Non-Abelian T-dualities in Two Dimensional $(0,2)$ Gauged Linear Sigma Models

Nana Geraldine Cabo Bizet,[a] **Josué Díaz-Correa**[b] **and Hugo García-Compeán**[b]

[a]*Departamento de Física, División de Ciencias e Ingenierías,*
Universidad de Guanajuato, Loma del Bosque 103,
C.P. 37150, León, Guanajuato, Mexico.

[b]*Departamento de Física,*
Centro de Investigación y de Estudios Avanzados del Instituto Politécnico Nacional,
P.O. Box 14-740, C.P. 07000, Ciudad de México, Mexico

E-mail: nana@fisica.ugto.mx, josue.diaz@cinvestav.mx,
hugo.compean@cinvestav.mx

ABSTRACT: Two dimensional gauged linear sigma models(GLSMs) with $(0,2)$ supersymmetry and $U(1)$ gauge group possesing global symmetries are considered. For the case obtained as a reduction from the $(2,2)$ supersymmetric GLSM, we find the Abelian T-dual, comparing with previous studies. Then, the Abelian T-dual model of the pure $(0,2)$ theory is found. Instanton corrections are also discussed in both situations. In the cases under study we explore the vacua for the scalar potential and we analyse the target space geometry of the dual model. A model with gauge symmetry $U(1) \times U(1)$ is also discussed as precursor of an interesting example with non-Abelian global symmetry. Non-Abelian T-dualization of $U(1)$ $(0,2)$ $2d$ GLSMs is implemented for models which arise as a reduction from $(2,2)$ supersymmetry; we study a concrete model with $U(1)$ gauge symmetry and $SU(2)$ global symmetry. It is shown that for a positive definite scalar potential, the dual vacua to $\mathbb{P}^1$ constitutes a disk. Instanton corrections to the superpotential are obtained and shown to be encoded in a shifting of the holomorphic function E. We conclude by analyzing an example with $SU(2) \times SU(2)$ global symmetry and we obtain that the space of dual vacua to $\mathbb{P}^1 \times \mathbb{P}^1$ consists of two copies of the disk, also for the case of positive definite potential. Here we are able to fully integrate the equations of motion leading to non-Abelian duality, improving with respect to previous $(2,2)$ studies.

KEYWORDS: T-Duality, Gauged Linear Sigma Model, $(0,2)$ Supersymmetry, non-Abelian T-duality.

Contents

1 Introduction

The study of suitable features of string theory compactifications leading to more realistic phenomenological scenarios has been always a great deal of interest in physics and mathematics [1–3]. The physical aspects involve the reproduction of important known features of the low energy physics and the possibility to have new predictions of phenomenological interest [4]. Some of these features are relevant, for instance, in the early universe or in microscopic aspects of black hole physics [5, 6], and moreover, in the derivation of the Standard Model of particle physics and beyond [7]. The study of string compactifications with fluxes have played a central role leading to more realistic relations of string theory with phenomenological phenomena at low energies [8]. A very important family of compactifications are described by a two-dimensional non-linear sigma model (NLSM) on target spaces consisting of Calabi-Yau manifolds. These models are superconformal field theories in two-dimensions

with a certain central charge with supersymmetry $(2, 2)$ [9, 10]. They have many interesting features, however they lead to low energy effective field theories consistent with Grand Unification Theories in the four-dimensional spacetime with exceptional gauge groups. Much more realistic compactifications leading to $SU(5)$ or $SO(10)$ GUTs are the non-linear sigma models with $(0, 2)$ supersymmetry [11]. This family represents a more general kind of compactifications than those of the $(2, 2)$ kind (for some reviews, see [12, 13]).

On the other hand, two-dimensional Gauged Linear Sigma Models (GLSMs) with $(2, 2)$ supersymmetry were introduced by Witten in [14], with the aim of studying solutions in string theory with the possibility to implement a change in the spacetime topology (target space) through a simple smooth variation of the parameters in the GLSM. These changes were observed in [15–18]. More importantly these models work as a theory which interpolate among different phases which involve topology change. In some models the GLSMs lead to transitions between the Calabi-Yau (CY) phase in the infrared (IR) to the Landau-Ginzburg phase in the ultraviolet (UV) and vice-versa. In the IR the CY phase is obtained by studying the supersymmetric space of vacua of the underlying effective scalar potential. This space may have different geometries depending the specific GLSM that one is considering i.e. one may vary the number of chiral superfields and the corresponding amount of charge which carry these fields and the Abelian or non-Abelian groups of the gauge sector. In the same reference [14], there were also introduced the $(0, 2)$ GLSMs in two dimensions. These models have similar properties as the $(2, 2)$ models but there are many features on which they differ. For instance, the $(0, 2)$ models are chiral. These models have been studied actively and very good sources can be found in [14, 19–21, 21–23].

Different aspects of $(2, 2)$ GLSMs have been worked out extensively, one of its prominent applications is the proof of Mirror Symmetry [24, 25]. The procedure of Hori and Vafa looks to implement the Buscher-Giveon-Roček-Verlinde target space (T-)duality algorithm (for some reviews, see [26–28]) to $(2, 2)$ GLSMs, gauging Abelian global symmetries. This procedure was successful to give a physical proof of the Mirror Symmetry correspondence. Localization properties of the partition function have been used to test Abelian T-duality in GLSMs leading to Mirror Symmetry [29]. Moreover, the duality algorithm can be generalized to consider the gauging of a non-Abelian group in contrast to the gauging of an Abelian group. This is termed the non-Abelian duality and many interesting traditional results have obtained in this direction [30–32]. More recently some interesting results involving the idea of non-Abelian duality can be found in Refs. [33–36].

As we mentioned before $(2, 2)$ GLSMs are an important tool to prove Mirror Symmetry of Calabi-Yau manifolds, in particular for the case of complete intersections of CY manifolds and toric varieties [24, 25], and there have been many studies in GLSMs and their applications [37–49]. However for GLSMs with $(0, 2)$ supersymmetry the realization of Mirror Symmetry was least apparent. Certain kind of Mirror

map can be defined for these models [50–55]. Other notions of the $(0,2)$ Mirror map are discussed in Refs. [21–23, 56–58]. In particular, in [56] it was studied the Abelian GLSM with a gauged Abelian global symmetry. The authors follow the duality algorithm mentioned previously and they obtained a dual action which is also a $(0,2)$ GLSM. In particular they found the instanton contributions in the dual action which are compatible with the instanton corrections of the original $(0,2)$ GLSM. Other developments of $(0,2)$ GLSMs in different contexts can be found in Refs. [59–64]. For a very recent overview of some important results of the GLSMs see [65, 66].

In the context of $(2,2)$ GLSMs with $U(1)$ gauge symmetry, the possibility of gauging up a non-Abelian global symmetry was explored in [67–69]. In this article, the dual action was given and the instanton corrections of the dual action were determined. Moreover, for the Calabi-Yau phase, there were given some models in where the geometry of the target space was found. One motivation to go beyond the realm of Abelian T-duality in [24] comes from the fact that there is a large set of Calabi-Yau manifolds that don't constitute complete intersections but rather Grassmanians, Pfaffians or determinantal; that can be studied as Non-Abelian GLSMs [70], and a description of the symmetries in these models is of interest [38, 40, 42], in particular the study of Mirror symmetry [44, 45, 58, 71]. Nevertheless one can also ask first the question, whether the T-dualities leading to mirror symmetry even for Abelian GLSMs can be generalized further, to obtain new geometric identifications. This is the question that we explore in this manuscript.

In the present article we start from the family of $(0,2)$ GLSMs considered in [56], arising from a reduction of the $(2,2)$ theory, and we study the gauging of a non-Abelian global symmetry. We also consider pure $(2,0)$ GLSMs models, not obtained from a supersymmetric reduction.

The present article is organized as follows: In Section 2 a brief overview of $(0,2)$ GLSMs is given. Our aim is not to provide an extensive review but to give the notation and conventions we will follow. We also discuss the field content and interactions of these models. In Section 3 we describe Abelian T-duality in $(0,2)$ GLSMs. We presented two examples, both of them with a single $U(1)$ gauge group and a pair of chiral superfields. In the first example we considered the case of a GLSM which is a reduction of a $(2,2)$ GLSM and in the second example we study the case of a pure $(0,2)$ GLSM. We find the equations of motion and the dual Lagrangian in both cases. Moreover, we compute the scalar potential and discuss the geometry of these vacua manifolds. The third example deals with a GLSM with $U(1) \times U(1)$ gauge group and $U(1)^4$ global symmetry group. We presented this model as a preliminary material which will be later generalized to non-Abelian T-duality in Section 5 . In Section 4 we perform non-Abelian dualization for a general global symmetry group G. In order to be specific we particularize to $SU(2)$ global group and give its dual Lagrangian and study its vacua manifold. Finally in Section 5 we consider an example with a $SU(2) \times SU(2)$ global symmetry. We conclude in Section 6 with our results

and conclusions. At the end of the article we added Appendix A devoted to carry out the algorithm of T-duality at the level of superfield components. We show that for the simpler example of Abelian duality in this paper, the dual action coincides with the dual action of the reduced GLSM in the superfields language presented in Section 3.

2 Field representations of $(0,2)$ supersymmetry

In this section, in order to be as self-contained as possible, we write an overview of the basic ingredients of two dimensional GLSMs with $(0,2)$ supersymmetry. Moreover, we will write down their corresponding Lagrangians and symmetries and we will describe the matter content and their interactions. Along the paper we follow the notation and conventions on supersymmetric field theory as the one in the references [72–74]. For reading background material regarding $(0,2)$ GLSMs it is useful to consult the references [14, 19, 20].

We start by reviewing the two-dimensional GLSM with $(0,2)$ supersymmetry and an Abelian gauge group, we will follow the original Witten's paper in GLSMs [14]. As usual, the coordinates of the $(0,2)$ superspace are given by $(y^0, y^1, \theta^+, \overline{\theta}^+)$. Where the first two are the space coordinates and the last two the fermionic counterparts. The covariant superderivatives are given by

$$D_+ = \partial_{\theta^+} - i\overline{\theta}^+ \partial_+ , \qquad \overline{D}_+ = -\partial_{\overline{\theta}^+} + i\theta^+ \partial_+ , \qquad (2.1)$$

where $\partial_+ := \frac{\partial}{\partial y^0} + \frac{\partial}{\partial y^1}$, $\partial_- := \frac{\partial}{\partial y^0} - \frac{\partial}{\partial y^1}$, $\partial_{\theta^+} := \frac{\partial}{\partial \theta^+}$ and $\partial_{\overline{\theta}^+} := \frac{\partial}{\partial \overline{\theta}^+}$.

The gauge covariant superderivatives $\mathcal{D}_+$, $\overline{\mathcal{D}}_+$, $\mathcal{D}_0$ and $\mathcal{D}_1$ are constructed with the following constraints:

$$\mathcal{D}_0 = D_0 , \qquad \mathcal{D}_1 = D_1 , \qquad (2.2)$$

$$\mathcal{D}_+ = e^{-\Psi} D_+ e^{\Psi} \;\; = \;\; (D_+ + D_+\Psi), \qquad (2.3)$$

$$\overline{\mathcal{D}}_+ = e^{\Psi} \overline{D}_+ e^{-\Psi} \;\; = \;\; (\overline{D}_+ - \overline{D}_+\Psi), \qquad (2.4)$$

$$\mathcal{D}_0 - \mathcal{D}_1 \;\; = \;\; \partial_- + iV, \qquad (2.5)$$

where Ψ and V are real functions, that constitute the gauge degrees of freedom. V is the $(0,2)$ vector superfield and in the Wess-Zumino gauge they can be expanded in components as follows:

$$V = v_- - 2i\theta^+ \overline{\lambda}_- - 2i\overline{\theta}^+ \lambda_- + 2\theta^+ \overline{\theta}^+ D, \qquad (2.6)$$

$$\Psi = v_+ \theta^+ \overline{\theta}^+ . \qquad (2.7)$$

The basic gauge invariant field strength Υ is defined as the field strength of V:

$$\begin{aligned}
\Upsilon &= [\overline{\mathcal{D}}_+, \mathcal{D}_0 - \mathcal{D}_1]V \\
&= \overline{D}_+(iV + \partial_-\Psi) \\
&= i\overline{D}_+V + \partial_-\overline{D}_+\Psi .
\end{aligned} \qquad (2.8)$$

In components, field strength is written as:

$$\Upsilon = -2\lambda_- + [2iD + (\partial_- v_+ - \partial_+ v_-)]\theta^+ + 2i\partial_+\lambda_-\theta^+\overline{\theta}^+ \,. \tag{2.9}$$

The $U(1)$ gauge theory has a natural Lagrangian given by

$$L_{\text{gauge}} = \frac{1}{8e^2}\int \mathrm{d}\theta^+ \mathrm{d}\overline{\theta}^+ \overline{\Upsilon}\Upsilon, \tag{2.10}$$

where e is the gauge coupling constant.

There are two kinds of matter fields: the chiral multiplets Φ and the Fermi multiplets Γ. The bosonic covariant chiral fields $\widetilde{\Phi}$ are defined by the following constraint:

$$\overline{\mathcal{D}}_+\widetilde{\Phi} = 0, \tag{2.11}$$

where $\overline{\mathcal{D}}_+$ is the covariant derivative and consequently it has the components expansion:

$$\widetilde{\Phi} = \phi + \sqrt{2}\theta^+\psi_+ - i\theta^+\overline{\theta}^+(\partial_+ + iv_+)\phi, \tag{2.12}$$

which is defined with $\widetilde{\Phi} := \Phi e^{\Psi}$, where the (uncharged) chiral superfield Φ fulfils the relation $\overline{D}_+\Phi = 0$. The corresponding gauge invariant Lagrangian is given by

$$\begin{aligned}
L_{\text{chiral}} &= -\frac{i}{2}\int \mathrm{d}\theta^+\mathrm{d}\overline{\theta}^+ \ \widetilde{\Phi}^\dagger(\mathcal{D}_0 - \mathcal{D}_1)\widetilde{\Phi} \\
&= \overline{\phi}\phi D + i\overline{\psi}_+(\partial_- + iv_-)\psi - \sqrt{2}i(\lambda_-\psi_+\overline{\phi} - \overline{\psi}_+\overline{\lambda}_-\phi) \\
&\quad - \frac{1}{2}[\overline{\phi}(\partial_- + iv_-)(\partial_+ + iv_+)\phi - (\partial_+ + iv_+)\overline{\phi}(\partial_- + iv_-)\phi].
\end{aligned} \tag{2.13}$$

In order to complete the rest of matter content let us introduce $\widetilde{\Gamma}$ which constitutes a $(0,2)$ Fermi multiplet. This multiplet satisfies the constraint:

$$\overline{\mathcal{D}}_+\widetilde{\Gamma} = \sqrt{2}\widetilde{E}\,, \qquad \text{then} \qquad \widetilde{E} = \frac{\sqrt{2}}{2}e^{\Psi}\overline{D}_+\Gamma, \tag{2.14}$$

where $\widetilde{E} = E(\widetilde{\Phi})$ is a holomorphic function of the superfield $\widetilde{\Phi}$. Similarly, we can define $\widetilde{\Gamma} := \Gamma e^{\Psi}$ and $\widetilde{E} := Ee^{\Psi}$, where $\overline{D}_+\Gamma = \sqrt{2}E$. Thus, the expansion for this Fermi multiplet and the field E are given by:

$$\widetilde{\Gamma} = \gamma - \sqrt{2}G\theta^+ - i\theta^+\overline{\theta}^+(\partial_+ + iv_+)\lambda - \sqrt{2}\widetilde{E}\overline{\theta}^+ \,. \tag{2.15}$$

$$\widetilde{E}(\Phi) = E(\phi) + \sqrt{2}\theta^+\frac{\partial E}{\partial\phi}\psi_+ - i\theta^+\overline{\theta}^+(\partial_+ + iv_+)E(\phi). \tag{2.16}$$

The dynamics of the Fermi field is given by the Lagrangian:

$$\begin{aligned}
L_{\text{Fermi}} &= -\frac{1}{2}\int \mathrm{d}\theta^+\mathrm{d}\overline{\theta}^+\widetilde{\overline{\Gamma}}\widetilde{\Gamma} \\
&= i\overline{\gamma}(\partial_+ + iv_+)\gamma + |G|^2 - |E|^2 - \left(\overline{\gamma}\frac{\partial E}{\partial\phi}\psi_+ + \frac{\partial\overline{E}}{\partial\overline{\phi}}\overline{\psi}_+\gamma\right). \tag{2.17}
\end{aligned}$$

In the case of $U(1)$ gauge theories (or in non-Abelian gauge theories with a gauge group with a $U(1)$ factor) we have an additional term in the Lagrangian given by the Fayet-Iliopoulos term

$$L_{D,\theta} = \frac{t}{4} \int \mathrm{d}\theta^+ \Upsilon|_{\overline{\theta}^+ = 0} + \text{h.c.}, \tag{2.18}$$

where $t = \frac{\theta}{2\pi} + ir$, with $\theta = 2\pi\Re(t)$ being an angular parameter and $r = \Im(t)$ is the Fayet-Iliopoulos parameter.

In (0,2) theories there is a superpotential Lagrangian L_J, which is the $(0,2)$ analog of the superpotential term of the $(2,2)$ model. L_J is of the form

$$L_J = -\frac{1}{\sqrt{2}} \int \mathrm{d}\theta^+ \left(\Gamma_p J^p(\Phi)|_{\overline{\theta}^+ = 0} \right) - \text{h.c.}, \tag{2.19}$$

where $J^p = J^p(\Phi)$ is a holomorphic function of the $(0,2)$ chiral superfield Φ, and Γ_p are Fermi superfields (different from the previous ones). Moreover J^p satisfies the relation $E_p(\Phi)J^p(\Phi) = 0$, where of course $\tilde{\mathcal{D}}_+\Gamma_p = \sqrt{2}E_p$.

The scalar potential can be obtained by the usual procedure in supersymmetric theories (integrating in the superspace) and it is given by

$$U(\phi_i) = \frac{e^2}{2} \left(\sum_i Q_i|\phi_i|^2 - r \right)^2 + \left(|E(\phi)|^2 + |J(\phi)|^2 \right), \tag{2.20}$$

where it is clear the contributions coming the D-terms from the $(0,2)$ gauge multiplet and from the FI term. The last two terms come from the E field and the last one, corresponds to the contribution from the superpotential.

In the present article it will be considered $(0,2)$ GLSMs with a $U(1)$ gauge group, and with non-Abelian global symmetries to be gauged. Thus the dynamics of the models studied is given by the addition of all of these Lagrangians, i.e.,

$$L = L_{\texttt{gauge}} + L_{\texttt{chiral}} + L_{\texttt{Fermi}} + L_{D,\theta} + L_J. \tag{2.21}$$

In the present article we will consider models without superpotential terms and therefore $L_J = 0$.

2.1 Reduction of the $(2,2)$ multiplets to $(0,2)$ superfields

It is known that certain $(0,2)$ GLSMs can be regarded as a supersymmetric reduction of $(2,2)$ GLSMs. The $(2,2)$ GLSM consists of chiral supefield $\Phi^{(2,2)}$, a vector superfield $V^{(2,2)}$ and its twisted field strength $\Sigma^{(2,2)}$.

Next, we enumerate the steps for the decomposing $(2,2)$ multiplets terms of $(0,2)$ multiplets:

- The $(2,2)$ chiral superfield $\Phi^{(2,2)}$ can be decomposed into the $(0,2)$ chiral superfield by: $\Phi = \Phi^{(2,2)}\big|_{\theta^-=\bar{\theta}^-=0}$; and the $(0,2)$ Fermi superfield can be decomposed by: $\Gamma = \frac{1}{\sqrt{2}}D_-\Phi^{(2,2)}\big|_{\theta^-=\bar{\theta}^-=0}$. Both matter fields are supersymmetry reductions of the single $(2,2)$ chiral superfield.

- The $(2,2)$ vector superfield $V^{(2,2)}$ gives the gauge field strength $\Upsilon = i\overline{D}_+(V - i\partial_-\Psi)$ by: $V - i\partial_-\Psi = -\overline{D}_-D_-V^{(2,2)}\big|_{\theta^-=\bar{\theta}^-=0}$. And also it gives a twisted chiral superfield Σ, which is identified as: $\overline{\theta}^+\Sigma = -\frac{1}{\sqrt{2}}D_-V\big|_{\theta^-=\bar{\theta}^-=0}$; which is also simply: $\Sigma = \Sigma^{(2,2)}\big|_{\theta^-=\bar{\theta}^-=0}$.

- It can be verified that if $\Phi^{(2,2)}$ has charge Q, then the E field can be written as $E = Q\sqrt{2}\Sigma\Phi$.

- The holomorphic function J can be obtained from the $(2,2)$ superpotential W in the form $J = \frac{\partial W}{\partial\widetilde{\Phi}}$.

3 Abelian T-duality in $(0,2)$ GLSMs

In this section we describe the T-duality for GLSMs with a $U^m(1)$ gauge group and a $U(1)^{k+s}$ global symmetry group to be gauged. We present the original model and find the T-dual model, by solving the equations of motion. For the sake of simplicity we consider the case when the superpotential J of the $(0,2)$ model vanishes, thus the underlying scalar potential consist only of the D-term and the Fayet-Iliopoulos term. We found the dual action and the geometry of the space of dual vacua. We study two separate cases, the first one is the case in which the $(0,2)$ GLSM can be obtained by reduction from a $(2,2)$ model. The second case is the general case of a pure $(0,2)$ GLSM which cannot be obtained from a reduction. In both cases we describe their corresponding instanton corrections. In the rest of the section we describe a particular reduced model with gauge group $U(1) \times U(1)$ and an Abelian global symmetry $U(1)^4$. This model was discussed in [56] and it will be analysed in the context of non-Abelian duality in Section 4.

3.1 GLSM with $U(1)^m$ gauge symmetry and $U(1)^{k+s}$ global symmetry

Here we describe the Abelian T-dualization for general $(0,2)$ $U(1)^m$ GLSM with $U(1)^k$ global symmetry related to the chiral fields and $U(1)^s$ global symmetry associated to the Fermi fields. We start by writing a Lagrangian with a given number of n chiral superfields Φ_i and $\widetilde{n}$ Fermi superfields Γ_j, and a given number m of $U(1)$

gauge symmetries

$$
\begin{aligned}
L &= \int \mathrm{d}\theta^+ \mathrm{d}\overline{\theta}^+ \left\{ \sum_{a=1}^{m} \frac{1}{8e_a^2} \overline{\Upsilon}_a \Upsilon_a - \sum_{i=1}^{n} \frac{i}{2} \overline{\Phi}_i e^{2\sum_{a=1}^{m} Q_i^a \Psi_a} \left(\partial_- + i \sum_{a=1}^{m} Q_i^a V_a \right) \Phi_i \right\} \\
&+ \int \mathrm{d}\theta^+ \mathrm{d}\overline{\theta}^+ \left\{ \sum_{i=1}^{n} \frac{i}{2} \overline{\Phi}_i \left(\overleftarrow{\partial}_- - i \sum_{a=1}^{m} Q_i^a V_a \right) e^{2\sum_{a=1}^{m} Q_i^a \Psi_a} \Phi_i \right\} \\
&- \int \mathrm{d}\theta^+ \mathrm{d}\overline{\theta}^+ \left\{ \sum_{j=1}^{\widetilde{n}} \frac{1}{2} e^{2\sum_{a=1}^{m} \widetilde{Q}_j^a \Psi_a} \overline{\Gamma}_j \Gamma_j \right\} \\
&+ \sum_{a=1}^{m} \frac{t_a}{4} \int \mathrm{d}\theta^+ \Upsilon_a |_{\overline{\theta}^+ = 0},
\end{aligned}
\tag{3.1}
$$

where Q_i^a are the charges of the chiral superfields Φ_i and $\widetilde{Q}_j^a$ are the charges of the Fermi superfields Γ_i^a.

There are m vector superfields V_a, Ψ_a with field strength Υ_a. In principle each kinetic term has a global phase symmetry, under which the chiral or the Fermi fields transform. As all the superfields are distinct, one can employ the m gauge symmetries to absorb m of these phases, giving a total of $k+s$ global symmetries where $k = n-m$ $(n > m)$ $U(1)$ global symmetries, these fields transform as follows:

$$
\delta_\Lambda V_a = -\partial_-(\Lambda_a + \overline{\Lambda}_a)/2, \qquad \delta\Psi = -i(\Lambda_a - \overline{\Lambda}_a)/2, \tag{3.2}
$$

$$
\Phi_i \rightarrow e^{i\sum_{a=1}^{m} Q_i^a \Lambda_a} \Phi_i, \qquad \Gamma_j \rightarrow e^{i\sum_{a=1}^{m} \widetilde{Q}_j^a \Lambda_a} \Gamma_j. \tag{3.3}
$$

In general we would have the possibility to absorb with the m Abelian gauge symmetries not only the global symmetries of the chiral superfields but the total amount of chiral and Fermi fields $n + \widetilde{n}$. The master Lagrangian will remain the same with some few modifications in the sum's indices. This case will be not considered in this work.

In general one can consider a generic number of Fermi multiplets, this is true because the general $(0,2)$ model, presented here, doesn't come necessarily from a SUSY reduction from the $(2,2)$ theory. Therefore the Fermi multiplets are not necessarily related or coupled to the chiral multiplets. In the opposite case when the chiral superfields and the Fermi superfields come both from the $(2,2)$ reduction, the number of Fermi and chiral fields and their charges need to match.

Starting from (3.1) one can construct the *master Lagrangian* (or also named intermediate Lagrangian) by gauging the global symmetries and adding terms with

Lagrange multipliers Λ_b related to field strengths Υ_b

$$
\begin{aligned}
L_{\texttt{master}} = & \int \mathrm{d}\theta^+ \mathrm{d}\overline{\theta}^+ \sum_{a=1}^{m} \frac{1}{8e_a^2} \overline{\Upsilon}_a \Upsilon_a \\
& - \int \mathrm{d}\theta^+ \mathrm{d}\overline{\theta}^+ \left\{ \sum_{i=1}^{k} \frac{i}{2} \overline{\Phi}_i e^{2\sum_{a=1}^{m} Q_i^a \Psi_a + 2\sum_{b=1}^{k} Q_{1i}^b \Psi_{1b}} \left(\partial_- + i\sum_{a=1}^{m} Q_i^a V_a + i\sum_{b=1}^{k} Q_{1i}^b V_{1b} \right) \Phi_i \right\} \\
& + \int \mathrm{d}\theta^+ \mathrm{d}\overline{\theta}^+ \left\{ \sum_{i=1}^{k} \frac{i}{2} \overline{\Phi}_i \left(\overleftarrow{\partial}_- - i\sum_{a=1}^{m} Q_i^a V_a - i\sum_{a=1}^{k} Q_{1i}^b V_{1b} \right) e^{2\sum_{a=1}^{m} Q_i^a \Psi_a + 2\sum_{b=1}^{k} Q_{1i}^b \Psi_{1b}} \Phi_i \right\} \\
& - \int \mathrm{d}\theta^+ \mathrm{d}\overline{\theta}^+ \left\{ \sum_{j=1}^{s} \frac{1}{2} e^{2\sum_{a=1}^{m} \widetilde{Q}_j^a \Psi_a + 2\sum_{c=1}^{s} \widetilde{Q}_{1j}^c \Psi_{1c}} \overline{\Gamma}_j \Gamma_j + \sum_{j=s+1}^{\widetilde{n}} \frac{1}{2} e^{2\sum_{a=1}^{m} \widetilde{Q}_j^a \Psi_a} \overline{\Gamma}_j \Gamma_j \right\} \\
& + \sum_{a=1}^{m} \frac{t_a}{4} \int \mathrm{d}\theta^+ \Upsilon_a |_{\overline{\theta}^+=0} + \sum_{b=1}^{k} \int \mathrm{d}\theta^+ \mathrm{d}\overline{\theta}^+ \Lambda_b \Upsilon_{1b} + \sum_{b=k+1}^{k+s} \int \mathrm{d}\theta^+ \mathrm{d}\overline{\theta}^+ \Lambda_b \Upsilon_{1b} + \text{h.c.} \\
& - \int \mathrm{d}\theta^+ \mathrm{d}\overline{\theta}^+ \sum_{i=k+1}^{n} \left\{ \frac{i}{2} \overline{\Phi}_i e^{2\sum_{a=1}^{n} Q_i^a \Psi_a} \left(\partial_- + i\sum_{a=1}^{m} Q_i^a V_a \right) \Phi_i \right. \\
& \left. - \frac{i}{2} \overline{\Phi}_i \left(\overleftarrow{\partial}_- - i\sum_{a=1}^{m} Q_i^a V_a \right) e^{2\sum_{a=1}^{m} Q_i^a \Psi_a} \Phi_i \right\}.
\end{aligned}
\tag{3.4}
$$

For simplicity in this expression we are assuming that the chiral superfields are not charged under the global symmetries that the Fermi superfields are charged, and vice-versa. One could choose that each of the chiral superfields to dualize it is charged only under a single $U(1)$ global, such that $Q_{1i}^b = \delta_i{}^b$, as it was done by Hori and Vafa in their fundamental work on Mirror Symmetry as a T-duality [24]. There are $U(1)^{k+s}$ global symmetries, where $k + s = n - m + s$. For models coming from supersymmetric reduction s is zero and the Fermi superfield will be gauged with the same global symmetry implemented by the chiral superfields. In the general case there will be additional global symmetries arising thank to the presence of Fermi superfields to those come from the chiral superfields in the $(2,2)$ GLSM.

Let us now analyze the equations of motion from this master Lagrangian when the gauged fields are integrated. Due to the Weiss-Zumino gauge (2.7), $e^{2\Psi} = 1 + 2\Psi$. In this way, the fields Ψ_1, V_1 and Γ_1 are linear and the variation is easy performed. Carrying out the variation of the Lagrangian with respect to ψ_{1b} we obtain for the field V_{1b}:

$$
V_{1b} = A_{bd}^{-1} \left(-\frac{i}{2} \partial_- Y_-^d - R^d \right),
\tag{3.5}
$$

where

$$
A_{bd} = \sum_{i=1}^{k} |\phi_i|^2 Q_{1i}^d Q_{1i}^b,
\tag{3.6}
$$

and

$$
R^d = \sum_{i=1}^{k} \left(-\frac{i}{2} \overline{\Phi}_i \delta_- \Phi_i Q_{1i}^d + |\Phi_i|^2 \sum_{a=1}^{m} Q_i^a V_a Q_{1i}^d \right).
\tag{3.7}
$$

Here the new dual variable is defined by: $Y_\pm^c \equiv i\overline{D}_+\Lambda^c \pm iD_+\overline{\Lambda}^c$, and for simplicity, it has been used $\delta_- = \partial_- - \overleftarrow{\partial}_-$.

Performing the variation of the Lagrangian with respect to V_{1b}, for the component ψ_{1b} we have

$$\psi_{1b} = A_{bd}^{-1}\left(-\frac{i}{2}\partial_- Y_+^d - S^d\right), \qquad (3.8)$$

and

$$S^d = \sum_{i=1}^k |\Phi_i|^2 Q_{1i}^d + 2\sum_{a=1}^m |\Phi_i|^2 Q_{1i}^d Q_{1i}^a \psi_a. \qquad (3.9)$$

The variation with respect the component ψ_{1d} for $d \in \{k+1, ..., 2k\}$ yields

$$Q_{1j}^d \overline{\Gamma}_j \Gamma_j = -i\partial_- Y_-^d \to \overline{\Gamma}_j \Gamma_j = -Q_{1jd}^{-1}\partial_- Y_-^d. \qquad (3.10)$$

These equations of motion are employed to find the dual model.

3.2 A T-duality algorithm from a model coming from (2,2) reduction

In this subsection we obtain explicit expressions for the equations of motion of a general master Lagrangian of a $(0,2)$ GLSM which is obtained by reduction of a $(2,2)$ model. In this case, there is a Fermi superfield for every chiral superfield and there could be also extra Fermi superfields. These Fermi fields have the same charges under the gauge group than the chiral superfields related to them i.e. $Q_i = \widetilde{Q}_i$ and we consider the case $s = 0$, such that there are no extra Fermi fields. Then all the global symmetries will affect equally the chiral superfields and the Fermi superfields. In this case, the duality procedure will be carry out in the fields Φ and Γ, and there are Lagrange multipliers $\overline{\chi}$ associated to E. So, the new dual field is $\widetilde{\mathcal{F}} = e^\Psi \mathcal{D}_+ \chi$.

We start from the following Lagrangian, with n chiral fields and then n Fermi fields (related to them) and without any extra Fermi field. This is $\widetilde{n} = n$:

$$
L_{\texttt{master}} = \int \mathrm{d}\theta^+ \mathrm{d}\overline{\theta}^+ \sum_{a=1}^{m} \frac{1}{8e_a^2} \overline{\Upsilon}_a \Upsilon_a
$$

$$
- \int \mathrm{d}\theta^+ \mathrm{d}\overline{\theta}^+ \left\{ \sum_{i=1}^{k} \frac{i}{2} \overline{\Phi}_i e^{2\sum_{a=1}^{m} Q_i^a \Psi_a + 2\sum_{b=1}^{k} Q_{1i}^b \Psi_{1b}} \left(\partial_- + i\sum_{a=1}^{m} Q_i^a V_a + i\sum_{b=1}^{k} Q_{1i}^b V_{1b} \right) \Phi_i \right\}
$$

$$
+ \int \mathrm{d}\theta^+ \mathrm{d}\overline{\theta}^+ \left\{ \sum_{i=1}^{k} \frac{i}{2} \overline{\Phi}_i (\overleftarrow{\partial}_- - i\sum_{a=1}^{m} Q_i^a V_a - i\sum_{b=1}^{k} Q_{1i}^b V_{1b}) e^{2\sum_{a=1}^{m} Q_i^a \Psi_a + 2\sum_{b=1}^{k} Q_{1i}^b \Psi_{1b}} \Phi_i \right\}
$$

$$
- \int \mathrm{d}\theta^+ \mathrm{d}\overline{\theta}^+ \left\{ \sum_{j=1}^{k} \frac{1}{2} e^{2\sum_{a=1}^{m} Q_j^a \Psi_a + 2\sum_{b=1}^{k} Q_{1j}^b \Psi_{1b}} (\overline{\Gamma}_j + \overline{\Gamma}_{1j})(\Gamma_j + \Gamma_{1j}) \right\}
$$

$$
+ \sum_{a=1}^{m} \frac{t_a}{4} \int \mathrm{d}\theta^+ \Upsilon_a \big|_{\overline{\theta}^+ = 0} + \sum_{b=1}^{k} \int \mathrm{d}\theta^+ \mathrm{d}\overline{\theta}^+ \Lambda_b \Upsilon_b + \sum_{b=1}^{k} \int \mathrm{d}\theta^+ \mathrm{d}\overline{\theta}^+ \overline{\chi}_b E_b + \text{h.c.}
$$

$$
- \frac{i}{2} \int \mathrm{d}\theta^+ \mathrm{d}\overline{\theta}^+ \sum_{i=k+1}^{n} \left\{ \overline{\Phi}_i e^{2\sum_{a=1}^{m} Q_i^a \Psi_a} \left(\partial_- + i\sum_{a=1}^{m} Q_i^a V_a \right) \Phi_i \right. \tag{3.11}
$$

$$
\left. - \overline{\Phi}_i \left(\overleftarrow{\partial}_- - i\sum_{a=1}^{m} Q_i^a V_a \right) e^{2\sum_{a=1}^{m} Q_i^a \Psi_a} \Phi_i \right\} + \int \mathrm{d}\theta^+ \mathrm{d}\overline{\theta}^+ \sum_{j=k+1}^{\widetilde{n}} \frac{1}{2} e^{2\sum_{a=1}^{k} Q_j^a \Psi_a} \overline{\Gamma}_j \Gamma_j .
$$

The terms in the last two lines are not charged under the global (gauged) symmetry so those fields behave as spectators. The main difference with the other case (without reduction) lies on the dualized fields. In this reduced model, the Fermi fields are also gauged and new Lagrange multipliers were added, these terms are located on the 5th line of previous equation. If exists $Q \in GL(k)$ such that $(Q)_i^c := Q_{1i}^c$, then let be $X := Q^{-1}$ to find the variations, which results in:

$$
\delta_{V_{1c}} S = 0 : \qquad \left(1 + 2\sum_{a=1}^{m} Q_j^a \Psi_a + 2\sum_{b=1}^{k} Q_{1j}^b \Psi_{1b} \right) = -\frac{X_c^j Y_+^c}{|\Phi_j|^2}
$$

$$
\text{or} \qquad \Psi_{1d} = \sum_{j=1}^{k} X_d^j \left(\frac{X_c^j Y_+^c}{2|\Phi_j|^2} + \frac{1}{2} + Q_j^a \Psi_a \right), \qquad (3.12)
$$

$$
\delta_{\Psi_{1c}} S = 0 : \qquad -i\overline{\Phi}_j \Phi \delta_- \Phi_j + 2\sum_{a=1}^{m} Q_j^a V_a |\Phi_j|^2 + 2\sum_{b=1}^{k} Q_{1j}^b V_{1b} |\Phi_j|^2 = -\frac{i}{2} X_c^j \partial_- Y_-^c,
$$

$$
\tag{3.13}
$$

$$
\delta_{\Gamma_{1j}} S = 0 : \qquad \left(1 + 2\sum_{a=1}^{m} Q_j^a \Psi_a + 2\sum_{b=1}^{k} Q_{1j}^b \Psi_{1b} \right) (\overline{\Gamma}_j + \overline{\Gamma}_{1j}) = -\sqrt{2} \widetilde{\mathcal{F}}_j^{\dagger}. \qquad (3.14)
$$

Notice that we have solved the equations for the gauged fields, we denote $\delta_f X$ to the equation of motion obtained for the field X.

From the derived equations of motion, one can obtain the dual Lagrangian for the $(0,2)$ models. In general it is involved to carry out this program, thus we shall consider in this work the simpler models, with some specific values of m, n and $\widetilde{n}$. This will be developed in the following section for the case of susy reduction and then for the pure $(2,0)$ case.

3.3 GLSMs with a $U(1)$ global symmetry

Now we consider a concrete model described by the GLSM Lagrangian with $m = 1$, two chiral multiplets $(n = 2)$ and some Fermi superfields $\widetilde{n}$. One of each kind will act as spectator field, besides it will be considered $Q = 1$.

For the implementation of the algorithm of duality we consider only the relevant terms in the Lagrangian; which we call partial Lagrangian ΔL, where the other terms as the kinetic energies of the gauge fields, the FI terms and the spectator fields are omitted. Thus the partial Lagrangian is given by:

$$\Delta L_{\texttt{original}} = \int \mathrm{d}\theta^+ \mathrm{d}\overline{\theta}^+ \left\{ \frac{1}{8e^2}\overline{\Upsilon}\Upsilon - \frac{i}{2}\overline{\Phi}e^{2\Psi}\left(\partial_- + iV\right)\Phi + \frac{i}{2}\overline{\Phi}\left(\overleftarrow{\partial}_- - iV\right)e^{2\Psi}\Phi \right.$$

$$\left. -\frac{1}{2}e^{2\Psi}\overline{\Gamma}\Gamma \right\} + \frac{t}{4}\int \mathrm{d}\theta^+ \Upsilon|_{\overline{\theta}^+=0} + \text{h.c.} \,. \tag{3.15}$$

From this common Lagrangian, the 2 cases: the reduction from $(2,2)$ and the pure $(0,2)$ case are taken.

3.3.1 $(0,2)$ GLSM from a reduction of a $(2,2)$ GLSM

As we mentioned before in the case when the $(0,2)$ model is obtained as a reduction from a $(2,2)$ model, all the chiral multiplets have associated an only Fermi field $(s = 0)$ and the E field has a special form with the reduced fields given by

$$E = iQ\sqrt{2}\Sigma'\Phi' \,, \tag{3.16}$$

where $\Sigma' = \Sigma|_{\theta^- = \overline{\theta}^- = 0}$, and $\Phi' = \Phi|_{\theta^- = \overline{\theta}^- = 0}$. Thus the gauged Lagrangian is written as

$$\Delta L_{\texttt{master}} = \int \mathrm{d}\theta^+ \mathrm{d}\overline{\theta}^+ \left\{ -\frac{i}{2}\overline{\Phi}e^{2(\Psi+\Psi_1)}\left(\partial_- + i(V+V_1)\right)\Phi \right. \tag{3.17}$$

$$+ \frac{i}{2}\overline{\Phi}\left(\overleftarrow{\partial}_- - i(V+V_1)\right)e^{2(\Psi+\Psi_1)}\Phi$$

$$\left. -\frac{1}{2}e^{2(\Psi+\Psi_1)}(\overline{\Gamma}+\overline{\Gamma}_1)(\Gamma+\Gamma_1) + \Lambda\Upsilon_1 + \overline{\Upsilon}_1\overline{\Lambda} + \overline{\chi}\widetilde{E}_1 + \overline{\widetilde{E}}_1\chi \right\}.$$

Thus, the Eqs. (3.13-3.14) substituted back into (3.17) lead to the following Lagrangian

$$\Delta L_{\text{master}} = \int d\theta^+ d\overline{\theta}^+ \left\{ -\frac{i}{2} \frac{Y_-\partial_-Y_+}{Y_+} + \frac{|\Phi|^2 \widetilde{\mathcal{F}}\overline{\widetilde{\mathcal{F}}}}{Y_+} - \left(\Lambda \Upsilon + \overline{\Upsilon\Lambda} + \bar{\chi}E + \overline{E}\chi \right) \right\}. \quad (3.18)$$

Then, after the gauge fixing $|\Phi|^2 = 1$ the dual Lagrangian is obtained and it can be written as:

$$\Delta L_{\text{Dual}} = \int d\theta^+ d\overline{\theta}^+ \left\{ -\frac{i}{2} \frac{Y_-\partial_-Y_+}{Y_+} + \frac{\widetilde{\mathcal{F}}\overline{\widetilde{\mathcal{F}}}}{Y_+} \right\} - \int d\overline{\theta}^+ \left(iY\Upsilon - \bar{E}F \right) + \text{h.c.}. \quad (3.19)$$

This process can also be realized by components, gauging up each component field to find the scalar potential; this procedure is carried out in Appendix A.

Now we describe the various contributions to the scalar potential coming from the complete dual action (3.19). This is we show the complete Lagrangian adding Fayet-Iliopoulos term, gauge action and the spectators fields. The kinetic term of the dual variable Y in the first term of the dual action $-\frac{i}{2}\frac{Y_-\partial_-Y_+}{Y_+}$, does not contribute to the scalar potential. The third term in the Lagrangian $iY\Upsilon_0$ leads to a scalar potential of the form $-2Dy_+ + 2iv_{01}y_- + h.c.$ Moreover, the Fayet-Iliopoulos term gives rise to a potential of the form $D(\frac{it}{2} - \frac{i\bar{t}}{2}) + v_{01}(\frac{t}{2} + \frac{\bar{t}}{2})$. The gauge sector L_{gauge} contributes with a term of the form $\frac{v_{01}^2}{2e^2} + \frac{D^2}{2e^2}$. We have two additional contributions from the terms $\frac{\widetilde{\mathcal{F}}\overline{\widetilde{\mathcal{F}}}}{Y_+}$ and $-\widetilde{\mathcal{F}}E + \text{h.c.}$ which lead to terms of the potential of the form $\frac{-2H\overline{H}}{y_+}$ and $-\sqrt{2}(HE + \overline{EH})$, respectively.

Thus, the scalar potential coming from (3.19) can be written as

$$U_{\text{dual}} = D\left(\frac{i}{2}(t - \bar{t}) - 2y_+ + |\phi_2|^2 \right) + \frac{D^2}{2e^2} + \frac{v_{01}^2}{2e^2} + v_{01}\left(\frac{i}{2}(t + \bar{t}) - 2iy_- \right)$$

$$+ \frac{H\overline{H}}{\Re(y)} + \sqrt{2}(HE + \overline{EH}). \quad (3.20)$$

After eliminating the auxiliary fields D and v_{01}, the potential is:

$$U_{\text{dual}} = \frac{e^2}{2}\left(-\Im(t) - \Re(y) + |\phi_2|^2 \right)^2 \quad (3.21)$$

$$+ \frac{e^2}{2}\left(\Re(t) + \Im(y) \right)^2 + \frac{H\overline{H}}{\Re(y)} + \sqrt{2}(HE + \overline{EH}) \quad (3.22)$$

which minimum condition with respect to H, E and $\Re(y)$ gives

$$E = 0, \qquad H = 0; \qquad \Re(y) = |\phi_2|^2 - \Im(t). \quad (3.23)$$

Thus the minimum for the dual scalar potential is

$$U_{\text{dual}} = 0 \rightarrow |\phi_2|^2 - \Re(y) = \Im(t), \quad (3.24)$$

while for the original theory is:

$$U_{\texttt{original}} = 0 \rightarrow |\phi|^2 + |\phi_2|^2 = \Im(t). \tag{3.25}$$

From ec. (3.24) one obtains a cone with vertex at $y_+ = -r$. Considering the $U(1)$ gauge symmetry this will lead to the line $\mathbb{R}^+$, such that the dual expected vacua is $\mathbb{R}^+ \times \mathbb{R}$ while for the original model is $\mathbb{P}^1$. Notice that this dual is not the Abelian T-dual leading to the mirror pair obtained in [24, 56]. That to obtain the mirror pair we would need to add an spectator chiral superfield and gauge two $U(1)$ symmetries, one for each chiral superfield.

The superpotential of the original theory is given by [20]:

$$W_{\texttt{original}} = \frac{\Upsilon}{4\pi\sqrt{2}} \ln\left(\frac{\Sigma}{q\mu}\right). \tag{3.26}$$

We propose the following ansatz for superpotential of the dual theory:

$$W_{\texttt{dual}} = iY\Upsilon - \overline{E}F + \beta F e^{\alpha Y}, \tag{3.27}$$

thus we have

$$\begin{aligned} W_{\texttt{dual}} &= \frac{i\Upsilon}{\alpha} \ln\left(\frac{\overline{E}}{\beta}\right) \\ &= \frac{i\Upsilon}{\alpha} \ln\left(\frac{-i\Upsilon_0}{\alpha\beta F}\right). \end{aligned} \tag{3.28}$$

For $(0, 2)$ theories coming from a reduction of a (2,2) model with $\overline{E} = -iQ\sqrt{2}\Sigma\Phi$, the nonperturbative dual superpotential is written as [20]

$$W_{\texttt{dual}} = \frac{i\Upsilon}{\alpha} \ln\left(\frac{\Sigma}{\beta/(-iQ\sqrt{2}\Phi)}\right), \tag{3.29}$$

where we can see by comparison with (3.28) the choices of:

$$\alpha = 4i\pi\sqrt{2} \qquad \text{and} \qquad \beta = -iq\mu\sqrt{2}Q\Phi. \tag{3.30}$$

3.3.2 The case of a pure $(0, 2)$ GLSM

In this section we consider a building block $(0, 2)$ model which is not coming from reduction of a $(2, 2)$ case, and study the Abelian T-duality on it. In this case we have $m = 1$, $n = 2$, $\widetilde{n} = 1$, $k = 1$ and $s = 0$.

For a pure $(0, 2)$ Abelian case the model is the same

$$\mathbf{\Delta}L_{\texttt{master}} = \int d\theta^+ d\overline{\theta}^+ \left\{ \frac{1}{2}(1 + 2\Psi)(-i\overline{\Phi}\delta_-\Phi + 2V\overline{\Psi}\Psi - \overline{\Gamma}\Gamma) \right\}. \tag{3.31}$$

But this time the field Γ is not dualized and the gauged fields are only V and Ψ. Thus the dual Lagrangian is given by

$$\boldsymbol{\Delta} L_{\mathtt{dual}} = \int \mathrm{d}\theta^+ \mathrm{d}\overline{\theta}^+ \left\{ -\frac{i}{2} \frac{Y_- \partial_- Y_+}{Y_+} + \frac{Y_+ \overline{\Gamma} \Gamma}{2|\Phi|^2} \right\} + \int \mathrm{d}\theta^+ (iY\Upsilon) + \frac{t}{4} \int \mathrm{d}\theta_+ \Upsilon + \text{h.c.} \quad (3.32)$$

The scalar potential is found to be the same that in the previous case discussed in Section 3.1.1, except for the term $\frac{Y_+ \overline{\Gamma}\Gamma}{2}$ which contributes to the scalar potential with a term of the form $y_+ \overline{G}G - y_+ \overline{E}E$. Gathering all that, it results that the scalar potential of the dual theory given by

$$U_{\mathtt{dual}} = D\left(\frac{i}{2}(t-\overline{t}) - 2y_+ + |\phi|^2 \right) + \frac{D^2}{2e^2} + \frac{v_{01}^2}{2e^2} + v_{01}\left(\frac{i}{2}(t+\overline{t}) - 2iy_- \right) + 2\Re(y)(E\overline{E} - G\overline{G}). \quad (3.33)$$

Thus, after eliminating D, the potential is:

$$U_{\mathtt{dual}} = \frac{e^2}{2}\left(-\Im(t) - \Re(y) + |\phi_2|^2 \right)^2 + \frac{e^2}{2}\left[\Re(t) + \Im(y) \right]^2 + 2\Re(y)(E\overline{E} - G\overline{G}), \quad (3.34)$$

which minimum condition with respect to G, E and $\Re(y)$ gives:

$$E = 0, \qquad G = 0; \qquad \Re(y) = |\phi_2|^2 - \Im(t). \quad (3.35)$$

Thus the minimum for the dual scalar potential is as in the previous case:

$$U_{\mathtt{dual}} = 0 \to |\phi_2|^2 - \Re(y) - \Im(t) = 0. \quad (3.36)$$

This is precisely the same equation found in the previous case (3.24) and consequently the topology of the manifold of vacua is a also $\mathbb{R}^+ \times \mathbb{R}$. Recall that for the original model the scalar potential reads:

$$U_{\mathtt{original}} = 0 \to |\phi|^2 + |\phi_2|^2 = \Im(t), \quad (3.37)$$

which together with the $U(1)$ gauge symmetry it constitutes a $\mathbb{P}^1$.

Thus, the ansatz for the superpotential in the dual model is

$$W_{\mathtt{dual}} = iY\Upsilon + \beta e^{\alpha(Y+1)}$$

$$= \frac{i\Upsilon}{\alpha}\left[\ln\left(\frac{-i\Upsilon}{\alpha\beta} \right) \right]. \quad (3.38)$$

In this case it was not possible to compare this to a corresponding case already discussed in the literature, but we plan to discuss is obtention elsewhere.

3.3.3 A model with two Abelian gauge symmetries

As an example, let us apply this T-dualization procedure to the case of a $(0, 2)$ GLSM coming from a reduction, as discussed in Ref. [56]. This is a GLSM with two gauge groups $U(1)$. We have to gauge 4 global symmetries, this is 3 chiral fields Φ and 3 Fermi fields Γ under a $U(1)$ gauge symmetry and another 3 chiral $\widetilde{\Phi}$ and Fermi $\widetilde{\Gamma}$ with the other $U(1)$. In this case it has to be taken $m = 2$, $n = 6$, $\widetilde{n} = 6$, $k = 2$ and $s = 0$. The former example has to give the same dual model, apart from the 2 spectator fields. Let us write the master Lagrangian in general:

$$
\begin{aligned}
L_{\mathtt{master}} = & -\frac{i}{2} e^{2\Psi_1 + 2\Psi_1'} \overline{\Phi}_1 \Big(\partial_- - i(V_1 + V_1') \Big) \Phi_1 + \text{h.c.} \\
& -\frac{i}{2} e^{2\Psi_1 + 2\Psi_2'} \bar{\Phi}_2 \Big(\partial_- - i(V_1 + V_2') \Big) \Phi_2 + \text{h.c.} \\
& -\frac{i}{2} e^{2\Psi_2 + 2\Psi_3'} \overline{\widetilde{\Phi}_1} \Big(\partial_- - i(V_2 + V_3') \Big) \widetilde{\Phi}_1 + \text{h.c.} \\
& -\frac{i}{2} e^{2\Psi_2 + 2\Psi_4'} \overline{\widetilde{\Phi}}_2 \Big(\partial_- - i(V_2 + V_4') \Big) \widetilde{\Phi}_2 + \text{h.c.} \\
& -\frac{1}{2} (\overline{\Gamma}_1 + \overline{\Gamma'}_1) e^{2\Psi_1 + 2\Psi_1'} (\Gamma_1 + \Gamma_1') - \frac{1}{2} (\overline{\Gamma}_2 + \overline{\Gamma'}_2) e^{2\Psi_1 + 2\Psi_2'} (\Gamma_2 + \Gamma_2') \\
& -\frac{1}{2} (\overline{\widetilde{\Gamma}}_1 + \overline{\widetilde{\Gamma}}_1') e^{2\Psi_2 + 2\Psi_3'} (\widetilde{\Gamma_1} + \widetilde{\Gamma_1'}) - \frac{1}{2} (\overline{\widetilde{\Gamma}}_2 + \overline{\widetilde{\Gamma}}_2') e^{2\Psi_2 + 2\Psi_4'} (\widetilde{\Gamma}_2 + \widetilde{\Gamma}_2') \\
& + \Lambda_1 \Upsilon_1' + \Lambda_2 \Upsilon_2' + \Lambda_3 \Upsilon_3' + \Lambda_4 \Upsilon_4' + \text{h.c.} + \overline{\chi}_1 E_1' + \overline{\chi}_2 E_2' + \overline{\chi}_3 E_3' + \overline{\chi}_4 E_4' + \text{h.c.} \\
& -\frac{i}{2} e^{2\Psi_1} \overline{\Phi}_3 \Big(\partial_- - iV_1 \Big) \Phi_3 + \text{h.c.} - \frac{i}{2} e^{2\Psi_2} \overline{\widetilde{\Phi}}_3 \Big(\partial_- - iV_2 \Big) \widetilde{\Phi}_3 + \text{h.c.} \\
& -\frac{1}{2} e^{2\Psi_1} \overline{\Gamma}_3 \Gamma_3 - \frac{1}{2} e^{2\Psi_1} \overline{\widetilde{\Gamma}}_3 \widetilde{\Gamma}_3 + \int \mathrm{d}\theta^+ \mathrm{d}\overline{\theta}^+ \Big\{ \frac{1}{8e_1^2} \overline{\Upsilon}_1 \Upsilon_1 + \frac{1}{8e_2^2} \overline{\Upsilon}_2 \Upsilon_2 \Big\} \\
& + \frac{t_1}{4} \int \mathrm{d}\theta^+ \Upsilon_1 |_{\overline{\theta}^+ = 0} + \frac{t_2}{4} \int \mathrm{d}\theta^+ \Upsilon_2 |_{\overline{\theta}^+ = 0} + \text{h.c.}
\end{aligned}
\tag{3.39}
$$

Dual fields to the Fermi multiplet are given by $F = \bar{D}_+ \chi$, $\mathcal{F} = e^{\psi} F$. The scalar potential, the analysis of the supersymmetric vacua and the instanton corrections will not discussed here; since for the case of the non-Abelian global symmetry it will be discussed in detail in Section 5. The Lagrangian previously obtained constitutes two copies of (3.19), and that this Lagrangian is exactly the one obtained by [56] excluding the spectator terms.

4 GLSMs with gauge group $U(1)^m$ and non-Abelian global symmetries

In this section we construct the non-Abelian dual models of $(0, 2)$ GLSMs. This can be implemented when there is a non-Abelian global symmetry present. Thus duality

algorithm is realized gauging this non-Abelian symmetry and adding Lagrange multipliers which take values in the Lie Algebra of the global group. The only models considered in the present section are assumed to come from a reduction of a $(2,2)$ supersymmetric model thus the number of chiral fields $\boldsymbol{\Phi}_i$ and the number of Fermi fields $\boldsymbol{\Gamma}_i$ coincide. And they are equally charged under the $U(1)^m$ gauge group and they are assumed to be also equally charged under the global group. Moreover in order to be as general as possible, we consider models where the total non-Abelian gauged group is $G = G_1 \times \cdots \times G_S$. The Lagrangian of this model can be written as

$$
\begin{aligned}
L_{\mathtt{master}} = & \int \mathrm{d}\theta^+ \mathrm{d}\overline{\theta}^+ \sum_{a=1}^m \frac{1}{8e_a^2} \overline{\Upsilon}_a \Upsilon_a \\
& - \int \mathrm{d}\theta^+ \mathrm{d}\overline{\theta}^+ \left\{ \sum_{I=1}^S \frac{i}{2} \boldsymbol{\Phi}_I^\dagger e^{2\sum_{a=1}^m Q_I^a \Psi_a + 2\Psi_{1I}} \left(\partial_- + i\sum_{a=1}^m Q_I^a V_a + iV_{1I} \right) \boldsymbol{\Phi}_I \right\} \\
& + \int \mathrm{d}\theta^+ \mathrm{d}\overline{\theta}^+ \left\{ \sum_{I=1}^S \frac{i}{2} \boldsymbol{\Phi}_I^\dagger \left(\overleftarrow{\partial}_- - i\sum_{a=1}^m Q_I^a V_a - iV_{1I} \right) e^{2\sum_{a=1}^m Q_I^a \Psi_a + 2\Psi_{1I}} \boldsymbol{\Phi}_I \right\} \\
& - \int \mathrm{d}\theta^+ \mathrm{d}\overline{\theta}^+ \left\{ \sum_{I=1}^S \frac{1}{2} (\boldsymbol{\Gamma}_I^\dagger + \boldsymbol{\Gamma}_{1I}^\dagger) e^{2\sum_{a=1}^m Q_I^a \Psi_a + 2\Psi_{1I}} (\boldsymbol{\Gamma}_I + \boldsymbol{\Gamma}_{1I}) \right\} \\
& + \int \mathrm{d}\theta^+ \sum_{a=1}^m \frac{t_a}{4} \Upsilon_a \big|_{\overline{\theta}^+=0} + \int \mathrm{d}\theta^+ \mathrm{d}\overline{\theta}^+ \sum_{I=1}^S \mathrm{Tr}\big(\Lambda_I \Upsilon_I \big) + \text{h.c.} \\
& + \int \mathrm{d}\theta^+ \mathrm{d}\overline{\theta}^+ \sum_{I=1}^S \big(\boldsymbol{\chi}_I^\dagger \widetilde{\boldsymbol{E}}_I \big) + \text{h.c.},
\end{aligned}
\tag{4.1}
$$

where $\boldsymbol{\Phi}_I = (\Phi_{In_1}, \ldots, \Phi_{In_I})$, with $I \in \{1, \ldots, S\}$, are vectors of chiral superfields, n_I is the dimension of the representation of the Lie algebra of G_I and $V_{1I} = V_{1Ia}\mathcal{T}_a$, $\Psi_{1I} = \Psi_{1Ia}\mathcal{T}_a$ are superfields for each factor gauged group G_I, and it is assumed an inner product. In these definitions $\mathcal{T}^a$ are the generators of the Lie algebra of G_I. In the notation of the Lagrangian it is understood an inner product on the vector space indexed by the number of factors of the global group, thus we have to sum over the S factors there. For the implementation of the duality algorithm we write

the partial Lagrangian given by

$$\mathbf{\Delta}L_{\texttt{master}} = \sum_{I=1}^{S}\int d\theta^{+}d\overline{\theta}^{+}\bigg\{ -\frac{i}{2}e_I\mathbf{\Phi}_I^{\dagger}\delta_-\mathbf{\Phi}_I + \mathbf{\Phi}^{\dagger}\mathbf{\Phi}e_IQ_I^{\beta}V_{\beta} + V_{1I}^{b}e_IZ_I^{b}$$

$$+ \Psi_{1I}^{a}\bigg(-i\mathbf{\Phi}_I^{\dagger}\mathcal{T}^{a}\delta_-\mathbf{\Phi}_I + 2Q_I^{\beta}V_{\beta}Z_I^{a} \bigg) + \Psi_{1I}^{a}V_{1I}^{b}a_I^{ab}$$

$$- \frac{1}{2}(\mathbf{\Gamma}_I^{\dagger} + \mathbf{\Gamma}_I^{a\dagger}\mathcal{T}^{a})(e_I + 2\Psi_{1I}^{a}\mathcal{T}^{a})(\mathbf{\Gamma}_I + \mathbf{\Gamma}_I^{a}\mathcal{T}^{a})$$

$$+ \bigg(V_{1I}^{b}Y_{+a} + i\Psi_{1I}^{b}\partial_-Y_{-I}^{a} \bigg)\text{Tr}(\mathcal{T}^{a}\mathcal{T}^{b})$$

$$- \frac{\sqrt{2}}{2}\bigg(\mathbf{\Gamma}_{1I}^{a\dagger}\mathcal{T}^{a}\mathcal{T}^{b}\widetilde{\mathcal{F}}_I^{b} + \widetilde{\mathcal{F}}_I^{a\dagger}\mathcal{T}^{a}\mathcal{T}^{b}\mathbf{\Gamma}_{1I}^{b} \bigg)\bigg\}, \tag{4.2}$$

which is basically the sum of Lagrangians corresponding to each factor of the global group G_I. In the previous Lagrangian we have the following definitions: $a_I^{ab} := \mathbf{\Phi}_I^{\dagger}\{\mathcal{T}^{a}, \mathcal{T}^{b}\}\mathbf{\Phi}_I$, $e_I := 1_I + 2\sum_{\alpha=1}^{m}Q_I^{\alpha}\Psi_{\alpha}$, $Z_I^{a} := \mathbf{\Phi}_I^{\dagger}\mathcal{T}^{a}\mathbf{\Phi}_I$.

The variations with respect to V_{1I}^{c}, Ψ_{1I}^{c} and Γ_{1I}^{c}, give the following equations of motion:

$$\delta_{V_{1I}^{c}}S = 0: \qquad \Psi_{1I}^{a}a^{ca} = -Y_{+Ia}\text{Tr}(\mathcal{T}^{a}\mathcal{T}^{c}) - e_IZ_I^{c} := K_I^{c}, \tag{4.3}$$

$$\delta_{\Psi_{1I}^{c}}S = 0: \qquad V_{1I}^{b}a_I^{bc} + 2Q_I^{\beta}V_{\beta}Z_I^{c} - i\mathbf{\Phi}_I^{\dagger}\mathcal{T}^{c}\delta_-\mathbf{\Phi}_I + i\partial_-Y_{-aI}\text{Tr}(\mathcal{T}^{a}\mathcal{T}^{c})$$
$$-(\mathbf{\Gamma}_I^{\dagger} + \mathbf{\Gamma}_I^{a\dagger}\mathcal{T}^{a})\mathcal{T}^{b}(\mathbf{\Gamma}_I + \mathbf{\Gamma}_I^{c}\mathcal{T}^{c}) = 0, \tag{4.4}$$

$$\delta_{\Gamma_{1I}^{c}}S = 0: \qquad -\frac{1}{2}(\mathbf{\Gamma}_I^{\dagger} + \mathbf{\Gamma}_{1I}^{a\dagger}\mathcal{T}^{a})(e_I + 2\Psi_{1I}^{b}\mathcal{T}^{b})\mathcal{T}^{c} - \frac{\sqrt{2}}{2}\widetilde{\mathcal{F}}_I^{a\dagger}\mathcal{T}^{a}\mathcal{T}^{c} = 0. \tag{4.5}$$

Thus the corresponding partial dual Lagrangian becomes

$$\mathbf{\Delta}L_{\texttt{dual}} = \sum_{I=1}^{S}\int d\theta^{+}d\overline{\theta}^{+}\bigg\{ -\frac{i}{2}e_I\mathbf{\Phi}_I^{\dagger}\delta_-\mathbf{\Phi}_I + \mathbf{\Phi}_I^{\dagger}\mathbf{\Phi}_Ie_IQ_I^{\beta}V_{\beta} + \widetilde{\mathcal{F}}_I^{\dagger}X_I^{-1}\widetilde{\mathcal{F}}_I$$

$$+ \frac{\sqrt{2}}{2}(\widetilde{\mathcal{F}}_I^{\dagger}\mathbf{\Gamma}_I + \mathbf{\Gamma}_I^{\dagger}\widetilde{\mathcal{F}}_I)\bigg(-i\mathbf{\Phi}_I^{\dagger}\delta_-\mathcal{T}^{a}\mathbf{\Phi}_I + 2Q_I^{\beta}V_{\beta}Z_I^{a} \bigg)$$

$$\times \bigg(-Y_{+Ib}\text{Tr}(\mathcal{T}^{b}\mathcal{T}^{c}) - e_IZ_I^{c} \bigg)b_{ac}\bigg\}, \tag{4.6}$$

where b^{ac} is the inverse of a^{cd}, and $X_I := e_I + 2\mathcal{T}^{a}K_I^{a} = e_I - 2\mathcal{T}^{a}e_IZ_I^{c}b^{ca} - 2\mathcal{T}^{a}Y_{+b}\text{Tr}(\mathcal{T}^{b}\mathcal{T}^{c})b^{ca}$. Still it is necessary to remove the original chiral fields $\mathbf{\Phi}$ from the Lagrangian, step which will be implemented through the process of gauge fixing. Up to this point we have used generic well behaved Lie groups G_i, it hasn't been necessary to specify the groups.

4.1   A model with $SU(2)$ global symmetry

Let us consider the case of a model with global symmetry $G = SU(2)$. Before proceeding, some definitions and conventions related to the $SU(2)$ group are intro-

duced for future reference. Hence, the following relations hold: $\text{Tr}(T^a T^b) = 2\delta^{ab}$, $\{T^a, T^b\} = 2\delta^{ab}\mathbf{I_d}$ for the generators of the Lie algeba of the group $SU(2)$; in this way we have the relations $a^{ab} = 2|\Phi|^2$ and $b^{bc} = \frac{1}{2|\Phi|^2}$. Consequently, with $e_I := 1_I + 2\sum_{\alpha=1}^m Q_I^\alpha \Psi_\alpha$, it is obtained that $X_I := e_I\mathbf{I_d} - T^a \frac{e_I Z_I^a + 2Y_+^a}{|\Phi_I|^2}$.

For the specific model with a $U(1)$ gauge group and an $SU(2)$ global symmetry the partial Lagrangian is given by

$$\boldsymbol{\Delta L}_{\texttt{master}} = \sum_{I=1}^s \int \mathrm{d}\theta^+ \mathrm{d}\overline{\theta}^+ \left\{ \left(-i\boldsymbol{\Phi}_I^\dagger \delta_- \mathcal{T}^a \boldsymbol{\Phi}_I + Q_\beta V^\beta \right) \right.$$
$$\times \left(e_I |\boldsymbol{\Phi}_I|^2 - \frac{e_I}{|\boldsymbol{\Phi}_I|^2} Z^a Z_a - Y_+^a \frac{Z^a}{|\boldsymbol{\Phi}_I|^2} \right)$$
$$\left. + \widetilde{\boldsymbol{\mathcal{F}}}^\dagger X^{-1} \widetilde{\boldsymbol{\mathcal{F}}} + \frac{\sqrt{2}}{2}(\widetilde{\boldsymbol{\mathcal{F}}}_I^\dagger \boldsymbol{\Gamma}_I + \boldsymbol{\Gamma}_I^\dagger \widetilde{\boldsymbol{\mathcal{F}}}_I) - \frac{i}{2} e_I \boldsymbol{\Phi}_I^\dagger \delta_- \boldsymbol{\Phi}_I \right\}. \qquad (4.7)$$

For future convenience the fields $\boldsymbol{\Phi} = \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix}$, which are 2 complex fields, can be redefined in terms of new fields Z_0, Z_1, Z_2, Z_3:

$$Z_0 = \overline{\Phi}_1 \Phi_1 + \overline{\Phi}_2 \Phi_2, \qquad Z_1 = 2\Re(\overline{\Phi}_1 \Phi_2), \qquad Z_2 = 2\Im(\overline{\Phi}_1 \Phi_2), \qquad Z_3 = \overline{\Phi}_1 \Phi_1 - \overline{\Phi}_2 \Phi_2,$$
$$(4.8)$$

Then, the original chiral fields can be eliminated by gauge fixing the Z's, these are 4 real constants and the products are written as sums of chiral fields

$$\overline{\Phi}_1 \Phi_1 = \frac{Z_0 + Z_3}{2}, \qquad \overline{\Phi}_1 \Phi_2 = \frac{Z_1 + iZ_2}{2}, \qquad \overline{\Phi}_2 \Phi_2 = \frac{Z_0 - Z_3}{2}. \qquad (4.9)$$

Thus, with the partial gauge fixing: $\Phi_I^\dagger T^b \partial_- \Phi_I = \partial_- \Phi_I^\dagger T^b \Phi_I$, for $b \in \{0, 1, 2, 3\}$, the partial dual Lagrangian has the following form

$$\boldsymbol{\Delta L}_{\texttt{dual}} = \sum_{I=1}^s \int \mathrm{d}\theta^+ \mathrm{d}\overline{\theta}^+ \left\{ Q_\beta V^\beta \left(e_I - e_I \frac{Z^a Z_a}{Z_0} - \frac{Y_+^a Z^a}{Z_0} \right) \right.$$
$$+ \widetilde{\boldsymbol{\mathcal{F}}}^\dagger \left(e_I \mathbf{I_d} - \frac{\mathcal{T}^a}{Z_0}(e_I Z_I^a + 2Y_+^a) \right)^{-1} \widetilde{\boldsymbol{\mathcal{F}}}$$
$$\left. + \frac{\sqrt{2}}{2}\left(\widetilde{\boldsymbol{\mathcal{F}}}_I^\dagger \boldsymbol{\Gamma}_I + \boldsymbol{\Gamma}_I^\dagger \widetilde{\boldsymbol{\mathcal{F}}}_I \right) \right\} + \frac{t}{4} \int \mathrm{d}\theta^+ \Upsilon|_{\overline{\theta}^+ = 0}. \qquad (4.10)$$

To write the scalar potential in a more convenient form, it can be defined the variable $u^a = 2\frac{y_+^a}{Z_0} + \frac{Z_a}{Z_0}$. Therefore, the new dual coordinate is denoted as u^a. As a result, considering all the corresponding terms, the scalar potential can be expressed as

follows

$$U_{\mathtt{dual}} = \frac{-2}{1 - u^a u_a - 2\frac{Z^a Z_a}{Z_0^2} + 2\frac{Z_a u^a}{Z_0}} \tag{4.11}$$

$$\times \left[\overline{H}_1 H_1 + \overline{H}_2 H_2 + \left(\overline{H}_1 H_1 - \overline{H}_2 H_2 \right) u^3 + \overline{H}_2 H_1 \overline{u}^{12} + \text{h.c.} \right]$$

$$+ \sqrt{2} \left[\overline{H}_1 E_1(\phi) + \overline{H}_2 E_2(\phi) + H_1 \overline{E}_1(\phi) + H_2 \overline{E}_2(\phi) \right] - \frac{iQv_- \partial_+ u^a_- Z_a}{Z_0^2}$$

$$+ 2Q^2 v_- v_+ \left(1 - \frac{Z^a Z_a}{Z_0^2} \right) + 2QD \left(1 - \frac{Z^a Z_a}{2Z_0^2} - \frac{u^a Z_a}{2Z_0^2} \right) + \frac{D^2}{2e} - rD.$$

After eliminating the auxiliary field D we have for the potential the following form

$$U_{\mathtt{dual}} = \frac{-2}{1 - u^a u_a - 2\frac{Z^a Z_a}{Z_0^2} + 2\frac{Z_a u^a}{Z_0}} \left[\overline{H}_1 H_1 + \overline{H}_2 H_2 + \left(\overline{H}_1 H_1 - \overline{H}_2 H_2 \right) u^3 \right.$$

$$\left. + \overline{H}_2 H_1 \overline{u}^{12} + \text{h.c.} \right] + \sqrt{2} \left[\bar{H}_1 E_1(\phi) + \bar{H}_2 E_2(\phi) + H_1 \bar{E}_1(\phi) + H_2 \bar{E}_2(\phi) \right]$$

$$- \frac{iQv_- \partial_+ u^a_- Z_a}{Z_0^2} + 2Q^2 v_- v_+ \left(1 - \frac{Z^a Z_a}{Z_0^2} \right)$$

$$- \frac{e}{2} \left(1 - \frac{Z^a Z_a}{2Z_0^2} - \frac{u^a Z_a}{2Z_0^2} + r \right)^2. \tag{4.12}$$

In order to find the minimum through derivatives of H's it is found the vacuum condition $U = 0$, and integrating the fields v_- and v_+ we have

$$0 = \frac{e}{2} \left(r + 1 + \frac{Z^a Z_a}{2Z_0^2} - \frac{u^a Z_a}{2Z_0^2} \right)^2$$

$$+ \frac{u^c u_c - \frac{2u^c Z_c}{Z_0} + \frac{2Z^a Z_a}{Z_0^2} - 1}{1 - u^c u_c}$$

$$\times \left[|E_1|^2 (1 - u_3) + |E_2|^2 (1 + u_3) - E_1 \overline{E}_2 \overline{u}_{12} - E_2 \overline{E}_1 u_{12} \right]. \tag{4.13}$$

If $A = \frac{-2y_+^b y_+^b}{1 - u_c u^c} \begin{pmatrix} u_3 - 1 & u_{12} \\ \overline{u}_{12} & -1 - u_3 \end{pmatrix}$, then this previous condition is rewritten in the following form

$$U_{\mathtt{dual}} = \frac{e}{2} \left(\Im(t) - y_+^a Z_a \right)^2 + (\overline{E}_1 \quad \overline{E}_2) A \begin{pmatrix} E_1 \\ E_2 \end{pmatrix} = 0. \tag{4.14}$$

This analysis is valid in the IR when the vector fields are integrated.

For future convenience let us write the original scalar potential:

$$U_{\mathtt{original}} = \frac{e^2}{2} \left(\sum_i Q_i |\phi_i|^2 - r \right)^2 + \sum_a |E_a|^2. \tag{4.15}$$

The vacua manifold is characterized by the 3 coordinates: y_+^a, and there's one equation for the vacua, thus it is a two dimensional surface. The Y_- term doesn't appear on the Lagrangian, then y_- isn't a coordinate in the potential. The eigenvalues of the matrix A are: $\lambda_\pm = \frac{2y_+^a y_+^a}{1 \mp \sqrt{u^a u^a}}$. So due to A be Hermitian, there exists a unitary matrix P such that $A = P^\dagger DP$, and D is the diagonal matrix with eigenvalues as entries; therefore:

$$\widetilde{\boldsymbol{E}}^\dagger A \widetilde{\boldsymbol{E}} = \widetilde{\boldsymbol{E}}^\dagger P^\dagger D P \widetilde{\boldsymbol{E}} = (P\widetilde{\boldsymbol{E}})^\dagger D(P\widetilde{\boldsymbol{E}}) = \lambda_+ |(P\widetilde{\boldsymbol{E}})_+|^2 + \lambda_- |(P\widetilde{\boldsymbol{E}})_-|^2, \qquad (4.16)$$

which is a quadratic form. Thus, the vacua is made up of 3 regions[1] depending on whether $y_+^a y_+^a$ is greater than, less than or equal to $Z^a y_+^a$, these regions are: the inside of a sphere, the outside of it and the sphere shell. Thus we have three cases:

- Region 1:

$$y^a y^a < Z^a y^a, \qquad \Im(t) = y_+^a Z^a \quad \text{and} \quad |\widetilde{\boldsymbol{E}}|^2 = 0. \qquad (4.17)$$

- Region 2:

$$y^a y^a = Z^a y^a, \qquad \Im(t) = y_+^a Z^a \quad \text{and} \quad |(P\widetilde{\boldsymbol{E}})_-|^2 = 0. \qquad (4.18)$$

- Region 3:

$$y^a y^a > Z^a y^a, \qquad \frac{e}{2}(\Im(t) - y_+^a Z_a)^2 + \lambda_- |(P\widetilde{\boldsymbol{E}})_-|^2 = -\lambda_+ |(P\widetilde{\boldsymbol{E}})_+|^2. \quad (4.19)$$

This can be also written $\Im(t) = y_+^a Z^a - \sqrt{\frac{-2}{e} \widetilde{\boldsymbol{E}}^\dagger A \widetilde{\boldsymbol{E}}}$, where

$$(P\widetilde{\boldsymbol{E}})_\pm = \frac{\mp u_{12} E_1 + (\sqrt{u_c u^c} \pm u_3) E_2}{\sqrt{2\sqrt{u_c u^c}(\sqrt{u_c u^c} \pm u_3)}}, \qquad (4.20)$$

and consequently

$$|(P\widetilde{\boldsymbol{E}})_\pm|^2 = \qquad (4.21)$$
$$= \frac{|u_{12}|^2 |E_1|^2 + (\sqrt{u_c u^c} \pm u_3)^2 |E_2|^2 \mp 2(\sqrt{u_c u^c} \pm u_3)[u_1 \Re(E_1 \overline{E}_2) - u_2 \Im(E_1 \overline{E}_2)]}{2\sqrt{u_c u^c}(\sqrt{u_c u^c} \pm u_3)}.$$

For the regions 1 and 2, the potential is semi-definite positive and the surface $y_+^a Z_a = \Im(t)$ is a plane inside the sphere, thus this is a disk $\mathbf{D}$, where the modulus $r = \Im(t)$ determines the size of the disk (as its relative position inside the sphere). While for the case outside of the sphere, one has a more sophisticated surface, whose boundary is the same as the one of the disk, with the topology $\mathbb{R}^2 \backslash \mathbf{D}$.

---

[1]The point $y^a = 0$ is also solution only if $\Im(t) = 0$.

Until this point the superfield E has been arbitrary and it is a function only on Φ (which is gauge fixed) and other parameters; however, it can be chosen a particular form of it that comes from the $(2,2)$ reduction:

$$\begin{pmatrix} E_1 \\ E_2 \end{pmatrix} = \Sigma \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix} , \qquad (4.22)$$

where $\Sigma = \sigma + \sqrt{2}\theta^+\overline{\lambda}_+ - i\theta^+\overline{\theta}^+\partial_+\sigma$, this means for region 1: $|\sigma|^2 = 0$.

It is remarkable that in this case the equations of motion (4.3),(4.4) and (4.5) can be exactly solved, without requiring to project out to an Abelian component (or to particularize to a semichiral vector field) as in the $(2,2)$ supersymmetric case [67].

4.1.1 Instanton correction

The Lagrangian with the instanton correction is given by

$$\begin{aligned} L_{\mathtt{dual}} = \sum_{I=1}^{s} \int \mathrm{d}\theta^+ \mathrm{d}\overline{\theta}^+ & \left\{ Q_\beta V^\beta \left(e_I - e_I \frac{Z^a Z_a}{Z_0} - \frac{Y_+^a Z^a}{Z_0} \right) \right. \\ & + \widetilde{\boldsymbol{\mathcal{F}}}^\dagger \left(e_I \mathbf{I_d} - \frac{\mathcal{T}^a}{Z_0}(e_I Z_I^a + 2Y_+^a) \right)^{-1} \widetilde{\boldsymbol{\mathcal{F}}} + \frac{\sqrt{2}}{2}\left(\widetilde{\boldsymbol{\mathcal{F}}}_I^\dagger \Gamma_I + \Gamma_I^\dagger \widetilde{\boldsymbol{\mathcal{F}}}_I \right) \right\} \\ & + \int \mathrm{d}\theta^+ \left\{ \frac{t}{4}\Upsilon|_{\overline{\theta}^+=0} + \widetilde{\boldsymbol{\mathcal{F}}}^\dagger \boldsymbol{\beta} e^{\alpha^b Y_b} \right\}, \qquad (4.23) \end{aligned}$$

where the last term is the instanton correction and its contribution to the bosonic scalar potential is:

$$\int \mathrm{d}\theta^+ \widetilde{\boldsymbol{\mathcal{F}}}^\dagger \boldsymbol{\beta} e^{\alpha^b Y_b} = -\sqrt{2}\left(\overline{H}_0 \beta^0 + \overline{H}_1 \beta^1 \right) e^{\alpha_b y_+^b}. \qquad (4.24)$$

Then, the new vacua equation is:

$$\frac{e}{2}\left(\Im(t) - y_+^a Z_a \right)^2 + \left(\widetilde{\boldsymbol{E}} - e^{\alpha_b y_+^b}\boldsymbol{\beta} \right)^\dagger A \left(\widetilde{\boldsymbol{E}} - e^{\alpha_b y_+^b}\boldsymbol{\beta} \right) = 0 , \qquad (4.25)$$

and similarly, when $0 > y_+^a y_+^a + y_+^a Z_a$ the solution gives:

$$\frac{e}{2}\left(\Im(t) - y_+^a Z_a \right)^2 = 0 \qquad \text{and} \qquad |\varepsilon_1|^2(1-u_3) + |\varepsilon_2|^2(1+u_3) - \varepsilon_1\overline{\varepsilon}_2\overline{u}_{12} - \varepsilon_2\overline{\varepsilon}_1 u_{12} = 0, \qquad (4.26)$$

where $\boldsymbol{\varepsilon} = \widetilde{\boldsymbol{E}} - e^{\alpha_b y_+^b}\boldsymbol{\beta}$. Notice that the effect of the instanton in the effective potential is just a displacement of the holomorphic function E. Therefore the dual geometry coincides with the analysis performed without instanton corrections. This is a common point with observations of the dualities in the $(2,2)$ GLSMs [67].

5   A model with global symmetry $SU(2) \times SU(2)$

In this section we study a generalization of the model presented in [56] which consist of a GLSM with gauge symmetry $U(1)_1 \times U(1)_2$, two chiral fields Φ_1, Φ_2 and two Fermi Γ_1, Γ_2 with charge 1 under the first factor of the gauge symmetry $U(1)_1$; as well as two chiral fields $\widetilde{\Phi}_1$, $\widetilde{\Phi}_2$ and two Fermi $\widetilde{\Gamma}_1$, $\widetilde{\Gamma}_2$ with charge 1 under the $U(1)_2$ gauge group. This a deformation of a $(2,2)$ model into a $(0,2)$ model, so the restrictions for the fields E's are:

$$
\begin{aligned}
E_1 &= \sqrt{2}\{\Phi_1 \Sigma + \widetilde{\Sigma}(\alpha_1 \Phi_1 + \alpha_2 \Phi_2)\}, \\
E_2 &= \sqrt{2}\{\Phi_2 \Sigma + \widetilde{\Sigma}(\alpha'_1 \Phi_1 + \alpha'_2 \Phi_2)\}, \\
\widetilde{E}_1 &= \sqrt{2}\{\widetilde{\Phi}_1 \widetilde{\Sigma} + \Sigma(\beta_1 \widetilde{\Phi}_1 + \beta_2 \widetilde{\Phi}_2)\}, \\
\widetilde{E}_2 &= \sqrt{2}\{\widetilde{\Phi}_2 \widetilde{\Sigma} + \Sigma(\beta'_1 \widetilde{\Phi}_1 + \beta'_2 \widetilde{\Phi}_2)\},
\end{aligned}
\tag{5.1}
$$

where α, α', β and β' are real parameters. In the limit when the α's and β's parameters vanish the reduced $(0,2)$ model is recovered. The Lagrangian of this model is given by [56]

$$
\Delta L_{\texttt{original}} = \sum_{i=1}^{2} \int d\theta^+ d\overline{\theta}^+ \left\{ -\frac{i}{2}\overline{\Phi}_i \left(e^{2\Psi}\partial_- - \overleftarrow{\partial}_- e^{2\Psi} \right)\Phi_i + V e^{2\Psi}|\Phi_i|^2 - \frac{1}{2}e^{2\Psi}\overline{\Gamma}_i\Gamma_i \right\}
$$
$$
+ \sum_{i=1}^{2} \int d\theta^+ d\overline{\theta}^+ \left\{ -\frac{i}{2}\overline{\widetilde{\Phi}}_i \left(e^{2\widetilde{\Psi}}\partial_- - \overleftarrow{\partial}_- e^{2\widetilde{\Psi}} \right)\widetilde{\Phi}_i + V e^{2\widetilde{\Psi}}|\widetilde{\Phi}_i|^2 - \frac{1}{2}e^{2\widetilde{\Psi}}\overline{\widetilde{\Gamma}}_i\widetilde{\Gamma}_i \right\}.
\tag{5.2}
$$

While the potential of this Lagrangian is given by

$$
U_{\texttt{original}} = \frac{e^2}{2}\left(|\phi_1|^2 + |\phi_2|^2 - r_1 \right)^2 + \frac{e^2}{2}\left(|\widetilde{\phi}_1|^2 + |\widetilde{\phi}_2|^2 - r_2 \right)^2 + |E_1|^2 + |E_2|^2 + |\widetilde{E}_1|^2 + |\widetilde{E}_2|^2.
\tag{5.3}
$$

The vacuum solution for this model is [56]:

$$
|\phi_1|^2 + |\phi_2|^2 = r_1, \qquad |\widetilde{\phi}_1|^2 + |\widetilde{\phi}_2|^2 = r_2,
\tag{5.4}
$$

i.e., the vacua manifold is a product of $\mathbb{P}^1 \times \mathbb{P}^1$ with Kähler classes r_1 and r_2 respectively, and

$$
E_i = \widetilde{E}_i = 0.
\tag{5.5}
$$

In the $SU(2) \times SU(2)$ generalization both chiral fields and Fermi fields are $SU(2)$

multiplets related a different $SU(2)$ sector. Let us write the dual Lagrangian

$$\mathbf{\Delta} L_{\mathtt{master}} = \int \mathrm{d}\theta^+ \mathrm{d}\overline{\theta}^+ \sum_{i=1}^{2} \frac{1}{8e_i^2} \overline{\Upsilon}_i \Upsilon_i + \int \mathrm{d}\theta^+ \sum_{i=1}^{2} \frac{t_i}{4} \Upsilon_i \big|_{\overline{\theta}^+ = 0}$$

$$- \int \mathrm{d}\theta^+ \mathrm{d}\overline{\theta}^+ \frac{i}{2} \overline{\Phi} e^{2\Psi_1 + 2\Psi_{1a} T_a} \left(\partial_- + iV_1 + iV_{1a} T_a \right) \Phi$$

$$- \int \mathrm{d}\theta^+ \mathrm{d}\overline{\theta}^+ \frac{i}{2} \overline{\widetilde{\Phi}} e^{2\Psi_2 + 2\Psi_{2a} T_a} \left(\partial_- + iV_1 + iV_{2a} T_a \right) \widetilde{\Phi}$$

$$+ \int \mathrm{d}\theta^+ \mathrm{d}\overline{\theta}^+ \left\{ \frac{i}{2} \overline{\Phi} \left(\overset{\leftarrow}{\partial}_- - iV_1 - iV_{1a} T_a \right) e^{2\Psi_1 + 2\Psi_{1a} T_a} \Phi \right\}$$

$$+ \int \mathrm{d}\theta^+ \mathrm{d}\overline{\theta}^+ \left\{ \frac{i}{2} \overline{\widetilde{\Phi}} \left(\overset{\leftarrow}{\partial}_- - iV_2 - iV_{2a} T_a \right) e^{2\Psi_2 + 2\Psi_{2a} T_a} \widetilde{\Phi} \right\}$$

$$- \int \mathrm{d}\theta^+ \mathrm{d}\overline{\theta}^+ \left\{ \frac{1}{2} (\overline{\Gamma} + \overline{\Gamma}_1) e^{2\Psi_1 + 2\Psi_{1a} T_a} (\Gamma + \Gamma_1) \right\}$$

$$- \int \mathrm{d}\theta^+ \mathrm{d}\overline{\theta}^+ \left\{ \frac{1}{2} (\overline{\widetilde{\Gamma}} + \overline{\widetilde{\Gamma}}_2) e^{2\Psi_2 + 2\Psi_{2a} T_a} (\widetilde{\Gamma} + \widetilde{\Gamma}_2) \right\}$$

$$+ \sum_{i=1}^{2} \int \mathrm{d}\theta^+ \mathrm{d}\overline{\theta}^+ \mathrm{Tr}(\Lambda_i \Upsilon_i) + \sum_{i=1}^{2} \int \mathrm{d}\theta^+ \mathrm{d}\overline{\theta}^+ \mathrm{Tr}(\widetilde{\Lambda}_i \Upsilon_i) + \mathrm{h.c.}$$

$$+ \sum_{i=1}^{2} \int \mathrm{d}\theta^+ \mathrm{d}\overline{\theta}^+ \overline{\chi}_i E_i + \sum_{i=1}^{2} \int \mathrm{d}\theta^+ \mathrm{d}\overline{\theta}^+ \overline{\widetilde{\chi}}_i E_i + \mathrm{h.c.} \qquad (5.6)$$

Let us consider the following ansatz for the deformation of the (2,2) model in which α and β are the parameters of the deformation:

$$\begin{pmatrix} E_1 \\ E_2 \end{pmatrix} = \Sigma_0 \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix} + \widetilde{\Sigma}_0 \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix} \alpha_1 + \Sigma \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix} \alpha_2,$$

$$\begin{pmatrix} \widetilde{E}_1 \\ \widetilde{E}_2 \end{pmatrix} = \widetilde{\Sigma}_0 \begin{pmatrix} \widetilde{\Phi}_1 \\ \widetilde{\Phi}_2 \end{pmatrix} + \Sigma_0 \begin{pmatrix} \widetilde{\Phi}_1 \\ \widetilde{\Phi}_2 \end{pmatrix} \beta_1 + \widetilde{\Sigma} \begin{pmatrix} \widetilde{\Phi}_1 \\ \widetilde{\Phi}_2 \end{pmatrix} \beta_2,$$

$$(5.7)$$

This implies that (E_1, E_2) and $(\widetilde{E}_1, \widetilde{E}_2)$ are vectors under $SU(2)_1$ and $SU(2)_2$ respectively. As well (E_1, E_2) and $(\widetilde{E}_1, \widetilde{E}_2)$ are charged with charges 1 under the $U(1)_1$ and $U(1)_2$ respectively. Then, the dual Lagrangian becomes:

$$\mathbf{\Delta} L_{\mathtt{dual}} = \int \mathrm{d}\theta^+ \mathrm{d}\overline{\theta}^+ \Bigg\{ \left[Ve - Ve \frac{Z^a Z_a}{Z_0} - \frac{VY_+^a Z^a}{Z_0} \right] + \widetilde{\mathcal{F}}^\dagger \left(e\mathbf{I_d} - \frac{T^a}{Z_0} (eZ^a + 2Y_+^a) \right)^{-1} \widetilde{\mathcal{F}}$$

$$+ \left[\widetilde{V}\widetilde{e} - \widetilde{V}\widetilde{e} \frac{\widetilde{Z}^a \widetilde{Z}_a}{\widetilde{Z}_0} - \frac{\widetilde{V}\widetilde{Y}_+^a \widetilde{Z}^a}{\widetilde{Z}_0} \right] + \widetilde{\mathcal{F}}^\dagger \left(\widetilde{e}\mathbf{I_d} - \frac{T^a}{\widetilde{Z}_0} (\widetilde{e}\widetilde{Z}^a + 2\widetilde{Y}_+^a) \right)^{-1} \widetilde{\mathcal{F}} \Bigg\}$$

$$+ \frac{t}{4} \int \mathrm{d}\theta^+ \Upsilon \big|_{\overline{\theta}^+ = 0} - \int \mathrm{d}\theta^+ \left[(\Sigma_0 + \alpha_1 \widetilde{\Sigma}_0) \widetilde{\mathcal{F}}^\dagger \Phi + \alpha 2 \widetilde{\mathcal{F}}^\dagger \Sigma \Phi \right]$$

$$+ \frac{\widetilde{t}}{4} \int \mathrm{d}\theta^+ \widetilde{\Upsilon} \big|_{\overline{\theta}^+ = 0} - \int \mathrm{d}\theta^+ \left[(\widetilde{\Sigma}_0 + \beta_1 \Sigma_0) \widetilde{\mathcal{F}}^\dagger \widetilde{\Phi} + \beta 2 \widetilde{\mathcal{F}}^\dagger \widetilde{\Sigma} \widetilde{\Phi} \right], \quad (5.8)$$

where the $\mathbf{\Phi}$ is fixed (in terms of the Z-parameters) with (4.9).

For the scalar potential we have:

$$U_{\texttt{dual}} = -e\left(-y_+^a Z_a + \Im(t)\right)^2 - \tilde{e}\left(-\tilde{y}_+^a \tilde{Z}_a + \Im(\tilde{t})\right)^2$$

$$+ \frac{1}{2y_+^a y_{a+}}\left[\overline{H}_1 H_1 + \overline{H}_2 H_2 + \left(\overline{H}_1 H_1 - \overline{H}_2 H_2\right)\left(Z^3 + 2y_+^3\right)\right.$$

$$\left. + \overline{H}_2 H_1\left(2\overline{w} + \overline{Z}^{12}\right) + \text{h.c.}\right]$$

$$+ \frac{1}{2\tilde{y}_+^a \tilde{y}_{a+}}\left[\overline{\tilde{H}}_1 \tilde{H}_1 + \overline{\tilde{H}}_2 \tilde{H}_2 + \left(\overline{\tilde{H}}_1 \tilde{H}_1 - \overline{\tilde{H}}_2 \tilde{H}_2\right)\left(\tilde{Z}^3 + 2\tilde{y}_+^3\right)\right.$$

$$\left. + \overline{\tilde{H}}_2 \tilde{H}_1\left(2\overline{\tilde{w}} + \overline{\tilde{Z}}^{12}\right) + \text{h.c.}\right]$$

$$+ \sqrt{2}\left[(\sigma_0 + \alpha_1\tilde{\sigma}_0)(\overline{H}_1\phi_1 + \overline{H}_2\phi_2) + \alpha_2\overline{H}_1(\sigma^{11}\phi_1 + \sigma^{12}\phi_2)\right.$$

$$\left. + \alpha_2\overline{H}_2(\sigma^{21}\phi_1 + \sigma^{22}\phi_2) + \text{h.c}\right]$$

$$+ \sqrt{2}\left[(\tilde{\sigma}_0 + \beta_1\sigma_0)\left(\overline{\tilde{H}}_1\tilde{\phi}_1 + \overline{\tilde{H}}_2\tilde{\phi}_2\right) + \beta_2\overline{\tilde{H}}_1\left(\tilde{\sigma}^{11}\tilde{\phi}_1 + \tilde{\sigma}^{12}\tilde{\phi}_2\right)\right.$$

$$\left. + \beta_2\overline{\tilde{H}}_2\left(\tilde{\sigma}^{21}\tilde{\phi}_1 + \tilde{\sigma}^{22}\tilde{\phi}_2\right) + \text{h.c}\right]. \tag{5.9}$$

Thus, the bosonic scalar potential depends of 6 coordinates y_+^a and $\tilde{y}_+^a$, and the vacua $U_{\texttt{dual}} = 0$ after the minimum condition for H's gives:

$$U_{\texttt{dual}} = \frac{e}{2}\left(\Im(t) - y_+^a Z_a\right)^2 + (\overline{E}_1 \quad \overline{E}_2)A\begin{pmatrix} E_1 \\ E_2 \end{pmatrix}$$

$$+ \frac{e}{2}\left(\Im(\overline{t}) - \overline{y}_+^a \tilde{Z}_a\right)^2 + (\overline{\tilde{E}}_1 \quad \overline{\tilde{E}}_2)\overline{A}\begin{pmatrix} \tilde{E}_1 \\ \tilde{E}_2 \end{pmatrix} = 0, \tag{5.10}$$

with $A = \frac{-2y_+^b y_+^b}{1 - u_c u^c}\begin{pmatrix} u_3 - 1 & u_{12} \\ \overline{u}_{12} & -1 - u_3 \end{pmatrix}$, $\tilde{A} = \frac{-2\tilde{y}_+^b \tilde{y}_+^b}{1 - \tilde{u}_c \tilde{u}^c}\begin{pmatrix} \tilde{u}_3 - 1 & \tilde{u}_{12} \\ \overline{\tilde{u}}_{12} & -1 - \tilde{u}_3 \end{pmatrix}$ and

$$E_1 = \left[\sigma_0 + \alpha_1\tilde{\sigma}_0 + \alpha_2(\sigma_{11} + \sigma_{12})\right]\phi_1, \; E_2 = \left[\sigma_0 + \alpha_1\tilde{\sigma}_0 + \alpha_2(\sigma_{21} + \sigma_{22})\right]\phi_2 \; .$$

$$\tilde{E}_1 = \left[\tilde{\sigma}_0 + \beta_1\sigma_0 + \beta_2(\tilde{\sigma}_{11} + \tilde{\sigma}_{12})\right]\tilde{\phi}_1, \; \tilde{E}_2 = \left[\tilde{\sigma}_0 + \beta_1\sigma_0 + \beta_2(\tilde{\sigma}_{21} + \tilde{\sigma}_{22})\right]\tilde{\phi}_2 \; , \tag{5.11}$$

which as in the previous case, each one is positive quadratic form when $0 < 1 - u^a u_a$ and $0 < 1 - \tilde{u}^a \tilde{u}_a$. If it is this case, the solution is

$$r = y_+^a Z^a, \qquad\qquad \tilde{r} = \tilde{y}_+^a \tilde{Z}^a,$$

$$\sigma_0 + \alpha_1\tilde{\sigma}_0 + \alpha_2(\sigma_{11} + \sigma_{12}) = 0, \qquad \sigma_0 + \alpha_1\tilde{\sigma}_0 + \alpha_2(\sigma_{21} + \sigma_{22}) = 0,$$

$$\tilde{\sigma}_0 + \beta_1\sigma_0 + \beta_2(\tilde{\sigma}_{11} + \tilde{\sigma}_{12}) = 0 \quad \text{and} \quad \tilde{\sigma}_0 + \beta_1\sigma_0 + \beta_2(\tilde{\sigma}_{21} + \tilde{\sigma}_{22}) = 0. \tag{5.12}$$

which is simply two copies of the potential with a single group factor. For the instanton correction, it is

$$\int d\theta^+ \left(\widetilde{\boldsymbol{\mathcal{F}}}^\dagger \boldsymbol{\beta} e^{\alpha^b Y_b} + \widetilde{\widetilde{\boldsymbol{\mathcal{F}}}}^\dagger \widetilde{\boldsymbol{\beta}} e^{\widetilde{\alpha}^b \widetilde{Y}_b} \right). \tag{5.13}$$

Thus the change for the scalar potential is given by: $\widetilde{\boldsymbol{E}} \to \widetilde{\boldsymbol{E}} - e^{\alpha_b y_+^b} \boldsymbol{\beta}$. This means that the last 4 equations in (5.12) are equal to $|e^{\alpha_b y_+^b} \boldsymbol{\beta}|^2$.

For the analyzed case the geometry of the dual model is the one of the product of two disks $\mathbf{D}_1 \times \mathbf{D}_2$, which are the building blocks of the duality in subsection 4.1. Other possible cases involve a not positive definite matrix A or $\tilde{A}$. Notice also that the inclusion of instanton corrections preserves the geometry.

6 Discussion and outlook

In this work we describe T-dualities of two dimensional $(0, 2)$ Abelian GLSMs. After a brief review on the basics of $(0, 2)$ GLSMs, we started by constructing Abelian T-duality. This is implemented in models with $U(1)$ global symmetries; by gauging them. We analyse two cases: models coming from a $(2, 2)$ supersymmetry reduction and pure $(0, 2)$ models. The fundamental difference is that in the first case (reduction) the Fermi multiplet is dualized, while in the second case it is not. We study the simple example of a model with two chiral superfields, the first chiral field is charged under the $U(1)$ global symmetry and the other remains as an spectator, which just assists to obtain the global symmetry. A master Lagrangian is obtained by promoting the global symmetry to be local, and adding Lagrange multiplier fields. The equations of motion for the gauge fields are obtained from the master Lagrangian leading to the dual action. The original chiral fields are eliminated by the gauge fixing procedure. We then compute the contributions to the scalar potential for all the terms in the dual Lagrangian. From the potential we determine the geometry of the space of supersymmetric vacua. The geometry of the vacua space for the original model in both cases is $\mathbb{P}^1$. The dual model, under a single $U(1)$ T-duality, has the topology of $\mathbb{R}^+ \times \mathbb{R}$ for both cases. Notice that this is very different to the standard Mirror Symmetry duality, which will be a T-dualization of both chiral superfields. The instanton contributions to the superpotential are known for $(0, 2)$ models coming from a $(2, 2)$ reduction [20, 56]. For the case of a pure $(0, 2)$ model, we still haven't obtained them, we leave this for future work. From our results it seems that there is a difference of considering $(0, 2)$ models and their dual counterparts, if they come from a reduction or not. Moreover, in Section 3 we carried out the duality algorithm for a model with two global Abelian symmetries [56]. This is a model which was later generalized in Section 5 to the non-Abelian T-duality case. It consists of a reduction $(0, 2)$ GLSM with gauge symmetry $U(1) \times U(1)$, six chiral superfields and six Fermi fields. The global Abelian symmetry is given by $U(1)^4$.

Furthermore we construct T-dualities for Abelian GLSMs which non-Abelian global symmetry. Here we considered only the case coming from the reduction of a $(2, 2)$ model. To be as general as possible, we obtain the master Lagrangian of a model with $U(1)^m$ gauge symmetry and non-Abelian global group $G_1 \times \cdots \times G_S$. Starting from an original

$(2, 0)$ model with chiral, fermi superfields and a gauge multiplet, we obtain the master Lagrangian by gauging the global symmetries. We find suitable variables to write down the original master Lagrangians as a sum over the S factors of the gauged symmetry group; then we find the equations of motion for the gauge fields. We considered a particular case with just one global $G = SU(2)$ factor and $U(1)$ gauge symmetry. The dual action is obtained by gauging the global symmetry $SU(2)$. It is observed that under a suitable redefinition of the chiral superfields in terms of new variables (fields) Z's and u^a, the dual action can be rewritten in a simpler form. In these variables also was found the scalar potential (4.14). We also identify the conditions for which the potential is definite positive. This lead us to consider 3 regions depending on whether $y_+^a y_+^a$ is less than, equal or greater to $Z^a y_+^a$. We argued that these regions correspond to the condition with the topology of a open ball, a two-sphere or to the outside part of the sphere, respectively. Thus the vacua manifold for a positive semidefinite scalar potential corresponds to the closed disk $\mathbf{D}$. If the potential is not definite positive the component of the vacua manifold is $\mathbb{R}^2 \setminus \mathbf{D}$. Furthermore, we discussed non-perturbative corrections to the superpotential via instantons. We find that if the instanton corrections are incorporated in the potential $U_{\mathtt{dual}}$ the effect is equivalent to shift $\widetilde{\boldsymbol{E}}$ function as $\widetilde{\boldsymbol{E}} - e^{\alpha_b y_+^b}\boldsymbol{\beta}$ in the potential without instanton corrections. This coincides with the observation in the $(2, 2)$ GLSMs non-Abelian T-duality were the instanton corrections preserve the dual geometry [67].

In Section 5 we present the example of GLSM discussed in [56], which comes from a continuous $(0, 2)$ deformation of a $(2, 2)$ model. This model is a genuine pure $(0, 2)$ GLSM. We worked out this model by gauging the global non-Abelian symmetry $SU(2) \times SU(2)$. We find the dual Lagrangian, and analyze the dual geometry of the vacua manifold. For the case of a positive definite potential the manifold is the Cartesian product of the vacua space of the $SU(2)$ simple model already discussed in Section 4, i.e. $\mathbf{D}_1 \times \mathbf{D}_2$. There are also instanton corrections affecting both sectors by a similar shifting of $\widetilde{\boldsymbol{E}}$.

To summarize we have constructed systematically non-Abelian T-duality in $(0, 2)$ gauged linear sigma models in two dimensions. In the future we would like to analyze more examples, given by realistic Calabi-Yau manifolds, and their Mirror duals. We also are interested in analyzing models with a non-zero superpotential $J \neq 0$, which will lead to compact Calabi-Yau. It would also be interesting to explore the connection of non-Abelian T-dualities in $(0, 2)$ models with Mirror Symmetry in more general Calabi-Yau constructions (as Pfaffians and determinantal varieties). We would like to explore the implications of these GLSMs non-Abelian T-dualities in string theory, as possible extensions of Mirror symmetry.

Acknowledgments

A   Abelian T-duality algorithm in superfield components

In this appendix, we implement the Abelian dualization of a $(0, 2)$ GLSM coming from a $(2, 2)$ reduction in terms of superfield components. This is as an alternative way to the superfield language, to carry out the duality. First, we write down the component expansion of the fields. The gauge superfield is given by

$$V = v_- - 2i\theta^+\overline{\lambda}_- - 2i\overline{\theta}^+\lambda_- + 2\theta^+\overline{\theta}^+ D, \qquad \Psi = v_+\theta^+\overline{\theta}^+. \tag{A.1}$$

The fields in the model, including the chiral superfield, the Fermi superfield and the superfield E are written as:

$$\widetilde{\Phi} = \phi + \sqrt{2}\theta^+\psi_+ - i\theta^+\overline{\theta}^+(\partial_+ + iv_+)\phi, \tag{A.2}$$

$$\Gamma = \gamma_- - \sqrt{2}G\theta^+ - i\theta^+\overline{\theta}^+\partial_+\gamma_- - \sqrt{2}E\overline{\theta}^+, \tag{A.3}$$

$$E(\Phi) = E(\phi) + \sqrt{2}\theta^+\frac{\partial E}{\partial\phi}\psi_+ - i\theta^+\overline{\theta}^+\partial_+E, \tag{A.4}$$

The field component expansion for the Lagrangian multipliers which have been used in the bulk of the article are:

$$\chi = x + \xi\theta^+ + \rho\overline{\theta}^+ + z\theta^+\overline{\theta}^+, \tag{A.5}$$

$$\Lambda = \omega + k\theta^+ + l\overline{\theta}^+ + \varepsilon\theta^+\overline{\theta}^+. \tag{A.6}$$

The new dual fields are given by

$$Y_\pm = y_\pm + \sqrt{2}(\theta^+\overline{\xi}_+ + \xi\overline{\theta}^+) - i\theta^+\overline{\theta}^+\partial_+y_\mp, \tag{A.7}$$

$$F = \eta_- - \sqrt{2}\theta^+ H_a - i\theta^+\overline{\theta}^+\partial_+\eta_-. \tag{A.8}$$

Then the appropriate Lagrangian of a single chiral field and a Fermi one with Abelian global symmetry in component fields is written as

$$\begin{aligned}
L_{\texttt{components}} &= \overline{\phi}\phi D + i\overline{\psi}_+(\partial_- + iv_-)\psi_+ - \sqrt{2}i(\lambda_-\psi_+\overline{\phi} - \overline{\psi}_+\overline{\lambda}_-\phi) \\
&\quad - \frac{1}{2}[\overline{\phi}(\partial_- + iv_-)(\partial_+ + iv_+)\phi - (\partial_+ + iv_+)\overline{\phi}(\partial_- + iv_-)\phi] + D\Im(t) + v_{01}\Re(t) \\
&\quad + i\overline{\gamma}(\partial_+ + iv_+)\gamma + |G|^2 - |E|^2 - \left(\overline{\gamma}\frac{\partial E}{\partial\phi}\psi_+ + \frac{\partial\overline{E}}{\partial\overline{\phi}}\overline{\psi}_+\gamma\right) + \frac{v_{01}^2 + D^2}{2e^2}. \tag{A.9}
\end{aligned}$$

To realize the T duality algorithm, gauging the global symmetry we add the fields $v_\pm$, λ_-, D, γ and G (components of V, Ψ and Γ) as well as the Lagrange multipliers. The original fields will be denoted with a subindex 0, the gauged ones with a subindex 1 and the sum of both without any subindex. Thus, for example: $a := a_0 + a_1$, etc. Besides it is convenient to define: $\delta_\pm := \partial_\pm - \overleftarrow{\partial}_\pm$, thus:

$$\begin{aligned}
L_{\texttt{master}} &= \overline{\phi}\phi D + i\overline{\psi}_+(\partial_- + iv_-)\psi_+ - \sqrt{2}i(\lambda_-\psi_+\overline{\phi} - \overline{\psi}_+\overline{\lambda}_-\phi) \\
&\quad - \frac{1}{2}\left[\overline{\phi}(\partial_- + iv_-)(\partial_+ + iv_+)\phi - (\partial_+ + iv_+)\overline{\phi}(\partial_- + iv_-)\phi\right] + D\Im(t) + v_{01}\Re(t) \\
&\quad + i\overline{\gamma}(\partial_+ + iv_+)\gamma + |G|^2 - |E|^2 - \left(\overline{\gamma}\frac{\partial E}{\partial\phi}\psi_+ + \frac{\partial\overline{E}}{\partial\overline{\phi}}\overline{\psi}_+\gamma\right) + \frac{v_{01}^2 + D^2}{2e^2} \\
&\quad + 2iD(\omega - \overline{\omega}) + v_{01}(l + l) + 2(\varepsilon\lambda_- + \overline{\lambda}_-\overline{\varepsilon}) - 2i(l\partial_+\lambda_- + \partial_+\overline{\lambda}_-\overline{l}) \\
&\quad + i(\overline{x}\partial_+ E - x\partial_+\overline{E}) - \sqrt{2}\left(\overline{\xi}\frac{\partial E}{\partial\phi}\psi_+ + \overline{\psi}_+\frac{\overline{\partial E}}{\partial\phi}\xi\right) - \overline{z}E - z\overline{E}. \tag{A.10}
\end{aligned}$$

Taking variations with respect to $v_{\pm,1}$, λ_1, D_1, γ_1 and G_1 one obtains the corresponding equations of motion:

For $\delta_{D_1}L$:
$$i(l - \overline{l}) = 2|\phi|^2. \tag{A.11}$$

For $\delta_{\lambda_1}L$:
$$\varepsilon + 2i\partial_+\omega = 4\overline{\phi}\psi_+. \tag{A.12}$$

For $\delta_{b_1}L$:
$$4v_+|\phi|^2 = 4\overline{\psi}_+\psi_+ + 2i(\overline{\phi}\delta_+\phi) + \partial_{++}(l + \overline{l}). \tag{A.13}$$

For $\delta_{a_1}L$:
$$4v_-|\phi|^2 = -\frac{i}{2}\overline{\gamma}\gamma + 2i(\overline{\phi}\delta_-\phi) - \partial_{--}(l + \overline{l}) + \overline{\gamma}\gamma. \tag{A.14}$$

For $\delta_{\gamma_1}L$:
$$(\partial_+ - iv_+ + 2v_+)(\overline{\gamma}_0 + \overline{\gamma}_1) = i(\mu\overline{\phi} + \overline{s}\psi_+). \tag{A.15}$$

For $\delta_{G_1}L$:
$$G_0 + G_1 = 0. \tag{A.16}$$

New variables can be defined in the form:

$$l_\pm := l \pm \overline{l}, \qquad y := -i\varepsilon + 2\partial_{++}\omega \qquad \text{and} \qquad f := z + 2i\partial_{++}x. \tag{A.17}$$

Thus, using Eqs. (A.11, A.12, A.13, A.14) in the Lagrangian (A.10) it results the dual Lagrangian:

$$\frac{1}{2}L_{\texttt{dual}} = -i(y\lambda_{0-} - \overline{\lambda}_{0-}\overline{y}) + \frac{1}{2}(fs_0 + \overline{f}\overline{s}_0) - iD_0l_- - b_0\partial_+l_+ + a_0\partial_-l_+$$

$$= -i(\mu_0\overline{\rho} - \rho\overline{\mu}_0) + \overline{\phi}\delta_-\delta_+\phi + 4ab|\phi|^2 - |G|^2 - 2i\overline{\psi}_+\delta_-\psi_+ - \frac{i}{2}\overline{\gamma}\delta_+\gamma. \qquad (A.18)$$

It is useful to emphasize that $a = a_0 + a_1$ and $b = b_0 + b_1$. It is easy to check that this dual Lagrangian coincides with the component field expansion of the dual Lagrangian (3.19). A similar procedure could be carried over in the case of models with non-Abelian T-duality.

References

[1] M. Dine, editor. *STRING THEORY IN FOUR-DIMENSIONS*. 1988.

[2] Philip Candelas and Xenia de la Ossa. Moduli Space of Calabi-Yau Manifolds. *Nucl. Phys. B*, 355:455–481, 1991. doi: 10.1016/0550-3213(91)90122-E.

[3] Victor V. Batyrev and Lev A. Borisov. On Calabi-Yau complete intersections in toric varieties. 12 1994.

[4] Brian Greene. String theory on calabi-yau manifolds, 1997.

[5] Andy Strominger. Les houches lectures on black holes, 1995.

[6] Ahmed Almheiri, Donald Marolf, Joseph Polchinski, and James Sully. Black Holes: Complementarity or Firewalls? *JHEP*, 02:062, 2013. doi: 10.1007/JHEP02(2013)062.

[7] Luis E. Ibanez, Jihn E. Kim, Hans Peter Nilles, and F. Quevedo. Orbifold Compactifications with Three Families of SU(3) x SU(2) x U(1)**n. *Phys. Lett. B*, 191:282–286, 1987. doi: 10.1016/0370-2693(87)90255-3.

[8] Steven B. Giddings, Shamit Kachru, and Joseph Polchinski. Hierarchies from fluxes in string compactifications. *Phys. Rev. D*, 66:106006, 2002. doi: 10.1103/PhysRevD.66.106006.

[9] S. J. Gates, Jr., C. M. Hull, and M. Rocek. Twisted Multiplets and New Supersymmetric Nonlinear Sigma Models. *Nucl. Phys. B*, 248:157–186, 1984. doi: 10.1016/0550-3213(84)90592-3.

[10] Paul S. Howe and G. Sierra. TWO-DIMENSIONAL SUPERSYMMETRIC NONLINEAR SIGMA MODELS WITH TORSION. *Phys. Lett. B*, 148:451–455, 1984. doi: 10.1016/0370-2693(84)90736-6.

[11] Tom Banks, Lance J. Dixon, Daniel Friedan, and Emil J. Martinec. Phenomenology and Conformal Field Theory Or Can String Theory Predict the Weak Mixing Angle? *Nucl. Phys. B*, 299:613–626, 1988. doi: 10.1016/0550-3213(88)90551-2.

[12] Tristan Hubsch. *Calabi-Yau manifolds: A Bestiary for physicists*. World Scientific, Singapore, 1994. ISBN 978-981-02-1927-7.

[13] Jacques Distler. "Notes on (0,2) superconformal field theories,". 1 1995.

[14] Edward Witten. Phases of N=2 theories in two-dimensions. *Nucl.Phys.*, B403: 159–222, 1993. doi: 10.1016/0550-3213(93)90033-L.

[15] Emil J. Martinec. CRITICALITY, CATASTROPHES AND COMPACTIFICATIONS. pages 389–433, 4 1989.

[16] Cumrun Vafa and Nicholas P. Warner. Catastrophes and the Classification of Conformal Theories. *Phys. Lett. B*, 218:51–58, 1989. doi: 10.1016/0370-2693(89)90473-5.

[17] Brian R. Greene, C. Vafa, and N. P. Warner. Calabi-Yau Manifolds and

Renormalization Group Flows. *Nucl. Phys. B*, 324:371, 1989. doi: 10.1016/0550-3213(89)90471-9.

[18] Wolfgang Lerche, Cumrun Vafa, and Nicholas P. Warner. Chiral Rings in N=2 Superconformal Theories. *Nucl. Phys. B*, 324:427–474, 1989. doi: 10.1016/0550-3213(89)90474-4.

[19] Jock McOrist. The Revival of (0,2) Linear Sigma Models. *Int. J. Mod. Phys. A*, 26: 1–41, 2011. doi: 10.1142/S0217751X11051366.

[20] Ilarion V. Melnikov. *Gauged Linear Sigma Models*, pages 237–394. Springer International Publishing, Cham, 2019. ISBN 978-3-030-05085-6. doi: 10.1007/978-3-030-05085-6_5. URL https://doi.org/10.1007/978-3-030-05085-6_5.

[21] Z. Chen, J. Guo, E. Sharpe, and R. Wu. More Toda-like (0,2) mirrors. *JHEP*, 08: 079, 2017. doi: 10.1007/JHEP08(2017)079.

[22] W. Gu and E. Sharpe. A proposal for (0,2) mirrors of toric varieties. *JHEP*, 11:112, 2017. doi: 10.1007/JHEP11(2017)112.

[23] Luis Álvarez-Cónsul, Andoni De Arriba de La Hera, and Mario Garcia-Fernandez. (0,2) Mirror Symmetry on homogeneous Hopf surfaces. 12 2020.

[24] Kentaro Hori and Cumrun Vafa. Mirror symmetry. 2 2000.

[25] K. Hori, S. Katz, A. Klemm, R. Pandharipande, R. Thomas, C. Vafa, R. Vakil, and E. Zaslow. *Mirror symmetry*, volume 1 of *Clay mathematics monographs*. AMS, Providence, USA, 2003. URL http://www.claymath.org/library/monographs/cmim01.pdf.

[26] Martin Rocek and Erik P. Verlinde. Duality, quotients, and currents. *Nucl. Phys.*, B373:630–646, 1992. doi: 10.1016/0550-3213(92)90269-H.

[27] Amit Giveon, Massimo Porrati, and Eliezer Rabinovici. Target space duality in string theory. *Phys. Rept.*, 244:77–202, 1994. doi: 10.1016/0370-1573(94)90070-1.

[28] Fernando Quevedo. Duality and global symmetries. *Nucl. Phys. B Proc. Suppl.*, 61: 23–41, 1998. doi: 10.1016/S0920-5632(97)00517-3.

[29] Francesco Benini and Stefano Cremonesi. Partition Functions of $\mathcal{N} = (2,2)$ Gauge Theories on S^2 and Vortices. *Commun. Math. Phys.*, 334(3):1483–1527, 2015. doi: 10.1007/s00220-014-2112-z.

[30] Xenia C. de la Ossa and Fernando Quevedo. Duality symmetries from nonAbelian isometries in string theory. *Nucl. Phys.*, B403:377–394, 1993. doi: 10.1016/0550-3213(93)90041-M.

[31] Amit Giveon and Martin Rocek. On nonAbelian duality. *Nucl. Phys.*, B421:173–190, 1994. doi: 10.1016/0550-3213(94)90230-5.

[32] Enrique Alvarez, Luis Alvarez-Gaume, and Yolanda Lozano. On nonAbelian duality. *Nucl. Phys.*, B424:155–183, 1994. doi: 10.1016/0550-3213(94)90093-0.

[33] Konstadinos Sfetsos and Daniel C. Thompson. On non-abelian T-dual geometries with Ramond fluxes. *Nucl. Phys.*, B846:21–42, 2011. doi: 10.1016/j.nuclphysb.2010.12.013.

[34] Yolanda Lozano, Eoin. O Colgain, Konstadinos Sfetsos, and Daniel C. Thompson. Non-abelian T-duality, Ramond Fields and Coset Geometries. *JHEP*, 06:106, 2011. doi: 10.1007/JHEP06(2011)106.

[35] Georgios Itsios, Yolanda Lozano, Jesus Montero, and Carlos Nunez. The AdS_5 non-Abelian T-dual of Klebanov-Witten as a $\mathcal{N} = 1$ linear quiver from M5-branes. 2017.

[36] Jeroen van Gorsel and Salomon Zacarias. A Type IIB Matrix Model via non-Abelian T-dualities. 2017.

[37] Kentaro Hori. Mirror symmetry and quantum geometry. In *International Congress of Mathematicians*, 7 2002.

[38] Kentaro Hori and David Tong. Aspects of Non-Abelian Gauge Dynamics in Two-Dimensional N=(2,2) Theories. *JHEP*, 05:079, 2007. doi: 10.1088/1126-6708/2007/05/079.

[39] Manfred Herbst, Kentaro Hori, and David Page. Phases Of N=2 Theories In 1+1 Dimensions With Boundary. 3 2008.

[40] Kentaro Hori. Duality In Two-Dimensional (2,2) Supersymmetric Non-Abelian Gauge Theories. *JHEP*, 10:121, 2013. doi: 10.1007/JHEP10(2013)121.

[41] Kentaro Hori and Johanna Knapp. Linear sigma models with strongly coupled phases - one parameter models. *JHEP*, 11:070, 2013. doi: 10.1007/JHEP11(2013)070.

[42] Kentaro Hori and Johanna Knapp. A pair of Calabi-Yau manifolds from a two parameter non-Abelian gauged linear sigma model. 12 2016.

[43] Wei Gu, Leonardo Mihalcea, Eric Sharpe, and Hao Zou. Quantum K theory of symplectic Grassmannians. *J. Geom. Phys.*, 177:104548, 2022. doi: 10.1016/j.geomphys.2022.104548.

[44] Wei Gu, Hadi Parsian, and Eric Sharpe. More non-Abelian mirrors and some two-dimensional dualities. *Int. J. Mod. Phys. A*, 34(30):1950181, 2019. doi: 10.1142/S0217751X19501811.

[45] W. Gu, E. Sharpe, and H. Zou. GLSMs for exotic Grassmannians. *JHEP*, 10:200, 2020. doi: 10.1007/JHEP10(2020)200.

[46] Wei Gu. Correlation functions in massive Landau-Ginzburg orbifolds and tests of dualities. *JHEP*, 12:180, 2020. doi: 10.1007/JHEP12(2020)180.

[47] Wei Gu, Jirui Guo, and Yaoxiong Wen. Nonabelian mirrors and Gromov-Witten invariants. 12 2020.

[48] Wei Gu, Eric Sharpe, and Hao Zou. Notes on two-dimensional pure supersymmetric gauge theories. *JHEP*, 04:261, 2021. doi: 10.1007/JHEP04(2021)261.

[49] Zhuo Chen, Jirui Guo, and Mauricio Romo. A GLSM View on Homological Projective Duality. *Commun. Math. Phys.*, 394(1):355–407, 2022. doi: 10.1007/s00220-022-04401-1.

[50] Ralph Blumenhagen and Andreas Wisskirchen. Exploring the moduli space of (0,2) strings. *Nucl. Phys. B*, 475:225–243, 1996. doi: 10.1016/0550-3213(96)00298-2.

[51] Ralph Blumenhagen, Rolf Schimmrigk, and Andreas Wisskirchen. (0,2) mirror symmetry. *Nucl. Phys. B*, 486:598–628, 1997. doi: 10.1016/S0550-3213(96)00698-0.

[52] Ralph Blumenhagen and Savdeep Sethi. On orbifolds of (0,2) models. *Nucl. Phys. B*, 491:263–278, 1997. doi: 10.1016/S0550-3213(97)00095-3.

[53] Ralph Blumenhagen and Michael Flohr. Aspects of (0,2) orbifolds and mirror symmetry. *Phys. Lett. B*, 404:41–48, 1997. doi: 10.1016/S0370-2693(97)00523-6.

[54] Ralph Blumenhagen. Target space duality for (0,2) compactifications. *Nucl. Phys. B*, 513:573–590, 1998. doi: 10.1016/S0550-3213(97)00721-9.

[55] Ralph Blumenhagen. (0,2) Target space duality, CICYs and reflexive sheaves. *Nucl. Phys. B*, 514:688–704, 1998. doi: 10.1016/S0550-3213(97)00842-0.

[56] Allan Adams, Anirban Basu, and Savdeep Sethi. (0,2) duality. *Adv. Theor. Math. Phys.*, 7(5):865–950, 2003. doi: 10.4310/ATMP.2003.v7.n5.a5.

[57] Wei Gu. *Gauged Linear Sigma Model and Mirror Symmetry*. PhD thesis, Virginia Tech., Blacksburg, 2019.

[58] W. Gu, J. Guo, and E. Sharpe. A proposal for nonabelian (0,2) mirrors. 8 2019.

[59] Hugo Garcia-Compean and Angel M. Uranga. Brane box realization of chiral gauge theories in two-dimensions. *Nucl. Phys. B*, 539:329–366, 1999. doi: 10.1016/S0550-3213(98)00725-1.

[60] Abhijit Gadde, Sergei Gukov, and Pavel Putrov. (0, 2) trialities. *JHEP*, 03:076, 2014. doi: 10.1007/JHEP03(2014)076.

[61] Sebastián Franco, Alessandro Mininno, Ángel M. Uranga, and Xingyang Yu. Spin(7) orientifolds and 2d $\mathcal{N} = (0, 1)$ triality. *JHEP*, 01:058, 2022. doi: 10.1007/JHEP01(2022)058.

[62] Sebastian Franco, Dongwook Ghim, and Rak-Kyeong Seong. Brane brick models for the Sasaki-Einstein 7-manifolds $Y^{p,k}(\mathbb{CP}^1 \times \mathbb{CP}^1)$ and $Y^{p,k}(\mathbb{CP}^2)$. *JHEP*, 03:050, 2023. doi: 10.1007/JHEP03(2023)050.

[63] Matteo Sacchi. New 2d $\mathcal{N} = (0, 2)$ dualities from four dimensions. *JHEP*, 12:009, 2020. doi: 10.1007/JHEP12(2020)009.

[64] J. de-la Cruz-Moreno and H. García-Compeán. Star-triangle type relations from 2d $\mathcal{N} = (0, 2)$ $USp(2N)$ dualities. *JHEP*, 01:023, 2021. doi: 10.1007/JHEP01(2021)023.

[65] E. Sharpe. On A survey of recent developments in GLSMs, Talk at GLSM@30 (Simons Center, Stony Brook, May 22-26, 2023),. URL availableathttps://scgp.stonybrook.edu/video/video.php?id=5930.

[66] Sebastián Franco. 2d Supersymmetric Gauge Theories, D-branes and Trialities. 1 2022.

[67] Nana Cabo Bizet, Aldo Martínez-Merino, Leopoldo A. Pando Zayas, and Roberto Santos-Silva. Non Abelian T-duality in Gauged Linear Sigma Models. *JHEP*, 04:054, 2018. doi: 10.1007/JHEP04(2018)054.

[68] Nana Cabo Bizet and Roberto Santos-Silva. A toolkit for twisted chiral superfields. *JHEP*, 01:019, 2020. doi: 10.1007/JHEP01(2020)019.

[69] Nana Geraldine Cabo Bizet, Yulier Jiménez Santana, and Roberto Santos Silva. Non Abelian dual of the resolved conifold gauged linear sigma model. 12 2021.

[70] Hans Jockers, Vijay Kumar, Joshua M. Lapan, David R. Morrison, and Mauricio Romo. Nonabelian 2D Gauge Theories for Determinantal Calabi-Yau Varieties. *JHEP*, 11:166, 2012. doi: 10.1007/JHEP11(2012)166.

[71] Wei Gu and Eric Sharpe. A proposal for nonabelian mirrors. 2018.

[72] J. Wess and J. Bagger. *Supesymmetry and Supergravity*. Princeton Series in Physics. Princeton University Press, Princeton, USA, 1992.

[73] S. J. Gates, Marcus T. Grisaru, M. Rocek, and W. Siegel. Superspace Or One Thousand and One Lessons in Supersymmetry. *Front. Phys.*, 58:1–548, 1983.

[74] Harald J W Müller-Kirsten and Armin Wiedemann. *Introduction to Supersymmetry*. WORLD SCIENTIFIC, 2nd edition, 2010. doi: 10.1142/7594. URL https://www.worldscientific.com/doi/abs/10.1142/7594.