be the corresponding set of simple roots, and choose a root order. We denote a parabolic subgroup P of G corresponding to a subset $\phi\subset R$ by P_ϕ, to express the correspondence and its Lie algebra by $\mathfrak{p}_\phi$. Then we can write $\mathfrak{p}=\mathfrak{p}_\phi=\mathfrak{r}+\mathfrak{n}^-$, where $\mathfrak{n}^-=\sum_{\phi(\mathfrak{n})}\mathfrak{g}_{-\alpha}$ is the nilradical. Then, the opposite parabolic is given as $\mathfrak{p}^-=\mathfrak{r}+\mathfrak{n}^+$, where the nilradical is $\mathfrak{n}^+=\sum_{\phi(\mathfrak{n})}\mathfrak{g}_{+\alpha}$. The dual parabolic is the complex conjugate (of $\mathfrak{g}$ with respect to a real form $\mathfrak{g}_0$) $\mathfrak{p}^*:=\overline{\mathfrak{p}^-}$ of the opposite.*

In our setup, we begin with the larger group $G=GL_n\mathbb{C}\times GL_m\mathbb{C}\times GL_p\mathbb{C}$, and $\mathfrak{g}=Lie(G)$ is its Lie algebra. We can write $D_\circ=G/P, D'_\circ=G/P'$, and $D''_\circ=G/P''$, where P' and P'', and $P=P'\cap P''$ are associated parabolics. Let $\mathfrak{p},\mathfrak{p}'$, and $\mathfrak{p}''$ be the respective Lie algebras. Our result is the following.

PROPOSITION 5.5. *The Lie algebras $\mathfrak{p}'$ and $\mathfrak{p}''$ are dual parabolics according to Definition 5.4. Also, there exists an involutive isomorphism $\alpha: G\to G$ such that the following hold:*

1. $\alpha^2=id, \alpha(P')=P'', \alpha(P'')=P'$.
2. *The induced map $\phi: D'_\circ\to D''_\circ$ is given as $xP'\mapsto\alpha(x)P''$, and similarly, we have that $\phi^{-1}: xP''\mapsto\alpha(x)P'$ are biholomorphic diffeomorphisms.*

Sketch of proof. We prove that $\mathfrak{p}'$ and $\mathfrak{p}''$ are dual parabolics. Items 1 and 2 of the proposition follow from the duality of $\mathfrak{p}'$ and $\mathfrak{p}''$ by the main result in [1]. Let G be a semisimple Lie group and $\mathfrak{g}=Lie(G)$ be its Lie algebra. One can choose a Cartan subalgebra $\mathfrak{h}$ and a triangular decomposition $\mathfrak{g}=\mathfrak{n}_+\bigoplus\mathfrak{h}\bigoplus\mathfrak{n}_-$, where $\mathfrak{n}_+=\bigoplus_{\alpha>0}\mathfrak{g}_\alpha$, $\mathfrak{n}_-=\bigoplus_{\alpha<0}\mathfrak{g}_\alpha$, and a basis of the Lie algebras $\mathfrak{n}_+$ and $\mathfrak{n}_-$ in the form $(e_1,e_2,\dots,e_n)$ and $(f_1,\dots,f_n)$, such that $\mathfrak{g}_\alpha\subset\mathfrak{n}_+$ if $\alpha>0$ and $\mathfrak{g}_\alpha\subset\mathfrak{n}_-$ whenever $\alpha<0$. Then, $\mathfrak{g}_\alpha$ is the linear span of the elements of the form $[\dots[[e_{i_1},e_{i_2}],e_{i_3}]\dots e_{i_s}]$ and $[\dots[[f_{i_1},f_{i_2}],f_{i_3}]\dots f_{i_s}]$ such that $\alpha_{i_1}+\cdots+\alpha_{i_s}=\alpha$ (resp., $-\alpha$); see, for instance, Chapter 1 in [21]. In our case of the Lie group $G=GL_n\mathbb{C}\times GL_m\mathbb{C}\times GL_p\mathbb{C}$ and its Lie algebra $\mathfrak{g}=Lie(G)$, the explanations of the elements e_α and f_α can be concretely expressed in terms of dual transition matrices.

On the other hand, one can naturally consider the pencil of equations (4.25) as a bundle of equations over $\mathcal{D}$ by a tautological construction. With this, we mean that the equation $S(\lambda)$ is associated with the corresponding point in its classifying space $\mathcal{D}$. This produces a natural bundle of equations parametrized by the points in $\mathcal{D}$. In this way, we can regard the vector fields and differential forms that are used in definitions of Brunovsky indices to be parametrized by the points on the moduli space $\mathcal{D}$ rather than $\lambda\in\mathbb{P}^1$. Also, one can define a period map $\wp:\widetilde{\mathbb{P}^1}\to\mathcal{D}$ sending λ to its associated flag in $\mathcal{D}$, defined on the universal cover of the most generic locus (see Remark 5.6). In this case, the analysis of the Brunovsky indices defined via the successive bracket operations of the vector fields associated with an equation $S(\lambda)$ can be equally considered over the base $\mathcal{D}$, at the point $\wp(\lambda)$.

Now, let's remove the degeneracies in the pencil and consider the most generic locus, which is the complement to the degeneracy locus. This corresponds to restricting the attention to the open subset $\mathcal{D}_\circ = K_\circ/P_\circ$ in the $\mathcal{D}$ complement to the lower-dimensional strata. The tangent space at a point in $\mathcal{D}_\circ$ is $\mathfrak{k}_\circ/\mathfrak{p}_\circ$. The vectors e_α can be considered as a basis for the tangent space at a fixed point in $\mathcal{D}_\circ$, while the dual basis of differential forms in $\mathfrak{g}^*$ can be identified with the vectors f_α. On the other hand, both of the tangent spaces to a point in $\mathcal{D}'_\circ$ and $\mathcal{D}''_\circ$ can be considered as subspaces of the corresponding point in $\mathcal{D}$, naturally. In this way, the bracket operations defining the Brunovsky indices in Definition 2.3 (resp., Definition 2.15) will correspond to the various brackets between the vectors e_α (resp., vectors f_α). Then, pushing forward everything over the domain D, the explanation of the distributions and codistributions in Definitions 2.3 and 2.15 can be identified by the bracket operations in the Lie algebra $\mathfrak{g}$. Then, the claim follows from Definition 5.4. ∎

Remark 5.6. In the proof of Proposition 5.5, we referred to a period type map period : $\mathbb{P}^1 \to \mathcal{D}$. We mention several points about this map here. The definition of this map is natural, i.e., we assign to a point s the associated flag associated with the canonical form of the ODE obtained from (4.25) at s. Natural questions about this map should be its continuity or smoothness on the generic locus and the existence of monodromies around degenerations. In general, one should assume that the period map is well defined on the universal cover of $\mathbb{P}^1 \setminus$degenerate points. But this fact does not affect the proof of Proposition 5.5. The major point in understanding the above proof is that the vector fields on the most generic locus in $\mathcal{D}$ can be pulled back over the linear pencil space and conversely. The matter for the lift to the universal cover just takes care of monodromies when walked around a degenerate point λ_0.

6. Conclusion. When the exact linearization problem is solvable for the general system (1.4), it can be linearized in two different canonical ways. In this case, one obtains two different sets of Brunovsky indices, which are invariants of the system. We have compared the two sets of Brunovsky indices in a controllable system of the canonical form (1.4) and the associated MIMO system (1.7) of dimension n. Several new results have been obtained in the course of the proof. For instance, we provide a new set of criteria equivalent to the ones in [27]. In this regard, our main result given in Theorems 4.1 and 4.4 is a complement to the result presented in [27]. The conclusion is that the two sets of invariants, namely the Brunovsky controllability indices and the Brunovsky observability indices, give two transpose partitions of n. The result has been analyzed in an example. Generally, these two sets of indices are not identical in the same system and may not have an equal number of elements. The paper's main result is beneficial in further studying the geometric properties of the MIMO systems and their observers. In addition, a potential application for further studies is analyzing the Lie group of symmetries of controllable systems and their observers. In this regard, the group of symmetries transfers a controllable system to an equivalent system. Furthermore, the dimensional duality described as the main result should demonstrate a duality between the corresponding group of symmetries for the two sets of indices. The analysis of symmetric duality in this form should come in future work.

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