Supersymmetric Casimir energy on $\mathcal{N} = 1$ conformal supergravity backgrounds

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Abstract

We provide a first principles derivation of the supersymmetric Casimir energy of $\mathcal{N} = 1$ SCFTs in four dimensions using the supercharge algebra on general conformal supergravity backgrounds that admit Killing spinors. The superconformal Ward identities imply that there exists a continuous family of conserved R-currents on supersymmetric backgrounds, as well as a continuous family of conserved currents for each conformal Killing vector. These continuous families interpolate between the consistent and covariant R-current and energy-momentum tensor. The resulting Casimir energy, therefore, depends on two continuous parameters corresponding to the choice of conserved currents used to define the energy and R-charge. This ambiguity is in addition to any possible scheme dependence due to local terms in the effective action. As an application, we evaluate the general expression for the supersymmetric Casimir energy we obtain on a family of backgrounds with the cylinder topology $\mathbb{R} \times S^3$ and admitting two supercharges of opposite R-charge. Our result is a direct consequence of the supersymmetry algebra, yet it resembles more known expressions for the non-supersymmetric Casimir energy on such backgrounds and differs from the supersymmetric Casimir energy obtained from the zero temperature limit of supersymmetric partition functions. We defer a thorough analysis of the relation between these results to future work.

Contents

1 Introduction and summary of results

The Casimir energy of a conformal field theory (CFT) is defined as its ground state energy. Placing the theory on the cylinder $\mathbb{R} \times S^{d-1}$ and in the absence of other background fields it is given by

$$\mathcal{E}_{\text{Casimir}} = \int_{S^{d-1}} d^{d-1}x\sqrt{-g}\,\langle \mathcal{T}_{tt}\rangle_{\text{g.s.}}\,, \tag{1.1}$$

where $\mathcal{T}_{\mu\nu}$ is the energy-momentum tensor and $\langle ...\rangle_{\text{g.s.}}$ stands for the expectation value in the ground state of the CFT on the cylinder. This is determined by the conformal anomaly

$$\langle \mathcal{T}^{\mu}_{\mu}\rangle = \frac{1}{(4\pi)^2}(aE - cW^2 + b\Box R)\,, \tag{1.2}$$

where E is the Euler density, W^2 the square of the Weyl tensor and the anomaly coefficients a and c depend on the specific CFT. The b-term is scheme-dependent and corresponds to the addition of a local counterterm proportional to R^4 in the effective action.

To evaluate the Casimir energy, one may integrate the conformal anomaly (1.2) in order to determine its contribution to the effective action, known as the Riegert action [1, 2]. Its derivative with respect to the background metric computes the expectation value of the energy-momentum tensor in the ground state, and hence the Casimir energy (1.1). Concequently, the Casimir energy of a CFT is in general determined by the conformal anomaly coefficients a and c, as well as any scheme-dependent terms, such as b. For example, the Casimir energy on the round unit sphere is given by (see e.g. [3])

$$\mathcal{E}_{\text{Casimir}} = \frac{3}{4}\left(a - \frac{b}{2}\right) . \tag{1.3}$$

The Casimir energy, therefore, is in general a scheme-dependent quantity.

The procedure outlined above applies to any CFT, including supersymmetric conformal field theories (SCFTs). However, supersymmetry allows several alternative approaches to computing the supersymmetric Casimir energy [3–9], while requiring that the renormalization scheme preserves supersymmetry reduces considerably the allowed scheme dependence. One common definition of the Casimir energy on $S^1_\beta \times M_3$ with M_3 a three-manifold admitting supersymmetry is given by the "zero temperature" limit, $\beta \to \infty$, of the supersymmetric partition function. Namely,

$$\mathcal{E}^{\text{susy}}_{\text{Casimir}} = -\lim_{\beta\to\infty} \frac{d}{d\beta} \log Z^{\text{susy}}_{S^1_\beta \times M_3} , \tag{1.4}$$

since the $\beta \to \infty$ limit projects the ground state energy.

For $\mathcal{N} = 1$ SCFTs on $S^1_\beta \times S^3_{b_1,b_2}$ with squashing parameters b_1 and b_2 it has been shown using supersymmetric localization that [5]

$$Z^{\text{susy}}_{S^1_\beta \times S^3} = e^{-\beta \mathcal{E}^{\text{susy}}_{\text{Casimir}}} \mathcal{I}^{\text{susy}}_{S^1_\beta \times S^3} , \tag{1.5}$$

where $\mathcal{I}_{S^1_\beta \times S^3}$ is the superconformal index and the exponential factor determines the Casimir energy through the definition (1.4), namely

$$\mathcal{E}^{\text{susy}}_{\text{Casimir}}(b_1, b_2) = \frac{4\pi}{3}(|b_1| + |b_2|)(\mathbf{a} - \mathbf{c}) + \frac{4\pi}{27}\frac{(|b_1| + |b_2|)^3}{|b_1||b_2|}(3\mathbf{c} - 2\mathbf{a}) . \tag{1.6}$$

For $b_1 = b_2 = \frac{\beta}{2\pi}$, (1.6) gives the supersymmetric Casimir energy on the round sphere

$$\mathcal{E}^{\text{susy}}_{\text{Casimir}}\left(\frac{\beta}{2\pi}, \frac{\beta}{2\pi}\right) = \frac{4}{27}(\mathbf{a} + 3\mathbf{c}) . \tag{1.7}$$

Another approach to computing the supersymmetric Casimir energy was undertaken in [3], by reducing an $\mathcal{N}=1$ theory on $S^1_\beta \times M_3$ to quantum mechanics. The $1d$ Hamiltonian is the sum of the Hamiltonian for chiral and Fermi multiplets and, therefore, the ground state energy can be computed through the expression

$$\langle H_{\text{susy}}\rangle = \sum_{\text{chiral}} \langle H_{\text{chiral}}\rangle + \sum_{\text{Fermi}} \langle H_{\text{Fermi}}\rangle \,. \tag{1.8}$$

The result for the Casimir energy $\mathcal{E}_{\text{Casimir}} = \langle H_{\text{susy}}\rangle$ so obtained agrees with (1.6) and (1.7). Yet another approach was considered in [6], where it was conjectured that the supersymmetric Casimir energy in d even dimensions is given by an equivariant integral of the anomaly polynomial in two higher dimensions.

The approach we follow in the preset work, however, is based on the supersymmetry algebra. On a curved background, the supercharges satisfy (schematically) the algebra

$$\{Q^\dagger, Q\} = H - Q_R - \mathcal{E}_{\text{Casimir}}\,, \tag{1.9}$$

where H is the Hamiltonian and Q_R is the R-charge operator. Since in a supersymmetric vacuum $\langle\{Q^\dagger, Q\}\rangle = 0$, it follows that

$$\langle H - Q_R\rangle_{\text{g.s.}} = \mathcal{E}_{\text{Casimir}}\,. \tag{1.10}$$

We determine the general form of the Casimir energy $\mathcal{E}_{\text{Casimir}}$ in (1.9) by coupling a generic $\mathcal{N}=1$ SCFT to background conformal supergravity and using the operator algebra obtained in [19]. This leads to the general expression (6.23) for the supersymmetric Casimir energy in terms of local curvatures and Killing spinor bilinears. Evaluating this general expression on backgrounds with the cylinder topology $\mathbb{R}\times S^3$ and admitting two supercharges of opposite R-charge we arrive at the expression (6.29). This result is consistent with (1.3), but differs significantly from (1.6). We defer a thorough analysis of the relation between these two results to a future publication.

The rest paper is organized as follows. In section 2 we review the $\mathcal{N}=1$ superconformal Ward identities of conformal supergravity, including the superconformal anomalies. In section 3 we introduce the relevant Bardeen-Zumino terms for the R-current and energy-momentum tensor, which allow us to rewrite the superconformal Ward identities in terms of the covariant R-current and the energy-momentum tensor. In section 4 we consider bosonic backgrounds with numerically zero anomalies, such as those admiting Killing spinors, and show the existence of continuous families of conserved R-charges and charges associated with conformal Killing vectors. Section 5 is devoted to the detailed evaluation of the Casimir energy on such backgrounds using the anomalous transformation of the supercurrent obtained in [19]. This leads to the general BPS relation (5.43) and the "local charge" (5.44). These are the main results of this work. As an illustrative example, in section 6, we evaluate the Casimir energy on $\mathbb{R} \times S^3$, obtaining the result (6.23).

2 $\mathcal{N} = 1$ superconformal Ward identities

The superconformal Ward identities that four-dimensional $\mathcal{N} = 1$ SCFTs satisfy can be determined by coupling the theory to $\mathcal{N} = 1$ off-shell conformal supergravity [10–13], whose field content consists of the vielbein e^a_μ, an Abelian gauge field A_μ, and a Majorana gravitino ψ_μ, comprising 5+3 bosonic and 8 fermionic off-shell degrees of freedom. Treating the supergravity fields as external background fields, the SCFT partition function takes the form

$$\mathcal{Z}[e, A, \psi] = \int [\mathcal{D}\Phi] e^{iS[\Phi,e,A,\psi]} \,, \tag{2.1}$$

where Φ denotes collectively all the microscopic degrees of freedom.[1]

The logarithm of the partition function on an arbitrary supergravity background

$$\mathcal{W}[e, A, \psi] = -i \log \mathcal{Z}[e, A, \psi] \,, \tag{2.2}$$

is often referred to as the 'quantum effective action' and amounts to the generating function of connected (renormalized) correlation functions of the current operators

$$\langle \mathcal{T}^\mu_a \rangle = e^{-1} \frac{\delta \mathcal{W}}{\delta e^a_\mu}, \qquad \langle \mathcal{J}^\mu_a \rangle = e^{-1} \frac{\delta \mathcal{W}}{\delta A_\mu}, \qquad \langle \mathcal{S}^\mu_a \rangle = e^{-1} \frac{\delta \mathcal{W}}{\delta \bar{\psi}_\mu} \,, \tag{2.3}$$

where $e \equiv \det(e^a_\mu)$ and the notation $\langle \cdot \rangle$ denotes one-point functions in the presence of arbitrary sources, i.e. on an arbitrary supergravity background. These operators comprise the conformal multiplet of currents and satisfy a set of superconformal Ward identities.

$\mathcal{N} = 1$ conformal supergravity is a gauge theory of the $\mathcal{N} = 1$ superconformal algebra [10–13] (see [14–17] and chapter 16 of [18] for pedagogical reviews). Its local symmetries consist of diffeomorphisms $\xi^\mu(x)$, Weyl transformations $\sigma(x)$, local frame rotations $\lambda^{ab}(x)$, $U(1)$ R-symmetry transformations $\theta(x)$, as well as $\mathcal{Q}$- and $\mathcal{S}$-supersymmetry transformations,

[1] Evaluating the path integral over the microscopic degrees of freedom Φ in general requires the introduction of a regulator that explicitly breaks conformal symmetry. Although the regulated theory cannot couple consistently to conformal supergravity, the *renormalized* theory can, even in the presence of superconformal anomalies. Throughout this article we refer exclusively to the renormalized observables, which can consistently couple to conformal supergravity.

parameterized respectively by the local spinors $\varepsilon(x)$ and $\eta(x)$. Under these, the fields of $\mathcal{N}=1$ conformal supergravity transform as

$$\begin{aligned}
\delta e^a_\mu &= \xi^\lambda\partial_\lambda e^a_\mu + e^a_\lambda\partial_\mu\xi^\lambda - \lambda^a{}_b e^b_\mu + \sigma e^a_\mu - \frac{1}{2}\overline{\psi}_\mu\gamma^a\varepsilon\,, \\
\delta\psi_\mu &= \xi^\lambda\partial_\lambda\psi_\mu + \psi_\lambda\partial_\mu\xi^\lambda - \frac{1}{4}\lambda_{ab}\gamma^{ab}\psi_\mu + \frac{1}{2}\sigma\psi_\mu + \mathcal{D}_\mu\varepsilon - \gamma_\mu\eta - i\gamma^5\theta\psi_\mu\,, \\
\delta A_\mu &= \xi^\lambda\partial_\lambda A_\mu + A_\lambda\partial_\mu\xi^\lambda + \frac{3i}{4}\overline{\phi}_\mu\gamma^5\varepsilon - \frac{3i}{4}\overline{\psi}_\mu\gamma^5\eta + \partial_\mu\theta\,,
\end{aligned} \tag{2.4}$$

where

$$\phi_\mu \equiv -\frac{1}{6}\Big(4\delta^{[\rho}_\mu\delta^{\sigma]}_\nu + i\gamma^5\epsilon_{\mu\nu}{}^{\rho\sigma}\Big)\gamma^\nu\mathcal{D}_\rho\psi_\sigma\,, \tag{2.5}$$

and the spinor covariant derivatives are given by

$$\begin{aligned}
\mathcal{D}_\mu\psi_\nu &\equiv \Big(\partial_\mu + \frac{1}{4}\omega_\mu{}^{ab}(e,\psi)\gamma_{ab} + i\gamma^5 A_\mu\Big)\psi_\nu - \Gamma^\rho_{\mu\nu}\psi_\rho \equiv \Big(\mathscr{D}_\mu + i\gamma^5 A_\mu\Big)\psi_\nu\,, \\
\mathcal{D}_\mu\varepsilon &\equiv \Big(\partial_\mu + \frac{1}{4}\omega_\mu{}^{ab}(e,\psi)\gamma_{ab} + i\gamma^5 A_\mu\Big)\varepsilon \equiv \Big(\mathscr{D}_\mu + i\gamma^5 A_\mu\Big)\varepsilon\,.
\end{aligned} \tag{2.6}$$

In these expressions $\omega_\mu{}^{ab}(e,\psi)$ denotes the torsion-full spin connection

$$\omega_\mu{}^{ab}(e,\psi) \equiv \omega_\mu{}^{ab}(e) + \frac{1}{4}\Big(\overline{\psi}_a\gamma_\mu\psi_b + \overline{\psi}_\mu\gamma_a\psi_b - \overline{\psi}_\mu\gamma_b\psi_a\Big)\,, \tag{2.7}$$

where $\omega_\mu{}^{ab}(e)$ is the torsion-free metric compatible connection.

The quantum effective action (2.2) of any $\mathcal{N}=1$ SCFT is invariant under the local symmetry transformations (2.4), up to local expressions in the background supergravity fields that comprise the multiplet of superconformal anomalies. In particular, there exists a renormalization scheme such that, under an infinitesimal local symmetry transformation with parameters $\Omega=(\xi,\sigma,\lambda,\theta,\varepsilon,\eta)$, the quantum effective action transforms as

$$\delta_\Omega\mathcal{W}[e,A,\psi] = \int d^4x\, e\Big(\sigma\mathcal{A}_W - \theta\mathcal{A}_R - \bar{\varepsilon}\mathcal{A}_Q + \bar{\eta}\mathcal{A}_S\Big)\,, \tag{2.8}$$

where the superconformal anomalies $\mathcal{A}_W$, $\mathcal{A}_R$, $\mathcal{A}_Q$ and $\mathcal{A}_S$ take the form [19]

$$\begin{aligned}
\mathcal{A}_W &= \frac{c}{16\pi^2}\Big(W^2 - \frac{8}{3}F^2\Big) - \frac{a}{16\pi^2}E + \mathcal{O}(\psi^2)\,,\\
\mathcal{A}_R &= \frac{(5a-3c)}{27\pi^2}\,\widetilde{F}F + \frac{(c-a)}{24\pi^2}\mathcal{P}\,,\\
\mathcal{A}_Q &= -\frac{(5a-3c)i}{9\pi^2}\widetilde{F}^{\mu\nu}A_\mu\gamma^5\phi_\nu + \frac{(a-c)}{6\pi^2}\nabla_\mu\Big(A_\rho\widetilde{R}^{\rho\sigma\mu\nu}\Big)\gamma_{(\nu}\psi_{\sigma)} - \frac{(a-c)}{24\pi^2}F_{\mu\nu}\widetilde{R}^{\mu\nu\rho\sigma}\gamma_\rho\psi_\sigma + \mathcal{O}(\psi^3)\,,\\
\mathcal{A}_S &= \frac{(5a-3c)}{6\pi^2}\widetilde{F}^{\mu\nu}\Big(\mathcal{D}_\mu - \frac{2i}{3}A_\mu\gamma^5\Big)\psi_\nu + \frac{ic}{6\pi^2}F^{\mu\nu}\Big(\gamma_\mu{}^{[\sigma}\delta_\nu^{\rho]} - \delta_\mu^{[\sigma}\delta_\nu^{\rho]}\Big)\gamma^5\mathcal{D}_\rho\psi_\sigma \qquad (2.9)\\
&\quad + \frac{3(2a-c)}{4\pi^2}P_{\mu\nu}g^{\mu[\nu}\gamma^{\rho\sigma]}\mathcal{D}_\rho\psi_\sigma + \frac{(a-c)}{8\pi^2}\Big(R^{\mu\nu\rho\sigma}\gamma_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}g^{\mu[\nu}\gamma^{\rho\sigma]}\Big)\mathcal{D}_\rho\psi_\sigma + \mathcal{O}(\psi^3)\,.
\end{aligned}$$

The anomaly coefficients a and c in these expressions depend on the specific SCFT and are normalized such that for N_χ free chiral and N_v free vector multiplets [20]

$$a = \frac{1}{48}(N_\chi + 9N_v)\,, \qquad c = \frac{1}{24}(N_\chi + 3N_v)\,. \qquad (2.10)$$

W^2 in (2.9) denotes the square of the Weyl tensor, while E and $\mathcal{P}$ are respectively the Euler and Pontryagin densities. Namely,

$$\begin{aligned}
W^2 &\equiv W_{\mu\nu\rho\sigma}W^{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 2R_{\mu\nu}R^{\mu\nu} + \frac{1}{3}R^2\,,\\
E &= R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2\,,\\
\mathcal{P} &\equiv \frac{1}{2}\epsilon^{\kappa\lambda\mu\nu}R_{\kappa\lambda\rho\sigma}R_{\mu\nu}{}^{\rho\sigma} = \widetilde{R}^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}\,, \qquad (2.11)
\end{aligned}$$

where

$$\widetilde{R}_{\mu\nu\rho\sigma} \equiv \frac{1}{2}\epsilon_{\mu\nu}{}^{\kappa\lambda}R_{\kappa\lambda\rho\sigma}\,. \qquad (2.12)$$

Finally, $P_{\mu\nu}$ in (2.9) denotes the Schouten tensor

$$P_{\mu\nu} = \frac{1}{2}\Big(R_{\mu\nu} - \frac{1}{6}Rg_{\mu\nu}\Big)\,, \qquad (2.13)$$

and we have defined

$$F^2 \equiv F_{\mu\nu}F^{\mu\nu}\,, \qquad F\widetilde{F} \equiv \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}\,, \qquad \widetilde{F}_{\mu\nu} \equiv \frac{1}{2}\epsilon_{\mu\nu}{}^{\rho\sigma}F_{\rho\sigma}\,. \qquad (2.14)$$

The anomalous transformation (2.8) of the quantum effective action, together with the definition of the currents (2.3), leads to the superconformal Ward identities [19]

$$
\begin{aligned}
& e^a_\mu \nabla_\nu \langle \mathcal{T}^\nu_a \rangle + \nabla_\nu (\overline{\psi}_\mu \langle \mathcal{S}^\nu \rangle) - \overline{\psi}_\nu \overleftarrow{\mathcal{D}}_\mu \langle \mathcal{S}^\nu \rangle - F_{\mu\nu} \langle \mathcal{J}^\nu \rangle \\
& \qquad + A_\mu \Big(\nabla_\nu \langle \mathcal{J}^\nu \rangle + i \overline{\psi}_\nu \gamma^5 \langle \mathcal{S}^\nu \rangle \Big) - \omega_\mu{}^{ab} \Big(e_{\nu[a} \langle \mathcal{T}^\nu_{b]} \rangle + \frac{1}{4} \overline{\psi}_\nu \gamma_{ab} \langle \mathcal{S}^\nu \rangle \Big) = 0 \, , \\
& e^a_\mu \langle \mathcal{T}^\mu_a \rangle + \frac{1}{2} \overline{\psi}_\mu \langle \mathcal{S}^\mu \rangle = \mathcal{A}_W \, , \\
& e_{\mu[a} \langle \mathcal{T}^\mu_{b]} \rangle + \frac{1}{4} \overline{\psi}_\mu \gamma_{ab} \langle \mathcal{S}^\mu \rangle = 0 \, , \\
& \nabla_\mu \langle \mathcal{J}^\mu \rangle + i \overline{\psi}_\mu \gamma^5 \langle \mathcal{S}^\mu \rangle = \mathcal{A}_R \, , \\
& \mathcal{D}_\mu \langle \mathcal{S}^\mu \rangle - \frac{1}{2} \gamma^a \psi_\mu \langle \mathcal{T}^\mu_a \rangle - \frac{3i}{4} \gamma^5 \phi_\mu \langle \mathcal{J}^\mu \rangle = \mathcal{A}_Q \, , \\
& \gamma_\mu \langle \mathcal{S}^\mu \rangle - \frac{3i}{4} \gamma^5 \psi_\mu \langle \mathcal{J}^\mu \rangle = \mathcal{A}_S \, . \qquad (2.15)
\end{aligned}
$$

The transformation (2.8) of the quantum effective action, and hence the local terms in the Ward identities (2.15), may be modified by adding local terms to the effective action $\mathcal{W}[e, A, \psi]$. For example, adding the term

$$
\int A \wedge \mathrm{Tr} \left(\Gamma \wedge d\Gamma + \frac{2}{3} \Gamma \wedge \Gamma \wedge \Gamma \right) , \qquad (2.16)
$$

with $\Gamma^\mu{}_\nu \equiv \Gamma^\mu_{\nu\rho} dx^\rho$ the Christoffel connection, breaks the diffeomorphism invariance of the effective action and modifies the form of its anomalous transformation under the rest of the local symmetries. In particular, the addition of this term with a specific coefficient eliminates the Pontryagin term from the R-symmetry anomaly $\mathcal{A}_R$, as is reviewed e.g. in [21]. Although the fermionic anomalies $\mathcal{A}_Q$ and $\mathcal{A}_S$ can be modified by the addition of such local terms, there exists no local term that sets them to zero within conformal supergravity. It is these fermionic anomalies $\mathcal{A}_Q$ and $\mathcal{A}_S$ that determine the supersymmetric Casimir energy [9].

3 Bardeen-Zumino terms and covariant currents

The anomalies (2.9) are solutions of the Wess-Zumino (WZ) consistency conditions for the local symmetry transformations (2.4) of $\mathcal{N} = 1$ conformal supergravity [19]. As such, they correspond to the so called *consistent* anomalies, while the operators defined in (2.3) are known as the *consistent currents*. As a consequence of the R-symmetry anomaly, the consistent currents are not gauge invariant, while if the local term (2.16) is added to the effective action, the consistent currents are not diffeomorphism covariant either.

The gauge and diffeomorphism covariance of the R-current $\mathcal{J}^\mu$ and of the stress tensor $\mathcal{T}^\mu_a$ can be restored by adding local Bardeen-Zumino (BZ) terms to the currents [22]. These terms are not related to the choice of renormalization scheme discussed above, since they cannot be expressed as derivatives of a local term in the effective action. Instead, they arise from a Chern-Simons action in five dimensions that cancels the R-symmetry/mixed anomaly through the mechanism of anomaly inflow (see e.g. [21] for an extensive discussion).

However, the BZ terms for the R-current and the energy-momentum tensor are already encoded in the form of the anomalies (2.9) [9, 19]. This observation was understood in [23] as a direct consequence of the WZ consistency conditions. As an example, let us consider the WZ condition[2]

$$(\delta_\xi \delta_\theta - \delta_\theta \delta_\xi)\mathcal{W} = 0\,. \tag{3.1}$$

In the absence of the local term (2.16), $\delta_\xi \mathcal{W} = 0$ and so $\delta_\xi \delta_\theta \mathcal{W} = 0$ and $\delta_\theta \delta_\xi \mathcal{W} = 0$ separately. These imply respectively that $\delta_\theta \mathcal{W}$ is diffeomorphism invariant and $\delta_\xi \mathcal{W}$ is gauge invariant. Focusing on $\delta_\xi \mathcal{W}$, it can be expressed in terms of the consistent currents (2.3) as [19]

$$\delta_\xi \mathcal{W} = -\int d^4x\, e\, \xi^\mu \Big(e^a_\mu \nabla_\nu \langle \mathcal{T}^\nu_a \rangle + \nabla_\nu (\overline{\psi}_\mu \langle \mathcal{S}^\nu \rangle) - \overline{\psi}_\nu \overleftarrow{\mathcal{D}}_\mu \langle \mathcal{S}^\nu \rangle - F_{\mu\nu} \langle \mathcal{J}^\nu \rangle + A_\mu \mathcal{A}_R \Big)\,, \tag{3.2}$$

whose bosonic part is not manifestly gauge invariant. As we will now demonstrate, the form of the R-symmetry anomaly $\mathcal{A}_R$ is such that this expression can be written in manifestly gauge invariant form in terms of the *covariant* R-current and energy-momentum tensor.

Writing the R-symmetry anomaly explicitly, the bosonic terms in (3.2) take the form

$$\begin{aligned} & e^a_\mu \nabla_\nu \langle \mathcal{T}^\nu_a \rangle - F_{\mu\nu} \langle \mathcal{J}^\nu \rangle + A_\mu \mathcal{A}_R \\ &= e^a_\mu \nabla_\nu \langle \mathcal{T}^\nu_a \rangle - F_{\mu\nu} \langle \mathcal{J}^\nu \rangle + \frac{(5a-3c)}{54\pi^2}\, \epsilon^{\kappa\lambda\rho\sigma} F_{\kappa\lambda} F_{\rho\sigma} A_\mu + \frac{(c-a)}{48\pi^2} \epsilon^{\kappa\lambda\rho\sigma} R_{\kappa\lambda\alpha\beta} R_{\rho\sigma}{}^{\alpha\beta} A_\mu\,. \end{aligned} \tag{3.3}$$

Antisymmetrizing five indices in four dimensions gives zero. Hence, $F_{[\mu\kappa} F_{\lambda\rho} A_{\sigma]} = 0$ and $R_{[\kappa\lambda\alpha\beta} R_{\rho\sigma}{}^{\alpha\beta} A_{\mu]} = 0$, which imply respectively the identities

$$\epsilon^{\kappa\lambda\rho\sigma} F_{\kappa\lambda} F_{\rho\sigma} A_\mu = -4 \epsilon^{\kappa\lambda\rho\sigma} F_{\mu\kappa} F_{\lambda\rho} A_\sigma\,, \tag{3.4}$$

and

$$\epsilon^{\kappa\lambda\rho\sigma} R_{\kappa\lambda\alpha\beta} R_{\rho\sigma}{}^{\alpha\beta} A_\mu = -4 \epsilon^{\kappa\lambda\rho\sigma} R_{\mu\kappa\alpha\beta} R_{\lambda\rho}{}^{\alpha\beta} A_\sigma\,. \tag{3.5}$$

[2] We adopt the convention of [19], where the transformations act of the transformation parameters as well. This is analogous to the BRST treatment of the WZ consistency conditions, where the transformation parameters are replaced by ghost fields that themselves transform. The same conclusions are reached in the convention that the transformation parameters do not transform – see section 3.2 of [23].

Therefore, the bosonic part of (3.2) becomes

$$
\begin{aligned}
&e^a_\mu \nabla_\nu \langle \mathcal{T}^\nu_a \rangle - F_{\mu\nu}\langle \mathcal{J}^\nu \rangle + A_\mu \mathcal{A}_R \\
&= e^a_\mu \nabla_\nu \langle \mathcal{T}^\nu_a \rangle - F_{\mu\nu}\langle \mathcal{J}^\nu_{\rm cov} \rangle - \frac{(c-a)}{12\pi^2}\epsilon^{\kappa\lambda\rho\sigma} R_{\mu\kappa\alpha\beta} R_{\lambda\rho}{}^{\alpha\beta} A_\sigma \,,
\end{aligned} \tag{3.6}
$$

where the covariant R-current is given by

$$
\langle \mathcal{J}^\mu_{\rm cov} \rangle = \langle \mathcal{J}^\mu \rangle + P^\mu_{BZ} \,, \tag{3.7}
$$

with the Bardeen-Zumino term

$$
P^\mu_{BZ} = \frac{2(5a-3c)}{27\pi^2}\, \epsilon^{\mu\lambda\rho\sigma} F_{\lambda\rho} A_\sigma \,. \tag{3.8}
$$

Since the local term (2.16) is absent, the consistent R-current $\mathcal{J}^\mu$ transforms as tensor under diffeomorphisms and so the relevant BZ term need only restore gauge invariance.

The BZ term for the energy-momentum tensor $\mathcal{T}^{\mu\nu} = \mathcal{T}^{\mu a} e^\nu_a$ takes the form [21]

$$
\begin{aligned}
P^{\mu\nu}_{BZ} &= -\frac{1}{2}\nabla_\lambda\Big(X^{\lambda\mu\nu} + X^{\lambda\nu\mu} - X^{\mu\nu\lambda}\Big)\,, \\
X^{\mu\lambda}{}_\nu &= -\frac{(c-a)}{12\pi^2}\Big(\epsilon^{\mu\rho\kappa\sigma} R^\lambda{}_{\nu\kappa\sigma} + \epsilon^{\lambda\rho\kappa\sigma} R^\mu{}_{\nu\kappa\sigma}\Big) A_\rho \,.
\end{aligned} \tag{3.9}
$$

With a bit of algebra this can be simplified to

$$
P^{\mu\nu}_{BZ} = \frac{(c-a)}{6\pi^2}\nabla_\lambda\Big(\epsilon^{\alpha\rho\kappa\sigma} R^{\lambda\beta}{}_{\kappa\sigma} A_\rho\Big)\delta^\mu_{(\alpha}\delta^\nu_{\beta)} \,. \tag{3.10}
$$

Evaluating its covariant divergence we find

$$
\begin{aligned}
\nabla_\mu P^{\mu\nu}_{BZ} &= \frac{(c-a)}{12\pi^2}\nabla_\mu\nabla_\lambda\Big(\epsilon^{\mu\rho\kappa\sigma} R^{\lambda\nu}{}_{\kappa\sigma} A_\rho + \epsilon^{\nu\rho\kappa\sigma} R^{\lambda\mu}{}_{\kappa\sigma} A_\rho\Big) \\
&= \frac{(c-a)}{12\pi^2}\bigg(\Big([\nabla_\mu,\nabla_\lambda] + \nabla_\lambda\nabla_\mu\Big)(\epsilon^{\mu\rho\kappa\sigma} R^{\lambda\nu}{}_{\kappa\sigma} A_\rho) + \frac{1}{2}[\nabla_\mu,\nabla_\lambda](\epsilon^{\nu\rho\kappa\sigma} R^{\lambda\mu}{}_{\kappa\sigma} A_\rho)\bigg) \\
&= \frac{(c-a)}{24\pi^2}\bigg([\nabla_\mu,\nabla_\lambda](2\epsilon^{\mu\rho\kappa\sigma} R^{\lambda\nu}{}_{\kappa\sigma} A_\rho - \epsilon^{\nu\rho\kappa\sigma} R^{\mu\lambda}{}_{\kappa\sigma} A_\rho) + \nabla_\lambda(\epsilon^{\mu\rho\kappa\sigma} R^{\lambda\nu}{}_{\kappa\sigma} F_{\mu\rho})\bigg)\,,
\end{aligned} \tag{3.11}
$$

where we made use of the second Bianchi identity $\nabla_{[\mu} R^{\kappa\lambda}{}_{\nu\rho]} = 0$.

Moreover, using the first Bianchi identity, $R^\kappa{}_{[\lambda\nu\rho]} = 0$, we obtain

$$
[\nabla_\mu,\nabla_\lambda](2\epsilon^{\mu\rho\kappa\sigma} R^{\lambda\nu}{}_{\kappa\sigma} A_\rho - \epsilon^{\nu\rho\kappa\sigma} R^{\mu\lambda}{}_{\kappa\sigma} A_\rho) = 2\epsilon^{\rho\alpha\kappa\sigma} R^\nu{}_{\alpha\mu\lambda} R^{\mu\lambda}{}_{\kappa\sigma} A_\rho \,, \tag{3.12}
$$

and hence

$$\nabla_\mu P^{\mu\nu}_{BZ} = \frac{(c-a)}{12\pi^2} \epsilon^{\rho\alpha\kappa\sigma} R^\nu{}_{\alpha\mu\lambda} R^{\mu\lambda}{}_{\kappa\sigma} A_\rho + \nabla_\mu L^{\mu\nu} \,, \tag{3.13}$$

where

$$L^{\mu\nu} = \frac{(c-a)}{12\pi^2} F_{\rho\sigma} \widetilde{R}^{\rho\sigma\mu\nu} \,. \tag{3.14}$$

It follows that the bosonic part of (3.2) takes the form

$$e^a_\mu \nabla_\nu \langle \mathcal{T}^\nu_a \rangle - F_{\mu\nu} \langle \mathcal{J}^\nu \rangle + A_\mu \mathcal{A}_R = \nabla_\nu \langle \mathcal{T}^\nu_{\text{cov}\,\mu} \rangle - F_{\mu\nu} \langle \mathcal{J}^\nu_{\text{cov}} \rangle - \nabla_\nu L^\nu{}_\mu \,, \tag{3.15}$$

where

$$\langle \mathcal{T}^{\mu\nu}_{\text{cov}} \rangle = \langle \mathcal{T}^{\mu\nu} \rangle + P^{\mu\nu}_{BZ} \,. \tag{3.16}$$

Since $L_{\mu\nu}$ is gauge invariant, we have demonstrated that all terms in (3.2) can be made manifestly gauge invariant in terms of the covariant R-current and energy-momentum tensor.

The covariant R-current and energy-momentum tensor help simplify not only the diffeomorphism and R-symmetry Ward identities, but also those for $\mathcal{Q}$- and $\mathcal{S}$-supersymmetry. In particular, in terms of the covariant currents, the Ward identities (2.15) take the form

$$\begin{aligned}
& e^a_\mu \nabla_\nu \langle \mathcal{T}^\nu_{\text{cov}\,a} \rangle + \nabla_\nu (\overline{\psi}_\mu \langle \mathcal{S}^\nu \rangle) - \overline{\psi}_\nu \overleftarrow{\mathcal{D}}_\mu \langle \mathcal{S}^\nu \rangle - F_{\mu\nu} \langle \mathcal{J}^\nu_{\text{cov}} \rangle = \mathcal{A}^{\text{cov}}_{D\,\mu} \,, \\
& e^a_\mu \langle \mathcal{T}^\mu_{\text{cov}\,a} \rangle + \frac{1}{2} \overline{\psi}_\mu \langle \mathcal{S}^\mu \rangle = \mathcal{A}^{\text{cov}}_W \,, \\
& e_{\mu[a} \langle \mathcal{T}^\mu_{\text{cov}\,b]} \rangle + \frac{1}{4} \overline{\psi}_\mu \gamma_{ab} \langle \mathcal{S}^\mu \rangle = 0 \,, \\
& \nabla_\mu \langle \mathcal{J}^\mu_{\text{cov}} \rangle + i \overline{\psi}_\mu \gamma^5 \langle \mathcal{S}^\mu \rangle = \mathcal{A}^{\text{cov}}_R \,, \\
& \mathcal{D}_\mu \langle \mathcal{S}^\mu \rangle - \frac{1}{2} \gamma^a \psi_\mu \langle \mathcal{T}^\mu_{a\,\text{cov}} \rangle - \frac{3i}{4} \gamma^5 \phi_\mu \langle \mathcal{J}^\mu_{\text{cov}} \rangle = \mathcal{A}^{\text{cov}}_Q \,, \\
& \gamma_\mu \langle \mathcal{S}^\mu \rangle - \frac{3i}{4} \gamma^5 \psi_\mu \langle \mathcal{J}^\mu_{\text{cov}} \rangle = \mathcal{A}^{\text{cov}}_S \,.
\end{aligned} \tag{3.17}$$

with the covariant anomalies given by the simple expressions

$$
\begin{aligned}
\mathcal{A}^{\text{cov}}_{D\,\mu} &= \nabla_\nu L^\nu{}_\mu = \frac{(a-c)}{12\pi^2}\nabla^\nu(F_{\rho\sigma}\widetilde{R}^{\rho\sigma}{}_{\mu\nu})\,,\\
\mathcal{A}^{\text{cov}}_{W} &= \mathcal{A}_W = \frac{c}{16\pi^2}\Big(W^2-\frac{8}{3}F^2\Big)-\frac{a}{16\pi^2}E+\mathcal{O}(\psi^2)\,,\\
\mathcal{A}^{\text{cov}}_{R} &= \frac{(5a-3c)}{9\pi^2}\,\widetilde{F}F+\frac{(c-a)}{24\pi^2}\mathcal{P}\,,\\
\mathcal{A}^{\text{cov}}_{Q} &= \frac{(c-a)}{24\pi^2}F_{\mu\nu}\widetilde{R}^{\mu\nu\rho\sigma}\gamma_\rho\psi_\sigma+\mathcal{O}(\psi^3)\,,\\
\mathcal{A}^{\text{cov}}_{S} &= \Big[\frac{(5a-3c)}{6\pi^2}\widetilde{F}^{\rho\sigma}+\frac{ic}{6\pi^2}F^{\mu\nu}\Big(\gamma_\mu{}^{[\sigma}\delta^{\rho]}_\nu-\delta^{[\sigma}_\mu\delta^{\rho]}_\nu\Big)\gamma^5+\frac{3(2a-c)}{4\pi^2}P_{\mu\nu}g^{\mu[\nu}\gamma^{\rho\sigma]}\\
&\quad+\frac{(a-c)}{8\pi^2}\Big(R^{\mu\nu\rho\sigma}\gamma_{\mu\nu}-\frac{1}{2}Rg_{\mu\nu}g^{\mu[\nu}\gamma^{\rho\sigma]}\Big)\Big]\mathcal{D}_\rho\psi_\sigma+\mathcal{O}(\psi^3)\,.
\end{aligned}
\tag{3.18}
$$

4 Conserved charges

The superconformal Ward identities determine the conserved charges and their algebra. In preparation for the derivation of the supersymmetric Casimir energy in the subsequent sections, we consider the R-charge and the charges associated with conformal Killing vectors and spinors on supergravity backgrounds that admit Killing spinors. A key property of such backgrounds is that the supercoformal anomalies are numerically zero, resulting in continuous families of conserved bosonic charges.

Starting with the R-charge, we define the one-parameter family of R-currents [9]

$$
\langle \mathcal{J}^\mu_{\omega_{\mathcal{J}}}\rangle \equiv \langle \mathcal{J}^\mu\rangle + \omega_{\mathcal{J}}P^\mu_{BZ}\,,
\tag{4.1}
$$

where P^μ_{BZ} is the BZ term given in (3.8). Clearly, $\langle\mathcal{J}^\mu_0\rangle$ is the consistent R-current, while $\langle\mathcal{J}^\mu_1\rangle$ is the covariant one. On a bosonic background the divergence of this current is

$$
\nabla_\mu\langle\mathcal{J}^\mu_{\omega_{\mathcal{J}}}\rangle = (1+2\omega_{\mathcal{J}})\frac{5a-3c}{27\pi^2}\widetilde{F}F+\frac{c-a}{24\pi^2}\mathcal{P}\,,
\tag{4.2}
$$

which is nonzero on a generic supergravity background. However, for conformal supergravity backgrounds that admit Killing spinors both terms in the R-symmetry anomaly are numerically zero, i.e. $\widetilde{F}F=0$ and $\mathcal{P}=0$. As a result, on supersymmetric backgrounds there exists a continuous family of conserved R-charges defined as

$$Q_R^{\omega_{\mathcal{J}}} = \int_{\mathcal{C}} d\sigma_\mu \langle \mathcal{J}^\mu_{\omega_{\mathcal{J}}} \rangle \,, \tag{4.3}$$

where $\mathcal{C}$ is a Cauchy surface and $d\sigma_\mu$ is the corresponding infinitesimal area element. Notice that although the anomalies are numerically zero on supersymmetric backgrounds, the BZ terms are not necessarily vanishing, and hence the value of the R-charge (4.3) in general depends on the parameter $\omega_{\mathcal{J}}$.

Let us now consider the conserved charges associated with conformal Killing vectors of supersymmetric backgrounds. A conformal Killing vector $\mathcal{K}^\mu$ satisfies the relations

$$\mathcal{L}_{\mathcal{K}}\, g_{\mu\nu} = \nabla_\mu \mathcal{K}_\nu + \nabla_\nu \mathcal{K}_\mu = \frac{1}{2}(\nabla_\rho \mathcal{K}^\rho) g_{\mu\nu} \,, \qquad \mathcal{L}_{\mathcal{K}} A_\mu = \partial_\mu \Lambda_{\mathcal{K}} \,, \tag{4.4}$$

where $\mathcal{L}_{\mathcal{K}}$ denotes the Lie derivative with respect to $\mathcal{K}^\mu$ and $\Lambda_{\mathcal{K}}$ is an arbitrary R-symmetry gauge parameter. The Killing condition on the R-symmetry gauge field A_μ is equivalent to the gauge-invariant condition $\mathcal{L}_{\mathcal{K}} F_{\mu\nu} = 0$.

As with the R-current, we define the continuous family of energy-momentum tensors

$$\langle \mathcal{T}^{\mu\nu}_{\omega_{\mathcal{T}}} \rangle \equiv \langle \mathcal{T}^{\mu\nu} \rangle + \omega_{\mathcal{T}} P^{\mu\nu}_{BZ} \,, \tag{4.5}$$

where $P^{\mu\nu}_{BZ}$ is the BZ term (3.10). Notice again that $\langle \mathcal{T}^{\mu\nu}_0 \rangle$ corresponds to the consistent energy-momentum tensor, while $\langle \mathcal{T}^{\mu\nu}_1 \rangle$ is the covariant one. The divergence of this current on a bosonic background is given by

$$\nabla_\mu \langle \mathcal{T}^\mu_{\omega_{\mathcal{T}}\,\nu} \rangle = F_{\nu\mu} \langle \mathcal{J}^\mu_{\text{cov}} \rangle + \omega_{\mathcal{T}} \frac{(a-c)}{12\pi^2} \nabla^\mu (F_{\rho\sigma} \widetilde{R}^{\rho\sigma}{}_{\nu\mu}) \,. \tag{4.6}$$

In order to determine the conserved charge associated with the conformal Killing vector $\mathcal{K}^\mu$ we evaluate the divergence

$$\begin{aligned}
&\nabla_\mu \Big((\langle \mathcal{T}^\mu_{\omega_{\mathcal{T}}\,\nu} \rangle + A_\nu \langle \mathcal{J}^\mu_{\omega_{\mathcal{J}}} \rangle) \mathcal{K}^\nu \Big) \\
&= \nabla_\mu \langle \mathcal{T}^\mu_{\omega_{\mathcal{T}}\,\nu} \rangle \mathcal{K}^\nu + \frac{1}{4} \langle \mathcal{T}^\mu_{\omega_{\mathcal{T}}\,\mu} \rangle \nabla_\rho \mathcal{K}^\rho + \partial_\mu (A_\nu \mathcal{K}^\nu) \langle \mathcal{J}^\mu_{\omega_{\mathcal{J}}} \rangle + A_\nu \mathcal{K}^\nu \nabla_\mu \langle \mathcal{J}^\mu_{\omega_{\mathcal{J}}} \rangle \\
&= \mathcal{K}^\nu F_{\nu\mu} \langle \mathcal{J}^\mu_{\text{cov}} \rangle + (F_{\mu\nu} \mathcal{K}^\nu + \partial_\mu \Lambda_{\mathcal{K}}) \langle \mathcal{J}^\mu_{\omega_{\mathcal{J}}} \rangle \\
&\quad + \omega_{\mathcal{T}} \frac{(a-c)}{12\pi^2} \mathcal{K}^\nu \nabla^\mu (F_{\rho\sigma} \widetilde{R}^{\rho\sigma}{}_{\nu\mu}) + \frac{1}{4} \mathcal{A}_W \nabla_\rho \mathcal{K}^\rho + A_\nu \mathcal{K}^\nu \nabla_\mu \langle \mathcal{J}^\mu_{\omega_{\mathcal{J}}} \rangle \\
&= (\omega_{\mathcal{J}} - 1) F_{\mu\nu} \mathcal{K}^\nu P^\mu_{BZ} + \nabla_\mu (\Lambda_{\mathcal{K}} \langle \mathcal{J}^\mu_{\omega_{\mathcal{J}}} \rangle) \\
&\quad + \omega_{\mathcal{T}} \frac{(a-c)}{12\pi^2} \mathcal{K}^\nu \nabla^\mu (F_{\rho\sigma} \widetilde{R}^{\rho\sigma}{}_{\nu\mu}) + \frac{1}{4} \mathcal{A}_W \nabla_\rho \mathcal{K}^\rho + (A_\nu \mathcal{K}^\nu - \Lambda_{\mathcal{K}}) \nabla_\mu \langle \mathcal{J}^\mu_{\omega_{\mathcal{J}}} \rangle \,.
\end{aligned} \tag{4.7}$$

The relation (3.4) implies that

$$F_{\mu\nu}\mathcal{K}^\nu P^\mu_{BZ} = \frac{(5a-3c)}{27\pi^2} F\widetilde{F} A_\mu \mathcal{K}^\mu \,, \tag{4.8}$$

and so we conclude that

$$\begin{aligned}\nabla_\mu\Big(\langle \mathcal{T}^\mu_{\omega_\mathcal{T}\,\nu}\rangle \mathcal{K}^\nu + (A_\nu\mathcal{K}^\nu - \Lambda_\mathcal{K})\langle \mathcal{J}^\mu_{\omega_\mathcal{J}}\rangle\Big) &= \frac{1}{4}\mathcal{A}_W \nabla_\rho \mathcal{K}^\rho + \omega_\mathcal{T}\frac{(a-c)}{12\pi^2}\mathcal{K}^\nu\nabla^\mu(F_{\rho\sigma}\widetilde{R}^{\rho\sigma}{}_{\nu\mu}) \\ + (\omega_\mathcal{J}-1)\frac{(5a-3c)}{27\pi^2}A_\nu\mathcal{K}^\nu F\widetilde{F} + (A_\nu\mathcal{K}^\nu - \Lambda_\mathcal{K})&\Big((1+2\omega_\mathcal{J})\frac{5a-3c}{27\pi^2}\widetilde{F}F + \frac{c-a}{24\pi^2}\mathcal{P}\Big)\,. \end{aligned} \tag{4.9}$$

It follows that if the anomalies $\widetilde{F}F$, $\mathcal{P}$, $\mathcal{A}_W$ as well as $\nabla_\mu L^{\mu\nu}$ vanish numerically, as is the case for supersymmetric backgrounds, then there exists a two-parameter family of conserved currents for any conformal Killing vector $\mathcal{K}^\mu$,

$$\nabla_\mu\Big(\langle \mathcal{T}^\mu_{\omega_\mathcal{T}\,\nu}\rangle \mathcal{K}^\nu + (A_\nu\mathcal{K}^\nu - \Lambda_\mathcal{K})\langle \mathcal{J}^\mu_{\omega_\mathcal{J}}\rangle\Big) = 0\,, \tag{4.10}$$

and hence a two-parameter family of conserved charges

$$Q^{\omega_\mathcal{T},\omega_\mathcal{J}}[\mathcal{K}] = \int_\mathcal{C} d\sigma_\mu\Big(\langle \mathcal{T}^\mu_{\omega_\mathcal{T}\,\nu}\rangle \mathcal{K}^\nu + (A_\nu\mathcal{K}^\nu - \Lambda_\mathcal{K})\langle \mathcal{J}^\mu_{\omega_\mathcal{J}}\rangle\Big)\,. \tag{4.11}$$

Finally, we consider the conserved charges associated with (conformal) Killing spinors of $\mathcal{N}=1$ conformal supergravity. These are solutions of the Killing spinor equation obtained by setting the local symmetry transformation of the gravitino in (2.4) to zero. On a bosonic background this leads to the Killing spinor equation

$$\mathcal{D}_\mu\varepsilon_0 = \gamma_\mu\eta_0\,. \tag{4.12}$$

In the following we will also need the Majorana conjugate equation

$$\bar\varepsilon_0\overleftarrow{\mathcal{D}}_\mu = -\bar\eta_0\gamma_\mu\,. \tag{4.13}$$

Expressing η_0 in terms of ε_0 through the algebraic relation

$$\eta_0 = \frac{1}{4}\gamma^\nu\mathcal{D}_\nu\varepsilon_0\,, \tag{4.14}$$

the Killing spinor equation can be written in the form

$$\mathcal{D}_\mu \varepsilon_0 = \frac{1}{4}\gamma_\mu \gamma^\nu \mathcal{D}_\nu \varepsilon_0 \,. \tag{4.15}$$

On a bosonic background the Ward identities (2.15) imply that the supercurrent is covariantly conserved and has zero γ-trace, i.e.

$$\mathcal{D}_\mu \langle \mathcal{S}^\mu \rangle = 0 \,, \qquad \gamma_\mu \langle \mathcal{S}^\mu \rangle = 0 \,. \tag{4.16}$$

These in turn imply that

$$\nabla_\mu (\bar{\varepsilon}_0 \langle \mathcal{S}^\mu \rangle) = \bar{\varepsilon}_0 \overleftarrow{\mathcal{D}}_\mu \langle \mathcal{S}^\mu \rangle = -\bar{\eta}\gamma_\mu \langle \mathcal{S}^\mu \rangle = 0 \,, \tag{4.17}$$

and, hence, the quantity

$$Q[\varepsilon_0] = \int_{\mathcal{C}} d\sigma_\mu \bar{\varepsilon}_0 \langle \mathcal{S}^\mu \rangle \,, \tag{4.18}$$

corresponds to the conserved supercharge associated with the (conformal) Killing spinor ε_0. Note that the Killing spinor ε_0 must be commuting in order for the supercharge (4.18) to be Grassmann-valued.

5 Supersymmetric Casimir energy

We are now in a position to obtain the general form of the supersymmetric Casimir energy on $\mathcal{N} = 1$ conformal supergravity backgrounds, generalizing the result of [9] to generic SCFTs with $a \neq c$. Our starting point is the anomalous transformation of the supercurrent $\mathcal{S}^\mu$ under local $\mathcal{Q}$- and $\mathcal{S}$-supersymmetries. These transformations follow directly from the anomalous superconformal Ward identities (2.15) and are given by [19]

$$\begin{aligned}
\delta_\varepsilon \langle \mathcal{S}^\mu \rangle &= \frac{1}{2}\gamma^a \varepsilon \langle \mathcal{T}^\mu_{\mathrm{cov}\, a} \rangle + \frac{i}{8}\Big(4\delta^{[\mu}_\nu \delta^{\rho]}_\sigma + i\gamma^5 \epsilon^\mu{}_\nu{}^\rho{}_\sigma\Big)\gamma^\nu\gamma^5 \mathcal{D}_\rho\Big(\varepsilon\langle \mathcal{J}^\sigma_{\mathrm{cov}}\rangle\Big) + \frac{(a-c)}{24\pi^2} F_{\rho\sigma}\widetilde{R}^{\rho\sigma\mu\nu}\gamma_\nu \varepsilon \,, \\
\delta_\eta \langle \mathcal{S}^\mu \rangle &= \frac{3i}{4}\gamma^5 \eta \langle \mathcal{J}^\mu_{\mathrm{cov}}\rangle + \frac{(5a-3c)}{6\pi^2}\mathcal{D}_\nu(\widetilde{F}^{\mu\nu}\eta) - \frac{ic}{6\pi^2}\Big(\gamma^{[\mu}{}_\rho \delta^{\nu]}_\sigma - \delta^{[\mu}_\rho \delta^{\nu]}_\sigma\Big)\gamma^5 \mathcal{D}_\nu (F^{\rho\sigma}\eta) \\
&\quad - \frac{3(2a-c)}{4\pi^2}\mathcal{D}_\nu\Big(P_{\rho\sigma} g^{\rho[\sigma}\gamma^{\mu\nu]}\eta\Big) - \frac{(a-c)}{8\pi^2}\mathcal{D}_\nu\Big[\Big(R^{\mu\nu\rho\sigma}\gamma_{\rho\sigma} - \frac{1}{2}R g_{\rho\sigma} g^{\rho[\sigma}\gamma^{\mu\nu]}\Big)\eta\Big] \,.
\end{aligned} \tag{5.1}$$

5.1 Supercharge algebra on curved backgrounds

Taking the transformation parameters ε, η to be the (commuting) components of a conformal Killing spinor, ε_0, $\eta_0 = \frac{1}{4}\gamma^\nu\mathcal{D}_\nu\varepsilon_0$, the transformations (5.1) compute the algebra of the corresponding supercharges through the relation

$$\langle\{Q[\varepsilon_0], Q[\varepsilon_0]\}\rangle = \int_{\mathcal{C}} d\sigma_\mu \bar{\varepsilon}_0(\delta_{\varepsilon_0} + \delta_{\eta_0})\langle\mathcal{S}^\mu\rangle \,. \tag{5.2}$$

Since the anticommutator of the supercharges vanishes on BPS states, this relation determines the BPS relation among the bosonic conserved charges on an arbitrary supersymmetric background, and hence the general form of the supersymmetric Casimir energy.

Our task, therefore, is to evaluate the r.h.s. of (5.2) using th supercurrent transformations (5.1). The term $\bar{\varepsilon}_0\delta_{\varepsilon_0}\langle\mathcal{S}^\mu\rangle$ takes the form

$$\frac{1}{2}\bar{\varepsilon}_0\gamma_\nu\varepsilon_0\Big(\langle\mathcal{T}^{\mu\nu}_{\text{cov}}\rangle + \frac{(a-c)}{24\pi^2}F_{\rho\sigma}\widetilde{R}^{\rho\sigma\mu\nu}\Big) + \frac{i}{8}\bar{\varepsilon}_0\Big(4\delta^{[\mu}_\nu\delta^{\rho]}_\sigma + i\gamma^5\epsilon^\mu{}_\nu{}^\rho{}_\sigma\Big)\gamma^\nu\gamma^5\mathcal{D}_\rho\Big(\varepsilon_0\langle\mathcal{J}^\sigma_{\text{cov}}\rangle\Big)\,, \tag{5.3}$$

Integrating by parts, the term proportional to the covariant R-current becomes

$$\nabla_\rho\mathcal{V}^{\mu\rho}_{\mathcal{Q}} - \frac{i}{8}\bar{\varepsilon}_0\overleftarrow{\mathcal{D}}_\rho\Big(4\delta^{[\mu}_\nu\delta^{\rho]}_\sigma + i\gamma^5\epsilon^\mu{}_\nu{}^\rho{}_\sigma\Big)\gamma^\nu\gamma^5\varepsilon_0\langle\mathcal{J}^\sigma_{\text{cov}}\rangle\,, \tag{5.4}$$

where

$$\mathcal{V}^{\mu\rho}_{\mathcal{Q}} = \frac{i}{8}\bar{\varepsilon}_0\Big(4\delta^{[\mu}_\nu\delta^{\rho]}_\sigma + i\gamma^5\epsilon^\mu{}_\nu{}^\rho{}_\sigma\Big)\gamma^\nu\gamma^5\varepsilon_0\langle\mathcal{J}^\sigma_{\text{cov}}\rangle\,. \tag{5.5}$$

The Killing spinor equation (4.13) and the identity (A.4) in Appendix A, allow us to simplify the second term in (5.4) to

$$-\frac{3i}{4}(\bar{\eta}_0\gamma^5\varepsilon_0)\langle\mathcal{J}^\mu_{\text{cov}}\rangle\,. \tag{5.6}$$

Therefore, we conclude that the term $\bar{\varepsilon}_0\delta_{\varepsilon_0}\langle\mathcal{S}^\mu\rangle$ takes the form

$$\bar{\varepsilon}_0\delta_{\varepsilon_0}\langle\mathcal{S}^\mu\rangle = \frac{1}{2}\bar{\varepsilon}_0\gamma_\nu\varepsilon_0\Big(\langle\mathcal{T}^{\mu\nu}_{\text{cov}}\rangle + \frac{(a-c)}{24\pi^2}F_{\rho\sigma}\widetilde{R}^{\rho\sigma\mu\nu}\Big) - \frac{3i}{4}(\bar{\eta}_0\gamma^5\varepsilon_0)\langle\mathcal{J}^\mu_{\text{cov}}\rangle + \nabla_\rho\mathcal{V}^{\rho\mu}_{\mathcal{Q}}\,. \tag{5.7}$$

Let us next focus on the term

$$\begin{aligned}\bar{\varepsilon}_0\delta_{\eta_0}\langle\mathcal{S}^\mu\rangle = {}& \frac{3i}{4}\bar{\varepsilon}_0\gamma^5\eta_0\langle\mathcal{J}^\mu_{\text{cov}}\rangle + \frac{(5a-3c)}{6\pi^2}\bar{\varepsilon}_0\mathcal{D}_\nu(\widetilde{F}^{\mu\nu}\eta_0) - \frac{ic}{6\pi^2}\bar{\varepsilon}_0\Big(\gamma^{[\mu}{}_\rho\delta^{\nu]}_\sigma - \delta^{[\mu}_\rho\delta^{\nu]}_\sigma\Big)\gamma^5\mathcal{D}_\nu(F^{\rho\sigma}\eta_0) \\ & - \frac{3(2a-c)}{4\pi^2}\bar{\varepsilon}_0\mathcal{D}_\nu\Big(P_{\rho\sigma}g^{\rho[\sigma}\gamma^{\mu\nu]}\eta_0\Big) - \frac{(a-c)}{8\pi^2}\bar{\varepsilon}_0\mathcal{D}_\nu\Big[\Big(R^{\mu\nu\rho\sigma}\gamma_{\rho\sigma} - \frac{1}{2}Rg_{\rho\sigma}g^{\rho[\sigma}\gamma^{\mu\nu]}\Big)\eta_0\Big]\,,\end{aligned} \tag{5.8}$$

and consider in turn all local terms.

$\frac{(5a-3c)}{6\pi^2}\bar{\varepsilon}_0\mathcal{D}_\rho(\widetilde{F}^{\mu\rho}\eta_0)$:

Integrating by parts and using the Killing spinor equation (4.13) this term becomes

$$\nabla_\rho\mathcal{V}^{\mu\rho}_{\mathcal{S}_1}+\frac{(5a-3c)}{6\pi^2}\widetilde{F}^{\mu\rho}(\bar{\eta}_0\gamma_\rho\eta_0)\,, \tag{5.9}$$

where

$$\mathcal{V}^{\mu\rho}_{\mathcal{S}_1}=\frac{(5a-3c)}{6\pi^2}\widetilde{F}^{\mu\rho}\bar{\varepsilon}_0\eta_0\,. \tag{5.10}$$

$-\frac{ic}{6\pi^2}\bar{\varepsilon}_0(\gamma^{[\mu}{}_\rho\delta^{\nu]}_\sigma-\delta^{[\mu}_\rho\delta^{\nu]}_\sigma)\gamma^5\mathcal{D}_\nu(F^{\rho\sigma}\eta_0)$:

Similarly, this term can be written in the form

$$\nabla_\rho\mathcal{V}^{\mu\rho}_{\mathcal{S}_2}-\frac{ic}{6\pi^2}\bar{\eta}_0\gamma_\rho\Big(\gamma^{[\mu}{}_\nu\delta^{\rho]}_\sigma-\delta^{[\mu}_\nu\delta^{\rho]}_\sigma\Big)\gamma^5\eta_0\,F^{\nu\sigma}\,, \tag{5.11}$$

with

$$\mathcal{V}^{\mu\rho}_{\mathcal{S}_2}=-\frac{ic}{6\pi^2}\bar{\varepsilon}_0\Big(\gamma^{[\mu}{}_\nu\delta^{\rho]}_\sigma-\delta^{[\mu}_\nu\delta^{\rho]}_\sigma\Big)\gamma^5\eta_0\,F^{\nu\sigma}\,. \tag{5.12}$$

Moreover, the second term in (5.11) gives

$$\begin{aligned}-\frac{ic}{6\pi^2}\bar{\eta}_0\gamma_\rho\Big(\gamma^{[\mu}{}_\nu\delta^{\rho]}_\sigma-\delta^{[\mu}_\nu\delta^{\rho]}_\sigma\Big)\gamma^5\eta_0\,F^{\nu\sigma}&=-\frac{ic}{6\pi^2}\bar{\eta}_0(\gamma_\sigma\gamma^\mu{}_\nu-\delta^\mu_\nu\gamma_\sigma)\gamma^5\eta_0F^{\nu\sigma}\\&=-\frac{ic}{6\pi^2}\Big(\bar{\eta}_0\gamma^\sigma\gamma^{\mu\nu}\gamma^5\eta_0\,F_{\nu\sigma}-\bar{\eta}_0\gamma_\sigma\gamma^5\eta_0F^{\mu\sigma}\Big)\,.\end{aligned} \tag{5.13}$$

The second term vanishes, since $\bar{\eta}_0\gamma_\sigma\gamma^5\eta_0=0$ for commuting spinors, while

$$\bar{\eta}_0\gamma^\sigma\gamma^{\mu\nu}\gamma^5\eta_0=\bar{\eta}_0\gamma^{\mu\nu}\gamma^\sigma\gamma^5\eta_0=\bar{\eta}_0(\gamma^{\mu\nu\sigma}+\gamma^\mu g^{\nu\sigma}-\gamma^\nu g^{\mu\sigma})\gamma^5\eta_0=i\epsilon^{\mu\nu\sigma\rho}\bar{\eta}_0\gamma_\rho\eta_0\,. \tag{5.14}$$

Hence,

$$-\frac{ic}{6\pi^2}\bar{\eta}_0\gamma_\rho\Big(\gamma^{[\mu}{}_\nu\delta^{\rho]}_\sigma-\delta^{[\mu}_\nu\delta^{\rho]}_\sigma\Big)\gamma^5\eta_0\,F^{\nu\sigma}=\frac{c}{6\pi^2}\epsilon^{\mu\nu\sigma\rho}F_{\nu\sigma}\bar{\eta}_0\gamma_\rho\eta_0=\frac{c}{3\pi^2}\widetilde{F}^{\mu\rho}\,\bar{\eta}_0\gamma_\rho\eta_0\,, \tag{5.15}$$

and therefore (5.11) can be simplified to

$$\nabla_\rho\mathcal{V}^{\mu\rho}_{\mathcal{S}_2}+\frac{2c}{6\pi^2}\widetilde{F}^{\mu\rho}\,\bar{\eta}_0\gamma_\rho\eta_0\,. \tag{5.16}$$

$-\frac{3(2a-c)}{4\pi^2}\bar{\varepsilon}_0\mathcal{D}_\nu(P_{\rho\sigma}g^{\rho[\sigma}\gamma^{\mu\nu]}\eta_0)$:

Using integration by parts and the Killing spinor equation (4.13) this term becomes

$$\nabla_\rho \mathcal{V}^{\mu\rho}_{\mathcal{S}_3} - \frac{3(2a-c)}{4\pi^2} P_{\nu\sigma}\, \bar{\eta}_0 \gamma_\rho\, g^{\nu[\sigma}\gamma^{\mu\rho]}\eta_0\,, \tag{5.17}$$

where

$$\mathcal{V}^{\mu\rho}_{\mathcal{S}_3} = -\frac{3(2a-c)}{4\pi^2}\bar{\varepsilon}_0 (P_{\nu\sigma} g^{\nu[\sigma}\gamma^{\mu\rho]}\eta_0)\,. \tag{5.18}$$

The second term in (5.17) can be simplified as

$$\begin{aligned} -\frac{3(2a-c)}{4\pi^2} P_{\nu\sigma}\, \bar{\eta}_0 \gamma_\rho\, g^{\nu[\sigma}\gamma^{\mu\rho]}\eta_0 &= -\frac{3(2a-c)}{4\pi^2}\frac{2}{3} P_{\nu\sigma}(-g^{\nu\sigma}\bar{\eta}_0\gamma^\mu\eta_0 + g^{\mu\nu}\bar{\eta}_0\gamma^\sigma\eta_0) \\ &= -\frac{(2a-c)}{2\pi^2}\left(-\frac{1}{6}R\,\delta^\mu_\sigma + P^\mu{}_\sigma\right)\bar{\eta}_0\gamma^\sigma\eta_0\,, \end{aligned} \tag{5.19}$$

where in the last step we used that $P \equiv P_{\nu\sigma}g^{\nu\sigma} = R/6$. Observing that

$$-\frac{1}{6}R\,\delta^\mu_\sigma + P^\mu{}_\sigma = \frac{1}{2}\left(R^\mu{}_\nu - \frac{1}{2}R\,\delta^\mu_\nu\right) \tag{5.20}$$

we arrive at the final form of the expression (5.17):

$$\nabla_\rho \mathcal{V}^{\mu\rho}_{\mathcal{S}_3} - \frac{(2a-c)}{4\pi^2}\left(R^\mu{}_\nu - \frac{1}{2}R\,\delta^\mu_\nu\right)\bar{\eta}_0\gamma^\nu\eta_0\,. \tag{5.21}$$

$-\frac{(a-c)}{8\pi^2}\bar{\varepsilon}_0\,\mathcal{D}_\nu\Big[\Big(R^{\mu\nu\rho\sigma}\gamma_{\rho\sigma} - \frac{1}{2}Rg_{\rho\sigma}g^{\rho[\sigma}\gamma^{\mu\nu]}\Big)\eta_0\Big]$:

This term is similarly written as

$$\nabla_\rho \mathcal{V}^{\mu\rho}_{\mathcal{S}_4} - \frac{(a-c)}{8\pi^2}\bar{\eta}_0\gamma_\rho\Big(R^{\mu\rho\nu\sigma}\gamma_{\nu\sigma} - \frac{1}{2}Rg_{\nu\sigma}g^{\nu[\sigma}\gamma^{\mu\rho]}\Big)\eta_0\,, \tag{5.22}$$

with

$$\mathcal{V}^{\mu\rho}_{\mathcal{S}_4} = -\frac{(a-c)}{8\pi^2}\bar{\varepsilon}_0\Big(R^{\mu\rho\nu\sigma}\gamma_{\nu\sigma} - \frac{1}{2}Rg_{\nu\sigma}g^{\nu[\sigma}\gamma^{\mu\rho]}\Big)\eta_0\,. \tag{5.23}$$

The first term in the square bracket in (5.22) gives

$$\begin{aligned} \bar{\eta}_0\gamma_\rho\,(R^{\mu\rho\nu\sigma}\gamma_{\nu\sigma})\,\eta_0 &= -R^\mu{}_{\rho\nu\sigma}\,\bar{\eta}_0\,(\gamma^{\nu\sigma\rho} + \gamma^\nu g^{\sigma\rho} - \gamma^\sigma g^{\nu\rho})\,\eta_0 \\ &= -2R^\mu{}_\nu\,\bar{\eta}_0\gamma^\nu\eta_0\,, \end{aligned} \tag{5.24}$$

where we used the property $\bar{\eta}_0\gamma_\rho\gamma_{\nu\sigma}\eta_0 = -\bar{\eta}_0\gamma_{\nu\sigma}\gamma_\rho\eta_0$ for commuting spinors. Moreover, the second term in the square bracket in (5.22) becomes

$$\frac{1}{2}R\, g_{\nu\sigma}(\bar{\eta}_0\gamma_\rho g^{\nu[\sigma}\gamma^{\mu\rho]}\eta_0) = \frac{1}{2}R\, g_{\nu\sigma}\left(-\frac{2}{3}\, g^{\nu\sigma}\bar{\eta}_0\gamma^\mu\eta_0 + \frac{2}{3}\, g^{\mu\nu}\bar{\eta}_0\gamma^\sigma\eta_0\right) = -R\,\bar{\eta}_0\gamma^\mu\eta_0\,, \tag{5.25}$$

where we used the identity (A.11) in Appendix A with $\varepsilon = \eta$.

We therefore conclude that

$$-\frac{(a-c)}{8\pi^2}\bar{\eta}_0\gamma_\rho\Big(R^{\mu\rho\nu\sigma}\gamma_{\nu\sigma} - \frac{1}{2}Rg_{\nu\sigma}g^{\nu[\sigma}\gamma^{\mu\rho]}\Big)\eta_0 = \frac{(a-c)}{4\pi^2}\left(R^\mu{}_\nu - \frac{1}{2}R\,\delta^\mu_\nu\right)\bar{\eta}_0\gamma^\nu\eta_0\,, \tag{5.26}$$

and hence (5.22) reduces to

$$\nabla_\rho\mathcal{V}^{\mu\rho}_{\mathcal{S}_4} + \frac{(a-c)}{4\pi^2}\left(R^\mu{}_\nu - \frac{1}{2}R\,\delta^\mu_\nu\right)\bar{\eta}_0\gamma^\nu\eta_0\,. \tag{5.27}$$

Returning to the overall variation (5.8), we observe that (5.9) and (5.16) combine to

$$\frac{(5a-c)}{6\pi^2}\widetilde{F}^{\mu\rho}(\bar{\eta}_0\gamma_\rho\eta_0)\,, \tag{5.28}$$

while (5.21) and (5.27) reduce to

$$-\frac{a}{4\pi^2}\Big(R^\mu{}_\nu - \frac{1}{2}R\,\delta^\mu_\nu\Big)\bar{\eta}_0\gamma^\nu\eta_0\,. \tag{5.29}$$

Therefore, gathering all terms, the variation (5.8) is written as

$$\bar{\varepsilon}_0\delta_{\eta_0}\langle\mathcal{S}^\mu\rangle = \frac{3i}{4}\bar{\varepsilon}_0\gamma^5\eta_0\langle\mathcal{J}^\mu_{\text{cov}}\rangle + \Big[\frac{(5a-c)}{6\pi^2}\widetilde{F}^\mu{}_\nu - \frac{a}{4\pi^2}\Big(R^\mu{}_\nu - \frac{1}{2}R\,\delta^\mu_\nu\Big)\Big]\bar{\eta}_0\gamma^\nu\eta_0 + \nabla_\rho\mathcal{V}^{\mu\rho}_{\mathcal{S}}\,, \tag{5.30}$$

with

$$\begin{aligned}\mathcal{V}^{\mu\rho}_{\mathcal{S}} = {}& \frac{(5a-3c)}{6\pi^2}\bar{\varepsilon}_0(\widetilde{F}^{\mu\rho}\eta_0) + \frac{ic}{6\pi^2}\varepsilon_0\Big(\gamma^{[\mu}{}_\nu\delta^{\rho]}_\sigma - \delta^{[\mu}_\nu\delta^{\rho]}_\sigma\Big)\gamma^5(F^{\nu\sigma}\eta_0)\\ &- \frac{3(2a-c)}{4\pi^2}\bar{\varepsilon}_0\Big(P_{\nu\sigma}g^{\nu[\sigma}\gamma^{\mu\rho]}\eta\Big) - \frac{(a-c)}{8\pi^2}\varepsilon_0\Big(R^{\mu\rho\nu\sigma}\gamma_{\nu\sigma} - \frac{1}{2}Rg_{\nu\sigma}g^{\nu[\sigma}\gamma^{\mu\rho]}\Big)\eta_0\,.\end{aligned} \tag{5.31}$$

Combining (5.7) and (5.30), and using the commuting spinor identity $\bar{\eta}_0\gamma^5\varepsilon_0 = -\bar{\varepsilon}_0\gamma^5\eta_0$, we arrive at the result

$$\bar{\varepsilon}_0(\delta_{\varepsilon_0}+\delta_{\eta_0})\langle\mathcal{S}^\mu\rangle = \frac{1}{2}\bar{\varepsilon}_0\gamma_\nu\varepsilon_0\Big(\langle\mathcal{T}^{\mu\nu}_{\rm cov}\rangle + \frac{(a-c)}{24\pi^2}F_{\rho\sigma}\widetilde{R}^{\rho\sigma\mu\nu}\Big) + \frac{3i}{2}(\bar{\varepsilon}_0\gamma^5\eta_0)\langle\mathcal{J}^\mu_{\rm cov}\rangle \\ + \Big[\frac{(5a-c)}{6\pi^2}\widetilde{F}^\mu{}_\nu - \frac{a}{4\pi^2}\Big(R^\mu{}_\nu - \frac{1}{2}R\,\delta^\mu_\nu\Big)\Big]\bar{\eta}_0\gamma^\nu\eta_0 + \nabla_\rho(\mathcal{V}^{\rho\mu}_{\mathcal{Q}}+\mathcal{V}^{\mu\rho}_{\mathcal{S}})\,. \tag{5.32}$$

Inserting this in (5.2) and dropping the total derivative terms we conclude that BPS states satisfy the integral constraint

$$\begin{aligned} 0 &= \int_{\mathcal{C}} d\sigma_\mu \bar{\varepsilon}_0(\delta_{\varepsilon_0}+\delta_{\eta_0})\langle\mathcal{S}^\mu\rangle \\ &= \int_{\mathcal{C}} d\sigma_\mu\Big\{\frac{1}{2}\bar{\varepsilon}_0\gamma_\nu\varepsilon_0\Big(\langle\mathcal{T}^{\mu\nu}_{\rm cov}\rangle + \frac{(a-c)}{24\pi^2}F_{\rho\sigma}\widetilde{R}^{\rho\sigma\mu\nu}\Big) + \frac{3i}{2}(\bar{\varepsilon}_0\gamma^5\eta_0)\langle\mathcal{J}^\mu_{\rm cov}\rangle \\ &\qquad + \Big[\frac{(5a-c)}{6\pi^2}\widetilde{F}^\mu{}_\nu - \frac{a}{4\pi^2}\Big(R^\mu{}_\nu - \frac{1}{2}R\,\delta^\mu_\nu\Big)\Big]\bar{\eta}_0\gamma^\nu\eta_0\Big\}\,. \end{aligned} \tag{5.33}$$

We will now show that this constraint corresponds to the BPS relation among the bosonic charges of supersymmetric states and determines the supersymmetric Casimir energy.

5.2 BPS relation and the supersymmetric Casimir energy

In order to relate the constraint (5.33) to the conserved R-charge and conformal Killing charges, respectively (4.3) and (4.11), we begin by showing that the spinor bilinear $\bar{\varepsilon}_0\gamma^\mu\varepsilon_0$ is a conformal Killing vector. In fact, for later use we show more generally that if ε_0 and ε_0' are conformal Killing spinors, then the spinor bilinear

$$\mathcal{K}^\mu(\varepsilon_0,\varepsilon_0') \equiv \bar{\varepsilon}_0'\gamma^\mu\varepsilon_0\,, \tag{5.34}$$

is a conformal Killing vector.

We have,

$$\begin{aligned} \nabla_\mu\mathcal{K}_\nu(\varepsilon_0,\varepsilon_0') &= \bar{\varepsilon}_0'\overleftarrow{\mathcal{D}}_\mu\gamma_\nu\varepsilon_0 + \bar{\varepsilon}_0'\gamma_\nu\mathcal{D}_\mu\varepsilon_0 \\ &= -\,\bar{\eta}_0'\gamma_\mu\gamma_\nu\varepsilon_0 + \bar{\varepsilon}_0'\gamma_\nu\gamma_\mu\eta_0 \\ &= \bar{\varepsilon}_0\gamma_\nu\gamma_\mu\eta_0' + \bar{\varepsilon}_0'\gamma_\nu\gamma_\mu\eta_0\,, \end{aligned} \tag{5.35}$$

where we used the Killing spinor equations (4.12) and (4.13) and the fact that for commuting spinors $\bar{\eta}_0'\gamma_\mu\gamma_\nu\varepsilon_0 = -\bar{\varepsilon}_0\gamma_\nu\gamma_\mu\eta_0'$. Hence,

$$\nabla_\mu\mathcal{K}_\nu(\varepsilon_0,\varepsilon_0') + \nabla_\nu\mathcal{K}_\mu(\varepsilon_0,\varepsilon_0') = g_{\mu\nu}(\bar{\varepsilon}_0\eta_0' + \bar{\varepsilon}_0'\eta_0) = \frac{1}{2}g_{\mu\nu}\nabla_\rho\mathcal{K}^\rho(\varepsilon_0,\varepsilon_0')\,, \tag{5.36}$$

which confirms that $\mathcal{K}^\mu(\varepsilon_0, \varepsilon_0')$ is a conformal Killing vector.

It follows that

$$\mathcal{K}_0^\mu \equiv \bar{\varepsilon}_0 \gamma^\mu \varepsilon_0 \,, \tag{5.37}$$

is a conformal Killing vector and so (5.33) can be written in the form

$$\begin{aligned} 0 = Q_{\omega_\mathcal{T},\omega_\mathcal{J}}[\mathcal{K}_0] + \int_\mathcal{C} d\sigma_\mu \Big\{ \Big(3i(\bar{\varepsilon}_0\gamma^5\eta_0) - A_\nu \mathcal{K}_0^\nu + \Lambda_{\mathcal{K}_0} \Big) \langle \mathcal{J}^\mu_{\omega_\mathcal{J}} \rangle \\ + \Big((1-\omega_\mathcal{T}) P^{\mu\nu}_{BZ} + \frac{(a-c)}{24\pi^2} F_{\rho\sigma} \widetilde{R}^{\rho\sigma\mu\nu} \Big) \mathcal{K}_{0\nu} + 3i(\bar{\varepsilon}_0\gamma^5\eta_0)(1-\omega_\mathcal{J}) P^\mu_{BZ} \\ + \Big[\frac{(5a-c)}{3\pi^2} \widetilde{F}^\mu{}_\nu - \frac{a}{2\pi^2} \Big(R^\mu{}_\nu - \frac{1}{2} R\, \delta^\mu_\nu \Big) \Big] \bar{\eta}_0 \gamma^\nu \eta_0 \Big\} \,. \end{aligned} \tag{5.38}$$

However, the coefficient multiplying the R-current $\langle \mathcal{J}^\mu_{\omega_\mathcal{J}} \rangle$ is a constant, since

$$\partial_\mu \Big(3i(\bar{\varepsilon}_0\gamma^5\eta_0) - A_\nu \mathcal{K}_0^\nu + \Lambda_{\mathcal{K}_0} \Big) = 3i\partial_\mu(\bar{\varepsilon}_0\gamma^5\eta_0) - F_{\mu\nu}\mathcal{K}_0^\nu \,, \tag{5.39}$$

and

$$\begin{aligned} 3i\partial_\mu(\bar{\varepsilon}_0\gamma^5\eta_0) &= 3i\bar{\varepsilon}_0 \overleftarrow{\mathcal{D}}_\mu \gamma^5 \eta_0 + 3i\bar{\varepsilon}_0\gamma^5 \mathcal{D}_\mu \eta_0 \\ &= -3i\bar{\eta}_0\gamma_\mu\gamma^5\eta_0 - \frac{3i}{2}\bar{\varepsilon}_0\gamma^5 \Big(P_{\mu\nu} + \frac{2i}{3} F_{\mu\nu}\gamma^5 - \frac{1}{3}\widetilde{F}_{\mu\nu} \Big) \gamma^\nu \varepsilon_0 \\ &= F_{\mu\nu}\mathcal{K}_0^\nu \,, \end{aligned} \tag{5.40}$$

where we used the fact that $\bar{\eta}_0\gamma_\mu\gamma^5\eta_0 = \bar{\varepsilon}_0\gamma_\mu\gamma^5\varepsilon_0 = 0$ for commuting spinors and [19]

$$\mathcal{D}_\mu \eta_0 = -\frac{1}{2} \Big(P_{\mu\nu} + \frac{2i}{3} F_{\mu\nu}\gamma^5 - \frac{1}{3}\widetilde{F}_{\mu\nu} \Big) \gamma^\nu \varepsilon_0 \,. \tag{5.41}$$

$$\Phi_{\mathcal{K}_0} \equiv 3i(\bar{\varepsilon}_0\gamma^5\eta_0) - A_\nu \mathcal{K}_0^\nu + \Lambda_{\mathcal{K}_0} = \text{const.}\,. \tag{5.42}$$

We conclude that (5.33) can be further simplified to

$$Q^{\omega_\mathcal{T},\omega_\mathcal{J}}[\mathcal{K}_0] + \Phi_{\mathcal{K}_0} Q_R^{\omega_\mathcal{J}} + Q^{\omega_\mathcal{T},\omega_\mathcal{J}}_{\text{local}}[\mathcal{K}_0] = 0 \,, \tag{5.43}$$

where

$$Q^{\omega_{\mathcal{T}},\omega_{\mathcal{J}}}_{\text{local}}[\mathcal{K}_0] \equiv \int_{\mathcal{C}} d\sigma_\mu \Big\{ \Big((1-\omega_{\mathcal{T}}) P^{\mu\nu}_{BZ} + \frac{(a-c)}{24\pi^2} F_{\rho\sigma} \widetilde{R}^{\rho\sigma\mu\nu} \Big) \mathcal{K}_{0\nu} \\ + (1-\omega_{\mathcal{J}})(\Phi_{\mathcal{K}_0} + A_\nu \mathcal{K}_0^\nu - \Lambda_{\mathcal{K}_0}) P^\mu_{BZ} + \Big[\frac{(5a-c)}{3\pi^2} \widetilde{F}^\mu{}_\nu - \frac{a}{2\pi^2} \Big(R^\mu{}_\nu - \frac{1}{2} R\, \delta^\mu_\nu \Big) \Big] \bar{\eta}_0 \gamma^\nu \eta_0 \Big\} , \tag{5.44}$$

is a *local* charge that depends only on the supergravity background. As we demonstrate in the next section, when applied to the global timelike Killing vector, the relation (5.43) determines the energy of a BPS state in terms of the conserved R-charge, with (5.44) corresponding to the supersymmetric Casimir energy.

6 Casimir energy on $\mathbb{R} \times S^3$

The BPS relation (5.43) and the local charge (5.44) hold for any supersymmetric background of $\mathcal{N} = 1$ conformal supergravity with numerically vanishing superonformal anomalies. Such backgrounds have been studied extensively [24–34] (see also [35, 36] for earlier work). In this section we will focus on a concrete background for which the supersymmetric Casimir energy has been computed in the literature by other means. In particular, we will apply our general result to an example in the class of backgrounds with topology $\mathbb{R} \times S^3$ (or $S^1 \times S^3$ in Euclidean signature) that admit two supercharges with opposite R-charge.

6.1 Supersymmetric backgrounds with $\mathbb{R} \times S^3$ topology

Following [5], we consider four-dimensional backgrounds of topology $\mathbb{R} \times S^3$ that admit a non-singular complete direct product metric of the form[3]

$$ds^2 = -\Omega^2(\rho) dt^2 + f^2(\rho) d\rho^2 + m_{IJ}(\rho)\, d\varphi_I\, d\varphi_J , \quad I, J = 1, 2 , \tag{6.1}$$

where $\varphi_I \in [0, 2\pi]$, $\rho \in [0, 1]$, while the functions $\Omega(\rho)$, $f(\rho)$ are positive definite, as is the symmetric matrix $m_{IJ}(\rho)$. This metric possesses an $\mathbb{R} \times U(1)^2$ isometry ($U(1)^3$ in its Euclidean form) corresponding to the commuting Killing vectors ∂_t, ∂_{φ_1} and ∂_{φ_2}.

Demanding that the background admits two supercharges with opposite R-charge requires the existence of a globally defined null Killing vector of the form

[3]The corresponding Euclidean backgrounds are obtained by setting $t = i\tau$, see e.g. [32, 37].

$$\mathcal{K} = \frac{1}{2}(\partial_t + b_I \partial_{\varphi_I}) = \frac{1}{2}(\partial_t + b_1 \partial_{\varphi_1} + b_2 \partial_{\varphi_2}) \,, \tag{6.2}$$

where b_1, b_2 are real parameters. The requirement that this is null fixes

$$\Omega^2 = b^I m_{IJ} b^J \,. \tag{6.3}$$

However, note that the metric on S^3, which is parameterized by $f(\rho)$ and $m_{IJ}(\rho)$, is not constrained by supersymmetry.

Finally, the globally well defined R-symmetry gauge field takes the form [5][4]

$$A = -\frac{\Omega}{8fc}\partial_\rho \Big[(c^2 + a_\chi^2)\Big(\frac{d\varphi_1}{b_1} - \frac{d\varphi_2}{b_2}\Big) + 2a_\chi \Big(\frac{d\varphi_1}{b_1} + \frac{d\varphi_2}{b_2} + dt\Big)\Big] - \frac{1}{2} d\omega \,, \tag{6.4}$$

where

$$a_\chi = \frac{1}{\Omega^2}(b_1^2 m_{11} - b_2^2 m_{22}) \,, \quad c = \frac{2|b_1 b_2|}{\Omega^2}\sqrt{\det(m_{IJ})} \,, \quad \omega = \mathrm{sgn}(b_1)\varphi_1 + \mathrm{sgn}(b_2)\varphi_2 \,. \tag{6.5}$$

These expressions describe a family of non-singular backgrounds that admit two supercharges of opposite R-charge. They are parameterized by the arbitrary non-singular metric on S^3 and the real parameters b_1, b_2. The squashed (Berger) three-sphere is a special case of these backgrounds [5]. However, for simplicity we will illustrate our results by considering the background with arbitrary b_1, b_2 and the round metric on S^3.

6.2 Round S^3 with arbitrary b_1, b_2

Defining the angular coordinate $\theta = \pi\rho$, the metric (6.1) corresponding to the round S^3 is

$$ds^2 = -\Omega^2 dt^2 + d\theta^2 + m_{IJ} d\varphi_I d\varphi_J \,, \tag{6.6}$$

with

$$m_{11} = 4\cos^2\frac{\theta}{2} \,, \qquad m_{22} = 4\sin^2\frac{\theta}{2} \,, \qquad m_{12} = 0 \,, \qquad f = \pi \,. \tag{6.7}$$

These expressions for m_{IJ} completely determine the background. In particular,

$$\Omega^2 = b^I m_{IJ} b^J = 2\Big(b_1^2 + b_2^2 + (b_1^2 - b_2^2)\cos\theta\Big) \,, \tag{6.8}$$

[4]This differs by an overall minus sign compared to [5] due to different conventions.

$$a_\chi = \frac{1}{\Omega^2}(b_1^2 m_{11} - b_2^2 m_{22}) = \frac{b_1^2 - b_2^2 + (b_1^2 + b_2^2)\cos\theta}{b_1^2 + b_2^2 + (b_1^2 - b_2^2)\cos\theta}\,, \tag{6.9}$$

and

$$c = \frac{2|b_1 b_2|}{\Omega^2}\sqrt{\det(m_{IJ})} = \frac{2|b_1 b_2|\sin\theta}{b_1^2 + b_2^2 + (b_1^2 - b_2^2)\cos\theta}\,. \tag{6.10}$$

Inserting these in (6.4) we obtain the R-symmetry gauge field

$$A = \frac{\mathrm{sgn}(b_1 b_2)}{\Omega}(b_2 d\varphi_1 + b_1 d\varphi_2) + \frac{|b_1 b_2|}{\Omega}\,dt - \frac{1}{2}d\omega\,. \tag{6.11}$$

Killing spinors

The Killing spinors are solutions of the conformal Killing equation (4.15). In order to solve this equation we introduce a suitable local frame for the background metric (6.6), namely

$$\begin{aligned}
e^0 &= \Omega dt\,,\\
e^1 &= \frac{\Omega}{2b_1 b_2}\Big((1+a_\chi)b_2 d\varphi_1 + (1-a_\chi)b_1 d\varphi_2\Big)\,,\\
e^2 &= d\theta\,,\\
e^3 &= \frac{\Omega}{2b_1 b_2}(b_2 d\varphi_1 - b_1 d\varphi_2)\,,
\end{aligned} \tag{6.12}$$

so that

$$ds^2 = -(e^0)^2 + (e^1)^2 + (e^2)^2 + (e^3)^2\,. \tag{6.13}$$

We also use the Weyl representation of the gamma matrices so that $\gamma^\mu = e_a^\mu \gamma^a$ with

$$\gamma^a = \begin{pmatrix} 0 & \sigma^a \\ \bar{\sigma}^a & 0 \end{pmatrix}, \qquad \sigma_a = (-\mathbb{1}, \sigma_i), \quad \bar{\sigma}_a = (\mathbb{1}, \sigma_i)\,, \tag{6.14}$$

and

$$\gamma^{ab} = \begin{pmatrix} \sigma^{ab} & 0 \\ 0 & \bar{\sigma}^{ab} \end{pmatrix}, \qquad \sigma_{ab} = \frac{1}{2}[\sigma_a, \bar{\sigma}_b], \quad \bar{\sigma}_{ab} = \frac{1}{2}[\bar{\sigma}_a, \sigma_b]\,. \tag{6.15}$$

Moreover, the chirality matrix takes the form

$$\gamma^5 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}\,. \tag{6.16}$$

Th commuting Majorana spinors ε_0 and η_0 can then be expressed as

$$\varepsilon_0 = \begin{pmatrix} \epsilon \\ \tilde{\epsilon} \end{pmatrix} , \qquad \eta_0 = \begin{pmatrix} \zeta \\ \tilde{\zeta} \end{pmatrix} , \tag{6.17}$$

where ϵ, ζ are left-handed two-component Weyl spinors and $\tilde{\epsilon} \equiv i\sigma_2 \epsilon^*$, $\tilde{\zeta} \equiv i\sigma_2 \zeta^*$. Moreover, the Killing spinor equation (4.15) is equivalent to the two-component spinor equations

$$\mathcal{D}^L_\mu \epsilon = \sigma_\mu \tilde{\zeta} , \quad \tilde{\zeta} = \frac{1}{4} \bar{\sigma}^\mu \, \mathcal{D}^L_\mu \epsilon , \qquad \mathcal{D}^R_\mu \tilde{\epsilon} = \bar{\sigma}_\mu \zeta , \quad \zeta = \frac{1}{4} \sigma^\mu \, \mathcal{D}^R_\mu \epsilon , \tag{6.18}$$

where the chiral derivatives are $\mathcal{D}^L_\mu = \partial_\mu + \frac{1}{4}\omega^{ab}_\mu \sigma_{ab} + iA_\mu$ and $\mathcal{D}^R_\mu = \partial_\mu + \frac{1}{4}\omega^{ab}_\mu \bar{\sigma}_{ab} - iA_\mu$.

For generic b_1 and b_2 the Killing spinor equation (4.15) admits the unique solution

$$\varepsilon_0 = e^{\frac{i}{2}\omega} \sqrt{\frac{\Omega}{2}} \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + e^{-\frac{i}{2}\omega} \sqrt{\frac{\Omega}{2}} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} , \tag{6.19}$$

while $\eta_0 = \frac{1}{4}\gamma^\nu \mathcal{D}_\nu \varepsilon_0$ takes the form

$$\eta_0 = -\frac{i}{(2\Omega)^{3/2}} e^{\frac{i}{2}\omega} \begin{pmatrix} 0 \\ 0 \\ -2|b_1 b_2| + \Omega\Omega' \\ 2|b_1 b_2| + \Omega\Omega' \end{pmatrix} - \frac{i}{(2\Omega)^{3/2}} e^{-\frac{i}{2}\omega} \begin{pmatrix} 2|b_1 b_2| + \Omega\Omega' \\ 2|b_1 b_2| - \Omega\Omega' \\ 0 \\ 0 \end{pmatrix} . \tag{6.20}$$

Having determined the conformal Killing spinor, we can evaluate the three spinor bilinears that enter in the BPS relation (5.43) and the local charge (5.44). We find

$$\begin{aligned} &\bar{\varepsilon}_0 \gamma^\mu \varepsilon_0 = -4\mathcal{K}^\mu , \\ &3i\bar{\varepsilon}_0 \gamma^\mu \eta_0 - A_\mu \bar{\varepsilon}_0 \gamma^\mu \varepsilon_0 = -(|b_1| + |b_2|) , \\ &\bar{\eta}_0 \gamma^\mu \eta_0 = \frac{1}{4}\mathcal{K}^\mu - \frac{1}{2\Omega^2}\Big(b_1^2 + b_2^2, 0, (b_1^2 - b_2^2) b_1, -(b_1^2 - b_2^2) b_2\Big) , \end{aligned} \tag{6.21}$$

where $\mathcal{K}$ is the globally defined null Killing vector (6.2). These expressions reaffirm the results we obtained earlier. In particular, $\bar{\varepsilon}_0 \gamma^\mu \varepsilon_0$ is a Killing vector, as it should, while the R-charge potential (5.42) is indeed constant (note that $\Lambda_{\mathcal{K}_0} = 0$ in this case since $\mathcal{L}_{\mathcal{K}_0} A_\mu = 0$).

Casimir energy

We now have all ingredients in order to evaluate the local charge (5.44) on this supersymmetric background. Since $\bar{\varepsilon}_0\gamma^\mu\varepsilon_0 = -4\mathcal{K}^\mu \sim -2\partial_t$, the Casimir energy corresponds to

$$\mathcal{E}^{\omega_\mathcal{T},\omega_\mathcal{J}}_{\text{Casimir}} \equiv -\frac{1}{2}Q^{\omega_\mathcal{T},\omega_\mathcal{J}}_{\text{local}}[\mathcal{K}_0] = Q^{\omega_\mathcal{T},\omega_\mathcal{J}}_{\text{local}}[2\mathcal{K}]\,, \tag{6.22}$$

and, therefore, it is given by

$$\begin{aligned}\mathcal{E}^{\omega_\mathcal{T},\omega_\mathcal{J}}_{\text{Casimir}} \equiv \int_\mathcal{C} d\sigma_\mu\Big\{&\Big(2(1-\omega_\mathcal{T})P^{\mu\nu}_{BZ} - L^{\mu\nu}\Big)\mathcal{K}_\nu - (1-\omega_\mathcal{J})\frac{3i}{2}(\bar{\varepsilon}_0\gamma^5\eta_0)P^\mu_{BZ} \\ &+\left[\frac{a}{4\pi^2}\Big(R^\mu{}_\nu - \frac{1}{2}R\,\delta^\mu_\nu\Big) - \frac{(5a-c)}{6\pi^2}\widetilde{F}^\mu{}_\nu\right]\bar{\eta}_0\gamma^\nu\eta_0\Big\}\,,\end{aligned} \tag{6.23}$$

where $\mathcal{K}^\mu$ is the globally defined Killing vector in (6.2), $L^{\mu\nu}$ is given in (3.14), while the BZ terms are given in (3.8) and (3.10).

Evaluating the R-current BZ term on the background specified by the metric (6.6) and R-symmetry gauge field (6.11) we find

$$P^\mu_{BZ} = \frac{2(5a-3c)}{27\pi^2}\frac{(b_1^2-b_2^2)}{2\Omega^4}\Big(-\operatorname{sgn}(b_1b_2)(|b_1|-|b_2|), 0, -|b_1|b_2, |b_2|b_1\Big)\,. \tag{6.24}$$

Similarly, the only nonzero components of the stress tensor BZ term (3.10) become

$$\begin{aligned}P^{t\varphi_1}_{BZ} &= \frac{(c-a)}{3\pi^2}\frac{(b_1^2-b_2^2)}{\Omega^8}(2\Omega-5|b_1|)\,b_1^2b_2^2\operatorname{sgn}(b_2)\,,\\ P^{t\varphi_2}_{BZ} &= -\frac{(c-a)}{3\pi^2}\frac{(b_1^2-b_2^2)}{\Omega^8}(2\Omega-5|b_2|)\,b_1^2b_2^2\operatorname{sgn}(b_1)\,.\end{aligned} \tag{6.25}$$

Moreover, the nonzero components of the tensors $L^{\mu\nu}$ and $\widetilde{F}_{\mu\nu}$ are respectively

$$\begin{aligned}L^{t\varphi_1} &= \frac{(a-c)}{24\pi^2}\frac{(b_1^2-b_2^2)^2}{\Omega^6}|b_1|\operatorname{sgn}(b_2)(1-\cos\theta)\,,\\ L^{t\varphi_2} &= \frac{(a-c)}{24\pi^2}\frac{(b_1^2-b_2^2)^2}{\Omega^6}|b_2|\operatorname{sgn}(b_1)(1+\cos\theta)\,,\\ L^{\varphi_1\varphi_2} &= -\frac{(a-c)}{48\pi^2}\frac{(b_1^2-b_2^2)}{\Omega^4}|b_1b_2|\,,\end{aligned} \tag{6.26}$$

and

$$
\begin{aligned}
\widetilde{F}_{t\varphi_1} &= -\frac{(b_1^2-b_2^2)}{\Omega^2}|b_1|\mathrm{sgn}(b_2)(1+\cos\theta)\,,\\
\widetilde{F}_{t\varphi_2} &= \frac{(b_1^2-b_2^2)}{\Omega^2}|b_2|\mathrm{sgn}(b_1)(1-\cos\theta)\,,\\
\widetilde{F}_{\varphi_1\varphi_2} &= \frac{2(b_1^2-b_2^2)}{\Omega^4}|b_1b_2|\sin^2\theta\,,
\end{aligned}
\tag{6.27}
$$

while the Einstein tensor on the supersymmetric background (6.6) takes the form

$$
R_{\mu\nu}-\frac{1}{2}Rg_{\mu\nu}=\mathrm{diag}\Big(0,\frac{b_1^2+b_2^2}{\Omega^2},\frac{b_2^2m_{11}}{\Omega^2}\Big(\frac{4b_1^2}{\Omega^2}+1\Big),\frac{b_1^2m_{22}}{\Omega^2}\Big(\frac{4b_2^2}{\Omega^2}+1\Big)\Big)-\frac{3}{4}g_{\mu\nu}\,. \tag{6.28}
$$

Putting everything together we can evaluate the integral (6.23) on the Cauchy surface defined by the constant time slices. The result is

$$
\mathcal{E}^{\omega_{\mathcal{T}},\omega_{\mathcal{J}}}_{\mathrm{Casimir}}=\frac{3a}{2}\frac{|b_1||b_2|}{(|b_1|+|b_2|)}+\frac{(|b_1|-|b_2|)^2}{9b_1b_2(|b_1|+|b_2|)}\Big(c_1(|b_1|^2+|b_2|^2)+c_2|b_1||b_2|\Big)\,, \tag{6.29}
$$

where the coefficients c_1 and c_2 are given by

$$
\begin{aligned}
c_1 &= (5a-3c)(\omega_{\mathcal{J}}-1)-(a-c)(2\omega_{\mathcal{T}}-1)+\frac{3}{2}(5a-c)\,,\\
c_2 &= 9a\,\mathrm{sgn}(b_1b_2)+2(5a-3c)(\omega_{\mathcal{J}}-1)-3(a-c)(\omega_{\mathcal{T}}-1)+\frac{3}{2}(5a-c)\,.
\end{aligned}
\tag{6.30}
$$

For $|b_1|=|b_2|$, in which case the metric (6.6) is conformally flat, (6.23) reduces to

$$
\mathcal{E}^{\mathrm{Casimir}}=\frac{3a}{4}|b|\,,\qquad |b_1|=|b_2|=|b|\,, \tag{6.31}
$$

independently of the values of $\omega_{\mathcal{T}}$ and $\omega_{\mathcal{J}}$. This result is in agreement with the expression (1.3) in the scheme where the coefficient of the R^4 counterterm is set to zero.

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A Conventions and spinor identities

Our spacetime and spinor conventions are those of [18]. The tangent space metric is $\eta = \mathrm{diag}\,(-1,1,1,1)$ and the Levi-Civita symbol $\varepsilon_{\mu\nu\rho\sigma} = \pm 1$ satisfies $\varepsilon_{0123} = 1$. Moreover, the Levi-Civita tensor is defined as $\epsilon_{\mu\nu\rho\sigma} = \sqrt{-g}\,\varepsilon_{\mu\nu\rho\sigma} = e\,\varepsilon_{\mu\nu\rho\sigma}$. Finally, the chirality matrix is

$$\gamma^5 = i\gamma_0\gamma_1\gamma_2\gamma_3\,. \tag{A.1}$$

Several gamma matrix and spinor identities we use in this manuscript are given in Appendix A of [19]. Here we quote only the gamma matrix identities

$$\gamma^{\mu\nu\rho\sigma} = i\epsilon^{\mu\nu\rho\sigma}\gamma^5, \qquad \gamma^{\mu\nu\rho} = i\epsilon^{\mu\nu\rho\sigma}\gamma_\sigma\gamma^5, \qquad \gamma^{\mu\nu} = \frac{i}{2}\epsilon^{\mu\nu}{}_{\rho\sigma}\gamma^{\rho\sigma}\gamma^5 \tag{A.2}$$

and the *anticommuting* spinor flip relations under Majorana conjugation

$$\begin{aligned}
&\bar\varepsilon\,\gamma^5\eta = \bar\eta\gamma^5\varepsilon\,,\\
&\bar\varepsilon\,\gamma_\mu\eta = -\bar\eta\gamma_\mu\varepsilon\,,\\
&\bar\varepsilon\,\gamma_\mu\gamma^5\eta = \bar\eta\,\gamma_\mu\gamma^5\,,\varepsilon\\
&\varepsilon\,\gamma^\sigma\gamma^{\mu\nu}\gamma^5\eta = -\bar\eta\,\gamma^{\mu\nu}\gamma^\sigma\gamma^5\varepsilon\,,\\
&\varepsilon\,\gamma^\sigma\gamma^{\mu\nu}\eta = \bar\eta\,\gamma^{\mu\nu}\gamma^\sigma\varepsilon\,.
\end{aligned} \tag{A.3}$$

However, the evaluation of the Casimir energy involves bilinears of *commuting* spinors, for which the flip relation (A.3) hold with an additional minus sign. Using these we now prove two identities commuting spinor identities that we use extensively in the our analysis.

$$\bar\eta\gamma_\rho\Big(4\delta^{[\mu}_\nu\delta^{\rho]}_\sigma + i\gamma^5\epsilon^\mu{}_\nu{}^\rho{}_\sigma\Big)\gamma^\nu\gamma^5\varepsilon = -6\delta^\mu_\sigma\,\bar\eta\gamma^5\varepsilon\,. \tag{A.4}$$

Proof

The first term can rewritten as follows:

$$\begin{aligned}\bar{\eta}\gamma_\rho(4\delta^{[\mu}_\nu\delta^{\rho]}_\sigma)\gamma^\nu\gamma^5\varepsilon &= 2\,\bar{\eta}\gamma_\rho(\delta^\mu_\nu\delta^\rho_\sigma - \delta^\rho_\nu\delta^\mu_\sigma)\gamma^\nu\gamma^5\epsilon \\ &= 2\,\bar{\eta}\gamma_\sigma\gamma^\mu\gamma^5\epsilon - 2\delta^\mu_\sigma\,\bar{\eta}\gamma_\nu\gamma^\nu\gamma^5\varepsilon \\ &= 2g_{\kappa\sigma}\bar{\eta}\gamma^\kappa\gamma^\mu\gamma^5\varepsilon - 8\,\delta^\mu_\sigma\,\bar{\eta}\gamma^5\varepsilon\,,\end{aligned} \tag{A.5}$$

where in the second step we used that $\gamma_\nu\gamma^\nu = 4$. The first term of eq. (A.5) becomes

$$\begin{aligned}2g_{\kappa\sigma}\bar{\eta}\gamma^\kappa\gamma^\mu\gamma^5\varepsilon &= 2g_{\kappa\sigma}\bar{\eta}(\gamma^{\kappa\mu} + g^{\kappa\mu})\gamma^5\varepsilon \\ &= 2g_{\kappa\sigma}\,\bar{\eta}\gamma^{\kappa\mu}\gamma^5\varepsilon + 2\delta^\mu_\sigma\,\bar{\eta}\gamma^5\varepsilon\,,\end{aligned} \tag{A.6}$$

and expressing $\gamma^{\mu\nu}$ in terms of the Levi-Civita symbol using eq. (A.2), we get

$$2g_{\kappa\sigma}\,\bar{\eta}\gamma^{\kappa\mu}\gamma^5\varepsilon = ig_{\kappa\sigma}\,\epsilon^{\kappa\mu}{}_{\nu\rho}\,\bar{\eta}\gamma^{\nu\rho}\varepsilon = -i\epsilon^\mu{}_{\sigma\nu\rho}\,\bar{\eta}\gamma^{\nu\rho}\varepsilon\,. \tag{A.7}$$

Inserting eq. (A.6) we obtain

$$2g_{\kappa\sigma}\bar{\eta}\gamma^\kappa\gamma^\mu\gamma^5\varepsilon = -i\epsilon^\mu{}_{\sigma\nu\rho}\,\bar{\eta}\gamma^{\nu\rho}\varepsilon + 2\delta^\mu_\sigma\,\bar{\eta}\gamma^5\varepsilon\,, \tag{A.8}$$

so that eq. (A.5) is written as

$$\bar{\eta}\gamma_\rho(4\delta^{[\mu}_\nu\delta^{\rho]}_\sigma)\gamma^\nu\gamma^5\varepsilon = -i\epsilon^\mu{}_{\sigma\nu\rho}\,\bar{\eta}\gamma^{\nu\rho}\varepsilon - 6\delta^\mu_\sigma\,\bar{\eta}\gamma^5\varepsilon\ . \tag{A.9}$$

The second term of the spinor part in eq. (5.4) is easily treated and gives

$$i\bar{\eta}\gamma_\rho\gamma^5\epsilon^\mu{}_\nu{}^\rho{}_\sigma\gamma^\nu\gamma^5\varepsilon = i\epsilon^\mu{}_{\sigma\rho\nu}\,\bar{\eta}\gamma^\rho\gamma^\nu\varepsilon_0 = i\epsilon^\mu{}_{\sigma\nu\rho}\,\bar{\eta}\gamma^{\nu\rho}\varepsilon_0\ . \tag{A.10}$$

Summing eq. (A.9) and eq. (A.10) the Levi-Civita part cancels out and get eq. (A.4). ■

$$\bar{\varepsilon}\gamma_\rho\,g^{\nu[\sigma}\gamma^{\mu\rho]}\eta = -\frac{i}{3}\epsilon^{\sigma\mu\nu\rho}\,\bar{\eta}\gamma_\rho\gamma^5\varepsilon - \frac{2}{3}\,g^{\nu\sigma}\bar{\varepsilon}\gamma^\mu\eta + \frac{2}{3}\,g^{\mu\nu}\bar{\varepsilon}\gamma^\sigma\eta\,. \tag{A.11}$$

Proof

Let us first consider the γ-terms. We have

$$\gamma_\rho\,g^{\nu[\sigma}\gamma^{\mu\rho]} = \frac{1}{3}\gamma_\rho\,(g^{\nu\sigma}\gamma^{\mu\rho} + g^{\nu\rho}\gamma^{\sigma\mu} + g^{\nu\mu}\gamma^{\rho\sigma}) = \frac{1}{3}(-3g^{\nu\sigma}\gamma^\mu + \gamma^\nu\gamma^{\sigma\mu} + 3\,g^{\mu\nu}\gamma^\sigma)\,, \tag{A.12}$$

where we used that $\gamma_\rho \gamma^{\rho\mu} = 3\gamma^\mu$. For commuting spinors $\bar{\varepsilon}\gamma^\nu\gamma^{\mu\rho}\eta = -\bar{\eta}\gamma^{\mu\rho}\gamma^\nu\varepsilon$ and hence

$$\bar{\varepsilon}\gamma_\rho\, g^{\nu[\sigma}\gamma^{\mu\rho]}\eta = -g^{\nu\sigma}\bar{\varepsilon}\gamma^\mu\eta + g^{\mu\nu}\bar{\varepsilon}\gamma^\sigma\eta - \frac{1}{3}\,\bar{\eta}\gamma^{\sigma\mu}\gamma^\nu\varepsilon\,. \tag{A.13}$$

Note that the last term may be written as

$$\bar{\eta}\gamma^{\sigma\mu}\gamma^\nu\varepsilon = \bar{\eta}\left(\gamma^{\sigma\mu\nu} + \gamma^\sigma g^{\mu\nu} - \gamma^\mu g^{\sigma\nu}\right)\varepsilon\,. \tag{A.14}$$

The first term in this expression gives $\bar{\eta}\gamma^{\sigma\mu\nu}\varepsilon = i\epsilon^{\sigma\mu\nu\rho}\,\bar{\eta}\gamma_\rho\gamma^5\varepsilon$. Inserting eq. (A.14) in eq. (A.13) we obtain eq. (A.11). ■

Note that for $\varepsilon = \eta$, the first term on the r.h.s of (A.14) vanishes and we get

$$\bar{\eta}\gamma_\rho\, g^{\nu[\sigma}\gamma^{\mu\rho]}\eta = -\frac{2}{3}\, g^{\nu\sigma}\bar{\eta}\gamma^\mu\eta + \frac{2}{3}\, g^{\mu\nu}\bar{\eta}\gamma^\sigma\eta\,. \tag{A.15}$$

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