HYPERBOLIZATION OF AFFINE LIE ALGEBRAS

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ABSTRACT. In 1983, Feingold and Frenkel posed a question about possible relations between affine Lie algebras, hyperbolic Kac–Moody algebras and Siegel modular forms. In this paper we give an automorphic answer to this question and its generalization. We classify hyperbolic Borcherds–Kac–Moody superalgebras whose super-denominators define reflective automorphic products of singular weight on lattices of type $2U \oplus L$. As a consequence, we prove that there are exactly 81 affine Lie algebras $\hat{\mathfrak{g}}$ which have nice extensions to hyperbolic BKM superalgebras for which the leading Fourier–Jacobi coefficients of super-denominators coincide with the denominators of $\hat{\mathfrak{g}}$. We find that 69 of them appear in Schellekens' list of semi-simple V_1 structures of holomorphic CFT of central charge 24, while 8 of them correspond to the $\mathcal{N} = 1$ structures of holomorphic SCFT of central charge 12 composed of 24 chiral fermions. The last 4 cases are related to exceptional modular invariants from nontrivial automorphisms of fusion algebras. This clarifies the relationship of affine Lie algebras, vertex algebras and hyperbolic BKM superalgebras at the level of modular forms.

CONTENTS

1. Introduction

Affine Lie algebras are the simplest class of infinite dimensional Kac–Moody Lie algebras, and they have numerous connections with other areas of mathematics and theoretical physics. The next simplest class of Kac–Moody algebras after the affine algebras are the hyperbolic Lie algebras. In 1983, as an extension of the affine Lie algebra of type A_1, Feingold and Frenkel [33] investigated the hyperbolic Kac–Moody algebra with Cartan matrix

$$\begin{pmatrix} 2 & -2 & 0 \\ -2 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

and found a connection with the Igusa cusp form [65] of weight 35 on $\mathrm{Sp}_2(\mathbb{Z})$. This suggests exploring more general relationships between hyperbolic Kac–Moody Lie algebras and modular forms of several variables.

In 1988, Borcherds [3] introduced generalized Kac–Moody algebras, which are now usually called Borcherds–Kac–Moody or simply BKM algebras. These infinite-dimensional Lie algebras are also defined in terms of Chevalley–Serre generators and relations that are encoded in a generalized Cartan matrix, and they differ from Kac–Moody algebra mainly by allowing the diagonal entries of the Cartan matrix to be non-positive. In other words, simple roots are allowed to be imaginary, whereas simple roots in a Kac–Moody algebra are always real. The best-known example of a BKM algebra is the monster Lie algebra. In 1992, Borcherds [6] constructed this algebra as the BRST cohomology related to the monster vertex algebra [4, 35] by means of the no-ghost theorem from string theory. By considering the action of the monster group on the denominator identity of the monster Lie algebra, Borcherds proved the celebrated monstrous moonshine conjecture. Furthermore, he observed that the denominator functions of some BKM algebras are modular forms on orthogonal groups of signature $(l, 2)$. In 1995 and 1998 Borcherds [7, 9] developed the theory of singular theta lift to construct modular forms for arithmetic subgroups of $\mathrm{O}(l, 2)$ which have infinite product expansions. These are called automorphic products, or Borcherds products, and they are natural candidates for the denominator functions of BKM algebras. Similarly to affine Lie algebras, BKM algebras and automorphic products also have many applications in physics. For example, Harvey and Moore [58, 59] proposed that BKM algebras should play as the underlying organizing structure of BPS states in string compactifications; in particular, the denominators of BKM algebras might be generating functions for BPS states.

In 1996 Gritsenko and Nikulin [42] constructed an automorphic correction of the rank-three hyperbolic Lie algebra considered previously by Feingold and Frenkel. More precisely, they extended

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this hyperbolic Lie algebra to a (hyperbolic) BKM algebra by adding infinitely many imaginary simple roots, such that the denominator of the resulting BKM algebra is exactly the Igusa cusp form of weight 35. Later, they constructed automorphic corrections of other hyperbolic Lie algebras in a series of papers [46, 48, 47, 43, 51]. A common feature of these corrections is that the resulting BKM algbera only has finitely many real simple roots, a Weyl chamber of finite volume and a Weyl vector of negative norm. Moreover, its denominator usually defines a cuspidal automorphic product on $O(l, 2)$. These corrections extend affine Lie algebras in a nice way, but they have several features that are not preferred from our perspective:

(1) for the automorphic correction $\mathfrak{G}$ of an affine Lie algebra $\hat{\mathfrak{g}}$, the multiplicity of an imaginary root of $\hat{\mathfrak{g}}$ in $\mathfrak{G}$ is strictly greater than its multiplicity in $\hat{\mathfrak{g}}$;

(2) the set of imaginary simple roots is very complicated, although the set of real simple roots is easy to describe;

(3) the role of the vertex algebra is unclear.

In this paper we extend affine Lie algebras to (hyperbolic) BKM algebras in a different way. Certain affine Kac–Moody algebras $\hat{\mathfrak{g}}$ will be extended to BKM algebras $\mathcal{G}_{\mathfrak{g}}$ which have infinitely many real simple roots and satisfy:

(a) for any root α of $\hat{\mathfrak{g}}$, the root multiplicities of α in $\hat{\mathfrak{g}}$ and $\mathcal{G}_{\mathfrak{g}}$ are the same;

(b) the imaginary simple roots of $\mathcal{G}_{\mathfrak{g}}$ are negative integral multiples of the Weyl vector;

(c) the Lie algebras $\hat{\mathfrak{g}}$ and $\mathcal{G}_{\mathfrak{g}}$ are closely related to some exceptional vertex algebras.

Our main results are about the classification and construction of such extensions, which are connected with various types of modular forms. In the rest of the introduction, we will explain the setup and state the main theorems.

1.1. **Denominators of affine Lie algebras and Jacobi forms.** Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra of rank r and let $\Delta_{\mathfrak{g}}^+$ be a set of positive roots. The product side of the denominator identity of the associated affine Lie algebra $\hat{\mathfrak{g}}$ is the holomorphic function

$$\vartheta_{\mathfrak{g}}(\tau, \mathfrak{z}) = \eta(\tau)^r \prod_{\alpha \in \Delta_{\mathfrak{g}}^+} \frac{\vartheta(\tau, \langle \alpha, \mathfrak{z} \rangle)}{\eta(\tau)},$$

where η and ϑ are the Dedekind eta function and the odd Jacobi theta function, respectively:

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n), \quad \tau \in \mathbb{H}, \ q = e^{2\pi i \tau},$$

$$\vartheta(\tau, z) = -q^{\frac{1}{8}} \zeta^{-\frac{1}{2}} \prod_{n=1}^{\infty} (1 - q^{n-1}\zeta)(1 - q^n\zeta^{-1})(1 - q^n), \quad z \in \mathbb{C}, \ \zeta = e^{2\pi i z}.$$

The function $\vartheta_{\mathfrak{g}}$ has nice modular properties and is an example of a Jacobi form in the sense of Eichler and Zagier [32]. Let L be an even positive definite lattice. A Jacobi form of integral weight k and index L is a holomorphic function $\varphi : \mathbb{H} \times (L \otimes \mathbb{C}) \to \mathbb{C}$ that is modular under $\mathrm{SL}_2(\mathbb{Z})$ and doubly quasi-periodic, namely

$$\varphi\left(\frac{a\tau + b}{c\tau + d}, \frac{\mathfrak{z}}{c\tau + d}\right) = (c\tau + d)^k \exp\left(i\pi t \frac{c(\mathfrak{z}, \mathfrak{z})}{c\tau + d}\right) \varphi(\tau, \mathfrak{z}), \quad A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}),$$

$$\varphi(\tau, \mathfrak{z} + x\tau + y) = \exp\left(-i\pi t\big((x, x)\tau + 2(x, \mathfrak{z})\big)\right) \varphi(\tau, \mathfrak{z}), \quad x, y \in L,$$

and whose Fourier expansion satisfies a boundary condition. The function $\vartheta_{\mathfrak{g}}$ is a Jacobi form of weight $r/2$ and index $P_{\mathfrak{g}}^\vee(h_{\mathfrak{g}}^\vee)$ for some character, where $P_{\mathfrak{g}}^\vee$ is the dual of the root lattice and $h_{\mathfrak{g}}^\vee$ is the dual Coxeter number. Note that Jacobi forms defined by an expression similar to $\vartheta_{\mathfrak{g}}$ are called theta blocks following Gritsenko–Skoruppa–Zagier [55].

1.2. Automorphic products of singular weight.

A modular form of integral weight k and trivial character for an arithmetic subgroup $\Gamma < \mathrm{O}(l, 2)$ is a holomorphic function on the associated type IV symmetric domain which satisfies

$$F(t\mathcal{Z}) = t^{-k}F(\mathcal{Z}), \quad t \in \mathbb{C}^{\times},$$
$$F(g\mathcal{Z}) = F(\mathcal{Z}), \qquad g \in \Gamma.$$

Let M be an even lattice of signature $(l, 2)$. The input into Borcherds' theta lift is a vector-valued modular form of weight $1 - l/2$ with integral Fourier expansion for the Weil representation of $\mathrm{SL}_2(\mathbb{Z})$ attached to the discriminant form M'/M, and the output is a meromorphic modular form for a certain subgroup of $\mathrm{O}(M)$ which has an infinite product expansion at any 0-dimensional cusp and whose divisors are linear combinations of hyperplanes.

Since the denominators of affine Lie algebras satisfy modularity, it is natural to focus on hyperbolic BKM algebras whose denominators are modular. Let $\mathcal{G}$ be a BKM algebra whose denominator function defines the Fourier expansion of an $\mathrm{O}(l, 2)$-modular form F at some 0-dimensional cusp. Since F has an infinite product expansion, by Bruinier's converse theorem [14, 15] one expects that it can be constructed by the Borcherds lift. In the case, the roots and their multiplicities are encoded in the Fourier expansion of the input. When F has singular weight, i.e. weight $l/2 - 1$, the Fourier expansion is supported only on isotropic vectors, which often forces the imaginary simple roots of $\mathcal{G}$ to be negative integral multiples of the Weyl vector. Moreover, it is conjectured in this particular case that $\mathcal{G}$ can be constructed as the BRST cohomology related to some vertex algebra, similarly to the monster algebra. This suggests focusing on BKM algebras whose denominators are automorphic products of singular weight. The zeros of F that contain the cusp are actually hyperplanes orthogonal to real roots of $\mathcal{G}$, hence F is anti-invariant under the reflections through these hyperplanes. It is natural to expect that F is anti-invariant under all reflections associated with zeros of F. This has been proven by the last two named authors [115]. It follows that F is a reflective modular form.

A non-constant modular form on $\Gamma < \mathrm{O}(M)$ is called reflective if it vanishes only on mirrors of reflections fixing the lattice M. Reflective modular forms were introduced in 1998 by Borcherds [7, 9] and Gritsenko–Nikulin [47], and their classification has been an active project for the past thirty years (see [43, 1, 94, 25, 97, 81, 82, 23, 89, 104, 106, 107, 30, 110, 111]), because they have nice applications to hyperbolic reflection groups [9, 11, 48, 44, 39], algebraic geometry [8, 12, 44, 82, 41] and free algebras of modular forms [105, 114] in addition to infinite-dimensional Lie algebras.

1.3. Main results.

BKM algebras whose denominator functions are reflective automorphic products of singular weight are exceptional. The main examples are the fake monster algebra [5] and their twists by the Conway group Co_0 [6, 93, 113]. There are conjecturally only finitely many such algebras and constructing and classifying them is an open problem. Many partial results have been proved towards such a classification [5, 6, 88, 1, 92, 93, 94, 97, 23, 113, 30]. In this paper we contribute some new results in this direction.

Let U be an even unimodular lattice of signature $(1, 1)$ and let L be an even, positive-definite lattice. The input of the Borcherds lift on $2U \oplus L$ can be identified with Jacobi forms of weight 0 and index L. We will identify affine Lie algebras, which naturally extends to BKM algebras or superalgebras whose denominators or super-denominators are reflective Borcherds product of singular weight on lattices of type $2U \oplus L$. The setting is inspired by the following result.

Theorem 1.1. *Let F be a reflective Borcherds product of singular weight on $2U \oplus L$ whose Jacobi form input has Fourier expansion*

$$\phi(\tau, \mathfrak{z}) = \sum_{n \in \mathbb{Z}} \sum_{\ell \in L'} f(n, \ell) q^n \zeta^\ell$$

satisfying that $f(0, \ell) \geq 0$ for all $\ell \in L'$. If L is the Leech lattice, then F is the denominator of the fake monster algebra and ϕ is the full character of the Leech lattice vertex operator algebra. Otherwise, the set

$$\mathcal{R} = \{\ell \in L' : \ell \neq 0, \ f(0, \ell) \neq 0\}$$

determines a finite-dimensional semi-simple Lie algebra

$$\mathfrak{g} = \bigoplus_{j=1}^{s} \mathfrak{g}_{j,k_j}$$

with the same rank as L such that the identity

$$(1.1) \qquad\qquad C := \frac{\dim \mathfrak{g}}{24} - a = \frac{h_j^\vee}{k_j}$$

holds for any $1 \leq j \leq s$ and such that the leading Fourier–Jacobi coefficient of F at the 1-dimensional cusp determined by $2U$ coincides with the denominator of the associated affine Lie algebra $\hat{\mathfrak{g}}$. For any $1 \leq j \leq s$, $\mathfrak{g}_j$ is a simple ideal of $\mathfrak{g}$, k_j is a positive integer indicating the level of $\mathfrak{g}_j$ and $h_j^\vee$ is the dual Coxeter number of $\mathfrak{g}_j$. The number a equals $f(-1, 0)$, which has to be 0 or 1. When $a = 0$, $k_j > 1$ for any $1 \leq j \leq s$. The cases $a = 0$ and $a = 1$ are called symmetric and anti-symmetric, respectively.

If the q^0-term of ϕ has negative Fourier coefficients, then $\mathcal{R}$ determines a semi-simple Lie super-algebra and the corresponding BKM superalgebra will have odd real roots. We will consider this case in a separate paper.

If the Fourier expansion of F defines the (super)-denominator of a BKM (super)-algebra $\mathcal{G}$, then the (super)-denominator has the form

$$e^\rho \prod_{\alpha > 0} \left(1 - e^{-\alpha}\right)^{f(nm, \ell)},$$

where ρ is the Weyl vector of F, and where (n, ℓ, m) are coordinates of positive roots $\alpha \in U \oplus L'$ with $n \in \mathbb{Z}$, $m \in \mathbb{N}$, $\ell \in L'$ and $\alpha^2 = \ell^2 - 2nm$. The above $\hat{\mathfrak{g}}$ is embedded into $\mathcal{G}$ as the sum of the root spaces associated with roots of type $\pm(n, \ell, 0)$. In this way, $\mathcal{G}$ can be regarded as a graded module over $\hat{\mathfrak{g}}$ graded by $m \in \mathbb{N}$. This leads us to define $\mathcal{G}$ as a *hyperbolization* of $\hat{\mathfrak{g}}$ and F as a *hyperbolization* of the denominator of $\hat{\mathfrak{g}}$. It turns out that there are only 81 affine Lie algebras with a hyperbolization:

Theorem 1.2. *There are* 81 *possibilities for the semi-simple Lie algebra* $\mathfrak{g}$ *in Theorem* 1.1 *and they fall into three categories:*

> *(1)* 69 *make up Schellekens' list of semi-simple* V_1 *structures of holomorphic vertex operator algebras of central charge* 24;
> *(2)* 8 *are the* $\mathcal{N} = 1$ *structures of holomorphic vertex operator superalgebras* F_{24} *of central charge* 12 *composed of* 24 *fermions;*
> *(3)* *The remaining* 4 *cases* $A_{1,16}$, $A_{1,8}^2$, $A_{1,4}^4$ *and* $A_{2,9}$ *possess an exceptional modular invariant that comes from a nontrivial automorphism of the fusion algebra.*

Case (1) is anti-symmetric, while Cases (2) and (3) are symmetric. Schellekens' list [98] was established using the solutions of Equation (1.1) with $a = 1$. Holomorphic vertex operator super-algebras of central charge 12 were classified by Creutzig, Duncan and Riedler [20], and the $\mathcal{N} = 1$ structures of F_{24} were determined in [57], corresponding to solutions of Equation (1.1) with $a = 0$ and $C = 1$. The exceptional modular invariants mentioned in (3) were discovered around the 1990s by Moore and Seiberg [87], Verstegen [103] and Gannon [37]. The 4 exotic cases satisfy Equation (1.1) with $a = 0$ and $C < 1$.

We now present hyperbolizations of these affine Lie algebras.

Theorem 1.3. *For any $\mathfrak{g}$ in Theorem 1.2 there exists a singular-weight reflective Borcherds product $\Psi_\mathfrak{g}$ on some lattice $2U \oplus L_\mathfrak{g}$ whose leading Fourier–Jacobi coefficient equals the denominator of $\hat{\mathfrak{g}}$. Moreover, the Jacobi form input $\phi_\mathfrak{g}$ can be expressed as a $\mathbb{Z}$-linear combination of full characters of the affine vertex operator algebra generated by $\hat{\mathfrak{g}}$.*

The construction will be briefly summarized here. If $\mathfrak{g}$ is in Schellekens' list, $L_\mathfrak{g}$ is the orbit lattice in Höhn's construction [63] of the holomorphic VOA of central charge 24 with $V_1 = \mathfrak{g}$, and $\phi_\mathfrak{g}$ is the full character of the VOA. If $\mathfrak{g}$ is of symmetric type, $L_\mathfrak{g}$ is the maximal even sublattice of the coweight lattice of $\mathfrak{g}$. If $\mathfrak{g}$ defines an $\mathcal{N} = 1$ structure of F_{24}, then the Jacobi form input can be expressed in terms of characters of F_{24} as

$$\phi_\mathfrak{g} = (\chi_{\mathrm{NS}} - \chi_{\widetilde{\mathrm{NS}}} - \chi_{\mathrm{R}})/2.$$

Finally, we will explain the relation between Jacobi form inputs and exceptional modular invariants for the remaining four $\mathfrak{g}$. The Jacobi form input for $\mathfrak{g} = A_{1,16}$ can be written in terms of affine characters as

$$\phi_{A_{1,16}} = \chi_{2,\frac{1}{9}}^{A_{1,16}} + \chi_{14,\frac{28}{9}}^{A_{1,16}} - \chi_{8,\frac{10}{9}}^{A_{1,16}},$$

and we find that the difference between the simple current modular invariant and the exceptional modular invariant [87] is given by $|\phi_{A_{1,16}}|^2$. Similar relations hold for the other three $\mathfrak{g}$. Note that these $L_\mathfrak{g}$ are chosen so that the resulting BKM superalgebra has root lattice $U \oplus L_\mathfrak{g}'$.

Clearly, the Borcherds products $\Psi_\mathfrak{g}$ in Theorem 1.3 are closely related to vertex algebras. We therefore expect that the BRST cohomology related to these vertex algebras define the BKM (super)-algebras with $\Psi_\mathfrak{g}$ as the (super)-denominators. This type of realization has been achieved in [5, 61, 19, 62, 84, 30, 57] under some technical assumptions for $\mathfrak{g}$ from Schellekens' list and the $\mathcal{N} = 1$ structure of F_{24}. However, such a realization is completely open for $\mathfrak{g}$ related to the four exceptional modular invariants.

Affine Lie algebras, vertex algebras and BKM (super)-algebras are therefore closely connected from the point of view of the attached modular forms. The connections are illustrated in Figure 1.
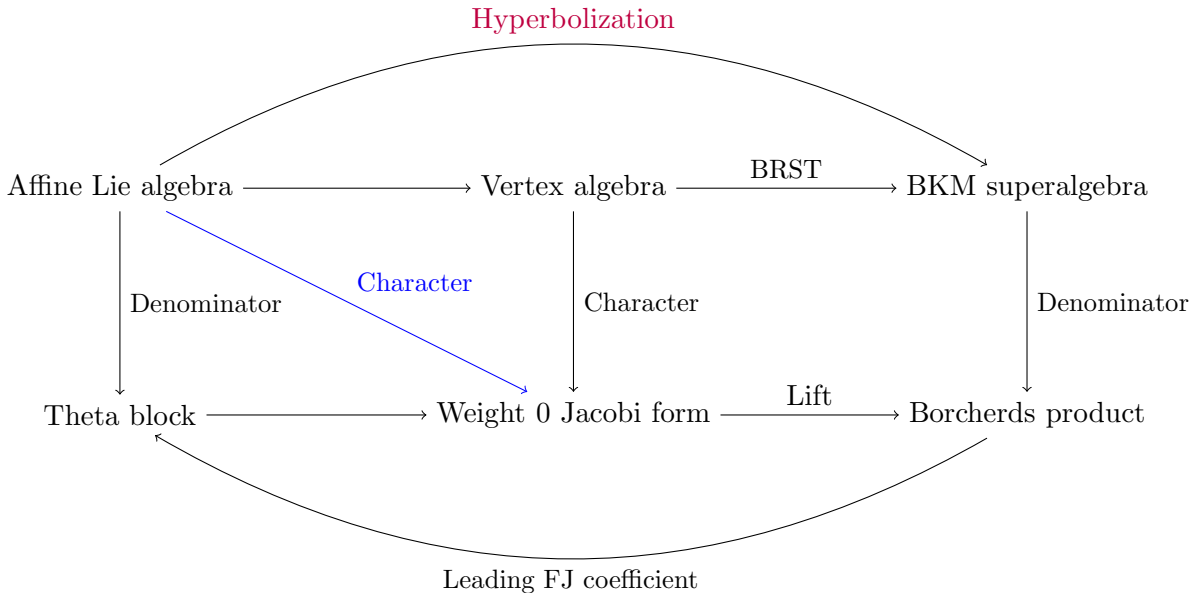

FIGURE 1. Hyperbolization of affine Lie algebras

1.4. **Outline of the proof.** The proof of Theorem 1.1 relies on some previous results. We know from [115] that singular-weight reflective Borcherds products have only simple zeros and are anti-invariant under reflections associated with their zeros. Therefore, the nonzero coefficients $f(0, \ell)$ of the Jacobi form input have to be 1 if $\ell \neq 0$, and $f(0, 0)$ equals the rank of $\mathfrak{g}$ by singular-weight. The theorem follows by applying an argument used by the second named author [108, 105, 110] to classify reflective modular forms. Theorem 1.1 further shows that the central charge of the affine vertex operator algebra generated by $\hat{\mathfrak{g}}$ is

$$c_{\mathfrak{g}} = \frac{24(C + a)}{C + 1}.$$

In particular, $c_{\mathfrak{g}} = 24$ if $a = 1$, and $c_{\mathfrak{g}} = 12$ if $a = 0$ and $C = 1$. This motivates the three groupings in Theorem 1.2.

To prove Theorem 1.2 we first solve equations of type (1.1). In the anti-symmetric case, Equation (1.1) was first derived by Schellekens [98] in the context of conformal field theories. Schellekens found 221 solutions to this equation and eliminated 152 of them to arrive at his list. We also have the same extra solutions to eliminate, but we have to use a completely different approach. In the symmetric case, there are 17 solutions to Equation (1.1) and we have to rule out 5 of them. In the setting of Theorem 1.1, we can prove that $L(C)$ is an integral lattice and that L is bounded by

$$\mathbf{Q}_{\mathfrak{g}} < L < \mathbf{P}_{\mathfrak{g}},$$

where $\mathbf{Q}_{\mathfrak{g}}$ and $\mathbf{P}_{\mathfrak{g}}$ are the coroot lattice and coweight lattice of $\mathfrak{g}$, respectively. On one hand, for every extraneous $\mathfrak{g}$, we will be able to find an even overlattice K of L for which there is no reflective Borcherds product on $2U \oplus K$ satisfying certain constraints. On the other hand, we prove that if $2U \oplus L$ has a singular-weight reflective Borcherds product then $2U \oplus K$ has also a reflective Borcherds product of the same type. Taken together, this allows us to rule out all 157 extraneous solutions of Equation (1.1).

We will now sketch the proof of Theorem 1.3, beginning with the anti-symmetric case. Let V be a holomorphic VOA of central charge 24 with semi-simple $V_1 = \mathfrak{g}$. The full character χ_V of V is known to be a Jacobi form of weight 0 and lattice index $\mathbf{Q}_{\mathfrak{g}}$ with non-negative integral Fourier expansion [117, 83, 69]. This immediately implies that the singular theta lift of χ_V, denoted $\mathbf{B}(\chi_V)$, defines a holomorphic Borcherds product of singular weight on $2U \oplus \mathbf{Q}_{\mathfrak{g}}$. It remains to find an extension L of $\mathbf{Q}_{\mathfrak{g}}$ for which $\mathbf{B}(\chi_V)$ is reflective on $2U \oplus L$. We recognize that L should be the Höhn' orbit lattice $L_{\mathfrak{g}}$. Höhn [63, 71, 2] proposed a construction of V as the simple current extension of the tensor product of the lattice VOA $V_{L_{\mathfrak{g}}}$ and a certain VOA $W_{\mathfrak{g}}$ of central charge $24 - \mathrm{rk}(\mathfrak{g})$ with trivial weight-one subspace. This construction corresponds to the theta decomposition of χ_V as a Jacobi form of index $L_{\mathfrak{g}}$. It is possible to prove directly that $\mathbf{B}(\chi_V)$ is reflective on $2U \oplus L_{\mathfrak{g}}$ by computing the Fourier expansion of χ_V; however, this is only feasible for certain specific $\mathfrak{g}$. We carry out the calculation for $\mathfrak{g} = B_{12,2}$, $A_{2,2}F_{4,6}$ and $C_{4,10}$, in which case $[L_{\mathfrak{g}} : \mathbf{Q}_{\mathfrak{g}}] \leq 2$.

To complete the proof for the remaining $\mathfrak{g}$, we relate $\mathbf{B}(\chi_V)$ to a twisted denominator of the fake monster algebra. By [63, 71], there exists a conjugacy class $[g]$ of the Conway group Co_0 such that $W_{\mathfrak{g}}$ is isomorphic to the orbifold $V_{\Lambda_g}^{\hat{g}}$, where Λ_g is the coinvariant sublattice of the Leech lattice Λ. Moreover, $\mathbf{B}(\chi_V)$ for distinct V define the same modular form on type IV symmetric domain if $V_1 = \mathfrak{g}$ correspond to the same $[g]$. Combining this fact with our previous calculation of χ_V, we prove that $\mathbf{B}(\chi_V)$ is reflective if V_1 corresponds to a class $[g]$ whose order is distinct from its level. The last two named authors [113] proved that the g-twisted denominator of the fake monster algebra defines a reflective Borcherds product Φ_g of singular weight on $U(n_g) \oplus U \oplus \Lambda^g$ if g has the same level and order n_g, where Λ^g is the sublattice of Λ fixed by g. We identify $\mathbf{B}(\chi_V)$ with the associated Φ_g and thus prove that $\mathbf{B}(\chi_V)$ is reflective if V_1 corresponds to some $[g]$ with the equal level and order. As a by-product, we find that the BKM algebra constructed as the BRST

cohomology related to V and the g-twist of the fake monster algebra are isomorphic if g has equal order and level, but they are not isomorphic if the order and level of g are distinct.

Then we consider the symmetric case with $C = 1$. For any $\mathcal{N} = 1$ structure of F_{24}, we show that there exists a Co_0-conjugacy class $[g]$ with level equal to its order such that the Borcherds product $\mathbf{B}(\phi_{\mathfrak{g}})$ is the g-twisted denominator of the fake monster algebra. This is proven by identifying their Jacobi form inputs, and relies on the construction of these g-twisted denominators as Gritsenko (additive) lifts due to Dittmann and the second named author [24]. We then confirm that $\mathbf{B}(\phi_{\mathfrak{g}})$ is a singular-weight reflective Borcherds product and that the BKM superalgebra constructed in [57] as the BRST cohomology related to F_{24} is isomorphic to the g-twist of the fake monster algebra.

In the final case, we determine the Jacobi form inputs in terms of affine characters and prove that their singular theta lifts are reflective by direct calculation.

1.5. **Outline of the paper.** In Section 2, we quickly introduce Jacobi forms, affine Kac–Moody algebras, reflective modular forms, automorphic products and Borcherds–Kac–Moody algebras. We also fix some notation that will be used frequently later on.

In Section 3 we define the hyperbolization of an affine Lie algebra and explain its motivation.

In Section 4 we present the proof of Theorem 1.1 and the solutions of Equation (1.1).

In Section 5 we state the full version of the main results.

In Section 6, we first introduce holomorphic VOA of central charge 24 and Höhn's construction, as well as their relations with twisted denominators of the fake monster algebra. We then construct hyperbolizations of the affine Lie algebras on Schellekens' list.

In Section 7, we begin by introducing holomorphic SVOA of central charge 12 and type F_{24}, and then construct hyperbolizations for the eight $\mathcal{N} = 1$ structures of F_{24}. As an application, we construct many exceptional modular invariants by considering the conformal embedding from the affine VOA generated by the $\mathcal{N} = 1$ structure to F_{24} and considering an automorphism of the $D_{12,1}$ fusion algebra.

In Section 8, we construct hyperbolizations of the remaining affine Lie algebras and explain their connection with some exceptional modular invariants.

Combining Sections 6-8 completes the proof of Theorem 1.3.

In Section 9, we present a uniform construction of the 12 symmetric singular-weight reflective Borcherds products as additive lifts.

In Section 10, we compute the Fourier expansions of these singular-weight reflective Borcherds products at the 0-dimensional cusp determined by one copy of U.

Section 11 is devoted to the proof of the anti-symmetric case of Theorem 1.2.

Section 12 contains the proof of the symmetric case of Theorem 1.2.

In Section 13 we give an application of our main results. We use the pull-back to construct a new infinite series of anti-symmetric Siegel paramodular forms of weight 3.

In Section 14 we raise some related questions and conjectures.

This article ends with several long tables.

2. Preliminaries

In this section, we will first review Jacobi forms and denominator identities of affine Lie algebras. We then define reflective modular forms on orthogonal groups $O(n, 2)$ and review Borcherds' theory of automorphic products, as well as the denominator identities of Borcherds–Kac–Moody algebras. This background is necessary to state and prove the main theorems.

2.1. Jacobi forms of lattice index.

Let $\mathbb{Z}$ and $\mathbb{N}$ denote the sets of integers and non-negative integers, respectively. Let L be an even integral positive-definite lattice with bilinear form $(-, -)$ and dual lattice

$$L' = \{v \in L \otimes \mathbb{Q} : (x, y) \in \mathbb{Z}, \text{ for all } y \in L\}.$$

For any nonzero integer a, the lattice with abelian group L and bilinear form $a(-, -)$ is denoted by $L(a)$. Let $\mathbb{H}$ be the complex upper half plane. The Dedekind eta function

$$\eta(\tau) = q^{1/24} \prod_{j=1}^{\infty} (1 - q^j), \quad q = e^{2\pi i \tau}, \ \tau \in \mathbb{H}$$

is a modular form of weight $\frac{1}{2}$ on $SL_2(\mathbb{Z})$ with a multiplier system of order 24. We denote this multiplier system by v_η.

Definition 2.1. Let D be a positive integer, $k \in \frac{1}{2}\mathbb{Z}$ and $t \in \frac{1}{2}\mathbb{N}$. A holomorphic function $\varphi : \mathbb{H} \times (L \otimes \mathbb{C}) \to \mathbb{C}$ is called a *weakly holomorphic* Jacobi form of weight k, index t and character (or multiplier system) v_η^D for L if it satisfies

$$\varphi\left(\frac{a\tau + b}{c\tau + d}, \frac{\mathfrak{z}}{c\tau + d}\right) = v_\eta^D(A)(c\tau + d)^k \exp\left(i\pi t \frac{(\mathfrak{z}, \mathfrak{z})}{c\tau + d}\right) \varphi(\tau, \mathfrak{z}),$$

$$\varphi(\tau, \mathfrak{z} + x\tau + y) = (-1)^{t\left((x,x)+(y,y)\right)} \exp\left(-i\pi t\left((x,x)\tau + 2(x,\mathfrak{z})\right)\right)\varphi(\tau, \mathfrak{z}),$$

for $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ and $x, y \in L$, and if its Fourier expansion takes the form

$$\varphi(\tau, \mathfrak{z}) = \sum_{\substack{-\infty \ll n \in \mathbb{Z} - \frac{D}{24} \\ \ell \in \frac{1}{2}L'}} f(n, \ell) q^n \zeta^\ell, \quad \zeta^\ell = e^{2\pi i (\ell, \mathfrak{z})},$$

where $n \gg -\infty$ means that n is bounded from below. If $f(n, \ell) = 0$ whenever $n < 0$, then φ is called a *weak* Jacobi form. If $f(n, \ell) = 0$ whenever $2nt - (\ell, \ell) < 0$, then φ is called a *holomorphic* Jacobi form.

We denote the vector spaces of weakly holomorphic, weak and holomorphic Jacobi forms of weight k, index t and character v_η^D for L by

$$J_{k,L,t}^!(v_\eta^D) \supset J_{k,L,t}^w(v_\eta^D) \supset J_{k,L,t}(v_\eta^D),$$

respectively. We simply write $J_{k,L,t}^! \supset J_{k,L,t}^w \supset J_{k,L,t}$ if the character is trivial. The spaces $J_{k,t}$ of classical Jacobi forms introduced by Eichler–Zagier [32] are simply $J_{k,A_1,t}$ for the lattice $A_1 = \mathbb{Z}$ with $(x, x) = 2x^2$. If $\varphi \in J_{k,L,t}^!(v_\eta^D)$ is nonzero for some $t > 0$ then $L(t)$ is necessarily an integral lattice [18]. Jacobi forms of index t for L are also called Jacobi forms of (lattice) index $L(t)$ and we sometimes write

$$J_{k,L(t)}^! = J_{k,L,t}^!, \quad J_{k,L(t)}^w = J_{k,L,t}^w, \quad J_{k,L(t)} = J_{k,L,t}.$$

We will use the following Hecke operators to construct Jacobi forms later.

Proposition 2.2 (Proposition 3.1 in [18]). *Let $\varphi \in J^!_{k,L,t}(v^D_\eta)$. Assume that $k \in \mathbb{Z}$ and D is an even divisor of 24. If $Q = 24/D$ is odd, we further assume that $t \in \mathbb{Z}$. Then for any positive integer m coprime to Q we have*

$$\varphi|_{k,t}T^{(Q)}_-(m)(\tau,\mathfrak{z}) = m^{-1}\sum_{\substack{ad=m,\,a>0 \\ 0\leq b<d}} a^k v^D_\eta(\sigma_a)\varphi\left(\frac{a\tau+bQ}{d},a\mathfrak{z}\right) \in J^!_{k,L,mt}(v^{Dx}_\eta),$$

where $x,y \in \mathbb{Z}$ such that $mx + Qy = 1$, and

$$\sigma_a = \begin{pmatrix} dx+Qdxy & -Qy \\ Qy & a \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}).$$

Moreover, the Fourier coefficients $f_m(-,-)$ of $\varphi|_{k,t}T^{(Q)}_-(m)$ are linear combinations of the Fourier coefficients $f(-,-)$ of φ, i.e.

$$f_m(n,\ell) = \sum_{a|(n,\ell,m)} a^{k-1}v^D_\eta(\sigma_a)f\left(\frac{nm}{a^2},\frac{\ell}{a}\right),$$

where $a\,|\,(n,\ell,m)$ means that $a\,|\,nQ$, $a^{-1}\ell \in \frac{1}{2}L'$ and $a\,|\,m$.

2.2. Affine Kac–Moody algebras and theta blocks.
We review untwisted affine Kac–Moody algebras following [66] and identify their denominators as important examples of Jacobi forms.

Let $\mathfrak{g}$ be a simple Lie algebra of rank r with Cartan subalgebra $\mathfrak{h}$ and root system $\Delta_\mathfrak{g}$. We fix a set of simple roots $\{\alpha_1,...,\alpha_r\} \subset \mathfrak{h}^*$ and denote the set of positive roots by $\Delta^+_\mathfrak{g}$ and the highest root by θ. Note that $\dim\mathfrak{g} = r + |\Delta_\mathfrak{g}|$. The invariant symmetric bilinear form $\langle -, - \rangle$ on $\mathfrak{h}^*$ is normalized such that long roots have (square) norm two; in particular, $\langle\theta,\theta\rangle = 2$. We identify $\mathfrak{h}$ with $\mathfrak{h}^*$ and define the coroot of a root α as

$$\alpha^\vee = 2\alpha/\langle\alpha,\alpha\rangle.$$

The fundamental weights $w_i \in \mathfrak{h}^*$ are defined by

$$\langle w_i, \alpha^\vee_i \rangle = \delta_{i,j}, \quad 1 \leq i,j \leq r,$$

where $\delta_{i,j} = 1$ if $i = j$ and 0 otherwise. The Weyl vector $\rho_\mathfrak{g}$ is defined as

$$\rho_\mathfrak{g} = \frac{1}{2}\sum_{\alpha\in\Delta^+_\mathfrak{g}}\alpha = \sum_{j=1}^r w_j.$$

Let $Q_\mathfrak{g}$ be the rational lattice generated by the roots of $\mathfrak{g}$ and let $Q^\vee_\mathfrak{g}$ be the even integral lattice generated by the coroots, or equivalently by the long roots of $\mathfrak{g}$. The weight lattice $P_\mathfrak{g}$, generated by the fundamental weights, is the dual of the coroot lattice:

$$P_\mathfrak{g} = (Q^\vee_\mathfrak{g})' = \{x \in Q_\mathfrak{g}\otimes\mathbb{Q} : \langle x,\alpha^\vee\rangle \in \mathbb{Z},\ \alpha\in\Delta_\mathfrak{g}\} \supset Q_\mathfrak{g}.$$

The coweight lattice $P^\vee_\mathfrak{g}$ is the dual of the root lattice:

$$P^\vee_\mathfrak{g} = Q'_\mathfrak{g} = \{x \in Q_\mathfrak{g}\otimes\mathbb{Q} : \langle x,\alpha\rangle \in \mathbb{Z},\ \alpha\in\Delta_\mathfrak{g}\} \supset Q^\vee_\mathfrak{g}.$$

The reflection associated with a root α is defined as

$$\sigma_\alpha(x) = x - \langle x,\alpha^\vee\rangle\alpha, \quad x \in Q_\mathfrak{g}\otimes\mathbb{R}.$$

The reflections associated with simple roots generate the Weyl group $W_\mathfrak{g}$. The coroot of θ can be written as an $\mathbb{N}$-linear combination of the coroots of simple roots,

$$\theta = \theta^\vee = \sum_{j=1}^r a^\vee_j\alpha^\vee_j.$$

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The integers $a_j^\vee$ are called comarks. The number

$$h_{\mathfrak{g}}^\vee = 1 + \sum_{j=1}^r a_j^\vee = \frac{1}{r} \sum_{\alpha \in \Delta_{\mathfrak{g}}^+} \langle \alpha, \alpha \rangle$$

is the dual Coxeter number and it satisfies the identity

$$\sum_{\alpha \in \Delta_{\mathfrak{g}}^+} \langle \alpha, \mathfrak{z} \rangle^2 = h_{\mathfrak{g}}^\vee \langle \mathfrak{z}, \mathfrak{z} \rangle, \quad \mathfrak{z} \in Q_{\mathfrak{g}} \otimes \mathbb{C}.$$

The above identity implies that the rescaled lattice $P_{\mathfrak{g}}^\vee(h_{\mathfrak{g}}^\vee)$ is integral.

The classification of irreducible root systems into types A_n for $n \geq 1$, B_n for $n \geq 2$, C_n for $n \geq 3$, D_n for $n \geq 4$, and the exceptional systems E_6, E_7, E_8, F_4 and G_2 is well-known. We use the same symbol to stand for the corresponding root lattice with its normalized bilinear form. Some useful data is summarized in Table 1 for convenience; for the coordinates of the fundamental weights and the values of the comarks, see [13].

$\Delta_{\mathfrak{g}}$	A_n	B_n	C_n	D_n	E_6	E_7	E_8	G_2	F_4
$|\Delta_{\mathfrak{g}}|$	$n(n+1)$	$2n^2$	$2n^2$	$2n(n-1)$	72	126	240	12	48
$h_{\mathfrak{g}}^\vee$	$n+1$	$2n-1$	$n+1$	$2(n-1)$	12	18	30	4	9
$Q_{\mathfrak{g}}^\vee$	A_n	D_n	nA_1	D_n	E_6	E_7	E_8	A_2	D_4
$P_{\mathfrak{g}}^\vee$	A_n'	$\mathbb{Z}^n$	$D_n'(2)$	D_n'	E_6'	E_7'	E_8	A_2	D_4

TABLE 1. Data related to the irreducible root systems

The untwisted affine Kac–Moody algebra $\hat{\mathfrak{g}}$ is an extension of $\mathfrak{g}$ defined by

$$\hat{\mathfrak{g}} = \mathbb{C}[t, t^{-1}] \otimes \mathfrak{g} \oplus \mathbb{C}K \oplus \mathbb{C}d,$$

where K is a central element and d is a derivative. The algebra $\hat{\mathfrak{g}}$ is an infinite-dimensional Lie algebra with affine Cartan subalgebra $\hat{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}K \oplus \mathbb{C}d$. We write $\hat{\mathfrak{h}}^* = \mathfrak{h}^* \oplus \mathbb{C}\hat{w}_0 \oplus \mathbb{C}\delta$ with

$$\hat{w}_0(\mathfrak{h} \oplus \mathbb{C}d) = 0, \qquad \hat{w}_0(K) = 1,$$
$$\delta(\mathfrak{h} \oplus \mathbb{C}K) = 0, \qquad \delta(d) = 1.$$

We embed $\mathfrak{h}^*$ into $\hat{\mathfrak{h}}^*$ and define

$$\alpha_0 = \delta - \theta \quad \text{and} \quad \alpha_0^\vee = K - \theta^\vee.$$

Then $\{\alpha_0, \alpha_1, \cdots, \alpha_r\}$ is a set of simple roots of $\hat{\mathfrak{g}}$ and $\{\alpha_0^\vee, \alpha_1^\vee, \cdots, \alpha_r^\vee\}$ is the corresponding set of coroots. Setting

$$\hat{w}_i = w_i + a_i^\vee \hat{w}_0, \quad 1 \leq i \leq r,$$

$\{\hat{w}_0, \hat{w}_1, \cdots, \hat{w}_r\}$ is a set of fundamental weights of $\hat{\mathfrak{g}}$. The Weyl vector of $\hat{\mathfrak{g}}$ is defined by

$$\hat{\rho}_{\mathfrak{g}} = \rho_{\mathfrak{g}} + h_{\mathfrak{g}}^\vee \hat{w}_0 - (\dim \mathfrak{g}/24)\delta,$$

such that the norm of $\hat{\rho}_{\mathfrak{g}}$ is zero; indeed, by the "strange formula" of Freudenthal–de Vries,

$$h_{\mathfrak{g}}^\vee \cdot \dim \mathfrak{g} = 12\langle \rho_{\mathfrak{g}}, \rho_{\mathfrak{g}} \rangle.$$

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The integrable highest weight representations of $\hat{\mathfrak{g}}$ are indexed by dominant integral weights, which are elements of

$$\hat{P}_+ = \sum_{j=0}^{r} \mathbb{N}\hat{w}_j + \mathbb{C}\delta.$$

The level of $\lambda = \sum_{j=0}^{r} x_j \hat{w}_j + c\delta \in \hat{P}_+$ is the integer

$$\lambda(K) = \lambda_0 + \sum_{j=1}^{r} x_j a_j^\vee.$$

The character of the irreducible highest weight module labeled by a weight $\lambda \in \hat{P}_+$ of level k is given by the *Weyl–Kac character formula*,

$$(2.1) \qquad \chi_\lambda^{\hat{\mathfrak{g}}} = \frac{\sum_{\sigma \in W_{\hat{\mathfrak{g}}}} (-1)^{\ell(\sigma)} e^{\sigma(\lambda + \rho_{\hat{\mathfrak{g}}})}}{e^{\rho_{\hat{\mathfrak{g}}}} \prod_{\alpha \in \Delta_{\hat{\mathfrak{g}}}^+} (1 - e^{-\alpha})^{\mathrm{mult}(\alpha)}},$$

where $W_{\hat{\mathfrak{g}}}$ is the Weyl group of $\hat{\mathfrak{g}}$, a semi-direct product of $W_{\mathfrak{g}}$ by a certain group of translations, $\ell(\sigma)$ is the length of σ, $\Delta_{\hat{\mathfrak{g}}}^+$ is the set of positive roots of $\hat{\mathfrak{g}}$, and $\mathrm{mult}(\alpha)$ is the multiplicity of α, i.e. the dimension of the root space $\hat{\mathfrak{g}}_\alpha$. When $\lambda = 0$, we have $\chi_\lambda^{\hat{\mathfrak{g}}} = 1$, which gives the *Macdonald–Weyl denominator identity*:

$$(2.2) \qquad e^{\rho_{\hat{\mathfrak{g}}}} \prod_{\alpha \in \Delta_{\hat{\mathfrak{g}}}^+} (1 - e^{-\alpha})^{\mathrm{mult}(\alpha)} = \sum_{\sigma \in W_{\hat{\mathfrak{g}}}} (-1)^{\ell(\sigma)} e^{\sigma(\rho_{\hat{\mathfrak{g}}})}.$$

The algebra $\hat{\mathfrak{g}}$ has real roots (with norm > 0) and imaginary roots (with norm $= 0$). The set of positive real roots is

$$\Delta_{\hat{\mathfrak{g}}}^{+,\mathrm{re}} = \{\alpha + n\delta : 0 < n \in \mathbb{Z}, \alpha \in \Delta_{\mathfrak{g}}\} \cup \Delta_{\mathfrak{g}}^+.$$

The set of positive imaginary roots is

$$\Delta_{\hat{\mathfrak{g}}}^{+,\mathrm{im}} = \{n\delta : 0 < n \in \mathbb{Z}\}.$$

Every real root has multiplicity one, but every imaginary root has multiplicity r.

Let $(\tau, \mathfrak{z}) \in \mathbb{H} \times \mathfrak{h}$. By interpreting the formal variable e^α as $e^{2\pi i \alpha(\mathfrak{z} - \tau d)}$, the character $\chi_\lambda^{\hat{\mathfrak{g}}}(\tau, \mathfrak{z})$ defines a holomorphic function on $\mathbb{H} \times \mathbb{C}^r$. The product side of the denominator identity can be written as a theta block (introduced by Gritsenko–Skoruppa–Zagier [55])

$$(2.3) \qquad \vartheta_{\mathfrak{g}}(\tau, \mathfrak{z}) = \eta(\tau)^r \prod_{\alpha \in \Delta_{\mathfrak{g}}^+} \frac{\vartheta(\tau, \langle \alpha, \mathfrak{z} \rangle)}{\eta(\tau)},$$

where the odd Jacobi theta function

$$\vartheta(\tau, z) = \sum_{n \in \mathbb{Z}} \left(\frac{-4}{n}\right) q^{\frac{n^2}{8}} \zeta^{\frac{n}{2}} = -q^{\frac{1}{8}} \zeta^{-\frac{1}{2}} \prod_{n=1}^\infty (1 - q^{n-1}\zeta)(1 - q^n\zeta^{-1})(1 - q^n), \quad z \in \mathbb{C}, \ \zeta = e^{2\pi i z}$$

is the denominator of $\chi_\lambda^{\hat{\mathfrak{g}}}$ for the A_1 Lie algebra $\mathfrak{g}$. Note that $\vartheta_{\mathfrak{g}}$ is a holomorphic Jacobi form of (singular) weight $r/2$ and lattice index $P_{\mathfrak{g}}^\vee(h_{\mathfrak{g}}^\vee)$ with multiplier system $\upsilon_\eta^{\dim \mathfrak{g}}$. By [66], all characters $\chi_\lambda^{\hat{\mathfrak{g}}}$ of fixed level k form a vector-valued weakly holomorphic Jacobi form of weight 0 and lattice index $Q_{\mathfrak{g}}^\vee(k)$ which is invariant under the action of the Weyl group $W_{\mathfrak{g}}$ on $\mathfrak{z}$. We refer to [66, Theorem 13.8] for the precise transformation laws with respect to the generators of $\mathrm{SL}_2(\mathbb{Z})$.

Let k be a positive integer. The irreducible $\hat{\mathfrak{g}}$-module $L_{\hat{\mathfrak{g}}}(k,0)$ associated with $k\hat{w}_0$ has a canonical structure as a simple rational vertex operator algebra. This is called the affine VOA generated by $\hat{\mathfrak{g}}$ at level k. Such VOAs are physically realized as the well-known Wess–Zumino–Witten (WZW)

models, which are nonlinear sigma models describing mapping fields from Riemann surfaces to Lie group manifolds. The central charge of $L_{\hat{\mathfrak{g}}}(k,0)$ is

$$(2.4) \qquad\qquad c_{\mathfrak{g}} = \frac{k \dim \mathfrak{g}}{k + h_{\mathfrak{g}}^{\vee}}.$$

The irreducible modules of $L_{\hat{\mathfrak{g}}}(k,0)$ are the irreducible $\hat{\mathfrak{g}}$-modules $L_{\hat{\mathfrak{g}}}(k,\lambda)$ associated with level k dominant integral weights $k\hat{w}_0 + \lambda \in \hat{P}_+$ for any $\lambda \in \sum_{j=1}^{r} \mathbb{N}w_j$ that satisfies

$$\langle \lambda, \theta^{\vee} \rangle = \sum_{j=1}^{r} a_j^{\vee} \lambda_j \le k, \quad \text{where} \quad \lambda = \sum_{j=1}^{r} \lambda_j w_j.$$

The conformal weight of $L_{\hat{\mathfrak{g}}}(k,\lambda)$ is

$$(2.5) \qquad\qquad h_\lambda = \frac{\langle \lambda, \lambda + 2\rho_{\mathfrak{g}} \rangle}{2(k + h_{\mathfrak{g}}^{\vee})}.$$

The full character of $L_{\hat{\mathfrak{g}}}(k,\lambda)$ is actually $\chi_{k\hat{w}_0+\lambda}^{\hat{\mathfrak{g}}}(\tau, \mathfrak{z})$ as defined above. When the root system of $\mathfrak{g}$ is R_r as in Table 1 and $\lambda = \sum_{j=1}^{r} \lambda_j w_j$, we often write $\chi_{k\hat{w}_0+\lambda}^{\hat{\mathfrak{g}}}$ as

$$(2.6) \qquad\qquad \chi_{\lambda_1 \cdots \lambda_r, h_\lambda}^{R_{r,k}}.$$

Notation 2.3. Let $\mathfrak{g} = \oplus_{j=1}^{s}\mathfrak{g}_j$ be a semi-simple Lie algebra. We will often have to associate to each simple ideal $\mathfrak{g}_j$ a positive integer k_j called the level. In this case we indicate the levels by writing $\mathfrak{g} = \oplus_{j=1}^{s}\mathfrak{g}_{j,k_j}$. For simplicity we write

$$h_j^{\vee} = h_{\mathfrak{g}_j}^{\vee}, \quad Q_j = Q_{\mathfrak{g}_j}, \quad Q_j^{\vee} = Q_{\mathfrak{g}_j}^{\vee}, \quad P_j = P_{\mathfrak{g}_j}, \quad P_j^{\vee} = P_{\mathfrak{g}_j}^{\vee}.$$

We further fix two lattices

$$\mathbf{Q}_{\mathfrak{g}} = \bigoplus_{j=1}^{s} Q_j^{\vee}(k_j) \quad \text{and} \quad \mathbf{P}_{\mathfrak{g}} = \bigoplus_{j=1}^{s} P_j^{\vee}(k_j).$$

Let $V_{\mathfrak{g}}$ denote the affine vertex operator algebra

$$\bigotimes_{j=1}^{s} L_{\hat{\mathfrak{g}}_j}(k_j, 0).$$

Clearly, the denominator of $\hat{\mathfrak{g}}$ is given by the theta block

$$(2.7) \qquad\qquad \vartheta_{\mathfrak{g}} := \vartheta_{\mathfrak{g}_1} \otimes \vartheta_{\mathfrak{g}_2} \otimes \cdots \otimes \vartheta_{\mathfrak{g}_s}.$$

Remark 2.4. Some remarks on the denominator identities are in order:

(1) Gritsenko, Skoruppa and Zagier [55] found a direct proof of the denominator identity based on their theory of theta blocks.
(2) The function $\vartheta_{\mathfrak{g}}(\tau, \mathfrak{z})/\eta(\tau)^{\dim \mathfrak{g}}$ equals the modular Jacobian of the generators of the ring of weak Jacobi forms for $Q_{\mathfrak{g}}^{\vee}$ invariant under $W_{\mathfrak{g}}$ (see [104]).
(3) If a theta block defines a holomorphic Jacobi form of singular weight, then it has to be the denominator of an affine Lie algebra ([109]). Therefore, there is an one-to-one correspondence between affine Lie algebras and singular-weight theta blocks.

2.3. **Reflective modular forms on orthogonal groups.** Let M be an even integral lattice of signature $(l, 2)$ with $l \geq 3$. We choose one of the two connected components of

$$\{\mathcal{Z} \in M \otimes \mathbb{C} : (\mathcal{Z}, \mathcal{Z}) = 0, (\mathcal{Z}, \bar{\mathcal{Z}}) < 0\}$$

and label it $\mathcal{A}(M)$. The symmetric domain of type IV attached to M is

$$\mathcal{D}(M) := \mathcal{A}(M)/\mathbb{C}^{\times} = \{[\mathcal{Z}] \in \mathbb{P}(M \otimes \mathbb{C}) : \mathcal{Z} \in \mathcal{A}(M)\}.$$

Let $\mathrm{O}^+(M)$ denote the subgroup of $\mathrm{O}(M \otimes \mathbb{R})$ which preserves M and $\mathcal{A}(M)$. Let Γ be a finite-index subgroup of $\mathrm{O}^+(M)$. The most important example of Γ will be the *discriminant kernel*

$$\widetilde{\mathrm{O}}^+(M) = \{g \in \mathrm{O}^+(M) : g(v) - v \in M, \text{ for all } v \in M'\},$$

where M' is the dual lattice of M.

Definition 2.5. Let $k \in \mathbb{Z}$ and $\chi : \Gamma \to \mathbb{C}^{\times}$ be a character. A holomorphic function $F : \mathcal{A}(M) \to \mathbb{C}$ is called a modular form of weight k and character χ on Γ if it satisfies

$$F(t\mathcal{Z}) = t^{-k}F(\mathcal{Z}), \quad \text{for all } t \in \mathbb{C}^{\times},$$
$$F(g\mathcal{Z}) = \chi(g)F(\mathcal{Z}), \quad \text{for all } g \in \Gamma.$$

Modular forms can be represented by their Fourier expansions. Let c be a primitive isotropic vector of M and choose $c' \in M'$ satisfying $(c, c') = 1$. Then $M_{c,c'} = M \cap c^{\perp} \cap (c')^{\perp}$ is an even lattice of signature $(l-1, 1)$. Around the cusp c, one can identify the symmetric domain $\mathcal{D}(M)$ with a tube domain $\mathbb{H}_{c,c'}$, a connected component of

$$\{Z = X + iY : X, Y \in M_{c,c'} \otimes \mathbb{R}, (Y, Y) < 0\}.$$

This induces an action of $\mathrm{O}^+(M)$ on $\mathbb{H}_{c,c'}$ and an automorphy factor, which allows us to define modular forms of half-integral weight. A modular form of trivial character on $\widetilde{\mathrm{SO}}^+(M)$ can be expanded on $\mathbb{H}_{c,c'}$ as

$$F(Z) = \sum_{\lambda \in M'_{c,c'}, (\lambda, \lambda) \leq 0} c(\lambda) e^{2\pi i (\lambda, Z)}.$$

Modular forms F on general Γ have similar expansions with $M_{c,c'}$ replaced by a finite-index sublattice. If F is nonzero, then either $k = 0$ in which case F is constant, or $k \geq l/2 - 1$. The smallest possible positive weight $l/2 - 1$ is called the *singular weight*. When F has singular weight, its Fourier coefficients $c(\lambda)$ are zero whenever $(\lambda, \lambda) \neq 0$.

Let $\lambda \in M \otimes \mathbb{Q}$ be a vector of positive norm. The *rational quadratic divisor* associated with λ is

$$\lambda^{\perp} = \{[\mathcal{Z}] \in \mathcal{D}(M) : (\mathcal{Z}, \lambda) = 0\}.$$

We define the associated reflection $\sigma_{\lambda} \in \mathrm{O}^+(M \otimes \mathbb{Q})$ as

$$\sigma_{\lambda}(x) = x - \frac{2(x, \lambda)}{(\lambda, \lambda)}\lambda, \quad x \in M \otimes \mathbb{Q}.$$

The divisor $\lambda^{\perp}$ is called *reflective* if σ_{λ} fixes M, i.e. $\sigma_{\lambda} \in \mathrm{O}^+(M)$. If λ is a primitive vector in M', then $\lambda^{\perp}$ is reflective if and only if there exists a positive integer d such that $(\lambda, \lambda) = 2/d$ and $d\lambda \in M$. More precisely, the order of λ in M'/M is either d, or $d/2$ in which case $d/2$ is necessarily even. A non-constant (holomorphic) modular form on Γ is called *reflective* if its zero divisor is a linear combination of reflective rational quadratic divisors.

2.4. Borcherds products.

Let M be an even lattice of signature (b^+, b^-) with discriminant form $D_M := (M'/M, \mathfrak{q})$, where $\mathfrak{q}(x) = (x, x)/2$ is the induced quadratic form. Let $\mathrm{Mp}_2(\mathbb{Z})$ be the metaplectic group, which consists of pairs $A = (A, \phi_A)$, where $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ and ϕ_A is a holomorphic square root of $\tau \mapsto c\tau + d$ on $\mathbb{H}$, with the standard generators

$$T = \left(\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right), 1 \right) \quad \text{and} \quad S = \left(\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right), \sqrt{\tau} \right).$$

The *Weil representation* ρ_M is the unitary representation of $\mathrm{Mp}_2(\mathbb{Z})$ on the group ring $\mathbb{C}[D_M] = \mathrm{span}(e_x : x \in D_M)$ defined by

$$\rho_M(T)e_x = \mathbf{e}(-\mathfrak{q}(x))e_x \quad \text{and} \quad \rho_M(S)e_x = \frac{\mathbf{e}(\mathrm{sign}(M)/8)}{\sqrt{|D_M|}} \sum_{y \in D_M} \mathbf{e}((x, y))e_y,$$

where $\mathbf{e}(t) = e^{2\pi i t}$ for $t \in \mathbb{C}$, and $\mathrm{sign}(M) = b^+ - b^- \mod 8$. The dual representation of ρ_M is the complex conjugate of ρ_M; moreover, $\bar{\rho}_M = \rho_{M(-1)}$.

Let $k \in \frac{1}{2}\mathbb{Z}$. A holomorphic function $f : \mathbb{H} \to \mathbb{C}[D_M]$ is called a *weakly holomorphic modular form* of weight k if f satisfies

$$\phi_A(\tau)^{-2k} f(A \cdot \tau) = \rho_M(A)f(\tau), \quad \text{for all } A \in \mathrm{Mp}_2(\mathbb{Z}),$$

and if f is represented by a Fourier series of the form

$$(2.8) \qquad\qquad f(\tau) = \sum_{x \in D_M} \sum_{\substack{n \in \mathbb{Z} - \mathfrak{q}(x) \\ n \gg -\infty}} c_x(n)q^n e_x.$$

The finite sum $c_x(n)q^n e_x$ with $n < 0$ is called the *principal part* of f. If f is holomorphic at infinity, i.e. its principal part is zero, then it is a *holomorphic modular form*. Note that $k + \mathrm{sign}(M)/2 \in \mathbb{Z}$ if non-zero f exist, and that if $\mathrm{sign}(M)$ is even then ρ_M factors through a representation of $\mathrm{SL}_2(\mathbb{Z})$. We denote the spaces of weakly holomorphic and holomorphic modular forms of weight k for ρ_M respectively by

$$M_k^!(\rho_M) \supset M_k(\rho_M).$$

There is a natural homomorphism $\mathrm{O}(M) \to \mathrm{O}(D_M)$ with kernel $\widetilde{\mathrm{O}}(M)$. We define an action of $\mathrm{O}(D_M)$ on modular forms $f = \sum c_x(n)q^n e_x$ by

$$\sigma(f) = \sum c_x(n)q^n e_{\sigma(x)}, \quad \sigma \in \mathrm{O}(D_M).$$

We now assume that M has signature $(l, 2)$ with $l \geq 3$. Let f be a weakly holomorphic modular form of weight $1 - l/2$ for ρ_M with integral principal part. The Borcherds singular theta lift [7, 9] produces a meromorphic modular form $\mathbf{B}(f)$ of weight $c_0(0)/2$ and some character (or multiplier system) on

$$\mathrm{O}^+(M, f) = \{\sigma \in \mathrm{O}^+(M) : \sigma(f) = f\} \supset \widetilde{\mathrm{O}}^+(M)$$

which has an infinite product expansion at any 0-dimensional cusp involving the Fourier coefficients of f (see below). Moreover, the divisor of $\mathbf{B}(f)$ is a linear combination of rational quadratic divisors $\lambda^\perp$, each with multiplicity

$$\sum_{d=1}^{\infty} c_{d\lambda}(-d^2\lambda^2/2),$$

where $\lambda \in M'$ are primitive vectors of positive norm.

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There is an identification between modular forms for the Weil representation and Jacobi forms. Let L be an even positive-definite lattice of rank $\mathrm{rk}(L)$. Then one has the isomorphism

$$M^!_{k-\frac{1}{2}\mathrm{rk}(L)}(\rho_L) \xrightarrow{\sim} J^!_{k,L}$$

(2.9)
$$f(\tau) = \sum_{\gamma \in L'/L} f_\gamma(\tau)e_\gamma \longmapsto \sum_{\gamma \in L'/L} f_\gamma(\tau)\Theta_{L,\gamma}(\tau,\mathfrak{z}),$$

where $\Theta_{L,\gamma}$ is the Jacobi theta function of $L + \gamma$, defined as

$$\Theta_{L,\gamma}(\tau,\mathfrak{z}) = \sum_{\ell \in L+\gamma} e^{\pi i (\ell,\ell)\tau + 2\pi i (\ell,\mathfrak{z})}, \quad (\tau,\mathfrak{z}) \in \mathbb{H} \times (L \otimes \mathbb{C}).$$

This map can be extended to modular forms with characters and it induces an isomorphism between holomorphic modular forms. It follows that a holomorphic Jacobi form of weight $\frac{1}{2}\mathrm{rk}(L)$ and index L is a $\mathbb{C}$-linear combination of $\Theta_{L,\gamma}$. We call $\frac{1}{2}\mathrm{rk}(L)$ the *singular weight* for Jacobi forms.

It will be convenient to represent Borcherds products in terms of Jacobi forms. Let $\phi \in J^!_{0,L,1}$ be a weakly holomorphic Jacobi form of trivial character with Fourier expansion

$$\phi(\tau,\mathfrak{z}) = \sum_{n \in \mathbb{Z}} \sum_{\ell \in L'} f(n,\ell) q^n \zeta^\ell.$$

The terms $f(n,\ell)q^n\zeta^\ell$ with $2n - (\ell,\ell) < 0$ are called *singular Fourier coefficients* and they correspond to the principal part of the preimage of ϕ under the above isomorphism.

Remark 2.6. The Fourier expansion of ϕ satisfies $f(n,\ell) = f(n,-\ell)$ and $f(n_1,\ell_1) = f(n_2,\ell_2)$ if $2n_1 - (\ell_1,\ell_1) = 2n_2 - (\ell_2,\ell_2)$ and if $\ell_1 - \ell_2 \in L$. Therefore, any singular Fourier coefficient of ϕ already appears as a summand $f(n,\ell)q^n\zeta^\ell$ where $n \leq \hat{\delta}_L$, $\ell \in L'$ and $2n < (\ell,\ell)$, where $\hat{\delta}_L$ is the largest integer less than $\delta_L/2$ and

(2.10)
$$\delta_L := \max\big\{\min\{(y,y) : y \in L + x\} : x \in L'\big\}.$$

Let U be a hyperbolic plane, i.e. an even unimodular lattice of signature $(1,1)$. We write

$$U = \mathbb{Z}e + \mathbb{Z}f, \quad \text{where} \quad e^2 = f^2 = 0 \quad \text{and} \quad (e,f) = -1.$$

Let $U_1 = \mathbb{Z}e_1 + \mathbb{Z}f_1$ be a second hyperbolic plane and define $M = U_1 \oplus U \oplus L$. We fix coordinates on the tube domain about U_1 by writing

$$\mathbb{H}(L) = \{Z = \tau f + \mathfrak{z} + \omega e : \tau, \omega \in \mathbb{H}, \ \mathfrak{z} \in L \otimes \mathbb{C}, \ 2\,\mathrm{Im}(\tau)\,\mathrm{Im}(\omega) - (\mathfrak{z},\mathfrak{z}) > 0\},$$

such that

$$-(\alpha, Z) = n\tau - (\ell,\mathfrak{z}) + m\omega \quad \text{for} \quad \alpha = ne + \ell + mf \in U \oplus L'.$$

These coordinates are chosen such that the infinite expansions below match the denominators of Borcherds–Kac–Moody algebras.

Notation 2.7. We write $v = x_1 e_1 + xe + \ell + yf + y_1 f_1 \in U_1 \oplus U \oplus L'$ in the coordinate (x_1, x, ℓ, y, y_1) and $\alpha = xe + \ell + yf \in U \oplus L'$ in the coordinate (x,ℓ,y). In particular, $v^2 = (\ell,\ell) - 2(xy + x_1y_1)$.

Theorem 2.8 ([41], Theorem 4.2). *Assume the above ϕ has integral singular Fourier coefficients. Then the Borcherds theta lift of ϕ is a meromorphic modular form of weight $f(0,0)/2$ on $\widetilde{\mathrm{O}}^+(M)$ and it can be expanded on an open subset of the tube domain $\mathbb{H}(L)$ as*

$$\mathbf{B}(\phi)(Z) = q^A \zeta^{\vec{B}} \xi^C \prod_{\substack{n,m \in \mathbb{Z},\ \ell \in L' \\ (n,\ell,m) > 0}} \left(1 - q^n \zeta^{-\ell} \xi^m\right)^{f(nm,\ell)},$$

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where $q = \exp(2\pi i \tau)$, $\zeta^\ell = \exp(2\pi i(\ell, \mathfrak{z}))$, $\xi = \exp(2\pi i \omega)$, where the positivity condition $(n, \ell, m) > 0$ means that either $m > 0$, or $m = 0$ and $n > 0$, or $m = n = 0$ and $\ell > 0$, and where the Weyl vector $\rho = (-A, \vec{B}, -C)$ of $\mathbf{B}(\phi)$ is defined by

$$A = \frac{1}{24} \sum_{\ell \in L'} f(0, \ell), \quad \vec{B} = \frac{1}{2} \sum_{\ell > 0} f(0, \ell)\ell, \quad C = \frac{1}{2 \operatorname{rk}(L)} \sum_{\ell \in L'} f(0, \ell)(\ell, \ell).$$

The Fourier–Jacobi expansion of $\mathbf{B}(\phi)$ on $\mathbb{H}(L)$ is

$$(2.11) \qquad \mathbf{B}(\phi)(Z) = \left(\Theta_{f(0,*)}(\tau, \mathfrak{z}) \cdot \xi^C \right) \exp\left(-\sum_{m=1}^{\infty} \left(\phi|_0 T_-^{(1)}(m) \right)(\tau, \mathfrak{z}) \cdot \xi^m \right),$$

where the leading Fourier–Jacobi coefficient is given by a generalized theta block

$$\Theta_{f(0,*)}(\tau, \mathfrak{z}) = \eta(\tau)^{f(0,0)} \prod_{\ell > 0} \left(\frac{\vartheta(\tau, (\ell, \mathfrak{z}))}{\eta(\tau)} \right)^{f(0,\ell)}.$$

We also need the following additive lifts with characters to construct orthogonal modular forms.

Theorem 2.9 (Theorem 3.2 in [18]). *Let D be an even divisor of 24, $k \in \mathbb{N}$, $t \in \frac{1}{2}\mathbb{N}$. If $Q = 24/D$ is odd, we further assume that $t \in \mathbb{N}$. Let $\varphi \in J_{k,L,t}(v_\eta^D)$. Let $G_k(\tau)$ be the normalized Eisenstein series of weight k on $\mathrm{SL}_2(\mathbb{Z})$ whose Fourier coefficient at q is 1. Then the function*

$$\mathbf{G}(\varphi)(Z) = f(0,0) G_k(\tau) + \sum_{0 < m \in 1 + Q\mathbb{Z}} \left(\varphi|_{k,t} T_-^{(Q)}(m) \right)(\tau, \mathfrak{z}) \cdot \xi^{m/Q}$$

is a (holomorphic) modular form of weight k and some character on $\widetilde{\mathrm{O}}^+(U_1 \oplus U \oplus L(Qt))$. The form $\mathbf{G}(\varphi)$ is always invariant under the involution $(\omega, \mathfrak{z}, \tau) \mapsto (\tau, \mathfrak{z}, \omega)$.

For a given lattice M, we will often need to determine all holomorphic, reflective Borcherds products and in some cases prove that no such products exist. To do this, we apply Borcherds's obstruction criterion [10] which states that a formal sum

$$\sum_{n < 0} \sum_{x \in D_M} c_x(n) q^n e_x$$

occurs as the principal part of a weakly-holomorphic modular form of weight κ for ρ_M if and only if the identity

$$\sum_{n < 0} \sum_{x \in D_M} c_x(n) a_x(-n) = 0$$

holds for every cusp form $\sum_{n > 0} \sum_{x \in D_M} a_x(n) q^n e_x$ of weight $2 - \kappa$ for the dual representation $\rho_{M(-1)}$. We construct a basis of cusp forms for $\rho_{M(-1)}$ following [116] and realize the problem of computing holomorphic reflective products as the problem of enumerating integral lattice points in a polyhedral cone defined by finitely many inequalities (non-negative order along reflective divisors) and finitely many linear equations. The solution can be conveniently expressed by the notion of a Hilbert basis, i.e. a minimal system of reflective products $F_1, ..., F_r$ such that every holomorphic reflective product can be written in the form

$$F = F_1^{n_1} \cdot ... \cdot F_r^{n_r}$$

with non-negative integers $n_1, ..., n_r$. Given the polyhedral cone, we used the software Normaliz [16] to find a Hilbert basis.

2.5. **Borcherds–Kac–Moody superalgebras.** Borcherds–Kac–Moody superalgebras are infinite dimensional Lie superalgebras introduced by Borcherds [3] in 1988 which generalize affine Kac–Moody algebras. They can be defined in terms of Chevalley–Serre generators and relations which are encoded in a generalized Cartan matrix. The restrictions on the generalized Cartan matrix are weaker; in particular, simple roots are allowed to have non-positive norm (i.e. they can be imaginary roots). BKM superalgebras also have a character formula for highest-weight modules and a denominator identity. We will review the denominator identity following [90] because it contains valuable information about the roots, root multiplicities and the Weyl group.

Let $\mathcal{G} = \mathcal{G}_0 \oplus \mathcal{G}_1$ be a BKM superalgebra with even and odd components $\mathcal{G}_0$ and $\mathcal{G}_1$, respectively. If $\mathcal{G}_1$ is trivial, $\mathcal{G}$ is a BKM algebra. Let $\mathcal{H}$ be the generalized Cartan subalgebra of $\mathcal{G}$. Let Δ^+ be the set of positive roots of $\mathcal{G}$. We have the root space decomposition

$$\mathcal{G} = \bigoplus_{\alpha \in \Delta^+} \mathcal{G}_\alpha \oplus \mathcal{H} \oplus \bigoplus_{\alpha \in \Delta^+} \mathcal{G}_{-\alpha}.$$

For $\alpha \in \Delta^+$, we define

$$\mathrm{mult}_0(\alpha) = \dim(\mathcal{G}_\alpha \cap \mathcal{G}_0), \quad \mathrm{mult}_1(\alpha) = \dim(\mathcal{G}_\alpha \cap \mathcal{G}_1), \quad \mathrm{mult}(\alpha) = \dim(\mathcal{G}_\alpha),$$

and define the super-multiplicity s-$\mathrm{mult}(\alpha)$ as

$$\mathrm{mult}_0(\alpha) - \mathrm{mult}_1(\alpha).$$

The sets of positive even roots and odd roots are respectively

$$\Delta_0^+ = \{\alpha \in \Delta^+ : \mathrm{mult}_0(\alpha) > 0\} \quad \text{and} \quad \Delta_1^+ = \{\alpha \in \Delta^+ : \mathrm{mult}_1(\alpha) > 0\}.$$

Let I be a countable set indexing the simple roots α_i and S be the subset of I indexing odd roots. By expanding a root as $\alpha = \sum_{i \in I} k_i \alpha_i$, we define the height and even height of α respectively as

$$\mathrm{ht}(\alpha) = \sum_{i \in I} k_i \quad \text{and} \quad \mathrm{ht}_0(\alpha) = \sum_{i \in I \setminus S} k_i.$$

We further define two formal sums

$$T = e^\rho \sum_\mu (-1)^{\mathrm{ht}(\mu)} e^{-\mu}, \quad T' = e^\rho \sum_\mu (-1)^{\mathrm{ht}_0(\mu)} e^{-\mu},$$

where ρ is the Weyl vector of $\mathcal{G}$ and the sums are taken over all sets μ of distinct pairwise orthogonal imaginary simple roots. Let W be the Weyl group generated by reflections associated with real simple roots. In this paper we assume that $\mathcal{G}$ has no odd real roots. Under this assumption, one has the denominator identity

$$(2.12) \qquad e^\rho \cdot \frac{\prod_{\alpha \in \Delta_0^+} (1 - e^{-\alpha})^{\mathrm{mult}_0(\alpha)}}{\prod_{\alpha \in \Delta_1^+} (1 + e^{-\alpha})^{\mathrm{mult}_1(\alpha)}} = \sum_{\sigma \in W} \det(\sigma)\sigma(T)$$

and the super-denominator identity

$$(2.13) \qquad e^\rho \cdot \frac{\prod_{\alpha \in \Delta_0^+} (1 - e^{-\alpha})^{\mathrm{mult}_0(\alpha)}}{\prod_{\alpha \in \Delta_1^+} (1 - e^{-\alpha})^{\mathrm{mult}_1(\alpha)}} = \sum_{\sigma \in W} \det(\sigma)\sigma(T').$$

An affine Kac–Moody algebra $\mathcal{G}$ is a BKM algebra with no imaginary simple roots. Therefore, $T = e^\rho$, and we recover the denominator identity for affine Kac–Moody algebras.

Following Borcherds, we study BKM superalgebras that have a hyperbolic root lattice (i.e. its signature is of type $(l, 1)$) and whose denominator or super-denominator is the Fourier expansion of a holomorphic modular form F on $\mathrm{O}(l+1, 2)$ at some 0-dimensional cusp. The product side of the denominator identity suggests that F can usually be constructed as a Borcherds product [14, 15], in which case the roots and their multiplicities are encoded in a weakly holomorphic modular form

for the Weil representation of $\mathrm{SL}_2(\mathbb{Z})$. The real roots of $\mathcal{G}$ correspond to the zeros of F through the cusp, so F is locally reflective. Usually, F is also globally reflective.

This is particularly interesting when F is of singular weight. In this case, imaginary simple roots are easy to describe, and in many cases they are just the multiples of the Weyl vector. Moreover, one can write the sum side of the denominator identity explicitly as in [9, 93, 94], and it is expected that such a BKM superalgebra has a natural construction as the BRST cohomology of a suitable vertex algebra. In this paper, we will classify this type of BKM superalgebra under some mild conditions and work out the corresponding vertex algebras. Our classification is closely related to the fake monster algebra and its twists, so we will review that theory as well.

In 1990 Borcherds [5] constructed the fake monster algebra $\mathbb{G}$ as the BRST cohomology related to the Leech lattice VOA. The root lattice of the BKM algebra $\mathbb{G}$ is $U \oplus \Lambda$, where Λ is the Leech lattice. The Weyl vector is $\rho = (-1, 0, 0)$ (see Notation 2.7) and the real roots are $\alpha \in U \oplus \Lambda$ with $\alpha^2 = 2$. The real simple roots are characterized by the equality $(\rho, \alpha) = 1$. The imaginary simple roots are $n\rho$ for negative integers n, each with multiplicity 24. The denominator of $\mathbb{G}$ defines a reflective Borcherds product Φ_{12} of singular weight on $2U \oplus \Lambda$ whose input is the full character of the Leech lattice VOA (see [7]). Let $\mathrm{Co}_0 = \mathrm{O}(\Lambda)$ be the Conway group. Borcherds [6] proved that the denominator identity of $\mathbb{G}$ is a cohomological identity and that Co_0 acts on $\mathbb{G}$ naturally. In this way, he obtained a twisted denominator identity for each conjugacy class $[g]$ of Co_0 and proved that it is the (untwisted) super-denominator identity of a BKM superalgebra, denoted $\mathbb{G}_g$. The root lattice of $\mathbb{G}_g$ is $L_g = U \oplus \Lambda^g$, where

$$\Lambda^g = \{v \in \Lambda : g(v) = v\}$$

is the fixed-point sublattice. The real simple roots are the roots α of L_g satisfying

$$(\rho, \alpha) = \alpha^2/2$$

and the imaginary simple roots are $m\rho$ with super-multiplicity $\sum_{k|(m,n_g)} b_k$ for all negative integers m, where n_g is the order of g and where b_k describe the cycle shape $\prod_{b_k \neq 0} k^{b_k}$ of g. The last two named authors [113] proved that the twisted denominator function associated with g defines a holomorphic Borcherds product (denoted Φ_g) of singular weight on $U(N_g) \oplus U \oplus \Lambda^g$, where N_g is the level of g, and that Φ_g is reflective on $U(N_g) \oplus U \oplus \Lambda^g$ if $N_g = n_g$, confirming a conjecture of Borcherds [7, §15, Example 3].

3. The hyperbolization of affine Kac–Moody algebras

It is well-known how to realize affine Kac–Moody algebras as extensions of semi-simple Lie algebras. In this section, we introduce extensions of affine Kac–Moody algebras to BKM superalgebras that we call their hyperbolizations.

Let L be an even positive-definite lattice of rank $\mathrm{rk}(L)$, let U and U_1 be two hyperbolic planes, and define $M = U_1 \oplus U \oplus L$. Let $\mathcal{G}$ be a BKM superalgebra with root lattice $U \oplus L'$ and generalized Cartan subalgebra $\mathcal{H}$ with no odd real roots. We assume that the super-denominator of $\mathcal{G}$ is given by the Fourier expansion of a Borcherds product F for $\widetilde{\mathrm{O}}^+(M)$ at the 0-dimensional cusp determined by U_1.

Every divisor $\lambda^\perp$ of F lies in the $\widetilde{\mathrm{O}}^+(M)$-orbit of the rational quadratic divisor associated with some primitive vector $\alpha \in U \oplus L'$. If $\alpha^\perp$ appears in the divisor of F, then α is a real root. Since we have assumed that $\mathcal{G}$ has no odd real roots, the divisor $\alpha^\perp$ is always a zero and therefore F is holomorphic. Moreover, the associated reflection σ_α lies in the Weyl group of $\mathcal{G}$. Therefore, σ_α fixes the root lattice $U \oplus L'$, so it also fixes $M' = U_1 \oplus U \oplus L'$, which implies that $\sigma_\alpha \in \mathrm{O}^+(M') = \mathrm{O}^+(M)$ and finally that F is a reflective modular form.

We denote by ϕ the Jacobi form input of F and write its expansion at ∞ as

$$\phi(\tau, \mathfrak{z}) = \sum_{n \in \mathbb{Z}, \, \ell \in L'} f(n, \ell) q^n \zeta^\ell \in J^!_{0,L}.$$

By Theorem 2.8, the product side of the denominator identity is (see Notation 2.7)

$$e^\rho \prod_{\alpha > 0} \left(1 - e^{-\alpha}\right)^{f(nm,\ell)}, \quad \alpha = (n, \ell, m) \in U \oplus L', \quad \rho = (-A, \vec{B}, -C).$$

In particular, if $f(nm, \ell) \neq 0$ then α is a root of $\mathcal{G}$ and

$$\mathrm{mult}_0(\alpha) - \mathrm{mult}_1(\alpha) = f(nm, \ell).$$

The weight of F is given by $f(0,0)/2$. The root α is positive if and only if either $m > 0$, or $m = 0$ and $n > 0$, or $m = n = 0$ and $\ell > 0$.

Let $\mathcal{R}_+$ be the set of positive roots of $\mathcal{G}$ of type $(n, \ell, 0)$. For any positive integer n, $(n, 0, 0)$ is an imaginary root in $\mathcal{R}_+$. All other roots in $\mathcal{R}_+$ are (even) real and have multiplicity $f(0, \ell)$. Since the multiplicity of a real root is always 1 (see e.g. [90, Corollary 2.3.42]), we have $f(0, \ell) = 1$.

Assume further that none of $(n, 0, 0)$ is odd. Then the even roots $(n, 0, 0)$ always have multiplicity $f(0, 0)$. Now consider the sum of root spaces

$$\mathcal{G}^0 := \bigoplus_{\alpha \in \mathcal{R}_+} \mathcal{G}_\alpha \oplus \mathcal{H} \oplus \bigoplus_{\alpha \in \mathcal{R}_+} \mathcal{G}_{-\alpha}.$$

If F is of singular weight, i.e. $f(0,0) = \mathrm{rk}(L)$, then $\mathcal{G}^0$ defines an affine Kac–Moody algebra $\hat{\mathfrak{g}}$ for which $\mathfrak{g}$ is a semi-simple Lie algebra whose positive roots are the roots of type $(0, \ell, 0)$ in $\mathcal{R}^+$ (here, we can view $K = \delta = (1, 0, 0)$ and $d = \hat{w}_0 = (0, 0, -1)$). The leading Fourier–Jacobi coefficient of F at the 1-dimensional cusp determined by $U_1 \oplus U$ is a holomorphic Jacobi form of singular weight given by the theta block

$$\eta(\tau)^{\mathrm{rk}(L)} \prod_{(0, \ell, 0) \in \mathcal{R}^+} \frac{\vartheta(\tau, (\ell, \mathfrak{z}))}{\eta(\tau)}, \quad \mathfrak{z} \in L \otimes \mathbb{C},$$

which equals the denominator $\vartheta_{\mathfrak{g}}$ of $\hat{\mathfrak{g}}$. In particular, $\mathcal{G}$ is an extension of $\hat{\mathfrak{g}}$. For $m > 0$ we define $\mathcal{G}^m$ to be the sum of root spaces associated with roots of type $\pm(*, *, m)$. Then

$$\mathcal{G} = \bigoplus_{m=0}^\infty \mathcal{G}^m$$

is a graded module over $\hat{\mathfrak{g}} = \mathcal{G}^0$. From this point of view, it is natural to regard $\mathcal{G}$ as a *hyperbolization* of $\hat{\mathfrak{g}}$.

Motivated by the discussion above, we introduce the following definition.

Definition 3.1. Let $\hat{\mathfrak{g}}$ be an affine Kac–Moody algebra. A BKM superalgebra $\mathcal{G}$ without odd real roots is called a *hyperbolization* of $\hat{\mathfrak{g}}$ if there exists an even positive-definite lattice L such that the root lattice of $\mathcal{G}$ is $U \oplus L'$ and the super-denominator of $\mathcal{G}$ defines a holomorphic Borcherds product F of (singular) weight $\mathrm{rk}(L)/2$ on $2U \oplus L$ whose leading Fourier–Jacobi coefficient is the denominator $\vartheta_{\mathfrak{g}}$ of $\hat{\mathfrak{g}}$. We also call F a *hyperbolization* of $\vartheta_{\mathfrak{g}}$.

The notion of hyperbolization was first introduced by Gritsenko and the second named author in [50, Theorem 1.2], [52, Section 6.3] and [53, Remark 3.11].

In this paper we will classify affine Kac–Moody algebras that admit hyperbolizations and construct a hyperbolization for every such affine Kac–Moody algebra. Moreover, we express every Jacobi form input ϕ as a $\mathbb{Z}$-linear combination of full characters of an associated vertex operator (super)algebra.

4. Root systems associated with singular-weight reflective modular forms

In this section, we classify reflective automorphic products of singular weight on lattices of type $2U \oplus L$ whose Jacobi form inputs have non-negative q^0-terms. Although Jacobi form inputs that have negative coefficients in their q^0-terms can lift to holomorphic Borcherds products, the corresponding BKM superalgebras in these cases will have odd real roots.

By an argument similar to [108, Theorem 6.2] and [105, Lemma 4.5], we will show that any such Borcherds product belongs to a root system satisfying certain strong restrictions. These restrictions yield a finite list of root systems and a finite list of candidates for affine Kac–Moody algebras that admit hyperbolizations.

Theorem 4.1. *Let L be an even positive-definite lattice of rank $\mathrm{rk}(L)$ and let U be a hyperbolic plane. Suppose there is a reflective Borcherds product F of singular weight on $\widetilde{\mathrm{O}}^+(2U \oplus L)$ whose Jacobi form input has non-negative q^0-term. Then either (a) or (b) below holds:*

  (a) *The lattice L is the Leech lattice, F is the denominator Φ_{12} of the fake monster algebra, and the Jacobi form input is the full character of the Leech lattice VOA.*
  (b) *There exists a semi-simple Lie algebra $\mathfrak{g} = \oplus_{j=1}^s \mathfrak{g}_{j,k_j}$ of rank $\mathrm{rk}(L)$ satisfying restrictions (1) or (1′) and (2) and (3), where $\mathfrak{g}_j$ are simple Lie algebras and k_j are positive integers for $1 \le j \le s$. We refer to Notation 2.3 for the symbols below. Let $\lambda \in U$ with $\lambda^2 = 2$.*
    (1) If F vanishes on $\lambda^\perp$, then

$$(4.1) \qquad C := \frac{\dim \mathfrak{g}}{24} - 1 = \frac{h_j^\vee}{k_j}, \quad for \ 1 \le j \le s.$$

  (1′) If F does not vanish on $\lambda^\perp$, then

$$(4.2) \qquad C := \frac{\dim \mathfrak{g}}{24} = \frac{h_j^\vee}{k_j} \quad and \quad k_j > 1, \quad for \ 1 \le j \le s.$$

  (2) The rescaled lattice $L(C)$ is integral and L is bounded by

$$(4.3) \qquad \mathbf{Q}_\mathfrak{g} < L < \mathbf{P}_\mathfrak{g}.$$

  (3) The leading Fourier–Jacobi coefficient of F at the 1-dimensional cusp determined by $2U$ coincides with the denominator of the affine Kac–Moody algebra $\hat{\mathfrak{g}}$.

Let $c_\mathfrak{g}$ denote the central charge of the affine VOA generated by $\hat{\mathfrak{g}} = \oplus_{j=1}^s \hat{\mathfrak{g}}_{j,k_j}$. Then $c_\mathfrak{g}$ is always 24 in case (1) and in case (1′) we have

$$(4.4) \qquad c_\mathfrak{g} = \frac{24C}{C+1}.$$

We call case (1) (resp. (1′)) the *anti-symmetric* (resp. *symmetric*) case, since F is anti-invariant (resp. invariant) under the involution $(\omega, \mathfrak{z}, \tau) \mapsto (\tau, \mathfrak{z}, \omega)$.

Proof. This is a more precise version of [110, Theorem 2.2] in the specific case of singular weight. We provide a detailed proof here as it combines the proof of [110, Theorem 2.2] with some recent results proved in [115], and because it explains the construction of the Lie algebra $\mathfrak{g}$.

Let $\phi \in J_{0,L}^!$ be the Jacobi form input of F, and write its Fourier expansion as

$$\phi(\tau, \mathfrak{z}) = \sum_{n \in \mathbb{Z}, \, \ell \in L'} f(n, \ell) q^n \zeta^\ell.$$

By assumption, $f(0, \ell)$ are non-negative integers. Each vector $v = (n, \ell, 1) \in U \oplus L'$ is primitive in $M := 2U \oplus L$ and satisfies $v^2 = (\ell, \ell) - 2n$. The multiplicity of $v^\perp$ in the divisor of F is

$$\sum_{d=1}^{\infty} f(d^2 n, d\ell).$$

By [110, Lemma 2.1 and Equation (2.1)], the Fourier expansion of F begins with

$$\phi(\tau, \mathfrak{z}) = f(-1, 0)q^{-1} + \sum_{\ell \in L'} f(0, \ell)\zeta^\ell + O(q).$$

Since F is of singular weight, $f(0,0) = \mathrm{rk}(L)$. We know from [115, Theorem 1.2] that a holomorphic Borcherds product of singular weight has only simple zeros. Therefore, $f(-1,0)$ and $f(0,\ell)$ can only be 0 or 1. By definition, $f(-1,0) = 1$ in the anti-symmetric case and $f(-1,0) = 0$ in the symmetric case. We define

$$\mathcal{R} = \{\ell \in L'\backslash\{0\} : f(0, \ell) = 1\}.$$

By [41, Proposition 2.6], we have

(4.5)
$$\frac{|\mathcal{R}| + \mathrm{rk}(L)}{24} - f(-1, 0) = \frac{1}{2\,\mathrm{rk}(L)} \sum_{\ell \in \mathcal{R}} (\ell, \ell) =: C,$$

(4.6)
$$\sum_{\ell \in \mathcal{R}} (\ell, \mathfrak{z})^2 = 2C(\mathfrak{z}, \mathfrak{z}).$$

If $\mathcal{R}$ is empty, then $C = 0$, $f(-1, 0) = 1$ and $\mathrm{rk}(L) = 24$. In this case, [81, Proposition 3.2] shows that M is isomorphic to the even unimodular lattice of signature $(26, 2)$, which then forces L to be the Leech lattice Λ and F to be the Borcherds form Φ_{12} (see [113, Theorem 3.5]). The Jacobi form input of Φ_{12} is given by

$$\Theta_{\Lambda, 0}(\tau, \mathfrak{z})/\eta^{24}(\tau) = q^{-1} + 24 + O(q),$$

which equals the full character of the Leech lattice VOA.

Now assume that $\mathcal{R}$ is non-empty. Equation (4.6) implies that $C > 0$ and that $L \otimes \mathbb{Q}$ is spanned by $\mathcal{R}$ over $\mathbb{Q}$. We will now show that $\mathcal{R}$ is a rescaled root system. Let $\ell_1, \ell_2 \in \mathcal{R}$. Then $-\ell_1 \in \mathcal{R}$ because $f(0, \ell) = f(0, -\ell)$. For any integer $a > 1$, $a\ell_1 \notin \mathcal{R}$, as otherwise $(0, \ell_1, 1)^\perp$ would have multiplicity at least 2 in the divisor of F, contradicting the fact that all zeros of F are simple. Since F is reflective, the reflection $\sigma_{(0,0,\ell_1,1,0)}$ fixes M (see Notation 2.7). From

$$\sigma_{(0,0,\ell_1,1,0)}\big((0, 0, \ell_2, 1, 0)\big) = \big(0, 0, \sigma_{\ell_1}(\ell_2), 1 - 2(\ell_1, \ell_2)/(\ell_1, \ell_1), 0\big) =: \lambda \in M$$

we obtain

$$\sigma_{\ell_1}(\ell_2) \in L' \quad \text{and} \quad 2(\ell_1, \ell_2)/(\ell_1, \ell_1) \in \mathbb{Z}.$$

We know from [110, Theorem 1.2] that F is anti-invariant under $\sigma_{(0,0,\ell_1,1,0)}$ and therefore vanishes on $\lambda^\perp$. Since $(0, 0, \sigma_{\ell_1}(\ell_2), 1, 0)$ is primitive in M', by the Eichler criterion (see e.g. [40, Proposition 3.3]) there exists $g \in \widetilde{\mathrm{O}}^+(M)$ such that $g(\lambda) = (0, 0, \sigma_{\ell_1}(\ell_2), 1, 0)$. Therefore, F also vanishes on $(0, 0, \sigma_{\ell_1}(\ell_2), 1, 0)^\perp$, from which it follows that $f(0, \sigma_{\ell_1}(\ell_2)) = 1$ and that $\sigma_{\ell_1}(\ell_2) \in \mathcal{R}$. This proves that $\mathcal{R}$ is a (rescaled) root system.

Now we can decompose $\mathcal{R}$ into rescaled irreducible root systems

$$\mathcal{R} = \bigoplus_{j=1}^{s} \mathcal{R}_j(a_j),$$

where $\mathcal{R}_j$ are standard irreducible root systems and a_j are rational numbers by which the (squared) norms of the roots in $\mathcal{R}_j$ are rescaled. Since $(0, 0, \ell, 1, 0)^\perp$ is reflective for any $\ell \in \mathcal{R}$, there exists a positive integer d such that $\ell^2 = 2/d$ and $d\ell \in L$. It follows that $a_j = 1/k_j$ for some positive

integers k_j. In the symmetric case, there are no 2-roots in $\mathcal{R}$, so every k_j is larger than 1. Let $\mathfrak{g}_{j,k_j}$ be the simple Lie algebra of type $\mathcal{R}_j$ with level k_j. We see from (4.5) and (4.6) that the semi-simple Lie algebra $\mathfrak{g} = \oplus_{j=1}^s \mathfrak{g}_{j,k_j}$ satisfies condition (1) or (1') of the theorem.

By [108, Corollary 4.5], $L(C)$ is an integral lattice. The set $\mathcal{R} = \oplus_{j=1}^s \mathcal{R}_j(1/k_j)$ generates a sublattice of L' over $\mathbb{Z}$, from which it follows that

$$\bigoplus_{j=1}^s Q_j(1/k_j) < L' \quad \text{and therefore} \quad L < \bigoplus_{j=1}^s P_j^\vee(k_j).$$

The vectors $\mathrm{ord}(\ell)\ell$ for $\ell \in \mathcal{R}$ generate a sublattice of L over $\mathbb{Z}$, where $\mathrm{ord}(\ell)$ is the order of ℓ in L'/L. It is easy to check that this lattice contains the lattice generated by long roots of $\oplus_{j=1}^s \mathcal{R}_j(k_j)$, and therefore $\oplus_{j=1}^s Q_j^\vee(k_j) < L$. This proves claim (2).

Finally, let Δ_j^+ be the set of positive roots of $\mathcal{R}_j$ and let $\langle -, - \rangle$ be the normalized bilinear form on $\mathcal{R}_j$ that was fixed in Section 2.2. For $\alpha \in \Delta_j^+$, we have $\alpha/k_j \in \mathcal{R} \subsetneq L'$ and

$$(\alpha/k_j, \mathfrak{z}) = k_j \langle \alpha/k_j, \mathfrak{z} \rangle = \langle \alpha, \mathfrak{z} \rangle.$$

Therefore, the leading Fourier–Jacobi coefficient of F is a Jacobi form of lattice index $L(C)$ given by

$$\eta(\tau)^{\mathrm{rk}(L)} \prod_{0 < \ell \in \mathcal{R}} \frac{\vartheta(\tau, (\ell, \mathfrak{z}))}{\eta(\tau)} = \prod_{j=1}^s \left(\eta(\tau)^{\mathrm{rk}(\mathfrak{g}_j)} \prod_{\alpha \in \Delta_j^+} \frac{\vartheta(\tau, \langle \alpha, \mathfrak{z} \rangle)}{\eta(\tau)} \right),$$

which is actually the denominator of $\hat{\mathfrak{g}}$. Note that

$$\bigoplus_{j=1}^s Q_j^\vee(h_j^\vee) < L(C) < \bigoplus_{j=1}^s P_j^\vee(h_j^\vee),$$

which matches the index $L(C)$ of the above leading Fourier–Jacobi coefficient of F.

The last assertion follows from the central charge formula

$$c_{\mathfrak{g}} = \sum_{j=1}^s \frac{k_j \dim \mathfrak{g}_j}{k_j + h_j^\vee} = \sum_{j=1}^s \frac{\dim \mathfrak{g}_j}{1 + C} = \frac{\dim \mathfrak{g}}{1 + C}$$

and because $\dim \mathfrak{g} = 24(C + 1)$ in the anti-symmetric case and $\dim \mathfrak{g} = 24C$ in the symmetric case. $\qquad\square$

Remark 4.2. In the above theorem, when $\mathcal{R}_j$ is of type E_8, F_4 or G_2, we have $Q_j^\vee = P_j^\vee$. Let J denote the subset of such j. Then we have (see [110, Theorem 2.2 (4)])

$$L = K \oplus \bigoplus_{j \in J} P_j^\vee(k_j), \quad \text{where} \quad \bigoplus_{j \neq J} Q_j^\vee(k_j) < K < \bigoplus_{j \neq J} P_j^\vee(k_j).$$

Proposition 4.3. *Equation* (4.1) *has* 221 *solutions, and they are listed in Table 4. Equation* (4.2) *has* 17 *solutions, and they are listed in Table 5.*

Proof. Equation (4.1) was first derived by Schellekens [98] in 1993 in the context of holomorphic vertex operator algebras of central charge 24, and he determined the solutions that are listed in Table 4. The solutions of Equation (4.2) can be found in a similar way. $\qquad\square$

5. The classification of affine Lie algebras with hyperbolizations

In this section we state the main theorems and explain the ideas behind the proofs.

Our first main result is the classification of root systems associated with reflective Borcherds products of singular weight on lattices of type $2U \oplus L$, which was introduced in Theorem 4.1.

Theorem 5.1. *Suppose $2U \oplus L$ has a reflective Borcherds product of singular weight whose Jacobi form input has non-negative q^0-term. Then the associated semi-simple Lie algebra $\mathfrak{g}$ defined in Theorem 4.1 is one of the 81 Lie algebras colored blue in Tables 4 and 5.*

To prove this, we have to rule out most of the solutions of Equation (4.1) and Equation (4.2). There are 238 root systems in Tables 4 and 5 altogether. A root system will be called *extraneous* if it is not one of the 81 root systems described in Theorem 5.1; there are 152 extraneous root systems of anti-symmetric type and 5 extraneous root systems of symmetric type.

Let $\mathfrak{g}$ be a semi-simple Lie algebra associated with a singular-weight reflective Borcherds product on $2U \oplus L$ that is extraneous. By Theorem 4.1 (2), the underlying lattice L satisfies the bounds

$$\mathbf{Q}_{\mathfrak{g}} < L < \mathbf{P}_{\mathfrak{g}}.$$

We will determine an explicit even overlattice K of L with the property that any singular-weight reflective Borcherds product on $2U \oplus L$ lifts to a reflective Borcherds product satisfying certain conditions on $2U \oplus K$, but for which no such reflective products on $2U \oplus K$ exist. This leads to a contradiction. The complete proof appears in Section 11 for anti-symmetric root systems and in Section 12 for symmetric root systems.

The second main result is the construction of hyperbolizations for the 81 semi-simple Lie algebras in the theorem above.

Theorem 5.2. *For each semi-simple Lie algebra $\mathfrak{g}$ in Theorem 4.1, there exists an even positive definite lattice $L_{\mathfrak{g}}$ with the same rank as $\mathfrak{g}$ that satisfies the following conditions:*

(1) *There is a singular-weight reflective Borcherds product $\Psi_{\mathfrak{g}}$ on $2U \oplus L_{\mathfrak{g}}$;*

(2) *The leading Fourier–Jacobi coefficient of $\Psi_{\mathfrak{g}}$ at the 1-dimensional cusp determined by $2U$ is the denominator of the affine Lie algbera $\hat{\mathfrak{g}}$;*

(3) *The Jacobi form input $\phi_{\mathfrak{g}}$ of $\Psi_{\mathfrak{g}}$ is a $\mathbb{Z}$-linear combination of full characters of the affine vertex operator algebra generated by $\hat{\mathfrak{g}}$;*

(4) *The lattice generated by $d\rho$ and by those $\lambda = (n, \ell, m) \in U \oplus L'_{\mathfrak{g}}$ for which $\Psi_{\mathfrak{g}}$ vanishes on $\lambda^{\perp}$ is $U \oplus L'_{\mathfrak{g}}$; that is, $U \oplus L'_{\mathfrak{g}}$ is the root lattice of the BKM (super)algebra with (super)-denominator $\Psi_{\mathfrak{g}}$. Here, ρ is the Weyl vector of $\Psi_{\mathfrak{g}}$ and d is the denominator of C as defined in Theorem 4.1.*

(5) *When $\mathfrak{g}$ is of symmetric type, the Borcherds product $\Psi_{\mathfrak{g}}$ coincides with the Gritsenko (additive) lift of the denominator of $\hat{\mathfrak{g}}$.*

The above lattices $L_{\mathfrak{g}}$ and Jacobi forms $\phi_{\mathfrak{g}}$ are constructed as follows.

(a) *Let $\mathfrak{g}$ be one of the 69 semi-simple Lie algebras of anti-symmetric type. In this case, $\mathfrak{g}$ appears as the semi-simple V_1 structure of a holomorphic vertex operator algebra V of central charge 24 in Schellekens's list. Then $L_{\mathfrak{g}}$ is the orbit lattice in Höhn's construction of V and $\phi_{\mathfrak{g}}$ is the unique full character of V.*

(b) *Let $\mathfrak{g}$ be one of the 8 semi-simple Lie algebras of symmetric type with $C = 1$. In this case, $\mathfrak{g}$ appears as the $\mathcal{N} = 1$ structure of a holomorphic vertex operator superalgebra of central charge 12 composed of 24 chiral fermions. Then $L_{\mathfrak{g}}$ is the maximal even sublattice of $\mathbf{P}_{\mathfrak{g}}$, and $\phi_{\mathfrak{g}}$ can be expressed as a $\mathbb{Z}$-linear combination of the full NS-, $\widetilde{\mathrm{NS}}$- and R-characters of the associated SVOA:*

$$\phi_{\mathfrak{g}} = (\chi_{\mathrm{NS}} - \chi_{\widetilde{\mathrm{NS}}} - \chi_{\mathrm{R}})/2.$$

(c) *Let $\mathfrak{g} = A_{1,16}$, $A_{1,8}^2$, $A_{1,4}^4$ or $A_{2,9}$. These semi-simple Lie algebras are of symmetric type with $C < 1$. In these cases, the affine Lie algebras $\hat{\mathfrak{g}}$ admit exceptional modular invariants coming from a nontrivial automorphism of the fusion algebra. Then $L_{\mathfrak{g}} = \mathbf{P}_{\mathfrak{g}}$, the expressions of $\phi_{\mathfrak{g}}$ in terms of affine characters are given in Theorem 8.2, and the relationship between $\phi_{\mathfrak{g}}$ and the exceptional modular invariants is explained in Remark 8.5.*

The proof of Theorem 5.2 is divided into four sections. In Sections 6-8 we prove parts (a), (b) and (c) respectively. In Section 9 we prove property (5). The connections between our construction and the twists of the fake monster algebra are also explained in Sections 6 and 7.

Our last main result is the complete classification of affine Lie algebras with hyperbolizations. This is a direct consequence of Theorem 5.1 and Theorem 5.2.

Theorem 5.3. *There are exactly* 81 *affine Kac–Moody algebras which have a hyperbolization in the sense of Definition 3.1. The* 81 *algebras are colored blue in Tables 4 and 5.*

Combining Theorem 5.1 and Theorem 5.2, we obtain the following constraint on the existence of singular automorphic products:

Corollary 5.4. *There are reflective Borcherds products of singular weight on lattices of type* $2U \oplus L$ *whose input has non-negative principal part if and only if*

$$\mathrm{rk}(L) \in \{1, 2, 4, 6, 8, 10, 12, 16, 24\}.$$

6. The antisymmetric case: holomorphic CFT of central charge 24

The 69 semi-simple Lie algebras $\mathfrak{g}$ in the anti-symmetric case of Theorem 5.1 coincide with the semi-simple V_1 structures of holomorphic vertex operator algebras (VOA) of central charge 24 in Schellekens' list [98]. In 2017, Höhn [63] found a uniform construction of the holomorphic VOAs. For each $\mathfrak{g}$, we will take Höhn's orbit lattice $L_\mathfrak{g}$ as the underlying lattice L and construct the reflective Borcherds product of singular weight as the Borcherds lift of the full character of the holomorphic VOA. We also explain the connection between these hyperbolizations and the twisted denominators of the fake monster algebra.

6.1. **Holomorphic CFTs of central charge 24 and Schellekens' list.** Let V be a holomorphic vertex operator algebra of central charge 24. The weight-one subspace V_1 has a natural Lie algebra structure and by [98, 26], V_1 is either trivial, abelian of dimension 24, or semi-simple. In the first case it was conjectured in [35] that V is isomorphic to the monster VOA, which was proved under some conditions in [29]. In the second case V is isomorphic to the Leech lattice VOA.

We focus on the third case where V_1 is a semi-simple Lie algebra $\mathfrak{g}$. Let $(-, -)$ be the unique symmetric, non-degenerate, invariant bilinear form on V normalized such that $(\mathbf{1}, \mathbf{1}) = -1$, where $\mathbf{1}$ is the vacuum vector. The restriction of $(-, -)$ to a simple ideal $\mathfrak{g}_j$ of $\mathfrak{g}$ satisfies $(-, -) = k_j \langle -, - \rangle$ for some positive integer k_j, where $\langle -, - \rangle$ is the normalized bilinear form of $\mathfrak{g}_j$ (see [98, 28]). We indicate these integers by writing

$$\mathfrak{g} = \mathfrak{g}_{1,k_1} \oplus \cdots \oplus \mathfrak{g}_{s,k_s}.$$

Then the affine vertex operator algebra

$$V_\mathfrak{g} \cong L_{\hat{\mathfrak{g}}_1}(k_1, 0) \otimes \cdots \otimes L_{\hat{\mathfrak{g}}_s}(k_s, 0)$$

generated by V_1 is a full vertex operator subalgebra of V. As a $V_\mathfrak{g}$-module, one can decompose V into finitely many irreducible $V_\mathfrak{g}$-modules

$$(6.1) \qquad V = \bigoplus_{\lambda_1, \ldots, \lambda_s} m_{\lambda_1, \ldots, \lambda_s} L_{\hat{\mathfrak{g}}_1}(k_1, \lambda_1) \otimes \cdots \otimes L_{\hat{\mathfrak{g}}_s}(k_s, \lambda_s),$$

where the sum runs over the dominant integral weights λ_j of $\mathfrak{g}_j$ that satisfy $\langle \lambda_j, \theta_j^\vee \rangle \leq k_j$, where θ_j is the highest root of $\mathfrak{g}_j$.

In 1993 Schellekens [98] established the equation (see Notation 2.3)

$$\frac{\dim \mathfrak{g}}{24} - 1 = \frac{h_j^\vee}{k_j}, \quad \text{for } 1 \leq j \leq s$$

and showed that it has exactly 221 solutions. By excluding 152 of them, Schellekens proved that V_1 is isomorphic to one of the 69 semi-simple Lie algebras in [98, Table 1] (named *Schellekens's list*) and further determined the induced decompositions (6.1). This result was reproved in [100] using similar arguments and in [102] by means of the "very strange formula".

Schellekens also conjectured that there exists a unique holomorphic VOA of central charge 24 with $V_1 = \mathfrak{g}$ for every $\mathfrak{g}$ on his list. By the work of many authors over the past three decades, this celebrated conjecture was finally proved (see [70, 73, 75, 76, 72, 91, 100] for the existence and [27, 101, 72, 68, 74, 77, 78, 79] for the uniqueness). There are at least three uniform proofs of the conjecture: the Leech lattice method of Höhn ([63, 71, 2]), the generalized deep hole method of Möller–Scheithauer [86, 85], and the Niemeier lattice method of Höhn–Möller [60].

6.2. Höhn's construction of holomorphic CFT of central charge 24.

Our construction of hyperbolizations relies heavily on Höhn's argument, which is reviewed below.

Let $\mathbf{Q}_{\mathfrak{g}}$ be the rescaled coroot lattice (see Notation 2.3). Then V contains the lattice VOA associated with $\mathbf{Q}_{\mathfrak{g}}$ denoted $V_{\mathbf{Q}_{\mathfrak{g}}}$. If we let $W_{\mathfrak{g}} = \mathrm{Com}_V(V_{\mathbf{Q}_{\mathfrak{g}}})$ be the commutant (or centralizer), then the double commutant

$$\mathrm{Com}_V(W_{\mathfrak{g}}) = \mathrm{Com}_V(\mathrm{Com}_V(V_{\mathbf{Q}_{\mathfrak{g}}}))$$

is a lattice VOA extending $V_{\mathbf{Q}_{\mathfrak{g}}}$. Therefore, there exists an even positive definite lattice $L_{\mathfrak{g}} \supset \mathbf{Q}_{\mathfrak{g}}$ such that $\mathrm{Com}_V(W_{\mathfrak{g}})$ is isomorphic to the lattice VOA $V_{L_{\mathfrak{g}}}$. It is well known that $V_{L_{\mathfrak{g}}}$ has group-like fusion: all irreducible $V_{L_{\mathfrak{g}}}$-modules are simple current modules. In this case, the set of all irreducible modules $R(V_{L_{\mathfrak{g}}})$ forms an abelian group with respect to the fusion product, and it also carries the quadratic form $\mathfrak{q} : R(V_{L_{\mathfrak{g}}}) \cong L'_{\mathfrak{g}}/L_{\mathfrak{g}} \to \mathbb{Q}/\mathbb{Z}$ defined via

$$\mathfrak{q}(V_{\alpha+L_{\mathfrak{g}}}) = \mathrm{wt}(V_{\alpha+L_{\mathfrak{g}}}) = (\alpha, \alpha)/2 \mod \mathbb{Z},$$

where $\mathrm{wt}(-)$ denotes the conformal weight of the module. The full character of the irreducible module labeled by $\alpha + L_{\mathfrak{g}} \in L'_{\mathfrak{g}}/L_{\mathfrak{g}}$ can be expressed in terms of the Jacobi theta function and η-function:

$$(6.2) \qquad \chi_{V_{\alpha+L_{\mathfrak{g}}}}(\tau, \mathfrak{z}) = \frac{\Theta_{L_{\mathfrak{g}}, \alpha}(\tau, \mathfrak{z})}{\eta(\tau)^{\mathrm{rk}(L_{\mathfrak{g}})}}, \quad (\tau, \mathfrak{z}) \in \mathbb{H} \times (L_{\mathfrak{g}} \otimes \mathbb{C}).$$

It is also known that $W_{\mathfrak{g}}$ is strongly rational and also has group-like fusion (see e.g. [21, 80]). The set $R(W_{\mathfrak{g}})$ of irreducible modules forms a quadratic space isomorphic to $(R(V_{L_{\mathfrak{g}}}), -\mathfrak{q})$, where the quadratic form is defined by reducing the conformal weight modulo $\mathbb{Z}$. Therefore, V is a simple current extension of $W_{\mathfrak{g}} \otimes V_{L_{\mathfrak{g}}}$, i.e. there is an isometry $\iota : L'_{\mathfrak{g}}/L_{\mathfrak{g}} \to R(W_{\mathfrak{g}})$ such that

$$V \cong \bigoplus_{\alpha+L_{\mathfrak{g}} \in L'_{\mathfrak{g}}/L_{\mathfrak{g}}} W_{\iota(\alpha+L_{\mathfrak{g}})} \otimes V_{\alpha+L_{\mathfrak{g}}}.$$

Note that $V_{L_{\mathfrak{g}}}$ has central charge $\mathrm{rk}(L_{\mathfrak{g}}) = \mathrm{rk}(V_1) = \mathrm{rk}(\mathfrak{g})$, that $W_{\mathfrak{g}}$ has central charge $24 - \mathrm{rk}(\mathfrak{g})$, and that the weight-one subspace of $W_{\mathfrak{g}}$ is zero. Höhn computed the lattice $L_{\mathfrak{g}}$ for each of the 69 semi-simple $\mathfrak{g}$ and called it the *orbit lattice* of $\mathfrak{g}$.

Let $\mathfrak{h}$ be the Cartan subalgebra of $\mathfrak{g}$. Recall that the full character of V, defined by

$$(6.3) \qquad \chi_V(\tau, \mathfrak{z}) = \mathrm{Tr}_V(e^{2\pi i \mathfrak{z}} q^{L_0 - 1}), \quad q = e^{2\pi i \tau}, \quad (\tau, \mathfrak{z}) \in \mathbb{H} \times \mathfrak{h},$$

is a weakly holomorphic Jacobi form of weight 0 and lattice index $\mathbf{Q}_{\mathfrak{g}}$ (with trivial character on $\mathrm{SL}_2(\mathbb{Z})$) (see e.g. [117, 83, 69]). The above construction implies that χ_V also defines a weakly holomorphic Jacobi form of weight 0 and lattice index $L_{\mathfrak{g}}$. Moreover, the character decomposition

$$(6.4) \qquad \chi_V(\tau, \mathfrak{z}) = \sum_{\alpha+L_{\mathfrak{g}} \in L'_{\mathfrak{g}}/L_{\mathfrak{g}}} \chi_{V_{\alpha+L_{\mathfrak{g}}}}(\tau, \mathfrak{z}) \cdot \chi_{W_{\iota(\alpha+L_{\mathfrak{g}})}}(\tau)$$

determines the theta decomposition of χ_V as a Jacobi form of lattice index $L_{\mathfrak{g}}$. In particular, the vector-valued function

$$(6.5) \qquad f_{\mathfrak{g}}(\tau) := \sum_{\alpha \in L_{\mathfrak{g}}'/L_{\mathfrak{g}}} \eta(\tau)^{-\mathrm{rk}(\mathfrak{g})} \cdot \chi_{W_{\iota(\alpha+L_{\mathfrak{g}})}}(\tau) e_{\alpha}$$

is a weakly holomorphic modular form of weight $-\mathrm{rk}(\mathfrak{g})/2$ for the Weil representation $\rho_{L_{\mathfrak{g}}}$.

Höhn described $W_{\mathfrak{g}}$ in an elegant way. Let Λ continue to be the Leech lattice and let

$$\Lambda_g := (\Lambda^g)^{\perp} = \{x \in \Lambda : (x,y) = 0 \quad \text{for all } y \in \Lambda \text{ satisfying } g(y) = y\}$$

be the coinvariant sublattice for $g \in \mathrm{Co}_0 = \mathrm{O}(\Lambda)$. Let $\hat{g} \in \mathrm{Aut}(V_{\Lambda})$ be a standard lift of g; that is, $\hat{g}$ is a lift of g to the Leech lattice VOA that acts trivially on Λ^g. Höhn [63] conjectured that $W_{\mathfrak{g}}$ is isomorphic to an orbifold VOA $V_{\Lambda_g}^{\hat{g}}$ where $[g]$ is one of 11 particular conjugacy classes $[g]$ of Co_0 (see Tables 6 and 7). This would imply that V is isomorphic to a simple-current extension of $V_{\Lambda_g}^{\hat{g}} \otimes V_{L_{\mathfrak{g}}}$. Lam [71] proved that the orbifold $V_{\Lambda_g}^{\hat{g}}$ with $\Lambda^g \neq \{0\}$ has group-like fusion and described the corresponding quadratic form explicitly, and in this way was able to confirm Höhn's conjecture. The simple-current extension of $V_{\Lambda_g}^{\hat{g}} \otimes V_{L_{\mathfrak{g}}}$ depends on the double cosets

$$(6.6) \qquad \mathrm{O}(L_{\mathfrak{g}}) \backslash \mathrm{O}(R(W_{\mathfrak{g}}), -\mathfrak{q}) / \mathrm{Aut}(W_{\mathfrak{g}}).$$

By counting the numbers of the above double cosets denoted $n(L_{\mathfrak{g}})$, Betsumiya, Lam and Shimakura [2] supplemented Höhn's proof of the uniqueness of holomorphic VOAs of central charge 24 with semi-simple V_1.

We will now describe the 69 simple current extensions explicitly. The construction involves 11 distinct Co_0-conjugacy classes $[g]$, which are characterized by the property that the discriminant form $(R(V_{\Lambda_g}^{\hat{g}}), -\mathfrak{q})$ of signature $\mathrm{rk}(\Lambda^g) \mod 8$ can be realized by an even positive definite lattice of rank $\mathrm{rk}(\Lambda^g)$. The orbit lattices $L_{\mathfrak{g}}$ lie in the 11 associated genera.

(1) Let $[g]$ be one of the 8 conjugacy classes in Table 6. For each orbit lattice of genus $[g]$, the number of double cosets (6.6) is always 1, so the number of inequivalent simple-current extensions equals to the number of classes in the genus. There are in total 58 classes (including the Leech lattice) in the 8 genera, so the 8 conjugacy classes $[g]$ induce 57 holomorphic VOAs of central charge 24 with semi-simple $V_1 = \mathfrak{g}$.

(2) Let $[g]$ be the conjugacy class of cycle shape 2^{12}. The associated (lattice) genus has 2 classes $D_{12}(2)$ and $E_8(2) \oplus D_4(2)$ and the corresponding numbers of double cosets are respectively 6 and 3. Therefore, this $[g]$ induces 9 simple-current extensions.

(3) Let $[g]$ be the conjugacy class of cycle shape $2^3 6^3$. The associated (lattice) genus has a unique class and the corresponding number of double cosets is 2. Therefore, this $[g]$ induces 2 simple current extensions.

(4) Let $[g]$ be the conjugacy class of cycle shape $2^2 10^2$. The associated (lattice) genus has a unique class and the corresponding number of double cosets is 1. Therefore, this $[g]$ induces only 1 simple current extension.

The relevant data appear in Tables 6 and 7.

6.3. Constructing hyperbolizations.

Theorem 6.1. *Let V be a holomorphic VOA of central charge 24 with semi-simple $V_1 = \mathfrak{g}$. Let $L_{\mathfrak{g}}$ be Höhn's orbit lattice of $\mathfrak{g}$ and let χ_V be the full character of V. Then the theta lift $\mathbf{B}(\chi_V)$ is a reflective Borcherds product of singular weight on $2U \oplus L_{\mathfrak{g}}$. Moreover, the leading Fourier–Jacobi coefficient of $\mathbf{B}(\chi_V)$ coincides with the denominator of the affine Kac–Moody algebra $\hat{\mathfrak{g}}$.*

Proof. Let $\Delta_{\mathfrak{g}}$ be the set of roots of $\mathfrak{g}$. By definition, the Fourier expansion of χ_V begins

$$\chi_V(\tau, \mathfrak{z}) = q^{-1} + \sum_{\alpha \in \Delta_{\mathfrak{g}}} e^{2\pi i \langle \alpha, \mathfrak{z} \rangle} + \mathrm{rk}(\mathfrak{g}) + O(q),$$

and all Fourier coefficients of χ_V are non-negative integers. It follows from Höhn's construction that χ_V is a weakly holomorphic Jacobi form of weight 0 and lattice index $L_{\mathfrak{g}}$. We conclude immediately that $\mathbf{B}(\chi_V)$ is a holomorphic Borcherds product of singular weight on $2U \oplus L_{\mathfrak{g}}$. It remains to show that $\mathbf{B}(\chi_V)$ is reflective.

Suppose first that $\mathfrak{g}$ comes from one of the conjugacy classes $[g]$ in Table 6. Then the level and order of g are the same (denoted n_g as in Section 2.5). The orbit lattice $L_{\mathfrak{g}}$ is determined by

$$U(n_g) \oplus \Lambda^g \cong U \oplus L_{\mathfrak{g}}.$$

As mentioned at the end of Section 2.5, it was proved in [113, Theorem 6.5, Remark 6.14] that the twisted denominator of the fake monster algebra associated with g defines a reflective Borcherds product Φ_g of singular weight on $U(n_g) \oplus U \oplus \Lambda^g$. Möller [84] calculated the characters of the orbifold VOA $V_{\Lambda_g}^{\hat{g}}$, and by comparing Möller's result and [113, Theorem 6.5], it was also proved [113, Remark 6.13] that the input of Φ_g equals the full character of $V_{\Lambda_g}^{\hat{g}}$ (up to a factor of $\eta^{-\mathrm{rk}(\mathfrak{g})}$) as a vector-valued modular form. By Höhn's construction, the input of Φ_g is actually $f_{\mathfrak{g}}$ as defined in Equation (6.5). We can then use Equation (6.4) to see that $\mathbf{B}(\chi_V) = \Phi_g$; in particular, $\mathbf{B}(\chi_V)$ is reflective.

Otherwise, $\mathfrak{g}$ comes from one of the conjugacy classes $[g]$ in Table 7, the level of g is twice its order, and the isomorphism $U(n_g) \oplus \Lambda^g \cong U \oplus L_{\mathfrak{g}}$ does not hold. In fact, $\mathbf{B}(\chi_V)$ is not obviously related to Φ_g in these cases. We will prove that $\mathbf{B}(\chi_V)$ is reflective by directly calculating the input forms χ_V. By Höhn's construction, it is sufficient to do this for any semi-simple $\mathfrak{g}$ from a fixed $[g]$, because the input for any other semi-simple Lie algebra from $[g]$ can be expressed as $\sigma(f_{\mathfrak{g}})$ for some $\sigma \in \mathrm{O}(L_{\mathfrak{g}}'/L_{\mathfrak{g}})$. The proof by cases is given in Lemmas 6.8, 6.9 and 6.10 below. $\square$

Remark 6.2. All products $\mathbf{B}(\chi_V)$ for $V_1 = \mathfrak{g}$ within a fixed conjugacy class $[g]$ define the same modular form on the type IV symmetric domain of dimension $2 + \mathrm{rk}(\mathfrak{g})$ up to automorphism. We denote this modular form by Ψ_g. The different products $\mathbf{B}(\chi_V)$ can be viewed as the distinct Fourier–Jacobi expansions of Ψ_g at different 1-dimensional cusps represented by a splitting of two hyperbolic planes. Letting D_g denote the discriminant group of $L_{\mathfrak{g}}$, the automorphism group $\mathrm{Aut}(V_{\Lambda_g}^{\hat{g}})$ equals the subgroup of $\mathrm{O}(D_g)$ that fixes $f_{\mathfrak{g}}$. If g is in Table 6, then

$$\mathrm{Aut}(V_{\Lambda_g}^{\hat{g}}) = \mathrm{O}(D_g),$$

and therefore Ψ_g is modular for the full orthogonal group. This property does not hold if g is in Table 7.

Remark 6.3. By the proof of Theorem 6.1, $\Psi_g = \Phi_g$ for $[g]$ in Table 6. The eight Φ_g were first constructed by Borcherds [7, 9] and Scheithauer in [93, 95], but they only considered the Fourier expansions at 0-dimensional cusps. Gritsenko [50] first calculated the 24 distinct Fourier–Jacobi expansions of Φ_g for g of cycle shape 1^{24} at the distinct 1-dimensional cusps, which correspond to the 24 Niemeier lattices. The Fourier–Jacobi expansions of Φ_g for g of cycle shapes $1^3 7^3$ and $1^4 5^4$ were first determined in [52]. We also remark that Scheithauer [96] calculated Ψ_g at 0-dimensional cusps for g of cycle shape 2^{12}.

Remark 6.4. Borcherds [5, 6] proved that the BRST cohomology related to the Leech lattice VOA naturally defines the fake monster algebra. It is natural to expect that the BRST cohomology related to any holomorphic VOA of central charge 24 defines a BKM algebra whose denominator

function is $\mathbf{B}(\chi_V)$. This type of construction has been realized for holomorphic vertex operator algebras with V_1 structures $A_{1,2}^{16}$, $A_{2,3}^8$, $A_{4,5}^2$, $A_{6,7}$ and $B_{12,2}$ by Creutzig, Höhn, Klauer and Scheithauer [61, 19, 62]. A uniform construction for all vertex operator algebras was given by Driscoll-Spittler in his thesis [30] under several technical assumptions. This (conditional) natural construction also implies that $\mathbf{B}(\chi_V)$ is reflective on $2U \oplus L_{\mathfrak{g}}$.

Remark 6.5. Let $\mathcal{G}_{\mathfrak{g}}$ be the BKM algebra that arises as the BRST cohomology of a holomorphic VOA with $V_1 = \mathfrak{g}$. Let $[g]$ be the Co_0-conjugacy class corresponding to $\mathfrak{g}$. Clearly $\mathcal{G}_{\mathfrak{g}}$ depends only on $[g]$.

(1) When $[g]$ is in Table 6, we have the two isomorphisms

$$U(n_g) \oplus \Lambda^g \cong U \oplus L_{\mathfrak{g}} \quad \text{and} \quad U \oplus \Lambda^g \cong (U \oplus L'_{\mathfrak{g}})(n_g),$$

since $(\Lambda^g)'(n_g) \cong \Lambda^g$. This induces the identifications

(6.7)
$$\mathrm{O}\big(U_1(n_g) \oplus U \oplus \Lambda^g\big) \cong \mathrm{O}\big(U_1(n_g) \oplus (U \oplus L'_{\mathfrak{g}})(n_g)\big)$$
$$= \mathrm{O}\big(U_1 \oplus U \oplus L'_{\mathfrak{g}}\big) = \mathrm{O}\big(U_1 \oplus U \oplus L_{\mathfrak{g}}\big).$$

It follows that the Fourier expansion of $\mathbf{B}(\chi_V)$ at U_1 equals the Fourier expansion of Φ_g at $U_1(n_g)$. Recall that the latter is also identical to the denominator of the twist $\mathbb{G}_g$ of the fake monster algebra by g (see Section 2.5). Therefore, the denominator of $\mathcal{G}_{\mathfrak{g}}$ is the same as the denominator of $\mathbb{G}_g$. The root lattice of $\mathbb{G}_g$ is always $U \oplus \Lambda^g$. From (6.7) we conclude that the root lattice of $\mathcal{G}_{\mathfrak{g}}$ is $U \oplus L'_{\mathfrak{g}}$, and moreover that the two BKM algebras $\mathcal{G}_{\mathfrak{g}}$ and $\mathbb{G}_g$ are isomorphic.

(2) When $[g]$ is in Table 7, $\mathcal{G}_{\mathfrak{g}}$ and $\mathbb{G}_g$ are not isomorphic because their root lattices are not even isomorphic up to scaling. In addition, the Weyl vector of $\mathbb{G}_g$ lies in the root lattice, but the Weyl vector of $\mathcal{G}_{\mathfrak{g}}$ does not lie in the root lattice (recall that C is half-integral and the Weyl vector is of type $(-C - 1, *, -C)$ in this case). For instance, when g has cycle shape 2^{12}, the root lattices of $\mathcal{G}_{\mathfrak{g}}$ and $\mathbb{G}_g$ are respectively

$$U \oplus D'_{12}(1/2) \quad \text{and} \quad U \oplus D_{12}^+(2).$$

In particular, we disagree with [62, Remark 3.11]: the denominator there is not a twisted denominator function of the fake monster algebra. We conjecture that the denominator Φ_g of $\mathbb{G}_g$ is given by the Fourier expansion of $\Psi_g = \mathbf{B}(\chi_V)$ at some other 0-dimensional cusp.

Remark 6.6. The $\mathbb{Z}$-lattice generated by $d\rho$ and the $\lambda = (n, \ell, m) \in U \oplus L'_{\mathfrak{g}}$ for which $\mathbf{B}(\chi_V)$ vanishes on $\lambda^\perp$ is exactly the dual lattice $U \oplus L'_{\mathfrak{g}}$. This follows from Remark 6.5 and a direct calculation of zero divisors (see Lemmas 6.8, 6.9 and 6.10 below). Here, ρ is the Weyl vector of $\mathbf{B}(\chi_V)$ and $d = 1$ or 2 if the corresponding $[g]$ lies in Table 6 or Table 7, respectively.

Remark 6.7. Let $[g]$ be a Co_0-conjugacy class of level N_g and order n_g. Let M_g be an even lattice of signature $(\mathrm{rk}(\Lambda^g) + 2, 2)$ whose discriminant form is isomorphic to $(R(V_{\Lambda_g}^{\hat{g}}), -\mathfrak{q})$, as determined by Lam [71]. Let Φ_g be the g-twisted denominator of the fake monster algebra as before, and recall that Φ_g is a holomorphic Borcherds product of singular weight on $U(N_g) \oplus U \oplus \Lambda^g$ (see [113, Theorem 6.5]). We have the following:

(1) If $N_g = n_g$, then $M_g \cong U(N_g) \oplus U \oplus \Lambda^g$ (see [71, Theorem 5.3]) and Φ_g is reflective on M_g (see [113, Remark 6.14]).

(2) If $N_g \neq n_g$, then M_g is not isomorphic to $U(N_g) \oplus U \oplus \Lambda^g$ and Φ_g is not reflective on the lattice $U(N_g) \oplus U \oplus \Lambda^g$. Motivated by Höhn's construction and the discussions above, we conjecture that the vector-valued characters of $V_{\Lambda_g}^{\hat{g}}$ (divided by $\eta^{\mathrm{rk}(\Lambda^g)}$) can be lifted to a reflective Borcherds product of singular weight on M_g. Obviously, the theta lift defines a holomorphic Borcherds product of singular weight on M_g. However, we have to compute

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the conformal weights to confirm that it is reflective. We further conjecture that the Fourier expansion of this product at a certain 0-dimensional cusp recovers the g-twisted denominator of the fake monster algebra. In other words, Φ_g defines a reflective Borcherds product on M_g, i.e. M_g plays the role of the lattice $\mathbb{M}$ in [115, Theorem 1.4].

We can now complete the proof of Theorem 6.1. By (6.1), we can express χ_V as an $\mathbb{N}$-linear combination of characters of the affine VOA generated by $V_1 = \mathfrak{g}$ which have integral conformal weight. Expressions of this type have been determined by Schellekens [98]. We will compute the characters χ_V for $V_1 = B_{12,2}$, $F_{4,6}A_{2,2}$ and $C_{4,10}$. In these cases, the index $[L_{\mathfrak{g}} : \mathbf{Q}_{\mathfrak{g}}]$ is 1 or 2. To determine the divisor of $\mathbf{B}(\chi_V)$, it suffices to compute the singular Fourier coefficients of χ_V, and by Remark 2.6 it is enough to consider Fourier coefficients of the form

$$(6.8) \qquad f(n,\ell)q^n\zeta^\ell, \quad n \le \hat{\delta}_{L_{\mathfrak{g}}}, \ \ell \in \mathbf{Q}'_{\mathfrak{g}}, \ 2n < (\ell,\ell).$$

Note that the above $f(n,\ell)$ are 0 or 1 because $f(n,\ell) \ge 0$ and $\mathbf{B}(\chi_V)$ has only simple zeros, hence $\mathbf{B}(\chi_V)$ is reflective on $2U \oplus L_{\mathfrak{g}}$ as soon as we can show that for every nonzero Fourier coefficient of form (6.8) there exists a positive integer t such that $(\ell,\ell) - 2n = 2/t$ and $t\ell \in L_{\mathfrak{g}}$.

In the three lemmas below, we will express the Fourier expansion of χ_V in terms of Weyl orbits. Suppose $\mathfrak{g} = \mathfrak{g}_{1,k_1} \oplus \mathfrak{g}_{2,k_2}$, where $\mathfrak{g}_{2,k_2}$ may be zero, and let W_j be the Weyl group of $\mathfrak{g}_j$. For a dominant integral weight $\lambda_j = \sum_i x_i w_i$ of $\mathfrak{g}_j$, we define

$$O_{\lambda_j,n_j} = \sum_{\ell \in W_j \cdot \lambda_j} e^{2\pi i \langle \ell, \mathfrak{z} \rangle}, \quad \mathfrak{z} \in L_{\mathfrak{g}} \otimes \mathbb{C},$$

where $n_j = \langle \lambda_j, \lambda_j \rangle / k_j$. Recall (from Notation 2.3) that

$$\mathbf{Q}_{\mathfrak{g}} = Q_1^\vee(k_1) \oplus Q_2^\vee(k_2), \quad \mathbf{Q}'_{\mathfrak{g}} = P_1(1/k_1) \oplus P_2(1/k_2)$$

and

$$\mathbf{Q}_{\mathfrak{g}} < L_{\mathfrak{g}} < L'_{\mathfrak{g}} < \mathbf{Q}'_{\mathfrak{g}}.$$

When we view O_{λ_j,n_j} as part of the Fourier series of χ_V, the vector $\ell \in W_j \cdot \lambda_j \subsetneq \mathbf{Q}'_{\mathfrak{g}}$ is identified with ℓ/k_j and its norm is $(\ell,\ell) = k_j \langle \ell/k_j, \ell/k_j \rangle = n_j$. Therefore, $q^n \cdot O_{\lambda_1,n_1} \otimes O_{\lambda_2,n_2}$ induces reflective zeros if and only if there exists a positive integer t such that

$$n_1 + n_2 - 2n = 2/t \quad \text{and} \quad t(\lambda_1/k_1 + \lambda_2/k_2) \in L_{\mathfrak{g}}.$$

For simplicity, we set

$$O_{\lambda_1,\lambda_2,n_1+n_2} = O_{\lambda_1,n_1} \otimes O_{\lambda_2,n_2}.$$

Lemma 6.8. *When $V_1 = \mathfrak{g}$ is of type $B_{12,2}$, the product $\mathbf{B}(\chi_V)$ is reflective on $2U \oplus D_{12}(2)$.*

Proof. χ_V can be expressed in terms of characters of the affine VOA generated by $\mathfrak{g}$ (using the notation of (2.6)) as

$$\chi_V = \chi_{0,0}^{B_{12,2}} + \chi_{w_1+w_{12},2}^{B_{12,2}} + \chi_{w_{10},3}^{B_{12,2}} + \chi_{w_5,2}^{B_{12,2}}.$$

In this case, $\mathbf{Q}_{\mathfrak{g}} = L_{\mathfrak{g}} = D_{12}(2)$. Note that $\delta_L = 6$ (see [62, Lemma 2.3]), so we only need to calculate χ_V up to its q^2-term. We first use SAGE to compute the Fourier coefficients of $\chi^{B_{12,2}}$ as representations of the simple Lie algebra of type B_{12}, and then we decompose those representations into Weyl orbits as defined above. We find

$$\chi_V = q^{-1} + (O_{w_2,1} + O_{w_1,\frac{1}{2}} + 12) + \sum_{i=1}^\infty c_i q^i,$$

where

$$c_1 = O_{2w_2,4} + O_{w_1+w_3,3} + O_{w_5,\frac{5}{2}} + O_{w_1+w_{12},\frac{5}{2}} + O_{w_1+w_2,\frac{5}{2}} + 4O_{w_4,2}$$
$$+ 12O_{2w_1,2} + 12O_{w_3,\frac{3}{2}} + 12O_{w_{12},\frac{3}{2}} + 44O_{w_2,1} + 90O_{w_1,\frac{1}{2}} + 300,$$
$$c_2 = O_{w_{10},5} + O_{2w_1+w_2,5} + O_{w_2+w_4,5} + O_{w_9,\frac{9}{2}} + O_{3w_1,\frac{9}{2}} + O_{w_2+w_3,\frac{9}{2}} + O_{2w_1+w_{12},\frac{9}{2}}$$
$$+ O_{w_3+w_{12},\frac{9}{2}} + O_{w_1+w_6,\frac{9}{2}} + 4O_{w_8,4} + 12O_{2w_2,4} + 4O_{w_1+w_5,4} + 12O_{w_7,\frac{7}{2}}$$
$$+ 12O_{w_2+w_{12},\frac{7}{2}} + 12O_{w_1+w_4,\frac{7}{2}} + 32O_{w_6,3} + 44O_{w_1+w_3,3} + 90O_{w_5,\frac{5}{2}}$$
$$+ 90O_{w_1+w_{12},\frac{5}{2}} + 90O_{w_1+w_2,\frac{5}{2}} + 224O_{w_4,2} + 288O_{2w_1,2}$$
$$+ 520O_{w_{12},\frac{3}{2}} + 520O_{w_3,\frac{3}{2}} + 1242O_{w_2,1} + 2535O_{w_1,\frac{1}{2}} + 5792.$$

The proof follows by verifying that every singular Weyl orbit is reflective. As an example, consider the orbit $q^2 \cdot O_{w_9,9/2}$. We write $w_9 = (1,1,1,1,1,1,1,1,1,0,0,0)$ as coordinates in the simple roots as in [13]. Then $9/2 - 2 \times 2 = 1/2 = 2/4$ and $4 \cdot (w_9/2) = 2w_9 \in D_{12}(2)$, so $q^2 \cdot O_{w_9,9/2}$ determines reflective divisors. $\square$

Lemma 6.9. *When $V_1 = \mathfrak{g}$ is of type $A_{2,2}F_{4,6}$, the product $\mathbf{B}(\chi_V)$ is reflective on $2U \oplus D_4(6) \oplus A_2(2)$.*

Proof. We express χ_V as the linear combination

$$\left(\chi^{F_{4,10}}_{0000,0} + \chi^{F_{4,10}}_{0004,2} + \chi^{F_{4,10}}_{0030,3} + \chi^{F_{4,10}}_{1100,2}\right) \otimes \chi^{A_{2,2}}_{00,0}$$
$$+ \left(\chi^{F_{4,10}}_{0101,\frac{26}{15}} + \chi^{F_{4,10}}_{1012,\frac{41}{15}}\right) \otimes \left(\chi^{A_{2,2}}_{10,\frac{4}{15}} + \chi^{A_{2,2}}_{01,\frac{4}{15}}\right)$$
$$+ \left(\chi^{F_{4,10}}_{0003,\frac{7}{5}} + \chi^{F_{4,10}}_{0006,\frac{17}{5}} + \chi^{F_{4,10}}_{0021,\frac{12}{5}} + \chi^{F_{4,10}}_{2010,\frac{12}{5}}\right) \otimes \chi^{A_{2,2}}_{11,\frac{3}{5}}$$
$$+ \left(\chi^{F_{4,10}}_{0102,\frac{7}{3}} + \chi^{F_{4,10}}_{2000,\frac{4}{3}}\right) \otimes \left(\chi^{A_{2,2}}_{02,\frac{2}{3}} + \chi^{A_{2,2}}_{20,\frac{2}{3}}\right).$$

Note that

$$\mathbf{Q}_{\mathfrak{g}} = L_{\mathfrak{g}} = A_2(2) \oplus D_4(6).$$

Since $\delta_L = 22/3$, we only need to calculate χ_V up to its q^3-term. We write

$$\chi_V = q^{-1} + \left(O_{11,0000,1} + O_{00,1000,1} + O_{10,0000,\frac{1}{3}} + O_{01,0000,\frac{1}{3}} + O_{00,0001,\frac{1}{2}} + 6\right) + \sum_{i=1}^{\infty} c_i q^i$$

and list the singular Weyl orbits in c_i for $i \leq 3$. There are 17 orbits with norm > 2 in c_1:

$$O_{22,0000,4}, O_{03,0000,3}, O_{30,0000,3}, O_{00,0004,\frac{8}{3}}, O_{02,2000,\frac{8}{3}}, O_{20,2000,\frac{8}{3}},$$
$$O_{11,0003,\frac{5}{2}}, O_{00,1100,\frac{7}{3}}, O_{02,0100,\frac{7}{3}}, O_{20,0100,\frac{7}{3}}, O_{00,0012,\frac{13}{6}}, O_{00,2001,\frac{13}{6}},$$
$$O_{01,0101,\frac{13}{6}}, O_{02,1001,\frac{13}{6}}, O_{10,0101,\frac{13}{6}}, O_{11,0011,\frac{13}{6}}, O_{20,1001,\frac{13}{6}}.$$

There are 26 orbits with norm > 4 in c_2:

$$O_{00,0030,\frac{9}{2}}, O_{03,0003,\frac{9}{2}}, O_{30,0003,\frac{9}{2}}, O_{00,1004,\frac{13}{3}}, O_{00,2100,\frac{13}{3}}, O_{02,0102,\frac{13}{3}}, O_{02,3000,\frac{13}{3}},$$
$$O_{20,0102,\frac{13}{3}}, O_{20,3000,\frac{13}{3}}, O_{22,1000,\frac{13}{3}}, O_{00,0005,\frac{25}{6}}, O_{00,0111,\frac{25}{6}}, O_{00,3001,\frac{25}{6}}, O_{01,1012,\frac{25}{6}},$$
$$O_{02,0110,\frac{25}{6}}, O_{02,1003,\frac{25}{6}}, O_{03,0011,\frac{25}{6}}, O_{10,1012,\frac{25}{6}}, O_{11,0021,\frac{25}{6}}, O_{11,2010,\frac{25}{6}}, O_{12,0101,\frac{25}{6}},$$
$$O_{20,0110,\frac{25}{6}}, O_{20,1003,\frac{25}{6}}, O_{21,0101,\frac{25}{6}}, O_{22,0001,\frac{25}{6}}, O_{30,0011,\frac{25}{6}}.$$

There are 41 orbits with norm > 6 in c_3:

$$O_{11,0006,7}, O_{14,0000,7}, O_{41,0000,7}, O_{00,2004,\frac{20}{3}}, O_{02,4000,\frac{20}{3}}, O_{04,2000,\frac{20}{3}}, O_{20,4000,\frac{20}{3}},$$
$$O_{22,0004,\frac{20}{3}}, O_{40,2000,\frac{20}{3}}, O_{00,0104,\frac{19}{3}}, O_{00,1200,\frac{19}{3}}, O_{02,1102,\frac{19}{3}}, O_{04,0100,\frac{19}{3}}, O_{20,1102,\frac{19}{3}},$$
$$O_{22,1100,\frac{19}{3}}, O_{40,0100,\frac{19}{3}}, O_{00,0031,\frac{37}{6}}, O_{00,1005,\frac{37}{6}}, O_{00,2012,\frac{37}{6}}, O_{01,0112,\frac{37}{6}}, O_{01,2101,\frac{37}{6}},$$
$$O_{02,1110,\frac{37}{6}}, O_{02,2003,\frac{37}{6}}, O_{03,0021,\frac{37}{6}}, O_{03,2010,\frac{37}{6}}, O_{04,1001,\frac{37}{6}}, O_{10,0112,\frac{37}{6}}, O_{10,2101,\frac{37}{6}},$$
$$O_{11,0014,\frac{37}{6}}, O_{11,1021,\frac{37}{6}}, O_{12,1012,\frac{37}{6}}, O_{13,0101,\frac{37}{6}}, O_{20,1110,\frac{37}{6}}, O_{20,2003,\frac{37}{6}}, O_{21,1012,\frac{37}{6}},$$
$$O_{22,0012,\frac{37}{6}}, O_{22,2001,\frac{37}{6}}, O_{30,0021,\frac{37}{6}}, O_{30,2010,\frac{37}{6}}, O_{31,0101,\frac{37}{6}}, O_{40,1001,\frac{37}{6}}.$$

We verify that each of these orbits is reflective by direct computation. $\qquad\square$

Lemma 6.10. *When $V_1 = \mathfrak{g}$ is of type $C_{4,10}$, the product $\mathbf{B}(\chi_V)$ is reflective on $2U \oplus D_4(10)$.*

Proof. We have the expression

$$
\chi_V = \sum_1 \Big(\chi_{0000,0}^{C_{4,10}} + \chi_{0024,4}^{C_{4,10}} + \chi_{0040,2}^{C_{4,10}} + \chi_{0044,6}^{C_{4,10}} + \chi_{00,10,0,8}^{C_{4,10}} + \chi_{0260,5}^{C_{4,10}}
$$
$$
+ \chi_{0321,3}^{C_{4,10}} + \chi_{0323,5}^{C_{4,10}} + \chi_{0500,2}^{C_{4,10}} + \chi_{0800,4}^{C_{4,10}} + \chi_{1051,4}^{C_{4,10}} + \chi_{1430,4}^{C_{4,10}}
$$
$$
+ \chi_{1431,5}^{C_{4,10}} + \chi_{2242,6}^{C_{4,10}} + \chi_{3031,3}^{C_{4,10}} + \chi_{4140,4}^{C_{4,10}} \Big) + 2\chi_{2222,4}^{C_{4,10}},
$$

where the glue vector 1 exchanges the affine Dynkin labels $\hat{w}_0, \hat{w}_1, \hat{w}_2, \hat{w}_3, \hat{w}_4$ and $\hat{w}_4, \hat{w}_3, \hat{w}_2, \hat{w}_1, \hat{w}_0$ (such that e.g. $\chi_{0000,0}^{C_{4,10}}$ becomes $\chi_{000,10,10}^{C_{4,10}}$ and $\chi_{0024,4}^{C_{4,10}}$ becomes $\chi_{2004,3}^{C_{4,10}}$; see (2.6) and the explanations there). In this case, we have

$$\mathbf{Q}_\mathfrak{g} = \mathbb{Z}^4(20) \quad \text{and} \quad L_\mathfrak{g} = D_4'(20) \cong D_4(10).$$

Since $\delta_L = 10$, we only need to calculate χ_V up to its q^4-term. We write

$$\chi_V = q^{-1} + \big(O_{2000,\frac{1}{5}} + O_{0100,\frac{1}{10}} + 4\big) + \sum_{i=1}^\infty c_i q^i$$

and find that c_1 equals

$$O_{0500,\frac{5}{2}} + O_{0040,\frac{12}{5}} + O_{4002,\frac{12}{5}} + O_{4020,\frac{11}{5}} + O_{0121,\frac{21}{10}} + O_{1310,\frac{21}{10}} + O_{2102,\frac{21}{10}}$$
$$+ O_{4101,\frac{21}{10}} + 2O_{0202,2} + 2O_{4200,2} + 2O_{1012,\frac{19}{10}} + 2O_{2120,\frac{19}{10}} + 2O_{5010,\frac{19}{10}} + 2O_{0003,\frac{9}{5}}$$
$$+ 4O_{0220,\frac{9}{5}} + 2O_{2201,\frac{9}{5}} + 4O_{6000,\frac{9}{5}} + 5O_{0301,\frac{17}{10}} + 5O_{1030,\frac{17}{10}} + 5O_{2300,\frac{17}{10}} + 5O_{3011,\frac{17}{10}}$$
$$+ 10O_{0400,\frac{8}{5}} + 10O_{1111,\frac{3}{2}} + 10O_{3110,\frac{3}{2}} + 12O_{0021,\frac{7}{5}} + 17O_{2002,\frac{7}{5}} + 12O_{4001,\frac{7}{5}}$$
$$+ 20O_{0102,\frac{13}{10}} + 20O_{1210,\frac{13}{10}} + 20O_{4100,\frac{13}{10}} + 32O_{2020,\frac{6}{5}} + 38O_{0120,\frac{11}{10}} + 38O_{2101,\frac{11}{10}}$$
$$+ 46O_{02011} + 56O_{22001} + 69O_{0300,\frac{9}{10}} + 69O_{1011,\frac{9}{10}} + 69O_{3010,\frac{9}{10}} + 101O_{0002,\frac{4}{5}}$$
$$+ 101O_{4000,\frac{4}{5}} + 120O_{1110,\frac{7}{10}} + 168O_{0020,\frac{3}{5}} + 148O_{2001,\frac{3}{5}} + 205O_{0101,\frac{1}{2}} + 205O_{2100,\frac{1}{2}}$$
$$+ 280O_{0200,\frac{2}{5}} + 340O_{1010,\frac{3}{10}} + 418O_{0001,\frac{1}{5}} + 456O_{2000,\frac{1}{5}} + 558O_{0100,\frac{1}{10}} + 748.$$

There are 10 orbits with norm > 4 in c_2:

$$O_{10,000,5}, O_{2004,\frac{21}{5}}, O_{4220,\frac{21}{5}}, O_{0104,\frac{41}{10}}, O_{0321,\frac{41}{10}},$$
$$O_{2302,\frac{41}{10}}, O_{3031,\frac{41}{10}}, O_{4301,\frac{41}{10}}, O_{5030,\frac{41}{10}}, O_{8100,\frac{41}{10}}.$$

There are 11 orbits with norm > 6 in c_3:

$$O_{0800,\frac{32}{5}}, O_{0024,\frac{31}{5}}, O_{6202,\frac{31}{5}}, O_{0105,\frac{61}{10}}, O_{10,100,\frac{61}{10}}, O_{1051,\frac{61}{10}},$$
$$O_{1430,\frac{61}{10}}, O_{3411,\frac{61}{10}}, O_{4140,\frac{61}{10}}, O_{5013,\frac{61}{10}}, O_{7012,\frac{61}{10}}.$$

There are 16 orbits with norm > 8 in c_4:

$$O_{0260,\frac{41}{5}}, O_{2800,\frac{41}{5}}, O_{8220,\frac{41}{5}}, O_{0323,\frac{81}{10}}, O_{0900,\frac{81}{10}}, O_{1070,\frac{81}{10}}, O_{1431,\frac{81}{10}}, O_{2304,\frac{81}{10}},$$
$$O_{3033,\frac{81}{10}}, O_{3412,\frac{81}{10}}, O_{4141,\frac{81}{10}}, O_{5411,\frac{81}{10}}, O_{6140,\frac{81}{10}}, O_{7013,\frac{81}{10}}, O_{8301,\frac{81}{10}}, O_{9030,\frac{81}{10}}.$$

To finish the proof, we verify that the required condition on the orders of vectors in the singular Weyl orbits is satisfied. We will work out $q^1 \cdot O_{4002,12/5}$ as an example. The associated dominant weight is

$$\lambda = 4w_1 + 2w_4 = (6, 2, 2, 2)$$

in coordinates as in [13]. The corresponding vector in $L'_{\mathfrak{g}} = D_4(1/20)$ is $\lambda/20$, which has norm $20 \times (12/100) = 12/5$, and we have $12/5 - 2 \times 1 = 2/5$. The order of $\lambda/20$ modulo $\mathbf{Q}_{\mathfrak{g}} = \mathbb{Z}^4(20)$ is 10, but its order modulo $L_{\mathfrak{g}} = D'_4(20)$ is 5. Hence the induced zero divisor is reflective on $2U \oplus L_{\mathfrak{g}}$ although it is not reflective on $2U \oplus \mathbf{Q}_{\mathfrak{g}}$. $\qquad\square$

7. The symmetric case: holomorphic SCFTs of central charge 12

In this section, we construct hyperbolizations of affine Kac–Moody algebras $\hat{\mathfrak{g}}$ for the eight $\mathfrak{g}$ of symmetric type with $C = 1$ in Theorem 5.1. These semi-simple $\mathfrak{g}$ first appeared in [24], where Dittmann and the second named author proved that the additive lifts of the denominators of these $\hat{\mathfrak{g}}$ are reflective Borcherds products of singular weight on the maximal even sublattices of the (integral) lattices $\mathbf{P}_{\mathfrak{g}}$. Shortly after [24] appeared on arXiv, Harrison, Paquette, Persson and Volpato [57] described these $\mathfrak{g}$ as the $\mathcal{N} = 1$ structures of holomorphic SCFT F_{24} of central charge 12, and gave a natural construction of a certain BKM superalgebra (denoted $\mathcal{G}_{\mathfrak{g}}$) as the BRST cohomology related to F_{24} with a fixed $\mathcal{N} = 1$ structure under some technical assumptions. They also determined the super-denominators of these BKM superalgebras.

We will prove that these super-denominators are actually the reflective Borcherds products of singular weight constructed in [24]. We will also describe the connection between these BKM super-algebras and the twists of the fake monster algebra, which resolves some open questions proposed in [57]. Additionally, we express the inputs of these Borcherds products as $\mathbb{Z}$-linear combinations of the full characters of the affine VOAs generated by $\mathfrak{g}$, and construct some exceptional modular invariants of $\hat{\mathfrak{g}}$ as an application.

7.1. Holomorphic SCFTs of central charge 12 composed of 24 chiral fermions.

Holomorphic vertex operator superalgebras (SVOA) of central charge 12 were classified by Creutzig, Duncan and Riedler [20], and they fall into three types: the Conway SCFT $V^{f\natural}$ with trivial weight-1/2 subspace, the theory V^{fE_8} of 8 chiral bosons and 8 chiral fermions, and the theory F_{24} of 24 chiral fermions. The hyperbolizations of affine Kac–Moody algebras are related to F_{24}. We review the theory of F_{24} following [57].

F_{24} is constructed from the lattice VOA associated with the D_{12} lattice

$$D_{12} = \left\{ (x_1, x_2, ..., x_{12}) \in \mathbb{Z}^{12} : \quad x_1 + x_2 + ... + x_{12} \in 2\mathbb{Z} \right\}.$$

The lattice VOA $V_{D_{12}}$ has 4 irreducible modules labeled by the 4 cosets of D'_{12}/D_{12},

$$0, \quad v = (1, 0, .., 0), \quad s = \left(\frac{1}{2}, ..., \frac{1}{2}, \frac{1}{2}\right) \quad \text{and} \quad c = \left(\frac{1}{2}, ..., \frac{1}{2}, -\frac{1}{2}\right).$$

The four modules V_0, V_v, V_s and V_c have conformal weights 0, 1/2, 3/2 and 3/2, respectively. Their full characters can be expressed as the quotients of Jacobi theta functions by η^{12}:

$$\chi_*(\tau, z) = \frac{\Theta_{D_{12},*}(\tau, z)}{\eta(\tau)^{12}}, \quad * = 0, v, s, c, \quad z \in D_{12} \otimes \mathbb{C}.$$

The sum $V_0 \oplus V_v$ is the $\mathbb{Z}_2$-graded superspace of F_{24} (i.e. the NS sector in 2d SCFTs). Let $(-1)^F$ be the canonical involution, such that

$$(-1)^F|_{V_0} = I \quad \text{and} \quad (-1)^F|_{V_v} = -I.$$

The SVOA F_{24} has a unique irreducible $(-1)^F$-stable canonically twisted module which is given by $V_s \oplus V_c$ (i.e. the R sector in 2d SCFTs). The weight-1/2 subspace $W_{1/2}$ of F_{24} has a natural semi-simple Lie algebra structure, which in physics is called the $\mathcal{N} = 1$ superconformal structure. We denote this Lie algebra by $\mathfrak{g}$ and its Cartan subalgebra by $\mathfrak{h}$. For $\mathfrak{z} \in \mathfrak{h}$, the full characters of F_{24} are defined by

$$\chi_{\mathrm{NS}}(\tau, \mathfrak{z}) = \mathrm{Tr}_{\mathrm{NS}}\big(q^{L_0 - \frac{1}{2}} e^{2\pi i \mathfrak{z}}\big), \qquad \chi_{\widetilde{\mathrm{NS}}}(\tau, \mathfrak{z}) = \mathrm{Tr}_{\mathrm{NS}}\big(q^{L_0 - \frac{1}{2}} e^{2\pi i \mathfrak{z}}(-1)^F\big),$$
$$\chi_{\mathrm{R}}(\tau, \mathfrak{z}) = \mathrm{Tr}_{\mathrm{R}}\big(q^{L_0 - \frac{1}{2}} e^{2\pi i \mathfrak{z}}\big), \qquad \chi_{\widetilde{\mathrm{R}}}(\tau, \mathfrak{z}) = \mathrm{Tr}_{\mathrm{R}}\big(q^{L_0 - \frac{1}{2}} e^{2\pi i \mathfrak{z}}(-1)^F\big).$$

The restriction of the unique normalized invariant bilinear form of F_{24} to a simple ideal $\mathfrak{g}_j$ of $\mathfrak{g}$ satisfies $(-, -) = k_j \langle -, - \rangle$, where $\langle -, - \rangle$ is the standard bilinear form of $\mathfrak{g}_j$ as before (see [57]). We indicate these integers by writing

$$\mathfrak{g} = \mathfrak{g}_{1,k_1} \oplus \cdots \oplus \mathfrak{g}_{s,k_s}.$$

The semi-simple Lie algebra $\mathfrak{g}$ is of dimension 24. There are exactly eight distinct possibilities for $\mathfrak{g}$ and they are characterized by the identity (see Notation 2.3)

$$1 = \frac{\dim \mathfrak{g}}{24} = \frac{h_j^\vee}{k_j}, \quad 1 \leq j \leq s.$$

The affine VOA $V_{\mathfrak{g}}$ generated by $W_{1/2} = \mathfrak{g}$ is a full sub-VOA of F_{24}. Let $\Delta_{\mathfrak{g}}^+$ denote the set of positive roots of $\mathfrak{g}$. We have the following embedding of integral lattices

$$\iota_{\mathfrak{g}} : \quad \mathbf{P}_{\mathfrak{g}} := \bigoplus_{j=1}^{s} P_j^\vee(h_j^\vee) \hookrightarrow \mathbb{Z}^{12}, \quad \mathfrak{z} \mapsto \iota_{\mathfrak{g}}(\mathfrak{z}) = \{\langle \mathfrak{z}, \alpha \rangle\}_{\alpha \in \Delta_{\mathfrak{g}}^+},$$

which induces an embedding of the maximal even sublattice $L_{\mathfrak{g}}$ of $\mathbf{P}_{\mathfrak{g}}$ into D_{12}. From the conformal embedding $V_{\mathfrak{g}} \hookrightarrow F_{24}$ we deduce that the full characters of F_{24} with $W_{1/2} = \mathfrak{g}$ can be expressed as

$$\chi_{\mathrm{NS}}(\tau, \mathfrak{z}) = \chi_0(\tau, \iota_{\mathfrak{g}}(\mathfrak{z})) + \chi_v(\tau, \iota_{\mathfrak{g}}(\mathfrak{z})),$$
$$\chi_{\widetilde{\mathrm{NS}}}(\tau, \mathfrak{z}) = \chi_0(\tau, \iota_{\mathfrak{g}}(\mathfrak{z})) - \chi_v(\tau, \iota_{\mathfrak{g}}(\mathfrak{z})),$$
$$\chi_{\mathrm{R}}(\tau, \mathfrak{z}) = \chi_s(\tau, \iota_{\mathfrak{g}}(\mathfrak{z})) + \chi_c(\tau, \iota_{\mathfrak{g}}(\mathfrak{z})),$$
$$\chi_{\widetilde{\mathrm{R}}}(\tau, \mathfrak{z}) = \chi_s(\tau, \iota_{\mathfrak{g}}(\mathfrak{z})) - \chi_c(\tau, \iota_{\mathfrak{g}}(\mathfrak{z})) \equiv 0,$$

where the last equality follows from the fact that $\iota(L_{\mathfrak{g}}) \hookrightarrow D_{10}$. It is easy to check that, up to scaling,

$$\phi_{\mathfrak{g}}(\tau, \mathfrak{z}) = \frac{1}{2}\Big(\chi_{\mathrm{NS}}(\tau, \mathfrak{z}) - \chi_{\widetilde{\mathrm{NS}}}(\tau, \mathfrak{z}) - \chi_{\mathrm{R}}(\tau, \mathfrak{z})\Big)$$

(7.1)
$$= \chi_v(\tau, \iota_{\mathfrak{g}}(\mathfrak{z})) - \chi_c(\tau, \iota_{\mathfrak{g}}(\mathfrak{z}))$$
$$= \mathrm{rk}(\mathfrak{g}) + \sum_{\alpha \in \Delta_{\mathfrak{g}}^+} \Big(e^{2\pi i \langle \alpha, \mathfrak{z} \rangle} + e^{-2\pi i \langle \alpha, \mathfrak{z} \rangle}\Big) + O(q)$$

is the unique non-trivial $\mathbb{C}$-linear combination of the full characters of F_{24} that is modular under $\mathrm{SL}_2(\mathbb{Z})$. In particular, $\phi_{\mathfrak{g}}(\tau, \mathfrak{z})$ is a weak Jacobi form of weight 0 and lattice index $L_{\mathfrak{g}}$ (with trivial character on $\mathrm{SL}_2(\mathbb{Z})$). It was proved in [57] under some assumptions that Fourier coefficients of $\phi_{\mathfrak{g}}$ determine the roots and root multiplicities of the BKM superalgebra $\mathcal{G}_{\mathfrak{g}}$, realized naturally as the BRST cohomology related to F_{24} with $W_{1/2} = \mathfrak{g}$. More precisely, the root $(n, \ell, m) \in U \oplus L'_{\mathfrak{g}}$ has multiplicity $a_{\mathfrak{g}}(nm, \ell)$ as an even root and has multiplicity $b_{\mathfrak{g}}(nm, \ell)$ as an odd root, where

$$\frac{1}{2}\big(\chi_{\mathrm{NS}}(\tau, \mathfrak{z}) - \chi_{\widetilde{\mathrm{NS}}}(\tau, \mathfrak{z})\big) = \chi_v(\tau, \iota_{\mathfrak{g}}(\mathfrak{z})) = \sum_{n,\ell} a_{\mathfrak{g}}(n, \ell) q^n \zeta^\ell,$$

$$\frac{1}{2}\chi_{\mathrm{R}}(\tau, \mathfrak{z}) = \chi_c(\tau, \iota_{\mathfrak{g}}(\mathfrak{z})) = \sum_{n,\ell} b_{\mathfrak{g}}(n, \ell) q^n \zeta^\ell.$$

7.2. Construction of hyperbolizations.
In this subsection we prove the following theorem.

Theorem 7.1. *Let $\mathfrak{g}$ be one of the $\mathcal{N} = 1$ structures of F_{24}. Then the theta lift $\mathbf{B}(\phi_{\mathfrak{g}})$ is a reflective Borcherds product of singular weight on $2U \oplus L_{\mathfrak{g}}$. Moreover, the leading Fourier–Jacobi coefficient of $\mathbf{B}(\phi_{\mathfrak{g}})$ coincides with the denominator of the affine Kac–Moody algebra $\hat{\mathfrak{g}}$.*

Proof. We will use (7.1) and Lemma 7.4 below to conclude that $\phi_{\mathfrak{g}}$ has integral Fourier coefficients and non-negative singular Fourier coefficients. Therefore, the theta lift of $\phi_{\mathfrak{g}}$ is a holomorphic Borcherds product of singular weight on $2U \oplus L_{\mathfrak{g}}$. We see from the q^0-term of $\phi_{\mathfrak{g}}$ that the leading Fourier–Jacobi coefficient is given by the denominator $\vartheta_{\mathfrak{g}}$ of $\hat{\mathfrak{g}}$. It remains to prove that $\mathbf{B}(\phi_{\mathfrak{g}})$ is reflective. This can be verified by calculating the Fourier expansions of $\phi_{\mathfrak{g}}$ up to the $q^{\hat{\delta}_L}$-term for $L = L_{\mathfrak{g}}$, where δ_L is defined in Remark 2.6. As we mentioned at the beginning of this section, this also follows from Lemma 7.3 and [24, Theorems 5.1 and 6.1]. $\qquad\square$

Lemma 7.2. *Let $z = (z_1, z_2, ..., z_{12}) \in \mathbb{C}^{12}$. We set*

$$\phi_*(\tau, z) = \eta(\tau)^{-12} \prod_{j=1}^{12} \vartheta_*(\tau, z_j), \quad * = 00, 01, 10, 11,$$

where $\vartheta_{11}(\tau, u) := \vartheta(\tau, u)$ and

$$\vartheta_{00}(\tau, u) = \prod_{n=1}^{\infty} (1 - q^n)(1 + q^{n-1/2}\zeta)(1 + q^{n-1/2}\zeta^{-1}),$$

$$\vartheta_{01}(\tau, u) = \prod_{n=1}^{\infty} (1 - q^n)(1 - q^{n-1/2}\zeta)(1 - q^{n-1/2}\zeta^{-1}),$$

$$\vartheta_{10}(\tau, u) = q^{1/8}(\zeta^{1/2} + \zeta^{-1/2}) \prod_{n=1}^{\infty} (1 - q^n)(1 + q^n\zeta)(1 + q^n\zeta^{-1}),$$

here $u \in \mathbb{C}$ and $\zeta = e^{2\pi i u}$. Then the following identities hold:

$$\chi_0 = (\phi_{00} + \phi_{01})/2, \quad \chi_v = (\phi_{00} - \phi_{01})/2, \quad \chi_s = (\phi_{10} + \phi_{11})/2, \quad \chi_c = (\phi_{10} - \phi_{11})/2,$$

$$\frac{\phi_{11}(2\tau, 2z)}{\phi_{11}(\tau, z)} = \phi_{10}(\tau, z), \quad \frac{\phi_{11}(\tau/2, z)}{\phi_{11}(\tau, z)} = \phi_{01}(\tau, z), \quad \frac{\phi_{11}(\tau/2 + 1/2, z)}{\phi_{11}(\tau, z)} = -\phi_{00}(\tau, z).$$

In particular, we have (see Proposition 2.2)

$$\phi_{D_{12}}(\tau, z) := \frac{1}{2}\big(\phi_{00}(\tau, z) - \phi_{01}(\tau, z) - \phi_{10}(\tau, z)\big) = -\frac{\phi_{11}|_0 T_-^{(1)}(2)(\tau, z)}{\phi_{11}(\tau, z)}.$$

Proof. The formulas for χ_* can be established by considering (χ_*) as a vector-valued Jacobi form for the dual Weil representation $\overline{\rho}_{D_{12}}$. The other formulas follow from the definition. $\qquad\square$

By restricting the above identities along the embedding $\iota_{\mathfrak{g}} : L_{\mathfrak{g}} \hookrightarrow D_{12}$, we obtain the following lemma:

Lemma 7.3. *The image $\iota_{\mathfrak{g}}(L_{\mathfrak{g}})$ is a sublattice of $D_{12-\mathrm{rk}(\mathfrak{g})/2} \subsetneq D_{12}$. By considering the quasi-pullback we can express the denominator of $\hat{\mathfrak{g}}$ as*

$$\vartheta_{\mathfrak{g}}(\tau,\mathfrak{z}) = \frac{1}{(2\pi i)^{\mathrm{rk}(\mathfrak{g})/2}}\left(\left(\prod_{j=1}^{\mathrm{rk}(\mathfrak{g})/2}\partial_{z_{12-\mathrm{rk}(\mathfrak{g})/2+j}}\right)\phi_{11}(\tau,z)\right)(\tau,\iota_{\mathfrak{g}}(\mathfrak{z})).$$

The full characters of F_{24} can be expressed as

$$\chi_{\mathrm{NS}}(\tau,\mathfrak{z}) = \phi_{00}(\tau,\iota_{\mathfrak{g}}(\mathfrak{z})) = -\frac{\vartheta_{\mathfrak{g}}(\tau/2+1/2,\mathfrak{z})}{\vartheta_{\mathfrak{g}}(\tau,\mathfrak{z})},$$

$$\chi_{\widetilde{\mathrm{NS}}}(\tau,\mathfrak{z}) = \phi_{01}(\tau,\iota_{\mathfrak{g}}(\mathfrak{z})) = \frac{\vartheta_{\mathfrak{g}}(\tau/2,\mathfrak{z})}{\vartheta_{\mathfrak{g}}(\tau,\mathfrak{z})},$$

$$\chi_{\mathrm{R}}(\tau,\mathfrak{z}) = \phi_{10}(\tau,\iota_{\mathfrak{g}}(\mathfrak{z})) = 2^{\mathrm{rk}(\mathfrak{g})/2}\frac{\vartheta_{\mathfrak{g}}(2\tau,2\mathfrak{z})}{\vartheta_{\mathfrak{g}}(\tau,\mathfrak{z})}.$$

The input form has the representation

$$\phi_{\mathfrak{g}}(\tau,\mathfrak{z}) = \phi_{D_{12}}(\tau,\iota_{\mathfrak{g}}(\mathfrak{z})) = -\frac{\vartheta_{\mathfrak{g}}|_{\mathrm{rk}(\mathfrak{g})/2}T_-^{(1)}(2)(\tau,\mathfrak{z})}{\vartheta_{\mathfrak{g}}(\tau,\mathfrak{z})}.$$

We can now prove directly that the BKM superalgebra $\mathcal{G}_{\mathfrak{g}}$ has no odd real roots.

Lemma 7.4. *The function $\chi_c(\tau,\iota_{\mathfrak{g}}(\mathfrak{z})) = O(q)$ is holomorphic at infinity, that is, its nonzero Fourier coefficients $f(n,\ell)q^n\zeta^\ell$ satisfy that $2n \geq (\ell,\ell)$.*

Proof. This can be read off of the Fourier expansion of $\chi_c(\tau,\iota_{\mathfrak{g}}(\mathfrak{z}))$ up to their $q^{\hat{\delta}_L}$-terms, since the analogue of Remark 2.6 holds. The numbers δ_L for $L = L_{\mathfrak{g}}$ are listed in Table 8. This claim can also be proved by a generalization of [54, Lemma 6.3]: if $\phi(\tau,\mathfrak{z})$ is a holomorphic Jacobi form of lattice index L which can be constructed as a theta block then $\phi(2\tau,2\mathfrak{z})/\phi(\tau,\mathfrak{z})$ is holomorphic at infinity. The claim then follows from Lemma 7.3. $\qquad\square$

Let us now explain the connection between $\mathbf{B}(\phi_{\mathfrak{g}})$ and the twisted denominators of the fake monster algebra. This is similar to Höhn's observation in the anti-symmetric case.

Theorem 7.5. *Let $\mathfrak{g}$ be one of the $\mathcal{N} = 1$ structures of F_{24}. Then there exists a unique Co_0-conjugacy class $[g]$ of the same order and level n_g (see Table 8) satisfying the following properties:*

(1) *The lattice $L_{\mathfrak{g}}$ is characterized by the isomorphism*

$$U \oplus L_{\mathfrak{g}} \cong U(n_g) \oplus (\Lambda^g)'(n_g).$$

(2) *Recall from Section 2.5 that the g-twisted denominator of the fake monster algebra is the Fourier expansion of a reflective Borcherds product Φ_g of singular weight on $U_1(n_g) \oplus U \oplus \Lambda^g$ at the 0-dimensional cusp related to U_1, and that the input form that lifts to Φ_g is given by the vector-valued characters of the orbifold VOA $V_{\Lambda_g}^{\hat{g}}$. The Fourier expansion of $\mathbf{B}(\phi_{\mathfrak{g}})$ at U_1 equals the Fourier expansion of Φ_g at $U_1(n_g)$ after making the identifications*

$$\mathrm{O}\big(U_1(n_g) \oplus U \oplus \Lambda^g\big) = \mathrm{O}\big(U_1(1/n_g) \oplus U \oplus (\Lambda^g)'\big)$$
$$= \mathrm{O}\big(U_1 \oplus U(n_g) \oplus (\Lambda^g)'(n_g)\big) \cong \mathrm{O}\big(U_1 \oplus U \oplus L_{\mathfrak{g}}\big).$$

(3) *The BKM superalgebra $\mathcal{G}_{\mathfrak{g}}$, i.e. the BRST cohomology related to F_{24} with $\mathcal{N} = 1$ structure $\mathfrak{g}$, is isomorphic to the BKM superalgebra $\mathbb{G}_g$ obtained as the twist of the fake monster algebra by g.*

(4) *The $\mathbb{Z}$-lattice generated by the Weyl vector ρ and the vectors $\lambda = (n, \ell, m) \in U \oplus L'_{\mathfrak{g}}$ for which $\mathbf{B}(\phi_{\mathfrak{g}})$ vanishes on $\lambda^{\perp}$, i.e. the root lattice of $\mathcal{G}_{\mathfrak{g}}$, is exactly $U \oplus L'_{\mathfrak{g}}$. Recall that the root lattice of $\mathbb{G}_g$ is $U \oplus \Lambda^g$. The two lattices are related by*

$$(U \oplus L'_{\mathfrak{g}})(n_g) \cong U \oplus \Lambda^g.$$

Proof. This follows from Theorem 7.1 and its proof and from [24, Theorem 6.1, Proposition 6.3]. $\square$

Remark 7.6. For each $\mathfrak{g}$, the lattice $L_{\mathfrak{g}}$ is the unique class in the corresponding genus. The Borcherds product $\mathbf{B}(\phi_{\mathfrak{g}}) = \Phi_g$ is modular under the full orthogonal group $\mathrm{O}^+(2U \oplus L_{\mathfrak{g}})$.

Remark 7.7. In 2000 Scheithauer [92] found a natural realization of the fake monster superalgebra (introduced by Borcherds [6]) as the BRST cohomology related to the holomorphic SCFT V^{fE_8}. This BKM superalgbera has no real roots and its denominator (not its super-denominator!) is given by the Fourier expansion of $\mathbf{B}(\phi_{\mathfrak{g}}) = \Phi_g$ at $U_1(2)$ for $\mathfrak{g} = A_{1,2}^8$ and g with cycle shape $1^{-8}2^{16}$, where we view $\mathbf{B}(\phi_{\mathfrak{g}})$ as a modular form on $U_1(2) \oplus U \oplus E_8 \cong U_1 \oplus U \oplus D_8$. By [113, Theorem 5.9] we can express the input as a pair of Jacobi forms (ϕ_1, ϕ_2), where ϕ_j is a weakly holomorphic Jacobi form of weight 0 and lattice index E_8 with trivial character on $\Gamma_0(2/j)$ for $j = 1, 2$. Let $\mathfrak{z} \in E_8 \otimes \mathbb{C}$. Note that $\chi_{\widetilde{R}} = 0$. The other full characters of V^{fE_8} have the expressions

$$\chi_{\mathrm{NS}}(\tau, \mathfrak{z}) = \frac{\Theta_{E_8,0}(\tau, \mathfrak{z})}{\eta(\tau)^8} \prod_{j=1}^{4} \frac{\vartheta_{00}(\tau, 0)}{\eta(\tau)},$$

$$\chi_{\widetilde{\mathrm{NS}}}(\tau, \mathfrak{z}) = \frac{\Theta_{E_8,0}(\tau, \mathfrak{z})}{\eta(\tau)^8} \prod_{j=1}^{4} \frac{\vartheta_{01}(\tau, 0)}{\eta(\tau)},$$

$$\chi_{\mathrm{R}}(\tau, \mathfrak{z}) = \frac{\Theta_{E_8,0}(\tau, \mathfrak{z})}{\eta(\tau)^8} \prod_{j=1}^{4} \frac{\vartheta_{10}(\tau, 0)}{\eta(\tau)}.$$

The input forms can be expressed as linear combinations of the characters:

$$\phi_1(\tau, \mathfrak{z}) = \left(\chi_{\mathrm{NS}} - \chi_{\widetilde{\mathrm{NS}}}\right)(\tau, \mathfrak{z}) = \sum_{n \in \mathbb{N}, \ell \in E_8} c_1(n, \ell) q^n \zeta^{\ell},$$

$$\phi_2(\tau, \mathfrak{z}) = \left(\chi_{\mathrm{NS}} - \chi_{\widetilde{\mathrm{NS}}} - \chi_{\mathrm{R}}\right)(\tau, \mathfrak{z}) = 0,$$

The denominator of the fake monster superalgebra can be written in the form

$$\prod_{0 < \alpha = (n, \ell, m) \in U \oplus E_8} \left(\frac{1 - e^{-\alpha}}{1 + e^{-\alpha}}\right)^{c_1(nm, \ell)/2}.$$

Remark 7.8. The Conway SCFT $V^{f\natural}$ was constructed by Duncan [31] in 2007. Harrison, Paquette and Volpato [56] proved in 2019 that the BRST cohomology related to $V^{f\natural}$ defines a BKM superalgebra with no real roots. Note that the denominator of this superalgebra is the Fourier expansion of a Borcherds product of weight 0 on $U_1(2) \oplus U$ at the 0-dimensional cusp determined by $U_1(2)$, whose input can be expressed as a pair (ϕ_1, ϕ_2) with

$$\phi_1(\tau) = (\chi_{\mathrm{NS}} - \chi_{\widetilde{\mathrm{NS}}})(\tau),$$
$$\phi_2(\tau) = (\chi_{\mathrm{NS}} - \chi_{\widetilde{\mathrm{NS}}} - \chi_{\mathrm{R}} + \chi_{\widetilde{\mathrm{R}}})(\tau) = -48,$$

and where the characters of $V^{f\natural}$ are given by

$$\chi_{\text{NS}}(\tau) = \frac{\eta(\tau)^{48}}{\eta(\tau/2)^{24}\eta(2\tau)^{24}} - 24, \qquad\qquad \chi_{\widetilde{\text{NS}}}(\tau) = \frac{\eta(\tau/2)^{24}}{\eta(\tau)^{24}} + 24,$$

$$\chi_{\text{R}}(\tau) = 2^{12}\frac{\eta(2\tau)^{24}}{\eta(\tau)^{24}} + 24, \qquad\qquad \chi_{\widetilde{\text{R}}}(\tau) = -24.$$

Let $[g]$ be the Co_0-conjugacy class of cycle shape $1^{-24}2^{24}$. The g-twisted denominator of the fake monster algebra is identical to the Fourier expansion of the above Borcherds product at the other 1-dimensional cusp determined by U through the identification

$$\text{O}(U \oplus U_1(2)) = \text{O}(U \oplus U_1(1/2)) = \text{O}(U(2) \oplus U_1).$$

Inspired by the two remarks above, we hope to systematically study the connection between vertex algebras and reflective Borcherds products of singular weight on lattices of type $U(N) \oplus U \oplus L$ in the near future.

7.3. Exceptional modular invariants.

The 2D rational CFT consists of both holomorphic and anti-holomorphic parts, which together can yield $\text{SL}_2(\mathbb{Z})$ modular invariants. The classification of modular invariants of affine Kac–Moody algebras was an interesting and vast topic particularly in the 1980s and 90s. We will reveal some new connections to the theory of Borcherds products here and in the next section.

Let us first recall some basics about modular invariants. Clearly, the CFT torus partition function $\sum |\chi_i|^2$, i.e. the summation of the norm-squares of the characters of all primaries, is a diagonal modular invariant. However there can exist other, non-diagonal modular invariants. In general, a $\text{SL}_2(\mathbb{Z})$ modular invariant of a 2D rational CFT can be written as a sesquilinear combination of characters

$$Z = \sum M_{ab}\chi_a\overline{\chi}_b, \qquad M_{ab} \in \mathbb{N}.$$

Two major approaches to constructing modular invariants are the *conformal embedding* and *simple currents extension*, see e.g. the textbook [22, Chapter 17]. Modular invariants that cannot be obtained from simple currents are often called *exceptional*. For example, $A_{1,k}$ is known to have three exceptional modular invariants at levels $k = 10, 16, 28$, respectively called E_6, E_7, E_8 modular invariants. Exceptional modular invariants are rare and can be constructed by either special conformal embeddings or via nontrivial automorphisms of fusion algebras. The full classification of modular invariants for simple affine Lie algebras of all levels has been achieved for A_1 [17, 67] and A_2 [36], see some recent progress in [38].

We can construct many exceptional modular invariants by means of the conformal embedding

$$V_{\mathfrak{g}} \hookrightarrow F_{24}$$

and the automorphism $\chi_v \leftrightarrow \chi_c$ of the $D_{12,1}$ fusion algebra. First we have the diagonal modular invariant of D_{12}, i.e.

$$Z_{D_{12}} = |\chi_0|^2 + |\chi_v|^2 + |\chi_s|^2 + |\chi_c|^2.$$

The diagonal modular invariant of F_{24} with $W_{1/2} = \mathfrak{g}$ is given by

(7.2)
$$Z_{F_{24}} = \frac{1}{2}\Big(|\chi_{\text{NS}}|^2 + |\chi_{\widetilde{\text{NS}}}|^2 + |\chi_{\text{R}}|^2 + |\chi_{\widetilde{\text{R}}}|^2\Big)$$
$$= |\chi_0(\tau, \iota_{\mathfrak{g}}(\mathfrak{z}))|^2 + |\chi_v(\tau, \iota_{\mathfrak{g}}(\mathfrak{z}))|^2 + |\chi_s(\tau, \iota_{\mathfrak{g}}(\mathfrak{z}))|^2 + |\chi_c(\tau, \iota_{\mathfrak{g}}(\mathfrak{z}))|^2.$$

By letting the automorphism $\chi_v \leftrightarrow \chi_c$ act on the holomorphic part, we obtain an exceptional modular invariant of $V_{\mathfrak{g}}$,

(7.3) $\quad Z^{\text{exc}}_{F_{24}} = |\chi_0(\tau, \iota_{\mathfrak{g}}(\mathfrak{z}))|^2 + |\chi_s(\tau, \iota_{\mathfrak{g}}(\mathfrak{z}))|^2 + \chi_v(\tau, \iota_{\mathfrak{g}}(\mathfrak{z}))\overline{\chi_c(\tau, \iota_{\mathfrak{g}}(\mathfrak{z}))} + \chi_c(\tau, \iota_{\mathfrak{g}}(\mathfrak{z}))\overline{\chi_v(\tau, \iota_{\mathfrak{g}}(\mathfrak{z}))}.$

The following relation is immediate:

$$(7.4) \qquad\qquad Z_{F_{24}} - Z_{F_{24}}^{\text{exc}} = |\phi_{\mathfrak{g}}|^2.$$

In order to write out the exceptional modular invariants (7.3) explicitly, it is enough to express the full characters of F_{24} as $\mathbb{Z}$-linear combinations of full characters of the affine VOA $V_{\mathfrak{g}}$. We have calculated these expressions by cases and have found that the full character χ_{R} is always given as follows (see (2.6) for the notation):

$$(7.5) \qquad\qquad \chi_{\text{R}} = 2^{\text{rk}(\mathfrak{g})/2} \bigotimes_{j=1}^{s} \chi_{\rho_j, 3/2}^{\mathfrak{g}_j, k_j},$$

where ρ_j is the Weyl vector of $\mathfrak{g}_j$. Note that if we write $\phi_{\mathfrak{g}}$ as a $\mathbb{Z}$-linear combination of characters of $V_{\mathfrak{g}}$ then there is a unique negative term, and it is given by

$$\phi_{\mathfrak{g},-} = s \bigotimes_{j=1}^{s} \chi_{\rho_j, 3/2}^{\mathfrak{g}_j, k_j}.$$

We will see that

$$\chi_{\text{R}} \neq 2\phi_{\mathfrak{g},-}$$

if and only if $\mathfrak{g}$ is of type $A_{4,5}$ or $A_{2,3}^3$, in which cases there are certain imaginary roots that are simultaneously even and odd. Note that the q-order of χ_{R} is always 1.

The expressions of χ_{NS} and $\chi_{\widetilde{\text{NS}}}$ for the eight $\mathcal{N} = 1$ structures are given below.

Case $A_{1,2}^8$: The affine VOA of type $A_{1,2}$ has nonzero conformal weights $\frac{3}{16}$ and $\frac{1}{2}$ associated with the weights w_1 and $2w_1$, respectively. The (formal) fermionization of $A_{1,2}$ is well-known to be

$$\chi_{\text{NS}}^{A_{1,2}} = \chi_{0,0}^{A_{1,2}} + \chi_{2,\frac{1}{2}}^{A_{1,2}}, \quad \chi_{\widetilde{\text{NS}}}^{A_{1,2}} = \chi_{0,0}^{A_{1,2}} - \chi_{2,\frac{1}{2}}^{A_{1,2}}, \quad \chi_{\text{R}}^{A_{1,2}} = \sqrt{2}\chi_{1,\frac{3}{16}}^{A_{1,2}},$$

This theory describes three 2D chiral fermions. For the holomorphic SVOA of type $A_{1,2}^8$, the fermionic characters are naturally given by

$$\chi_{\text{NS}} = \bigotimes \chi_{\text{NS}}^{A_{1,2}}, \quad \chi_{\widetilde{\text{NS}}} = \bigotimes \chi_{\widetilde{\text{NS}}}^{A_{1,2}}, \quad \chi_{\text{R}} = \bigotimes \chi_{\text{R}}^{A_{1,2}}.$$

where the tensor products take over all eight copies of $A_{1,2}$. We remark that (7.3) for $A_{1,2}^8$ gives exactly the exceptional modular invariant found in [37, Equation (5.12b)].

Case $A_{2,3}^3$: We use the well-known conformal embedding $A_{2,3} \subset D_{4,1}$. Denote

$$\phi_0 = \chi_{00,0}^{A_{2,3}} + \chi_{03,1}^{A_{2,3}} + \chi_{30,1}^{A_{2,3}}, \qquad \phi_1 = \chi_{11,\frac{1}{2}}^{A_{2,3}}.$$

As an SVOA, the fermionization of $A_{2,3}$ has the fermionic characters

$$\chi_{\text{NS}}^{A_{2,3}} = \phi_0 + \phi_1, \quad \chi_{\widetilde{\text{NS}}}^{A_{2,3}} = \phi_0 - \phi_1, \quad \chi_{\text{R}}^{A_{2,3}} = 2\phi_1.$$

Therefore, for the holomorphic SVOA $A_{2,3}^3$, we have the fermionic characters

$$\chi_{\text{NS}} = \bigotimes \chi_{\text{NS}}^{A_{2,3}}, \quad \chi_{\widetilde{\text{NS}}} = \bigotimes \chi_{\widetilde{\text{NS}}}^{A_{2,3}}, \quad \chi_{\text{R}} = \bigotimes \chi_{\text{R}}^{A_{2,3}}.$$

where the tensor products run over the three copies of $A_{2,3}$.

Case $A_{4,5}$: Several modular invariants of $A_{4,5}$ were found in [99, Appendix B]. The one related to the conformal embedding $A_{4,5} \subset D_{12,1}$ is actually the fermionic modular invariant. Let us denote

$$\phi_0 = \sum_1 \chi^{A_{4,5}}_{0000,0}, \quad \phi_1 = \sum_1 \chi^{A_{4,5}}_{0102,1}, \quad \phi_2 = \sum_1 \chi^{A_{4,5}}_{1001,\frac{1}{2}},$$

$$\phi_3 = \sum_1 \chi^{A_{4,5}}_{0021,\frac{6}{5}}, \quad \phi_4 = \sum_1 \chi^{A_{4,5}}_{0110,\frac{4}{5}}, \quad \phi_5 = \chi^{A_{4,5}}_{1111,\frac{3}{2}}.$$

Here the glue vector 1 describes $\mathbb{Z}_5$ permutation on the affine Dynkin labels, i.e., $\hat{w}_i \to \hat{w}_{i-1}$ for $1 \leq i \leq 4$ and $\hat{w}_0 \to \hat{w}_4$. For the holomorphic SVOA, we find that the fermionic characters can be expressed as

$$\chi_{\mathrm{NS}} = \phi_0 + \phi_1 + \phi_2 + \phi_5, \quad \chi_{\widetilde{\mathrm{NS}}} = \phi_0 + \phi_1 - \phi_2 - \phi_5, \quad \chi_{\mathrm{R}} = 4\phi_5.$$

The modular invariant (7.2) is exactly the (B.6) modular invariant in [99]. The exceptional modular invariant (7.3) coincides with the (B.5) modular invariant in [99]. Moreover, the summation

$$Z_{F_{24}} + |\phi_{\mathfrak{g}}|^2 = |\phi_0 + \phi_1|^2 + 2|\phi_2|^2 + 10|\phi_5|^2$$

gives the (B.3) modular invariant in [99] (compare (7.4)). Let us comment on the extra modular invariants. The simple current extended modular invariant was given in [99, Equation (B.2)] as

$$Z_{(\mathrm{B.2})} = \sum_{i=0}^{4} |\phi_i|^2 + 5|\phi_5|^2.$$

In addition, a different exceptional modular invariant was given in [99, Equation (B.4)] as

$$Z_{(\mathrm{B.4})} = |\phi_0|^2 + |\phi_1|^2 + |\phi_3|^2 + |\phi_4|^2 + 4|\phi_5|^2 + (\phi_2\bar{\phi}_5 + c.c.).$$

These two modular invariants are related by

$$Z_{(\mathrm{B.2})} - Z_{(\mathrm{B.4})} = |\phi_{\mathfrak{g}}|^2.$$

(Compare (7.4).)

Case $A_{3,4}A_{1,2}^3$: In this case we use the conformal embedding $A_{3,4} \subset B_{7,1}$. We find that the fermionic characters can be expressed as

$$\chi_{\mathrm{NS}} = \left(\sum_1 \left(\chi^{A_{3,4}}_{000,*} + \chi^{A_{3,4}}_{101,*} \right) \right) \otimes \bigotimes \left(\chi^{A_{1,2}}_{0,0} + \chi^{A_{1,2}}_{2,\frac{1}{2}} \right),$$

$$\chi_{\widetilde{\mathrm{NS}}} = \left(\sum_{1,even} \left(\chi^{A_{3,4}}_{000,*} + \chi^{A_{3,4}}_{101,*} \right) - \sum_{1,odd} \left(\chi^{A_{3,4}}_{000,*} + \chi^{A_{3,4}}_{101,*} \right) \right) \otimes \bigotimes \left(\chi^{A_{1,2}}_{0,0} - \chi^{A_{1,2}}_{2,\frac{1}{2}} \right).$$

The tensor products run over the 3 copies of $A_{1,2}$. The glue vector 1 describes the $\mathbb{Z}_4$ permutation on the affine Dynkin labels, i.e., $\hat{w}_i \to \hat{w}_{i-1}$ and $\hat{w}_0 \to \hat{w}_3$. The subscript *even* means the summation is taken over all primaries generated by the simple current 1 with integral conformal weights, while *odd* means all those generated by the simple current 1 with half-integral conformal weights.

Case $B_{2,3}G_{2,4}$: Consider the decomposition $D_{5,1} \times D_{7,1} \subset D_{12,1}$ and the conformal embeddings $B_{2,3} \subset D_{5,1}$ and $G_{2,4} \subset D_{7,1}$. We find the following formulas for the fermionic characters:

$$\chi_{\mathrm{NS}} = \left(\sum_{even} \chi^{B_{2,3}} + \sum_{odd} \chi^{B_{2,3}} \right) \otimes \left(\sum_{even} \chi^{G_{2,4}} + \sum_{odd} \chi^{G_{2,4}} \right),$$

$$\chi_{\widetilde{\mathrm{NS}}} = \left(\sum_{even} \chi^{B_{2,3}} - \sum_{odd} \chi^{B_{2,3}} \right) \otimes \left(\sum_{even} \chi^{G_{2,4}} - \sum_{odd} \chi^{G_{2,4}} \right).$$

Here *even* means the summation is taken over all primaries with integral conformal weights, while *odd* means all those with half-integral conformal weights. For both $B_{2,3}$ and $G_{2,4}$, the even part

contains two primaries with conformal weights 0 and 1, while the odd part contains two primaries with conformal weights $1/2$ and $3/2$.

Case $B_{2,3}A_{2,3}A_{1,2}^2$: We use the conformal embeddings from the previous cases and find

$$\chi_{\mathrm{NS}} = \Big(\sum_{even} \chi^{B_{2,3}} + \sum_{odd} \chi^{B_{2,3}} \Big) \otimes \big(\chi_{00,0}^{A_{2,3}} + \chi_{03,1}^{A_{2,3}} + \chi_{30,1}^{A_{2,3}} + \chi_{11,\frac{1}{2}}^{A_{2,3}} \big) \otimes \bigotimes \big(\chi_{0,0}^{A_{1,2}} + \chi_{2,\frac{1}{2}}^{A_{1,2}} \big),$$

$$\chi_{\widetilde{\mathrm{NS}}} = \Big(\sum_{even} \chi^{B_{2,3}} - \sum_{odd} \chi^{B_{2,3}} \Big) \otimes \big(\chi_{00,0}^{A_{2,3}} + \chi_{03,1}^{A_{2,3}} + \chi_{30,1}^{A_{2,3}} - \chi_{11,\frac{1}{2}}^{A_{2,3}} \big) \otimes \bigotimes \big(\chi_{0,0}^{A_{1,2}} - \chi_{2,\frac{1}{2}}^{A_{1,2}} \big).$$

Here *even* and *odd* are defined as in the case of $B_{2,3}G_{2,4}$.

Case $B_{3,5}A_{1,2}$: We use the conformal embedding $B_{3,5} \subset B_{10,1}$ and find

$$\chi_{\mathrm{NS}} = \Big(\sum_{even} \chi^{B_{3,5}} + \sum_{odd} \chi^{B_{3,5}} \Big) \otimes \big(\chi_{0,0}^{A_{1,2}} + \chi_{2,\frac{1}{2}}^{A_{1,2}} \big),$$

$$\chi_{\widetilde{\mathrm{NS}}} = \Big(\sum_{even} \chi^{B_{3,5}} - \sum_{odd} \chi^{B_{3,5}} \Big) \otimes \big(\chi_{0,0}^{A_{1,2}} - \chi_{2,\frac{1}{2}}^{A_{1,2}} \big).$$

Here the even part of $B_{3,5}$ contains four primaries with conformal weights $0, 1, 2, 2$, while the odd part contains four primaries with conformal weights $1/2, 3/2, 3/2, 5/2$.

Case $C_{3,4}A_{1,2}$: This case is very similar to $B_{3,5}A_{1,2}$. We find

$$\chi_{\mathrm{NS}} = \Big(\sum_{even} \chi^{C_{3,4}} + \sum_{odd} \chi^{C_{3,4}} \Big) \otimes \big(\chi_{0,0}^{A_{1,2}} + \chi_{2,\frac{1}{2}}^{A_{1,2}} \big),$$

$$\chi_{\widetilde{\mathrm{NS}}} = \Big(\sum_{even} \chi^{C_{3,4}} - \sum_{odd} \chi^{C_{3,4}} \Big) \otimes \big(\chi_{0,0}^{A_{1,2}} - \chi_{2,\frac{1}{2}}^{A_{1,2}} \big).$$

Here the even part of $C_{3,4}$ contains four primaries with conformal weights $0, 1, 2, 3$, while the odd part contains four primaries with conformal weights $1/2, 3/2, 3/2, 5/2$.

8. The symmetric case: four exotic CFTs

In this section, we construct hyperbolizations of affine Kac–Moody algebras $\hat{\mathfrak{g}}$ for the remaining 4 semi-simple Lie algebras $\mathfrak{g}$ of symmetric type with $C < 1$ in Theorem 5.1. The data for these 4 cases is given in Table 2. Note that in these cases the lattices $\mathbf{P}_{\mathfrak{g}}$ (see Notation 2.3) are even.

$\mathfrak{g}$	$A_{1,16}$	$A_{1,8}^2$	$A_{1,4}^4$	$A_{2,9}$
$\mathbf{P}_{\mathfrak{g}}$	$A_1(4)$	$2A_1(2)$	$4A_1$	$A_2(3)$
C	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{3}$
$c_{\mathfrak{g}}$	$\frac{8}{3}$	$\frac{24}{5}$	8	6

Table 2. The 4 exotic cases related to exceptional modular invariants.

The 4 exceptional $\mathfrak{g}$ correspond to 4 exotic 2D CFTs that already made appearances in the physics literature around three decades ago. The common feature of these 4 exotic 2D CFTs is that they possess certain exceptional modular invariants that come from the nontrivial automorphisms of the fusion algebra of the simple current modular invariants. This accidental phenomenon was first found by Moore and Seiberg for $A_{1,16}$ and $A_{2,9}$ [87], later by Verstegen for $A_{1,8}^2$ [103] and finally

by Gannon for $A_{1,4}^4$ [37]. Our new observation here is that the two stories are actually deeply connected; that is, both the existence of reflective Borcherds products of singular weight and the existence of exceptional modular invariants come from the same identities among the characters of the affine Kac–Moody algebras.

Let $\Delta_{\mathfrak{g}}^+$ be the set of positive roots of $\mathfrak{g}$. Note that $\dim \mathfrak{g}/C = 24$. Similarly to the previous section, we work with the lattice embedding

$$\iota_{\mathfrak{g}}: \quad \mathbf{P}_{\mathfrak{g}} \hookrightarrow D_{12}, \quad \mathfrak{z} \mapsto \iota_{\mathfrak{g}}(\mathfrak{z}) = \{1/C \text{ copies of } \langle \mathfrak{z}, \alpha \rangle\}_{\alpha \in \Delta_{\mathfrak{g}}^+},$$

such that the following identity holds:

$$(\mathfrak{z}, \mathfrak{z}) = (\iota_{\mathfrak{g}}(\mathfrak{z}), \iota_{\mathfrak{g}}(\mathfrak{z})) = \frac{1}{C} \sum_{\alpha \in \Delta_{\mathfrak{g}}^+} \langle \alpha, \mathfrak{z} \rangle^2.$$

Theorem 8.1. *For each $\mathfrak{g}$ in Table 2, we define the pullback*

$$\phi_{\mathfrak{g}}(\tau, \mathfrak{z}) := C \cdot \phi_{D_{12}}(\tau, \iota_{\mathfrak{g}}(\mathfrak{z})),$$

where $\phi_{D_{12}}$ is defined in Lemma 7.2. Then the theta lift $\mathbf{B}(\phi_{\mathfrak{g}})$ is a reflective Borcherds product of singular weight on $2U \oplus \mathbf{P}_{\mathfrak{g}}$. Morover, the leading Fourier–Jacobi coefficient of $\mathbf{B}(\phi_{\mathfrak{g}})$ coincides with the denominator of $\hat{\mathfrak{g}}$.

Proof. By definition, we find that

$$\phi_{\mathfrak{g}}(\tau, \mathfrak{z}) = \mathrm{rk}(\mathfrak{g}) + \sum_{\alpha \in \Delta_{\mathfrak{g}}^+} \left(e^{2\pi i \langle \alpha, \mathfrak{z} \rangle} + e^{-2\pi i \langle \alpha, \mathfrak{z} \rangle} \right) + O(q)$$

and this is a weak Jacobi form of weight 0 and lattice index $\mathbf{P}_{\mathfrak{g}}$. It is easy to check that $\delta_L = 2$ for these four cases. From Remark 2.6, it follows that all singular Fourier coefficients of $\phi_{\mathfrak{g}}(\tau, \mathfrak{z})$ appear in the q^0-term given above. Therefore, the theta lift is reflective and of singular weight, and its leading Fourier–Jacobi coefficient has the desired form. $\square$

The next theorem expresses the inputs of the Borcherds products as $\mathbb{Z}$-linear combinations of the full characters of the affine VOA generated by $\hat{\mathfrak{g}}$.

Theorem 8.2. *The following identities hold:*

$$\phi_{A_{1,16}} = \chi_{2,\frac{1}{9}}^{A_{1,16}} + \chi_{14,\frac{28}{9}}^{A_{1,16}} - \chi_{8,\frac{10}{9}}^{A_{1,16}},$$

$$\phi_{A_{1,8}^2} = \left(\chi_{0,0}^{A_{1,8}} + \chi_{8,2}^{A_{1,8}} \right) \otimes \left(\chi_{2,\frac{1}{5}}^{A_{1,8}} + \chi_{6,\frac{6}{5}}^{A_{1,8}} \right) + \left(\chi_{2,\frac{1}{5}}^{A_{1,8}} + \chi_{6,\frac{6}{5}}^{A_{1,8}} \right) \otimes \left(\chi_{0,0}^{A_{1,8}} + \chi_{8,2}^{A_{1,8}} \right) - 2\chi_{4,\frac{3}{5}}^{A_{1,8}} \otimes \chi_{4,\frac{3}{5}}^{A_{1,8}},$$

$$\phi_{A_{2,9}} = \chi_{11,\frac{1}{4}}^{A_{2,9}} + \chi_{17,\frac{9}{4}}^{A_{2,9}} + \chi_{71,\frac{9}{4}}^{A_{2,9}} - \chi_{33,\frac{5}{4}}^{A_{2,9}},$$

$$\phi_{A_{1,4}^4} = \sum_{cyclic} \left(\chi_{0,0}^{A_{1,4}} + \chi_{4,1}^{A_{1,4}} \right) \otimes \left(\chi_{0,0}^{A_{1,4}} + \chi_{4,1}^{A_{1,4}} \right) \otimes \left(\chi_{0,0}^{A_{1,4}} + \chi_{4,1}^{A_{1,4}} \right) \otimes \chi_{2,\frac{1}{3}}^{A_{1,4}} - 4 \bigotimes \chi_{2,\frac{1}{3}}^{A_{1,4}},$$

where the sum contains 4 terms by permutation. By taking $\mathfrak{z} = 0$ we obtain the identities among unflavored characters given by

$$\phi_{\mathfrak{g}}(\tau, 0) = \dim \mathfrak{g}.$$

Proof. First one proves that the $\mathbb{Z}$-linear combinations of affine characters above define Jacobi forms by checking their transformations under the action of the generators of $\mathrm{SL}_2(\mathbb{Z})$. To prove the identities, it is enough to check that their q^0-terms match, since a weak Jacobi form of weight -12 and lattice index $\mathbf{P}_{\mathfrak{g}}$ has to be zero (see e.g. [104]). $\square$

The embedding $\iota_{\mathfrak{g}}$ for $\mathfrak{g} = A_{2,9}$ induces an embedding $A_2(3) \hookrightarrow 3A_2$. By considering the corresponding pullback we obtain an identity between affine characters of type A_2 at levels 3 and 9:

$$\left(\chi_{00,0}^{A_{2,3}} + \chi_{03,1}^{A_{2,3}} + \chi_{30,1}^{A_{2,3}}\right)^2 \chi_{11,\frac{1}{2}}^{A_{2,3}} - \left(\chi_{11,\frac{1}{2}}^{A_{2,3}}\right)^3 = \chi_{11,\frac{1}{4}}^{A_{2,9}} + \chi_{17,\frac{9}{4}}^{A_{2,9}} + \chi_{71,\frac{9}{4}}^{A_{2,9}} - \chi_{33,\frac{5}{4}}^{A_{2,9}}.$$

Both sides of this identity reduce to the constant 8 if we set $\mathfrak{z} \in A_2 \otimes \mathbb{C}$ equal to 0. Similarly, by considering the embedding

$$A_{1,16} \to A_{1,8}^2 \to A_{1,4}^4 \to A_{1,2}^8,$$
$$A_1(4) \hookrightarrow 2A_1(2) \hookrightarrow 4A_1 \hookrightarrow D_8,$$

we obtain identities between affine characters of type A_1 at levels 2, 4, 8 and 16:

$$\chi_{2,\frac{1}{9}}^{A_{1,16}} + \chi_{14,\frac{28}{9}}^{A_{1,16}} - \chi_{8,\frac{10}{9}}^{A_{1,16}}$$
$$= \left(\chi_{0,0}^{A_{1,8}} + \chi_{8,2}^{A_{1,8}}\right)\left(\chi_{2,\frac{1}{5}}^{A_{1,8}} + \chi_{6,\frac{6}{5}}^{A_{1,8}}\right) - \left(\chi_{4,\frac{3}{5}}^{A_{1,8}}\right)^2$$
$$= \left(\chi_{0,0}^{A_{1,4}} + \chi_{4,1}^{A_{1,4}}\right)^3 \chi_{2,\frac{1}{3}}^{A_{1,4}} - \left(\chi_{2,\frac{1}{3}}^{A_{1,4}}\right)^4$$
$$= \left(\chi_{0,0}^{A_{1,2}}\right)^7 \chi_{2,\frac{1}{2}}^{A_{1,2}} + 7\left(\chi_{0,0}^{A_{1,2}}\right)^5 \left(\chi_{2,\frac{1}{2}}^{A_{1,2}}\right)^3 + 7\left(\chi_{0,0}^{A_{1,2}}\right)^3 \left(\chi_{2,\frac{1}{2}}^{A_{1,2}}\right)^5 + \chi_{0,0}^{A_{1,2}} \left(\chi_{2,\frac{1}{2}}^{A_{1,2}}\right)^7 - \left(\chi_{1,\frac{3}{16}}^{A_{1,2}}\right)^8.$$

All four expressions reduce to the constant 3 if we set $\mathfrak{z} \in A_1 \otimes \mathbb{C}$ equal to 0.

Remark 8.3. The $\mathbb{Z}$-lattice generated by those $\lambda = (n, \ell, m) \in U \oplus \mathbf{P}'_{\mathfrak{g}}$ for which $\mathbf{B}(\phi_{\mathfrak{g}})$ vanishes on $\lambda^\perp$ is exactly $U \oplus \mathbf{P}'_{\mathfrak{g}}$.

Remark 8.4. The Borcherds product $\mathbf{B}(\phi_{\mathfrak{g}})$ for $\mathfrak{g} = A_{1,16}$ was first constructed by Gritsenko and Nikulin [47, §1.4] in 1998. They showed that this product equals a certain even theta constant when viewed as a Siegel paramodular form of genus 2 and level 4. They also studied the corresponding BKM superalgebra [47, Section 5.1].

Remark 8.5. As we mentioned at the beginning of this section, the inputs $\phi_{\mathfrak{g}}$ are related to exceptional modular invariants.

(1) For $A_{1,16}$, there exist the D_8 modular invariant [87, Equation (5.8)] by simple current extension and the E_7 exceptional modular invariant [87, Equation (5.11)]. They satisfy the identity

$$Z_{D_{10}} - Z_{E_7} = |\phi_{A_{1,16}}|^2$$

(2) For $A_{2,9}$, there exist the $\mathcal{D}_9$ modular invariant [87, Equation (5.12)] by simple current extension and the $\mathcal{E}'_9$ exceptional modular invariant [87, Equation (5.14)]. They are related by the identity

$$Z_{\mathcal{D}_9} - Z_{\mathcal{E}'_9} = |\phi_{A_{2,9}}|^2.$$

(3) For $A_{1,8}^2$, there exist the $2D_6$ modular invariant by simple current extension for each $A_{1,8}$ [87, Equation (5.8)] and the exceptional modular invariant [103, Equation (4.1a)].

(4) For $A_{1,4}^4$, there exist the $4D_4$ modular invariant by simple current extension for each $A_{1,4}$ [87, Equation (5.8)] and the exceptional modular invariant [37, Equation (5.12a)].

(5) In cases (3)-(4), the difference of the simple current modular invariant and the exceptional modular invariant is given by $\sum_j |\phi_{\mathfrak{g},j}|^2$ for a suitable finite decomposition $\phi_{\mathfrak{g}} = \sum_j \phi_{\mathfrak{g},j}$.

Remark 8.6. Theorem 8.2 relates the Borcherds products of Theorem 8.1 to vertex algebras. However, we do not know whether they are super-denominators of BKM superalgebras that can be constructed as BRST cohomology related to vertex algebras.

9. The 12 symmetric cases: the construction as additive lifts

Some Borcherds products are known to have constructions as additive lifts. Since additive lifts are always invariant under the involution $(\omega, \mathfrak{z}, \tau) \to (\tau, \mathfrak{z}, \omega)$, we cannot hope to express anti-symmetric Borcherds products as additive lifts. In this section, we will give a uniform construction of the 12 symmetric Borcherds products as additive lifts. The inputs into the additive lift turn out to be the denominators of the associated affine Kac–Moody algebras $\hat{\mathfrak{g}}$.

Recall that for each $\mathfrak{g}$ in Table 8 or Table 2 we have constructed a reflective Borcherds product $\mathbf{B}(\phi_{\mathfrak{g}})$ of singular weight on $2U \oplus L_{\mathfrak{g}}$ in the previous two sections, where $L_{\mathfrak{g}}$ is the maximal even sublattice of $\mathbf{P}_{\mathfrak{g}}$. The product $\mathbf{B}(\phi_{\mathfrak{g}})$ has a Fourier–Jacobi expansion of the form

$$\mathbf{B}(\phi_{\mathfrak{g}})(\omega, \mathfrak{z}, \tau) = \vartheta_{\mathfrak{g}}(\tau, \mathfrak{z}) \cdot e^{2\pi i C \omega} \cdot \exp\left(-\sum_{m=1}^{\infty} \left(\phi_{\mathfrak{g}}|_0 T_-^{(1)}(m)\right)(\tau, \mathfrak{z}) \cdot e^{2\pi i m \omega} \right).$$

When $\mathfrak{g}$ is not of type $A_{1,16}$, the weight of $\vartheta_{\mathfrak{g}}$ is integral, so we can define the additive lift of $\vartheta_{\mathfrak{g}}$ following Theorem 2.9:

$$\mathbf{G}(\vartheta_{\mathfrak{g}})(\omega, \mathfrak{z}, \tau) = \sum_{0 < m \in 1 + \frac{1}{C}\mathbb{Z}} \left(\vartheta_{\mathfrak{g}}|_{\frac{1}{2}\mathrm{rk}(\mathfrak{g})} T_-^{(\frac{1}{C})}(m) \right)(\tau, \mathfrak{z}) \cdot e^{2\pi i m C \omega}$$

When $\mathfrak{g} = A_{1,16}$, we have $\vartheta_{\mathfrak{g}}(\tau, \mathfrak{z}) = \vartheta(\tau, z)$. In this case, we use Gritsenko's "trivial lifting" of ϑ, defined in [47, Theorem 1.11] to be

$$\mathbf{G}(\vartheta)(\omega, z, \tau) = \sum_{m=1}^{\infty} \left(\frac{-4}{m} \right) \vartheta(\tau, mz) \cdot e^{2\pi i m^2 \omega / 8}.$$

Theorem 9.1. *Let $\mathfrak{g}$ be one of the semi-simple Lie algebras in Table 8 or Table 2. Then we have*

$$\mathbf{B}(\phi_{\mathfrak{g}}) = \mathbf{G}(\vartheta_{\mathfrak{g}}).$$

Proof. The identity follows from Koecher's principle if we can show that the divisor of the additive lift contains the divisor of the Borcherds product. The 8 cases with $C = 1$ were proved in [24, Theorem 5.1], and a much simpler proof can be found in [112, Corollary 4.5]. In particular, the identities for $\mathfrak{g} = A_{1,2}^8$ and $A_{2,3}^3$ were first established by Gritsenko in [49, Theorems 3.2, 4.2].

The case $\mathfrak{g} = A_{1,16}$ was proved in [47, Example 2.3], the case $\mathfrak{g} = A_{1,4}^4$ was proved in [49, Theorem 5.1], and the case $\mathfrak{g} = A_{2,9}$ was proved in [55, Theorem 13.5 (2)]. The remaining case $\mathfrak{g} = A_{1,8}^2$ can be proved in a similar way; we will give a sketch. Here we have $L_{\mathfrak{g}} = 2A_1(2)$ and the number δ_L defined in Remark 2.6 is 2. Therefore, all of the singular Fourier coefficients of $\phi_{\mathfrak{g}}$ already appear in its q^0-term. By an argument analogous to [53, Remark 3.4, Lemma 3.5], $\mathbf{G}(\vartheta_{\mathfrak{g}})/\mathbf{B}(\phi_{\mathfrak{g}})$ is holomorphic and therefore constant. $\square$

Compare the first two Fourier–Jacobi coefficients of the Borcherds products and additive lifts,

$$\mathbf{B}(\phi_{\mathfrak{g}}) = \vartheta_{\mathfrak{g}}(\tau, \mathfrak{z}) \cdot \zeta^C - \vartheta_{\mathfrak{g}}(\tau, \mathfrak{z})\phi_{\mathfrak{g}}(\tau, \mathfrak{z}) \cdot \zeta^{C+1} + O(\zeta^{C+2}),$$

$$\mathbf{G}(\vartheta_{\mathfrak{g}}) = \vartheta_{\mathfrak{g}}(\tau, \mathfrak{z}) \cdot \zeta^C + \left(\vartheta_{\mathfrak{g}}|_{\frac{1}{2}\mathrm{rk}(\mathfrak{g})} T_-^{(\frac{1}{C})}(1/C + 1) \right)(\tau, \mathfrak{z}) \cdot \zeta^{C+1} + O(\zeta^{C+2}),$$

where for $\mathfrak{g} = A_{1,16}$ we define

$$\left(\vartheta|_{\frac{1}{2}} T_-^{(8)}(9) \right)(\tau, z) = -\vartheta(\tau, 3z).$$

This implies the following uniform expression.

Corollary 9.2. *Let $\mathfrak{g}$ be one of the semi-simple Lie algebras in Table 8 or Table 2. Then the input can be expressed as*

$$\phi_{\mathfrak{g}}(\tau,\mathfrak{z}) = -\frac{\left(\vartheta_{\mathfrak{g}}|_{\frac{1}{2}\,\mathrm{rk}(\mathfrak{g})}T_{-}^{(\frac{1}{C})}(1/C+1)\right)(\tau,\mathfrak{z})}{\vartheta_{\mathfrak{g}}(\tau,\mathfrak{z})}.$$

Remark 9.3. This corollary shows that in the symmetric cases the input, which was originally constructed as a $\mathbb{Z}$-linear combination of characters of some vertex algebra, can be read off of the denominator of $\hat{\mathfrak{g}}$. There is no expression of this kind in the anti-symmetric cases.

Remark 9.4. Recall from Corollary 4.3 that Equation (4.2) have 17 solutions. The 12 solutions of Equation (4.2) that correspond to symmetric cases with hyperbolizations are characterized by $1/C$ being integral. This is related to the expression of the input $\phi_{\mathfrak{g}}$ as a $\mathbb{Z}$-linear combination of characters of the affine VOA generated by $\hat{\mathfrak{g}}$. This expression contains a unique term with negative coefficients, namely

$$s \cdot \bigoplus_{j=1}^{s} \chi_{k_j \rho_j/h_j^{\vee}}^{\mathfrak{g}_{j,k_j}}.$$

The condition that $1/C$ is integral guarantees that the dominant integral weight $k_j \rho_j/h_j^{\vee}$ is well-defined. The equality $\dim\mathfrak{g}/24 = C = h_j^{\vee}/k_j$ implies that the first nonzero Fourier coefficient in the above character is q^1. We do not have a conceptual explanation for this. A similar phenomenon for exceptional modular invariants has been observed by Schellekens and Yankielowicz [99, p.100].

10. Fourier expansions of singular-weight reflective Borcherds products

In this section we will work out the Fourier expansions of singular-weight reflective Borcherds products as in [9, Example 13.7], [93, Section 9] and [25, Section 5].

Let F be a reflective Borcherds product of singular weight on $U_1 \oplus U \oplus L$ and set $K = U \oplus L$. Assume that the input as a Jacobi form,

$$\phi(\tau,\mathfrak{z}) = \sum_{n\in\mathbb{Z}}\sum_{\ell\in L'} f(n,\ell)q^n\zeta^{\ell},$$

has only non-negative singular Fourier coefficients. We will consider the Fourier expansion of F at the 0-dimensional cusp related to U_1. Let W be the Weyl group, i.e. the subgroup of $\mathrm{O}(U \oplus L)$ generated by reflections associated with vectors $(n,\ell,m) \in U \oplus L'$ for which $2n < \ell^2$ and $f(nm,\ell) > 0$. We fix a Weyl chamber $\mathcal{C}$ of F and denote its closure by $\overline{\mathcal{C}}$. Let ρ be the corresponding Weyl vector.

We will view F as a function on the associated tube domain. This is anti-invariant under the Weyl group W, i.e.

$$F(\sigma(Z)) = \det(\sigma)F(Z), \quad \text{for all } \sigma \in W.$$

It is known that W acts simply transitively on the Weyl chambers of the negative cone of $K \otimes \mathbb{R}$. Therefore, the Fourier expansion of F has the form

$$F(Z) = \sum_{\sigma\in W}\det(\sigma)\sum_{\substack{\lambda\in K',\,\lambda+\rho\in\overline{\mathcal{C}}\\(\lambda,\mathcal{C})<0}} c(\lambda)\mathbf{e}\Big(\sigma(\lambda+\rho),Z\Big),$$

where $\mathbf{e}(t) = e^{2\pi i t}$ as before. Since F has singular weight, $c(\lambda) = 0$ whenever $(\lambda+\rho,\lambda+\rho)\neq 0$. Note that $(\lambda,\mathcal{C}) < 0$ and $\rho\in\overline{\mathcal{C}}$ together imply that $(\lambda,\rho)\leq 0$. But on the other hand, if $(\lambda+\rho,\lambda+\rho) = 0$, then we have

$$2(\lambda,\rho) = (\lambda+\rho,\lambda+\rho) - (\lambda,\lambda) - (\rho,\rho) = -\lambda^2$$

and
$$(\lambda, \rho) = (\lambda, \lambda + \rho) - (\lambda, \lambda) \leq -\lambda^2,$$
since $(\lambda, \lambda + \rho) \leq 0$. Since $-\lambda^2/2 \leq -\lambda^2$, $\lambda^2 \leq 0$ and therefore $(\lambda, \rho) \geq 0$, so in this case $(\lambda, \rho) = 0$.

Write λ and ρ in coordinates as $(x_1, x_2, ..., x_l, x_0) \in \mathbb{R}^{l,1}$ and $(y_1, y_2, ..., y_r, y_0) \in \mathbb{R}^{l,1}$ respectively, such that
$$x_1^2 + x_2^2 + ... + x_r^2 - x_0^2 = (\lambda, \lambda) \leq 0,$$
$$y_1^2 + y_2^2 + ... + y_r^2 - y_0^2 = (\rho, \rho) = 0,$$
$$x_1 y_1 + x_2 y_2 + ... + x_r y_r - x_0 y_0 = (\lambda, \rho) = 0.$$

By the Cauchy–Schwarz inequality, we have
$$x_0^2 y_0^2 = \Big(\sum_{j=1}^{r} x_j y_j \Big)^2 \leq \sum_{j=1}^{r} x_j^2 \sum_{j=1}^{r} y_j^2 \leq x_0^2 y_0^2,$$
from which it follows that $\lambda = c\rho \in K'$ for some $c \in \mathbb{Q}$. Note that $(v, \mathcal{C}) < 0$ for $v \in \mathcal{C}$. Since $(\lambda, \mathcal{C}) < 0$ and $\rho \in \overline{\mathcal{C}}$, we find $c \geq 0$.

Write $\lambda = c\rho = \sum \lambda_j$ with $\lambda_j \in K'$ and $(\lambda_j, \mathcal{C}) < 0$, and suppose that these λ_j contribute to $c(\lambda)$ in the product expansion of F at U_1. From $(\lambda_j, \rho) \leq 0$ and $\sum (\lambda_j, \rho) = (c\rho, \rho) = 0$ we obtain $(\lambda_j, \rho) = 0$.

If $(\lambda_j, \lambda_j) \leq 0$, then a similar argument as before shows that λ_j is a positive rational multiple of ρ. Otherwise, suppose $(\lambda_j, \lambda_j) > 0$ for some j. Then F vanishes on $\lambda_j^\perp$ and the reflection σ_{λ_j} lies in the Weyl group W. Since σ_{λ_j} maps $\mathcal{C}$ to a new Weyl chamber, it maps ρ to the corresponding new Weyl vector, i.e. $\sigma_{\lambda_j}(\rho) \neq \rho$. It follows that $(\lambda_j, \rho) \neq 0$, leading to a contradiction. Therefore, every λ_j is a positive rational multiple of ρ and $\lambda_j \in K'$. If we choose a positive $c \in \mathbb{Q}$ such that $c\rho$ is a primitive vector in K', then every λ_j is a positive integral multiple of $c\rho$.

This gives us the following expression:

$$F(Z) = \sum_{\sigma \in W} \det(\sigma) \, \mathbf{e}\big((\sigma(\rho), Z)\big) \prod_{n=1}^{\infty} \Big(1 - \mathbf{e}\big((\sigma(nc\rho), Z)\big) \Big)^{f(n^2 c^2 \rho^2/2, \, nc\rho)}$$

(10.1)
$$= \sum_{\sigma \in W} \det(\sigma) \, \sigma \left(\mathbf{e}\big((\rho, Z)\big) \prod_{n=1}^{\infty} \Big(1 - \mathbf{e}\big(nc(\rho, Z)\big) \Big)^{f(n^2 c^2 \rho^2/2, \, nc\rho)} \right)$$

$$=: \sum_{\sigma \in W} \det(\sigma) \, \sigma \Big(\Delta_F \big((\rho, Z)\big) \Big).$$

The value of $f(n^2 c^2 \rho^2/2, \, nc\rho)$ depends only on the coset of $nc\rho$ in K'/K, and
$$f(*, nc\rho) = f(*, -nc\rho) = f(*, mc\rho)$$
if $(n+m)c\rho \in K$. Note that the function Δ_F above is often an eta quotient.

We will now work out the expressions of type (10.1) for the singular-weight reflective Borcherds products constructed in the previous sections. Let $\mathfrak{g} = \oplus_j \mathfrak{g}_{j,k_j}$ be one of the 81 semi-simple Lie algebras with a hyperbolization. Let $\Psi_{\mathfrak{g}}$ be the corresponding singular-weight reflective Borcherds product on $2U \oplus L_{\mathfrak{g}}$. Using the notation of Theorem 2.8 and Theorem 4.1, the Weyl vector of $\Psi_{\mathfrak{g}}$ is a norm zero vector given by
$$(-C - 1, \, \rho_{\mathfrak{g}}, \, -C)$$
in the anti-symmetric cases, and by
$$(-C, \, \rho_{\mathfrak{g}}, \, -C)$$

in the symmetric cases, where $\rho_{\mathfrak{g}}$ is the normalized Weyl vector of $\mathfrak{g}$ defined by

$$\rho_{\mathfrak{g}} = \sum_j \rho_j / k_j,$$

where ρ_j is the Weyl vector of the simple Lie algebra $\mathfrak{g}_j$. The function Δ_F that appears in (10.1) for $\Psi_{\mathfrak{g}}$ can be described as follows.

(1) Let $\mathfrak{g}$ be of anti-symmetric type. Then $\mathfrak{g}$ corresponds to a Co_0-conjugacy class denoted $[g]$. Let $\prod_{b_k \neq 0} k^{b_k}$ be the cycle shape of g and define the associated eta quotient by

$$\eta_g(\tau) := \prod_{k \geq 1} \eta(k\tau)^{b_k}, \quad \eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n).$$

Then

$$\Delta_{\Psi_{\mathfrak{g}}}\big((\rho, Z)\big) = \eta_g\big((\rho, Z)\big).$$

Here the number c in (10.1) is 1 if g has order equal to its level, and it is 2 otherwise.

(2) Let $\mathfrak{g}$ be of symmetric type with $C = 1$. Then $\mathfrak{g}$ also corresponds to a Co_0-conjugacy class denoted $[g]$. The number c in (10.1) is always 1, and

$$\Delta_{\Psi_{\mathfrak{g}}}\big((\rho, Z)\big) = \eta_g\big((\rho, Z)\big).$$

(3) Let $\mathfrak{g}$ be of symmetric type with $C < 1$. In this case, $\mathfrak{g}$ does not correspond to any Co_0-conjugacy class. However, we obtain a similar expression for Δ_F as an eta quotient associated to a (fake) cycle shape. More precisely,

$$\Delta_{\Psi_{\mathfrak{g}}}\big((\rho, Z)\big) = \eta_g\big((\rho, Z)\big),$$

where the (fake) cycle shapes of g are given in Table 3. The number c in (10.1) equals $1/C$, which also appears in Table 3.

$\mathfrak{g}$	$A_{1,16}$	$A_{1,8}^2$	$A_{1,4}^4$	$A_{2,9}$
c	8	4	2	3
g	$8^{-1}16^2$	$4^{-2}8^4$	$2^{-4}4^8$	$3^{-1}9^3$

TABLE 3. The (fake) cycle shapes of the 4 exceptional semi-simple Lie algebras

11. THE CLASSIFICATION OF ANTI-SYMMETRIC ROOT SYSTEMS

In this section we prove Theorem 5.1 in the anti-symmetric cases.

Theorem 11.1. *If $2U \oplus L$ has an anti-symmetric reflective Borcherds product of singular weight whose Jacobi form input has non-negative q^0-term, then the associated semi-simple Lie algebra $\mathfrak{g}$ defined in Theorem 4.1 lies in Schellekens' list of 69 semi-simple V_1 structures of holomorphic vertex operator algebras of central charge 24.*

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11.1. **The main argument.** To explain the proof of Theorem 11.1, we begin by introducing a useful concept.

Definition 11.2. An even positive definite lattice K is called a *forbidden component* if there is no reflective Borcherds product of any weight on $2U \oplus K$. By considering its symmetrization, we find that K is forbidden if and only if there is no reflective Borcherds product of any weight that is modular for the full orthogonal group $\mathrm{O}^+(2U \oplus K)$.

Let $F = \mathbf{B}(f)$ be the potential Borcherds product in Theorem 11.1. By [110, Lemma 3.3], for any even overlattice L_1 of L, there exists an anti-symmetric reflective Borcherds product of some weight on $2U \oplus L_1$. Moreover, if there is a decomposition

$$2U \oplus L_1 \cong 2U \oplus L_2 \oplus L_3,$$

then the quasi-pullback will yield an anti-symmetric reflective Borcherds product on $2U \oplus L_2$.

To rule out the existence of a singular, reflective, anti-symmetric Borcherds product on $2U \oplus L$, it is therefore sufficient to find an overlattice L_1 of L that admits a direct sum decomposition $L_1 \cong K \oplus L_0$ with a forbidden component K.

Argument 11.3. To exclude the 152 extraneous root systems, we use the bounds

$$\mathbf{Q}_{\mathfrak{g}} < L < \mathbf{P}_{\mathfrak{g}}$$

and the property that $L(C)$ is integral to determine an even overlattice $K = K_1 \oplus K_2$ of L that contains a forbidden component K_1.

   (a) In many cases, the upper bound $\mathbf{P}_{\mathfrak{g}}$ is integral, and we can take K to be the maximal even sublattice $\mathbf{P}_{\mathfrak{g}}^{\mathrm{ev}}$ of $\mathbf{P}_{\mathfrak{g}}$ or a certain even overlattice of $\mathbf{P}_{\mathfrak{g}}^{\mathrm{ev}}$.
   (b) When $\mathbf{P}_{\mathfrak{g}}$ is not integral, L is contained in a maximal even overlattice Q of the lower bound $\mathbf{Q}_{\mathfrak{g}}$, and we usually find K as a suitable sublattice of Q.

We will use the following forbidden components:

Rank 2:

(1) $2A_1(16)$;   (2) $2A_1(18)$;   (3) $2A_1(20)$;   (4) $A_2(16)$;   (5) $A_2(24)$;

Rank 3:

(6) $A_1 \oplus 2A_1(8)$;   (7) $2A_1(2) \oplus A_1(5)$;   (8) $A_1(3) \oplus 2A_1(9)$;   (9) $A_1(3) \oplus A_2(5)$;
(10) $A_1 \oplus A_2(6)$;   (11) $A_1 \oplus A_2(8)$;   (12) $A_1(2) \oplus A_2(8)$;   (13) $A_1 \oplus A_2(12)$;
(14) $A_1(2) \oplus A_2(12)$;   (15) $A_3'(24)$;

Rank 4:

(16) $2A_1(6) \oplus A_2$;   (17) $A_1(2) \oplus A_3(3)$;   (18) $A_1 \oplus A_3'(16)$;
(19) $A_1(7) \oplus L_1$, where L_1 is the rank three lattice with Gram matrix $\left(\begin{smallmatrix} 4 & 4 & 6 \\ 4 & 14 & 0 \\ 6 & 0 & 14 \end{smallmatrix} \right)$. The lattice L_1 is a maximal even overlattice of $3A_1(7)$ with discriminant 56.

Rank 5:

(20) $3A_1 \oplus 2A_1(5)$;   (21) $A_1 \oplus 2A_1(2) \oplus A_2(4)$;   (22) $A_1(3) \oplus A_2 \oplus A_2(4)$;
(23) $A_1(3) \oplus A_4$;   (24) $A_1(4) \oplus A_4(2)$;

Rank 6:

(25) $2A_1(3) \oplus A_2 \oplus A_2(3)$;   (26) $A_2 \oplus A_2(2) \oplus A_2(4)$;   (27) $A_1 \oplus A_2 \oplus A_3'(8)$;
(28) $A_3 \oplus A_3'(16)$;   (29) $2A_1(2) \oplus A_4$;   (30) $2A_1 \oplus A_4(2)$;
(31) $A_2(6) \oplus D_4$;
(32) $A_2 \oplus L_2$, where L_2 is a maximal even overlattice of $A_4(3)$ (such that $\mathrm{discr}(L_2) = 45$);
(33) $2L_3 \oplus L_3(2)$, where L_3 is the rank two lattice with Gram matrix $\left(\begin{smallmatrix} 2 & 1 \\ 1 & 4 \end{smallmatrix} \right)$;

Rank 7:

(34) $2A_2 \oplus A_3'(8)$;

Rank 8:

(35) $4A_2(3)$;   (36) $2A_2 \oplus A_4$;   (37) $A_1 \oplus A_3'(8) \oplus D_4$;
(38) $3A_1 \oplus A_1(4) \oplus D_4$;   (39) $2A_2(2) \oplus D_4$;   (40) $A_4 \oplus D_4$;
(41) $2D_4(3)$;   (42) $A_1 \oplus A_2(2) \oplus D_5$;   (43) $A_2(4) \oplus D_6$;
(44) $A_4'(5) \oplus L_4$, where L_4 is the maximal even sublattice of $\mathbb{Z} \oplus \mathbb{Z}^3(5)$ of genus $2_{II}^2 5^3$;

(45) $D_4 \oplus L_5$, where L_5 is the rank four lattice with Gram matrix $\begin{pmatrix} 2 & 0 & 1 & -1 \\ 0 & 2 & -1 & -1 \\ 1 & -1 & 6 & 0 \\ -1 & -1 & 0 & 6 \end{pmatrix}$ of genus $2_{II}^{-2} 5^2$;

Rank 9:

(46) $3A_1 \oplus 3A_2$;   (47) $2A_1 \oplus A_1(3) \oplus A_2 \oplus D_4$;   (48) $A_1 \oplus 3A_1(2) \oplus D_5$;

Rank 10:

(49) $2A_1 \oplus 2A_2 \oplus D_4$;   (50) $2A_2 \oplus A_2(2) \oplus D_4$;   (51) $2A_2(2) \oplus D_6$;   (52) $A_4 \oplus D_6$;
(53) $3A_1(2) \oplus D_7$;   (54) $A_3 \oplus D_7$;   (55) $A_2 \oplus D_8(3)$;
(56) $3A_2 \oplus L_7$, where L_7 is the maximal even sublattice of $\mathbb{Z}^2 \oplus \mathbb{Z}^2(3)$ in the genus $2_{II}^2 3^2$;

Rank 11:

(57) $3A_2 \oplus D_5$;   (58) $A_1 \oplus A_2(2) \oplus E_8$;

(59) $D_4 \oplus L_6$, where L_6 is the rank seven lattice with Gram matrix $\begin{pmatrix} 2 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 2 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 2 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 4 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 4 \end{pmatrix}$ of genus $2_7^1 4_{II}^2$;

Rank 12:

(60) $D_6 \oplus E_6$;   (61) $2A_2 \oplus D_8$;   (62) $A_4 \oplus D_8$;   (63) $A_1 \oplus A_3 \oplus E_8$;   (64) $3A_1 \oplus D_9$;
(65) The maximum even sublattice of $\mathbb{Z}^6 \oplus 2A_3'(4)$ in the genus $2_{II}^{-2} 4_{II}^4$;

Rank 13:

(66) $3A_1 \oplus A_2 \oplus E_8$;

Rank 14:

(67) $2A_2 \oplus A_2(2) \oplus E_8$;   (68) $A_2(2) \oplus D_4 \oplus E_8$;   (69) $A_1 \oplus D_5 \oplus E_8$;   (70) $E_6 \oplus E_8$;

(71) $E_8 \oplus L$, where L is the rank six lattice with Gram matrix $\begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & -1 & -1 & -1 \\ 0 & -1 & 2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 4 & 2 & 2 \\ 0 & -1 & 0 & 2 & 4 & 0 \\ 0 & -1 & 0 & 2 & 0 & 4 \end{pmatrix}$ in the genus $2_6^2 4_{II}^2$;

(72) The maximal even sublattice of $\mathbb{Z}^8 \oplus 2A_3'(4)$;

Rank 16:

(73) $A_2(2) \oplus D_6 \oplus E_8$;   (74) $A_1 \oplus D_7 \oplus E_8$;

Rank 18:

(75) $A_2 \oplus 2D_4 \oplus E_8$;   (76) $2A_1 \oplus D_8 \oplus E_8$;   (77) $A_1 \oplus A_1(2) \oplus 2E_8$;   (78) $2A_1(2) \oplus 2E_8$.

The forbidden components above can be verified using the algorithm described at the end of Section 2.4 except for the following three cases, which would have been prohibitively time-consuming to check directly due to the large lattice discriminant.

Lemma 11.4. *The lattice $4A_2(3)$ is a forbidden component.*

Proof. We use the overlattice

$$2U \oplus 4A_2(3) \cong 2U(9) \oplus 4A_2 < U \oplus U(9) \oplus E_8$$

and can verify quickly that there are no reflective Borcherds products on $U \oplus U(9) \oplus E_8$. By [110, Lemma 3.3], $2U \oplus 4A_2(3)$ has no anti-symmetric reflective Borcherds product of any weight. Suppose that $2U \oplus 4A_2(3)$ has a symmetric reflective Borcherds product, and denote its input as a vector-valued modular form by f. Since the components of its divisor must be of the form $v^\perp$ for primitive vectors $v \in 2U \oplus 4A_2'(1/3)$ with $v^2 = 2/3$ or $2/9$, the function $\eta^8 f$ defines a holomorphic vector-valued modular form of weight 0, so it is constant, i.e. an invariant of the Weil representation $\rho_{2U \oplus 4A_2(3)}$. This is impossible because the constant coefficient $c(0,0)$ of f and therefore the coefficient of $q^{1/3} e_0$ of $\eta^8 f$ is nonzero, which proves the lemma. $\qquad\square$

Lemma 11.5. *The lattice $L_4 \oplus A_4'(5)$ is a forbidden component.*

Proof. There is a positive definite even lattice K of rank 4 such that

$$2U \oplus L_4 \oplus A_4'(5) \cong U \oplus U(10) \oplus K \oplus A_4'(5).$$

It follows from the bound of [23, Lemma 4.5 and Table 2] with $N = 10$ that there is no reflective Borcherds product on $2U \oplus L_4 \oplus A_4'(5)$. $\qquad\square$

Lemma 11.6. *The lattice $A_2 \oplus D_8(3)$ is a forbidden component.*

Proof. Use the isometry

$$2U \oplus A_2 \oplus D_8(3) \cong U \oplus U(6) \oplus 2A_2 \oplus E_6'(3).$$

By [23, Lemma 4.5 and Table 2] with $N = 6$, there is no reflective Borcherds product of any weight on $2U \oplus A_2 \oplus D_8(3)$. $\qquad\square$

11.2. **Excluding the extraneous root systems.** In this subsection we rule out the 152 extraneous anti-symmetric root systems using Argument 11.3 and the 78 forbidden components that were listed above.

11.2.1. *Rank 4.* There are **3** extraneous root systems of rank 4:

(1) $D_{4,36}$: $C = 1/6$, $\mathbf{P_g} = D_4(18)$. Here,

$$L < D_4(18) < A_2(6) \oplus A_2(12) < A_1(3) \oplus A_1(9) \oplus A_2(6) < A_1 \oplus A_1(3) \oplus A_2(6)$$

using the overlattices $D_4(3) < A_2 \oplus A_2(2)$, $A_2(4) < A_1 \oplus A_1(3)$ and $A_1(9) < A_1$. This overlattice of L contains the forbidden component $A_1 \oplus A_2(6)$ labelled (10).

(2) $G_{2,24}^2$: $C = 1/6$, $L = 2A_2(24)$. This lattice contains the forbidden component $A_2(24)$ labelled (5).

(3) $B_{4,14}$: $C = 1/2$, $\mathbf{P_g} = 4A_1(7)$. Then

$$L < 4A_1(7) < A_1(7) \oplus L_1,$$

where L_1 is a certain maximal even overlattice of $3A_1(7)$. This is the forbidden component of rank 4 labelled (19).

11.2.2. *Rank 5.* There are **6** extraneous root systems of rank 5.

(4) $A_{1,48} A_{2,72} G_{2,96}$: $C = 1/24$, $\mathbf{P_g} = A_1(12) \oplus A_2(24) \oplus A_2(96)$. This lattice contains the forbidden component $A_2(24)$ labelled (5).

(5) $A_{3,96} B_{2,72}$: $C = 1/24$, $\mathbf{P_g} = A_3'(96) \oplus 2A_1(36)$. Then

$$L < A_3'(96) \oplus 2A_1(36) < A_1(12) \oplus 2A_1(24) \oplus 2A_1(36) < A_1(3) \oplus 2A_1(9) \oplus 2A_1(24)$$

using the rules $A_3'(8) < A_1 \oplus 2A_1(2)$ and $A_1(4) < A_1$. This overlattice contains the forbidden component $A_1(3) \oplus 2A_1(9)$ that we labelled (8).

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(6) $A_{1,16}B_{2,24}G_{2,32}$: $C = 1/8$, $\mathbf{P_g} = A_1(4) \oplus 2A_1(12) \oplus A_2(32)$. Then

$$L < A_1(4) \oplus 2A_1(12) \oplus A_2(32) < A_1 \oplus 2A_1(12) \oplus A_2(8)$$

using the rules $A_1(4) < A_1$ and $A_2(4) < A_2$. This overlattice contains forbidden component (11), i.e. $A_1 \oplus A_2(8)$.

(7) $A_{1,16}^2 C_{3,32}$: $C = 1/8$, $\mathbf{P_g} = 2A_1(4) \oplus A_3'(64)$. Then

$$L < 2A_1(4) \oplus A_3'(64) < 2A_1(4) \oplus A_1(8) \oplus 2A_1(16)$$

using the overlattice $A_3'(8) < A_1 \oplus 2A_1(2)$. This contains forbidden component (1), i.e. $2A_1(16)$.

(8) $A_{1,16}^2 B_{3,40}$: $C = 1/8$, $\mathbf{P_g} = 2A_1(4) \oplus 3A_1(20)$. Then

$$L < 2A_1(4) \oplus 3A_1(20).$$

This overlattice contains forbidden component (3), i.e. $2A_1(20)$.

(9) $A_{1,16}A_{4,40}$: $C = 1/8$, $\mathbf{P_g} = A_1(4) \oplus A_4'(40)$. Then

$$L < A_1(4) \oplus A_4'(40) < A_1(4) \oplus A_4(8) < A_1(4) \oplus A_4(2)$$

using the overlattice $A_4'(5) < A_4$ and $A_4(4) < A_4$. This case is ruled out because the overlattice contains forbidden component (24), i.e. $A_1(4) \oplus A_4(2)$.

11.2.3. *Rank 6.* There are **22** extraneous root systems of rank 6.

(10) $A_{1,24}^4 G_{2,48}$: $C = 1/12$, $\mathbf{P_g} = 4A_1(6) \oplus A_2(48)$. Then

$$L < 4A_1(6) \oplus A_2(48) < 4A_1(6) \oplus A_2$$

using the overlattices $A_2(4) < A_2$ and $A_2(3) < A_2$. This case can be excluded because the overlattice contains forbidden component (16), i.e. $2A_1(6) \oplus A_2$.

(11) $A_{1,24}^2 B_{2,36}^2$: $C = 1/12$, $\mathbf{P_g} = 2A_1(6) \oplus 4A_1(18)$. Then

$$L < 2A_1(6) \oplus 4A_1(18).$$

This overlattice contains forbidden component (2), i.e. $2A_1(18)$.

(12) $A_{2,36}^2 B_{2,36}$: $C = 1/12$, $\mathbf{P_g} = 2A_2(12) \oplus 2A_1(18)$. Then

$$L < 2A_2(12) \oplus 2A_1(18).$$

This overlattice contains forbidden component (2), i.e. $2A_1(18)$.

(13) $A_{1,24}A_{2,36}A_{3,48}$: $C = 1/12$, $\mathbf{P_g} = A_1(6) \oplus A_2(12) \oplus A_3'(48)$. Then

$$L < A_1(6) \oplus A_2(12) \oplus A_3'(48) < 4A_1(6) \oplus A_2(12) < 4A_1(6) \oplus A_2,$$

using the overlattices $A_3'(8) < 3A_1$, $A_2(4) < A_2$ and $A_2(3) < A_2$. This overlattice contains forbidden component (16), i.e. $2A_1(6) \oplus A_2$.

(14) $A_{1,12}^2 A_{2,18}G_{2,24}$: $C = 1/6$, $\mathbf{P_g} = 2A_1(3) \oplus A_2(6) \oplus A_2(24)$. Then

$$L < 2A_1(3) \oplus A_2(6) \oplus A_2(24).$$

This overlattice contains forbidden component (5), i.e. $A_2(24)$.

(15) $A_{1,12}A_{3,24}B_{2,18}$: $C = 1/6$, $\mathbf{P_g} = A_1(3) \oplus A_3'(24) \oplus 2A_1(9)$. Then

$$L < A_1(3) \oplus A_3'(24) \oplus 2A_1(9).$$

This overlattice contains the forbidden component (8), i.e. $A_1(3) \oplus 2A_1(9)$.

(16) $A_{2,18}B_{2,18}^2$: $C = 1/6$, $\mathbf{P_g} = A_2(6) \oplus 4A_1(9)$. Then

$$L < A_2(6) \oplus 4A_1(9) < A_1 \oplus 3A_1(9) \oplus A_2(6).$$

This overlattice contains forbidden component (10), i.e. $A_1 \oplus A_2(6)$.

(17) $A_{1,8}^3 C_{3,16}$: $C = 1/4$, $\mathbf{P_g} = 3A_1(2) \oplus A_3'(32)$. Then

$$L < 3A_1(2) \oplus A_3'(32) < 3A_1(2) \oplus A_1(4) \oplus 2A_1(8) < A_1 \oplus 3A_1(2) \oplus 2A_1(8)$$

using $A_3'(8) < A_1 \oplus 2A_1(2)$. This overlattice contains forbidden component (6), i.e. $A_1 \oplus 2A_1(8)$.

(18) $A_{1,8}^2 B_{2,12} G_{2,16}$: $C = 1/4$, $\mathbf{P_g} = 2A_1(2) \oplus 2A_1(6) \oplus A_2(16)$. Then

$$L < 2A_1(2) \oplus 2A_1(6) \oplus A_2(16).$$

This overlattice contains forbidden component (4), i.e. $A_2(16)$.

(19) $A_{1,8}^3 B_{3,20}$: $C = 1/4$, $\mathbf{P_g} = 3A_1(2) \oplus 3A_1(10)$. Then

$$L < 3A_1(2) \oplus 3A_1(10) < 3A_1(2) \oplus 2A_1(5) \oplus A_1(10)$$

using $2A_1(2) < 2A_1$. This overlattice contains forbidden component (7), i.e. $2A_1(2) \oplus A_1(5)$.

(20) $A_{3,16}^2$: $C = 1/4$, $\mathbf{P_g} = 2A_3'(16)$. Then

$$L < 2A_3'(16) < A_3 \oplus A_3'(16)$$

using the rule $A_3'(16) < A_3$. The overlattice $A_3 \oplus A_3'(16)$ is forbidden component (28).

(21) $A_{1,8}^2 A_{4,20}$: $C = 1/4$, $\mathbf{P_g} = 2A_1(2) \oplus A_4'(20)$. Then

$$L < 2A_1(2) \oplus A_4'(20) < 2A_1(2) \oplus A_4(4) < 2A_1(2) \oplus A_4$$

using the rule $A_4'(5) < A_4$. The overlattice $2A_1(2) \oplus A_4$ is forbidden component (29).

(22) $A_{2,12}^2 G_{2,16}$: $C = 1/4$, $\mathbf{P_g} = 2A_2(4) \oplus A_2(16)$. Then

$$L < 2A_2(4) \oplus A_2(16).$$

This overlattice contains forbidden component (4), i.e. $A_2(16)$.

(23) $B_{2,12}^3$: $C = 1/4$, $\mathbf{P_g} = 6A_1(6)$. Then

$$L < 6A_1(6) < 3A_1(6) \oplus A_1(2) \oplus A_2$$

using the rule $3A_1(6) < A_1(2) \oplus A_2$. This overlattice contains forbidden component (16), i.e. $2A_1(6) \oplus A_2$.

(24) $A_{2,9} B_{2,9} G_{2,12}$: $C = 1/3$, $\mathbf{P_g} = A_2(3) \oplus \mathbb{Z}^2(9) \oplus A_2(12)$. Since L is an even sublattice of $\mathbf{P_g}$,

$$L < A_2(3) \oplus 2A_1(9) \oplus A_2(12) < A_1(3) \oplus 3A_1(9) \oplus A_2(3)$$

using $A_2(4) < A_1 \oplus A_1(3)$. This overlattice contains forbidden component (8), i.e. $A_1(3) \oplus 2A_1(9)$.

(25) $A_{1,6} A_{2,9} C_{3,12}$: $C = 1/3$, $\mathbf{P_g} = \mathbb{Z}(3) \oplus A_2(3) \oplus A_3'(24)$. Since L is an even sublattice of $\mathbf{P_g}$,

$$L < A_1(6) \oplus A_2(3) \oplus A_3'(24).$$

This overlattice contains forbidden component (15), i.e. $A_3'(24)$.

(26) $A_{2,9} A_{4,15}$: $C = 1/3$, $\mathbf{P_g} = A_2(3) \oplus A_4'(15)$. Then

$$L < A_2(3) \oplus A_4'(15) < A_2 \oplus A_4(3) < A_2 \oplus L_2,$$

where L_2 is a maximal overlattice of $A_4(3)$, using $A_2(3) < A_2$ and $A_4'(5) < A_4$. This overlattice contains forbidden component (32), i.e. $A_2 \oplus L_2$.

(27) $A_{1,6} A_{2,9} B_{3,15}$: $C = 1/3$, $\mathbf{P_g} = \mathbb{Z}(3) \oplus A_2(3) \oplus \mathbb{Z}^3(15)$. The maximal even sublattice of $\mathbb{Z}(3) \oplus \mathbb{Z}^3(15)$ is an index two sublattice in $A_4'(15)$, and it is contained in $A_4(3)$. Since L is even, we have

$$L < A_2(3) \oplus A_4(3) < A_2 \oplus L_2$$

as in the previous case. This overlattice is forbidden component (32).

(28) $A_{1,6}A_{3,12}G_{2,12}$: $C = 1/3$, $\mathbf{P_g} = \mathbb{Z}(3) \oplus A_3'(12) \oplus A_2(12)$. Using the rule $A_3'(4) < \mathbb{Z} \oplus 2A_1$,

$$\mathbf{P_g} < A_2(12) \oplus \mathbb{Z}^2(3) \oplus 2A_1(3).$$

Since L is an even sublattice of $\mathbf{P_g}$, we have

$$L < 4A_1(3) \oplus A_2(12) < A_1(3) \oplus A_1 \oplus A_2(2) \oplus A_2(12)$$

using $3A_1(3) < A_1 \oplus A_2(2)$. This overlattice contains forbidden component (13), i.e. $A_1 \oplus A_2(12)$.

(29) $A_{2,6}G_{2,8}^2$: $C = 1/2$, $\mathbf{P_g} = A_2(2) \oplus 2A_2(8)$. Then

$$L < A_2(2) \oplus 2A_2(8) < A_1(2) \oplus A_1(6) \oplus A_2(2) \oplus A_2(8)$$

using $A_2(4) < A_1 \oplus A_1(3)$. This overlattice contains forbidden component (12), i.e. $A_1(2) \oplus A_2(8)$.

(30) $A_{3,8}B_{3,10}$: $C = 1/2$, $\mathbf{P_g} = A_3'(8) \oplus 3A_1(5)$. Then

$$L < A_3'(8) \oplus 3A_1(5) < A_1 \oplus 2A_1(2) \oplus 3A_1(5)$$

using $A_3'(8) < A_1 \oplus 2A_1(2)$. This overlattice contains forbidden component (7), i.e. $2A_1(2) \oplus A_1(5)$.

(31) $A_{3,8}C_{3,8}$: $C = 1/2$, $\mathbf{P_g} = A_3'(8) \oplus A_3'(16)$. Then

$$L < A_3'(8) \oplus A_3'(16) < A_1 \oplus 2A_1(2) \oplus A_3'(16)$$

using the rule $A_3'(8) < A_1 \oplus 2A_1(2)$. This overlattice contains forbidden component (18), i.e. $A_1 \oplus A_3'(16)$.

11.2.4. *Rank 7.* There are **5** extraneous root systems of rank 7.

(32) $A_{1,48}^3 A_{2,72}^2$: $C = 1/24$, $\mathbf{P_g} = 3A_1(12) \oplus 2A_2(24)$. Then

$$L < 3A_1(12) \oplus 2A_2(24).$$

This overlattice contains forbidden component (5), i.e. $A_2(24)$.

(33) $A_{1,48}^5 B_{2,72}$: $C = 1/24$, $\mathbf{P_g} = 5A_1(12) \oplus 2A_1(36)$. Then

$$L < 5A_1(12) \oplus 2A_1(36) < 5A_1(3) \oplus 2A_1(9),$$

using $A_1(4N) < A_1(N)$. This overlattice contains forbidden component (8), i.e. $A_1(3) \oplus 2A_1(9)$.

(34) $A_{1,16}A_{2,24}^3$: $C = 1/8$, $\mathbf{P_g} = A_1(4) \oplus 3A_2(8)$. Then

$$L < A_1(4) \oplus 3A_2(8) < A_1 \oplus 3A_2(8).$$

This overlattice contains forbidden component (11), i.e. $A_1 \oplus A_2(8)$.

(35) $A_{1,16}^3 A_{2,24}B_{2,24}$: $C = 1/8$, $\mathbf{P_g} = 3A_1(4) \oplus A_2(8) \oplus 2A_1(12)$. Then

$$L < 3A_1(4) \oplus A_2(8) \oplus 2A_1(12) < 3A_1 \oplus 2A_1(12) \oplus A_2(8).$$

This overlattice contains forbidden component (11), i.e. $A_1 \oplus A_2(8)$.

(36) $A_{1,16}^4 A_{3,32}$: $C = 1/8$, $\mathbf{P_g} = 4A_1(4) \oplus A_3'(32)$. Then

$$L < 4A_1(4) \oplus A_3'(32) < 5A_1(4) \oplus 2A_1(8) < 5A_1 \oplus 2A_1(8)$$

using $A_3'(8) < A_1 \oplus 2A_1(2)$ and $A_1(4) < A_1$. This overlattice contains forbidden component (6), i.e. $A_1 \oplus 2A_1(8)$.

11.2.5. *Rank 8.* There are **38** extraneous root systems of rank 8.

(37) $A_{1,24}^6 A_{2,36}$: $C = 1/12$, $\mathbf{P_g} = 6A_1(6) \oplus A_2(12)$. Then
$$L < 6A_1(6) \oplus A_2(12) < 6A_1(6) \oplus A_2$$
using $A_2(4) < A_2$ and $A_2(3) < A_2$. This overlattice contains forbidden component (16), i.e. $2A_1(6) \oplus A_2$.

(38) $A_{1,12}^6 B_{2,18}$: $C = 1/6$, $\mathbf{P_g} = 6A_1(3) \oplus 2A_1(9)$. Then
$$L < 6A_1(3) \oplus 2A_1(9).$$
This overlattice contains forbidden component (8), i.e. $A_1(3) \oplus 2A_1(9)$.

(39) $A_{1,12}^4 A_{2,18}^2$: $C = 1/6$, $\mathbf{P_g} = 4A_1(3) \oplus 2A_2(6)$. By the rule $4A_1(3) < 4A_1$ we obtain
$$L < 4A_1(3) \oplus 2A_2(6) < 4A_1 \oplus 2A_2(6).$$
This overlattice contains forbidden component (10), i.e. $A_1 \oplus A_2(6)$.

(40) $A_{1,8}^2 A_{2,12}^3$: $C = 1/4$, $\mathbf{P_g} = 2A_1(2) \oplus 3A_2(4)$. Then
$$L < 2A_1(2) \oplus 3A_2(4) < 2A_1(2) \oplus A_2(4) \oplus A_2 \oplus A_1 \oplus A_1(3)$$
using $A_2(4) < A_1 \oplus A_1(3) < A_2$. This overlattice contains forbidden component (22), i.e. $A_1(3) \oplus A_2 \oplus A_2(4)$.

(41) $A_{1,8}^5 A_{3,16}$: $C = 1/4$, $\mathbf{P_g} = 5A_1(2) \oplus A_3'(16)$. Then
$$L < 5A_1(2) \oplus A_3'(16) < 3A_1(2) \oplus 2A_1 \oplus A_3'(16)$$
using $2A_1(2) < 2A_1$. This overlattice contains forbidden component (18), i.e. $A_1 \oplus A_3'(16)$.

(42) $A_{1,8}^4 A_{2,12} B_{2,12}$: $C = 1/4$, $\mathbf{P_g} = 4A_1(2) \oplus A_2(4) \oplus 2A_1(6)$. Then
$$L < 4A_1(2) \oplus A_2(4) \oplus 2A_1(6) < 2A_1 \oplus 2A_1(2) \oplus A_2(4) \oplus 2A_1(6)$$
using $2A_1(2) < 2A_1$. This overlatice contains forbidden component (21), i.e. $A_1 \oplus 2A_1(2) \oplus A_2(4)$.

(43) $A_{1,6}^6 G_{2,12}$: $C = 1/3$, $\mathbf{P_g} = \mathbb{Z}^6(3) \oplus A_2(12)$. Since L is an even sublattice of $\mathbf{P_g}$, we have
$$L < D_6(3) \oplus A_2(12) < 2A_1 \oplus 2A_2 \oplus A_2(12)$$
using $D_6(3) < 2A_1 \oplus 2A_2$. This overlattice contains forbidden component (13), i.e. $A_1 \oplus A_2(12)$.

(44) $A_{1,6}^2 A_{2,9}^2 B_{2,9}$: $C = 1/3$, $\mathbf{P_g} = \mathbb{Z}^2(3) \oplus 2A_2(3) \oplus \mathbb{Z}^2(9)$. The maximal even sublattice of $\mathbb{Z}^2(3) \oplus \mathbb{Z}^2(9)$ is the rescaling by 3 of a lattice in the genus $2_{\mathrm{II}}^2 3^2$, and it is contained in $2A_2(3)$. Therefore, $L < 4A_2(3)$. This case is excluded because it is forbidden component (35).

(45) $A_{1,6}^4 B_{2,9}^2$: $C = 1/3$, $\mathbf{P_g} = \mathbb{Z}^4(3) \oplus \mathbb{Z}^4(9)$. The maximal even sublattice of $\mathbf{P_g}(1/3)$ is
$$2A_2(3) \oplus M_4(3),$$
where M_4 is a maximal even overlattice of $\mathbb{Z}^2 \oplus \mathbb{Z}^2(3)$ (such that M_4 has discriminant 36) with $M_4 < 2A_2$. Therefore we have
$$L < 2A_2(3) \oplus M_4(3) < 4A_2(3).$$
This overlattice is forbidden component (35).

(46) $A_{1,6}^3 A_{2,9} A_{3,12}$: $C = 1/3$, $\mathbf{P_g} = \mathbb{Z}^3(3) \oplus A_2(3) \oplus A_3'(12)$. Using the rule $A_3'(4) < \mathbb{Z} \oplus 2A_1$,
$$\mathbf{P_g} < A_2(3) \oplus 2A_1(3) \oplus \mathbb{Z}^4(3).$$
Since L is an even sublattice of $\mathbf{P_g}$, we have
$$L < A_2(3) \oplus 2A_1(3) \oplus D_4(3) < A_2(3) \oplus 2A_1(3) \oplus A_2 \oplus A_2(2)$$

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using $D_4(3) < A_2 \oplus A_2(2)$. This overlattice contains forbidden component (25), i.e. $2A_1(3) \oplus A_2 \oplus A_2(3)$.

(47) $A_{2,9}^4$: $C = 1/3$, $\mathbf{P_g} = 4A_2(3)$. Then

$$L < 4A_2(3).$$

This overlattice is the forbidden component (35).

(48) $A_{1,4}^2 A_{3,8}^2$: $C = 1/2$, $\mathbf{P_g} = 2A_1 \oplus 2A_3'(8)$. Then

$$L < 2A_1 \oplus 2A_3'(8) < A_3'(8) \oplus 5A_1 < A_3'(8) \oplus D_4 \oplus A_1$$

using $A_3'(8) < 3A_1$ and $4A_1 < D_4$. This overlattice is the forbidden component (37).

(49) $A_{1,4}^4 A_{4,10}$: $C = 1/2$, $\mathbf{P_g} = 4A_1 \oplus A_4'(10)$. Using $A_4'(5) < A_4$ we have

$$L < 4A_1 \oplus A_4'(10) < 4A_1 \oplus A_4(2)$$

This overlattice contains forbidden component (30), i.e. $2A_1 \oplus A_4(2)$.

(50) $A_{1,4} A_{2,6} A_{3,8} B_{2,6}$: $C = 1/2$, $\mathbf{P_g} = A_1 \oplus A_2(2) \oplus A_3'(8) \oplus 2A_1(3)$. Then

$$L < A_1 \oplus A_2(2) \oplus A_3'(8) \oplus 2A_1(3) < A_3'(8) \oplus A_2 \oplus 3A_1$$

using $A_1(3) \oplus A_2(2) < 3A_1$ and $A_1 \oplus A_1(3) < A_2$. This overlattice contains forbidden component (27), i.e. $A_1 \oplus A_2 \oplus A_3'(8)$.

(51) $A_{1,4}^5 B_{3,10}$: $C = 1/2$, $\mathbf{P_g} = 5A_1 \oplus 3A_1(5)$. Then

$$L < 5A_1 \oplus 3A_1(5).$$

This overlattice contains forbidden component (20), i.e. $3A_1 \oplus 2A_1(5)$.

(52) $A_{2,6}^2 B_{2,6}^2$: $C = 1/2$, $\mathbf{P_g} = 2A_2(2) \oplus 4A_1(3)$. Then

$$L < 2A_2(2) \oplus 4A_1(3) < 2A_2(2) \oplus D_4$$

using the rule $4A_1(3) < D_4$. This overlattice is forbidden component (39).

(53) $A_{1,4}^2 A_{2,6}^2 G_{2,8}$: $C = 1/2$, $\mathbf{P_g} = 2A_1 \oplus 2A_2(2) \oplus A_2(8)$. Then

$$L < 2A_1 \oplus 2A_2(2) \oplus A_2(8).$$

This overlattice contains forbidden component (11), i.e. $A_1 \oplus A_2(8)$.

(54) $A_{1,4}^4 B_{2,6} G_{2,8}$: $C = 1/2$, $\mathbf{P_g} = 4A_1 \oplus 2A_1(3) \oplus A_2(8)$. Then

$$L < 4A_1 \oplus 2A_1(3) \oplus A_2(8).$$

This overlattice contains forbidden component (11), i.e. $A_1 \oplus A_2(8)$.

(55) $A_{1,4}^5 C_{3,8}$: $C = 1/2$, $\mathbf{P_g} = 5A_1 \oplus A_3'(16)$. Then

$$L < 5A_1 \oplus A_3'(16).$$

This overlattice contains forbidden component (28), i.e. $A_1 \oplus A_3'(16)$.

(56) $A_{1,4}^2 B_{2,6}^3$: $C = 1/2$, $\mathbf{P_g} = 2A_1 \oplus 6A_1(3)$. Then

$$L < 2A_1 \oplus 6A_1(3) < 4A_1 \oplus 2A_2(2) < D_4 \oplus 2A_2(2)$$

using $3A_1(3) < A_1 \oplus A_2(2)$ and $4A_1 < D_4$. This overlattice is forbidden component (39).

(57) $A_{1,3}^4 G_{2,6}^2$: $C = 2/3$, $L = K \oplus 2A_2(6)$ with $4A_1(3) < K < \mathbb{Z}^4(3/2)$. Then

$$\mathbb{Z}^4(4) < K(2/3) < \mathbb{Z}^4.$$

Since $K(2/3)$ is integral and $K = K(2/3)(3/2)$ is even, $K(2/3)$ is the 2-scaling of an even lattice. Therefore, $K(2/3) < D_4(2)$. Using the overlattice $D_4(3) < D_4$ we then have

$$L < 2A_2(6) \oplus D_4(3) < 2A_2(6) \oplus D_4$$

This overlattice contains forbidden component (31), i.e. $A_2(6) \oplus D_4$.

(58) $A_{1,3}^4 D_{4,9}$: $C = 2/3$, $\mathbf{P_g} = \mathbb{Z}^4(3/2) \oplus D_4(9/2)$ and $\mathbf{Q_g} = 4A_1(3) \oplus D_4(9)$. Then $L(2/3)$ is integral and

$$4A_1(2) \oplus D_4(6) < L(2/3) < \mathbb{Z}^4 \oplus D_4(3).$$

We see from $L = L(2/3)(3/2)$ that $L(2/3)$ is the 2-scaling of an even lattice. Therefore, $L(1/3)$ is an even lattice bounded by

$$4A_1 \oplus D_4(3) < L(1/3) < D_4(1/2) \oplus D_4(3/2) \cong D_4' \oplus D_4'(3).$$

It follows that $L(1/3) < D_4 \oplus D_4(3)$. Then

$$L < D_4(3) \oplus D_4(9) < A_2(3) \oplus A_2(6) \oplus D_4,$$

using $D_4(3) < A_2 \oplus A_2(2) < D_4$. This overlattice contains forbidden component (31), i.e. $A_2(6) \oplus D_4$.

(59) $A_{1,2} A_{4,5} C_{3,4}$: $C = 1$, $\mathbf{P_g} = \mathbb{Z} \oplus A_4'(5) \oplus A_3'(8)$. Since L is an even sublattice of $\mathbf{P_g}$,

$$L < A_1(2) \oplus A_4'(5) \oplus A_3'(8) < A_1 \oplus 3A_1(2) \oplus A_4,$$

where used $A_4'(5) < A_4$ and $A_3'(8) < A_1 \oplus 2A_1(2)$. This overlattice contains forbidden component (29), i.e. $2A_1(2) \oplus A_4$.

(60) $A_{1,2}^4 B_{4,7}$: $C = 1$, $\mathbf{P_g} = \mathbb{Z}^4 \oplus \mathbb{Z}^4(7)$. Since L is even, it is contained in the maximal even sublattice of $\mathbf{P_g}$:

$$L < 3L_3 \oplus L_3(2)$$

where L_3 is $\mathbb{Z}^2$ with Gram matrix $\left(\begin{smallmatrix} 2 & 1 \\ 1 & 4 \end{smallmatrix}\right)$. This overlattice contains forbidden component (33), i.e. $2L_3 \oplus L_3(2)$.

(61) $B_{2,3}^2 D_{4,6}$: $C = 1$, $\mathbf{P_g} = \mathbb{Z}^4(3) \oplus D_4(3)$. Since L is an even sublattice of $\mathbf{P_g}$, we have $L < 2D_4(3)$. This overlattice is forbidden component (41).

(62) $A_{1,2} B_{2,3} B_{3,5} G_{2,4}$: $C = 1$, $\mathbf{P_g} = \mathbb{Z} \oplus \mathbb{Z}^2(3) \oplus \mathbb{Z}^3(5) \oplus A_2(4)$. Since L is even, it is contained in the maximal even sublattice

$$Q \oplus A_2(4) \oplus A_2(5)$$

of $\mathbf{P_g}$, where Q has genus $[2^2]_6 3^1 5^{-1}$ and discriminant 60. Therefore,

$$L < Q \oplus A_2(4) \oplus A_2(5) < Q \oplus A_1 \oplus A_1(3) \oplus A_2(5)$$

using $A_2(4) < A_1 \oplus A_1(3)$. This overlattice contains forbidden component (9), i.e. $A_1(3) \oplus A_2(5)$.

(63) $A_{1,2} A_{4,5} B_{3,5}$: $C = 1$, $\mathbf{P_g} = \mathbb{Z} \oplus A_4'(5) \oplus \mathbb{Z}^3(5)$. Let L_4 denote the maximal even sublattice of $\mathbb{Z} \oplus \mathbb{Z}^3(5)$. The overlattice $L_4 \oplus A_4'(5)$ of L is forbidden component (44).

(64) $A_{1,2}^2 B_{3,5}^2$: $C = 1$, $\mathbf{P_g} = \mathbb{Z}^2 \oplus \mathbb{Z}^6(5)$. The maximal even sublattice of $\mathbf{P_g}$ is $L_4 \oplus A_4'(5)$, which is forbidden component (44) and is an even overlattice of L.

(65) $A_{1,2} B_{2,3} C_{3,4} G_{2,4}$: $C = 1$, $\mathbf{P_g} = \mathbb{Z} \oplus \mathbb{Z}^2(3) \oplus A_3'(8) \oplus A_2(4)$. Since L is even, it is contained in the maximal even sublattice

$$L < Q \oplus A_2(4) \oplus A_3'(8)$$

of $\mathbf{P_g}$, where Q has genus $[4^1]_7 3^2$. Then

$$L < A_1 \oplus 2A_1(2) \oplus A_2(4) \oplus Q$$

using the rule $A_3'(8) < A_1 \oplus 2A_1(2)$. This overlattice contains forbidden component (21), i.e. $A_1 \oplus 2A_1(2) \oplus A_2(4)$.

(66) $A_{1,2}^2 G_{2,4}^3$: $C = 1$, $\mathbf{P_g} = \mathbb{Z}^2 \oplus 3A_2(4)$. Since L is an even sublattice of $\mathbf{P_g}$, we have

$$L < 2A_1 \oplus 3A_2(4) < 3A_1 \oplus A_1(3) \oplus A_2 \oplus A_2(4)$$

using $A_2(4) < A_1 \oplus A_1(3) < A_2$. This overlattice contains forbidden component (22), i.e. $A_1(3) \oplus A_2 \oplus A_2(4)$.

(67) $A_{1,2}^2 C_{3,4}^2$: $C = 1$, $\mathbf{P_g} = \mathbb{Z}^2 \oplus 2A_3'(8)$. The maximal even sublattice of $\mathbf{P_g}$ is $2A_1 \oplus 2A_3'(8)$. Then
$$L < 2A_1 \oplus 2A_3'(8) < A_3'(8) \oplus 5A_1 < A_3'(8) \oplus D_4 \oplus A_1$$
using $A_3'(8) < 3A_1$ and $4A_1 < D_4$. The overlattice $A_1 \oplus A_3'(8) \oplus D_4$ is forbidden component (37).

(68) $A_{1,2}^2 B_{3,5} C_{3,4}$: $C = 1$, $\mathbf{P_g} = \mathbb{Z}^2 \oplus \mathbb{Z}^3(5) \oplus A_3'(8)$. Since L is even, it is contained in the maximal even sublattice $A_3'(8) \oplus Q$ of $\mathbf{P_g}$, where Q has genus $[4^1]_1 5^3$. Then
$$L < A_1(2) \oplus A_3'(8) \oplus A_4 < A_1 \oplus 3A_1(2) \oplus A_4$$
using $Q < A_1(2) \oplus A_4$ and $A_3'(8) < A_1 \oplus 2A_1(2)$. This overlattice contains forbidden component (29), i.e. $2A_1(2) \oplus A_4$.

(69) $B_{2,3}^2 G_{2,4}^2$: $C = 1$, $\mathbf{P_g} = \mathbb{Z}^4(3) \oplus 2A_2(4)$. Since L is an even sublattice of $\mathbf{P_g}$, we have
$$L < D_4(3) \oplus 2A_2(4) < D_4 \oplus 2A_2(2)$$
using $2A_2(2) < 2A_2$ and $D_4(3) < D_4$. The overlattice $D_4 \oplus 2A_2(2)$ is forbidden component (39).

(70) $A_{4,5} B_{2,3} G_{2,4}$: $C = 1$, $\mathbf{P_g} = A_4'(5) \oplus \mathbb{Z}^2(3) \oplus A_2(4)$. Since L is an even sublattice of $\mathbf{P_g}$ and $A_4'(5) < A_4$ we have
$$L < A_4'(5) \oplus 2A_1(3) \oplus A_2(4) < A_4 \oplus 2A_1(3) \oplus A_2(4).$$
This overlattice contains forbidden component (23), i.e. $A_1(3) \oplus A_4$.

(71) $A_{1,2}^4 C_{4,5}$: $C = 1$, $\mathbf{P_g} = \mathbb{Z}^4 \oplus D_4(5)$. Since L is even,
$$L < D_4 \oplus D_4(5) < D_4 \oplus L_5,$$
where L_5 is an even overlattice of $D_4(5)$ of genus $2_{\mathrm{II}}^{-2} 5^2$. This overlattice is forbidden component (45).

(72) $A_{1,2}^2 D_{4,6} G_{2,4}$: $C = 1$, $\mathbf{P_g} = \mathbb{Z}^2 \oplus D_4(3) \oplus A_2(4)$. Since L is even,
$$L < 2A_1 \oplus D_4(3) \oplus A_2(4) < 2A_1 \oplus A_2 \oplus A_2(2) \oplus A_2(4)$$
using $D_4(3) < A_2 \oplus A_2(2)$. This overlattice contains forbidden component (26), i.e. $A_2 \oplus A_2(2) \oplus A_2(4)$.

(73) $C_{3,3}^2 G_{2,3}$: $C = 4/3$, $L = A_2(3) \oplus K$ with $6A_1(3) < K < 2A_3'(6)$. Then
$$6A_1 < K(1/3) < 2A_3'(2).$$
Since K is even and $K(4/3)$ is integral, $K(1/3)$ is even. It follows that $K(1/3) = 6A_1$, and thus
$$L < A_2(3) \oplus 6A_1(3) < A_2(3) \oplus A_1 \oplus A_2(2) \oplus 3A_1(3) < A_2(2) \oplus 2A_1(3) \oplus A_2 \oplus A_2(3)$$
using $3A_1(3) < A_1 \oplus A_2(2)$ and $A_1 \oplus A_1(3) < A_2$. This overlattice contains forbidden component (25), i.e. $2A_1(3) \oplus A_2 \oplus A_2(3)$.

(74) $G_{2,3}^4$: $C = 4/3$, $L = 4A_2(3)$ is forbidden component (35).

11.2.6. *Rank 9.* There is only **1** extraneous root system of rank 9.

(75) $A_{1,16}^9$: $C = 1/8$, $\mathbf{P_g} = 9A_1(4)$. Then
$$L < 9A_1(4) < 4A_1 \oplus A_1(4) \oplus D_4.$$
This overlattice contains forbidden component (38), i.e. $3A_1 \oplus A_1(4) \oplus D_4$.

11.2.7. *Rank 10.* There are **24** extraneous root systems of rank 10.

(76) $A_{1,8}^{10}$: $C = 1/4$, $\mathbf{P_g} = 10A_1(2)$. Then

$$L < 10A_1(2) < 3A_1(2) \oplus D_7.$$

This overlattice is forbidden component (53).

(77) $A_{1,6}^8 A_{2,9}$: $C = 1/3$, $\mathbf{P_g} = \mathbb{Z}^8(3) \oplus A_2(3)$. Since L is even, we have

$$L < D_8(3) \oplus A_2(3) < D_8(3) \oplus A_2.$$

The overlattice $D_8(3) \oplus A_2$ is forbidden component (55).

(78) $A_{1,4}^4 A_{2,6}^3$: $C = 1/2$, $\mathbf{P_g} = 4A_1 \oplus 3A_2(2)$. Using $4A_1 < D_4$ we obtain

$$L < 4A_1 \oplus 3A_2(2) < 3A_2(2) \oplus D_4$$

This overlattice contains forbidden component (39), i.e. $2A_2(2) \oplus D_4$.

(79) $A_{1,4}^7 A_{3,8}$: $C = 1/2$, $\mathbf{P_g} = 7A_1 \oplus A_3'(8)$. Then

$$L < 7A_1 \oplus A_3'(8) < 3A_1 \oplus D_4 \oplus A_3'(8).$$

This overlattice contains forbidden component (37), i.e. $A_1 \oplus A_3'(8) \oplus D_4$.

(80) $A_{1,4}^6 A_{2,6} B_{2,6}$: $C = 1/2$, $\mathbf{P_g} = 6A_1 \oplus A_2(2) \oplus 2A_1(3)$. Then

$$L < 6A_1 \oplus A_2(2) \oplus 2A_1(3) < D_4 \oplus 2A_2 \oplus A_2(2)$$

using the rules $4A_1 < D_4$ and $A_1 \oplus A_1(3) < A_2$. The overlattice $D_4 \oplus 2A_2 \oplus A_2(2)$ is forbidden component (50).

(81) $A_{2,4}^4 B_{2,4}$: $C = 3/4$, $\mathbf{P_g} = 4A_2'(4) \oplus 2A_1(2)$, $\mathbf{Q_g} = 4A_2(4) \oplus 2A_1(4)$. Then

$$2A_1 \oplus 4A_2 < L(1/4) < \mathbb{Z}^2 \oplus 4A_2'.$$

Since $L(3/4)$ is integral and L is even, $L(1/4)$ is integral. It follows that $L(1/4) < \mathbb{Z}^2 \oplus E_8$, and therefore that $L < 2A_1(2) \oplus E_8(4)$. Note that

$$2U \oplus 2A_1(2) \oplus E_8(4) \cong 2U \oplus 10A_1(2)$$

and

$$10A_1(2) < 4A_1(2) \oplus 6A_1 < 4A_1(2) \oplus D_4 \oplus 2A_1 < 3A_1(2) \oplus D_5 \oplus 2A_1$$

using $2A_1(2) < 2A_1$ and $4A_1 < D_4$ and $D_4 \oplus A_1(2) < D_5$. This overlattice contains forbidden component (48), i.e. $A_1 \oplus 3A_1(2) \oplus D_5$.

(82) $A_{1,2} A_{2,3}^3 B_{3,5}$: $C = 1$, $\mathbf{P_g} = \mathbb{Z} \oplus 3A_2 \oplus \mathbb{Z}^3(5)$. Since L is even, it is contained in the maximal even sublattice Q of $\mathbf{P_g}$, where Q is an index two sublattice of A_4 in the genus $2_{\mathrm{II}}^2 5^3$. Therefore we have

$$L < Q \oplus 3A_2 < A_4 \oplus 3A_2.$$

This overlattice contains forbidden component (36), i.e. $2A_2 \oplus A_4$.

(83) $A_{1,2}^3 A_{2,3} B_{2,3} C_{3,4}$: $C = 1$, $\mathbf{P_g} = \mathbb{Z}^3 \oplus A_2 \oplus \mathbb{Z}^2(3) \oplus A_3'(8)$. The maximal even sublattice of $\mathbb{Z}^3 \oplus \mathbb{Z}^2(3)$ is $A_1(2) \oplus 2A_2$. Since L is an even sublattice of $\mathbf{P_g}$, we have

$$L < A_1(2) \oplus 3A_2 \oplus A_3'(8).$$

This overlattice contains forbidden component (34), i.e. $2A_2 \oplus A_3'(8)$.

(84) $A_{1,2} A_{3,4} B_{2,3}^3$: $C = 1$, we have

$$\mathbf{P_g} = \mathbb{Z} \oplus A_3'(4) \oplus \mathbb{Z}^6(3) < \mathbb{Z}^4 \oplus 2A_1 \oplus 2A_2$$

using the rules $A_3'(4) < \mathbb{Z} \oplus 2A_1$ and $\mathbb{Z}^3(3) < \mathbb{Z} \oplus A_2$. Therefore, $L < D_4 \oplus 2A_1 \oplus 2A_2$. The overlattice $D_4 \oplus 2A_1 \oplus 2A_2$ is forbidden component (49).

(85) $A_{1,2}^4 A_{2,3} G_{2,4}^2$: $C = 1$, $\mathbf{P_g} = \mathbb{Z}^4 \oplus A_2 \oplus 2A_2(4)$. Since L is even, we have

$$L < D_4 \oplus A_2 \oplus 2A_2(4) < A_2 \oplus 2A_2(2) \oplus D_4$$

using $2A_2(2) < 2A_2$. This overlattice contains forbidden component (39), i.e. $2A_2(2) \oplus D_4$.

(86) $A_{1,2}^2 A_{2,3} B_{2,3}^2 G_{2,4}$: $C = 1$, $\mathbf{P_g} = \mathbb{Z}^2 \oplus A_2 \oplus \mathbb{Z}^4(3) \oplus A_2(4)$. The maximal even sublattice of $\mathbf{P_g}$ is $2A_1(3) \oplus 3A_2 \oplus A_2(4)$, and it is an even overlattice of L. This overlattice contains forbidden component (22), i.e. $A_1(3) \oplus A_2 \oplus A_2(4)$.

(87) $A_{2,3}^3 A_{4,5}$: $C = 1$, $\mathbf{P_g} = 3A_2 \oplus A_4'(5)$. Then

$$L < 3A_2 \oplus A_4'(5) < 3A_2 \oplus A_4.$$

This overlattice contains forbidden component (36), i.e. $2A_2 \oplus A_4$.

(88) $A_{1,2}^2 A_{2,3} A_{4,5} B_{2,3}$: $C = 1$, $\mathbf{P_g} = \mathbb{Z}^2 \oplus A_2 \oplus A_4'(5) \oplus \mathbb{Z}^2(3)$. The maximal even sublattice of $\mathbf{P_g}$ decomposes as $Q \oplus A_2 \oplus A_4'(5)$, where Q has genus $2_{\mathrm{II}}^2 3^2$. Since L is even, we have

$$L < Q \oplus A_2 \oplus A_4'(5) < 3A_2 \oplus A_4,$$

using $Q < 2A_2$ and $A_4'(5) < A_4$. This overlattice contains forbidden component (36), i.e. $2A_2 \oplus A_4$.

(89) $A_{1,2} A_{2,3}^3 C_{3,4}$: $C = 1$, $\mathbf{P_g} = \mathbb{Z} \oplus 3A_2 \oplus A_3'(8)$. Since L is even,

$$L < A_1(2) \oplus 3A_2 \oplus A_3'(8).$$

This overlattice contains forbidden component (34), i.e. $A_3'(8) \oplus 2A_2$.

(90) $A_{2,3}^3 B_{2,3} G_{2,4}$: $C = 1$, $\mathbf{P_g} = 3A_2 \oplus \mathbb{Z}^2(3) \oplus A_2(4)$. Since L is even, we have

$$L < 2A_1(3) \oplus 3A_2 \oplus A_2(4).$$

This overlattice contains forbidden component (22), i.e. $A_1(3) \oplus A_2 \oplus A_2(4)$.

(91) $A_{1,2}^4 A_{2,3} D_{4,6}$: $C = 1$, $\mathbf{P_g} = \mathbb{Z}^4 \oplus A_2 \oplus D_3$. Since L is even and $D_4(3) < A_2 \oplus A_2(2)$ we have

$$L < A_2 \oplus D_4 \oplus D_4(3) < 2A_2 \oplus A_2(2) \oplus D_4$$

The overlattice $2A_2 \oplus A_2(2) \oplus D_4$ is forbidden component (50).

(92) $A_{1,2}^4 A_{3,4} C_{3,4}$: $C = 1$, $\mathbf{P_g} = \mathbb{Z}^4 \oplus A_3'(4) \oplus A_3'(8)$. The maximal even sublattice of $\mathbf{P_g}$ is $2A_1(2) \oplus D_5 \oplus A_3'(8)$. Since L is even,

$$L < 2A_1(2) \oplus D_5 \oplus A_3'(8) < A_1 \oplus 4A_1(2) \oplus D_5$$

using the rule $A_3'(8) < A_1 \oplus 2A_1(2)$. This overlattice contains forbidden component (48), i.e. $A_1 \oplus 3A_1(2) \oplus D_5$.

(93) $A_{1,2}^3 A_{3,4} A_{4,5}$: $C = 1$, we have

$$\mathbf{P_g} = \mathbb{Z}^3 \oplus A_3'(4) \oplus A_4'(5) < \mathbb{Z}^4 \oplus 2A_1 \oplus A_4'(5)$$

using $A_3'(4) < \mathbb{Z} \oplus 2A_1$. Since L is even,

$$L < D_4 \oplus 2A_1 \oplus A_4'(5) < D_4 \oplus 2A_1 \oplus A_4.$$

This overlattice contains forbidden component (40), i.e. $A_4 \oplus D_4$.

(94) $A_{2,3} A_{3,4}^2 B_{2,3}$: $C = 1$, $\mathbf{P_g} = A_2 \oplus 2A_3'(4) \oplus \mathbb{Z}^2(3)$. The maximal even sublattice of $\mathbf{P_g}$ decomposes as $A_1(2) \oplus A_3(3) \oplus Q$, where Q has genus $[4^3]_5$. The overlattice $A_1(2) \oplus A_3(3) \oplus Q$ of L contains forbidden component (17), i.e. $A_1(2) \oplus A_3(3)$.

(95) $A_{2,3} B_{2,3}^4$: $C = 1$, $\mathbf{P_g} = A_2 \oplus \mathbb{Z}^8(3)$. Since L is even, we have

$$L < A_2 \oplus D_8(3).$$

This overlattice is forbidden component (55).

(96) $A_{1,2}^4 A_{3,4} B_{3,5}$: $C = 1$, $\mathbf{P_g} = \mathbb{Z}^4 \oplus A_3'(4) \oplus \mathbb{Z}^3(5)$. The maximal even sublattice of $\mathbf{P_g}$ splits as $A_4 \oplus Q$, where Q is a sublattice of D_6 with genus $[2^2]_6 4^2 5^2$ and discriminant 1600. Then

$$L < A_4 \oplus Q < A_4 \oplus D_6.$$

The overlattice $A_4 \oplus D_6$ is forbidden component (52).

(97) $A_{1,2} A_{2,3}^2 A_{3,4} G_{2,4}$: $C = 1$, $\mathbf{P_g} = \mathbb{Z} \oplus 2A_2 \oplus A_3'(4) \oplus A_2(4)$. The maximal even sublattice of $\mathbf{P_g}$ is $2A_2 \oplus A_2(4) \oplus D_4(2)$. Therefore,

$$L < 2A_2 \oplus A_2(4) \oplus D_4(2) < 4A_1 \oplus 3A_2,$$

using $A_2(4) < A_2$ and $D_4(2) < 4A_1$. This overlattice contains forbidden component (46), i.e. $3A_1 \oplus 3A_2$.

(98) $A_{1,2}^3 A_{2,3} B_{2,3} B_{3,5}$: $C = 1$, $\mathbf{P_g} = \mathbb{Z}^3 \oplus A_2 \oplus \mathbb{Z}^2(3) \oplus \mathbb{Z}^3(5)$. The maximal even sublattice of $\mathbf{P_g}$ splits as $2A_2 \oplus Q$ where Q has genus $2_{II}^2 3^{-1} 5^3$. Therefore,

$$L < 2A_2 \oplus Q < 3A_2 \oplus A_4$$

since $Q < A_2 \oplus A_4$. This overlattice contains forbidden component (36), i.e. $2A_2 \oplus A_4$.

(99) $A_{1,2}^3 A_{3,4} B_{2,3} G_{2,4}$: $C = 1$, $\mathbf{P_g} = \mathbb{Z}^3 \oplus A_3'(4) \oplus \mathbb{Z}^2(3) \oplus A_2(4)$. The maximal even sublattice of $\mathbf{P_g}$ is $A_2 \oplus 2A_2(4) \oplus D_4$. Therefore,

$$L < A_2 \oplus 2A_2(4) \oplus D_4 < 2A_1 \oplus 2A_1(3) \oplus A_2 \oplus D_4$$

using $A_2(4) < A_1 \oplus A_1(3)$. This overlattice contains forbidden component (47), i.e. $2A_1 \oplus A_1(3) \oplus A_2 \oplus D_4$.

11.2.8. *Rank 12.* There are **21** extraneous root systems of rank 12.

(100) $A_{1,2}^6 A_{2,3}^2 G_{2,4}$: $C = 1$, $\mathbf{P_g} = \mathbb{Z}^6 \oplus 2A_2 \oplus A_2(4)$. Since L is even,

$$L < 2A_2 \oplus A_2(4) \oplus D_6.$$

This overlattice contains forbidden component (43), i.e. $A_2(4) \oplus D_6$.

(101) $A_{1,2}^9 B_{3,5}$: $C = 1$, $\mathbf{P_g} = \mathbb{Z}^9 \oplus \mathbb{Z}^3(5)$. The maximal even sublattice of $\mathbf{P_g}$ is $A_4'(5) \oplus D_8$. Therefore,

$$L < A_4'(5) \oplus D_8 < A_4 \oplus D_8.$$

The overlattice $A_4 \oplus D_8$ is forbidden component (62).

(102) $A_{1,2}^3 A_{2,3}^3 A_{3,4}$: $C = 1$, $\mathbf{P_g} = \mathbb{Z}^3 \oplus 3A_2 \oplus A_3'(4)$. The maximal even sublattice of $\mathbf{P_g}$ splits as $Q \oplus 3A_2$, where Q is an index two sublattice of $2A_1 \oplus D_4$ in the genus $2_6^2 4^2$. Therefore,

$$L < Q \oplus 3A_2 < 2A_1 \oplus 3A_2 \oplus D_4.$$

This overlattice contains forbidden component (49), i.e. $2A_1 \oplus 2A_2 \oplus D_4$.

(103) $A_{1,2}^5 A_{2,3} A_{3,4} B_{2,3}$: $C = 1$, $\mathbf{P_g} = \mathbb{Z}^5 \oplus A_2 \oplus A_3'(4) \oplus \mathbb{Z}^2(3)$. The maximal even sublattice of $\mathbf{P_g}$ is $2A_2 \oplus A_2(4) \oplus D_6$. This is an even overlattice of L containing forbidden component (43), i.e. $A_2(4) \oplus D_6$.

(104) $A_{1,2}^8 B_{2,3} G_{2,4}$: $C = 1$, $\mathbf{P_g} = \mathbb{Z}^8 \oplus \mathbb{Z}^2(3) \oplus A_2(4)$. The maximal even sublattice of $\mathbf{P_g}$ is $2A_2 \oplus A_2(4) \oplus D_6$. This is an even overlattice of L containing forbidden component (43), i.e. $A_2(4) \oplus D_6$.

(105) $A_{1,2}^6 A_{3,4}^2$: $C = 1$, $\mathbf{P_g} = \mathbb{Z}^6 \oplus 2A_3'(4)$. The maximal even sublattice of $\mathbf{P_g}$ is a sublattice of $A_1 \oplus D_4 \oplus L_6$, where L_6 is a rank seven lattice in the genus $2_7^1 4_{II}^2$. Therefore,

$$L < A_1 \oplus D_4 \oplus L_6.$$

This overlattice contains forbidden component (59), i.e. $L_6 \oplus D_4$.

(106) $A_{1,2}^2 A_{2,3}^4 B_{2,3}$: $C = 1$, $\mathbf{P_g} = \mathbb{Z}^2 \oplus 4A_2 \oplus \mathbb{Z}^2(3)$. The maximal even sublattice of $\mathbf{P_g}$ is $4A_2 \oplus L_7$, where L_7 has genus $2_{II}^2 3^2$. The overlattice $4A_2 \oplus L_7$ of L contains forbidden component (56), i.e. $3A_2 \oplus L_7$.

(107) $A_{1,2}^8 A_{4,5}$: $C = 1$, $\mathbf{P_g} = \mathbb{Z}^8 \oplus A_4'(5)$. The maximal even sublattice of $\mathbf{P_g}$ is $A_4'(5) \oplus D_8$. Therefore,
$$L < A_4'(5) \oplus D_8 < A_4 \oplus D_8.$$
The overlattice $A_4 \oplus D_8$ is forbidden component (62).

(108) $A_{1,2}^4 A_{2,3}^2 B_{2,3}^2$: $C = 1$, $\mathbf{P_g} = \mathbb{Z}^4 \oplus 2A_2 \oplus \mathbb{Z}^4(3)$. The maximal even sublattice of $\mathbf{P_g}$ is $4A_2 \oplus L_7$, where L_7 has genus $2_{II}^2 3^2$ as before. The overlattice $4A_2 \oplus L_7$ of L contains forbidden component (56), i.e. $3A_2 \oplus L_7$.

(109) $A_{1,2}^9 C_{3,4}$: $C = 1$, $\mathbf{P_g} = \mathbb{Z}^9 \oplus A_3'(8)$. Since L is even,
$$L < A_3'(8) \oplus D_9 < 3A_1 \oplus D_9,$$
using the rule $A_3'(8) < 3A_1$. The overlattice $D_9 \oplus 3A_1$ is forbidden component (64).

(110) $A_{1,2}^6 B_{2,3}^3$: $C = 1$, $\mathbf{P_g} = \mathbb{Z}^6 \oplus \mathbb{Z}^6(3)$. The maximal even sublattice of $\mathbf{P_g}$ is $4A_2 \oplus L_7$, where L_7 is the maximal even sublattice of $\mathbb{Z}^2 \oplus \mathbb{Z}^2(3)$ as before. The overlattice $4A_2 \oplus L_7$ of L contains forbidden component (56), i.e. $3A_2 \oplus L_7$.

(111) $A_{3,2}^2 G_{2,2}^3$: $C = 2$, $L = 3A_2(2) \oplus K$ with $2A_3(2) < K < 2A_3'(2)$. Then $K < \mathbb{Z}^2(1/2) \oplus \mathbb{Z}^4$. Thus $K < \mathbb{Z}^6$ and further $K < D_6$. Therefore, $L < 3A_2(2) \oplus D_6$. This overlattice contains forbidden component (51), i.e. $2A_2(2) \oplus D_6$.

(112) $A_{1,1}^3 C_{3,2} D_{4,3} G_{2,2}$: $C = 2$, $L = A_2(2) \oplus K$ with
$$3A_1 \oplus 3A_1(2) \oplus D_4(3) < K < \mathbb{Z}^3(1/2) \oplus A_3'(4) \oplus D_4'(3).$$
Then K is contained in a maximal even overlattice of $3A_1 \oplus 3A_1(2) \oplus D_4(3)$ in the genus of $A_1 \oplus A_1(2) \oplus E_8$. Therefore, L is contained in a lattice in the genus of $A_1 \oplus A_1(2) \oplus A_2(2) \oplus E_8$ which contains forbidden component (58), i.e. $A_1 \oplus A_2(2) \oplus E_8$.

(113) $A_{1,1}^3 C_{3,2} G_{2,2}^3$: $C = 2$, $L = 3A_2(2) \oplus K$ with
$$3A_1 \oplus 3A_1(2) < K < \mathbb{Z}^3(1/2) \oplus A_3'(4).$$
Then K is contained in a maximal even overlattice of $3A_1 \oplus 3A_1(2)$ in the genus of $A_1 \oplus D_5$. Therefore, L is contained in a lattice in the genus of $A_1 \oplus D_5 \oplus 3A_2(2)$ which contains forbidden component (42), i.e. $A_1 \oplus A_2(2) \oplus D_5$.

(114) $A_{1,1}^3 C_{3,2}^3$: $C = 2$, $\mathbf{Q_g} = 3A_1 \oplus 9A_1(2)$. Then L is contained in a maximal even overlattice of $3A_1 \oplus 9A_1(2)$ in the genus of $A_1 \oplus A_3 \oplus E_8$ which is forbidden component (63).

(115) $A_{3,2}^2 D_{4,3} G_{2,2}$: $C = 2$, $L = A_2(2) \oplus K$ with $2A_3(2) \oplus D_4(3) < K$. Then K is contained in a maximal even overlattice of $2A_3(2) \oplus D_4(3)$ in the genus of $2A_1 \oplus E_8$. Therefore, L is contained in a lattice in the genus of $2A_1 \oplus A_2(2) \oplus E_8$ which contains forbidden component (58), i.e. $A_1 \oplus A_2(2) \oplus E_8$.

(116) $A_{3,2}^2 C_{3,2}^2$: $C = 2$,
$$2A_3(2) \oplus 6A_1(2) < L < 2A_3'(2) \oplus 2A_3'(4).$$
We see from the upper bound that $L < \mathbb{Z}^2(1/2) \oplus \mathbb{Z}^4 \oplus 2A_3'(4)$. Therefore, $L < \mathbb{Z}^6 \oplus 2A_3'(4)$. The maximal even sublattice of $\mathbb{Z}^6 \oplus 2A_3'(4)$ is forbidden component (65).

(117) $A_{1,1}^3 A_{5,3} G_{2,2}^2$: $C = 2$, $L = 2A_2(2) \oplus K$ with
$$3A_1 \oplus A_5(3) < K.$$
Then K is contained in a maximal even overlattice of $3A_1 \oplus A_5(3)$ in the genus of $2A_2 \oplus D_4$. Therefore, L is contained in a lattice in the genus of $2A_2 \oplus D_4 \oplus 2A_2(2)$ which contains forbidden component (39), i.e. $2A_2(2) \oplus D_4$.

(118) $A_{1,1}^4 A_{3,2} D_{5,4}$: $C = 2$, $\mathbf{Q_g} = 4A_1 \oplus A_3(2) \oplus D_5(4) < L$. Then L is contained in a maximal even overlattice of $4A_1 \oplus A_3(2) \oplus D_5(4)$ in the genus of $A_1 \oplus A_3 \oplus E_8$ which is forbidden component (63).

(119) $A_{2,1}^2 D_{4,2} F_{4,3}$: $C = 3$, $L = D_4(3) \oplus K$ with

$$2A_2 \oplus D_4(2) < K < 2A_2' \oplus D_4.$$

Thus $K < 2A_2 \oplus D_4$. Using the rule $D_4(3) < A_2 \oplus A_2(2)$, we have

$$L < 2A_2 \oplus D_4 \oplus D_4(3) < 3A_2 \oplus D_4 \oplus A_2(2).$$

This overlattice contains forbidden component (50), i.e. $D_4 \oplus 2A_2 \oplus A_2(2)$.

(120) $C_{3,1}^2 E_{6,3}$: $C = 4$, L is bounded by

$$6A_1 \oplus E_6(3) < L < 2A_3'(2) \oplus E_6'(3).$$

Therefore, L is contained in a maximal even overlattice of $6A_1 \oplus E_6(3)$ in the genus of $D_6 \oplus E_6$, which is forbidden component (60).

11.2.9. *Rank 14.* There are **10** extraneous root systems of rank 14.

(121) $A_{1,2}^8 A_{2,3}^3$: $C = 1$, $\mathbf{P_g} = \mathbb{Z}^8 \oplus 3A_2$. Since L is even, we have

$$L < 3A_2 \oplus D_8.$$

This overlattice contains forbidden component (61), i.e. $2A_2 \oplus D_8$.

(122) $A_{1,2}^{11} A_{3,4}$: $C = 1$, $\mathbf{P_g} = \mathbb{Z}^{11} \oplus A_3'(4)$. The maximal even sublattice of $\mathbf{P_g}$ is $L_8 \oplus E_8$, where L_8 has genus $2_6^2 4_{\mathrm{II}}^2$. The overlattice $L_8 \oplus E_8$ of L is forbidden component (71).

(123) $A_{1,2}^{10} A_{2,3} B_{2,3}$: $C = 1$, $\mathbf{P_g} = \mathbb{Z}^{10} \oplus A_2 \oplus \mathbb{Z}^2(3)$. The maximal even sublattice of $\mathbf{P_g}$ is $A_2 \oplus E_8 \oplus L_7$, where L_7 has genus $2_{\mathrm{II}}^2 3^2$ as before. Then

$$L < A_2 \oplus E_8 \oplus L_7 < 3A_2 \oplus E_8 < E_6 \oplus E_8$$

using $L_7 < 2A_2$ and $3A_2 < E_6$. The overlattice $E_6 \oplus E_8$ of L is forbidden component (70).

(124) $A_{2,2}^5 B_{2,2}^2$: $C = 3/2$, L is bounded by

$$5A_2(2) \oplus 4A_1(2) < L < 5A_2'(2) \oplus 4A_1.$$

The projection K of L to the first component $5A_2'(2)$ is even and integral, because L itself is even and $4A_1$ is even. Therefore, $L < K \oplus 4A_1$, where K is an even sublattice of $5A_2'(2)$ containing $5A_2(2)$. Then K is contained in the maximal even overlattice $A_2 \oplus E_8$ of $5A_2(2)$. Therefore,

$$L < A_2 \oplus 4A_1 \oplus E_8.$$

This overlattice contains forbidden component (66), i.e. $3A_1 \oplus A_2 \oplus E_8$.

(125) $A_{1,1}^5 A_{3,2} C_{3,2}^2$: $C = 2$, L is bounded by

$$5A_1 \oplus A_3(2) \oplus 6A_1(2) < L < \mathbb{Z}^5(1/2) \oplus A_3'(2) \oplus 2A_3'(4).$$

We see from the upper bound that $L < \mathbb{Z}^6(1/2) \oplus \mathbb{Z}^2 \oplus 2A_3'(4)$. Then L is contained in a maximal integral lattice of $\mathbb{Z}^6(1/2) \oplus \mathbb{Z}^2 \oplus 2A_3'(4)$ which is in the genus of $\mathbb{Z}^8 \oplus 2A_3'(4)$. Since L is even, it is contained in the maximal even sublattice of that, which is forbidden component (72).

(126) $A_{1,1}^9 D_{5,4}$: $C = 2$, $\mathbf{Q_g} = 9A_1 \oplus D_5(4) < L$. Then L is contained in a maximal even overlattice of $9A_1 \oplus D_5(4)$, in the genus of $A_1 \oplus D_5 \oplus E_8$. This is forbidden component (69).

(127) $A_{1,1}^5 A_{3,2} G_{2,2}^3$: $C = 2$, $L = 3A_2(2) \oplus K$ with

$$5A_1 \oplus A_3(2) < K < \mathbb{Z}^5(1/2) \oplus A_3'(2).$$

The maximal even overlattice of $5A_1 \oplus A_3(2)$ is E_8. Using $2A_2(2) < 2A_2$, we have

$$L < E_8 \oplus 3A_2(2) < E_8 \oplus 2A_2 \oplus A_2(2)$$

This overlattice is forbidden component (67).

(128) $A_{1,1}^5 A_{3,2} D_{4,3} G_{2,2}$: $C = 2$, $L = A_2(2) \oplus K$ with $5A_1 \oplus A_3(2) \oplus D_4(3) < K$. The maximal even overlattice of $5A_1 \oplus A_3(2) \oplus D_4(3)$ is in the genus of $E_8 \oplus D_4$. Therefore,

$$L < A_2(2) \oplus E_8 \oplus D_4.$$

This overlattice contains forbidden component (68), i.e. $E_8 \oplus D_4 \oplus A_2(2)$.

(129) $A_{1,1}^2 A_{3,2}^3 C_{3,2}$: $C = 2$, $\mathbf{Q_g} = 2A_1 \oplus 3A_3(2) \oplus 3A_1(2) < L$. Therefore, L is contained in a maximal even overlattice of $2A_1 \oplus 3A_3(2) \oplus 3A_1(2)$ in the genus $A_1 \oplus D_5 \oplus E_8$. This overlattice is forbidden component (69).

(130) $A_{2,1}^3 B_{2,1}^2 F_{4,3}$: $C = 3$, $L = D_4(3) \oplus K$ with $3A_2 \oplus 4A_1 < K$. The maximal even overlattice of $3A_2 \oplus 4A_1$ is $E_6 \oplus D_4$. Therefore,

$$L < E_6 \oplus D_4 \oplus D_4(3) < E_6 \oplus 2D_4 < E_6 \oplus E_8$$

using the rules $D_4(3) < D_4$ and $2D_4 < E_8$. The overlattice $E_8 \oplus E_6$ of L is forbidden component (70).

11.2.10. *Rank 16.* There are **5** extraneous root systems of rank 16.

(131) $A_{1,1}^{10} C_{3,2}^2$: $C = 2$, L is bounded by

$$10A_1 \oplus 6A_1(2) < L < \mathbb{Z}^{10}(1/2) \oplus 2A_3'(4).$$

Then L is contained in a maximal integral sublattice of $\mathbb{Z}^{10}(1/2) \oplus 2A_3'(4)$ which is of genus $\mathbb{Z}^{10} \oplus 2A_3'(4)$. Therefore, L is contained in the maximal even sublattice of that, which is in the genus of $2A_3 \oplus D_7$ and contains forbidden component (54), i.e $A_3 \oplus D_7$.

(132) $A_{1,1}^{10} G_{2,2}^3$: $C = 2$, $L = 3A_2(2) \oplus K$ with $10A_1 < K$. Any maximal even overlattice of $10A_1$ is in the genus of $E_8 \oplus 2A_1$. Therefore,

$$L < E_8 \oplus 2A_1 \oplus 3A_2(2).$$

This overlattice contains forbidden component (58), i.e. $A_1 \oplus A_2(2) \oplus E_8$.

(133) $A_{1,1}^7 A_{3,2}^2 C_{3,2}$: $C = 2$, $\mathbf{Q_g} = 7A_1 \oplus 2A_3(2) \oplus 3A_1(2)$. Then L is contained in a maximal even overlattice of $\mathbf{Q_g}$ which lies in the genus of $A_1 \oplus D_7 \oplus E_8$. This overlattice is forbidden component (74).

(134) $A_{1,1}^{10} D_{4,3} G_{2,2}$: $C = 2$, $L = A_2(2) \oplus K$ with $10A_1 \oplus D_4(3) < K$. The maximal even overlattices of $10A_1 \oplus D_4(3)$ are all in the genus of $E_8 \oplus D_6$. Therefore,

$$L < E_8 \oplus D_6 \oplus A_2(2).$$

This is forbidden component (73).

(135) $A_{3,1} A_{7,2} G_{2,1}^3$: $C = 4$, $L = 3A_2 \oplus K$ with $A_3 \oplus A_7(2) < K$. The maximal even overlattices of $A_3 \oplus A_7(2)$ are all in the genus of $2A_1 \oplus E_8$. Therefore,

$$L < 2A_1 \oplus 3A_2 \oplus E_8 < 2A_1 \oplus E_6 \oplus E_8.$$

This overlattice contains forbidden component (70), i.e. $E_8 \oplus E_6$.

11.2.11. *Rank 18.* There are **7** extraneous root systems of rank 18.

(136) $A_{1,1}^9 A_{3,2}^3$: $C = 2$, $\mathbf{Q_g} = 9A_1 \oplus 3A_3(2)$, and $\mathbf{Q_g} < L$. The maximal even overlattice of $\mathbf{P_g}$ is in the genus $2A_1 \oplus 2E_8$. Then L is contained in a certain lattice K in the genus $2A_1 \oplus 2E_8$. Since there is a unique (up to powers) reflective Borcherds product on

$$2U \oplus K \cong 2U \oplus 2A_1 \oplus 2E_8$$

and this product has weight 42, we have $L \neq K$. Note that $\mathrm{discr}(\mathbf{Q_g}) = 2^{24}$, and therefore $\mathrm{discr}(L) = 2^a$ with an integer a. Therefore, L is contained in an index-two sublattice of K, which is either in the genus of $2E_8 \oplus 2A_1(2)$ or in the genus of $E_8 \oplus D_8 \oplus 2A_1$. These two lattices are respectively forbidden components (78) and (76), so neither case can occur.

(137) $A_{1,1}^{12} A_{3,2} C_{3,2}$: $C = 2$, $\mathbf{Q_g} = 12A_1 \oplus A_3(2) \oplus 3A_1(2)$. Since $\mathbf{Q_g} < L$, the lattice L is contained in a maximal even overlattice of $\mathbf{Q_g}$ which lies in the genus of $A_1 \oplus A_1(2) \oplus 2E_8$. This overlattice is forbidden component (77).

(138) $A_{2,1}^5 D_{4,2}^2$: $C = 3$, $\mathbf{Q_g} = 5A_2 \oplus 2D_4(2)$, $\mathbf{P_g} = 5A_2' \oplus 2D_4$, and $\mathbf{Q_g} < L < \mathbf{P_g}$. The maximal even overlattices of $\mathbf{Q_g}$ lie in the genus of $A_2 \oplus 2E_8$. Since L is contained in a certain lattice K of this genus and also in $\mathbf{P_g}$, it follows that L is contained in an index 4 sublattice of K in the genus of $A_2 \oplus 2D_4 \oplus E_8$. This overlattice is forbidden component (75).

(139) $A_{2,1} B_{2,1}^6 D_{4,2}$: $C = 3$, $\mathbf{P_g} = A_2' \oplus \mathbb{Z}^{12} \oplus D_4$. Since L is an even sublattice of $\mathbf{P_g}$, we obtain

$$L < A_2 \oplus D_{12} \oplus D_4.$$

Note that

$$2U \oplus A_2 \oplus D_{12} \oplus D_4 \cong 2U \oplus A_2 \oplus 2D_4 \oplus E_8.$$

This case can then be excluded using forbidden component (75), i.e. $A_2 \oplus 2D_4 \oplus E_8$.

(140) $A_{3,1} C_{3,1}^5$: $C = 4$, $\mathbf{Q_g} = A_3 \oplus 15A_1$, and $\mathbf{Q_g} < L$. Then L is contained in a maximal even overlattice of $\mathbf{Q_g}$ which lies in the genus of $A_1 \oplus A_1(2) \oplus 2E_8$. This overlattice is forbidden component (77).

(141) $A_{3,1} C_{3,1} G_{2,1}^6$: $C = 4$, $L = 6A_2 \oplus K$ with $A_3 \oplus 3A_1 < K$. The maximal even overlattice of $A_3 \oplus 3A_1$ is $D_5 \oplus A_1$. Therefore,

$$L < D_5 \oplus A_1 \oplus 6A_2.$$

This overlattice contains forbidden component (57), i.e. $D_5 \oplus 3A_2$.

(142) $A_{3,1} C_{3,1}^3 G_{2,1}^3$: $C = 4$, $L = 3A_2 \oplus K$ with $A_3 \oplus 9A_1 < K$. The maximal even overlattices of $A_3 \oplus 9A_1$ are all in the genus of $E_8 \oplus A_3 \oplus A_1$. Therefore,

$$L < E_8 \oplus A_3 \oplus A_1 \oplus 3A_2 < E_8 \oplus E_6 \oplus A_3 \oplus A_1.$$

This overlattice contains forbidden component (70), i.e. $E_8 \oplus E_6$.

11.2.12. *Rank larger than 18.* There are **10** extraneous root systems of rank greater than 18, but they cannot admit reflective Borcherds products of singular weight by [115, Theorem 1.5].

12. The classification of symmetric reflective modular forms

In this section we prove the symmetric cases of Theorem 5.1:

Theorem 12.1. *If $2U \oplus L$ has a symmetric reflective Borcherds product of singular weight whose Jacobi form input has non-negative q^0-term, then the associated semi-simple Lie algebra $\mathfrak{g}$ defined in Theorem 4.1 satisfies $1/C \in \mathbb{Z}$.*

The Lie algebras $\mathfrak{g}$ as above are listed in Tables 8 and 2.

To prove the theorem we need the following generalization of [110, Lemma 3.3].

Lemma 12.2. *Let $U \oplus K \oplus L$ be an even lattice of signature $(l, 2)$ with $l \geq 3$ and set $M = U \oplus K$. Suppose that $M \oplus L$ has a reflective Borcherds product which vanishes on some $\lambda^{\perp}$ with $\lambda \in M'$. Let L_1 be an even overlattice of L. Then $M \oplus L_1$ also has a reflective Borcherds product vanishing on $\lambda^{\perp}$.*

Proof. Let f be the input of the Borcherds product on $M \oplus L$ as a vector-valued modular form. Recall that the principal part of f was described in [110, Lemma 2.1]. Note that

$$M \oplus L < M \oplus L_1 < M' \oplus L_1' < M' \oplus L'.$$

By applying the arrow operator of [14, Lemma 5.6] to f, we obtain a weakly holomorphic modular form of weight $1 - l/2$ for the Weil representation $\rho_{M \oplus L_1}$ that is given by

$$f| \uparrow_{M \oplus L}^{M \oplus M_1} = \sum_{\gamma \in (M' \oplus L_1')/(M \oplus L_1)} \sum_{n \in \mathbb{Z} - \gamma^2/2} \sum_{x \in (\gamma + M \oplus L_1)/(M \oplus L)} c(x, n) q^n e_{\gamma},$$

where $c(x, n)$ are the coefficients of $q^n e_x$ in the Fourier expansion of f. We will write

$$f| \uparrow_{M \oplus L}^{M \oplus M_1} = \sum_{\gamma \in (M' \oplus L_1')/(M \oplus L_1)} c'(\gamma, n) q^n e_{\gamma}.$$

It is not hard to see that the nonzero coefficients of $f| \uparrow_{M \oplus L}^{M \oplus M_1}$ are reflective (similarly to the proof of [110, Lemma 3.3]); it is less obvious that $f| \uparrow_{M \oplus L}^{M \oplus M_1}$ is not identically zero. Suppose without loss of generality that λ is primitive in M', and write $\lambda^2 = 2/d$ where $d \in \mathbb{N}$. We have two cases to consider.

(a) If the order of λ in the discriminant group of $M \oplus L$ is $\mathrm{ord}(\lambda) = d$, then we know by [110, Section 2.1 and Lemma 3.3] that $c(\lambda, -1/d) > 0$ and thus

$$c'(\lambda, -1/d) = \sum_{\substack{x \in (\lambda + M \oplus L_1)/(M \oplus L) \\ \mathrm{ord}(x) = d}} c(x, -1/d) \geq c(\lambda, -1/d) > 0,$$

because every $c(x, -1/d)$ is non-negative.

(b) If $\mathrm{ord}(\lambda) \neq d$, then $\mathrm{ord}(\lambda) = d/2$ and $d/2$ is even. In this case,

$$c(2\lambda, -4/d) + c(\lambda, -1/d) > 0$$

and therefore

$$c'(2\lambda, -4/d) + c'(\lambda, -1/d) = \sum_{\substack{x \in (\lambda + M \oplus L_1)/(M \oplus L) \\ \mathrm{ord}(x) = d}} c(x, -1/d)$$

$$+ \sum_{\substack{y \in (\lambda + M \oplus L_1)/(M \oplus L) \\ \mathrm{ord}(y) = d/2}} \Big(c(2y, -4/d) + c(y, -1/d) \Big)$$

$$\geq c(2\lambda, -4/d) + c(\lambda, -1/d) > 0,$$

because $c(x, -1/d) \geq 0$ and $c(2y, -4/d) + c(y, -1/d) \geq 0$.

In particular, $\mathbf{B}(f| \uparrow_{M \oplus L}^{M \oplus M_1})$ is a reflective Borcherds product on $M \oplus L_1$ that vanishes on $\lambda^{\perp}$. Note that the coefficients of $q^{-1} e_0$ in the Fourier expansions of f and $f| \uparrow_{M \oplus L}^{M \oplus M_1}$ are the same, so $\mathbf{B}(f| \uparrow_{M \oplus L}^{M \oplus M_1})$ remains symmetric if $\mathbf{B}(f)$ was. $\qquad \square$

Proof of Theorem 12.1. To prove the theorem we have to rule out the 5 extraneous semi-simple $\mathfrak{g}$ of symmetric type with non-integral $1/C$. We do this by cases.

(1) $A_{2,4}B_{2,4}$: The lattice L is bounded by
$$A_2(4) \oplus 2A_1(4) < L < A_2'(4) \oplus 2A_1(2).$$
Since L is even, we conclude
$$A_2(4) \oplus 2A_1(4) < L < A_2(4) \oplus 2A_1(2).$$
It follows that $L = A_2(4) \oplus 2A_1(4)$ or $A_2(4) \oplus 2A_1(2)$. By a direct calculation, we can prove that $2U \oplus A_2(4) \oplus 2A_1(2)$ has no symmetric reflective Borcherds product of any weight. Lemma 12.2 then shows that $2U \oplus L$ has no symmetric reflective Borcherds products.

(2) $A_{2,2}D_{4,4}$: The lattice L is bounded by
$$A_2(2) \oplus D_4(4) < L < A_2'(2) \oplus D_4(2),$$
and further by
$$A_2(2) \oplus D_4(4) < L < A_2(2) \oplus D_4(2).$$
Since both L and $L(3/2)$ are integral, $L(1/2)$ is also integral. Since
$$A_2 \oplus D_4(2) < L(1/2) < A_2 \oplus D_4,$$
$L(1/2)$ can only be $A_2 \oplus D_4(2)$, $A_2 \oplus 4A_1$ or $A_2 \oplus D_4$, and therefore $L = A_2(2) \oplus D_4(4)$, $A_2(2) \oplus 4A_1(2)$ or $A_2(2) \oplus D_4(2)$. We were able to check by a direct calculation that there are no symmetric reflective Borcherds products of singular weight on $2U \oplus L$ in the latter two cases; in the first case, the discriminant was prohibitively large and we needed a more subtle argument. Let v_2 be a 2-root of D_4 and write K for the lattice generated by D_4 and $v_2/2$, such that $K(4)$ is an even lattice of discriminant $4^4 = 256$. We were able to compute that there are no symmetric reflective Borcherds products of any weights on $2U \oplus A_2(2) \oplus K(4)$. By Lemma 12.2, there are also no symmetric reflective Borcherds products of any weight on $2U \oplus A_2(2) \oplus D_4(4)$.

(3) $A_{2,2}^2 B_{2,2}^2$: The lattice L is bounded by
$$2A_2(2) \oplus 4A_1(2) < L < 2A_2'(2) \oplus 4A_1.$$
Since both L and $L(3/2)$ are integral, $L(1/2)$ is also integral. From
$$2A_2 \oplus 4A_1 < L(1/2) < 2A_2' \oplus \mathbb{Z}^4,$$
and the integrality of $L(1/2)$, it follows that
$$2A_2 \oplus 4A_1 < L(1/2) < 2A_2 \oplus \mathbb{Z}^4.$$
This forces $L(1/2)$ to be one of $2A_2 \oplus 4A_1$, $A_2 \oplus 2A_1 \oplus \mathbb{Z}^2$, $2A_2 \oplus D_4$ or $2A_2 \oplus \mathbb{Z}^4$. Therefore, $L = 2A_2(2) \oplus 4A_1(2)$, $2A_2(2) \oplus 2A_1(2) \oplus 2A_1$, $2A_2(2) \oplus D_4(2)$ or $2A_2(2) \oplus 4A_1$. Here, $2A_2(2) \oplus D_4$ was the forbidden component (39) - there is no reflective Borcherds product (symmetric or anti-symmetric) of any weight on $2U \oplus 2A_2(2) \oplus D_4$. Since L is of type $2A_2(2) \oplus K$ and $K < D_4$, we can use Lemma 12.2 to conclude that $2U \oplus L$ has no symmetric reflective Borcherds products of any weight.

(4) $A_{4,2}C_{4,2}$: The lattice L is bounded by
$$A_4(2) \oplus 4A_1(2) < L < A_4'(2) \oplus D_4(2).$$
Since L is even, we have
$$A_4(2) \oplus 4A_1(2) < L < A_4(2) \oplus D_4(2).$$
It follows that $L = A_4(2) \oplus 4A_1(2)$ or $A_4(2) \oplus D_4(2)$. By direct calculation, we were able to verify that there are no reflective Borcherds products of any weight on $2U \oplus A_4(2) \oplus D_4$. Lemma 12.2 then shows that $2U \oplus L$ also has no symmetric reflective Borcherds products of any weight.

(5) $A_{6,2}B_{4,2}$: The lattice L is bounded by

$$A_6(2) \oplus D_4(2) < L < A_6'(2) \oplus 4A_1.$$

Since L is even, we have

$$A_6(2) \oplus D_4(2) < L < A_6(2) \oplus 4A_1.$$

It follows that $L = A_6(2) \oplus D_4(2)$ or $A_6(2) \oplus 4A_1$. By direct calculation, we were able to verify that there are no symmetric reflective Borcherds products of any weight on $2U \oplus A_6(2)$. Using the pullback from $2U \oplus L$ to $2U \oplus A_6(2)$, this rules out the root system $A_{6,2}B_{4,2}$.

□

13. Application: anti-symmetric paramodular forms of weight 3

The restrictions of reflective Borcherds products of singular weight on appropriate lattices yield anti-symmetric Siegel paramodular forms of degree 2 and weight 3.

Siegel paramodular forms of degree two and level t are holomorphic functions on the genus-two Siegel upper half space

$$\mathbb{H}_2 = \{Z = (\begin{smallmatrix} \tau & z \\ z & \omega \end{smallmatrix}) \in M(2, \mathbb{C}) : \operatorname{Im} Z > 0\}$$

which are modular under the level t paramodular group

$$\Gamma_t = \begin{pmatrix} * & t* & * & * \\ * & * & * & */t \\ * & t* & * & * \\ t* & t* & t* & * \end{pmatrix} \cap \operatorname{Sp}_2(\mathbb{Q}), \quad \text{all } * \in \mathbb{Z}.$$

These can be realized as modular forms on orthogonal groups of the lattice $2U \oplus A_1(t)$ of signature $(3, 2)$. More precisely, modular forms on $\widetilde{\operatorname{O}}^+(2U \oplus A_1(t))$ correspond exactly to Siegel paramodular forms that are modular under the normal extension

$$\Gamma_t^+ = \Gamma_t \cup \Gamma_t V_t, \qquad V_t = \frac{1}{\sqrt{t}} \begin{pmatrix} 0 & t & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -t & 0 \end{pmatrix}.$$

Let $\chi_t : \Gamma_t^+ \to \{\pm 1\}$ be the unique nontrivial character with kernel Γ_t. Then $M_k(\Gamma_t)$ is decomposed into the direct sum of plus and minus V_t-eigenspaces, that is,

$$M_k(\Gamma_t) = M_k(\Gamma_t^+) \oplus M_k(\Gamma_t^+, \chi_t).$$

Moreover, symmetric modular forms of weight k on $\widetilde{\operatorname{O}}^+(2U \oplus A_1(t))$ correspond to $M_k(\Gamma_t^+, \chi_t^k)$. Therefore, we call Siegel paramodular forms in $M_k(\Gamma_t^+, \chi_t^k)$ *symmetric* and Siegel paramodular forms in $M_k(\Gamma_t^+, \chi_t^{k+1})$ *anti-symmetric*. For a paramodular eigenform, the distinction between symmetry and anti-symmetry is exactly the sign in the functional equation of the associated L-function.

Anti-symmetric Siegel paramodular forms of weight 3 have applications to birational geometry. Let t be squarefree. By [45, Proposition 1.5], the modular variety

$$\mathcal{A}_t^+ = \Gamma_t^+ \backslash \mathbb{H}_2$$

is isomorphic to *the moduli space of polarized $K3$ surfaces with a polarisation of type* $\langle 2t \rangle \oplus 2E_8(-1)$. [45, Theorem 1.5] further yields that if t is prime then the above modular variety is actually *the moduli space of Kummer surfaces associated to* $(1, t)$-*polarised abelian surfaces*. We know from [34] that the geometric genus of the variety $\mathcal{A}_t^+$ is given by

$$h^{3,0}\left(\overline{\mathcal{A}_t^+}\right) = \dim_{\mathbb{C}} S_3(\Gamma_t^+).$$

Recently, Ibukiyama [64] found a dimension formula for $S_3(\Gamma_t^+)$ for squarefree t. In particular, he proved [64, Proposition 6.1] that $\dim_{\mathbb{C}} S_3(\Gamma_t^+) > 0$ for prime t if and only if $t > 163$ and $t \neq 179, 181, 191, 193, 199, 211, 229, 241$. In particular the moduli space $\mathcal{A}_t^+$ always has positive geometric genus if $t > 241$ is prime.

In this section we construct a large infinite family of anti-symmetric Siegel paramodular forms of weight 3 that seems to be new.

Recall that there are exactly two semi-simple Lie algebras $\mathfrak{g}$ of rank 6 with integral C in Schellekens' list. Let $z \in \mathbb{C}$. For any nonzero $v \in L_{\mathfrak{g}}$ with $\vartheta_{\mathfrak{g}}(\tau, vz) \neq 0$, the pullback of the corresponding singular-weight reflective Borcherds product $\mathbf{B}(\chi_V)$ along the embedding

$$2U \oplus \mathbb{Z}v \hookrightarrow 2U \oplus L_{\mathfrak{g}}$$

defines a nonzero anti-symmetric paramodular form of weight 3 and level $v^2/2$. The case $\mathfrak{g} = A_{6,7}$ was considered in [52, Theorem 2.1]. Here we use the other Lie algebra, $\mathfrak{g} = A_{1,2}D_{5,8}$. In this case, the orbit lattice is

$$L_{\mathfrak{g}} = A_1(2) \oplus D_5'(8).$$

By restricting to vectors in $L_{\mathfrak{g}}$ we obtain the following theorem.

Theorem 13.1. *For* $\mathbf{a} = (a_1, a_2, a_3, a_4, a_5, a_6) \in \mathbb{Z}^6$*, the theta block*

(13.1)
$$\begin{aligned}
\Theta_{\mathbf{a}} = {}& \vartheta_{a_1}\vartheta_{a_2}\vartheta_{a_3}\vartheta_{a_4}\vartheta_{a_5}\vartheta_{a_1+a_2}\vartheta_{a_1+a_2+a_3}\vartheta_{a_1+a_2+a_3+a_4}\vartheta_{a_2+a_3}\vartheta_{a_2+a_3+a_4}\vartheta_{a_3+a_4} \\
& \vartheta_{a_1+a_2+a_3+a_5}\vartheta_{a_2+a_3+a_5}\vartheta_{a_3+a_5}\vartheta_{a_1+2a_2+2a_3+a_4+a_5}\vartheta_{a_1+a_2+2a_3+a_4+a_5} \\
& \vartheta_{a_1+a_2+a_3+a_4+a_5}\vartheta_{a_2+2a_3+a_4+a_5}\vartheta_{a_2+a_3+a_4+a_5}\vartheta_{a_3+a_4+a_5}\vartheta_{2a_6}/\eta^{15} \\
= {}& q^2(\cdots) \in J_{3,N(\mathbf{a})}
\end{aligned}$$

of type $\frac{21-\vartheta}{15-\eta}$ *is a holomorphic Jacobi form of weight 3 and index $N(\mathbf{a})$ for A_1, where*

$$\vartheta_b(\tau, z) := \vartheta(\tau, bz), \quad b \in \mathbb{Z},$$

and the index $N(\mathbf{a})$ is half the sum of squares of the subscripts in ϑ_*. If this theta block is not identically zero, then there exists an anti-symmetric Siegel paramodular form $F_{\mathbf{a}}$ of weight 3 and level $N(\mathbf{a})$ with trivial character whose leading Fourier–Jacobi coefficient is exactly $\Theta_{\mathbf{a}}$. Moreover, if $N(\mathbf{a})$ is squarefree then $F_{\mathbf{a}}$ is a cusp form.

In squarefree levels $t < 300$, Theorem 13.1 produces 64 anti-symmetric paramodular forms of weight 3 that are listed in Table 9. Using [52, Tables 1-3] and Table 9, we obtain a basis of $S_3(\Gamma_t^+)$ consisting of Borcherds products for the prime levels

$$t = 167,\ 173,\ 197,\ 223,\ 227,\ 239,\ 251,\ 257,\ 263,\ 269,\ 271,\ 277,\ 283,\ 293.$$

Note that Ibukiyama [64] also showed that $\dim_{\mathbb{C}} S_3(\Gamma_t^+) = 1$ for $t = 233, 281$; however, the corresponding paramodular forms cannot be obtained from the two infinite series [52, Theorem 2.2] and Theorem 13.1.

Anti-symmetric Siegel paramodular forms of weight 2 and trivial character are very interesting due to their role in the paramodular conjecture. Unfortunately, there is no semi-simple V_1 structure $\mathfrak{g}$ of rank 4 with integral C, so the above method does not work. However, there does exist a unique semi-simple V_1 structure $\mathfrak{g}$ of rank 4 with non-integral C. In particular, we can use a similar argument to construct anti-symmetric Siegel paramodular forms of weight 2 with a character of order 2 by considering the pullbacks of $\mathbf{B}(\chi_V)$ where $V_1 = \mathfrak{g} = C_{4,10}$.

14. Remarks, questions and conjectures

In this section we raise some questions and conjectures that are related to our work in this paper.

14.1. Uniqueness of hyperbolizations.

We have proved that there are exactly 81 affine Kac–Moody algebras $\hat{\mathfrak{g}}$ which have hyperbolizations. For each of these algebras, we have constructed a natural hyperbolization with underlying lattice $L_{\mathfrak{g}}$.

Question 14.1. Does every affine Lie algebra have a unique hyperbolization? More concretely, let $\mathfrak{g}$ be one of the 81 affine Lie algebras with a hyperbolization, and suppose that L is an even positive-definite lattice for which there is a singular-weight reflective Borcherds product F on $2U \oplus L$ whose associated semi-simple Lie algebra is $\mathfrak{g}$.

(a) Is the lattice L uniquely determined?
(b) Is the modular form F uniquely determined?

Note in (b) that a single function can have interpretations as a modular form on $\mathrm{O}^+(2U \oplus L)$ for different lattices L.

Recall that the lattice L must satisfy the bounds $\mathbf{Q}_{\mathfrak{g}} < L < \mathbf{P}_{\mathfrak{g}}$. When $\mathfrak{g}$ is of symmetric type, there are examples for which the lattice L is not unique:

(1) $\mathfrak{g} = A_{1,16}$: In this case, $\mathbf{Q}_{\mathfrak{g}} = A_1(16)$ and $\mathbf{P}_{\mathfrak{g}} = A_1(4)$. Then L has only two possibilities: $A_1(16)$ and $A_1(4)$. We verify by direct calculation that both $2U \oplus A_1(16)$ and $2U \oplus A_1(4)$ have unique reflective Borcherds products of singular weight; however, the two products are expansions of the same modular form.
(2) $\mathfrak{g} = A_{1,4}^4$: We have constructed a reflective Borcherds product of singular weight on $2U \oplus L_{\mathfrak{g}}$ for $L_{\mathfrak{g}} = 4A_1$. This product can also be viewed as a reflective Borcherds product on $2U \oplus D_4(2)$.
(3) $\mathfrak{g} = A_{1,8}^2$: the singular Borcherds product we constructed can be defined on $2U \oplus L$ for both $L = L_{\mathfrak{g}} = 2A_1(2)$ and $L = 2A_1(4)$.
(4) $\mathfrak{g} = A_{1,2}^8$: besides $L_{\mathfrak{g}} = D_8$, we can also take any of $L = 2D_4, D_8'(2)$, or $E_8(2)$.
(5) $\mathfrak{g} = A_{2,3}^3$: besides $L_{\mathfrak{g}} = 3A_2$, we can also take $L = E_6'(3)$.

When $\mathfrak{g} = B_{2,3}G_{2,4}$, we have $\mathbf{Q}_{\mathfrak{g}} = 2A_1(3) \oplus A_2(4)$ and $\mathbf{P}_{\mathfrak{g}} = \mathbb{Z}^2(3) \oplus A_2(4)$. Since L is even, it is contained in the maximal even sublattice of $\mathbf{P}_{\mathfrak{g}}$, which coincides with $\mathbf{Q}_{\mathfrak{g}}$. Therefore, $L = 2A_1(3) \oplus A_2(4) = L_{\mathfrak{g}}$. It follows that L is unique.

Regarding the question above, we propose the following conjecture.

Conjecture 14.2. *The singular Borcherds product F is always unique as functions on symmetric domains. For any semi-simple $\mathfrak{g}$ of anti-symmetric type, the lattice L is unique, i.e. $L = L_{\mathfrak{g}}$. Furthermore, the hyperbolization of $\mathfrak{g}$ is unique.*

Some cases of this conjecture can be proved easily, but in general it appears difficult to check. For example, [106, Theorem 4.7] implies the uniqueness of L and F for the 23 semi-simple $\mathfrak{g}$ of rank 24. When $\mathfrak{g} = E_{8,2}B_{8,1}$, we have $\mathbf{Q}_{\mathfrak{g}} = E_8(2) \oplus D_8$ and $\mathbf{P}_{\mathfrak{g}} = E_8(2) \oplus \mathbb{Z}^8$, which uniquely determines $L = E_8(2) \oplus D_8$. The uniqueness of F follows from [110, Lemma 3.2].

We proved in Section 6 that the 69 singular Borcherds products on $2U \oplus L_{\mathfrak{g}}$ for anti-symmetric $\mathfrak{g}$ come from only 11 different modular forms. The above conjecture would therefore imply the following classification.

Conjecture 14.3. *There are exactly 23 distinct modular forms that can be realized as reflective Borcherds products of singular weight on a lattice of type $2U \oplus L$ whose input forms have non-negative principal parts.*

14.2. Other questions and conjectures.

We have the following conjecture related to Remark 6.7.

Conjecture 14.4. *Let $[g]$ be a Co_0-conjugacy class such that Λ^g is nonzero. Let M_g be an even lattice of signature $(\mathrm{rk}(\Lambda^g)+2, 2)$ whose discriminant form is isomorphic to $(R(V_{\Lambda_g}^{\hat{g}}), -\mathfrak{q})$ as introduced*

by Lam [71]. Then the multiplicative theta lift of the (vector-valued) characters of $V_{\Lambda_g}^{\hat{g}}$ (divided by $\eta^{\mathrm{rk}(\Lambda^g)}$) is a reflective Borcherds product of singular weight on M_g. Moreover, the Fourier expansion of this product at a certain 0-dimensional cusp is the g-twisted denominator of the fake monster algebra.

As we mentioned in Section 6, this conjecture was proved in [113] whenever g has level equal to its order. However, the proof is indirect.

Remarks 7.7 and 7.8 lead us to ask the following questions:

Question 14.5. Let $[g]$ be a Co_0-conjugacy class whose level n_g is equal to its order. By [113], the g-twisted denominator of the fake monster algebra is the Fourier expansion of a singular-weight reflective Borcherds product Φ_g for $\mathrm{O}^+(U_1(n_g) \oplus U \oplus \Lambda^g)$ at the 0-dimensional cusp determined by $U_1(n_g)$. The function Φ_g can also be viewed as a singular-weight reflective Borcherds product on $U_1 \oplus U(n_g) \oplus (\Lambda^g)'(n_g)$, and we can consider its Fourier expansion there at the 0-dimensional cusp determined by $U(n_g)$. When $(\Lambda^g)'(n_g) \neq \Lambda^g$, this Fourier expansion may define the denominator of a new BKM superalgebra, denoted $\mathbb{G}_g'$, which is different from the g-twist of the fake monster algebra. Is $\mathbb{G}_g'$ related to a vertex algebra? If so, can we express its input Jacobi form

$$\phi_d(\tau, \mathfrak{z}) \in J_{0,(\Lambda^g)'(n_g)}^!(\Gamma_0(n_g/d)), \quad d|n_g$$

in terms of characters of the vertex algebra? Does $\mathbb{G}_g'$ have a natural construction by the BRST cohomology? Note that the corresponding vertex algebra is V^{fE_8} when g has cycle shape $1^{-8}2^{16}$ and it is the Conway SCFT $V^{f\natural}$ when g has cycle shape $1^{-24}2^{24}$.

We have the following question concerning Section 8.

Question 14.6. Let $\mathcal{G}_{\mathfrak{g}}$ denote the BKM superalgebra corresponding to any of the four affine Lie algebras possessing exceptional modular invariants. Can we realize $\mathcal{G}_{\mathfrak{g}}$ as the BRST cohomology related to a vertex algebra? Our expression of the Jacobi form input in terms of affine characters may be a hint towards finding such a desirable vertex algebra.

C	$\mathfrak{g}$
1/24	$A^3_{1,48} A^2_{2,72}$
1/24	$A^5_{1,48} B_{2,72}$
1/24	$A_{1,48} A_{2,72} G_{2,96}$
1/24	$A_{3,96} B_{2,72}$
1/12	$A^6_{1,24} A_{2,36}$
1/12	$A^4_{1,24} G_{2,48}$
1/12	$A^2_{1,24} B^2_{2,36}$
1/12	$A^2_{2,36} B_{2,36}$
1/12	$A_{1,24} A_{2,36} A_{3,48}$
1/8	$A_{1,16} B_{2,24} G_{2,32}$
1/8	$A_{1,16} A^3_{2,24}$
1/8	$A^2_{1,16} C_{3,32}$
1/8	$A^3_{1,16} A_{2,24} B_{2,24}$
1/8	$A^9_{1,16}$
1/8	$A^4_{1,16} A_{3,32}$
1/8	$A^2_{1,16} B_{3,40}$
1/8	$A_{1,16} A_{4,40}$
1/6	$A^2_{1,12} A_{2,18} G_{2,24}$
1/6	$A^6_{1,12} B_{2,18}$
1/6	$A^4_{1,12} A^2_{2,18}$
1/6	$D_{4,36}$
1/6	$G^2_{2,24}$
1/6	$A_{1,12} A_{3,24} B_{2,18}$
1/6	$A_{2,18} B^2_{2,18}$
1/4	$A^3_{1,8} C_{3,16}$
1/4	$A^{10}_{1,8}$
1/4	$A^2_{1,8} A^3_{2,12}$
1/4	$A^5_{1,8} A_{3,16}$
1/4	$A^2_{1,8} B_{2,12} G_{2,16}$
1/4	$A^3_{1,8} B_{3,20}$
1/4	$A^2_{3,16}$
1/4	$A^2_{1,8} A_{4,20}$
1/4	$A^4_{1,8} A_{2,12} B_{2,12}$
1/4	$A^2_{2,12} G_{2,16}$
1/4	$B^3_{2,12}$
1/3	$A_{2,9} B_{2,9} G_{2,12}$
1/3	$A^8_{1,6} A_{2,9}$
1/3	$A_{1,6} A_{2,9} C_{3,12}$
1/3	$A_{2,9} A_{4,15}$
1/3	$A_{1,6} A_{2,9} B_{3,15}$
1/3	$A^6_{1,6} G_{2,12}$

C	$\mathfrak{g}$
1/3	$A^2_{1,6} A^2_{2,9} B_{2,9}$
1/3	$A^4_{1,6} B^2_{2,9}$
1/3	$A^3_{1,6} A_{2,9} A_{3,12}$
1/3	$A^4_{2,9}$
1/3	$A_{1,6} A_{3,12} G_{2,12}$
1/2	$A_{2,6} G^2_{2,8}$
1/2	$A^2_{1,4} A^2_{3,8}$
1/2	$A^4_{1,4} A^3_{2,6}$
1/2	$A^4_{1,4} A_{4,10}$
1/2	$A_{1,4} A_{2,6} A_{3,8} B_{2,6}$
1/2	$A^7_{1,4} A_{3,8}$
1/2	$\color{blue}{C_{4,10}}$
1/2	$A^5_{1,4} B_{3,10}$
1/2	$A^2_{2,6} B^2_{2,6}$
1/2	$A_{3,8} B_{3,10}$
1/2	$A^6_{1,4} A_{2,6} B_{2,6}$
1/2	$A^2_{1,4} A^2_{2,6} G_{2,8}$
1/2	$B_{4,14}$
1/2	$\color{blue}{A^{12}_{1,4}}$
1/2	$A^4_{1,4} B_{2,6} G_{2,8}$
1/2	$\color{blue}{A_{2,6} D_{4,12}}$
1/2	$A^5_{1,4} C_{3,8}$
1/2	$A^2_{1,4} B^3_{2,6}$
1/2	$A_{3,8} C_{3,8}$
2/3	$A^4_{1,3} G^2_{2,6}$
2/3	$A^4_{1,3} D_{4,9}$
3/4	$A^4_{2,4} B_{2,4}$
1	$A_{1,2} A^3_{2,3} B_{3,5}$
1	$A^3_{1,2} A_{2,3} B_{2,3} C_{3,4}$
1	$A_{1,2} A_{3,4} B^3_{2,3}$
1	$A_{1,2} A_{4,5} C_{3,4}$
1	$\color{blue}{A_{6,7}}$
1	$A^4_{1,2} A_{2,3} G^2_{2,4}$
1	$A^4_{1,2} B_{4,7}$
1	$\color{blue}{A^{16}_{1,2}}$
1	$\color{blue}{A_{1,2} A_{5,6} B_{2,3}}$
1	$\color{blue}{A_{1,2} D_{5,8}}$
1	$A^6_{1,2} A^2_{2,3} G_{2,4}$
1	$A^2_{1,2} A_{2,3} B^2_{2,3} G_{2,4}$
1	$A^9_{1,2} B_{3,5}$
1	$B^2_{2,3} D_{4,6}$

C	$\mathfrak{g}$
1	$A_{1,2} B_{2,3} B_{3,5} G_{2,4}$
1	$A^3_{2,3} A_{4,5}$
1	$A_{1,2} A_{4,5} B_{3,5}$
1	$A^2_{1,2} A_{2,3} A_{4,5} B_{2,3}$
1	$A^2_{1,2} B^2_{3,5}$
1	$A_{1,2} B_{2,3} C_{3,4} G_{2,4}$
1	$A^3_{1,2} A^3_{2,3} A_{3,4}$
1	$\color{blue}{A_{1,2} A^3_{3,4}}$
1	$A^8_{1,2} A^3_{2,3}$
1	$\color{blue}{A^6_{2,3}}$
1	$A^{11}_{1,2} A_{3,4}$
1	$A^5_{1,2} A_{2,3} A_{3,4} B_{2,3}$
1	$A^8_{1,2} B_{2,3} G_{2,4}$
1	$A_{1,2} A^3_{2,3} C_{3,4}$
1	$A^3_{2,3} B_{2,3} G_{2,4}$
1	$A^4_{1,2} A_{2,3} D_{4,6}$
1	$A^4_{1,2} A_{3,4} C_{3,4}$
1	$A^3_{1,2} A_{3,4} A_{4,5}$
1	$A^2_{1,2} G^3_{2,4}$
1	$A^2_{1,2} C^2_{3,4}$
1	$A^6_{1,2} A^2_{3,4}$
1	$A_{2,3} A^2_{3,4} B_{2,3}$
1	$A^2_{1,2} A^4_{2,3} B_{2,3}$
1	$A^2_{1,2} B_{3,5} C_{3,4}$
1	$A^8_{1,2} A_{4,5}$
1	$\color{blue}{A^2_{4,5}}$
1	$A^4_{1,2} A^2_{2,3} B^2_{2,3}$
1	$A_{2,3} B^4_{2,3}$
1	$B^2_{2,3} G^2_{2,4}$
1	$A^9_{1,2} C_{3,4}$
1	$A^4_{1,2} A_{3,4} B_{3,5}$
1	$A_{4,5} B_{2,3} G_{2,4}$
1	$A_{1,2} A^2_{2,3} A_{3,4} G_{2,4}$
1	$A^4_{1,2} C_{4,5}$
1	$A^3_{1,2} A_{2,3} B_{2,3} B_{3,5}$
1	$A^{10}_{1,2} A_{2,3} B_{2,3}$
1	$A^3_{1,2} A_{3,4} B_{2,3} G_{2,4}$
1	$A^2_{1,2} D_{4,6} G_{2,4}$
1	$A^6_{1,2} B^3_{2,3}$
4/3	$C^2_{3,3} G_{2,3}$
4/3	$G^4_{2,3}$

TABLE 4. The 221 solutions of Equation (4.1) in the order of increasing C. Those allowing hyperbolization are colored blue (continued on next page).

C	$\mathfrak{g}$
$3/2$	$A_{2,2}^5 B_{2,2}^2$
$3/2$	$A_{2,2}^4 D_{4,4}$
$3/2$	$B_{2,2}^6$
$3/2$	$A_{2,2} F_{4,6}$
2	$A_{1,1}^2 C_{3,2} D_{5,4}$
2	$A_{1,1}^{10} C_{3,2}^2$
2	$A_{1,1}^5 A_{3,2} C_{3,2}^2$
2	$A_{1,1}^9 D_{5,4}$
2	$A_{3,2}^2 G_{2,2}^3$
2	$A_{1,1}^{14} A_{3,2}^2$
2	$A_{1,1}^{10} G_{2,2}^3$
2	$A_{1,1}^{19} A_{3,2}$
2	$A_{1,1}^5 A_{3,2} G_{2,2}^3$
2	$A_{1,1}^3 A_{5,3} D_{4,3}$
2	$A_{1,1}^9 A_{3,2}^3$
2	$A_{1,1}^{24}$
2	$A_{1,1}^3 C_{3,2} D_{4,3} G_{2,2}$
2	$A_{1,1}^7 A_{3,2}^2 C_{3,2}$
2	$A_{1,1}^3 A_{7,4}$
2	$A_{1,1}^2 D_{6,5}$
2	$A_{1,1}^5 A_{3,2} D_{4,3} G_{2,2}$
2	$A_{1,1}^3 C_{3,2} G_{2,2}^3$
2	$A_{1,1}^3 C_{3,2}^3$
2	$A_{3,2}^2 D_{4,3} G_{2,2}$
2	$A_{1,1}^4 A_{3,2}^4$
2	$A_{1,1}^{17} C_{3,2}$
2	$A_{1,1} C_{5,3} G_{2,2}$
2	$A_{3,2}^2 C_{3,2}^2$
2	$A_{1,1}^2 A_{3,2}^3 C_{3,2}$
2	$A_{1,1}^3 A_{5,3} G_{2,2}^2$
2	$A_{1,1}^{12} A_{3,2} C_{3,2}$
2	$A_{1,1}^{10} D_{4,3} G_{2,2}$
2	$A_{1,1}^4 A_{3,2} D_{5,4}$

C	$\mathfrak{g}$
$5/2$	$A_{4,2}^2 C_{4,2}$
$5/2$	$B_{3,2}^4$
3	$A_{2,1}^2 A_{8,3}$
3	$A_{2,1} B_{2,1} E_{6,4}$
3	$A_{2,1}^5 D_{4,2}^2$
3	$A_{2,1}^2 B_{2,1}^8$
3	$A_{2,1}^7 B_{2,1}^4$
3	$A_{2,1}^6 B_{2,1}^2 D_{4,2}$
3	$A_{2,1}^{12}$
3	$A_{2,1}^2 D_{4,2} F_{4,3}$
3	$A_{2,1}^3 B_{2,1}^2 F_{4,3}$
3	$B_{2,1}^4 D_{4,2}^2$
3	$A_{2,1}^2 A_{5,2}^2 B_{2,1}$
3	$A_{2,1} B_{2,1}^6 D_{4,2}$
$7/2$	$B_{4,2}^3$
4	$A_{3,1} C_{7,2}$
4	$A_{3,1}^5 D_{5,2}$
4	$A_{3,1} C_{3,1}^5$
4	$A_{3,1}^2 D_{5,2}^2$
4	$A_{3,1} A_{7,2} C_{3,1}^2$
4	$C_{3,1}^2 E_{6,3}$
4	$A_{3,1}^8$
4	$A_{3,1} C_{3,1} G_{2,1}^6$
4	$A_{3,1} A_{7,2} G_{2,1}^3$
4	$A_{3,1} D_{7,3} G_{2,1}$
4	$A_{3,1} C_{3,1}^3 G_{2,1}^3$
4	$E_{6,3} G_{2,1}^3$
$9/2$	$A_{8,2} F_{4,2}$
5	$A_{4,1} B_{3,1}^4 C_{4,1}$
5	$A_{4,1}^6$
5	$A_{4,1}^3 C_{4,1}^2$
5	$A_{4,1} A_{9,2} B_{3,1}$
5	$C_{4,1}^4$

C	$\mathfrak{g}$
5	$B_{3,1}^2 C_{4,1} D_{6,2}$
$11/2$	$B_{6,2}^2$
6	$D_{4,1}^6$
6	$A_{5,1} C_{5,1} E_{6,2}$
6	$A_{5,1} E_{7,3}$
6	$A_{5,1}^4 D_{4,1}$
7	$B_{4,1}^2 D_{8,2}$
7	$B_{4,1} C_{6,1}^2$
7	$A_{6,1}^4$
7	$A_{6,1} B_{4,1}^4$
8	$A_{7,1} D_{9,2}$
8	$A_{7,1}^2 D_{5,1}^2$
9	$C_{8,1} F_{4,1}^2$
9	$B_{5,1} E_{7,2} F_{4,1}$
9	$A_{8,1}^3$
10	$D_{6,1}^4$
10	$A_{9,1}^2 D_{6,1}$
11	$B_{6,1} C_{10,1}$
$23/2$	$B_{12,2}$
12	$E_{6,1}^4$
12	$A_{11,1} D_{7,1} E_{6,1}$
13	$A_{12,1}^2$
14	$D_{8,1}^3$
15	$B_{8,1} E_{8,2}$
16	$A_{15,1} D_{9,1}$
18	$D_{10,1} E_{7,1}^2$
18	$A_{17,1} E_{7,1}$
22	$D_{12,1}^2$
25	$A_{24,1}$
30	$E_{8,1}^3$
30	$D_{16,1} E_{8,1}$
46	$D_{24,1}$

TABLE 4. (continued).

C	$\mathfrak{g}$
1/8	$A_{1,16}$
1/4	$A_{1,8}^2$
1/3	$A_{2,9}$
1/2	$A_{1,4}^4$
3/4	$A_{2,4}B_{2,4}$
1	$A_{1,2}^8$

C	$\mathfrak{g}$
1	$A_{2,3}^3$
1	$A_{4,5}$
1	$A_{3,4}A_{1,2}^3$
1	$B_{2,3}G_{2,4}$
1	$B_{2,3}A_{2,3}A_{1,2}^2$
1	$B_{3,5}A_{1,2}$

C	$\mathfrak{g}$
1	$C_{3,4}A_{1,2}$
3/2	$A_{2,2}D_{4,4}$
3/2	$A_{2,2}^2 B_{2,2}^2$
5/2	$A_{4,2}C_{4,2}$
7/2	$A_{6,2}B_{4,2}$

TABLE 5. The 17 solutions of Equation (4.2) in the order of increasing C. Those allowing hyperbolization are colored blue.

g	genus	$\mathfrak{g}$	C
1^{24}	$\text{II}_{24,0}(1)$	$D_{24,1}$	46
		$D_{16,1}E_{8,1}$	30
		$E_{8,1}^3$	30
		$A_{24,1}$	25
		$D_{12,1}^2$	22
		$A_{17,1}E_{7,1}$	18
		$D_{10,1}E_{7,1}^2$	18
		$A_{15,1}D_{9,1}$	16
		$D_{8,1}^3$	14
		$A_{12,1}^2$	13
		$A_{11,1}D_{7,1}E_{6,1}$	12
		$E_{6,1}^4$	12
		$A_{9,1}^2 D_{6,1}$	10
		$A_{8,1}^3$	9
		$A_{7,1}^2 D_{5,1}^2$	8
		$A_{6,1}^4$	7
		$A_{5,1}^4 D_{4,1}$	6
		$D_{4,1}^6$	6
		$A_{4,1}^6$	5
		$A_{3,1}^8$	4
		$A_{2,1}^{12}$	3
		$A_{1,1}^{24}$	2
		$\mathbb{C}^{24}$	1
$1^8 2^8$	$\text{II}_{16,0}(2_{\text{II}}^{+10})$	$E_{8,2}B_{8,1}$	15
		$C_{10,1}B_{6,1}$	11
		$C_{8,1}F_{4,1}^2$	9
		$E_{7,2}B_{5,1}F_{4,1}$	9
		$D_{9,2}A_{7,1}$	8
		$D_{8,2}B_{4,1}^2$	7

g	genus	$\mathfrak{g}$	C
$1^8 2^8$	$\text{II}_{16,0}(2_{\text{II}}^{+10})$	$C_{6,1}^2 B_{4,1}$	7
		$E_{6,2}C_{5,1}A_{5,1}$	6
		$A_{9,2}A_{4,1}B_{3,1}$	5
		$D_{6,2}C_{4,1}B_{3,1}^2$	5
		$C_{4,1}^4$	5
		$A_{7,2}C_{3,1}^2 A_{3,1}$	4
		$D_{5,2}^2 A_{3,1}^2$	4
		$A_{5,2}^2 B_{2,1}A_{2,1}^2$	3
		$D_{4,2}^2 B_{2,1}^4$	3
		$A_{3,2}^4 A_{1,1}^4$	2
		$A_{1,2}^{16}$	1
$1^6 3^6$	$\text{II}_{12,0}(3^{-8})$	$E_{7,3}A_{5,1}$	6
		$D_{7,3}A_{3,1}G_{2,1}$	4
		$E_{6,3}G_{2,1}^3$	4
		$A_{8,3}A_{2,1}^2$	3
		$A_{5,3}D_{4,3}A_{1,1}^3$	2
		$A_{2,3}^6$	1
$1^4 2^4 4^4$	$\text{II}_{10,0}(2_2^{+2}4_{\text{II}}^{+6})$	$C_{7,2}A_{3,1}$	4
		$E_{6,4}B_{2,1}A_{2,1}$	3
		$A_{7,4}A_{1,1}^3$	2
		$D_{5,4}C_{3,2}A_{1,1}^2$	2
		$A_{3,4}^3 A_{1,2}$	1
$1^4 5^4$	$\text{II}_{8,0}(5^{+6})$	$D_{6,5}A_{1,1}^2$	2
		$A_{4,5}^2$	1
$1^2 2^2 3^2 6^2$	$\text{II}_{8,0}(2_{\text{II}}^{+6}3^{-6})$	$C_{5,3}G_{2,2}A_{1,1}$	2
		$A_{5,6}B_{2,3}A_{1,2}$	1
$1^3 7^3$	$\text{II}_{6,0}(7^{-5})$	$A_{6,7}$	1
$1^2 2^1 4^1 8^2$	$\text{II}_{6,0}(2_5^{+1}4_1^{+1}8_{\text{II}}^{+4})$	$D_{5,8}A_{1,2}$	1

TABLE 6. Höhn's construction: the 8 conjugacy classes with the same order and level.

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g	genus	$L_{\mathfrak{g}}$	$n(L_{\mathfrak{g}})$	$\mathfrak{g}$	C
2^{12}	$\mathrm{II}_{12,0}(2_{\mathrm{II}}^{-10}4_{\mathrm{II}}^{-2})$	$D_{12}(2)$	6	$B_{12,2}$	$23/2$
				$B_{6,2}^2$	$11/2$
				$B_{4,2}^3$	$7/2$
				$B_{3,2}^4$	$5/2$
				$B_{2,2}^6$	$3/2$
				$A_{1,4}^{12}$	$1/2$
		$E_8(2) \oplus D_4(2)$	3	$A_{8,2}F_{4,2}$	$9/2$
				$C_{4,2}A_{4,2}^2$	$5/2$
				$D_{4,4}A_{2,2}^4$	$3/2$
$2^3 6^3$	$\mathrm{II}_{6,0}(2_{\mathrm{II}}^{+4}4_{\mathrm{II}}^{-2}3^{+5})$	$D_4(6) \oplus A_2(2)$	2	$F_{4,6}A_{2,2}$	$1/2$
				$D_{4,12}A_{2,6}$	$1/2$
$2^2 10^2$	$\mathrm{II}_{4,0}(2_{\mathrm{II}}^{-2}4_{\mathrm{II}}^{-2}5^{+4})$	$D_4(10)$	1	$C_{4,10}$	$1/2$

TABLE 7. Höhn's construction: the 3 conjugacy classes with distinct order and level.

g	$\mathfrak{g}$	genus of $L_{\mathfrak{g}}$	$L_{\mathfrak{g}}$	$\delta_{L_{\mathfrak{g}}}$
$1^{-1}5^5$	$A_{4,5}$	$\mathrm{II}_{4,0}(5^{+3})$	$A_4'(5)$	2
$1^{-2}2^35^210^1$	$A_{1,2}B_{3,5}$	$\mathrm{II}_{4,0}(2_{\mathrm{II}}^{+2}5^{+3})$	L_1	4
$1^{-2}2^34^18^2$	$A_{1,2}C_{3,4}$	$\mathrm{II}_{4,0}(2_3^{-1}4_1^{+1}8_{\mathrm{II}}^{-2})$	$A_1(2) \oplus A_3'(8)$	$7/2$
$1^{-2}2^23^24^112^1$	$B_{2,3}G_{2,4}$	$\mathrm{II}_{4,0}(2_6^{+2}4_{\mathrm{II}}^{-2}3^{-3})$	$2A_1(3) \oplus A_2(4)$	$17/3$
$1^{-3}3^9$	$A_{2,3}^3$	$\mathrm{II}_{6,0}(3^{-3})$	$3A_2$	2
$1^{-4}2^64^4$	$A_{1,2}^3A_{3,4}$	$\mathrm{II}_{6,0}(2_6^{+2}4_{\mathrm{II}}^{-2})$	L_2	2
$1^{-4}2^53^46^1$	$A_{1,2}^2A_{2,3}B_{2,3}$	$\mathrm{II}_{6,0}(2_{\mathrm{II}}^{+2}3^{-3})$	L_3	$8/3$
$1^{-8}2^{16}$	$A_{1,2}^8$	$\mathrm{II}_{8,0}(2_{\mathrm{II}}^{+2})$	D_8	2

TABLE 8. The hyperbolizations related to F_{24}

$$L_1 = \begin{pmatrix} 4 & 2 & 2 & 2 \\ 2 & 6 & 1 & 1 \\ 2 & 1 & 6 & 1 \\ 2 & 1 & 1 & 6 \end{pmatrix} \qquad L_2 = \begin{pmatrix} 2 & 0 & 1 & 1 & 1 & 0 \\ 0 & 2 & 1 & 1 & 1 & 0 \\ 1 & 1 & 4 & 2 & 2 & 3 \\ 1 & 1 & 2 & 4 & 0 & 1 \\ 1 & 1 & 2 & 0 & 4 & 1 \\ 0 & 0 & 3 & 1 & 1 & 4 \end{pmatrix} \qquad L_3 = \begin{pmatrix} 4 & 2 & 0 & 0 & -2 & 0 \\ 2 & 4 & 0 & 0 & -1 & 0 \\ 0 & 0 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & 0 & 0 \\ -2 & -1 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{pmatrix}$$

$N(\mathbf{a})$	$\mathbf{a} = (a_1,...,a_6)$	Theta block $\cdot\, \eta^{15}$
122	$(1,1,1,1,1,1)$	$1^5 2^5 3^4 4^3 5^2 6\, 7$
138	$(-7,1,1,1,1,3)$	$1^5 2^4 3^4 4^3 5^2 6^2 7$
158	$(-8,1,1,1,1,1)$	$1^4 2^5 3^4 4^2 5^3 6\, 7\, 8$
167	$(-8,1,1,1,2,1)$	$1^4 2^5 3^3 4^3 5^2 6^2 7\, 8$
170	$(-7,1,1,1,1,5)$	$1^5 2^4 3^4 4^3 5^2 6\, 7(10)$
173	$(-8,1,1,1,2,2)$	$1^4 2^4 3^3 4^4 5^2 6^2 7\, 8$
174	$(-8,1,1,1,1,3)$	$1^4 2^4 3^4 4^2 5^3 6^2 7\, 8$
183	$(-8,1,1,1,2,3)$	$1^4 2^4 3^3 4^3 5^2 6^3 7\, 8$
186	$(-9,2,1,1,1,1)$	$1^4 2^4 3^3 4^3 5^3 6^2 7\, 9$
197	$(-8,1,1,1,2,4)$	$1^4 2^4 3^3 4^3 5^2 6^2 7\, 8^2$
202	$(-9,1,1,1,1,1)$	$1^4 2^4 3^4 4^2 5^2 6^2 7\, 8\, 9$
202	$(-9,2,1,1,1,3)$	$1^4 2^3 3^3 4^3 5^3 6^3 7\, 9$
206	$(-9,1,2,1,1,1)$	$1^4 2^3 3^3 4^5 5^3 6\, 7\, 8\, 9$
206	$(-8,1,1,1,1,5)$	$1^4 2^4 3^4 4^2 5^3 6\, 7\, 8(10)$
213	$(-9,1,1,1,2,2)$	$1^3 2^4 3^3 4^4 5^2 6^2 7\, 8\, 9$
215	$(-8,1,1,1,2,5)$	$1^4 2^4 3^3 4^3 5^2 6^2 7\, 8(10)$
218	$(-9,1,1,1,1,3)$	$1^4 2^3 3^4 4^2 5^2 6^3 7\, 8\, 9$
218	$(-9,1,1,2,2,1)$	$1^3 2^5 3^3 4^2 5^3 6\, 7^2 8\, 9$
218	$(-7,1,1,1,1,7)$	$1^5 2^4 3^4 4^3 5^2 6\, 7(14)$
222	$(-9,1,1,1,3,1)$	$1^4 2^4 3^3 4^2 5^2 6^2 7^2 8\, 9$
222	$(-9,1,2,1,1,3)$	$1^4 2^2 3^3 4^4 5^3 6^2 7\, 8\, 9$
223	$(-9,1,1,1,2,3)$	$1^3 2^4 3^3 4^3 5^2 6^3 7\, 8\, 9$
230	$(-10,2,1,1,1,1)$	$1^3 2^5 3^2 4^3 5^2 6^3 7\, 8(10)$
237	$(-9,1,1,1,2,4)$	$1^3 2^4 3^3 4^3 5^2 6^2 7\, 8^2 9$
237	$(-8,1,1,1,2,6)$	$1^4 2^4 3^3 4^3 5^2 6^2 7\, 8(12)$
238	$(-9,1,1,1,3,3)$	$1^4 2^3 3^3 4^2 5^2 6^3 7^2 8\, 9$
239	$(-10,2,1,1,2,1)$	$1^3 2^4 3^3 4^3 5^2 6^2 7^2 8(10)$
246	$(-10,2,1,1,1,3)$	$1^3 2^4 3^2 4^3 5^2 6^4 7\, 8(10)$
254	$(-10,1,1,1,1,1)$	$1^4 2^4 3^3 4^2 5^2 6\, 7^2 8\, 9(10)$
254	$(-9,1,2,1,1,5)$	$1^4 2^2 3^3 4^4 5^3 6\, 7\, 8\, 9(10)$
254	$(-8,1,1,1,1,7)$	$1^4 2^4 3^4 4^2 5^3 6\, 7\, 8(14)$
255	$(-10,1,1,1,2,1)$	$1^3 2^4 3^3 4^3 5^2 6^2 7\, 8\, 9(10)$

TABLE 9. The 64 antisymmetric paramodular cusp forms of weight 3 and squarefree level < 300 constructed from Theorem 13.1 (continued on the next page).

$N(\mathbf{a})$	$\mathbf{a} = (a_1, ..., a_6)$	Theta block $\cdot\, \eta^{15}$
255	$(-10, 2, 1, 1, 2, 3)$	$1^3 2^3 3^3 4^3 5^2 6^3 7^2 8 (10)$
262	$(-10, 1, 1, 2, 2, 1)$	$1^2 2^5 3^3 4^3 5 6^3 7 8 9 (10)$
263	$(-10, 1, 2, 1, 2, 1)$	$1^3 2^4 3^2 4^3 5^3 6^2 7 8 9 (10)$
263	$(-8, 1, 1, 1, 2, 7)$	$1^4 2^4 3^3 4^3 5^2 6^2 7 8 (14)$
266	$(-10, 1, 1, 1, 3, 1)$	$1^3 2^4 3^3 4^2 5^3 6 7^2 8 9 (10)$
266	$(-10, 1, 2, 1, 1, 3)$	$1^3 2^2 3^4 4^3 5^2 6^3 7^2 9 (10)$
266	$(-8, -3, 4, 1, 1, 1)$	$1^4 2^3 3^2 4^2 5^3 6^3 7^2 8 (11)$
269	$(-10, 1, 2, 1, 2, 2)$	$1^3 2^3 3^2 4^4 5^3 6^2 7 8 9 (10)$
269	$(-10, 2, 1, 1, 2, 4)$	$1^3 2^3 3^3 4^3 5^2 6^2 7^2 8^2 (10)$
271	$(-10, 1, 1, 1, 2, 3)$	$1^3 2^3 3^3 4^3 5^2 6^3 7 8 9 (10)$
277	$(-9, 1, 1, 1, 2, 6)$	$1^3 2^4 3^3 4^3 5^2 6^2 7 8 9 (12)$
278	$(-10, 1, 1, 2, 2, 3)$	$1^2 2^4 3^3 4^3 5 6^4 7 8 9 (10)$
278	$(-10, 2, 1, 1, 1, 5)$	$1^3 2^4 3^2 4^3 5^2 6^3 7 8 (10)^2$
282	$(-10, 1, 1, 1, 3, 3)$	$1^3 2^3 3^3 4^2 5^3 6^2 7^2 8 9 (10)$
282	$(-9, -2, 3, 1, 1, 1)$	$1^3 2^4 3^3 4^2 5^2 6^2 7^2 8 9 (11)$
282	$(-9, 2, 1, 1, 1, 7)$	$1^4 2^3 3^3 4^3 5^3 6^2 7 9 (14)$
282	$(-8, -3, 4, 1, 1, 3)$	$1^4 2^2 3^2 4^2 5^3 6^4 7^2 8 (11)$
282	$(-7, 1, 1, 1, 1, 9)$	$1^5 2^4 3^4 4^3 5^2 6 7 (18)$
285	$(-10, 1, 1, 2, 3, 2)$	$1^3 2^3 3^3 4^3 5^2 6^2 7 8^2 9 (10)$
286	$(-9, -2, 4, 1, 1, 1)$	$1^3 2^3 3^3 4^2 5^3 6^3 7 8 9 (11)$
287	$(-10, 1, 1, 1, 4, 1)$	$1^4 2^4 3^2 4^2 5 6^2 7^2 8^2 9 (10)$
287	$(-10, 2, 1, 1, 2, 5)$	$1^3 2^3 3^3 4^3 5^2 6^2 7^2 8 (10)^2$
287	$(-9, -2, 3, 1, 2, 1)$	$1^2 2^5 3^2 4^3 5^2 6^2 7^2 8 9 (11)$
293	$(-10, 1, 1, 1, 4, 2)$	$1^4 2^3 3^2 4^3 5 6^2 7^2 8^2 9 (10)$
293	$(-10, 1, 2, 1, 2, 4)$	$1^3 2^3 3^2 4^3 5^3 6^2 7 8^2 9 (10)$
293	$(-9, -2, 3, 1, 2, 2)$	$1^2 2^4 3^2 4^4 5^2 6^2 7^2 8 9 (11)$
293	$(-8, 1, 1, 1, 2, 8)$	$1^4 2^4 3^3 4^3 5^2 6^2 7 8 (16)$
295	$(-10, 1, 1, 2, 3, 3)$	$1^3 2^3 3^3 4^2 5^2 6^3 7 8^2 9 (10)$
298	$(-10, 1, 2, 1, 1, 5)$	$1^3 2^2 3^4 4^3 5^2 6^2 7^2 9 (10)^2$
298	$(-9, -2, 3, 1, 1, 3)$	$1^3 2^3 3^3 4^2 5^2 6^3 7^2 8 9 (11)$
298	$(-9, -2, 3, 2, 2, 1)$	$1^2 2^4 3^4 4 5^3 6^2 7 8^2 9 (11)$
298	$(-9, 1, 1, 1, 1, 7)$	$1^4 2^3 3^4 4^2 5^2 6^2 7 8 9 (14)$

TABLE 10. (continued).

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