arXiv:2401.00296v1 [hep-th] 30 Dec 2023

LTH 1362

T-duality across non-extremal horizons

M. Médevielle[1] and T. Mohaupt[2]

[1] Graduate School of Arts and Sciences,
The University of Tokyo,
Tokyo 153-8902, Japan
mmedevielle@g.ecc.u-tokyo.ac.jp

[2]Department of Mathematical Sciences
University of Liverpool
Peach Street
Liverpool L69 7ZL, UK
Thomas.Mohaupt@liv.ac.uk

January 2, 2024

Abstract

When applying T-duality to a generic, non-extreme Killing horizon, T-duality is spacelike on one side and timelike on the other. We show, using simple examples from four-dimensional Einstein-Maxwell theory, that the image of the horizon is a singularity which can be understood as an interface between two different T-dual theories and their solutions. Using an embedding into type-II string theory, we show that the singularity occurs when scalars reach the boundary of moduli space, resulting in a breakdown of the effective field theory due to the presence of tensionless strings.

Contents

1 Introduction

T-duality is one of the features which makes string theory distinct. It states that a perturbative string theory A, when compactified on a circle of radius R, is equivalent to a perturbative string theory B, compactified on a circle of radius $1/R$, where length is measured in string units $\sqrt{\alpha'}$. While the bosonic string is 'T-selfdual,' $A = B$, we will be interested in type-II superstrings, where type-IIA and type-IIB are T-dual to each other [1], [2]. The ten-dimensional type-II theories can be seen as two decompactification limits, $R \to \infty$ and $R \to 0$, of a one-parameter family of type-II theories on $\mathbb{R}^{1,8} \times S^1_R$. While the ten-dimensional theories are not equivalent, one still refers to them as 'being T-dual' to each other. In this case T-duality is not an equivalence between theories, but a solution-generating technique, relating, for example, D-p-branes to D-$(p \pm 1)$-branes. By taking several directions to be compact one generates a larger discrete T-duality group. We refer to [3] for a review of T-duality.

T-duality can be formulated at the level of the effective (super-)gravity theories of the massless modes. Here it can be derived by compactifying a theory and decompactifying it in an alternative way ('taking $R \to 0$ in string units'). As discussed above, when relating decompactified theories in this way, one does not necessarily have an equivalence of theories, but a powerful solution generating technique: by application of the 'Buscher rules' to a solution of theory A we obtain a solution of theory B.

T-duality shows that the framework of (pseudo-)Riemannian geometry used in general relativity is too narrow to provide a satisfactory geometric framework for string theory. It suggests that $\sqrt{\alpha'}$ is a minimal length scale if we probe spacetime with strings. It also mixes the metric $g_{(10)}$ with the Kalb-Ramond field $b_{(10)}$ and the dilaton $\phi_{(10)}$. This has motivated various approaches to define generalized 'stringy' geometries, where these data, and possibly also other massless string excitations, are treated on the same footing: generalized geometry [4], [5], double field theory [6], [7], as well as their 'exceptional' and 'heterotic' extensions which include fields outside the universal (NS-NS) sector [8], [9], [10], [11]. T-duality has been used to construct various classes of string backgrounds which go outside the remit of Riemannian geometry [12],[6].

In this paper we investigate an issue which arises when applying T-duality to a very important class of Riemannian spacetimes, namely those which contain Killing horizons. While it was already observed in [13] that T-duality maps horizons to singularities, not much further work seems to have been done since then to follow up on this observation.[1] We will study generic, non-extremal Killing horizons, where the square-norm of the Killing vector field has a simple zero at the horizon and thus changes between being timelike and being spacelike. From the viewpoint of a $(d-1)$-dimensional effective theory, obtained by dimensional reduction with respect to the Killing vector field, a non-extremal Killing horizon corresponds to an interface where spacetime signature changes: for timelike reduction the lower-dimensioal theory has Euclidean signature, while for spacelike reduction it has Lorentzian signature. The T-dual of the d-dimensional solution is obtained by identifying and applying alternative dimensional lifts of the $(d-1)$-dimensional solutions. The resulting d-dimensional configuration solves the field equations of the T-dual theory, which in general is different from the original d-dimensional theory. Moreover, the T-dual theory is in general different for spacelike and timelike T-duality, so that in the T-dual solution the horizon is mapped to an interface between solutions of two different theories. While all these operations can be carried out for any gravitational theory coupled to matter, an embedding into string theory provides one with additional insights, as we will see in detail later.

Timelike T-duality was introduced in [14] where it was shown to map the type-IIA/IIB string theories to two new superstring theories called type-IIB* and type-IIA*. The partition functions of type-II* theories differ from those of the conventional theories by certain phases. At the level of the effective supergravity theories, the signs of the kinetic terms of all R-R fields are reversed, while D-branes are replaced by E-branes which satisfy Dirichlet boundary conditions in real time. While this is highly unusual, as long as the T-duality circle is kept finite, type-II* theories are equivalent

[1] For certain two-dimensional backgrounds T-duality exchanges singularities and horizons, see Section 4 of [3].

to type-II theories and all apparent pathologies are taken care of by stringy gauge symmetries. When assuming that the decompactified type-II* are well defined limits, and when adding S-duality into the mix, one can generate a web of type-II string theories that extends over all ten-dimensional space-time signatures [15]. Further features and potential pathologies of exotic type-II theories, as well as ways to cure them have been discussed in [16] and [17]. Lower-dimensional supergravity theories in non-Lorentizan signature, which arise from compactification of exotic string theories, as well as their solutions have been studied recently in [18],[19],[20],[21],[22],[23]. As we will discuss later, one application of our work is to help clarify the physical validity of dynamical transitions between conventional and exotic string theories.

We will carry out our analysis from a four-dimensional perspective, and the reduction-lifting procedure we apply makes sense in the context of Einstein-Maxwell theory coupled to scalar fields. However, to properly interprete this procedure as T-duality, we need an embedding into string theory, which identifies one of the scalars as the four-dimensional dilaton and another as the universal string axion, dual to the four-dimensional Kalb-Ramond field. By performing the reduction-lifting procedure in the string conformal frame, we obtain four-dimensional Buscher rules which take the same form as in ten dimensions. In particular, T-duality corresponds to inverting the radius of a compactification circle in string units, or, when decompactifying, inverting of the corresponding metric coefficient.

We will use type-II and type-II* theories to provide a string embedding, and pick a particular solution of Einstein-Maxwell theory, the planar version of the non-extreme Reissner-Nordstrom solution, for concreteness. The geometric and thermodynamic properties of this solution, as well as its lift to ten-dimensional string theories and eleven-dimensional M-theory have been studied in detail in [24], [25] and we will draw on this material where needed. It will be obvious that the behaviour that we find is generic for spacetimes containing a non-extremal horizon. The planar Reissner-Nordstrom solution describes a bouncing cosmology interpolating between a contracting and an expanding Kasner solution. The bounce is caused by the presence of timelike singularities, which reside in static regions, meeting the expanding and contracting regions along Killing horizons. We pick one static and one non-static region and apply timelike and spacelike T-duality, respectively. The T-dual theories are Einstein-dilaton-axion theories, which differ by a sign-flip in the axion kinetic term. To provide an embedding into string theory, we first observe that the Einstein-Maxwell and Einstein-dilaton-axion system both sit inside $\mathcal{N} = 2$ supergravity coupled to a single hypermultiplet, with target space $U(1,2)/U(1) \times U(2)$ or $U(1,2)/U(1)\times U(1,1)$ depending on the sign of the axion kinetic term. These models are in turn what we may call the universal sector of type-II and type-II* superstring theory compactified on a Calabi-Yau threefold [26]. They are universal in the sense that they are what remains if we consistently truncate out all supermultiplets which contain moduli of the Calabi-Yau manifold. The scalars remaining in the 'universal hypermultiplet' (UHM) are the four-dimensional dilaton φ, the universal axion $\tilde{\varphi}$, and two R-R scalars $\zeta, \tilde{\zeta}$. The embedding also shows that the Maxwell field is the universal vector field present in any type-II/type-II* compactification, namely the one residing in the $\mathcal{N} = 2$ supergravity multiplet.

We will refer to the combined system of Einstein-Maxwell theory and the four scalars of the UHM with target $U(1,2)/U(1)\times U(2)$ as the EM-UHM system. We will show that this system is self-dual under spacelike T-duality (with certain subsystems mapped to each other), while under timelike T-duality it gets mapped to a system where the signs of the Maxwell term, and as well those of the kinetic terms of the R-R scalar $\zeta, \tilde{\zeta}$ have been flipped. This changes the scalar target geometry to $U(1,2)/U(1) \times U(1,1)$. We will refer to this theory as the twisted EM-UHM system, and use the notation (EM-UHM)$_-$ to indicate the sign flips. The twisted EM-UHM system is the bosonic part of a twisted $\mathcal{N} = 2$ supergravity theory, which realises a twisted version of the standard four-dimensional $\mathcal{N} = 2$ supersymmetry algebra with R-symmetry group $U(1,1)$ [27]. In this theory, the target space geometry of hypermultiplets is para-quaternionic Kähler instead of quaternionic Kähler. The twisted EM-UHM system is the universal sector of type-II* compactifications on Calabi-Yau threefolds [26]. It is self-dual under spacelike T-duality and gets mapped to the untwisted version under timelike T-duality. By performing spacelike and timelike reductions and liftings for both models, we will obtain the explicit Buscher rules relating the fields.

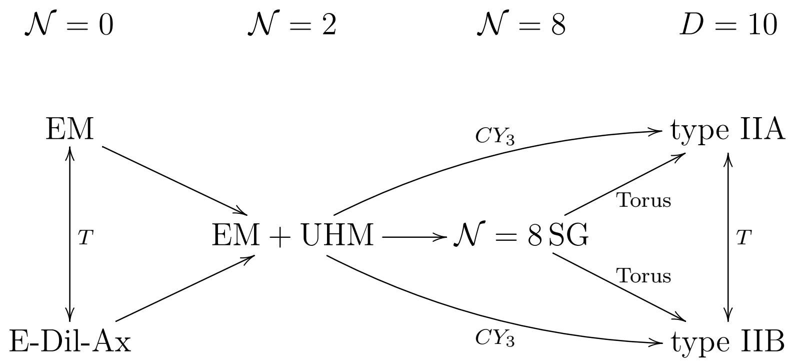

Figure 1: Action of spacelike T-duality on various models discussed in the text. Double arrows with a 'T' denote relations by spatial T-duality, the other arrays are embeddings and dimensional liftings. Absence of a T-duality arrow indicates that the model is self-dual under spatial T-duality. A similar diagram (not shown) relates twisted versions of the various models, which differ from the models shown here by sign flips for the kinetic terms of some of the fields.

We remark that instead of using Calabi-Yau compactifications, we can also embed our models using toroidal compactifications. A toroidal compactification of type-II string theory leads to $\mathcal{N} = 8$ supegravity, which can be truncated to $\mathcal{N} = 2$ supergravity. The toroidal reduction of the type-IIA/IIB theories leads to the standard $\mathcal{N} = 8$ supergravity theory with scalar target space $E_{7(7)}/SU(8)$, while toroidal reduction of the type-IIA*/IIB* theories leads to a twisted version with scalar target space $E_{7(7)}/SU(4,4)$ [14]. A toroidal embedding has the advantage that one can explicitly lift four-dimensional solutions to ten or eleven dimensions, and identify the corresponding brane configurations. The planar Reissner-Nordstrom solution lifts to the same brane configuration that underlies the 'STU-black hole', namely a D4-D4-D4-D0 system in type-IIA and and D5-D1 system augmented with a pp-wave and a Taub-NUT soution in type-IIB. The relations between the various models discussed above, as well as the action of spacelike and timelike T-duality on them, is summarized in Figure 1 and 2.

We will see that the T-dual solutions become singular at both the positions of the horizon and of the singularity of the original solution. While from the purely gravitational point of view, these are just naked singularities, a string theory em-

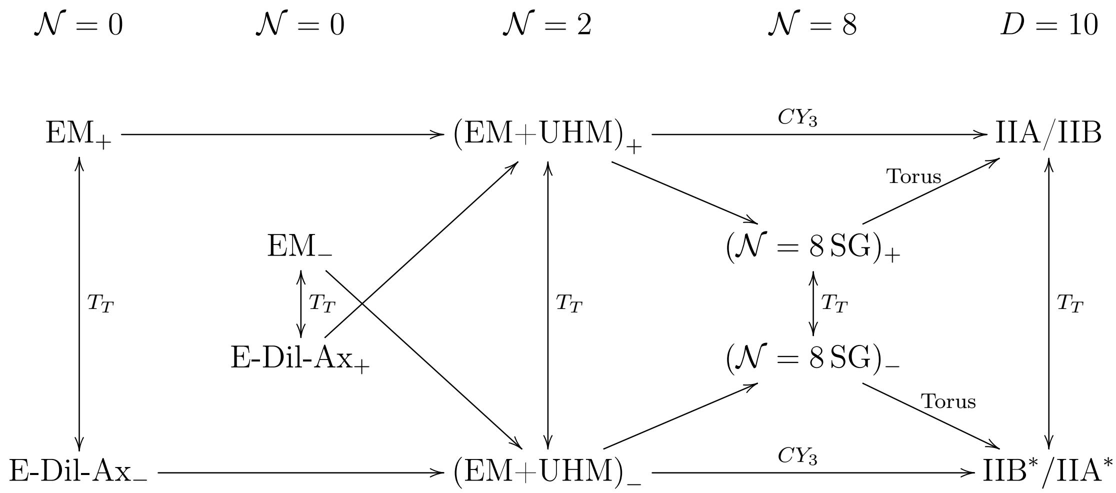

Figure 2: Action of timelike T-duality on various models discussed in the text. Arrows with a T_T indicate the action of timelike T-duality, the other arrows indicate embeddings and dimensional lifts. Subscripts $\pm$ distinguish between the standard and twisted versions of the models.

bedding will allow us to make sense out of them: they are related to the run-away behaviour of scalar fields, which suggests an interpretation in terms of the swampland distance conjecture [28]. This conjecture states that whenever scalar fields ('moduli') in a string-effective field theory approach the boundary of their target space, there should be an infinite tower of states in the full string theory which becomes massless in this limit, resulting in a breakdown of the effective field theory. By choosing a suitable string embedding and duality frame, we will argue that all singularities observed in the T-dual solution are related to limits where such a tower arises through strings becoming tensionless. While the singularities appear at infinite distance in moduli space, they are at finite distance in spacetime, which raises the question of what happens at these points. We will make some concrete proposals how these questions can be addressed based on the results of our work in the conclusions section.

This paper is organised as follows. In Section 2 we provide the necessary background on T-duality and its relation to the supergravity c-map. In Section 3 we derive the four-dimensional Buscher rules for our EM-UHM model using the reduction/lifting method. In Section 4 we apply these results to T-dualize the planar cosmological solution and to investigate its properties in detail. Using an embedding of the Einstein-Maxwell solution into type-IIA string theory, its T-dual image embeds into type-IIB* and type-IIB depending on which side of the horizon we are. We use these embeddings to argue that the singularities of the T-dual solution are related to tensionless strings. In Section 5 we apply the same procedure to the planar black hole solution of a twisted Einstein-Maxwell theory where the sign of the Maxwell field has been flipped. This model can be embedded into type-IIA* and we briefly discuss its relation to the planar cosmological solution, when embedded into type-IIA. We conclude with an outlook on further research. Some details have been relegated to appendices.

2 Background

2.1 Solution-generation through dimensional reduction/lifting

This paper can be read from two perspectives. The first is the one of classical gravity, where we provide and apply a solution-generating technique for four-dimensional Einstein-Maxwell theory coupled to a non-linear sigma model with four real scalar fields $\varphi, \tilde{\varphi}, \zeta, \tilde{\zeta}$ parametrising the symmetric space $U(2,1)/(U(2) \times U(1))$. The corresponding action is

$$
\begin{aligned}
\mathcal{S}_4 = \int d^4x \sqrt{\hat{g}_E} \Big(& \hat{R}_E - \frac{1}{2} \hat{g}_E^{\mu\rho} \hat{g}_E^{\nu\lambda} \hat{F}_{\mu\nu} \hat{F}_{\rho\lambda} - 2 \hat{g}_E^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi \\
& - 2 e^{4\varphi} \, [\, \partial^\mu \tilde{\varphi} + \frac{1}{2} (\zeta \partial^\mu \tilde{\zeta} - \tilde{\zeta} \partial^\mu \zeta)] \, [\, \partial_\mu \tilde{\varphi} + \frac{1}{2} (\zeta \partial_\mu \tilde{\zeta} - \tilde{\zeta} \partial_\mu \zeta)] \\
& - e^{2\varphi} \left[\partial_\mu \zeta \partial^\mu \zeta + \partial^\rho \tilde{\zeta} \partial_\rho \tilde{\zeta} \right] \Big) .
\end{aligned} \tag{1}
$$

It is a special case of the class of actions considered in [26]. Apart from truncating out all but one vector field and all but four scalars, we have rescaled the field φ (which from the string perspective considered below is the four-dimensional dilaton) by $\varphi \to -2\varphi$ and there is an overall factor of 2. Anticipating that we will apply dimensional reduction, in this section the four-dimensional metric and vector field carry a 'hat', and the letter 'E' denotes that we are in the Einstein conformal frame.

The class of solutions we start with has trivial scalars and thus is a solution to Einstein-Maxwell theory. It takes the form

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)} + r^2(dx^2 + dy^2)\ , \quad F = -\frac{e}{r^2}dt \wedge dr\ , \tag{2}$$

where

$$f(r) = -\frac{2M}{r} + \frac{e^2}{r^2}\ , \tag{3}$$

with a mass-like parameter $M > 0$ and electric charge e, such that $M^2 > e^2$. For later reference we note that a gauge potential is given by $A = -\frac{e}{r}dt$. The solution has planar symmetry in the two spacelike coordinates x, y. For $0 < r < r_* = \frac{e^2}{2M}$ the coordinate r is spacelike, while t is timelike and the solution is static with timelike Killing vector field $\xi = \partial_t$. For $r_* < r < \infty$, r is a timelike coordinate and the isometry ∂_t is spacelike. Thus $r = r_*$ is a Killing horizon of the non-extremal type, where square-norm of the Killing vector field has a simple zero and changes between spacelike and timelike. For $r \to \infty$ the solution is asymptotic to a Kasner cosmological solution, while $r = 0$ is a curvature singularity. This cosmological solution is the planar symmetric cousin of the familiar non-extremal Reissner-Nordström solution. Its global geometric structure and the thermodynamics of its horizon are described in detail in [24, 25], where this solution appeared as a special case of a class of solutions for the so-called STU-model, an $\mathcal{N} = 2$ supergravity theory with three vector supermultiplets. Within this class of models, solutions to Einstein-Maxwell theory can be obtained by setting its three complex scalars constant and by setting its four vector fields proportional to each other [24], [25].

Compared to the spherically symmetric Reissner-Nordström solution the planar version lacks the exterior, asymptotically flat region, so that the intermediate dynamical region between the inner and outer horizon has now become the exterior region, making this a cosmological rather than a black hole solution. In the extremal limit of the planar solution the horizon and the exterior region are pushed off to infinity, leaving a static solution with a naked singularity, see Figure 5 of [24]. We only consider the non-extremal solution in the following, since we are interested in the effects in T-dualizing in the presence of a Killing horizon. By maximal analytical continuation one obtains two copies of the static interior and two copies of the dynamical exterior region. The resulting Penrose diagram is the one of the maximally extended Schwarzschild spacetime, rotated by 90 degrees, see Figure 3 of [24]. After picking a global time orientation, it describes a bouncing cosmology, which is complete for timelike geodesics. The static regions, sourced by time-like curvature singularities (which can be related to certain branes upon embedding into string theory) connect an early contracting to a late expanding cosmology, making it a cosmological bounce which is timelike geodesically complete.

To this solution we will apply a solution-generating technique which uses dimensional reduction with respect to the Killing vector field ∂_t, followed by an 'alternative dimensional lift' or 'oxidation', which exchanges the degrees of freedom of metric and Maxwell field with those of the four scalars. In the dynamical region, where the Killing vector field is spacelike, the reduced $(1+2)$-dimensional theory is the bosonic part of $\mathcal{N}=4$ supergravity coupled to two hypermultiplets. The dynamical degrees of freedom are eight scalar fields, which parametrize the symmetric space

$$\mathcal{N} = \frac{U(2,1)}{U(2)\times U(1)} \times \frac{U(2,1)}{U(2)\times U(1)} \,. \tag{4}$$

This follows immediately from the results of [27] and [26]. By inspection it is obvious that there exists an alternative lift to $1+3$ dimensions, with the roles of four-dimensional scalar and non-scalar fields exchanged. Since the Killing vector field changes type at the horizon, the solution generating reduction in the static region is timelike, leading to an effective three-dimensional Euclidean theory. We will come back to this below.

2.2 T-duality and the c-map

The second perspective that we can take is the one of ten-dimensional type-II superstring theory, into which the solutions can be embedded. Specifically, an explicit uplift was given in [24] for the static part of the solution (2), with the type-IIB D5-D1-pp-wave system and the M-theory M5-M5-M5-pp-wave system among the possible lifts. These lifts use that the EM-UHM system (using terminology and acronyms introduced in the introduction) can be embedded into the maximal (ungauged) four-dimensional $N = 8$ supergravity theory, which in turn is the low-energy effective field theory of the massless modes for type-IIA/IIB string theory compactified on a six-torus T^6. In this context, the solution generating technique described above amounts to applying the T-duality between type-IIA string theory compactified on $T^6 \times S^1_R$ and type-IIB string theory compactified on $T^6 \times S^1_{1/R}$, where R is the radius of an additional circle, measured in string length units, followed by taking the alternative compactification limit $R \to 0$ to obtain a duality between two effective four-dimensional theories and their solutions.

Apart from being a subsector of $\mathcal{N} = 8$ supergravity, the EM-UHM system is the bosonic part of $\mathcal{N} = 2$ supergravity coupled to a single hypermultiplet, and can be considered as the universal sector that any type-II compactification on a Calabi-Yau threefold contains. In general, compactification of type-IIA superstring theory on a Calabi-Yau threefold with Hodge numbers $(h_{1,1}, h_{2,1})$ results in $\mathcal{N} = 2$ supergravity coupled to $n_V = h_{1,1}$ vector multiplets and $h_{2,1} + 1$ hypermultiplets [29]. To leading order in α', one of the hypermultiplets parametrizes a totally geodesic submanifold of the hypermultiplet moduli space which is isomorphic to $U(2,1)/(U(2) \times U(1))$, so that a consistent truncation to this single hypermultiplet is possible.[2] This hypermultiplet is called the universal hypermultiplet (UHM) and contains the IIA dilaton φ, the universal axion $\tilde{\varphi}$ obtained by Hodge-dualizing the four-dimensional Kalb-Ramond B-field, and two scalars $\zeta, \tilde{\zeta}$ descending from the ten-dimensional R-R sector (Ramond-Ramond sector). These scalars are universal in the sense that even in the case of a rigid complex structure there are at least two homological three-cycles associated with the holomorphic top form and its complex conjugate.[3] Reducing the R-R three-form potential C_3 over these cycles gives rise to two real scalars. The scalars in the other (non-universal) hypermultiplets parametrize deformations of the complex structure of the Calabi-Yau manifold, together with further R-R scalars obtained by reducing C_3 on the other $2h_{2,1}$ homological three-cycles. Truncating these fields out thus amounts to freezing the complex structure and the non-universal R-R moduli. The theory also contains $h_{1,1} + 1 = n_V + 1$ vector fields one of which,

[2]See [30] for subtleties arising through α'-corrections.

[3]In terms of Betti numbers b_i and Hodge numbers $h_{i,j}$: $b_3 = 2 + 2h_{1,2}$ for Calabi-Yau threefolds.

dubbed the graviphoton, belongs to the $\mathcal{N}=2$ supergravity multiplet while the others sit in the $n_V = h_{1,1}$ vector multiplets. The (complex) scalar fields in IIA vector multiplets parametrize the (complexified) Kähler structures of the Calabi-Yau threefold. Freezing them out corresponds to fixing the Kähler structure as well as the moduli of the ten-dimensional Kalb-Ramond field. After truncating out the vector multiplets, a single vector field remains which we refer to as the Maxwell field. We remark that while our model with $n_V = 0, n_H = 1$ formally corresponds to the case $h_{1,1} = h_{2,1} = 0$, there is no such Calabi-Yau threefold, since one can always at least change the overall size and hence $h_{1,1} \geq 1$. However, the EM-UHM system is a consistent truncation where all non-universal scalars have been frozen and only one vector field has been kept. We have focused on the type-IIA case. If one compactifies type-IIB string theory on the same Calabi-Yau manifold one obtains a different $\mathcal{N}=2$ supergravity theory which has $n'_V = h_{2,1}$ vector multiplets and $n'_H = h_{1,1}+1$ hypermultiplets. It is these two theories that are related by T-duality over a circle transverse to the Calabi-Yau manifold [31].[4]

In the context of four-dimensional $\mathcal{N}=2$ supergravity and of type-II Calabi-Yau compactifications, the action of T-duality on a circle transverse to the Calabi-Yau space acts on the scalar fields by the so-called c-map [31], [32]. Given type-IIA and type-IIB theories compactified on the same Calabi Yau manifold X_6, T-duality states that type-IIA on $X_6 \times S^1_R$ is equivalent to type-IIB on $X_6 \times S^1_{1/R}$, where R is the radius of a circle orthogonal to X_6 measured in string length units. By taking the decompactification limits $R, 1/R \to \infty$, one obtains a map between type-IIA and type-IIB compactified on X_6, which can be used to generate solutions of the type-IIB effective supergravity theory from those of the type-IIA theory, and vice versa. Starting from type-IIA with $n_V = h_{1,1}$ vector multiplets and $n_H = h_{2,1}+1$ hypermultiplets, spacelike dimensional reduction results in a theory with $(h_{1,1}+1)+(h_{2,1}+1)$ hypermultiplets: after reduction to three dimensions, the $h_{1,1}$ vector multiplets can be dualized into $h_{1,1}$ hypermultiplets, while the four local degrees of freedom of the

[4]Note that this needs to be distinguished from mirror symmetry, which is an equivalence between type-IIA and type-IIB compactified on two different Calabi-Yau threefolds X_6, $\tilde{X}_6$ with Hodge numbers related by $\tilde{h}_{1,1} = h_{2,1}$, $\tilde{h}_{2,1} = h_{1,1}$.

four-dimensional Einstein-Maxwell system organise themselves into a hypermultiplet with scalar manifold $U(2,1)/(U(2)\times U(1))$, which is a totally geodesic subspace of a combined scalar manifold $N_{4h_{1,1}+4}$ hosting all degrees of freedom descending from the four-dimensional supergravity and vector multiplets. This manifold is quaternionic-Kähler [32]. The four-dimensional hypermultiplet manifold $N'_{4h_{2,1}+4}$ which also is a quaternionic Kähler manifold reduces trivially, so that the resulting total scalar manifold of the reduced three-dimensional theory is a metric product

$$\mathcal{N} = N_{4h_{1,1}+4} \times N'_{4h_{2,1}+4}$$

of two quaternionic-Kähler manifolds, as required for consistent coupling to three-dimensional supergravity. This in turn allows an alternative dimensional uplift to a four-dimensional $\mathcal{N}=2$ supergravity theory with $n'_V = h_{2,1}$ vector multiplets and $n'_H = h_{2,1}$ hypermultiplets, which is the low energy effective field theory of type-IIB string theory compactified on X_6.

In this process the four-dimensional supergravity multiplet gets mapped to the universal hypermultiplet. Therefore the EM-UHM model is the smallest theory for which this duality makes sense, and, moreover, the EM-UHM model is self-dual since the total numbers $n_V = 0, n_H = 1$ of vector and hypermultiplets does not change and the two factors of three-dimensional scalar manifold are isometric to each other.

2.3 Timelike T-duality and the image of a Killing horizon

Since the spacetime (2) has a non-extremal Killing horizon at $r = r_*$, the region $0 < r < r_*$ has to be treated separately. The Killing vector field is now timelike, so that by reduction we obtain a three-dimensional Euclidean effective theory. Timelike dimensional reduction is a standard technique for generating stationary solutions [33], [34]. The maximal Euclidean supergravity theories are known from dimensional reduction of eleven-dimensional supergravity and ten-dimensional type-IIB supergravity on Lorentzian tori [35], [14]. Euclidean $\mathcal{N}=2$ supergravity in four and three dimensions features a variant of special geometry where complex structures are replaced by para-complex structures. These geometries were developed in depth in [36], [37], [38], [27],see also [39] for a comprehensive review. This work has been extended to all four-dimensional and three-dimensional signatures in [26]. Scalar geometries depend on the dimension and signature in a way that is encoded in the corresponding supersymmetry algebra, specifically in its R-symmetry group [40]. In the following we will rely on this work for background and will sometimes quote results from it. For the specific purpose of this paper the relevant point is the variation of various relative signs between terms in the Lagrangian.

The special geometry of four-dimensional $N = 2$ vector multiplets coupled to supergravity is projective special Kähler, see [39] for a review which uses our conventions. While spacelike reduction leads, after inclusion of the reduced degrees of freedom of the supergravity multiplet, to a quaternionic-Kähler manifold [32], a timelike reduction leads to a so called para-quaternionic Kähler manifold, where the signs of the kinetic of half the scalars have been flipped [19] . From the perspective of type-II Calabi-Yau compactifications, these are precisely the R-R scalars. If one just reduces the four-dimensional supergravity multiplet over time, the resulting manifold is one of the two Euclidean versions of the UHM, namely $U(2,1)/(U(1,1 \times U(1))$. Since the four-dimensional hypermultiplet reduces trivially, timelike reduction of the EM-UHM theory results in a three-dimensional Euclidean theory with scalar manifold

$$\mathcal{N}' = \frac{U(2,1)}{U(1,1) \times U(1)} \times \frac{U(2,1)}{U(2) \times U(1)} \, . \tag{5}$$

This follows immediately from the results of [27] and [26]. If one performs an alternative dimensional uplift over time it is the second factor which will give rise to the four-dimensional graviton and Maxwell field, while the four-dimensional scalar fields now parametrize the first factor. Thus the EM-UHM model is not self-dual under timelike T-duality but is mapped to a dual theory where various signs have been flipped. According to [26] the resulting four-dimensional action is

$$\begin{aligned} \mathcal{S}_4 = \int d^4x \sqrt{\hat{g}_E} \Big(& \hat{R}_E + \frac{1}{2} \hat{g}_E^{\mu\rho} \hat{g}_E^{\nu\lambda} \hat{F}_{\mu\nu} \hat{F}_{\rho\lambda} - 2 \hat{g}_E^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi \\ & - 2 e^{4\varphi} \, [\, \partial^\mu \tilde{\varphi} + \frac{1}{2} (\zeta \partial^\mu \tilde{\zeta} - \tilde{\zeta} \partial^\mu \zeta)] \, [\, \partial_\mu \tilde{\varphi} + \frac{1}{2} (\zeta \partial_\mu \tilde{\zeta} - \tilde{\zeta} \partial_\mu \zeta)] \\ & + e^{2\varphi} \left[\partial_\mu \zeta \partial^\mu \zeta + \partial^\rho \tilde{\zeta} \partial_\rho \tilde{\zeta} \right] \Big). \end{aligned} \tag{6}$$

Observe the sign flips, indicated in red, of the kinetic terms of the Maxwell field and of the scalars $\zeta, \tilde{\zeta}$, compared to the EM-UHM theory (1). We refer to (6) as the twisted EM-UHM theory, denoted (EM-UHM)$_-$, see Figure 2. It comprises the universal sector (supergravity multiplet and universal hypermultiplet) of type-II* superstring theories compactified on a Calabi-Yau threefold. Due to the sign flips the bosonic Lagrangian (6) completes into a Lagrangian invariant under a twisted version of the standard $\mathcal{N} = 2$ supersymmetry algebra, which has R-symmetry group $U(1,1) \simeq SU(1,1) \times U(1)$ rather than $U(2) \simeq SU(2) \times U(1)$ [19]. In this twisted Lorentz signature theory, vector multiplet geometry is still projective special Kähler, but the vector fields have inverted kinetic terms, while the hypermultiplet geometry is para-quaternionic Kähler. The twisted EM-UHM model (6) is self-dual under spacelike T-duality, since dimensional reduction over space leads to a three-dimensional theory with scalar target space

$$\mathcal{N}'' = \frac{U(2,1)}{U(1,1) \times U(1)} \times \frac{U(2,1)}{U(1,1) \times U(1)} , \tag{7}$$

as follows immediately from the results of [27] and [26].

2.4 Summary of this section

In this section we have reviewed the c-map, including its extension to timelike reductions and lifts. When applied to Einstein-Maxwell theory coupled to the UHM, T-duality amounts to exchanging the roles of the two UHMs which carry the dynamical degrees of freedom after dimensional reduction. We have taken the supergravity perspective, using the Einstein conformal frame. In this parametrization of the four-dimensional actions, the assignments of fields to supermultiplets is clear. However, while T-duality can be viewed as a reduction/lifting procedure applicable to any gravitational theory, it fundamentally is a 'stringy' duality, and these feature only become manifest when using the string conformal frame. We will turn to this perspective in the next section.

3 Four-dimensional Buscher rules

We now turn to the derivation for the Buscher rules for the EM-UHM model and its twisted version. In order to work out the T-dual of the solution (2), we could in principle work with the actions (1) and (6). The $\mathcal{N} = 2$ Lagrangians resulting from Calabi-Yau compactifications of all type-II string theories, including those of non-Lorentzian signature, can be found in [26] together with those of all possible spacelike and timelike reductions to three dimensions. From this information one can build a dictionary between the fields of any two four-dimensional theories that are related by a spacelike, timelike or mixed (signature changing) T-duality. By applying these rules to any solution of one theory, one obtains a corresponding solution of the T-dual theory.

However, the Lagrangians used in [26] were all given in the Einstein frame, while T-duality looks more natural in the string frame. By committing to embedding the solution (2) into a type-II string theory (which we take to be IIA for concreteness), with φ as four-dimensional dilaton and $\tilde{\varphi}$ as the universal axion, we know how to rewrite the four-dimensional effective theory in terms of the four-dimensional string frame. Then, by going through the reduction/lifting process with the dual theory again be written in terms of the four-dimensional string frame, we expect to obtain four-dimensional Buscher rules which resemble the ten-dimensional ones. In particular, since T-duality corresponds to inversion of the radius of a compactified isometric direction '*', we expect that the corresponding four-dimensional string frame metric coefficient transforms as $g_{**} \mapsto g_{**}^{-1}$. While the bosonic string frame NS-NS action takes the same form in any dimension, the R-R sector is non-universal, and therefore we have to go through the reduction/lifting procedure to obtain the full set of Buscher rules.

3.1 Spacelike reduction of the Einstein-Maxwell-UHM sytem

In this section we derive the four-dimensional Buscher rules for T-dualizing the EM-UHM system over space. Before reducing to three dimensions, we transform (1) from the Einstein frame with metric $\hat{g}^{E}_{\mu\nu}$ to the four-dimensional string frame with metric

$\hat{g}^S_{\mu\nu}$, where

$$\hat{g}^E_{\mu\nu} = e^{-2\varphi}\hat{g}^S_{\mu\nu} \Rightarrow \sqrt{\hat{g}^E} = \sqrt{\det(\hat{g}^S e^{-2\varphi})} = \sqrt{e^{-8\varphi}\det(\hat{g}^S)} = e^{-4\varphi}\sqrt{\hat{g}^S}\ . \tag{8}$$

To re-write the Einstein-Hilbert term, we use the transformation rule of the Ricci scalar under a conformal rescaling, see for example [41], appendix D. After some manipulations and integration by parts, we obtain the following string frame action:

$$\begin{aligned}\mathcal{S}_4 = \int d^4x\sqrt{\hat{g}_S}\Big[&e^{-2\varphi}\Big(\hat{R}_S + 4\hat{g}_S^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi\Big) - \frac{1}{2}\hat{g}_S^{\mu\rho}\hat{g}_S^{\nu\lambda}\hat{F}_{\mu\nu}\hat{F}_{\rho\lambda}\\ &- 2e^{2\varphi}\,[\,\partial^\mu\tilde{\varphi} + \frac{1}{2}(\zeta\partial^\mu\tilde{\zeta} - \tilde{\zeta}\partial^\mu\zeta)]\,[\,\partial_\mu\tilde{\varphi} + \frac{1}{2}(\zeta\partial_\mu\tilde{\zeta} - \tilde{\zeta}\partial_\mu\zeta)]\\ &- \Big(\partial_\mu\zeta\partial^\mu\zeta + \partial^\rho\tilde{\zeta}\partial_\rho\tilde{\zeta}\Big)\Big]\ .\end{aligned} \tag{9}$$

Next, we restore the Kalb-Ramond B-field by Hodge-dualizing the universal axion $\tilde{\varphi}$. This procedure is standard, but we give some details in the appendix, in particular to be explicit about our conventions.

$$\begin{aligned}\mathcal{S}_4 = \int d^4x\sqrt{\hat{g}_S}\Big[&e^{-2\varphi}\Big(\hat{R}_S + 4\hat{g}_S^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi\Big) - \frac{1}{2}\hat{g}_S^{\mu\rho}\hat{g}_S^{\nu\lambda}\hat{F}_{\mu\nu}\hat{F}_{\rho\lambda}\\ &- \frac{1}{12}e^{-2\varphi}\hat{H}_{\mu\nu\rho}\hat{H}^{\mu\nu\rho} - \frac{1}{6}(\zeta\partial^\lambda\tilde{\zeta} - \tilde{\zeta}\partial^\lambda\zeta)\epsilon^{\mu\nu\rho\lambda}\hat{H}_{\mu\nu\rho}\\ &- \Big(\partial_\mu\zeta\partial^\mu\zeta + \partial^\rho\tilde{\zeta}\partial_\rho\tilde{\zeta}\Big)\Big]\ .\end{aligned} \tag{10}$$

While the Einstein frame action (6) organises the eight dynamical on-shell degrees of freedom into the bosonic components $\hat{g}_{\mu\nu}, \hat{A}_\mu$ of the $\mathcal{N} = 2$ supergravity multiplet and the bosonic components $\varphi, \tilde{\varphi}, \zeta, \tilde{\zeta}$ of the universal hypermultiplet, in the string frame action (10) these fields have been re-packaged into the NS-NS degrees of freedom $\hat{g}^S_{\mu\nu}$, $\hat{B}_{\mu\nu}$, φ and the R-R degrees of freedom $\hat{A}_\mu, \zeta, \tilde{\zeta}$.

We are now ready to perform the dimensional reduction. The coordinates are split into the coordinate y along the direction we reduce over, and the remaining coordinates x^μ, where μ from now on only takes three values. The four-dimensional line element is decomposed according to

$$d\hat{s}_4^2 = e^{2\alpha\sigma} ds_3^2 + e^{2\beta\sigma}(V+dy)^2 \ ,$$

where ds_3^2 is the three-dimensional line element, $V = V_\mu dx^\mu$ the Kaluza-Klein vector, σ the Kaluza-Klein scalar, and where α, β are numerical parameters. For a string frame to string frame reduction these parameter take the values $\alpha = 0$, $\beta = 1$ (in any dimension), so that

$$d\hat{s}^2_{(4),(S)} = ds^2_{(3),(S)} + e^{2\sigma}(V+dy)^2 \ .$$

Under dimensional reduction, part of the four-dimensional diffeomorphisms become three-dimensional $U(1)$ gauge transformations. The associated gauge field is the Kaluza-Klein vector V_μ. It is well known that in order to obtain a dimensionally reduced Lagrangian which is manifestly invariant under three-dimensional gauge transformations, one needs to include certain 'shifts' proportional to the KK-vector $V = V_\mu dx^\mu$ when decomposing four-dimensional into three-dimensional gauge fields. The four-dimensional gauge field is decomposed as

$$\hat{A} = \xi dy + (A_\mu + \xi V_\mu) dx^\mu$$

into a three-dimensional gauge field A_μ and a scalar ξ. The four-dimensional Kalb-Ramond field strength is decomposed as $\hat{H}_3 = H_3 + F' \wedge dy$ into a three-dimensional three-form $H_3 = dB_2$ and two-form $F' = dV'$. The corresponding two-form potential decomposes as $\hat{B}_2 = B + V' \wedge dy$. These expressions are re-arranged as follows:

$$\begin{aligned} \hat{H}_3 &= H_3 - F' \wedge V + F' \wedge (dy + V) =: \tilde{H}_3 + F' \wedge (dy+V) \\ \hat{B}_2 &= B_2 - V \wedge V' + V' \wedge (dy + V) \ . \end{aligned}$$

The modified three-dimensional Kalb-Ramond field strength

$$\tilde{H}_3 := H_3 - F' \wedge V$$

is invariant under $U(1)$ transformations associated with V', since the transformation of $H_3 = dB_2$ (induced by diffeomorphisms around the Kaluza-Klein circle) is compensated by the transformation of the 'transgression term' $-F' \wedge V$. While the scalars reduce trivially, we need to introduce the three-dimensional dilaton

$$\bar{\varphi} := \varphi - \frac{1}{2}\sigma$$

in order to obtain the standard form of three-dimensional string frame action

$$\begin{aligned}
\mathcal{S}_3 = \int d^3x \sqrt{g_S} \Bigg(& e^{-2\bar{\varphi}} R_S + 4 e^{-2\bar{\varphi}} g_S^{\mu\nu} \partial_\mu \bar{\varphi} \partial_\nu \bar{\varphi} - e^{-2\bar{\varphi}} g_S^{\mu\nu} \partial_\mu \sigma \partial_\nu \sigma - \frac{1}{4} e^{-2(\bar{\varphi}-\sigma)} V_{\mu\nu} V^{\mu\nu} \\
& - \frac{1}{2} e^{\sigma} (F^*_{\mu\nu} + \xi V_{\mu\nu})(F^{*\mu\nu} + \xi V^{\mu\nu}) - e^{-\sigma} \partial_\mu \xi \partial^\mu \xi \\
& - \frac{1}{12} e^{-2\bar{\varphi}} \tilde{H}_{\mu\nu\rho} \tilde{H}^{\mu\nu\rho} - \frac{1}{4} e^{-2(\bar{\varphi}+\sigma)} F'_{\mu\nu} F'^{\mu\nu} \\
& - \frac{1}{2} \epsilon^{\mu\nu y\lambda} F'_{\mu\nu} (\zeta \partial_\lambda \tilde{\zeta} - \tilde{\zeta} \partial_\lambda \zeta) \\
& - e^{\sigma} \left[\partial_\mu \zeta \partial^\mu \zeta + \partial^\rho \tilde{\zeta} \partial_\rho \tilde{\zeta} \right] \Bigg) \, . \qquad (11)
\end{aligned}$$

The independent dynamical degrees of freedom are the three-dimensional dilaton $\bar{\varphi}$, the Kaluza-Klein scalar σ, the two R-R scalars $\zeta, \tilde{\zeta}$, the Kaluza-Klein gauge field V_μ, the vector field A_μ and scalar field ξ descending from the Maxwell field, and the vector V'_μ descending from the Kalb-Ramond field. The three-dimensional metric $g^S_{\mu\nu}$ and Kalb-Ramond field $B_{\mu\nu}$ do not carry local on-shell degrees of freedom.

We expect T-duality to manifest itself as a symmetry of the three-dimensional action, where two groups of fields are exchanged for each other. The present form of the action is not quite suitable for exhibiting this symmmetry, since the dynamical degrees of freedom are 5 scalars and 3 vector fields. To obtain a symmetric expression, we dualize the reduced Maxwell field $F_{\mu\nu}$ into a scalar $\tilde{\xi}$. We add the following Lagrange multiplier:

$$\mathcal{L}_m = -\epsilon^{\mu\nu\rho} F^{\mu\nu} \partial_\rho \tilde{\xi} \, . \qquad (12)$$

Then we eliminate $F_{\mu\nu}$ using its algebraic equation of motion

$$F^{\mu\nu} = -e^{-\sigma} \epsilon^{\mu\nu\rho} \partial_\rho \tilde{\xi} - \xi V^{\mu\nu} \, . \qquad (13)$$

After integration by parts we obtain the final form of the three-dimensional action,

$$\begin{aligned}\mathcal{S}_3 = \int d^3x\sqrt{g_S}\Bigg(& e^{-2\bar{\varphi}}R_S + 4e^{-2\bar{\varphi}}g_S^{\mu\nu}\partial_\mu\bar{\varphi}\partial_\nu\bar{\varphi} - e^{-2\bar{\varphi}}g_S^{\mu\nu}\partial_\mu\sigma\partial_\nu\sigma \\ & -\frac{1}{4}e^{-2(\bar{\varphi}-\sigma)}V_{\mu\nu}V^{\mu\nu} - \frac{1}{4}e^{-2(\bar{\varphi}+\sigma)}F'_{\mu\nu}F'^{\mu\nu} \\ & - e^{-\sigma}\left[\partial_\mu\xi\partial^\mu\xi + \partial^\rho\tilde{\xi}\partial_\rho\tilde{\xi}\right] - e^{\sigma}\left[\partial_\mu\zeta\partial^\mu\zeta + \partial^\rho\tilde{\zeta}\partial_\rho\tilde{\zeta}\right] \\ & + \epsilon^{\mu\nu\rho}\xi V_{\mu\nu}\partial_\rho\tilde{\xi} - \epsilon^{\mu\nu y\lambda}F'_{\mu\nu}\zeta\partial_\lambda\tilde{\zeta} \\ & - \frac{1}{12}e^{-2\bar{\varphi}}\tilde{H}_{\mu\nu\rho}\tilde{H}^{\mu\nu\rho}\Bigg) , \end{aligned} \tag{14}$$

whose dynamical fields are the six scalars $\bar{\varphi}, \sigma, \xi, \tilde{\xi}, \zeta, \tilde{\zeta}$ and two gauge fields $V_{\mu\nu}, F'_{\mu\nu}$. This action is invariant under the following non-trivial involutive field transformation:

$$\bar{\varphi} \to \bar{\varphi}, \sigma \to -\sigma, \tilde{\xi} \leftrightarrow \tilde{\zeta}, \xi \leftrightarrow \zeta, V'_\mu \leftrightarrow -V_\mu, B_2 \to B_2 - V' \wedge V \ . \tag{15}$$

Since $\sigma \to -\sigma$ amounts to inverting the radius of the internal circle in string units, this $\mathbb{Z}_2$-symmetry is indeed T-duality. Using the relation between four- and three-dimensional fields, we can lift the three-dimensional transformation to four dimensions and obtain a four-dimensional version of the Buscher rules. We find that for the four-dimensional NS-NS fields, that is, for the four-dimensional dilaton, Kalb-Ramond form and string frame metric, the Buscher rules take exactly the same form as in ten dimensions. For example, the standard transformation rule for the four-dimensional dilaton follows from the invariance of the three-dimensional dilaton and the transformation of the KK-scalar:

$$\varphi' = \bar{\varphi}' + \frac{1}{2}\sigma' \to \bar{\varphi} - \frac{1}{2}\sigma = \varphi - \frac{1}{2}\sigma - \frac{1}{2}\sigma = \varphi - \sigma = \varphi - \frac{1}{2}\ln|\hat{g}_{yy}| \ . \tag{16}$$

We also obtain Buscher rules for the four-dimensional R-R fields $\hat{A}_\mu$, ζ and $\tilde{\zeta}$. We see easily that

$$\zeta \to \xi = \hat{A}_y \ . \tag{17}$$

For the derivative of the scalar field $\tilde{\zeta}$ we obtain

$$\partial_\mu\zeta \to \partial_\mu\tilde{\zeta} = \frac{1}{2}\hat{\epsilon}_{y\mu\nu\lambda}\sqrt{\hat{g}_{yy}}\left[\hat{F}^{\mu\nu} + \hat{A}_y\left(\partial^\mu\left(\frac{\hat{g}^{\nu y}}{\hat{g}_{yy}}\right) - \partial^\nu\left(\frac{\hat{g}^{\mu y}}{\hat{g}_{yy}}\right)\right)\right] , \tag{18}$$

see the appendix for details.

Note that there is no need to obtain a transformation law for $\tilde{\zeta}$ itself. As is well known [32, 27], the hypermultiplet manifold obtained by dimensional reduction of a theory of n vector multiplets coupled to $\mathcal{N} = 2$ supergravity has an isometry group which contains the Iwasawa subgroup of $\mathrm{SU}(1, n+2)$, acting by affine transformations on the scalar fields. While we work in the string frame, and have not dualized all vector fields into scalars, the effect of dualization is to convert axionic shift transformations into gauge transformations, so that symmetries are preserved.

Similarly, we obtain the transformation for the field strength of the Maxwell field:

$$\hat{F}_{\mu\nu} \to \hat{F}'_{\mu\mu} = -\sqrt{\hat{g}_{yy}}\hat{\epsilon}^{y\mu\nu\rho}\partial_\rho\tilde{\zeta} + \zeta\left(\partial^\mu(\hat{B}_{\nu y}) - \partial^\nu(\hat{B}_{\mu y})\right) , \tag{19}$$

see again the appendix for details.

To conclude this section, let us summarize the four-dimensional Buscher rules that we have derived.

NS-NS sector:

$$g'_{yy} = \frac{1}{g_{yy}} , \tag{20}$$

$$g'_{y\mu} = \frac{B_{y\mu}}{g_{yy}} , \tag{21}$$

$$g'_{\mu\nu} = g_{\mu\nu} + \frac{B_{y\mu}B_{y\nu} - g_{y\mu}g_{y\nu}}{g_{yy}} \tag{22}$$

$$B_{y\mu} = \frac{g_{y\mu}}{g_{yy}} , \tag{23}$$

$$B'_{\mu\nu} = B_{\mu\nu} + \frac{g_{y\mu}B_{y\nu} - B_{y\mu}g_{y\nu}}{g_{yy}} , \tag{24}$$

$$\varphi' = \varphi - \frac{1}{2}\ln|g_{yy}| . \tag{25}$$

R-R sector:

$$\zeta' = \hat{A}_y \,, \tag{26}$$

$$\partial_\lambda \tilde{\zeta}' = \frac{1}{2}\hat{\epsilon}_{y\mu\nu\lambda}\sqrt{|\hat{g}_{yy}|}\left[\hat{F}^{\mu\nu} + \hat{A}_y\left(\partial^\mu\left(\frac{\hat{g}^{\nu y}}{\hat{g}_{yy}}\right) - \partial^\nu\left(\frac{\hat{g}^{\mu y}}{\hat{g}_{yy}}\right)\right)\right] \,, \tag{27}$$

$$\hat{A}'_y = \zeta \,, \tag{28}$$

$$\hat{F'}^{\mu\nu} = -\sqrt{|\hat{g}_{yy}|}\hat{\epsilon}^{y\mu\nu\rho}\partial_\rho\tilde{\zeta} + \zeta\left(\partial^\mu(\hat{B}_{\nu y}) - \partial^\nu(\hat{B}_{\mu y})\right) \,. \tag{29}$$

3.2 Generalization to the twisted model and to timelike reduction

So far we have formulated the standard space-like T-duality for the EM-UHM model in the string frame and obtained the explicit relation between the fields. There are two modifications of this procedure which lead to relative sign flips. Firstly, we can change the starting point and apply a space-like T-duality to the twisted version of the theory, where the kinetic terms of the R-R fields have been flipped. Starting from the Einstein frame action obtained in [26],

$$\begin{aligned}\mathcal{S}_4 = \int d^4x\sqrt{\hat{g}_E}\Big(&\hat{R}_E + \frac{1}{2}\hat{g}_E^{\mu\rho}\hat{g}_E^{\nu\lambda}\hat{F}_{\mu\nu}\hat{F}_{\rho\lambda} - 2\hat{g}_E^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi \\ &- 2e^{4\varphi}\left[\,\partial^\mu\tilde{\varphi} + \frac{1}{2}(\zeta\partial^\mu\tilde{\zeta} - \tilde{\zeta}\partial^\mu\zeta)\right]\left[\,\partial_\mu\tilde{\varphi} + \frac{1}{2}(\zeta\partial_\mu\tilde{\zeta} - \tilde{\zeta}\partial_\mu\zeta)\right] \\ &+ e^{2\varphi}\left[\partial_\mu\zeta\partial^\mu\zeta + \partial^\rho\tilde{\zeta}\partial_\rho\tilde{\zeta}\right]\Big) \,,\end{aligned} \tag{30}$$

conversion to the string frame

$$\begin{aligned}\mathcal{S}_4 = \int d^4x\sqrt{\hat{g}_S}\Big[&e^{-2\varphi}\left(\hat{R}_S + 4\hat{g}_S^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi\right) + \frac{1}{2}\hat{g}_S^{\mu\rho}\hat{g}_S^{\nu\lambda}\hat{F}_{\mu\nu}\hat{F}_{\rho\lambda} \\ &- \frac{1}{12}e^{-2\varphi}\hat{H}_{\mu\nu\rho}\hat{H}^{\mu\nu\rho} - \frac{1}{6}(\zeta\partial^\lambda\tilde{\zeta} - \tilde{\zeta}\partial^\lambda\zeta)\epsilon^{\mu\nu\rho\lambda}\hat{H}_{\mu\nu\rho} \\ &+ \left(\partial_\mu\zeta\partial^\mu\zeta + \partial^\rho\tilde{\zeta}\partial_\rho\tilde{\zeta}\right)\Big] \,,\end{aligned} \tag{31}$$

followed by reduction over space gives

$$\mathcal{S}_3 = \int d^3x\sqrt{g_S}\Bigg(e^{-2\bar\varphi}R_S + 4e^{-2\bar\varphi}g_S^{\mu\nu}\partial_\mu\bar\varphi\partial_\nu\bar\varphi - e^{-2\bar\varphi}g_S^{\mu\nu}\partial_\mu\sigma\partial_\nu\sigma - \frac{1}{4}e^{-2(\bar\varphi-\sigma)}V_{\mu\nu}V^{\mu\nu}$$
$$+ e^{-\sigma}\left[\partial_\mu\xi\partial^\mu\xi + \partial^\rho\tilde\xi\partial_\rho\tilde\xi\right] + \epsilon^{\mu\nu\rho}\xi V_{\mu\nu}\partial_\rho\tilde\xi - \frac{1}{12}e^{-2\varphi}\tilde H_{\mu\nu\rho}\tilde H^{\mu\nu\rho}$$
$$- \frac{1}{4}e^{-2(\bar\varphi+\sigma)}F'_{\mu\nu}F'^{\mu\nu} - \epsilon^{\mu\nu y\lambda}F'_{\mu\nu}\zeta\partial_\lambda\tilde\zeta + e^{\sigma}\left[\partial_\mu\zeta\partial^\mu\zeta + \partial^\rho\tilde\zeta\partial_\rho\tilde\zeta\right]\Bigg) \tag{32}$$

This is again invariant under (15), so that lifting back over space after the transformation gives back the original Langrangian. This is as expected from the Einstein frame point of view, since we know from [26] that the three-dimensional Einstein frame theory has the scalar manifold

$$\mathcal{N}'' = \frac{U(1,2)}{U(1,1)\times U(1)} \times \frac{U(1,2)}{U(1,1)\times U(1)} \ .$$

This has two isomorphic factors which get exchanged by T-duality. Thus the two versions of the EM-UHM theory get mapped to themselves under spacelike T-duality.

The second modification is to perform a timelike T-duality, that is to reduce over time, perform an involutive field transformation, and lift back over time. Starting from the standard EM-UHM theory, reduction over time gives

$$\mathcal{S}_3 = \int d^3x\sqrt{g_S}\Bigg(e^{-2\bar\varphi}R_S + 4e^{-2\bar\varphi}g_S^{\mu\nu}\partial_\mu\bar\varphi\partial_\nu\bar\varphi - e^{-2\bar\varphi}g_S^{\mu\nu}\partial_\mu\sigma\partial_\nu\sigma$$
$$+ \frac{1}{4}e^{-2(\bar\varphi-\sigma)}V_{\mu\nu}V^{\mu\nu} + \frac{1}{4}e^{-2(\bar\varphi+\sigma)}F'_{\mu\nu}F'^{\mu\nu}$$
$$+ e^{-\sigma}\left[\partial_\mu\xi\partial^\mu\xi + \partial^\rho\tilde\xi\partial_\rho\tilde\xi\right] - e^{\sigma}\left[\partial_\mu\zeta\partial^\mu\zeta + \partial^\rho\tilde\zeta\partial_\rho\tilde\zeta\right]$$
$$+ \epsilon^{\mu\nu\rho}\xi V_{\mu\nu}\partial_\rho\tilde\xi - \epsilon^{\mu\nu y\lambda}F'_{\mu\nu}\zeta\partial_\lambda\tilde\zeta$$
$$- \frac{1}{12}e^{-2\varphi}\tilde H_{\mu\nu\rho}\tilde H^{\mu\nu\rho}\Bigg) \ . \tag{33}$$

The reduction of the twisted EM-UHM theory over time gives

$$\mathcal{S}_3 = \int d^3x\sqrt{g_S}\Bigg(e^{-2\bar{\varphi}}R_S + 4e^{-2\bar{\varphi}}g_S^{\mu\nu}\partial_\mu\bar{\varphi}\partial_\nu\bar{\varphi} - e^{-2\bar{\varphi}}g_S^{\mu\nu}\partial_\mu\sigma\partial_\nu\sigma$$
$$+ \frac{1}{4}e^{-2(\bar{\varphi}-\sigma)}V_{\mu\nu}V^{\mu\nu} + \frac{1}{4}e^{-2(\bar{\varphi}+\sigma)}F'_{\mu\nu}F'^{\mu\nu}$$
$$- e^{-\sigma}\left[\partial_\mu\xi\partial^\mu\xi + \partial^\rho\tilde{\xi}\partial_\rho\tilde{\xi}\right] + e^{\sigma}\left[\partial_\mu\zeta\partial^\mu\zeta + \partial^\rho\tilde{\zeta}\partial_\rho\tilde{\zeta}\right]$$
$$+ \epsilon^{\mu\nu\rho}\xi V_{\mu\nu}\partial_\rho\tilde{\xi} - \epsilon^{\mu\nu y\lambda}F'_{\mu\nu}\zeta\partial_\lambda\tilde{\zeta}$$
$$- \frac{1}{12}e^{-2\varphi}\tilde{H}_{\mu\nu\rho}\tilde{H}^{\mu\nu\rho}\Bigg)\,. \tag{34}$$

Since these two actions get mapped to each other under (15), timelike T-duality maps the standard EM-UHM theory to the twisted one, and vice versa. This is again as expected, since the time-like reduction of both theories gives, in the Einstein frame, a three-dimensional Euclidean theory with a scalar manifold (5) which has two non-isometric factors.

By tracing the effects of sign flips compared to the previous section, we obtain Buscher rules for all cases. It turns out that the rules for the NS-NS fields are universal. The rules for the R-R fields take the form

$$\zeta' = \hat{A}_y \tag{35}$$
$$\hat{A}'_y = \zeta \tag{36}$$
$$\partial_\lambda\tilde{\zeta}' = \alpha_1\frac{1}{2}\hat{\epsilon}_{y\mu\nu\lambda}\sqrt{\hat{g}_{yy}}\left[\hat{F}^{\mu\nu} + \hat{A}_y\left(\partial^\mu\left(\frac{\hat{g}^{\nu y}}{\hat{g}_{yy}}\right) - \partial^\nu\left(\frac{\hat{g}^{\mu y}}{\hat{g}_{yy}}\right)\right)\right] \tag{37}$$
$$\hat{F'}^{\mu\nu} = -\alpha_2\sqrt{\hat{g}_{yy}}\hat{\epsilon}^{y\mu\nu\rho}\partial_\rho\tilde{\zeta} + \zeta\left(\partial^\mu(\hat{B}_{\nu y}) - \partial^\nu(\hat{B}_{\mu y})\right) \tag{38}$$
$$\alpha_1 = \begin{cases} +1 & \text{if original theory untwisted,} \\ -1 & \text{if original theory twisted,} \end{cases} \qquad \alpha_2 = \begin{cases} +1 & \text{if dual theory untwisted,} \\ -1 & \text{if dual theory twisted.} \end{cases}$$

4 Dualizing a cosmological solution (from type-IIA to type-IIB*/IIB)

4.1 The dual solution

With the Buscher rules in place, we can now T-dualize the cosmological solution (2) of the EM-UHM model (1), (10), which we recall for reference. Since all quantities are four-dimensional, we omit 'hats' in this section. The indices μ, ν take values $0, 1, 2, 3$, equivalently t, r, x, y (where the direction indexed by 0 or t is timelike for $0 < r < r_*$). The metric is

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)} + r^2(dx^2 + dy^2) \ , \quad f(r) = -\frac{2M}{r} + \frac{e^2}{r^2} \ , \tag{39}$$

while B-field and dilaton are trivial and have been set to zero. The Einstein frame coincides with the string frame. Since the only non-trivial fields are the metric and the Maxwell field, this is a solution to Einstein-Maxwell theory. The mass-like parameter M and electric charge e have to satisfy $M > 0$ and $M^2 > e^2 > 0$ to avoid naked singularities. For these values there is a Killing horizon at $r = r_* = \frac{e^2}{2M}$, which separates a static region $0 < r < r_*$ from a non-static region $r > r_*$. The solution has a curvature singularity for $r \to 0+$ and is asymptotic to a Kasner cosmological solution for $r \to \infty$, see [24, 25] for details.

We T-dualize with respect to the Killing vector field $\xi = \partial_t$ which is timelike for $0 < r < r_* = \frac{e^2}{2M}$ and spacelike for $r_* < r < \infty$. Applying the NS-NS Buscher rules, the T-dual string frame metric is

$$ds'^2_{\rm S} = \frac{-dt^2 + dr^2}{f(r)} + r^2(dx^2 + dy^2) \ , \tag{40}$$

for all positive $r \neq r_*$. Since the initial metric is diagonal, the B-field remains trivial. However, in the dual solution we have a non-constant dilaton

$$\varphi' = -\frac{1}{2} \ln |f(r)| = -\frac{1}{2} \ln \left| -\frac{2M}{r} + \frac{e^2}{r^2} \right| , \tag{41}$$

and therefore the dual string frame and Einstein frame metric differ by a conformal factor.

Besides the metric, the only non-trivial field in the original solution is the Maxwell field

$$F = -\frac{e}{r^2} dt \wedge dr \Leftarrow A = -\frac{e}{r} dt \;, \tag{42}$$

where we have made a choice for the gauge potential, which is only determined up to an additive constant.

For the Maxwell field we have to apply the R-R Buscher rules, which distinguish between the standard and the twisted EM-UHM model. The initial model is untwisted, $\alpha_1 = 1$, while for the dual models this depends on the value of r. For $r < r_*$, T-duality is timelike, $\alpha_2 = -1$, while for $r > r_*$ it is spacelike, $\alpha_2 = 1$. From (35),(36),(38) we immediately find

$$\zeta' = -\frac{e}{r} \;, \quad A'_y = 0 \quad \text{and} \quad F'_{\mu\nu} = 0 \;. \tag{43}$$

Moreover, since we dualize over the direction $y = t$, and since the metric is diagonal, (37) implies

$$\partial_\lambda \tilde{\zeta}' = 0 \;. \tag{44}$$

The freedom of choosing a constant value for $\tilde{\zeta}'$ reflects the freedom to perform gauge transformations on the Maxwell potential A_μ. Since the parameter α_2 does not enter into the solution, the solution takes the same form for $r < r_*$ and $r > r_*$. While it is not required (since T-duality is a solution generating transformation), it is straightforward to check that the dual solution satisfies the equations of motion of the twisted and of the standard EM-UHM model for $r < r_*$ and $r > r_*$, respectively.

The non-trivial fields in the T-dual solution are the string frame metric $g'^{(S)}_{\mu\nu}$, the dilaton φ' and the axion ζ'. Truncating the actions (10), (31) to these fields we obtain a dilaton-axion system coupled to gravity, with string frame action

$$\mathcal{S}_4 = \int d^4x \sqrt{g_S} \Big[e^{-2\varphi'} \Big(R_S + 4 g_S^{\mu\nu} \partial_\mu \varphi' \partial_\nu \varphi' \Big) \mp g_S^{\mu\nu} \partial_\mu \zeta' \partial_\nu \zeta' \Big] \;. \tag{45}$$

The Einstein frame version of this action

$$\mathcal{S}_4 = \int d^4x \sqrt{g_E} \left(R_E - 2 g^E_{\mu\nu} \partial_\mu \varphi' \partial_\nu \varphi' \mp e^{2\varphi'} g^E_{\mu\nu} \partial_\mu \zeta' \partial_\nu \zeta' \right) \;,$$

is a nonlinear sigma model with target space $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$ and $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(1,1)$, respectively, coupled to gravity. The lower (upper) sign applies for timelike (spacelike) T-duality, thus establishing the relations we have provided in Figures 1 and 2.

The sign flip in the dilaton-axion action implies that the fields (φ', ζ') have to 'move' from one target space to the other. We will see below that the scalars run off to a boundary of their respective scalar manifolds when approaching $r = r_*$. With our embedding of the cosmological solution into type-IIA string theory, $r = r_*$ is an interface between its timelike and spacelike T-duals, type-IIB* for $r < r_*$ and type-IIB for $r > r_*$.

4.2 Behaviour of the T-dual solution at special points

We will now take a closer look at the behaviour of the dual solution at the image of the singularity $r = 0$ and at the image of the horizon $r = r_*$. In Figure 3 we display the graph of the function $h(r) := \frac{1}{f(r)}$, which is extremely useful for understanding the behaviour of the solution. While specific values for e, M have been used in this graph, it is straightforward to show analytically that the qualitative behaviour shown there is universal for $M > |e| > 0$. The function $h = 1/f$ has a pole at $r = r_*$ and is analytic otherwise. It is negative for $r > r_*$ with a local maximum at $r = 2r_*$ and non-negative for $r < r_*$ with a unique zero (which therefore is the unique local minimum) at $r = 0$. It approaches $\pm\infty$ for $r \to \mp\infty$ and $r \to r_* \mp 0$, as indicated in the graph.

At $r = 0$ all string frame metric coefficients go to zero, so that the metric completely degenerates (it has rank zero as a matrix):

$$g^{(S)}_{\mu\nu} \xrightarrow[r\to 0]{} 0 \ .$$

At the image $r = r_*$ of the horizon $f(r)$ has its zero, so that $h(r)$ and, hence, $g^{(S)}_{tt}$ and $g^{(S)}_{rr}$ diverge:

$$g^{(S)}_{tt} \xrightarrow[r\to r_*\mp]{} \pm\infty \ , \quad g^{(S)}_{rr} \xrightarrow[r\to r_*\mp]{} \mp\infty \ .$$

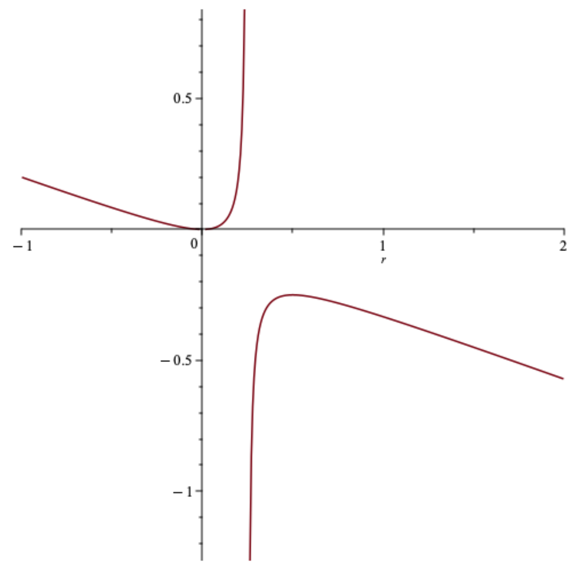

Figure 3: Graph of the function $h(r) = 1/f(r)$ with parameter values $M = 2$, $e = 1$. The shown behaviour applies to any choice of parameters where $M > 0$ and $M^2 > e^2$. The points $r = 0$ and $r = r_* = 0.25$ on the horizontal axis correspond to the T-duality images of the singularity and of the horizon of the cosmological solution. We have displayed the graph for negative values $r < 0$ to discuss a possible extension of the solution to these values.

The Ricci tensor of the string frame metric is

$$R^{(S)}_{\mu\nu} = \text{diag} \left(\frac{e^4 - 2e^2Mr + 2M^2r^2}{r^2(e^2 - 2Mr)^2}, \frac{3e^4 - 10e^2Mr + 6M^2r^2}{r^2(e^2 - 2Mr)^2}, -\frac{e^2 - 2Mr}{r^2}, -\frac{e^2 - 2Mr}{r^2} \right) , \tag{46}$$

and the corresponding Ricci scalar is:

$$R^{(S)} = \frac{4M^2}{r^2(-e^2 + 2Mr)} . \tag{47}$$

The string frame metric has singular Ricci curvature at both $r = 0$ and $r = r_*$.

Next, let us look at the two scalars. The axion $\zeta' = -e/r$ becomes singular at $r \to 0$ but is regular at $r = r_*$.The dilaton

$$\varphi' = -\frac{1}{2} \ln |f(r)| = -\frac{1}{2} \ln \left| \frac{-2M}{r} + \frac{e^2}{r^2} \right| \tag{48}$$

has a more complicated profile. It becomes infinite at the T-dual images of the singularity and of the horizon:

$$\varphi' \xrightarrow[r\to 0\pm]{} -\infty \ , \ \ \varphi' \xrightarrow[r\to r_*\pm]{} \infty \ , \tag{49}$$

The dilaton φ' has four zeros

$$\varphi'(r) = 0 \Rightarrow f(r) = -\frac{2M}{r} + \frac{e^2}{r^2} = \pm 1 \Rightarrow r = -M \pm \sqrt{M^2+e^2} \ , M \pm \sqrt{M^2-e^2} \ . \tag{50}$$

One of the zeros, $r_0 = -\sqrt{M^2+e^2} - M < 0$ is 'on the other side' of the singularity $r=0$, while the other three zeros occur for positive $r>0$. Since φ' is continuous between $r=0$ and $r=r_*$ and approaches $\pm\infty$ at these points, it must have an odd number of zeros between these values. Since $r=r_*$ is the unique zero of $f(r)$, two of the zero of φ' must be on either side of r_*. Thus φ' has one zero between 0 and r_* and two zeros between r_* and ∞. For illustration, we display the graph of φ' for the image of the interior solution, $0 < r < r_*$ in Figure 4 and as well for an extended range which displays all zeros and singularities in Figure 5.

Due to the non-constant dilaton, the Einstein frame metric behaves differently from the string frame metric. Transforming (40) to the Einstein frame, we obtain

$$\begin{aligned} ds_E'^2 &= e^{-2\varphi'} ds_S'^2 = |f(r)| ds_S'^2 \\ &= \begin{cases} -dt^2 + dr^2 + (e^2 - 2Mr)(dx^2+dy^2) \ , & 0 < r < r_* \ , \\ +dt^2 - dr^2 - (e^2 - 2Mr)(dx^2+dy^2) \ , & r_* < r < \infty \ . \end{cases} \end{aligned} \tag{51}$$

At the image of the singularity, $r=0$, all metric coefficients are finite, and can be continued analytically to $-\infty < r < r_*$. At $r=r_*$ the metric coefficients $g_{tt}^{(E)}, g_{rr}^{(E)}$ have a finite discontinuity and change sign, while $g_{xx}^{(E)}, g_{yy}^{(E)}$ have a first order zero and change sign without discontinuity. The signature of the metric is the same for $r < r_*$ and $r > r_*$.

The Einstein frame Ricci tensor and Ricci scalar are:

$$R_{\mu\nu}^{(E)} = \pm \frac{2M^2}{(e^2-2Mr)^2} \delta_{rr} \tag{52}$$

and

$$R^{(E)} = \pm \frac{2M^2}{(e^2-2Mr)^2} \ , \tag{53}$$

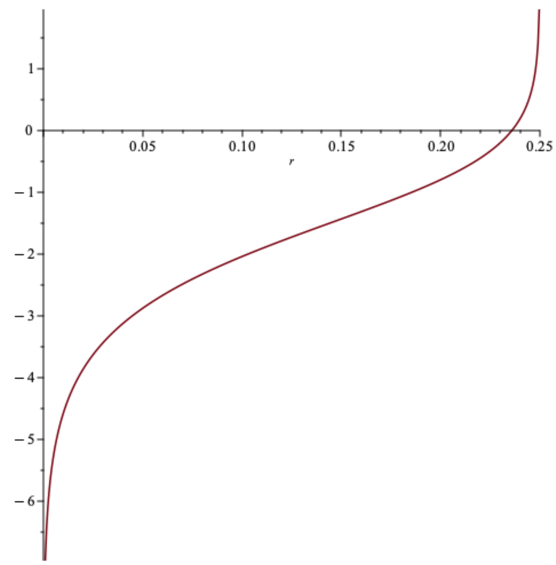

Figure 4: Graph of dilaton $\varphi'(r)$ for the range $0 < r < r_*$. The T-dual of the singularity and of the horzion are located at $r = 0$ and $r = r_* = 0.25$, respectively. The behaviour shown by this graph is universal for $M > 0$, $M^2 > e^2$.

where the upper (lower) sign refers to $r < r_*$ $(r > r_*)$. The Einstein frame Ricci curvature is finite at $r = 0$, as it of course must be, since all metric coefficients are analytical and the metric is extendible to negative r. This means that the conformal structure of the T-dual solution is extendible to $r \leq 0$, since this only requires that there is a representative of the conformal class which can be extended. However, the conformal factor relating Einstein and string frame is the dilaton, which is a dynamical field of our theory. Therefore the T-dual solution as a whole is not extendible to $r \leq 0$, since the dilaton and the axion diverge for $r \to 0+$. At the image of the horizon, the Einstein frame Ricci curvature diverges, like it does in the string frame.

4.3 Probing the singular points with geodesics

Since the T-dual solution is singular at $r = 0, r_*$ we now investigate whether these singularities are at finite or infinite 'distance.' There are different meaningful con-

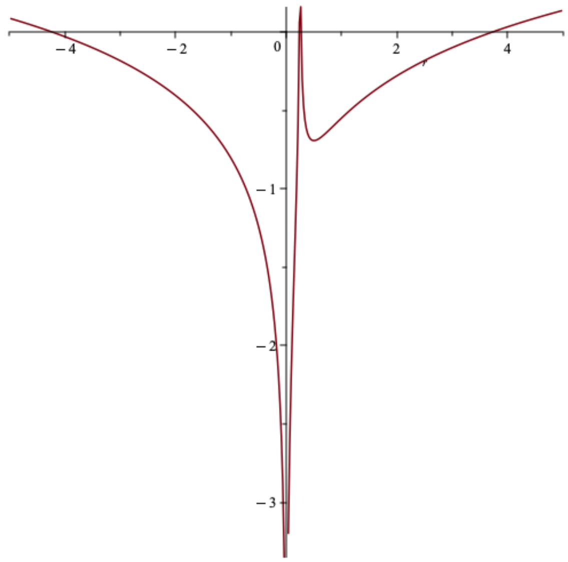

Figure 5: Graph of the dilaton $\varphi'(r)$ for an extended which displays its four zeros. The behaviour shown by this graph is universal for $M > 0$, $M^2 > e^2$.

cepts of distance that we can apply: we have two different conformal frames, and we can probe the singularity with either spacelike or with null geodesics. For the scalar fields we also have the concept of 'distance in moduli space,' to which we will come back in a later section.

We start by computing the length of a transverse spacelike geodesic connecting a point at position $r = R$ to a point at $r = 0$. Here 'transverse' means constant x, y coordinates, so that we can set $dx = dy = 0$ in the line element. In the Einstein frame, transverse curves locally see a two-dimensional Minkowski metric, with a discontinuity at $r = r_*$:

$$ds'^2_{(E)\text{transv.}} = \pm \left(-dt^2 + dr^2\right) \ .$$

Therefore any point at $0 < R < r_*$ can be connected to $r = 0$ and to $r = r_*$ by a spacelike geodesic of finite length. This becomes more interesting in the string frame, where the transverse metric is

$$ds'^2_{(S)\text{transv.}} = h(r) \left(-dt^2 + dr^2\right) \ , \ h(r) = \frac{1}{f(r)} = \frac{1}{-\frac{2M}{r} + \frac{e^2}{r^2}} \ .$$

We choose a point at transverse position R in the interior region, $0 < R < r_*$. Its spacelike distance to $r = 0$ is given by the length of the curve which connects $r = R$ to $r = 0$ at constant t, x, y. Along such transverse spacelike geodesics, the line element further reduces to

$$ds'^2_{(S)SL} = h(r)dr^2 \ . \tag{54}$$

Choosing the geodesic length as curve parameter λ we obtain

$$h(r)\left(\frac{dr}{d\lambda}\right)^2 = +1 \ . \tag{55}$$

This can be solved for the infinitesimal geodesic length:

$$d\lambda = \pm\frac{dr}{\sqrt{h}} = \pm\frac{dr}{\sqrt{-2M/r + e^2/r^2}} \ , \tag{56}$$

where the upper sign corresponds to outgoing and the lower sign to ingoing curves. We compute the length of an ingoing curve, thus obtaining the spacelike (SL) string frame (SF) distance between $r = R$ and $r = 0$:

$$\begin{aligned} D_{\mathrm{SL,SR}}(0,R) &= -\int_R^0 \frac{dr}{\sqrt{-2M/r + e^2/r^2}} = \left[\frac{\sqrt{e^2 - 2Mr}(Mr + e^2)}{3M^2}\right]_R^0 \\ &= \frac{e^2\sqrt{e^2}}{3M^2} - \left(\frac{\sqrt{e^2 - 2MR}(e^2 + MR)}{3M^2}\right) \\ &= \frac{e^2\sqrt{e^2}}{3M^2}\left(1 - (1 + \frac{RM}{e^2})\sqrt{1 - \frac{2MR}{e^2}}\right) \ . \end{aligned} \tag{57}$$

This expression is finite for $0 < R < r_*$.[5] Moreover, the distance remains finite for $R = r_*$, where it takes its maximal value (within the interval $[0, r_*]$):

$$D_{SL,SR}(0, r_*) = \frac{e^2\sqrt{e^2}}{3M^2} \ .$$

It follows without need for a separate computation that the spacelike distance from any point at $r = R$, with $0 < R < r_*$ to $r = r_*$ is finite. Both singularities are at finite spacelike distance from any interior point.

[5] By construction, it is also positive. This is straightforward to check by expanding the square root.

For null geodesics there is no concept of distance, but the distinguished class of affine curve parameters. We will use an affine curve parameter as substitute for the length, but since such parameters are only unique up to affine transformations the only relevant information is whether this 'affine length' is zero, finite, or infinite. Consider the transverse null curves

$$x^\mu(r) = (-r + R, r, 0, 0) \quad \text{and} \quad \dot{x}^\mu = (-1, 1, 0, 0) \;, \tag{58}$$

with curve parameter r, which start at time $t = 0$ at transverse position $r = R$ and propagate inwards. The non vanishing Christoffel symbols of the line element are

$$\Gamma^t_{rt} = \frac{1}{2}h^{-1}\dot{h}, \quad \Gamma^x_{rx} = r^{-1}, \quad \Gamma^r_{tt} = \frac{1}{2}h^{-1}\dot{h}, \quad \Gamma^r_{rr} = \frac{1}{2}h^{-1}\dot{h}, \quad \Gamma^r_{xx} = \frac{1}{2}h^{-1}(-2r). \tag{59}$$

We compute the absolute covariant derivatives

$$(\nabla_{\dot{x}}\dot{x})^\mu = \dot{x}^\nu \nabla_\nu \dot{x}^\mu = \ddot{x}^\mu + \Gamma^\mu_{\nu\rho}\dot{x}^\nu \dot{x}^\rho$$

of the tangent vectors $\dot{x}^\mu(r)$ along the curve:

$$\begin{aligned}
(\nabla_{\dot{x}}\dot{x})^t &= \Gamma^t_{rt}\dot{x}^r\dot{x}^t + \Gamma^t_{tr}\dot{x}^t\dot{x}^r = 2 \cdot \frac{1}{2}h^{-1}\dot{h}(-1)(1) = -h^{-1}\dot{h} = h^{-1}\dot{h}\dot{x}^t, \\
(\nabla_{\dot{x}}\dot{x})^r &= \Gamma^r_{tt}\dot{x}^t\dot{x}^t + \Gamma^r_{rr}\dot{x}^r\dot{x}^r + \Gamma^r_{xx}\dot{x}^x\dot{x}^x + \Gamma^x_{yy}\dot{x}^y\dot{x}^y = \Gamma^r_{tt} + \Gamma^r_{rr} = h^{-1}\dot{h}\dot{x}^r \;.
\end{aligned}$$

So all in all we have

$$\ddot{x}^\mu + \Gamma^\mu_{\nu\rho}\dot{x}^\nu\dot{x}^\rho = F(r)\dot{x}^\mu \;, \tag{60}$$

with $F(r) = h^{-1}\dot{h}$. This shows that the curves $x^\mu(r)$ are geodesics, but the curve parameter r is not an affine parameter. As usual, we can obtain an affine parameter λ by insisting on a parametrization where the right hand side of the geodesic equation becomes zero. This leads to the relation

$$\frac{d\lambda}{dr} = e^{\int^r F(s)ds} = h(r) \;, \tag{61}$$

which can be integrated to obtain the affine parameter. For an ingoing transverse null geodesic starting at $t = 0, r = R$, the point $r = 0$ is reached at affine distance

$$D_A(0, R) = \int_R^0 h(r)dr = \int_R^0 \frac{dr}{-2M/r + e^2/r^2} \tag{62}$$

$$= \left[-\frac{2Mr(Mr + e^2) + e^4 \ln(|e^2 - 2Mr|)}{8M^3} \right]_R^0 \;. \tag{63}$$

The point $r = 0$ has finite affine distance:

$$0 < D_A(0, R) < \infty \ .$$

Thus this point is at 'finite distance' for both spacelike and null curves.

Next we observe that if we take the limit $R \to r_*$ in (63) we obtain a divergent result due to the logarithmic term. This shows, without need for an additional computation that the affine distance from any point at $r = R$, with $0 < R < r_*$ is infinite:

$$D_A(R, r_*) = \infty \ .$$

Moreover, since the solution takes the same form for $r > r_*$ we will find the same divergence when computing the affine distance from any $r = R$ with $r > R$ to r_*. By the same logic, the spacelike distance between any such point and the image of the horizon is finite.

4.4 Interpretation of the singularities

We will now attempt to give a physical interpretation of the behaviour of the T-dual solution at $r = 0$ and $r = r_*$.

4.4.1 The T-dual of the singularity, $r = 0$

Let us first summarize our findings:

- The string frame metric degenerates (all metric coefficients go to zero), while the Einstein frame metric is analytical at $r = 0$ and can be extended analytically to any negative r.
- The string frame Ricci curvature diverges, while in the Einstein frame curvature is finite (as is implied by the analyticity of the metric). The Einstein equations imply that the energy momentum tensor is finite (as can of course be checked directly).

- The dilaton and the axion approach infinity, which is the asymptotic boundary of their field space:

$$\varphi' \xrightarrow[r\to 0+]{} -\infty\ , \quad \zeta' \xrightarrow[r\to 0+]{} \pm\infty\ .$$

  The sign of ζ' depends on the charge e, but positive and negative charge are physically equivalent. The finiteness of the energy mometum tensor, and of the Ricci curvature, results from a cancellation between the contributions of the dilaton and the axion. This is possible since due to the timelike T-duality the axion field has a kinetic energy term with a flipped sign.

- The position $r = 0$ is at finite spacelike and affine distance from points at positions $0 < r < r_*$.

Since in one conformal frame the metric has a curvature singularity while in the other it has not, one might wonder which metric is the 'right one.' We take the view that the relevant question is whether the solution taken as a whole, that is involving all fields, is regular or not. In fact, one of the implications of T-duality is that we should see the metric on equal footing with all other massless closed string modes. As a whole, the T-dual solution is singular at $r = 0$, because the dilaton and axion diverge. The Einstein frame metric happens to be non-singular, because the conformal transformation between string and Einstein frame, which is given by the dilaton, isolates the singularity. From a gravitational physicist's point of view we can still say that the conformal structure of spacetime is extendable to arbitrary negative r, though at the expense of singular behaviour of matter fields. This singular behaviour is not leading to singular stress-energy, but the run-off of scalars to infinity signals, taking the viewpoint of effective field theory, nevertheless the breakdown of our model, and the occurance of 'new physics'.

From a string theorist's point of view we can say more about this singularity, by using an embedding of the EM-UHM models into string theory. According to the swampland programme, more specifically the swampland distance conjecture [28], whenever a scalar field ('modulus') approaches the asymptotic boundary of its field space ('moduli space'), we should expect that an infinite tower of states becomes massless. We will therefore now analyze the T-dual solution from this perspective. In the region where $r < r_*$ the situation does not quite fit the standard assumptions of the swampland programme, since the scalar manifold of the dilaton-axion system obtained by timelike T-duality has indefinite signature. But we can still use its metric to compute the length of the parametrized curve $(\varphi'(r), \zeta'(r))$ which corresponds to the T-dual solution. The metric on the scalar manifold is

$$ds^2 = 2d\varphi'^2 - e^{2\varphi'}d\zeta'^2 = 2\frac{d\phi^2}{\phi^2} - \phi^2 d\zeta'^2 = 2\left(\frac{d\psi^2 - da^2}{\psi^2}\right) ,$$

where we have set $\phi = e^{\varphi'}$,$\psi = 1/\phi$ and $a = \zeta'/\sqrt{2}$ to make explicit that this is the (indefinite) Riemannian metric on the symmetric space $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(1,1) \cong \mathrm{AdS}_2$. For $r \to 0$ the pull back of this line element along the curve $(\varphi'(r), \zeta'(r))$ takes the form $ds \propto \frac{dr}{r}$, (up to terms of order r^0), which shows that the length of the curve diverges logarithmically in this limit. Thus the field values attained (asymptotically) at $r = 0$ are at infinite distance.[6] Next, we observe that with our string embedding the field φ' is the four-dimensional type-IIB* dilaton, which determines the four-dimensional string coupling:

$$g_{S,(4)} \propto e^{\varphi'} .$$

Here and in the following we ignore irrelevant constant factors. Therefore, the singular behaviour for $r \to 0$ corresponds to vanishing string coupling:

$$\varphi' \xrightarrow[r\to 0+]{} -\infty \implies g_{S,(4)} \xrightarrow[r\to 0+]{} 0 .$$

The relation between the four-dimensional and ten-dimensional string coupling is

$$e^{-2\varphi'} \propto e^{-2\varphi'_{(10)}}\mathcal{V} ,$$

where the volume $\mathcal{V}$ of the internal compact space depends on various moduli field. With our embedding of the dilaton-axion system as the universal part of a Calabi-Yau compactification (or torus), all geometric moduli have been consistently frozen

[6]We remark that is not unexpected that the curve $(\varphi'(r), \zeta'(r))$ has turned out to be non-null and thus to have a well defined length, because the solution we have T-dualized is non-extremal [42].

to constant values. Threrefore the ten-dimenional dilaton and string coupling show the same behaviour as their four-dimensional counterparts

$$\varphi_{(10)} \xrightarrow[r\to 0+]{} -\infty \quad \Longrightarrow \quad g_{S,(10)} \propto e^{\varphi_{(10)}} \xrightarrow[r\to 0+]{} 0 \; .$$

The ten-dimensional gravitational coupling κ, the Regge slope parameter α', the string tension T and the ten-dimensional dilaton $\varphi'_{(10)}$ are related by

$$\kappa \propto (\alpha')^2 e^{\varphi'_{(10)}} \; , \quad \alpha' = \frac{1}{2\pi T} \; .$$

Thus $\varphi'_{(10)} \to -\infty$ at fixed κ implies that $\alpha' \to \infty$ and $T \to 0$. In this limit the string tension vanishes, and a look at the string mass formula

$$M^2 \propto T(N_L + N_R + \cdots) \; ,$$

where $N_{L/R}$ are contributions of left- and right-moving excitations,[7] shows that in this limit an infinite tower of states becomes massless. This is consistent with the refined version of the swampland distance conjecture [43], which predicts a collapsing towers of massive states resulting from either Kaluza-Klein modes (decompatification due to run-away of geometric moduli) or tensionless strings (run-away behaviour of the dilaton).

We conclude that the singularity of the T-dual solution is related to the breakdown of the effective field theory, and in fact of perturbative string theory, due to strings becoming tensionless. This is a new phase of string theory with infinitely many massless exitations and enlarged unbroken gauge symmetries. One description which has been proposed for this phase are higher-spin field theories, [44],[45], see [46]for a more recent review. It would be interesting to investigate whether these theories can shed light on what happens at the singularity, in particular whether it can be resolved. In our context this question is even more pressing than in the standard swampland context, because the singularity, while at infinite distance in moduli space, is at finite distance in space, and is reached by light rays at finite

[7] The ellipses represent further contributions, like normal ordering constants or contributions from winding modes if we make some dimensions compact.

affine parameter. It is also tempting to speculate that understanding the singularity of the T-dual solution may tell us something about the singularity of the original solution. Having an equivalence of string theories (rather than a solution generating technique) requires one to keep the compactification circle finite, and to consider the effects of momentum and winding modes. For timelike circles this raises additional issues, see [16], [17]. It will be interesting to see whether this is a tractable problem.

4.4.2 The T-dual of the horizon, $r = r_*$

We now turn to our main point of interest, the T-dual image of the horizon at $r = r_*$. Let us summarize the previous findings

- The components g^S_{tt} and g^S_{rr} of the string frame metric diverge, while g^S_{xx} and g^S_{yy} are finite. In the Einstein frame, g^E_{tt} and g^E_{rr} have a discontinuity (sign flip at finite value), whereas g^E_{xx} and g^E_{yy} have a zero (thus preserving the signature).

- The Ricci curvature diverges in both frames. As a consequence of the Einstein equations, the stress energy tensor diverges.

- The dilaton runs off to infinity

  $$\varphi' \xrightarrow[r \to r_* \mp]{} +\infty$$

  while the axion ζ' is continuous and in fact analytic at $r = r_*$.

- In the Einstein frame, $r = r_*$ is at finite spacelike and affine distance from any finite points $r \neq 0, r_*$. In the string frame the distance is finite for spacelike geodesics but infinite for light rays.

At face value, the situation looks more singular than at $r = 0$, since the metric and curvature are singular in both conformal frames. On the other hand, we know that the original solution that we have T-dualized is completely regular at this point. This makes the behaviour at $r = r_*$ a testing ground for whether T-duality with respect to a null direction is meaningful, and whether dynamical transitions between standard and exotic type-II string theories are possible. The running off of the dilaton, which takes the same form from both sides, $\varphi' \xrightarrow[r\to r_*\pm]{} \infty$ is consistent with the idea that the scalar fields have to transfer between the associated scalar manifolds $SL(2,\mathbb{R})/SO(1,1)$ for $r < r_*$ and $SL(2,\mathbb{R})/SO(2)$ for $r > r_*$. The axion ζ' whose kinetic term gets flipped in this process is analytic at $r = r_*$, and it is manifest that the dilaton travels an infinite distance on its scalar manifold on either side of the interface.

When repeating the analysis we did for $r = 0$, we find that this time the four-dimensional string coupling becomes infinite:

$$\varphi' \xrightarrow[r\to r_*\pm]{} +\infty \quad \Longrightarrow \quad g_{S,(4)} \propto e^{\varphi'} \xrightarrow[r\to r_*\pm]{} +\infty \ .$$

At this point we may invoke S-duality to re-interpret this as a zero coupling limit of an S-dual theory, leading again to tensionless strings. For four-dimensional models with $\mathcal{N} = 2$ supersymmetry we don't have, in general a (four-dimensional) S-duality and thus no way to control strong coupling behaviour from a purely four-dimensional perspective [26]. However, since due to our embedding the volume of the internal space is constant, the ten-dimensional dilaton and string coupling grow as well. We can therefore interprete what happens at $r = r_*$ using ten-dimensional S-duality. Alternatively, we could choose a toroidal embedding, or choose one of the non-generic $\mathcal{N} = 2$ compactifications which exhibit S-duality. For $r > r_*$, the T-dual solution is embedded into type-IIB string theory which is self-dual under S-duality. S-duality acts as $g_{S,(10)} \to \frac{1}{g_{S,(10)}}$ on the string coupling and exchanges fundamental strings with D-strings (D1-branes). Thus the singular behaviour for $r \to r_*+$ is related to tensionless D-strings. For $r < r_*$, the T-dual solution is embedded into type-IIB*. This theory is not self-dual under S-duality, but gets mapped to a different exotic string theory, dubbed type-IIB'. The couplings are related by $g_S^{IIB^*} = \frac{1}{g_S^{IIB'}}$ and fundamental strings are replaced by E2-branes, which can be viewed as strings with a Euclidean worldsheet. Thus the singular behaviour for $r \to r_*-$ is related to tensionless fundamental IIB'-strings, equivalently, to tensionless type-IIB* E2 branes. From this point of view, any proposed continuation from $r < r_*$ to $r > r_*$ will need to explain how, in a tensionless limit, D1-strings can somehow morph into E2-branes. We note that according to the refined swampland distance conjecture (emergent string conjecture) [43] there is no alternative explanation: due to our choice of string embedding all geometric moduli are frozen in, preventing an explanation in terms of a collapsing Kaluza-Klein tower. Therefore tensionless strings (in a suitable duality frame) are the only remaining possible behaviour at the asymptotic boundary of moduli space.

5 Dualizing a black hole solution (from type-IIA* to type-IIB/IIB*)

In this section we consider another pair of T-dual solutions which is closely related to the previous. In [25] it was observed that the EM_- theory, that is Einstein-Maxwell theory with a flipped sign for the Maxwell term, admits a planar black hole solution which is related to the planar cosmological solution of the standard EM_+ theory by analytic continuation in the charge parameter e. According to [26], we can embedd this solution into the twisted EM-UHM model, and further into toroidal or Calabi-Yau compactifications of type-II* string theories, see Figure 2. For definiteness we assume an embedding into type-IIA*. Since scalars, including the dilaton, are trivial, string frame and Einstein frame coincide. The non-trivial fields are the metric and Maxwell field, given by

$$\begin{gathered} ds^2 = +f(r)dt^2 - \frac{dr^2}{f(r)} + r^2\left(dx^2+dy^2\right) , \quad f(r) = \left(-\frac{2M}{r} + \frac{e^2}{r^2}\right) , \\ F = -\frac{e}{r^2} dt \wedge dr \ . \end{gathered} \tag{64}$$

As discussed in detail in [25] the maximal analytic extension of this metric has the same conformal structure as the maximal analytic extension of the Schwarzschild metric. The geometry is of course different: the spatial symmetry is planar rather than spherical, and the solution is not asymptotically flat, but asymptotic to the static version of the type-AIII solution of Einstein gravity, which describes a vacuum outside an infinitely extended homogeneous planar mass distribution. The non-static version of the type-AIII solution is the Kasner cosmological solution which desribes the timelike asymptotics of the planar Einstein-Maxwell solution see [47] and also [48] for a detailed description of these solutions.

For $r > r_*$ the Killing vector field $\xi = \partial_t$ is timelike, so that the exterior region is static. For $r < r_*$ the Killing vector field becomes spacelike, and all future pointing timelike and null geodesic reach the spacelike singularity $r = 0$ at finite affine parameter. To be precise, $-\infty < t, x, y < \infty$, $0 < r < \infty$ covers half of the maximally extended solution, and we have chosen the time direction such that this part corresponds to a black hole rather than a white hole. The remaining half of the maximally extended spacetime is a time reversed version of the black hole, that is, a white hole.

We now T-dualize the black hole solution with respect to $\xi = \partial_t$. In the internal, non-static patch $0 < r < \infty$, the T-duality is spacelike. The dual theory is therefore again the twisted gravity-dilaton-axion system, which embeds into the twisted EM-UHM model, and further into type-IIB* string theory. The dual string frame metric is

$$\begin{aligned} ds'^2_{Str} &= h(r)\left(dt^2 - dr^2\right) + r^2\left(dx^2 + dy^2\right) \\ &= \frac{dt^2 - dr^2}{\left(-\frac{2M}{r} + \frac{e^2}{r^2}\right)} + r^2\left(dx^2 + dy^2\right) \ , \end{aligned} \tag{65}$$

and the R-R sector is the same as for the dualized cosmological solution discussed in the previous section. In the Einstein frame the metric takes the form:

$$ds'^2 = dt^2 - dr^2 + (e^2 - 2Mr)(dx^2 + dy^2) \ . \tag{66}$$

As previously one can check explicitly that this is a solution to the equations of motion. The dilaton has the same profile as before. In the Einstein frame we have

$$R^{(E)}_{\mu\nu} = \frac{2M^2}{(e^2 - 2Mr)^2}\delta_{rr} \ . \tag{67}$$

The Einstein frame Ricci scalar is

$$R^{(E)} = -\frac{2M^2}{(e^2 - 2Mr)^2} \ , \tag{68}$$

and the string frame Ricci tensor and scalar are

$$\begin{aligned} R^{(S)}_{\mu\nu} &= \mathrm{diag}\Big(\frac{e^4-2e^2Mr+2M^2r^2}{r^2(e^2-2Mr)^2}, \frac{3e^4-10e^2Mr+6M^2r^2}{r^2(e^2-2Mr)^2}, \\ &\quad -\frac{e^2-2Mr}{r^2}, -\frac{e^2-2Mr}{r^2}\Big) , \qquad (69) \\ R^{(S)} &= \frac{4M^2}{r^2(e^2-2Mr)} . \qquad (70) \end{aligned}$$

The equations for spacelike and affine distances remain essentially unchanged, and one finds the same behaviour as in the dualized cosmological solution. Therefore singularities at $r = 0$ and $r = r_*$ are again related to the presence of tensionless strings. We note that the locus $r = 0$ is T-dual to a spacelike rather than timelike singularity, a type of singularity that is poorly understood in string theory.

6 Relating cosmologies to black holes by T-duality (from type-IIA to type-IIA*)?

We now turn our attention to the relation between the cosmological and the black hole solution. Type-IIA and type-IIA* string theory are related by the compostion of two T-dualities, one timelike, the other spacelike. The effect on the Einstein-Maxwell subsector in the four-dimensional effective field theory is a sign flip of the Maxwell term. As discussed in [25] one can write static, planar symmetric solutions of the EM and twisted EM-theory uniformly as

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2(dx^2+dy^2) , \quad f(r) = \frac{2c}{r} + \frac{\epsilon e^2}{r^2} ,$$

with $\epsilon = -1$ for EM_+ and $\epsilon = 1$ for EM_-. The parameter c is an integration constant, which a priori can be positive or negative. For fixed c the solutions are related by analytic continuation of the charge $e \to ie$, reflecting that the sign flip of the Maxwell term is equivalent to making gauge fields imaginary.

Imposing the absence of naked singularities requires that $f(r)$ must have a zero at some point $r_* > 0$, which implies that

$$f(r) = \epsilon\left(\frac{2M}{r} - \frac{e^2}{r^2}\right) ,$$

where $M > 0$ and $M^2 > e^2$. Thus the type-IIA and type-IIA* solutions with horizons are related by $f(r) \to -f(r)$. We observe that when restricting the parameters M, e, in order to guarantee the existence of a horizon, flipping the sign ϵ is no longer equivalent to the analytic continuation $e \to ie$ of the charge, but requires in addition to flip the sign of the mass $M \to -M$. Equivalently, we could flip the sign of the transverse coordinate $r \to -r$. Viewed this way, we can formally join the cosmological and black hole solution at the singularity $r = 0$. This raises the question whether we could use a chain of dualities and analytic continuations to glue both solutions together into a single spacetime. Such a construction would be similar in spirit to the proposal to put a cosmological solution inside a black hole, using an S-brane as interface [49]. In our case the interface would be provided by a type-II* E-brane instead of an S-brane.

The obvious candidate for such a chain of dualities is combining a spacelike with a timelike T-duality, the minimum required to get from type-IIA to type-IIA*. While this relates the two actions to one another, the solutions themselves only have one (relevant) isometry, which is spacelike in some regions and timelike in others. This leaves one to apply spacelike T-duality in a static region, continue through the horizon, and then apply timelike T-duality. Equivalently, we can use that we have already applied T-duality to both solutions in both regions, and compare the resulting type-IIB/IIB* solutions with one another. The string frame metrics that we have obtained are:

$$
\begin{aligned}
ds^{2,(S)}_{IIA\to IIB^*/IIB} &= \frac{-dt^2 + dr^2}{f(r)} + r^2(dx^2 + dy^2) \;, \\
ds^{2,(S)}_{IIA^*\to IIB/IIB^*} &= \frac{dt^2 - dr^2}{f(r)} + r^2(dx^2 + dy^2) \;.
\end{aligned}
$$

where $r > 0$ and $f(r) = -\frac{2M}{r} + \frac{e^2}{r^2}$ with $M > 0$ and $M^2 > e^2$. As real solutions, these are two distinct families that cannot be combined into a single spacetime (even when allowing to patch solutions formally across singularities). Like for the corresponding IIA/IIA* solutions, gluing along $r = 0$ requires an additional analytic continuation in the charge. Therefore, the solutions are not connected by a combination of T-duality and continuation in spacetime.

There is one further option to be explored in the future, namely that both IIB/IIB* solutions arise as special members of a larger family of solutions (with generic members having less isometries). To motivate this idea, recall that when saying that Dp branes 'are' T-dual to D$(p-1)$ branes, one actually refers to a process where one first T-dualizes a Dp-brane along an isometric direction, and then allows that the dual solutions depends explicitly on the coordinate of that direction, thus obtaining a localized D$(p-1)$ brane solution with a reduced number of isometries. Conversely, going back from a localized D$(p-1)$ brane to a Dp-brane involves as a first step to create an isometry by de-localization, resulting in a smeared D$(p-1)$ which can then be T-dualized into a Dp-brane.[8] We should therefore investigate whether the type-IIB/IIB* solutions obtained by T-dualizing type-IIA/IIA* solutions admit localized versions, and whether one can move from IIA to IIA* by a chain of T-dualities combined with localizations/delocalizations and extensions in spacetime.

While it is not obvious how to obtain localized versions of the type-IIB/IIB* solutions, we can give an argument why they should exist: one of the dimensional uplifts of the IIA solution is the D4-D4-D4-D0 system. It was shown in [51] that this brane configuration can be T-dualized, after Wick rotation, into the D3-D3-D3-D(-1) brane system of type-IIB. This involves a localization of precisely the type mentioned before. Adapting the procedure of [51] requires two modifications: (i) applying timelike T-duality instead of a combination of Wick rotation and spacelike T-duality, which is straightforward, thus obtaining a IIB* E-brane system, which can be further T-dualized into a IIA* E-brane system, (ii) generalizing the localization/delocalization procedure to non-extremal solutions. As mentioned before, the IIA/IIA* solutions only have horizons as long as they are non-extremal, $M > |e| > 0$. The creation of an isometry by smearing can be viewed of the continuous limit of a periodic array of localized solutions, but this makes use of the no-force property of BPS (hence, extremal) solutions. One way to adapt the construction to non-extremal solutions is to first take the extremal limit, perform (de-)localization, and then to make the solution non-extremal again. We leave working out the details to future work.

[8] See for example [50] for a pedagogical review.

7 Conclusions and Outlook

In this paper we have derived four-dimensional Buscher rules for spacelike and timelike T-duality for Einstein-Maxwell theory coupled to the universal hypermultiplet as well as for the twisted version of this theory. We have applied these results to study the effect of T-duality on solutions which contain a non-extremal Killing horizon as well as a curvature singularity. While we have considered specific examples for concreteness, our findings result from qualitative properties of the function $f(r)$ and therefore are expected to hold for a large class of spacetimes. Using embeddings into type-II/II* string theory, we found that the singularities of the T-dual solutions occur at the boundary of the underlying moduli spaces, and are related to tensionless strings. This opens various venues for the further study of such singularities, and, possibly, also the singularities of the original solutions that we have T-dualized.

We think there are three promising directions to explore.

1. Using the embedding into type-II/II* string theory, we can study the higher-dimensional brane configurations underlying our solutions, as well as others. This is the most direct way to explore their microscopic, stringy nature. It will also allow to decide whether type-II cosmological and type-II* black holes solutions can be related by a chain of dualities. More generally, one could extend this to investigating the global geometric structures of non-extremal branes of standard and of exotic type-II theories, as well as the action of T-duality on them.

2. Since the behaviour of the T-dual solutions at their singularities involves tensionless strings, higher spin field theories may be helpful to understand the behaviour in the phase the solutions are driven into.

3. Generalized geometry, as well as double field theory are formalisms which allow for a fully T-duality covariant description, without the need to take the full string spectrum into account. The simplest option, and the one which is arguably the best understood mathematically, is generalized geometry, which is based on an extension of the tangent bundle, while leaving spacetime as is. In

this formalism one works with a finite compactification circle, which may allow one to draw conclusions on a solution using its T-dual. It will be interesting to explore whether some of the singular behaviour seen in a Riemannian geometry set-up gets mitigated when working with generalized geometry data.

Work along these lines will be able to shed light on the status of timelike T-duality and of type-II* string theories, as well as type-II string theories in non-Lorentzian signatures. While we have been able to argue that the singular behaviour of the T-dual solution at the image of the horizon is due to the presence of tensionless strings, it is not clear whether the solution can be continued through the 'tensionless string interface', which would amount to a dynamical transition from type-IIB to type-IIB* string theory. One option would be that the type-IIB moduli space ends in a tensionless string phase, without an extension into the exotic type-IIB* branch. This would support the case that type-IIB* theories, while they can be generated by formally applying T-duality, are not dynamically connected to standard string theory. Conversely, establishing a dynamical connection in a single example would force one to address the various exotic features, potential benefits, but also potential pathologies of the exotic type-II theories [16],[17].

The issue of dynamical type-II/type-II* transitions is different from, but closely related to the issue of dynamical signature change and thus transitions between type-II theories in different signatures. We noted before that, by means of dimensional reduction with respect to the Killing vector field, any non-extermal Killing horizon provides an example of 'smooth signature change': the dimensionally reduced effective field theories have Euclidean and Lorentzian signature, respectively, and while the transition between them cannot be smooth, there is a perfectly well behaved dimensional lift where the singular interface becomes a horizon. Similarly, [16] have argued that dynamical signature change between ten-dimensional type-II superstrings is possible, by relating them through eleven dimensional M-theory (which also has two exotic versions with non-Lorentzian signature [15]). Thus the same methodology can be applied for transitions between type-II/type-II* and for signature change. Establishing or refuting the viability of either type of dynamical transition is important for understanding 'how big' the configuration space and symmetry group underlying a non-perturbative formulation of string theory needs to be. Some approaches to uncovering a unifying symmetry principle, such as compactification of all direction including time [52], exceptional field theory and the E_{11}-approach [53], [54], [55], naturally include timelike reductions and exotic type-II theories.

It is also interesting to note that in the stretched horizon approach to black hole entropy, a gas of high temperature, and therefore effectively tensionless strings play a central role [56]. This raises the question whether the tensionless strings associated to T-dual horizons describe the same physics in a different duality frame.

Acknowledgements

We thank Vicente Cortés, Chris Hull and Lárus Thorlacius for illuminating discussions. T.M. would like to thank the Isaac Newton Institute for Mathematical Sciences for support and hospitality during the programme 'Black Holes: a bridge between number theory and holographic quantum information' when work for this paper was undertaken. This work was supported by EPSRC grant number EP/R014604/1. T.M. would like to thank the Department of Mathematics of the University of Hamburg for support and hospitality during several visits when work for this paper was undertaken. The work of M.M. is supported by an EPSRC DTP International Doctoral Scholarship (project number 2271092) and JSPS Postdoctoral Fellowships for Research in Japan (Standard) No. P23774. The work of T.M. was supported by the STFC Consolidated Grant ST/T000988/1.

A Details of selected computations

A.1 Details for the Hodge duality between universal axion and Kalb-Ramond field

We start from the action (9) and introduce a two-form potential

$$\hat{B} = \frac{1}{2!}\hat{B}_{\mu\nu}dx^{\mu} \wedge dx^{\nu} \ ,$$

with associated three-form field strength[9]

$$\hat{H} = \frac{1}{3!}\hat{H}_{\mu\nu\rho}dx^\mu \wedge dx^\nu \wedge dx^\rho = dB \Rightarrow H_{\mu\nu\rho} = 3\partial_{[\mu}B_{\nu\rho]} \ .$$

In order to replace the gradient $\partial_\mu \tilde{\varphi}$ by a general one-form $\hat{F} = \hat{F}_\mu dx^\mu$, we use the two-form $\hat{B}$ subject to the Bianchi identity $d\hat{F} = 0 \Leftrightarrow \partial_{[\mu}\hat{F}_{\nu]}$ which guarantees the (local) existence of a potential $\tilde{\varphi}$. This is done by adding the term

$$S_{\text{mult}} = -(3!)K \int \hat{B} \wedge d\hat{F} = (3!)K \int \hat{H} \wedge \hat{F}$$

to the action, where K is a numerical constant that we can set to a convenient value later. In terms of components:

$$S_{\text{mult}} = (3!)K \int \frac{1}{3!}\hat{H}_{\mu\nu\rho}\hat{F}_\lambda \ dx^\mu \wedge dx^\mu \wedge dx^\rho \wedge dx^\lambda \ .$$

Note that as an integral of a four-form over space-time, this is defined independently of the metric. We now re-express this as an integral of a function against the volume element assocated with the string frame metric:

$$\begin{aligned} S_{\text{mult}} &= K \int \hat{H}_{\mu\nu\rho}\hat{F}_\lambda \, \delta^{\mu\nu\rho\sigma}_{0123} \, dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \\ &= K \int d^4x \sqrt{\hat{g}_S} \, \epsilon^{\mu\nu\rho\sigma} \hat{H}_{\mu\nu\rho}\hat{F}_\lambda \ . \end{aligned}$$

Here

$$\delta^{\mu\nu\rho\sigma}_{0123} = 4!\delta_0^{[\mu}\delta_1^{\mu}\delta_2^{\rho}\delta_3^{\lambda]} = \begin{cases} 1, & \text{if } (\mu,\nu,\rho,\sigma) \text{ is an even permutation of } (0,1,2,3) \ , \\ -1, & \text{if } (\mu,\nu,\rho,\sigma) \text{ is an odd permutation of } (0,1,2,3) \ , \\ 0, & \text{else} \ , \end{cases}$$

is the numerical permutation symbol (which transforms as a tensor density), while

$$\epsilon^{\mu\nu\rho\sigma} = \frac{1}{\sqrt{\hat{g}_S}}\delta^{\mu\nu\rho\sigma}_{0123}$$

is the associated tensor.

[9] $A_{[\mu_1 \cdots \mu_n]}$ denotes antisymmetrization of indices, that is summing over all permutations, weighted by the signum of the permutation, and normalized by applying a factor $(n!)^{-1}$.

Now we replace (8) by

$$S[\hat{F},\hat{B}] = \int d^4x\sqrt{\hat{g}_S}(-2)e^{2\varphi}\,[\,\hat{F}_\mu + \frac{1}{2}(\zeta\partial^\mu\tilde{\zeta} - \tilde{\zeta}\partial^\mu\zeta)]\,[\,\hat{F}^\mu + \frac{1}{2}(\zeta\partial_\mu\tilde{\zeta} - \tilde{\zeta}\partial_\mu\zeta)] \tag{71}$$

$$+K\int d^4\sqrt{\hat{g}_S}\,\epsilon^{\mu\nu\rho\lambda}\hat{H}_{\mu\nu\rho}\hat{F}_\lambda\ . \tag{72}$$

Variation with respect to $\hat{B}_{\mu\nu}$ imposes the Bianchi identity $\partial_{[\mu}\hat{F}_{\nu]} = 0$, which is solved by $\hat{F}_\nu = \partial_\nu\tilde{\varphi}$. Upon substituting this into (71) we recover (8). To obtain the dualized action, we instead vary with respect to $\hat{F}_\mu$ and obtain

$$\hat{F}^\sigma = -\frac{1}{2}(\zeta\partial^\sigma\tilde{\zeta} - \tilde{\zeta}\partial^\sigma\zeta) + \frac{1}{4}Ke^{-2\varphi}\epsilon^{\mu\nu\rho\sigma}\hat{H}_{\mu\nu\rho}\ . \tag{73}$$

Plugging this back into the action, upon choosing $K = \frac{1}{3}$, we finally arrive at the dual action (10).

A.2 Details for the T-duality transformation of $\partial_\mu \tilde{\zeta}$

$$
\begin{aligned}
\partial_\mu \tilde{\zeta} &\to \partial_\mu \tilde{\xi} \\
&\Rightarrow \underbrace{\epsilon_{\mu\nu\lambda}\epsilon^{\mu\nu\rho}}_{=-2\delta^\rho_\lambda} \partial_\rho \tilde{\xi} = -\epsilon_{\mu\nu\rho} e^\sigma (F^{*\mu\nu} + \xi V^{\mu\nu}) \\
&\Rightarrow \partial_\lambda \tilde{\xi} = \frac{1}{2}\epsilon_{\mu\nu\lambda} e^\sigma (F^{*\mu\nu} + \xi V^{\mu\nu}) \\
&= \frac{1}{2}\epsilon_{\mu\nu\lambda} e^\sigma (F^{\mu\nu} - 2\partial_{[\mu}\xi V_{\nu]}) \\
&= \frac{1}{2}\epsilon_{\mu\nu\lambda} e^\sigma \left(\partial^\mu A^\nu - \partial^\nu A^\mu - 2\partial_{[\mu}\xi V_{\nu]}\right) \\
&= \frac{1}{2}\epsilon_{\mu\nu\lambda} e^\sigma \left(\partial^\mu (A^\nu + \xi V^\nu) - \partial^\nu (A^\mu + \xi V^\mu) - \partial_\mu \xi V_\nu + \partial_\nu \xi V_\mu\right) \\
&= \frac{1}{2}\epsilon_{\mu\nu\lambda} e^\sigma \left(\partial^\mu \hat{A}^\nu + \partial^\mu(\xi V^\nu) - \partial^\nu \hat{A}^\mu - \partial^\nu(\xi V^\mu) - \partial_\mu \xi V_\nu + \partial_\nu \xi V_\mu\right) \\
&= \frac{1}{2}\epsilon_{\mu\nu\lambda} e^\sigma \left(\hat{F}^{\mu\nu} + \xi \partial^\mu V^\nu - \xi \partial^\nu V^\mu\right) \\
&= \frac{1}{2}\epsilon_{\mu\nu\lambda} e^\sigma \left(\hat{F}^{\mu\nu} + \xi V^{\mu\nu}\right) \\
&= \frac{1}{2}\epsilon_{\mu\nu\lambda} e^\sigma \left(\hat{F}^{\mu\nu} + \hat{A}_y (\partial^\mu (e^{-2\sigma}\hat{g}^{\nu y}) - \partial^\nu (e^{-2\sigma}\hat{g}^{\mu y}))\right) \\
&= \frac{1}{2}\hat{\epsilon}_{y\mu\nu\lambda}\sqrt{\hat{g}_{yy}} \left[\hat{F}^{\mu\nu} + \hat{A}_y \left(\partial^\mu \left(\frac{\hat{g}^{\nu y}}{\hat{g}_{yy}}\right) - \partial^\nu \left(\frac{\hat{g}^{\mu y}}{\hat{g}_{yy}}\right)\right)\right] \, .
\end{aligned} \tag{74}
$$

A.3 Details for the T-duality transformation of the Maxwell field

$$
\begin{aligned}
\hat{F}_{\mu\nu} &= \partial^\mu \hat{A}^\nu - \partial^\nu \hat{A}^\mu = \partial^\mu (A^\nu - \xi V^\nu) - \partial^\nu (A^\mu - \xi V^\mu) \\
&= F^{*\mu\nu} = -e^{-\sigma}\epsilon^{\mu\nu\rho}\partial_\rho \tilde{\xi} - \xi V^{\mu\nu} \\
&\to -e^\sigma \epsilon^{\mu\nu\rho}\partial_\rho \tilde{\zeta} - \zeta(-F'^{\mu\nu}) \\
&= -e^\sigma \epsilon^{\mu\nu\rho}\partial_\rho \tilde{\zeta} + \zeta\left(\partial^\mu A'^\nu - \partial^\nu A'^\mu\right) \\
&= -e^\sigma \epsilon^{\mu\nu\rho}\partial_\rho \tilde{\zeta} + \zeta\left(\partial^\mu (\hat{B}_{\nu y}) - \partial^\nu (\hat{B}_{\mu y})\right) \\
&= -\sqrt{\hat{g}_{yy}}\hat{\epsilon}^{y\mu\nu\rho}\partial_\rho \tilde{\zeta} + \zeta\left(\partial^\mu (\hat{B}_{\nu y}) - \partial^\nu (\hat{B}_{\mu y})\right) \, .
\end{aligned} \tag{75}
$$

References

[1] J. Dai, R.G. Leigh and J. Polchinski, *New Connections Between String Theories*, *Mod. Phys. Lett. A* **4** (1989) 2073.

[2] M. Dine, P.Y. Huet and N. Seiberg, *Large and Small Radius in String Theory*, *Nucl. Phys. B* **322** (1989) 301.

[3] A. Giveon, M. Porrati and E. Rabinovici, *Target space duality in string theory*, *Phys.Rept.* **244** (1994) 77 [hep-th/9401139].

[4] M. Gualtieri, *Generalized complex geometry*, Ph.D. thesis, Oxford U., 2003. math/0401221.

[5] N. Hitchin, *Lectures on generalized geometry*, 1008.0973.

[6] C.M. Hull, *Doubled Geometry and T-Folds*, *JHEP* **07** (2007) 080 [hep-th/0605149].

[7] C. Hull and B. Zwiebach, *Double Field Theory*, *JHEP* **09** (2009) 099 [0904.4664].

[8] K. Koepsell, H. Nicolai and H. Samtleben, *An Exceptional geometry for D = 11 supergravity?*, *Class. Quant. Grav.* **17** (2000) 3689 [hep-th/0006034].

[9] C.M. Hull, *Generalised Geometry for M-Theory*, *JHEP* **07** (2007) 079 [hep-th/0701203].

[10] P. Pires Pacheco and D. Waldram, *M-theory, exceptional generalised geometry and superpotentials*, *JHEP* **09** (2008) 123 [0804.1362].

[11] A. Ashmore, C. Strickland-Constable, D. Tennyson and D. Waldram, *Heterotic backgrounds via generalised geometry: moment maps and moduli*, *JHEP* **11** (2020) 071 [1912.09981].

[12] C.M. Hull, *A Geometry for non-geometric string backgrounds*, *JHEP* **10** (2005) 065 [hep-th/0406102].

[13] M. Rocek and E.P. Verlinde, *Duality, quotients, and currents*, *Nucl. Phys. B* **373** (1992) 630 [hep-th/9110053].

[14] C.M. Hull, *Timelike T duality, de Sitter space, large N gauge theories and topological field theory*, *JHEP* **07** (1998) 021 [hep-th/9806146].

[15] C.M. Hull, *Duality and the signature of space-time*, *JHEP* **11** (1998) 017 [hep-th/9807127].

[16] R. Dijkgraaf, B. Heidenreich, P. Jefferson and C. Vafa, *Negative Branes, Supergroups and the Signature of Spacetime*, *JHEP* **02** (2018) 050 [1603.05665].

[17] R. Blumenhagen, M. Brinkmann, A. Makridou, L. Schlechter and M. Traube, *dS Spaces and Brane Worlds in Exotic String Theories*, *JHEP* **06** (2020) [2002.11746].

[18] L. Gall and T. Mohaupt, *Five-dimensional vector multiplets in arbitrary signature*, *JHEP* **09** (2018) 053 [1805.06312].

[19] V. Cortés, L. Gall and T. Mohaupt, *Four-dimensional vector multiplets in arbitrary signature*, *International Journal of Geometric Methods in Modern Physics* **Vol 17,** (2019) 2050150 and 2050151, [1907.12067].

[20] W.A. Sabra, *Flow equations in arbitrary signature*, *Phys. Lett. B* **829** (2022) 137084 [2111.05141].

[21] W.A. Sabra, *Hypersymplectic geometry and supersymmetric solutions in (t,s) 5D supergravity*, *Phys. Rev. D* **104** (2021) 046012 [2105.00806].

[22] W.A. Sabra, *Kasner metrics and very special geometry*, *Phys. Lett. B* **833** (2022) 137380 [2206.00467].

[23] D. Farotti, J.B. Gutowski and W.A. Sabra, *Real Supersymmetric Solutions of (3,2) Signature Five-Dimensional Supergravity*, 2305.03023.

[24] J. Gutowski, T. Mohaupt and G. Pope, *From static to cosmological solutions of $\mathcal{N} = 2$ supergravity*, *JHEP* **08** (2019) 172 [1905.09167].

[25] J. Gutowski, T. Mohaupt and G. Pope, *Cosmological Solutions, a New Wick-Rotation, and the First Law of Thermodynamics*, *JHEP* **03** (2021) 293 [2008.06929].

[26] M. Medevielle, T. Mohaupt and G. Pope, *Type-II Calabi-Yau compactifications, T-duality and special geometry in general spacetime signature*, *JHEP* **02** (2022) 048 [2111.09017].

[27] V. Cortés, P. Dempster, T. Mohaupt and O. Vaughan, *Special Geometry of Euclidean Supersymmetry IV: the local c-map*, *JHEP* **10** (2015) 066 [1507.04620].

[28] H. Ooguri and C. Vafa, *On the Geometry of the String Landscape and the Swampland*, *Nucl. Phys. B* **766** (2007) 21 [hep-th/0605264].

[29] M. Bodner, A.C. Cadavid and S. Ferrara, *(2,2) vacuum configurations for type IIA superstrings: N=2 supergravity Lagrangians and algebraic geometry*, *Class. Quant. Grav.* **8** (1991) 789.

[30] P.S. Aspinwall, *Compactification, geometry and duality: N=2*, in *Theoretical Advanced Study Institute in Elementary Particle Physics (TASI 99): Strings, Branes, and Gravity*, pp. 723–805, 1, 2000 [hep-th/0001001].

[31] S. Cecotti, S. Ferrara and L. Girardello, *Geometry of Type II Superstrings and the Moduli of Superconformal Field Theories*, *Int. J. Mod. Phys.* **A4** (1989) 2475.

[32] S. Ferrara and S. Sabharwal, *Quaternionic Manifolds for Type II Superstring Vacua of Calabi-Yau Spaces*, *Nucl. Phys. B* **332** (1990) 317.

[33] D. Gal'tsov and O. Rytchkov, *Generating branes via sigma-models*, *Phys. Rev. D* **58** (1998) 122001.

[34] K.S. Stelle, *BPS branes in supergravity*, in *ICTP Summer School in High-energy Physics and Cosmology*, 3, 1998 [hep-th/9803116].

[35] E. Cremmer et al., *Euclidean-signature supergravities, dualities and instantons*, *Nucl. Phys.* **B534** (1998) 40 [hep-th/9803259].

[36] V. Cortés, C. Mayer, T. Mohaupt and F. Saueressig, *Special geometry of Euclidean supersymmetry. I: Vector multiplets*, *JHEP* **03** (2004) 028 [hep-th/0312001].

[37] V. Cortés, C. Mayer, T. Mohaupt and F. Saueressig, *Special geometry of Euclidean supersymmetry. II: Hypermultiplets and the c-map*, *JHEP* **06** (2005) 025 [hep-th/0503094].

[38] V. Cortés and T. Mohaupt, *Special Geometry of Euclidean Supersymmetry III: the local r-map, instantons and black holes*, *JHEP* **07** (2009) 066 [0905.2844].

[39] G. Lopes Cardoso and T. Mohaupt, *Special Geometry, Hessian Structures and Applications*, *Phys. Rept.* **855** (2020) 1 [1909.06240].

[40] L. Gall and T. Mohaupt, *Supersymmetry algebras in arbitrary signature and their R-symmetry groups*, *JHEP* **08** (2021) 203 [2108.05109].

[41] R.M. Wald, *General Relativity*, The University of Chicago Press (1984).

[42] D. Errington, T. Mohaupt and O. Vaughan, *Non-extremal black hole solutions from the c-map*, *JHEP* **05** (2014) 052 [1408.0923].

[43] S.-J. Lee, W. Lerche and T. Weigand, *Emergent strings from infinite distance limits*, *JHEP* **02** (2022) 190 [1910.01135].

[44] A. Sagnotti and M. Tsulaia, *On higher spins and the tensionless limit of string theory*, *Nucl. Phys. B* **682** (2004) 83 [hep-th/0311257].

[45] M.A. Vasiliev, *Nonlinear equations for symmetric massless higher spin fields in (A)dS(d)*, *Phys. Lett. B* **567** (2003) 139 [hep-th/0304049].

[46] A. Sagnotti, *Notes on Strings and Higher Spins*, *J. Phys. A* **46** (2013) 214006 [1112.4285].

[47] J.B. Griffith and J. Podolsky, *Exact Space-Times in Einstein's General Relativity*, Cambridge University Press (2009).

[48] P. Aniceto, M. Câmara, G. Cardoso and M. Rosselló, *Weyl metrics and Wiener-Hopf factorization*, *JHEP* **05** (2020) 124 [1910.10632].

[49] R. Brandenberger, L. Heisenberg and J. Robnik, *Through a black hole into a new universe*, *Int. J. Mod. Phys. D* **30** (2021) 2142001 [2105.07166].

[50] T. Mohaupt, *Black holes in supergravity and string theory*, *Class. Quant. Grav.* **17** (2000) 3429 [hep-th/0004098].

[51] K. Behrndt, I. Gaida, D. Lust, S. Mahapatra and T. Mohaupt, *From type IIA black holes to T dual type IIB D instantons in N=2, D = 4 supergravity*, *Nucl. Phys. B* **508** (1997) 659 [hep-th/9706096].

[52] G.W. Moore, *Finite in all directions*, hep-th/9305139.

[53] A. Keurentjes, *E(11): Sign of the times*, *Nucl. Phys. B* **697** (2004) 302 [hep-th/0402090].

[54] O. Hohm and H. Samtleben, *The many facets of exceptional field theory*, *PoS* **CORFU2018** (2019) 098 [1905.08312].

[55] A.G. Tumanov and P. West, *E11 and exceptional field theory*, *Int. J. Mod. Phys. A* **31** (2016) 1650066 [1507.08912].

[56] L. Susskind, L. Thorlacius and J. Uglum, *The Stretched horizon and black hole complementarity*, *Phys. Rev. D* **48** (1993) 3743 [hep-th/9306069].