Prepared for submission to JHEP

The Schwarzian from gauge theories

Alejandro Cabo-Bizet [a]

[a] *Università del Salento, Dipartimento di Matematica e Fisica Ennio De Giorgi, and I.N.F.N. - sezione di Lecce, Via Arnesano, I-73100 Lecce, Italy*

E-mail: acbizet@gmail.com

Abstract: The continuum of holographic dual gravitational charges is recovered out of the discrete spectrum of $U(N)$ $\mathcal{N}=4$ SYM on $\mathbb{R}\times S^3$. In such a limit, the free energy of the free gauge theory is computed up to logarithmic contributions and exponentially suppressed contributions. Assuming the supergravity dual prediction to correctly capture strong-coupling results in field theory, the answer is bound to encode a complete low-temperature expansion of the Gibbons-Hawking gravitational on-shell action, valid well beyond the vicinity of supersymmetric black hole solutions. The formula recovers the long awaited Schwarzian contribution at low enough temperatures. The computed mass-gap matches the conjectured strong-coupling result obtained by Boruch, Heydeman, Iliesiu and Turiaci in supergravity. The emergent reparameterizations, broken by the Schwarzian, correspond to redefinitions of the relevant cutoff scale. Observations are made regarding the existence of $\frac{1}{8}$-BPS black holes and how this is in tension with BPS inequalities. The RG-flow procedure leading to these results opens the way to understanding the emergence of chaos in gauge theories and its relation to non-extremal and non-supersymmetric black hole physics.

Contents

1 Introduction

In a quantum system with a semi-positive Hamiltonian $\Delta \geq 0$ and a discrete spectrum with degeneracies $d(\Delta)$ the Taylor expansion of the partition function

$$Z[\beta] = \mathrm{Tr} e^{-\beta\Delta} = e^{-\mathcal{F}[\beta]} \tag{1.1}$$

at zero temperature $\beta = \infty$ is trivially equal to the number of ground states. [1] Equivalently, the Taylor expansion of the free energy $\mathcal{F}$ at zero temperature $\beta = \infty$ is trivially equal to minus the logarithm of the number of ground states. [2] If the discrete system has a holographic dual description [1–4][5] such that in an RG flow procedure [3] – denoted as $\Lambda \to$

[1] This is because every finite-temperature correction to the partition function beyond the vacuum degeneracy $d(0)$, $Z = d(0) + d(\Delta_{\min})e^{-\beta\Delta_{\min}} + \ldots$ is exponentially suppressed.

[2] $\mathcal{F}$ " $=$ " $-\log d(0)$.

[3] Very much in the mathematical spirit of [6, 7][8, 9].

∞ in this introduction – [4] it reduces to a semiclassical theory of gravity with black hole solutions at arbitrary Bekenstein-Hawking temperature $1/\beta$ [10–17][18, 19], then one reaches a contradiction, as the gravitational free energy $\mathcal{F}_g$

$$\mathcal{F} \underset{\Lambda\to\infty}{\to} \mathcal{F}_g \tag{1.2}$$

of any such black hole is bound to have a non-trivial perturbative expansion around $\beta = \infty$. For example, the Gibbons-Hawking gravitational on-shell action [20] of rotating and electrically charged black holes in AdS_5 [21, 22], and in particular its low temperature expansion, has been recently studied, with varied motivations, for example, in [23–29]. The first goal of this paper is to solve this apparent contradiction.

That $\mathcal{F}[\beta]$, the free energy of the fundamental theory has a trivial Taylor expansion at zero temperature while $\mathcal{F}_g[\beta]$, the free energy of the infrared effective theory has a non-trivial one, strongly suggests that the RG-flow procedure $\Lambda \to \infty$, which must be applied before expanding in low-temperatures $\beta \to \infty$, [5] should map the discrete spectrum of the fundamental theory (and its stringy dual formulation) to a continuum spectrum in the infrared.

In the grand-canonical ensemble (with fixed chemical potentials), we will find that the operation $\Lambda \to \infty$ corresponds to localizing the free energy $\mathcal{F}$ of the fundamental system to a $\frac{1}{\Lambda}$-vicinity of its singular locus. By localization, we mean discarding exponentially suppressed contributions at large values of the scaling parameter Λ, in the spirit of [30].

In a generic gauge theory there are several such singularities, [6] and disconnected families of them are expected to correspond to inequivalent $\Lambda \to \infty$ RG-flow procedures. It is also expected that they correspond to different saddle points [35–54][55, 56] which may or may not be intersected by the complex contour one wishes to integrate over in order to work at fixed and large charges. In this paper, we will focus on one such leading singularity, that is, one that corresponds to the potentially most dominant saddle point(s). [7] The test theory will be $U(N)$ $\mathcal{N}=4$ SYM on $\mathbb{R}\times S^3$ where there is convincing evidence that such a leading saddle point determines the analytic part of the asymptotic expansion of the free energy in the $\Lambda \to \infty$ RG flow procedure at zero temperature $\beta = \infty$. This is, of the free energy of the superconformal index [30]. We will further confirm that a smooth deformation of it continues to dominate the perturbative expansion around $\Lambda \to \infty$ at $\beta \neq \infty$.

This leading localization procedure may seem abstract at first. It has a clear physical meaning though. In the microcanonical ensemble it amounts to ignoring charge eigenvalues that are larger than a certain large energy scale $\Lambda^{n+1}R^n$, where R is the radius of

[4] To make contact with supergravity $N \gg 1$ is also required. The scale Λ, whose meaning we will explain below, is a conceptually different scale than N (first of all it is dimensionful). Whenever we connect with the theory obtained in the RG-expansion $\Lambda \to \infty$ to supergravity it is implicitly assumed that $N \to \infty$ in a particular way. For $\mathcal{N}=4$ SYM the scaling with N will be given in the main body of the paper. In this introduction to keep the presentation simple, we avoid explicitly mentioning the scaling of N to match with supergravity.

[5] As it corresponds to a semiclassical gravitational limit with black hole solutions at any temperature.

[6] For example, already at $\beta = \infty$, for the superconformal index of $\mathcal{N}=4$ SYM, this is known to be the case [30, 31] [32–34].

[7] These dominating saddle points come in representations of discrete groups [45, 48][57].

the S^3. Then requiring Λ to be much larger than $\frac{1}{R}$, implies that the spacing among contiguous eigenvalues $\delta E = \frac{1}{R}$ becomes infinitesimally small compared to the hierarchy of eigenvalues explored $\frac{\delta E}{E} \sim \frac{1}{(\Lambda R)^{n+1}} \ll 1$, and one effectively obtains a continuum spectrum. The integer power n may be conveniently selected depending on the landing point of the selected $\Lambda R \to \infty$ limit. The effective theory associated with such a continuum spectrum is called the *gravitational infrared theory* or simply the *infrared theory*.

Using this procedure, the complete analytic part of the asymptotic expansion of $\mathcal{F}$ around a reference $\frac{1}{16}$-BPS locus located at $\alpha_0 = 0$ [8] will be computed at any value of N. [9] The answer takes the form ($n = 2$)

$$\mathcal{F}_\infty = \beta_0 \, \mathcal{E}_0 + \sum_{p=-1}^{n} \sum_{q=0}^{4} \sum_{r=0}^{\infty} (\Lambda R)^p \, L_{p+1;q,r}[\varphi_v, \varphi_w, \underline{u}] \, \frac{F_{p;q;r}\alpha_0^q}{(\beta_0)^r \omega_{1,0}\omega_{2,0}} \,, \tag{1.3}$$

[10] where $\beta_0 = \beta\Lambda$. More details on this formula will be given in section 3, equation (3.28), here we just notice that it includes terms that range from low-temperature corrections including Casimir-energy like contributions of order $\mathcal{O}(\beta)$, and arbitrary high positive powers of temperature $\frac{1}{\beta}$. (1.3) will be called the free energy of the holographic infrared theory or simply *the infrared free energy*. Assuming supergravity predictions are correctly capturing strong-coupling results in field theory, and in virtue of analyticity, (1.3) is bound to recover a complete low-temperature expansion of the Gibbons-Hawking onshell action of the black holes of [58], valid even well beyond the vicinity of supersymmetric black hole solutions. [11]

In this paper by low-temperature expansion we mean an expansion that takes us to zero-temperature and that precisely at zero-temperature, it reduces the partition function to a superconformal index located as $\alpha_0 = 0$. [12] There are as many such families of expansions as independent superconformal indices (sectors) in $\mathcal{N} = 4$ SYM and each such family is defined by a set of *boundary conditions* or precisely linear constraints upon chemical potentials. $\frac{1}{4}$, $\frac{1}{8}$ and $\frac{1}{16}$-BPS boundary conditions and their corresponding expansions are all encoded in (1.3).

Given an index, there are infinitely many ways to reach it as $\Lambda R \to \infty$. The reparameterization group invariance that the Schwarzian breaks [64–66][67] is one that generates motion within the family of limits leading to the reference index. It corresponds, as well, to a set of chemical potential-dependent redefinitions of the cutoff $\Lambda \mapsto \Lambda'(\Lambda)$ such that $\frac{\Lambda}{\Lambda'(\Lambda)} \to 1$ as $\Lambda R \to \infty$.

[8] Sometimes it will be called the *perturbative part*.

[9] We will not pay attention to chemical potential independent $\mathcal{O}((\Lambda R)^0)$ contributions, such as those coming from counting the number of equally contributing saddle points like the ones mentioned in footnote 7.

[10] The sum over powers of temperature $\sum_{r=0}^{\infty}$ can be solved analytically as a rational function of β_0. We will not report such expressions here because they are too large, but their complete form can be found in the shared Mathematica notebook. Many of the function coefficients denoted as $F_{p;q;r}$ vanish trivially, for example, $F_{p;0;r\geq 1} = F_{p;q\geq 1;0} = 0$. Also, as it will be elaborated upon in due time there is implicit dependence on ΛR in the *auxiliary chemical potentials*, $\{\beta_0, \alpha_0, \omega_{1,0}, \omega_{2,0}\}$.

[11] Using their $\frac{1}{16}$-BPS solutions [59, 60] as reference point for the expansion [23, 61–63].

[12] Thus, the $\beta \to \infty$ limits studied in this paper are not counting ground states but BPS gauge invariant states in $\mathcal{N} = 4$ SYM.

To test (1.3) the goal will be to derive the free energy of the Schwarzian theory. For this we will select to work in a near $\frac{1}{8}$-BPS sector, and in a near-$\frac{1}{16}$-BPS sector.

The first Schwarzian contribution we will study, the one localized around a $\frac{1}{8}$-BPS sector, and infinitely many other higher low-temperature corrections upon it, will be shown to be protected by superconformal symmetry. The second Schwarzian contribution we will study, the one localized around a $\frac{1}{16}$-BPS sector, cannot a priori be argued to be protected by superconformal symmetry. [13] For both the *protected* and the *a priori unprotected* Schwarzian contributions the mass gap will be computed.

The protected contributions are bound to compute exact finite-temperature corrections of the gravitational onshell action of AdS_5 solutions [58, 68, 69] about their $\frac{1}{8}$-BPS locus. [14]

From this protected infrared limit we will learn something about near $\frac{1}{8}$-BPS black holes. For near-$\frac{1}{8}$-BPS black holes with at least two almost-equal electric charges $Q_1 \approx Q_3$ (but unequal), it will be shown that semipositivity bounds of the fundamental theory imply that the mixed-ensemble free energy of the infrared theory should come solely from the semiclassical Schwarzian contribution. Thus, it scales bilinearly with temperature T and a chemical potential $\left(\alpha - \frac{1}{2}\right)$. [15] This result strongly suggests that these black hole solutions have a near-vanishing horizon area, a conclusion that resonates with the results of [25] (in gravity) and of [74, 76, 77] (in field theory).

The Schwarzian contributions which we cannot argue (a priori) to be protected against coupling corrections are the ones corresponding – at strong coupling – to the breaking of re-parameterizations of the time coordinate in the near-horizon AdS_2 geometry of black holes [58, 68, 69] that are near-$\frac{1}{16}$-BPS, but not near $\frac{1}{8}$-BPS. These include the ones studied in [28] in the context of minimally gauged five-dimensional supergravity.

Remarkably, the free-field theory computation of the Schwarzian mass gap will exactly match the gravitational result obtained in reference [28]. In this near-$\frac{1}{16}$-BPS case logarithmic divergencies will be understood to come from the reference superconformal index ($\alpha_0 = 0$) confirming the absence of $\log T/T_{breakdown}$ corrections, as expected from the supergravity perspective [28][29]. [16]

The RG-flow procedure leading to these results opens the way to realize, analytically, how chaos may emerge [78–80] within higher dimensional non-averaged systems, such as four-dimensional $\mathcal{N} = 4$ SYM [81, 82][83].

This paper focuses on $\mathcal{N} = 4$ SYM, but the proposed RG-flow procedure can be applied

[13] At least we do not see a clean way of doing so, although we do see hints of an underlying anomaly protection argument, which currently we do not understand.

[14] The solutions of [58, 68, 69] include $\frac{1}{16}$-BPS black holes [59, 60] [23, 35, 36] but also horizon-less solutions such as BPS solitons [58, 68]. So, even in the case $\frac{1}{8}$-BPS black holes do not exist, this point may have a physical meaning, i.e., a dual horizonless geometry. $\frac{1}{16}$-BPS rotating solitons have been already found in, e.g., [58, 70]. Examples of $\frac{1}{8}$-BPS solitons with electric charge and no rotation have been already found in [71]. Other horizonless geometries, such as LLM geometries [72] have been shown to correspond to the $\frac{1}{2}$-BPS sector [73–75]. So, more general horizonless geometries may very well correspond to the $\frac{1}{4}$- and $\frac{1}{8}$-BPS sectors.

[15] This is the chemical potential controlling the limit to the supersymmetric locus [23].

[16] For the near-$\frac{1}{8}$-BPS case there may be subtleties that spoil this last conclusion. We leave for the future to study that case.

to any other example of superconformal gauge theories with a known discrete spectrum. Thinking not only in the context of gauge/gravity dualities, this way of triggering a flow between the discretruum and the continuum can be applied to any gauge theory with a known discrete spectrum, which can be enforced by placing the theory in a box with appropriate boundary conditions, for example. It can also be applied to systems that are already known to be realized in nature.

The content of this paper is organized as follows. Section 2 revisits the computation of the partition function of four-dimensional $U(N)$ $\mathcal{N}=4$ SYM on $\mathbb{R}\times S^3$ at zero gauge coupling and sets up conventions. Special emphasis is put on illustrating the constraints (boundary conditions) that reduce the partition function to $\frac{1}{16}$-BPS indices and $\frac{1}{8}$-BPS indices. Section 3 introduces the RG flow procedure that maps the discrete field theory spectrum to the continuum of charges in supergravity. It also summarizes the derivation of the infrared free energy (1.3), and explains its relation to the Gibbons-Hawking on-shell action well beyond supersymmetry and extremality. Section 4 proceeds to compute the Schwarzian contribution around the $\frac{1}{8}$-BPS locus and predicts its mass gap. Section 5 proceeds to compute the Schwarzian contribution around the generic $\frac{1}{16}$-BPS locus and its mass gap. It also compares results against the conjectured supergravity duals. Section 6 ends the main body of the article with a summary of the results, open questions, and observations. Some technical details and results are relegated to the Appendices.

2 The partition function

The space of states of $\mathcal{N}=4$ SYM on S^3 can be constructed with a set of 16 raising and lowering operators and an auxiliary vacuum state $|0\rangle$ [84]. These operators can be divided into 8 bosons and 8 fermions. The bosons, which we will denote as $a^{\pm}$, $a_{\pm}$, $b^{\pm}$, $b_{\pm}$ form an 8-dimensional spinoral representation of the conformal group in four-dimensions $SO(2,4)$. The fermions, which we will denote as $\mathfrak{f}^{1,2,3,4}$, $\mathfrak{f}_{1,2,3,4}$, form an 8-dimensional spinorial representation of the R-symmetry group $SO(6)$. These operators obey canonical commutation rules,

$$\begin{aligned} &[a^{\eta},a_{\gamma}]=\delta^{\eta}_{\gamma}\,, \qquad [b^{\dot\eta},b_{\dot\gamma}]\,=\,\delta^{\dot\eta}_{\dot\gamma}\,, \qquad \{\mathfrak{f}^{n},\mathfrak{f}_{m}\}\,=\,\delta^{n}_{m}\,,\\ &\eta,\,\dot\eta\,,\,\gamma\,,\dot\gamma\,=\,-,+ \qquad n,m=1,2,3,4\,. \end{aligned} \tag{2.1}$$

The Fock vacuum is defined by the conditions

$$a^{\pm}\,|0\rangle\,=\,0\,,\,b^{\pm}\,|0\rangle\,=\,0\,,\,\mathfrak{f}^{1}\,|0\rangle\,=\,0\,,\,\mathfrak{f}^{2}\,|0\rangle\,=\,0\,,\,\mathfrak{f}_{3}\,|0\rangle\,=\,0\,,\,\mathfrak{f}_{4}\,|0\rangle\,=\,0\,. \tag{2.2}$$

The $\mathfrak{f}^{1,2}$ are lowering operators and the $\mathfrak{f}^{3,4}$ are rising operators. Operators with supraindices are complex conjugated to operators with subindices . The scalar single-states operators in the theory X_1, X_2, X_3, $\overline{X}_1$, $\overline{X}_2$, and $\overline{X}_3$ are isomorphic to the states [17]

$$\begin{aligned} X_1 &\leftrightarrow \mathfrak{f}_1\mathfrak{f}_2\mathfrak{f}^3\mathfrak{f}^4|0\rangle\,, & \overline{X}_1 &\leftrightarrow |0\rangle\,,\\ X_2 &\leftrightarrow \mathfrak{f}_1\mathfrak{f}^4|0\rangle\,, & \overline{X}_2 &\leftrightarrow \mathfrak{f}_2\mathfrak{f}^3|0\rangle\,,\\ X_3 &\leftrightarrow \mathfrak{f}_2\mathfrak{f}^4|0\rangle\,, & \overline{X}_3 &\leftrightarrow \mathfrak{f}_1\mathfrak{f}^3|0\rangle\,. \end{aligned} \tag{2.3}$$

[17] For more details of this construction please refer to [85] and [84]. In this section, we will use conventions which are closely related to the ones in Appendix A of [84].

The symmetry generators in this theory, e.g. dilations E, the two independent angular momenta J_1 and J_2 and R-charges R_1, R_2 and R_3

$$\{E, J_1^3, J_2^3, R_1, R_2, R_3\} \tag{2.4}$$

can be expressed as quadratic combinations of oscillators a, b and $\mathfrak{f}$. [18] In particular, the 32 supercharges of the theory are

$$\mathcal{Q}^{n,\pm} = \widetilde{\mathfrak{f}}^n a_\pm,\; \mathcal{S}_{n,\pm} = \widetilde{\mathfrak{f}}_n a^\pm,\; \overline{\mathcal{Q}}^{n,\pm} = \widetilde{\mathfrak{f}}_n b_\pm\,,\; \overline{\mathcal{S}}_{n,\pm} = \widetilde{\mathfrak{f}}^n b^\pm\,. \tag{2.5}$$

where

$$\begin{aligned} \widetilde{\mathfrak{f}}^n &= \mathfrak{f}_n \qquad \text{if} \qquad n = 1\,, 2\,, \\ \widetilde{\mathfrak{f}}^n &= \mathfrak{f}^n \qquad \text{if} \qquad n = 3\,, 4\,. \end{aligned} \tag{2.6}$$

The $\dagger$-operation raises/lowers indices of single oscillators $\mathfrak{f}$, a and b. Thus $\mathcal{S}_{n,\pm} = (\mathcal{Q}^{n,\pm})^\dagger$, $\overline{\mathcal{S}}_{n,\pm} = (\overline{\mathcal{Q}}^{n,\pm})^\dagger$. In our conventions the R-charge generators are

$$\begin{aligned} R_1 &:= \widetilde{\mathfrak{f}}_2\widetilde{\mathfrak{f}}^2 - \widetilde{\mathfrak{f}}_1\widetilde{\mathfrak{f}}^1\,, \\ R_2 &:= \widetilde{\mathfrak{f}}_3\widetilde{\mathfrak{f}}^3 - \widetilde{\mathfrak{f}}_2\widetilde{\mathfrak{f}}^2\,, \\ R_3 &:= \widetilde{\mathfrak{f}}_4\widetilde{\mathfrak{f}}^4 - \widetilde{\mathfrak{f}}_3\widetilde{\mathfrak{f}}^3\,. \end{aligned} \tag{2.7}$$

Using these expressions one obtains the R-charges of the scalar single states, which are summarized in Table 1.

Scalars	R_1	R_2	R_3
X_1	0	-1	0
X_2	-1	$+1$	-1
X_3	$+1$	0	-1

Table 1. R-charges of the scalar single-states $X_{1,2,3}$. Barred scalars have opposite charges.

We will be interested in two sets of complex-conjugated supercharges. The first couple

$$\mathcal{Q} := \mathcal{Q}^{4,-}\,, \qquad \mathcal{S} := \mathcal{Q}^\dagger = \mathcal{S}_{4,-}\,, \tag{2.8}$$

whose anti-commutation relation, following from (2.1), is

$$\Delta = 2\{\mathcal{Q}, \mathcal{S}\} = H - 2J_1^3 - 2\sum_{k=1}^{3} \frac{k}{4} R_k \geq 0\,. \tag{2.9}$$

To obtain this commutation relation one must use the fact that the central element of the oscillator algebra

$$C := b^+ b_+ + b^- b_- - a^+ a_+ - a^- a_- - \widetilde{\mathfrak{f}}_n \widetilde{\mathfrak{f}}^n = -Z_1 - B_1 - 2 = -2\,, \tag{2.10}$$

[18] We will use the definitions of charges given in Appendix A of [84], equation (A.2), with the relations $\widetilde{\mathfrak{f}}^n = \alpha^n_{\text{there}}$, $(a^\eta)_{here} = (a_\eta)_{there}$, $(b^{\dot\eta})_{here} = (b_{\dot\eta})_{there}$. Moreover, $(R_a)_{\text{here}} = (-R_a)_{\text{there}}$.

equals -2, and that the $U(1)_J$ and $U(1)_B$ charges [85]

$$-Z_1 \,=\, N_b \,-\, N_a\,, \qquad\qquad -B_1 \,=\, -\widetilde{\mathfrak{f}}_n\widetilde{\mathfrak{f}}^n \,+\, 2 \,=\, N_\beta \,-\, N_\alpha\,, \tag{2.11}$$

vanish on the physical states of the theory. [19] In our conventions, the number operators are [85]

$$\begin{aligned} N_b \,&=\, b_+ b^+ \,+\, b_- b^-\,, \qquad & N_a \,&=\, a_+ a^+ \,+\, a_- a^-\,, \\ N_\beta \,&=\, \mathfrak{f}^3\mathfrak{f}_3 \,+\, \mathfrak{f}^4\mathfrak{f}_4\,, \qquad & N_\alpha \,&=\, \mathfrak{f}_1\mathfrak{f}^1 + \mathfrak{f}_2\mathfrak{f}^2\,. \end{aligned} \tag{2.12}$$

Q and S will be the supercharges preserved by the BPS sector around which we want to compute low-temperature corrections. The other two supercharges we will work with are, either

$$\overline{\mathcal{Q}}^{2,+}\,, \qquad \overline{\mathcal{S}}_{2,+} = (\overline{\mathcal{Q}}^{2,+})^\dagger\,, \tag{2.13}$$

or

$$\overline{\mathcal{Q}}^{2,-}\,, \qquad \overline{\mathcal{S}}_{2,-} = (\overline{\mathcal{Q}}^{2,-})^\dagger\,, \tag{2.14}$$

and their anticommutation relations are

$$\begin{aligned} 2\{\overline{\mathcal{Q}}^{2,+}, \overline{\mathcal{S}}_{2,+}\} \,&=\, H \,+\, 2J_2^3 \,+\, \frac{R_1}{2} - R_2 - \frac{R_3}{2} \\ &= \Delta_+ \,:=\, \Delta \,+\, \Delta_+^{(2)} \,\geq\, 0 \\ 2\{\overline{\mathcal{Q}}^{2,-}, \overline{\mathcal{S}}_{2,-}\} \,&=\, H \,-\, 2J_2^3 \,+\, \frac{R_1}{2} - R_2 - \frac{R_3}{2} \\ &= \Delta_- \,:=\, \Delta \,+\, \Delta_-^{(2)} \,\geq\, 0 \end{aligned} \tag{2.15}$$

where

$$\Delta_\pm^{(2)} \,:=\, 2\big(J_1^3 \,\pm\, J_2^3\big) \,+\, R_1 + R_3\,. \tag{2.16}$$

The weights of these supercharges under the action of the bosonic symmetries (2.4) are

$$\mathcal{Q}^{4,-} \,\to\, \big\{\frac{1}{2}, -\frac{1}{2}, 0, 0, 1\big\}\,, \qquad \overline{\mathcal{Q}}^{2,\pm} \,\to\, \big\{\frac{1}{2}, 0, \pm\frac{1}{2}, -1, +1, 0\big\}\,. \tag{2.17}$$

For latter reference we note that the set of bosonic charges that commute simultaneously with $\mathcal{Q}^{4,-}$, $\mathcal{S}_{4,-}$, $\mathcal{Q}^{2,+}$, and $\mathcal{S}_{2,+}$ is generated by

$$\Delta\,,\; R_1 + R_2\,,\; H \,+\, J_1^3 \,-\, R_2/2\,,\; + J_2^3 \,-\, R_2/2\,. \tag{2.18}$$

The set of those that commute simultaneously with $\mathcal{Q}^{4,-}$, $\mathcal{S}_{4,-}$, $\mathcal{Q}^{2,-}$, and $\mathcal{S}_{2,-}$ is generated instead by

$$\Delta\,,\; R_1 + R_2\,,\; H \,+\, J_1^3 \,-\, R_2/2\,,\; - J_2^3 \,-\, R_2/2\,. \tag{2.19}$$

[19]For example, these contraints imply that at the level of eigenvalues $\widetilde{\mathfrak{f}}_4\widetilde{\mathfrak{f}}^4 \,=\, -\sum_{n=1}^{3}\widetilde{\mathfrak{f}}_n\widetilde{\mathfrak{f}}^n \,+\, 2$. The oscilators $\widetilde{\mathfrak{f}}_n$ are the ones denoted as α_n in reference [84].

2.1 The matrix integral at zero gauge coupling

In the language of the $\mathcal{N} = 1$ superconformal symmetry corresponding to the $U(1)$ R-charge generator

$$R_3 \tag{2.20}$$

the fundamental field content of the theory organizes in a vector, three chiral and three anti-chiral multiplets. The vector multiplet is composed of a vector field with $R_3 = 0$ and *flavour* charges $R_1 = R_2 = 0$, and a gaugino with a chiral (and antichiral) components with $R_3 = +1$ (and -1) and flavour charges $R_1 = R_2 = 0$. The R-charges of the chiral multiplets can be found in table 2. The R-charges of the anti-chiral multiplets take the opposite values.

chiral mutliplet I	R_1	R_2	R_3
1	0	-1	0
2	-1	$+1$	-1
3	$+1$	0	-1

Table 2. R-charges of the chiral multiplets. The same ones as their corresponding scalar components quoted in table 1. The antichiral multiplets have opposite charges. The scalars in the chiral(antichiral)-multiplets have the same R-charges as their multiplets. The fermions in the chiral(resp. antichiral)-multiplets have the same charges as the scalars under R_1 and R_2. Their charge under R_3 increases by $+1$ (resp.-1) with respect to the R-charge of the scalar in the same multiplet.

For later use we note that

$$(-1)^F = e^{\pi \mathrm{i}(R_1+R_3)}\,. \tag{2.21}$$

After a straightforward computation the partition function Z of the free theory [86]

$$Z = e^{-\mathcal{F}} = Z[x,u,v,w,t,y] := \mathrm{Tr}_{\mathcal{H}} x^{\Delta} u^{-R_1-R_3} v^{-R_1} w^{-R_2} t^{2(H+J_1^3)} y^{2J_2^3} \tag{2.22}$$

reduces to

$$Z = e^{-\mathcal{F}} = \int [DU] e^{-\mathcal{F}_{sl}^{\infty}[x,u,v,w,t,y;U]} \tag{2.23}$$

where

$$\begin{aligned} -\mathcal{F}_{sl}^{\Lambda}[x,u,v,w,t,y;U] := \sum_{j=1}^{\Lambda^{n+1}} \frac{1}{j} \big(& f_{Bos}[x^j,u^j,v^j,w^j,t^j,y^j] \\ & + (-1)^{j+1} f_{Fer}[x^j,u^j,v^j,w^j,t^j,y^j]\big)\,\mathrm{Tr}U^j \mathrm{Tr}U^{\dagger j}\,, \end{aligned} \tag{2.24}$$

and

$$
\begin{aligned}
f_{Bos} &:= f_{Bos}^{(V)} + \sum_{I=1}^{3} f_{Bos}^{(I)} \\
&= -\frac{t^6 y^2 \left(u^2 x^4 \left(v \left(u^2 v + w^2\right) + w\right) + x^2 \left(u^2 v (vw+1) + w^2\right)\right)}{u^2 vw \left(t^3 - y\right)\left(t^3 y - 1\right)\left(t x^2 - y\right)\left(t x^2 y - 1\right)} \\
&\quad + \frac{t^5 y \left(2t^3 x^4 y - 2t^2 x^2 \left(y^2+1\right) + ty - 2x^4 \left(y^2+1\right)\right)}{\left(t^3 - y\right)\left(t^3 y - 1\right)\left(t x^2 - y\right)\left(t x^2 y - 1\right)} \\
&\quad + \frac{t^4 u^2 vwx^2 \left(y^2+1\right)^2 + t^2 y^2 \left(u^4 v^2 x^2 + u^2 \left(v^2 w + vwx^2 \left(w + x^2\right) + v + wx^2\right) + w^2\right)}{u^2 vw \left(t^3 - y\right)\left(t^3 y - 1\right)\left(t x^2 - y\right)\left(t x^2 y - 1\right)} ,
\end{aligned}
\tag{2.25}
$$

$$
\begin{aligned}
f_{Fer} &:= f_{Fer}^{(V)} + \sum_{I=1}^{3} f_{Fer}^{(I)} \\
&= \frac{t^3 y \left(x^2 \left(u^2 v \left(t^2(-v) + vw + 1\right) + w \left(w - t^2(vw+1)\right)\right) + vw\right)}{uvw \left(t^3 - y\right)\left(t^3 y - 1\right)\left(t x^2 - y\right)\left(t x^2 y - 1\right)} \\
&\quad + \frac{t^2 y \left(t^4(-v) y \left(u^2 x^2 + w\right) - t^3 u^2 vwx^4 + t^2 y \left(v \left(u^2 v + w^2\right) + w\right) + u^2 v^2 x^2 y\right)}{uvw \left(t^3 - y\right)\left(t^3 y - 1\right)\left(t x^2 - y\right)\left(t x^2 y - 1\right)}
\end{aligned}
\tag{2.26}
$$

$$
\begin{aligned}
&+ \frac{t^2 x^2 y^2 \left(-\left(t^4 \left(u^2 v^2 + w\right)\right) + t^2 \left(u^2 - 1\right) v + vw + 1\right)}{uv \left(t^3 - y\right)\left(t^3 y - 1\right)\left(t x^2 - y\right)\left(t x^2 y - 1\right)} \\
&+ \frac{t^2 y^2 \left(-t^2 x^4 \left(u^2 v (vw+1) + w^2\right) + tvy \left(u^2 x^2 + w\right) + u^2 vwx^4\right)}{uvw \left(t^3 - y\right)\left(t^3 y - 1\right)\left(t x^2 - y\right)\left(t x^2 y - 1\right)} \\
&- \frac{t^3 x^2 y^3 \left(t^2 \left(v \left(u^2 \left(v + wx^2\right) + w^2\right) + w\right) - w \left(u^2 v^2 + w\right)\right)}{uvw \left(t^3 - y\right)\left(t^3 y - 1\right)\left(t x^2 - y\right)\left(t x^2 y - 1\right)} .
\end{aligned}
\tag{2.27}
$$

Let us give some details on how this expressions above were derived. The contributions coming from vector multiplets are

$$
\begin{aligned}
f_{Bos}^{(V)} = &\sum_{j=1}^{\infty} \sum_{j_1^3 = -\frac{j+1}{2}}^{\frac{j+1}{2}} \sum_{j_2^3 = -\frac{j-1}{2}}^{\frac{j-1}{2}} \left(x^{\Delta_+} t^{2(\epsilon_j^{(1)} + j_1^3)} y^{2 j_2^3} u^{-R_1 - R_3} v^{-R_1} w^{-R_2}\right) \\
&+ \sum_{j=1}^{\infty} \sum_{j_1^3 = -\frac{j-1}{2}}^{\frac{j-1}{2}} \sum_{j_2^3 = -\frac{j+1}{2}}^{\frac{j+1}{2}} \left(x^{\Delta_-} t^{2(\epsilon_j^{(1)} - j_1^3)} y^{-2 j_2^3} u^{+R_1 + R_3} v^{+R_1} w^{+R_2}\right)
\end{aligned}
\tag{2.28}
$$

and

$$
\begin{aligned}
f_{Fer}^{(V)} = &\sum_{j=1}^{\infty} \sum_{j_1^3 = -\frac{j}{2}}^{\frac{j}{2}} \sum_{j_2^3 = -\frac{j-1}{2}}^{\frac{j-1}{2}} x^{\Delta^+} t^{2(\epsilon_j^{(\frac{1}{2})} + j_1^3)} y^{2 j_2^3} u^{-R_1 - R_3} v^{-R_1} w^{-R_2} \\
&+ \sum_{j=1}^{\infty} \sum_{j_1^3 = -\frac{j-1}{2}}^{\frac{j-1}{2}} \sum_{j_2^3 = -\frac{j}{2}}^{\frac{j}{2}} x^{\Delta^-} t^{2(\epsilon_j^{(\frac{1}{2})} - j_1^3)} y^{-2 j_2^3} u^{+R_1 + R_3} v^{+R_1} w^{+R_2} .
\end{aligned}
\tag{2.29}
$$

The contributions coming from chiral+antichiral multiplets are

$$
\begin{aligned}
f_{Bos}^{(I)} &= \sum_{j=0}^{\infty}\sum_{j_1^3=-\frac{j}{2}}^{\frac{j}{2}}\sum_{j_2^3=-\frac{j}{2}}^{\frac{j}{2}} x^{\Delta^+}t^{2(\epsilon_j^{(0)}+j_1^3)}y^{2j_2^3}u^{-R_1-R_3}v^{-R_1}w^{-R_2} \\
&+\sum_{j=0}^{\infty}\sum_{j_1^3=-\frac{j}{2}}^{\frac{j}{2}}\sum_{j_2^3=-\frac{j}{2}}^{\frac{j}{2}} x^{\Delta^-}t^{2(\epsilon_j^{(0)}-j_1^3)}y^{-2j_2^3}u^{+R_1+R_3}v^{+R_1}w^{+R_2}
\end{aligned} \tag{2.30}
$$

and

$$
\begin{aligned}
f_{Fer}^{(I)} &= \sum_{j=1}^{\infty}\sum_{j_1^3=-\frac{j}{2}}^{\frac{j}{2}}\sum_{j_2^3=-\frac{j-1}{2}}^{\frac{j-1}{2}} x^{\Delta^+}t^{2(\epsilon_j^{(\frac{1}{2})}+j_1^3)}y^{2j_2^3}u^{-R_1-R_3}v^{-R_1}w^{-R_2} \\
&+\sum_{j=1}^{\infty}\sum_{j_1^3=-\frac{j-1}{2}}^{\frac{j-1}{2}}\sum_{j_2^3=-\frac{j}{2}}^{\frac{j}{2}} x^{\Delta^-}t^{2(\epsilon_j^{(\frac{1}{2})}-j_1^3)}y^{-2j_2^3}u^{+R_1+R_3}v^{+R_1}w^{+R_2}\,.
\end{aligned} \tag{2.31}
$$

In these expressions we have used the following definitions

$$
\Delta^{\pm} = \epsilon_j \mp 2j_1^3 \mp \frac{1}{2}R_1 \mp R_2 \mp \frac{3}{2}R_3\,,\ \epsilon_j^{(1)} = j+1\,,\ \epsilon_j^{(\frac{1}{2})} = j+\frac{1}{2}\,,\ \epsilon_j^{(0)} = j+1\,, \tag{2.32}
$$

where j_1^3 and j_2^3 are the eigenvalues of J_1^3 and J_2^3 respectively.

After resumming the series in these contributions above, and summing the result over the R-charge values of the vector multiplet components, we obtain

$$
\begin{aligned}
f_{Bos}^{(V)} &= \frac{2t^8x^4y^2-2t^7x^2y\left(y^2+1\right)+t^6y^2-2t^5x^4y\left(y^2+1\right)+t^4x^2\left(y^2+1\right)^2+t^2x^4y^2}{\left(t^3-y\right)\left(t^3y-1\right)\left(tx^2-y\right)\left(tx^2y-1\right)} \\
f_{Fer}^{(V)} &= \frac{t^2y\left(-t^4y-t^3u^2x^4\left(y^2+1\right)+t^2\left(u^2-1\right)x^2y+t\left(y^2+1\right)+u^2x^4y\right)}{u\left(t^3-y\right)\left(t^3y-1\right)\left(tx^2-y\right)\left(tx^2y-1\right)}\,.
\end{aligned} \tag{2.33}
$$

After summing

$$
\begin{aligned}
f_{Bos}^{(I)} &= \frac{t^2y^2u^{R_1+R_3}v^{R_1}w^{R_2}x^{\frac{R_1}{2}+R_2+\frac{3R_3}{2}+1}\left(1-t^4x^2\right)\left(u^{-2(R_1+R_3)}v^{-2R_1}w^{-2R_2}x^{-R_1-2R_2-3R_3}+1\right)}{\left(t^3-y\right)\left(t^3y-1\right)\left(tx^2-y\right)\left(tx^2y-1\right)} \\
f_{Fer}^{(I)} &= \frac{t^2yu^{-R_1-R_3}v^{-R_1}w^{-R_2}x^{\frac{1}{2}(-R_1-2R_2-3R_3+1)}\left(-t^3x^2\left(y^2+1\right)+t^2y+x^2y\right)}{\left(t^3-y\right)\left(t^3y-1\right)\left(tx^2-y\right)\left(tx^2y-1\right)} \\
&+\frac{t^3yu^{+R_1+R_3}v^{+R_1}w^{+R_2}x^{\frac{1}{2}(+R_1+2R_2+3R_3+3)}\left(y\left(-t^3-tx^2+y\right)+1\right)}{\left(t^3-y\right)\left(t^3y-1\right)\left(tx^2-y\right)\left(tx^2y-1\right)}\,,
\end{aligned} \tag{2.34}
$$

over the R-charges $I=1,2,3$ reported in the table 2 and summing over contributions coming from bosons and fermions in the vector multiplets, we obtain the total contributions from bosons (2.25) and fermions (2.26) to $\mathcal{F}_{sl}^{\Lambda}$.

For completeness, we report the translation to the writing of the partition function given in equation (2.5) of [35],

$$\begin{aligned} e^{-\beta_{\text{there}}} &= x\,t^2\,,\; e^{-\omega_{1\text{there}}} = \frac{t}{yx}\,,\; e^{-\omega_{2\text{there}}} = \frac{ty}{x}\,,\\ e^{\Delta_{1\text{there}}} &= xw\,,\; e^{\Delta_{2\text{there}}} = x\frac{u^2 v}{w}\,,\; e^{\Delta_{3\text{there}}} = \frac{x}{v}\,,\\ e^{2\Delta_{\text{there}}} &= e^{\Delta_{1\text{there}}+\Delta_{2\text{there}}+\Delta_{3\text{there}}} = x^3 u^2\,. \end{aligned} \tag{2.35}$$

Eventually, we will use the following definitions of rapidities in terms of chemical potentials

$$\widetilde{x}^2 := \frac{tx^2}{y}\,,\; x = e^{-\beta}\,,\; \frac{t^3}{y} = e^{-\omega_1}\,,\; t^3 y = e^{-\omega_2}\,,\; t^2 v = e^{-\varphi_v}\,,\; \frac{w}{t^2} = e^{\varphi_w} \tag{2.36}$$

which can be equivalently written as follows

$$t = e^{-\frac{1}{6}(\omega_1+\omega_2)}\,,\; y = e^{\frac{\omega_1-\omega_2}{2}}\,,\; v = e^{\frac{1}{3}(\omega_1+\omega_2-3\varphi_v)}\,,\; w = e^{-\frac{1}{3}(\omega_1+\omega_2-3\varphi_w)}\,. \tag{2.37}$$

The BPS locus $\alpha = \pm\frac{1}{2}$ Note that from (2.21) it follows that at

$$u := e^{2\pi \mathrm{i}\alpha} = e^{\pm\pi \mathrm{i}} \tag{2.38}$$

cancellations happen and the dependence on x dissapears in

$$f_{Bos}[x^j,\ldots] + (-1)^{j+1} f_{Fer}[x^j,\ldots] = \mathcal{I}^{4,-}_{sl}[v^j, w^j, t^j, y^j] \tag{2.39}$$

where

$$\mathcal{I}^{4,-}_{sl}[v,w,t,y] := 1 - \frac{(t^2 v - 1)\left(\frac{t^2}{w} - 1\right)\left(\frac{t^2 w}{v} - 1\right)}{\left(\frac{t^3}{y} - 1\right)(1 - t^3 y)}\,. \tag{2.40}$$

This implies that at the value of chemical potential (2.38) the partition function equals

$$Z[x, u=-1, v, w, t, y] = e^{-\mathcal{F}[x,u=-1,v,w,t,y]} = \mathcal{I}^{4,-}(v,w,t,y)\,, \tag{2.41}$$

where

$$\begin{aligned} \mathcal{I}^{4,-} &:= \int [DU] e^{\sum_{j=1}^{\infty}\frac{1}{j}\mathcal{I}^{4,-}_{sl}(v^j,w^j,t^j,y^j)\,\mathrm{Tr}U^j\,\mathrm{Tr}U^{\dagger j}}\\ &= \mathrm{Tr}_{\mathcal{H}}(-1)^F\, x^{2\{Q,S\}} v^{-R_1} w^{-R_2} t^{2(H+J_1^3)} y^{2J_2^3} =: \mathcal{I}_1\,, \end{aligned} \tag{2.42}$$

is the $\frac{1}{16}$-BPS superconformal index counting states in the cohomologies of $Q = \mathcal{Q}^{4,-}$ and $S = \mathcal{S}_{4,-} = Q^\dagger$, for which $\Delta = 0$ [84, 87][88–90]. States which are not in such cohomology do not contribute to this index and thus $\mathcal{I}^{4,-}$ does not depend on x, as it has been explicitly shown in (2.40). We will also denote this index as $\mathcal{I}_1$.

Further imposing

$$w = vty \qquad \text{or} \qquad \varphi_w = -\varphi_v + \omega_1 + \omega_2\,, \tag{2.43}$$

on (2.40) we find that

$$\mathcal{I}_{sl}^{4,-;2,+}[v,t,y] \;=\; 1 - \frac{\left(1-t^2 v\right)\left(1-\frac{t}{vy}\right)}{1-\frac{t^3}{y}}\,. \tag{2.44}$$

Indeed, (2.43) implies, more generally, that

$$Z[x,u\;=\;-1,v,w\;=\;vty,t,y] \;=\; \mathcal{I}^{4,-;2,+}[v,t,y]\,. \tag{2.45}$$

where

$$\begin{aligned}\mathcal{I}^{4,-;2,+} &:= \int [DU] e^{\sum_{j=1}^{\infty}\frac{1}{j}\mathcal{I}_{sl}^{4,-;2,+}(v^j,t^j,y^j)\,\mathrm{Tr}U^j\mathrm{Tr}U^{\dagger j}} \\ &= \mathrm{Tr}_{\mathcal{H}}(-1)^F\, x^{2\{Q,S\}} v^{-R_1-R_2} t^{2(H+J_1^3)-R_2} y^{2J_2^3-R_2}\end{aligned} \tag{2.46}$$

is the $\frac{1}{8}$-BPS superconformal index counting states in the cohomologies of $\mathcal{Q}=\mathcal{Q}^{4,-}$, $\mathcal{S}=\mathcal{S}_{4,-}$, $\mathcal{Q}^{2,+}$ and $\mathcal{S}_{2,+}$: the Macdonald index [91] associated to the latter four supercharges.

2.2 An index to compute higher order thermal corrections at strong coupling

To compute physical thermal corrections around the point at which supersymmetric cancellations occur

$$\alpha = \frac{1}{2} \tag{2.47}$$

one can define the following restriction of the fully refined partition function (2.23)

$$Z[x,u=e^{2\pi\mathrm{i}\alpha},e^{-2\pi\mathrm{i}(\alpha-1/2)}\widetilde{v},w=\widetilde{v}ty\,,\,t,\,y] \;=\; \mathcal{I}_2[x,e^{2\pi\mathrm{i}\alpha},\widetilde{v},t,y] \;=\; e^{-\mathcal{F}_{\frac{1}{16}\text{near}\frac{1}{8}}}\,. \tag{2.48}$$

Equation (2.45) implies that at $\alpha=\frac{1}{2}$ the partition function $\mathcal{I}_2$ reduces to a $1/8$-BPS index

$$\mathcal{I}_2[x,-1,\widetilde{v},t,y] \;=\; \mathcal{I}^{4,-;2,+}[\widetilde{v},t,y]\,. \tag{2.49}$$

Remarkably, for any $\alpha \neq \frac{1}{2}$, the restricted partition function $\mathcal{I}_2$ remains a superconformal index. [20] That follows from the fact that the Taylor coefficients of (2.48) at $\alpha=\frac{1}{2}$

$$\mathrm{Tr}_{\mathcal{H}}(R_3)^n(-1)^F x^{\Delta}\widetilde{v}^{-R_1-R_2}t^{2(H+J_1^3)-R_2}y^{2J_2^3-R_2} \tag{2.50}$$

are protected observables under the supercharges $\mathcal{Q}^{2,+}$ and $\mathcal{S}_{2,+}$. [21] Indeed, starting from the fully refined partition function (2.22) and using the definition (2.48), a straightforward computation gives us

$$\mathcal{I}_2 \;=\; \int [DU] e^{\sum_{j=1}^{\infty}\frac{1}{j}\mathcal{I}_{2,sl}(x^j,e^{2\pi\mathrm{i}j\alpha},\widetilde{v}^j,t^j,y^j)\,\mathrm{Tr}U^j\mathrm{Tr}U^{\dagger j}} \tag{2.51}$$

where

$$\mathcal{I}_{2,sl}(x,e^{2\pi\mathrm{i}\alpha},\widetilde{v},t,y) \;=\; 1 \;+\; \frac{\left(1+e^{-2i\pi\alpha}t^2\widetilde{v}\right)\left(\frac{t}{\widetilde{v}y}-1\right)\left(1+e^{2i\pi\alpha}\widetilde{x}^2\right)}{\left(\frac{t^3}{y}-1\right)\left(\widetilde{x}^2-1\right)}\,. \tag{2.52}$$

[20] That is why we have attached the subindex $\frac{1}{16}near\frac{1}{8}$ to the free energy in equation (2.48).
[21] Not under $\mathcal{Q}$ and $\mathcal{S}$ because they do not commute with R_3 (See (2.17)).

Note that (2.52) is the known expression for the single-letter maximally refined superconformal index. The interesting feature of (2.52) is that its rapidites depend explicitly on the physical temperature $\frac{1}{\beta}$ of the system. For latter convenience we recall that

$$\widetilde{v} := e^{2\pi \mathrm{i}(\alpha-1/2)} v = \frac{e^{-\varphi_{\widetilde{v}}}}{t^2}\,, \qquad \widetilde{x}^2 := \frac{t x^2}{y}\,. \tag{2.53}$$

2.3 Protected near-1/8-BPS low-temperature corrections

Let us explain what has just been found. If we define inverse temperature β as the chemical potential dual to the twisted Hamiltonian obtained from the anticommutation of $\mathcal{Q}_1 = \mathcal{Q}$ and its complex conjugated supercharge $\mathcal{S}_1 = \mathcal{S}$, then states in the cohomology of $\mathcal{Q}_1$ and $\mathcal{S}_1$ are *zero-temperature states* in the sense that their contribution to the partition function does not depend on temperature. Indeed, the restricted partition function receiving contribution only from states in the $\mathcal{Q}_1$ and $\mathcal{S}_1$-cohomology, which happens to be the superconformal index $\mathcal{I}_1 = \mathcal{I}^{4,-}$, does not depend on β .

On the other hand, states in the cohomology of $\mathcal{Q}_2 = \overline{\mathcal{Q}}^{2,+}$ and its complex conjugated supercharge $\mathcal{S}_2 = \overline{\mathcal{S}}_{2,+}$, which are not in the cohomology of $\mathcal{Q}_1$ and $\mathcal{S}_1$, are *finite-temperature states* in the sense that their contribution to the physical partition function depends explicitly on temperature. The latter finite-temperature corrections are protected by $\mathcal{Q}_2, \mathcal{S}_2$-supersymmetry and thus are computed by another superconformal index $\mathcal{I}_2$ counting states in the cohomologies of $\mathcal{Q}_2$ and $\mathcal{S}_2$. Consequently, they do not receive corrections in the gauge coupling and can be computed, exactly, at zero gauge coupling.

In virtue of AdS/CFT conjecture, this predicts that $\mathcal{I}_2$ encodes perturbatively low-temperature corrections of the gravitational on-shell action of the solutions of [69], when the latter family of solutions is expanded around its $\frac{1}{8}$-BPS locus ($\Delta = \Delta_+ = 0$).

3 The holographic low-temperature expansion

As explained in the introduction, to obtain perturbative corrections in $\frac{1}{\mathfrak{b}} = \frac{1}{\beta R}$ [22] consistent with the dual gravitational picture, it is necessary to implement an RG flow mechanism by which the discrete spectrum of the gauge theory effectively becomes dense. Otherwise, the Taylor expansion of $\mathcal{F}$ at $\mathfrak{b} = \infty$ trivializes (as one can explicitly check from (2.23)).

This section defines such RG flow procedure to the continuum. It also derives the infrared free energy emerging after such flow, and it explains why it is bound to encode the low-temperature expansion of the Gibbons-Hawking onshell action of the black holes of [58, 68] even well beyond their BPS locus [59, 60].

3.1 The expansion to the continuum

The discreteness of the spectrum of $\mathcal{N} = 4$ SYM is controlled by the radius of the S^3, R . In the discussion above we have fixed $R = 1$. The dependence in R can recovered by

[22] Meaning by $\mathfrak{b}$ the dimensionful inverse temperature obtained by substituting $\beta \to \frac{\mathfrak{b}}{R}$ in the previous equations.

substituting

$$\beta = \frac{\mathfrak{b}}{R}\,, \qquad \omega_1 = \frac{\mathfrak{w}_1}{R}\,, \qquad \omega_2 = \frac{\mathfrak{w}_2}{R}\,, \qquad \alpha - \frac{1}{2} = \frac{\mathfrak{a}}{R}\,, \tag{3.1}$$

in the partition function Z in (2.23) (after using the relations (2.37)).

The naive limit to the continuum, $R \to \infty$, with the dimensionless ratios in the left-hand sides of (3.1) fixed and finite is not the limit we are looking for.

For reasons that were explained in the introduction and that we will comeback to discuss below, we need another length scale, say $1/\Lambda$. Then we define

$$\mathfrak{b} =: \frac{\beta_0[\Lambda R]}{\Lambda}\,, \qquad \mathfrak{a} =: \frac{\alpha_0[\Lambda R]}{\Lambda}\,, \qquad \mathfrak{w}_1 =: \frac{\omega_{1,0}[\Lambda R]}{\Lambda}\,, \qquad \mathfrak{w}_2 =: \frac{\omega_{2,0}[\Lambda R]}{\Lambda}\,. \tag{3.2}$$

The parameter functions in the numerators, from now on called *auxiliary potentials*,

$$\underline{\mu} := \{\mu_i\} = \{\beta_0\,,\,\alpha_0\,,\,\omega_{1,0}\,,\,\omega_{2,0}\} \tag{3.3}$$

are dimensionless, and around $\Lambda R = \infty$ are assumed to behave as follows

$$\mu[\Lambda R] = \mu_i^{(0)}\Big(1 + \sum_{p=1}^{\infty} \frac{\mu_i^{(p)}}{(\Lambda R)^p}\Big)\,. \tag{3.4}$$

The functions

$$\mu_i^{(p)} = \mu_i^{(p)}[\underline{\mu}^{(0)}]\,, \qquad p \geq 1\,, \tag{3.5}$$

are meromorphic functions of the *leading behaviors* $\mu_i^{(0)}$,

$$\mu_i^{(p)} = \mu_i^{(p)}[\underline{\mu}^{(0)}]\,. \tag{3.6}$$

Different choices of functions $\mu_i^{(p>0)}$ represent different ways to RG-flow towards the continuum. Sometimes we will call these $\mu_i^{(p>0)}$, *moduli of the space of limits* or *of the space of RG-flows.*

For example, let us assume two choices of moduli

$$\mu_i^{(p>0)} \to \mu_i^{\prime(p>0)} \implies \underline{\mu} \to \underline{\mu}' := \underline{f}(\Lambda R, \mu) \tag{3.7}$$

and define the invariant object

$$\mathcal{F} = F[\mathfrak{b}, \mathfrak{a}, \mathfrak{w}_1, \mathfrak{w}_2]\,. \tag{3.8}$$

Let us compute the expansions of the latter with both choices of moduli. From (3.2), (3.7), and (3.8) one obains

$$\begin{aligned} \mathcal{F}_\infty &= F_{\Lambda=\infty}[\frac{\underline{\mu}}{\Lambda}]\,, \\ \mathcal{F}'_\infty &= F_{\Lambda=\infty}[\frac{\underline{\mu}'}{\Lambda}] = F_{\Lambda=\infty}[\frac{f(\Lambda, \underline{\mu})}{\Lambda}]\,, \end{aligned} \tag{3.9}$$

where the subindex $\Lambda = \infty$ means asymptotic expansion around $\Lambda = \infty$ of the indexed quantity. Then, if $\underline{f}(\Lambda, \underline{\mu})$ does not generate isometries of F [23]

$$\mathcal{F}_\infty \neq \mathcal{F}'_\infty \,. \tag{3.10}$$

Different choices of (3.5) can generate the same form of infrared free energy. There is redundancy in their choice. For example, all possible changes of $\mathcal{F}_\infty$ can be generated by redefinitions of one out of the two angular velocities, $i = 3$ or $i = 4$. Namely, in complex redefinitions of either

$$\omega^{(p)}_{1,0}[\underline{\mu}^{(0)}] \qquad \text{or} \qquad \omega^{(p)}_{2,0}[\underline{\mu}^{(0)}]. \tag{3.11}$$

[24] Alternatively, they can be generated by reparameterizations of the cutoff scale

$$\Lambda \to \Lambda\Big(1 + \sum_{p=1}^{\infty} \frac{\Lambda^{(p)}[\underline{\mu}^{(0)}]}{\Lambda R}\Big)\,. \tag{3.12}$$

at fixed $\underline{\mu}^{(p)}$, i.e., by redefinitions of the $\Lambda^{(p)}[\underline{\mu}^{(0)}]$'s, which are generic meromorphic functions of the $\underline{\mu}^{(0)}$. The physical conditions that fix the Schwarzian action, i.e. the reality condition on charges and entropy, will happen to break part of these complex reparameterizations, loosely speaking their "imaginary" part. We will comeback to illustrate this in section 5.

From now on and until the end of this section the implicit dependence of the μ_i's, on ΛR will be ignored. Thus, by *expanding at large-*ΛR it will be meant expanding in every other dependence on ΛR, which is not the one implicit in the auxiliary potentials μ's. An important role in our discussion will be played by the following infinitesimal vicinities (at large enough Λ)

$$\mathfrak{b} \sim \frac{\beta_0}{\Lambda}\,, \qquad \mathfrak{a} \sim \frac{\alpha_0}{\Lambda}\,, \qquad \mathfrak{w}_1 \sim \frac{\omega_{1,0}}{\Lambda}\,, \qquad \mathfrak{w}_2 \sim \frac{\omega_{2,0}}{\Lambda}\,. \tag{3.13}$$

They correspond, in the sense explained in the introduction, to the leading RG-flow procedure to the continuum.

3.2 The RG flow procedure: the infrared free energy

Let us move on to compute the holographic low-temperature expansion of the free energy $\mathcal{F}$. To do so we follow the RG-flow procedure below:

[23] The problem of systematically classifying isometries of F or even simpler, of $F_{\Lambda=\infty}$, will be left for future work.

[24] In particular, there are choices of these functions that generate the same form of $\mathcal{F}_\infty$ as the $\mathcal{O}(\Lambda^0)$ scalings of $\mu_2 := \alpha_0 \to C\alpha_0 s$. For instance, the change $\omega_{a,0} \to \omega_{a,0}(1 + C_1 \frac{\alpha_0}{\Lambda})$ with C_1 being an $\mathcal{O}((\Lambda R)^0)$ meromorphic function of the physical chemical potentials $\{\beta, \alpha, \omega_1, \omega_2\}$, generates the same change in $\mathcal{F}_\infty$ as the change induced by keeping fixed the $\omega_{a,0}$ and scaling the $\alpha_0 \to C\alpha_0$ for some meromorphic $\mathcal{O}((\Lambda R)^0)$ function C. We will also use the latter kind of reparameterization, which is not of the kind (3.7), without explicitly invoking the former one, which induces it, and it is of the kind (3.7).

Step 1 Truncate $\mathcal{F}$ at a power LR which eventually we will assume to be $\Lambda^{n+1}R^{n+1}$, e.g., $n = 2$, as follows

$$\mathcal{F} \to \mathcal{F}_\Lambda := \mathcal{F}_{sl}^{(LR)^{\frac{1}{n+1}}}[x,u,v,w,t,y;e^{2\pi \mathrm{i} u_i}] + \sum_{j=1}^{LR}\sum_{\rho\neq 0}\frac{e^{2\pi \mathrm{i} j\rho(u)}}{j}\,. \tag{3.14}$$

$\mathcal{F}_{sl}^{\Lambda}$ was defined in (2.24). The truncation (3.14) is justified because we are interested in probing the physics of states with charges below the energy scale $\mathcal{O}(1)\Lambda^{n+1}R^n$, as $\Lambda R\to\infty$, forgetting about *heavier* states. We will comeback below to further illustrate the physical relevance of this truncation. [25]

The u_i's, with $i = 1,\ldots,N$, are the N gauge potentials. They are related to the N eigenvalues U_i of the unitary matrix U as follows $U_i = e^{2\pi \mathrm{i} u_i}$. The term

$$\sum_{j=1}^{LR}\sum_{\rho\neq 0}\frac{e^{2\pi \mathrm{i} j\rho(u)}}{j} \tag{3.15}$$

is the truncated contribution coming from the Vandermonde determinant. $\rho\neq 0$ denotes the non-vanishing adjoint weights of $U(N)$, namely $\sum_{\rho\neq 0} = \sum_{i\neq j=1}^{N}$ and $\rho(u) = u_i - u_j$. The goal is to compute the effective off-shell potential for the u_i's at large-ΛR . The next step is to extremize the effective potential with respect to the u_i's and find its leading saddle point. We already know that at any order in the low-temperature expansion

$$u_i = u_i^\star = \mathcal{O}\Big(\frac{1}{\Lambda R}\Big)\,. \tag{3.16}$$

This is because the effective potential for the u_i's equals the one of the superconformal index at very leading order in the $\frac{1}{\Lambda R}$-expansion and all β , by definition. That is, at $\Lambda R = \infty$, $\alpha = \frac{1}{2}$ and the partition function truncates to the index which does not depend on β .

(3.16) is enough to obtain the saddle-point approximation to $\mathcal{F}$ at order $\mathcal{O}(\Lambda^n R^{n-1})$ and $\mathcal{O}(\Lambda^{n-1}R^{n-2})$. To compute the exact saddle point approximation to $\mathcal{F}$ at order $\mathcal{O}(\Lambda^{n-2}R^{n-3})$ it is necessary to compute the $\mathcal{O}(\frac{1}{\Lambda R})$. We will develop all necessary tools to compute all such perturbative corrections to the gauge saddle point. That said, $\mathcal{O}(\Lambda^{n-2}R^{n-3})$ corrections to $\mathcal{F}$ (and all its subleading analytic corrections), depend on the choice of moduli $\mu_i^{(p)}$, e.g., we can always choose a moduli representative for which these corrections vanish. Thus, in order to compare to semiclassical gravity the question we need to answer is whether there exists a limit that is isomorphic to the one studied in supergravity [28]. To address that question, (3.16) is enough. To

[25] In concrete expansions it is convenient to think of the cutoff LR as independent of ΛR while expanding the summand at large-ΛR. Then one obtains an effective potential for the u_i 's. Then one can extemize such potential at $LR \gg 1$ (at this stage it is already safe to take $LR = \infty$) and find all the perturbative corrections to the saddle-point $u^\star$ in the $\frac{1}{\Lambda R}$-expansion. Corrections coming from the dependence of LR on ΛR will be exponentially suppressed, not perturbatively suppressed, and thus for the purposes of this paper they are not essential.

find features of quantum gravity (beyond higher derivative corrections) within the gauge theory, subleading analytic and non-analytic corrections to $u^\star$ at large ΛR are needed. Computing those lies beyond the scope of this paper. We plan to address this question in forthcoming work. For the present exposition it will be enough to work with (3.16).

Step 2 Substitute the relation between chemical potentials and the scale Λ (3.2) in $\mathcal{F}_\Lambda$.

Step 3 Expand the summand around

$$(\Lambda R) \,=\, \infty\,. \tag{3.17}$$

(ignoring the implicit dependence on ΛR in the auxiliary potentials) and then expand each term in the precedent expansion around

$$\beta_0 \,=\, \infty\,, \tag{3.18}$$

keeping the other dimensionless quantities finite.

The result of this two-step expansion has the form [26]

$$\begin{aligned} &\mathcal{O}((\Lambda R)^n) + \ldots + \mathcal{O}((\Lambda R)^0) \\ &\qquad\qquad + \mathcal{O}((\Lambda R)^{-1}) \\ &\qquad\qquad\quad + \mathrm{Li}_1^\Lambda\Big(1+\mathcal{O}(\frac{1}{\Lambda R})\Big) - \text{contributions} \\ &\qquad\qquad\qquad + \mathrm{Li}_{p<0}^\Lambda\Big(1+\mathcal{O}(\frac{1}{\Lambda R})\Big) - \text{contributions} \\ &\qquad\qquad\qquad\quad + \mathrm{Li}_{p<0}^\Lambda\Big(x\Big) - \text{contributions}\,. \end{aligned} \tag{3.19}$$

where

$$\mathrm{Li}_p^\Lambda(z) := \sum_{j=1}^{LR} \frac{z^j}{j^p}\,. \tag{3.20}$$

The monomials with powers of order

$$\mathcal{O}((\Lambda R)^n)\,,\,\ldots\,,\mathcal{O}((\Lambda R)^0) \tag{3.21}$$

entering in this expansion we denote as *Type I* contributions. The monomials with powers

$$\mathcal{O}((\Lambda R)^{-1})\,. \tag{3.22}$$

[26]Each individual (series) term in this expansion that may potentially diverge in the limit $\Lambda \to \infty$ we compute, whenever it is possible, in the region of its arguments $\{\varphi_v, \varphi_w, u_i\}$ where its limit converges, and then we analytically extend such finite answer to the complex domain. Some singularities, like the power-like or the $\log(\Lambda)$ ones will be physical and of course, it will not be possible to remove them using the previous regularization trick. There is a more elegant and general way to compute this asymptotic expansion, allowing to obtain also exact expressions at finite Λ, e.g., in the spirit of [92]. In this paper we will not try to compute exponentially suppressed D-instantonic contributions, for us it will be enough to use the pragmatic method above enunciated which leads to the analytic tail and allows to conclude that the reminder is a combination of either logarithmic singularities (independent of temperature) and exponentially suppressed terms. The finite-Λ completion of $\mathcal{F} - \mathcal{F}_\infty$, will be addressed in future work.

we denote as *Type II* contributions. Type I and II contributions to $\mathcal{F}$ can be further organized in powers of α_0. Their powers up to order $\mathcal{O}(\alpha_0^4)$ we collect in

$$\mathcal{F}_\infty := \mathcal{O}((\Lambda R)^n) + \ldots + \mathcal{O}((\Lambda R)^{-1}) . \tag{3.23}$$

In particular, the powers $\mathcal{O}(\alpha_0^3)$ are of order

$$\mathcal{O}((\Lambda R)^0) , \mathcal{O}((\Lambda R)^{-1}) . \tag{3.24}$$

The powers $\mathcal{O}(\alpha_0^4)$ are only of Type II

$$\mathcal{O}((\Lambda R)^{-1}) . \tag{3.25}$$

In this expansion of $\mathcal{F}$, there are no powers of order higher or equal than $\mathcal{O}(\alpha_0^5)$ entering in Type I and Type II contributions. All such contributions turn out to be exponentially suppressed around $\Lambda R = \infty$. Indeed, we advance that all perturbative contributions to $\mathcal{F}$ beyond Type I and Type II vanish in this expansion. This is, all possible corrections to $\mathcal{F}$ beyond Type I and Type II are either logarithmic (i.e. non meromorphic) or exponentially suppressed at $\Lambda R = \infty$, as we will proceed to explain below. The conclusion will be that in the $\Lambda R \to \infty$ expansion

$$\mathcal{F} = \mathcal{F}_\infty + \text{log-corrections} + \text{exp-suppresed corrections} . \tag{3.26}$$

Before moving on to explain this, let us note that there are Casimir energy-like terms, i.e. contributions that grow as β_0 around $\beta_0 = \infty$, in this expansion. We collect them in a term denoted as

$$\beta_0 \, \mathcal{E}_0 = \beta \mathcal{E} , \qquad \mathcal{E} := \Lambda \mathcal{E}_0 . \tag{3.27}$$

These contributions enclose Type I contributions of order $\mathcal{O}(\alpha_0 \Lambda^0)$, and Type II contributions of order $\mathcal{O}(\alpha_0^2 \Lambda^{-1})$. There are no other contributions emerging at powers higher that β_0 around $\beta_0 = \infty$ in this expansion. [27]

Summarizing, all Type I and Type II contributions can be organized in the form ($n = 2$)

$$\mathcal{F}_\infty = \beta_0 \, \mathcal{E}_0 + \sum_{p=-1}^{n} \sum_{q=0}^{4} \sum_{r=0}^{\infty} (\Lambda R)^p \, L_{p+1;q,r}[\varphi_v, \varphi_w, \underline{u}] \, \frac{F_{p;q;r} \alpha_0^q}{(\beta_0)^r \omega_{1,0} \omega_{2,0}} \tag{3.28}$$

where the $F_{p;q;r}$ are N-dependent homogenous polynomials of order $(-p-q+r+2) \geq 0$ in the variables $\omega_{1,0}$ and $\omega_{2,0}$. Some subsets of them vanish, e.g., $F_{2;3;r} = F_{1;3;r} = F_{2;4;r} = F_{1;4;r} = F_{0;4;r} = 0$ as it was noted in the previous paragraph. Also $F_{p;0;r\geq 1} = F_{p;q\geq 1;0} = 0$. The $L_{p+1;q;r}$ denotes a linear and finite combination of regularized poly-Logs Li^{Λ}_{p+1} of order $p+1$ combined symmetrically into periodic Bernoulli polynomials

[27] What happens is that all of them are exponentially suppressed, and are of the type IV and V to be defined below.

of the same order. [28] The details of formula (3.28) up to order $\mathcal{O}(\frac{1}{\beta})$ are reported in equation (D.1). A Mathematica notebook testing the derivation of (3.28) has been shared.

We note that all Type II contributions are proportional to linear combinations of terms of the form

$$\mathrm{Li}_0^{\Lambda}(e^{\varphi}) + \mathrm{Li}_0^{\Lambda}(e^{-\varphi}), \tag{3.29}$$

which, after using analytic continuation, converge exponentially fast as $\Lambda \to \infty$ to

$$\mathrm{Li}_0(e^{\varphi}) + \mathrm{Li}_0(e^{-\varphi}) = -1\,. \tag{3.30}$$

Let us pause and reiterate an important partial conclusion:

– Contributions which are not organized in $\mathcal{F}_{\infty}$, are defined to be either *Type III*, *Type IV*, or *Type V*. They are either logarithmic corrections or exponentially suppressed corrections. This is, Type I and Type II contributions encode all the possible perturbative corrections to $\mathcal{F}^{\Lambda}$ in the large-ΛR expansion just enunciated. [29] Let us proceed to explain this. The detailed answer is given in appendix D.

The Type III-contributions

$$\mathrm{Li}_1^{\Lambda}\Big(1+\mathcal{O}(\frac{1}{\Lambda R})\Big) - \text{contributions} \tag{3.31}$$

come only from $\mathcal{O}(\alpha_0^0)$, i.e., from the superconformal index $\mathcal{I}_1$. Some of these contributions come from the N zero modes $\rho = 0$. When combined they become proportional to a linear combination of terms of the form [30]

$$\mathrm{Li}_1^{\Lambda}\big(1-\omega_{1,2}\big) \to -\log \omega_{1,2}\,, \tag{3.32}$$

and half of

$$-\mathrm{Li}_1^{\Lambda}\big(1-\omega_1-\omega_2\big) \to +\log\Big(\omega_1+\omega_2\Big), \tag{3.33}$$

with $\omega_{1,2} = \frac{\omega_{1,2;0}}{\Lambda R}$. These are physical $\log(\Lambda)$ divergencies. They should correspond to logarithmic quantum corrections in the bulk. As they come solely from $\mathcal{O}(\alpha^0)$ contributions (the BPS ones) then they are independent on β. There are other logarithmic divergencies associated to middle-dimensional walls of non-analyticities [31].

[28] For the superconformal index, $r = q = 0$, these truncation of the perturbative expansion in the large-Λ-expansion up to logarithmic and exponentially suppressed contributions is implicit in previous results and expectations, at least in the simplest case $\omega_1 = \omega_2$ [31, 32, 38, 45, 49]. (3.28) is the generalization of tis previous observation from the index to the partition function.

[29] Recall that we are ignoring the arbitrary dependence on ΛR implicit in the auxiliary potentials. Such dependence can generate infinitelly many subleading $\mathcal{O}(\frac{1}{\Lambda R})$ corrections but at the moment we are ignoring corrections coming in that way. They will be essential to consider in due time though.

[30] Their expressions, using a specific regularization scheme have been given in equations (3.64)+(3.65) of [30].

We will not study these logarithmic contributions in this paper, but we advance that they are also independent of β , as expected.

The Type IV contributions

$$\mathrm{Li}^{\Lambda}_{p<0}\Big(1+\mathcal{O}(\frac{1}{\Lambda R})\Big) - \text{contributions} \tag{3.34}$$

organize in linear combinations of terms of the form ($y=\mathcal{O}(\frac{1}{\Lambda R})$)

$$\mathrm{Li}^{\Lambda}_{-p}(1+y) \,+\, (-1)^p\,\mathrm{Li}^{\Lambda}_{-p}(1-y)\,, \qquad p\,\geq\,1\,, \tag{3.35}$$

which, using analytic continuation, converge exponentially fast as $\Lambda R\to\infty$ to

$$\mathrm{Li}_{-p}(1+y) \,+\, (-1)^p\,\mathrm{Li}_{-p}(1-y) \,=0\,. \tag{3.36}$$

The Type V contributions organize in linear combinations of terms of the form

$$\mathrm{Li}^{\Lambda}_{-p}(e^{\Phi}) \,+\, (-1)^p\,\mathrm{Li}^{\Lambda}_{-p}(e^{-\Phi})\,, \qquad p\,=\,0\,,\,1\,,\,\ldots \tag{3.37}$$

which, using analytic continuation, converge exponentially fast as $\Lambda R\to\infty$ to

$$\mathrm{Li}_{-p}(e^{\Phi}) \,+\, (-1)^p\,\mathrm{Li}_{-p}(e^{-\Phi}) \,=\, 0\,, \tag{3.38}$$

where Φ can be one of the elements in the list $\{\varphi_v\,,\,\varphi_w\,,\,-\varphi_v-\varphi_w\,\}$ added to $\pm 2\pi i u_i$'s.

Step 3 Substitute the dimensionless auxiliary parameters back in terms of the physical quantities

$$\beta_0\,\to\,\mathfrak{b}\Lambda_1\,,\quad \alpha_0\,\to\,\mathfrak{a}\Lambda_2\,,\quad \omega_{1,0}\,\to\,\mathfrak{w}_1\Lambda_3\,,\quad \omega_{2,0}\,\to\,\mathfrak{w}_2\Lambda_4\,. \tag{3.39}$$

The obtained expression for the complete perturbative asymptotic expansion of the free energy, in terms of the physical chemical potentials

$$\mathcal{F}_\infty[\frac{\mathfrak{b}}{R},\frac{1}{2}+\frac{\mathfrak{a}}{R},\frac{\mathfrak{w}_1}{R}\,,\,\frac{\mathfrak{w}_2}{R}\,,\ldots]\,=\,\mathcal{F}_\infty[\beta\,,\,\alpha\,,\omega_1,\omega_2\,,\ldots]\,, \tag{3.40}$$

extends naturally to the physical low-temperature region

$$\frac{\mathfrak{b}}{R}\,=\,\beta\,\gg\,1\,. \tag{3.41}$$

Step 4. **The proposal:** We propose that the holographic low-temperature expansion of the Gibbons-Hawking free energy (in minimal gauged-supergravity) is $\mathcal{F}_\infty$

$$\begin{aligned}\mathcal{F}_g[\beta_g\,,\,\alpha_g\,,\omega_{g1},\omega_{g2}] \equiv{}& \mathcal{F}_\infty[\beta\,,\,\alpha\,,\omega_1,\omega_2\,,\,\varphi_v\,,\,\varphi_w;\underline{u}^\star] \\ &+\Big(\lambda=\infty \text{ meromorphic corrections}\Big)\,,\end{aligned} \tag{3.42}$$

where $\varphi_v\,=\varphi_v[\beta,\alpha,\omega_1,\omega_2]\,=\,\varphi_w$ is fixed by the zero R-charge condition $R_1=R_2=0$, and $u^\star\,=u^\star[\beta,\alpha,\omega_1,\omega_2,\varphi_v,\varphi_w]$ is fixed by the Gauss-constraint. The expansion of $\mathcal{F}_\infty$ up to $\mathcal{O}(\frac{1}{\beta})$ is reported in (D.1).

It should be recalled that there is an arbitrary implicit dependence on ΛR in the auxiliary potentials $\underline{\mu}$ entering in the $\{\beta, \alpha, \omega_1, \omega_2\}$, as detailed in (3.2), and after imposing the natural holographic dictionary relation [31]

$$\beta_g \doteq \beta\,, \quad \alpha_g \doteq \alpha\,, \quad \omega_{g1} \doteq \omega_1\,, \quad \omega_{g2} \doteq \omega_2\,. \tag{3.43}$$

In the bulk, the infinitelly many choices of functions $\underline{\mu} = \underline{\mu}[\Lambda R]$, which are implicit in the right-hand side of (3.42), correspond to different ways to reach the BPS limit $\alpha = \frac{1}{2}$ [23]. We will comeback to comment on this below.

The identification $\equiv$ is understood as an equality up to $\mathcal{O}((\Lambda R)^0)$ ambiguities in the choice of holographic renormalization scheme in the gravitational side [93] (the left-hand side). These ambiguities include the holographic dual of the field-theoretic Casimir factor (3.27). In the field theory side these ambiguities can be understood as deformations of the moduli (3.5)

$$\mu_i^{(p)} \to \mu_i^{(p)} = \mu_i^{(p)}\Big(1 + \mathcal{O}(\frac{1}{\Lambda R})\Big)\,, \qquad p \geq 1\,, \tag{3.44}$$

that generate $\mathcal{O}((\Lambda R)^0)$ deformations of the free energy $\mathcal{F}_\infty$. An explicit example will be given in (5.11). There, the order $\mathcal{O}\Big((\Lambda R)^0\Big)$ Casimir energy prefactor (3.27) will be removed by one such deformation.

Our goal in the following sections will be to compare the right hand side of this proposal against the left hand side in the near-BPS region. One conclusion of our analysis will be that the

$$\Big(\lambda = \infty \text{ meromorphic corrections}\Big) \tag{3.45}$$

vanish in the near BPS region.

Gibbons-Hawking action from an effective free-field computation: supersymmetry and analyticity Supersymmetry implies that the strong coupling corrections

$$\Big(\lambda = \infty \text{ meromorphic corrections}\Big) \tag{3.46}$$

must be subleading in the $\frac{1}{\Lambda R}$-expansion above introduced. Otherwise, the index $\mathcal{I}_1$ would receive $\lambda = \infty$ corrections, in order to match the BPS limit of the Gibbons-Hawking onshell action [23, 35, 36]. The expansion of the gravitational onshell about the BPS point has only meromorphic contributions. Thus, any strong coupling correction within it must be not only subleading in the $\frac{1}{\Lambda R}$-expansion, but also meromorphic in the chemical potentials.

On the other hand, any possible meromorphic and subleading corrections in the $\frac{1}{\Lambda R}$-expansion above is bound to be encoded in a change in the moduli space of limits to the BPS locus

$$\mu_i^{(p)} \to \mu_{i,\lambda=\infty}^{(p)} = \mu_i^{(p)}\Big(1 + \mathcal{O}(\frac{1}{\Lambda R})\Big) \tag{3.47}$$

[31] The symbol $\doteq$ denotes equal up to periodic relation implied by quantization of charges i.e. relations such as $\alpha \leftrightarrow \alpha + j$ for any integer j.

or equivalently, in a chemical-potential dependent redefinition of the cutoff Λ (3.11). That is predicting that strong coupling corrections within the gravitational onshell action are bound to be reproducible from an effective free-gauge theory computation, where the effect of the coupling is to renormalize the chemical potentials [32]

$$\mathcal{F}_g[\beta_g\,,\,\alpha_g\,,\omega_{g1},\omega_{g2}]\equiv\mathcal{F}_\infty[\beta^{\lambda=\infty}\,,\,\alpha^{\lambda=\infty}\,,\omega_1^{\lambda=\infty},\omega_2^{\lambda=\infty}]\,, \tag{3.48}$$

with

$$\beta^{\lambda=\infty}\;=\;\beta(1+\mathcal{O}(\frac{1}{\Lambda R}))\,,\;\alpha^{\lambda=\infty}\;=\;\alpha(1+\mathcal{O}(\frac{1}{\Lambda R}))\,,\;\omega_{1,2}^{\lambda=\infty}\;=\;\omega_{1,2}(1+\mathcal{O}(\frac{1}{\Lambda R}))\,,$$

and keeping the holographic dictionary, – the chemical potentials –, defined at zero-gauge coupling, or equivalently, independent of the coupling. Relation (3.48), which is bound to be true in virtue of analyticity and correcteness of the supergravity predictions, calls for deeper understanding in the field theory side. In the following sections we will test it at order $\mathcal{O}(\frac{1}{\beta})$.

The continuum limit in microcanonical picture As previously announced, the parameter Λ is the energy scale controlling the extension

$$L\;=\;\Lambda^{n'}R^{n'-1} \tag{3.49}$$

of a domain of charges (states) that we are interested in doing physics at. When ΛR is large enough, then the distance among contiguous charge eigenvalues:

$$\delta L\;=\;\frac{1}{R} \tag{3.50}$$

becomes, necessarily

$$\frac{\delta L}{L}\;=(\Lambda R)^{-n'}\ll 1 \tag{3.51}$$

and thus the spectrum becomes effectively continuous.

But, where is this proposed scaling of L with Λ coming from? In the spirit of the large-charge localization approach discussed in [30], we can ask for the hierarchy of charges for which the truncated free energy can be safely localized to its asymptotic behaviour within the singular vicinity

$$\beta\;\sim\;\frac{\beta_0}{\Lambda R}\,,\qquad\alpha\;\sim\;\frac{1}{2}\;+\;\frac{\alpha_0}{\Lambda R}\,,\qquad\omega_1\;\sim\;\frac{\omega_{1,0}}{\Lambda R}\,,\qquad\omega_2\;\sim\;\frac{\omega_{2,0}}{\Lambda R}\,, \tag{3.52}$$

defined by the double expansion

$$\Lambda R\;\gg\;1\qquad\text{and}\qquad\beta_0\;\gg\;1\,, \tag{3.53}$$

which is consistent with the hierarchy

$$\Lambda\;\gg\;\frac{\beta_0}{R}\;\gg\;\frac{|\alpha_0|}{R}\,,\;\frac{|\omega_{1,0}|}{R}\,,\;\frac{|\omega_{2,0}|}{R}\,, \tag{3.54}$$

[32] Or equivalently, to a shift of source terms.

and for which the following dimensionless quantities are kept finite

$$|\alpha_0|\,,|\omega_{1,0}|\,,|\omega_{2,0}|\;=\;\mathcal{O}(1)\,,\qquad \frac{|\alpha_0|}{|\omega_{1,0}|}\;=\;\mathcal{O}(1)\,,\qquad \frac{|\omega_{1,0}|}{|\omega_{2,0}|}\;=\;\mathcal{O}(1)\,. \tag{3.55}$$

In terms of the dimensionful chemical potentials (3.54) looks like

$$R\gg\mathfrak{b}\gg\mathfrak{w}_1\,,\,\mathfrak{w}_2\,,\,\mathfrak{a}\,. \tag{3.56}$$

In the expansion (3.54) we will find (in the following sections) that the free energy happens to scale as

$$\Lambda^n R^{n-1} \tag{3.57}$$

where the scaling power $n\;=\;1$ or 2, is determined by the theory and by how much supersymmetry is preserved by the states whose contributions to the partition function do not cancel in the limit $\alpha\;\to\;\frac{1}{2}$. The charges, instead, scale as

$$\Lambda^{n'}R^{n'-1} \tag{3.58}$$

where the positive integer n' is fixed by demanding that the source-term of the corresponding charge, scales as the free energy near its leading singularity. More concretely, if the charges generate isometries in the S_3 then $n'\;=\;n+1$, if not, then $n'\;=\;n$. [33]

For example, the n''s associated to the charges

$$\sqrt{2}J_{\pm}\;:=\;J_1^3\pm J_2^3+\frac{R_1+R_3}{2}+\frac{\Delta}{3}\;=\;\frac{\Delta_{\pm}}{2}-\frac{\Delta}{6}\,, \tag{3.59}$$

are fixed as $n'=n+1$ by demanding that their source terms (in the ensemble described in table 4)

$$\omega_1\sqrt{2}J_-\qquad\text{and}\qquad\omega_2\sqrt{2}J_+ \tag{3.60}$$

scale as

$$\delta L^{\frac{1}{n'}}L^{\frac{n'-1}{n'}}\;\sim\;\frac{L}{\Lambda R}\;\sim\;\Lambda^n\,R^{n-1}\,, \tag{3.61}$$

in the continuum limit

$$\frac{\delta L}{L}\;\to\;0\,. \tag{3.62}$$

From the scaling of chemical potentials (3.52) and that of free energy as $\Lambda R\;\to\;\infty$ (at any R), there follows the definning properties of the hierarchy of charges associated to its singularity (3.52)

$$\begin{aligned}\sqrt{2}J_{\pm}\;&=\;\mathcal{O}((\Lambda R)^0)\,L_{J_{\pm}}\,,\\ R_3\;&=\;\mathcal{O}((\Lambda R)^0)\,L_{J_{\pm}}\,,\\ R_{1,2}\;&=\;\mathcal{O}((\Lambda R)^0)\,L_{R_{1,2}}\,,\\ \Delta\;&=\;\mathcal{O}((\Lambda R)^0)\,L_{\Delta}\,,\end{aligned} \tag{3.63}$$

[33] Recall that due to twisting some R-charges can also generate rotations in S^3.

where the scales determining the extension of the truncations are

$$
\begin{aligned}
L_{J_\pm} &= N^2\,\Lambda^{n+1}R^n\,,\\
L_{R_{1,2}} &= N^2\,\Lambda^n R^{n-1}\,,\\
L_\Delta &= N^2\,|\alpha_0|\frac{\Lambda^{n+1}R^n}{\beta_0^2}\,.
\end{aligned}
\tag{3.64}
$$

These are the image domains obtained by Legendre-dualizing the singular vicinity (3.52). Namely, this will come from the computations presented in the forthcoming sections, following the Steps 1 to Step 4 above described.

Note that Δ, the semi-positive charge that measures distance from the $\mathcal{Q}$, and $\mathcal{S}$ supersymmetric locus ($\Delta = 0$), may be also large in the expansion (3.54), although much smaller than $J_\pm$. This means that the expansion (3.54) also probes states that are not necessarily close to be $\mathcal{Q}$ and $\mathcal{S}$-supersymmetric. [34]

That said, we note that at first order in the auxiliary temperature $\mathcal{O}(\frac{1}{\beta_0}) = \mathcal{O}(\frac{\Lambda R}{\beta})$

$$
L_\Delta = 0\,. \tag{3.65}
$$

This implies that contributions at first order in temperature to the free energy can be recovered working solely with the BPS partition function ($\Delta = 0$). This is, with the Hamiltonian trace definition of partition function withouth the $(-1)^F$ inserted over the space of BPS states $\Delta = 0$. [35] To go beyond $\mathcal{O}(\frac{1}{\beta})$ the complete non-BPS partition function is necessary.

An alternative understanding of this RG flow proposal using the microscopic picture just described, will be presented in upcoming work.

A comment on conventions Following the holographic dictionary we find natural to identify [23]

$$
R = \ell_{AdS_5} = \frac{1}{g}\,. \tag{3.66}
$$

From now on we work in natural units and fix

$$
R = 1 \implies \mathfrak{b} = \beta\,. \tag{3.67}
$$

The many ways to reach the BPS locus As explained before, there are infinitely many ways to deform the RG-flow limits (3.52). These deformations correspond to changes in the choice of functions (3.5) $\underline{\mu}^{(p)}[\underline{\mu}(0)]$. This is, the auxiliary chemical potentials $\underline{\mu}$ can have arbitrary implicit dependence on Λ, as long as the $\underline{\mu}$ obey the boundary conditions

[34] The large-charge expansion studied in [30] only supersymmetric states contribute ($\Delta = 0$).

[35] This confirms the expectations of [94] in AdS_4/CFT_3. Our analysis concerns AdS_5/CFT_4 but we expect an analogous RG-flow reduction will be implemented in AdS_4/CFT_3, eventually.

declared in (3.4),

$$\begin{aligned}\beta_0 &\to \widetilde{\beta}_0\,(1 + \frac{\widetilde{\beta}^{(1)}}{\Lambda} + \ldots) \\ \alpha_0 &\to \widetilde{\alpha}_0\Big(C_{0,0} + \frac{C_{1,0}}{\widetilde{\beta}_0} + \frac{\widetilde{\alpha}_0}{\widetilde{\beta}_0}C_{1,1} + \ldots\Big) \\ \omega_{a,0} &\to \widetilde{\omega}_{a,0}\Big(1 + \frac{\widetilde{\omega}_a^{(1)}}{\Lambda} + \ldots\Big)\end{aligned} \tag{3.68}$$

[36] with

$$\begin{aligned}&C_{0,0} \,=\, C_{0,0}(\omega_a) \,=\, \mathcal{O}(\Lambda^0)\,, \\ &C_{1,0} \,=\, C_{1,0}(\omega_a) = \mathcal{O}(\Lambda^{-1})\,, \quad C_{1,1} \,=\, C_{1,1}(\omega_a) \,=\, \mathcal{O}(\Lambda^{-1})\,,\end{aligned} \tag{3.69}$$

as $\Lambda = \infty$. As we will illustrate below, given

$$\beta \,=\, \frac{\widetilde{\beta}_0}{\Lambda}\,, \qquad \alpha \,=\, \frac{1}{2} + \frac{\widetilde{\alpha}_0}{\Lambda}\,, \qquad \omega_1 \,=\, \frac{\widetilde{\omega}_{1,0}}{\Lambda}\,, \qquad \omega_2 \,=\, \frac{\widetilde{\omega}_{2,0}}{\Lambda}\,, \tag{3.70}$$

the dependence of the free energy $\mathcal{F}$ on the physical variables

$$\beta\,, \quad \alpha\,, \quad \omega_1\,, \quad \omega_2\,, \tag{3.71}$$

in the large-Λ, large-$\widetilde{\beta}_0$ expansion at fixed $\{\widetilde{\alpha}_0, \widetilde{\omega}_{1,0}, \widetilde{\omega}_{2,0}\}$ (called Expansion 2) changes with respect to one obtained via the large-Λ, large-β_0 expansion (3.54) at fixed $\{\alpha_0, \omega_{1,0}, \omega_{2,0}\}$ (called Expansion 1).

Active transformation trick The same change in the choice of limits, in the sense of (3.5) i.e. choices of $\underline{\mu}^{(p)}[\underline{\mu}^{(0)}]$, can be implemented also by redefining chemical potentials. Take one of them, for instance α, and redefine it as follows

$$\alpha \,=\, C\,(\widetilde{\alpha} - \frac{1}{2}) \,+\, \frac{1}{2}. \tag{3.72}$$

Then localize the free energy which is at this point a function of the tilded chemical potentials, around the small vicinities

$$\widetilde{\beta} \sim \frac{\beta_0}{\Lambda}\,, \quad \widetilde{\alpha} \sim \frac{1}{2} + \frac{\alpha_0}{\Lambda}\,, \quad \widetilde{\omega}_1 \sim \frac{\omega_{1,0}}{\Lambda}\,, \qquad \widetilde{\omega}_2 \sim \frac{\omega_{2,0}}{\Lambda}\,, \tag{3.73}$$

where $C = C(\widetilde{\alpha})$ is the origin preserving reparameterization defined as

$$\begin{aligned}C \,&=\, C_{0,0} \,+\, \frac{C_{1,0}}{\beta_0} \,+\, \frac{\alpha_0}{\beta_0}\,C_{1,1} \,+\, \ldots \\ &=\, C_{0,0} \,+\, \frac{1}{\widetilde{\beta}}\,C_{1,0} \,+\, \frac{(\widetilde{\alpha} - \frac{1}{2})}{\widetilde{\beta}}\,C_{1,1} \,+\, \ldots\,,\end{aligned} \tag{3.74}$$

[36] Even the case $C_{0,0} \underset{\Lambda\to\infty}{\to} c \neq 1$ can be induced by a redefinition of the functions $\omega_{1,2}$ as explained in footnote 24.

(with the same functions $C_{i,j}$'s as in (3.68)) and write its expansion in terms of the tilded physical potentials. At last, replace back

$$\widetilde{\alpha} \to \alpha\,, \qquad \widetilde{\beta} \to \beta\,, \qquad \ldots \tag{3.75}$$

(i.e. drop the tildes). The answer obtained for $\mathcal{F}_\infty$ after this procedure, as a function of the variables without tildes $\{\alpha, \beta, \omega_1, \omega_2\}$, is the same one obtained after implementing the Expansion 2 in the physical chemical potentials $\{\alpha, \beta, \omega_1, \omega_2\}$ (i.e. without modifying the background geometry, and the backgound flavour potentials, in which the field theory is quantized).

This trick will be used to go from the results obtained with a reference large-Λ expansion to the results obtained with another expansion. We stress, though, that the chemical potentials of the fundamental theory will remain the ones to be identified with the gravitational ones. The abstract object that changes with different choices is the way to approach the BPS locus, not the identification of chemical potentials between the fundamental theory and the near horizon (gravitational) one.

Expansions to roots-of-unity Before moving on we point out there are many possible limits that we forsee may be relevant in forthcoming developments. They are limits to roots of unity

$$\beta \sim \frac{\beta_0}{\Lambda} + \mathfrak{r}_1\,, \quad \alpha - \frac{1}{2} \sim \frac{\alpha_0}{\Lambda} + \mathfrak{r}_2\,, \quad \omega_1 \sim \frac{\omega_{1,0}}{\Lambda} + \mathfrak{r}_3\,, \quad \omega_2 \sim \frac{\omega_{2,0}}{\Lambda} + \mathfrak{r}_4\,, \tag{3.76}$$

where

$$\mathfrak{r}_1\,,\, \mathfrak{r}_2\,,\, \mathfrak{r}_3\,,\, \mathfrak{r}_4 \in \mathbb{Q}\,, \tag{3.77}$$

and, again, the auxiliary chemical potentials $\underline{\mu}$ can have arbitrary implicit dependence on Λ, as long as it respects the boundary conditions at $\Lambda \to \infty$ imposed by the ansatz (3.4). In those limits one also obtains Schwarzian contributions with mass gap parameter depending, generically, on $\mathfrak{r}_{1,2,3,4}$. We leave the study of this for the future.

4 The near-1/8-BPS Schwarzian mass gap

As mentioned before the leading saddle-point of the Gauss-constraint is $U \sim e^{2\pi i u^\star} \times 1_{N\times N}$ (details on this are presented in appendix A). Thus, from now on

$$\mathrm{Tr}U^n \mathrm{Tr}U^{-n} \to N^2 + \mathcal{O}\Big(\frac{\alpha_0}{\Lambda}, \frac{\beta_0}{\Lambda}, \frac{\alpha_0}{\beta_0}, \frac{\omega_{1,0}}{\beta_0}, \frac{\omega_{2,0}}{\beta_0}\Big)\,, \tag{4.1}$$

where

$$\mathcal{O}\Big(\frac{\alpha_0}{\Lambda}, \frac{\beta_0}{\Lambda}, \frac{\alpha_0}{\beta_0}, \frac{\omega_{1,0}}{\beta_0}, \frac{\omega_{2,0}}{\beta_0}\Big) \tag{4.2}$$

stands for corrections that are first-order in at least one of the small dimensionless parameters in the expansion (3.52).

To compute low-temperature corrections of the free energy $\mathcal{F} = \mathcal{F}_{\frac{1}{16}\text{near}\frac{1}{8}}$ we focus on the asymptotic behaviour of its Taylor coefficients around the point of cancellations $\alpha = \frac{1}{2}$

$$\mathcal{F}^{(p)} := \frac{1}{p!}\,\partial_\alpha^p \mathcal{F}\Big|_{\alpha=\frac{1}{2}}\,, \qquad p = 0\,,\,1\,,\,2\,. \tag{4.3}$$

[37] In the naive zero-temperature limit $\beta \to \infty$ all the $\mathcal{F}^{(p\geq 1)}_{\frac{1}{16}\text{near}\frac{1}{8}}$ vanish exponentially fast. Instead, in the holographic low-temperature expansion obtained after implementing (3.52), or more precisely after following the Steps 1 to 4 in 3.1, we obtain [38]

$$
\begin{aligned}
\mathcal{F}^{(0)}_{\frac{1}{16}\text{near}\frac{1}{8}} &= -N^2\,\frac{\Lambda}{\omega_{1,0}}\,L_2(\varphi_{\widetilde{v}}) + \mathcal{O}(\Lambda^0) \,\sim\, -N^2\,\frac{L_2(\varphi_{\widetilde{v}})}{\omega_1} \\
\mathcal{F}^{(1)}_{\frac{1}{16}\text{near}\frac{1}{8}} &= \pi \mathrm{i} N^2\,\frac{\Lambda^2}{\beta_0\omega_{1,0}}\,\Big(1 \,+\, \mathcal{O}\Big(\frac{\omega_{1,0}}{\beta_0},\frac{\omega_{2,0}}{\beta_0}\Big)\Big)L_2(\varphi_{\widetilde{v}}) \,+\, \mathcal{O}(\Lambda) \\
&\sim\, \pi \mathrm{i} N^2\,\frac{1}{\beta\,\omega_1}\,\Big(1 \,+\, \mathcal{O}\Big(\frac{\omega_1}{\beta},\frac{\omega_2}{\beta}\Big)\Big)\,L_2(\varphi_{\widetilde{v}})\,, \\
\mathcal{F}^{(2)}_{\frac{1}{16}\text{near}\frac{1}{8}} &= \pi^2 N^2\frac{\Lambda^2}{\beta_0\omega_{1,0}}\Big(1+\mathcal{O}\Big(\frac{\omega_{1,0}}{\beta_0},\frac{\omega_{2,0}}{\beta_0}\Big)\Big)\,L_1(\varphi_{\widetilde{v}}) \,+\, \mathcal{O}(\Lambda) \\
&\sim\, \pi^2 N^2\frac{1}{\beta\omega_1}\Big(1+\mathcal{O}\Big(\frac{\omega_1}{\beta},\frac{\omega_2}{\beta}\Big)\Big)\,L_1(\varphi_{\widetilde{v}})
\end{aligned}
\tag{4.4}
$$

where

$$
\begin{aligned}
L_1(\varphi_{\widetilde{v}}) &:= -\,\mathrm{Li}_1^{\Lambda}\left(e^{\varphi_{\widetilde{v}}}\right) \,+\, \mathrm{Li}_1^{\Lambda}\left(e^{-\varphi_{\widetilde{v}}}\right)\,, \\
L_2(\varphi_{\widetilde{v}}) &:= -\,2\,\mathrm{Li}_2^{\Lambda}(1) \,+\, \mathrm{Li}_2^{\Lambda}\left(e^{-\varphi_{\widetilde{v}}}\right) \,+\, \mathrm{Li}_2^{\Lambda}\left(e^{\varphi_{\widetilde{v}}}\right)\,.
\end{aligned}
\tag{4.5}
$$

The equivalence relation $\sim$ indicates that the quantities in the left-hand and right-hand sides are equal in the $1/\Lambda$-expansion up to the lowest order in Λ obtained after expanding the right-hand side, but it assumes nothing about the asymptotic behaviour in the large-β_0 expansion. For completeness we note that the $\mathcal{O}(\Lambda^0)$ term in $\mathcal{F}^{(1)}_{\frac{1}{16}\text{near}\frac{1}{8}}$ [39] is proportional to

$$
-\frac{L_1(\varphi_{\widetilde{v}})}{2}\,. \tag{4.6}
$$

For later reference we note that the $\mathcal{O}(\Lambda)$ term in $\mathcal{F}^{(1)}_{\frac{1}{16}\text{near}\frac{1}{8}}$ is

$$
N^2\frac{\pi \mathrm{i} L_1(\varphi_{\widetilde{v}})}{2\beta} + \frac{\pi \mathrm{i} L_1(\varphi_{\widetilde{v}})}{\omega_1}\,. \tag{4.7}
$$

C was defined in (3.74). Below we will comment more on it. The term linear in temperature, is essential to compute subleading corrections to the mass gap, not for the leading ones we are looking to compute in this subsection. After extremization with respect to $\varphi_{\widetilde{v}}$, the contribution of this term vanishes exponentially fast as $\Lambda \to \infty$.

The functions L_2, and L_1 can be recast as combinations of periodic Bernoulli polynomials. Such representation can be straightforwardly recovered by expanding the most general answer (D.1), evaluated at (3.16)) and

$$
\varphi_w \,=\, \omega_1 \,-\, \varphi_v\,, \qquad \varphi_v \,=\, \varphi_{\widetilde{v}} \,+\, 2\pi \mathrm{i}(\alpha - \frac{1}{2})\,. \tag{4.8}
$$

[37]However, in the end they are irrelevant as they can be always removed by an allowed deformation of the large-Λ expansion.

[38]These expansions can be computed directly by expanding (D.1). We have chosen to explain how they can be derived from scratch. They can be also derived with the shared Mathematica file.

[39]Ignoring logarithmic contributions.

at large Λ.

Collecting these expansions, we find that the free energy (2.48) grows as $\mathcal{O}(\Lambda^1)$ and (up to corrections of order $\mathcal{O}(\frac{1}{\beta^2})$)

$$\begin{aligned} \mathcal{F}_{\frac{1}{16}\text{near}\frac{1}{8}} \,=\, \mathcal{F}_{\infty} \,=\, & -N^2\,\frac{1}{\omega_1}\,L_2(\varphi_{\widetilde{v}}) \\ & + \pi \mathrm{i} N^2\, C\,\frac{(\widetilde{\alpha}-\frac{1}{2})}{\beta\,\omega_1}\,\Big(1\,+\,\mathcal{O}\Big(\frac{\omega_1}{\beta},\frac{\omega_2}{\beta}\Big)\Big)\,L_2(\varphi_{\widetilde{v}})\,+\,\mathcal{O}(\Lambda^0)\,, \end{aligned} \tag{4.9}$$

where C has been defined in (3.74). A more complete expression of (4.9) up-to quadratic order in $\widetilde{\alpha}-\frac{1}{2}$ and with C expanded as indicated in (3.74) is given in (B.22).

Note, that the first asymptotic correction in temperature to the free energy $\mathcal{F}_{\frac{1}{16}\text{near}\frac{1}{8}}$ is

$$\pi \mathrm{i} N^2\, C\,\frac{(\widetilde{\alpha}-\frac{1}{2})}{\beta\,\omega_1}\,\Big(1\,+\,\mathcal{O}\Big(\frac{\omega_1}{\beta},\frac{\omega_2}{\beta}\Big)\Big)\,L_2(\varphi_{\widetilde{v}})\,, \tag{4.10}$$

which is, essentially, a Schwarzian contribution in grand-canonical ensemble. Let us show this.

In the mixed ensemble defined by taking the Legendre transform, i.e., after extremizing

$$-\mathcal{F}_{\frac{1}{16}\text{near}\frac{1}{8}}\,+\,(\text{Some of the source terms in Table 3}) \tag{4.11}$$

with respect to the chemical potentials $\varphi_{\widetilde{v}}$ under the condition that the charge obtained after such transform

$$R_{\widetilde{v}}\,:=\,-R_1-R_2=N^2\,\frac{\delta}{\omega_1}\,,\qquad \delta\,\approx\,0\,,\quad \delta\,\neq\,0 \tag{4.12}$$

remains <u>independent of the chemical potentials $\{\widetilde{\alpha}\,,\,\beta\}$</u>, we obtain

$$\begin{aligned} \varphi_{\widetilde{v}}\,&\sim\,\varphi^{\star}_{\widetilde{v}}\left(1+\frac{(\alpha-\frac{1}{2})\,(\delta C_{0,0})}{\beta}\right)+O\left(\delta^1\right) \\ \frac{\varphi^{\star}_{\widetilde{v}}}{2\pi\mathrm{i}}\,&=\,\frac{1}{2}\,\mathrm{mod}\,1\,, \end{aligned} \tag{4.13}$$

where we assume

$$C_{0,0}\,=:\,\frac{\chi_0}{\delta}\,, \tag{4.14}$$

with χ_0 a constant independent of δ.

As a result of the intermediate extremization just mentioned we obtain (keeping only leading terms in the large-Λ and small-δ expansion)

$$-\mathcal{F}_{eff}\,:=\,-\mathcal{F}_0\,-\,\frac{2(C_{0,0}\delta)\,\mathrm{i}\,(\widetilde{\alpha}-\frac{1}{2})}{\pi}\mathcal{F}_0\,-\,\frac{C_{0,0}\pi\mathrm{i}\,(\widetilde{\alpha}\,-\frac{1}{2})}{\beta}\mathcal{F}_0\,-\,\frac{C_{1,0}\,\pi\mathrm{i}\,(\widetilde{\alpha}-\frac{1}{2})^2}{\beta}\mathcal{F}_0 \tag{4.15}$$

where

$$\mathcal{F}_0\,:=\,N^2\,\frac{\pi^2}{2\omega_1}\,. \tag{4.16}$$

Then we proceed to compute the mixed-ensemble free energy (up to order $\mathcal{O}(\Lambda^0)$) [28]

$$\begin{aligned} -I_{\rm ME}(\beta, J_+, J_-, \widetilde{\alpha}) \; &= \; S_{\frac{1}{16}\,\text{near}\,\frac{1}{8}} \\ &:= \; \underset{\varphi_{\widetilde{v}},\omega_1,\omega_2}{\text{ext}} \left(-\mathcal{F}_{\frac{1}{16}\text{near}\frac{1}{8}} \, + \, \omega_1 \sqrt{2} J_- \, + \, \omega_2 \sqrt{2} J_+ \, + \, \varphi_{\widetilde{v}} R_{\widetilde{v}} \right) \\ &= \; \underset{\omega_1,\omega_2}{\text{ext}} \left(-\mathcal{F}_{\text{eff}} \, + \, \omega_1 \sqrt{2} J_- \, + \, \omega_2 \sqrt{2} J_+ \right) \, + \, \mathcal{O}(\delta^1) \, . \end{aligned} \tag{4.17}$$

At leading order, this functional is independent of ω_2 . This imposes a further constraint on charges – on top of the ones declared in equation (3.60) –

$$\sqrt{2} J_+ \; := \; J_1^3 + J_2^3 + \frac{R_1 + R_3}{2} \, + \, \frac{\Delta}{3} \; = \; \frac{\Delta_+}{2} \, - \, \frac{\Delta}{6} \; = \; - \, \frac{\Delta}{6} \; = \; 0 \, + \, \mathcal{O}(\frac{1}{\beta^2}) \tag{4.18}$$

The next-to-last equation follows because by definition $\mathcal{I}_2$ only counts states with $\Delta_+ = 0$. The first corrections in ω_2 come at the order we have reincorporated in the right-hand side.

Next, we enforce the physical charge

$$\sqrt{2} J_- \; := \; J_1^3 - J_2^3 + \frac{R_1 + R_3}{2} \, + \, \frac{\Delta}{3} \; = \; \frac{\Delta_-}{2} \, - \, \frac{\Delta}{6} \; = \; \frac{\Delta_-}{2} \, + \, \mathcal{O}(\frac{1}{\beta^2}) \tag{4.19}$$

to be

$$\sqrt{2} J_- \; = \; \sqrt{2} J_-^\star \, + \, \mathcal{O}(\frac{1}{\beta^2}), \tag{4.20}$$

with $J_-^\star$ being a fixed value independent of $\{\alpha, \beta\}$. Under this constraint, the extremization procedure (4.17) fixes

$$\omega_1 \; = \; \omega_1^\star \left(1 \, - \, \frac{i\pi \left(\alpha - \frac{1}{2}\right) C_{0,0}}{2\beta} \, + \, \frac{i \left(\alpha - \frac{1}{2}\right) C_{0,0} \delta}{\pi} \, + \, \mathcal{O}\Big((\widetilde{\alpha} - \frac{1}{2})^2 \Big) \right) \, + \, \mathcal{O}(\delta^1) \, , \tag{4.21}$$

where $\omega_1^\star$ is a function of the extremal charges $J_\pm^\star$ fixed by the auxiliary extremization problem

$$\underset{\omega_1^\star}{\text{ext}} \left(-\mathcal{F}_0[\omega_1^\star] \, + \, \omega_1^\star \sqrt{2} J_-^\star \right) \; = \; \pm \sqrt{2} \pi \sqrt{-\sqrt{2} J_-^\star} \; = \; - \frac{\pi^2}{\omega_1^\star} \, . \tag{4.22}$$

Collecting results we obtain (up to order $\mathcal{O}(\Lambda^0)$)

$$S_{\frac{1}{16}near\frac{1}{8}} \; = \; S_0 \, + \, 2\pi \mathrm{i} \widetilde{\alpha} R_0 \, - \, \frac{8\pi^2}{M} \frac{\left(\widetilde{\alpha} - \frac{1}{2}\right) \, + \, \mathcal{O}\Big((\widetilde{\alpha} - \frac{1}{2})^2 \Big)}{\beta} \, + \, \mathcal{O}(\frac{1}{\beta^2}) \, , \tag{4.23}$$

with

$$\begin{aligned} \frac{S_0}{N^2} \; &= \; \frac{2\pi^2}{2\omega_1^\star} \, - \, \frac{i\pi C_{0,0} \delta}{2\omega_1^\star}, \\ \frac{R_0}{N^2} \; &= \; - \frac{C_{0,0} \delta}{2\omega_1^\star}, \\ \frac{1}{MN^2} \; &= \; \frac{C_{0,0} \pi}{16 i \omega_1^\star} \, . \end{aligned} \tag{4.24}$$

Now we move on to impose the reality conditions that select the *physical* near BPS-limit

$$\mathrm{Im}(R_{\widetilde{v}}) = \mathrm{Im}(J_-) = \mathrm{Im}(S_0) = \mathrm{Im}(R_0) = \mathrm{Im}(M) \; = \; 0\,. \tag{4.25}$$

The semi-positivity condition

$$\Delta_- \; \geq \; 0\,, \tag{4.26}$$

implies (omiting $\mathcal{O}(\frac{1}{\beta_0^2})$)

$$-\sqrt{2}J_- \;=\; -\sqrt{2}J_-^\star \;=\; -\,\Delta_- \;\leq\; 0 \;\Longrightarrow\; \mathrm{Re}\Big(\frac{1}{\omega_1^\star}\Big) \;=\; 0\,. \tag{4.27}$$

Together with this identity,

$$\mathrm{Im}(R_{\widetilde{v}}) = 0 \;\;\Longrightarrow\;\; \mathrm{Re}(\delta) \;=\; 0\,. \tag{4.28}$$

Together with the previous conclusions

$$\mathrm{Im}(R_0) = \mathrm{Im}(M) \;=\; 0 \;\Longrightarrow\; \mathrm{Im}(C_{00}) = \mathrm{Re}(\chi_0) \;=\; 0\,. \tag{4.29}$$

Together with the previous conclusions

$$\mathrm{Im}(S_0) \;=\; 0 \;\Longrightarrow\; \chi_0 \;=\; -2\pi\mathrm{i}\,. \tag{4.30}$$

Collecting conclusions, we obtain (at leading order in the large-Λ, large β_0 expansion)

$$\begin{aligned} S_0 \;&=\; 0\,, \\ R_0 \;&=\; \mp\, N\sqrt{2\Delta_-}\,, \\ \frac{1}{M} \;&=\; \pm\Big(\frac{\pi}{4\sqrt{2}\mathrm{i}\delta}\Big)\, N\sqrt{\Delta_-} \;=\; N^2\,\frac{4R_{\widetilde{v}}}{J_-}\,. \end{aligned} \tag{4.31}$$

The signs in the second and third line are correlated [40]. If i$\delta \; > \; 0$ (resp. < 0) we have picked up the $+$ (resp. $-$) in the third line, in such a way $M \; > \; 0$, but the other choice is a priori equally relevant, as it will become evident in the more general case that will be analyzed in the following section. The two sign choices in (4.31) correspond to the two saddle points of (4.22).

At last, we update (4.17) with $S_0 \,=\, 0$. Then substituting $\widetilde{\alpha} \to \alpha$ we obtain (up to order-Λ^0)

$$S_{\frac{1}{16}near\frac{1}{8}} \;=\; 2\pi\mathrm{i}\alpha R_0 \;-\; \frac{8\pi^2}{M}\,\frac{\big(\alpha-\frac{1}{2}\big)\;+\;(\alpha-\frac{1}{2})^2}{\beta} \;+\; \mathcal{O}\big(\frac{1}{\beta^2}\big)\,. \tag{4.32}$$

This is the effective low-temperature infrared action consistent with reality of charges and BPS entropy. The latter being constrained to vanish in this case.

In the last line of (4.32) we have reinstated the canonical $\mathcal{O}\big((\alpha-\frac{1}{2})^2\big)$ contribution to $S_{\frac{1}{16}\mathrm{near}\frac{1}{8}}$, up to order $\frac{1}{\beta}$, which is subleading at large-Λ, i.e. $\mathcal{O}(\Lambda^0)$, and comes from a repetition of the procedure above reported considering $C_{1,0} \neq 0$.

Chemical potential	Dual charge	Source term
β	Δ	$+\beta\Delta$
ω_1	$J_1^3 - J_2^3 + \frac{R_1}{2} + R_2 + \frac{R_3}{2} + \frac{\Delta}{3}$	$+\omega_1 \cdot \left(J_1^3 - J_2^3 + \frac{R_1}{2} + R_2 + \frac{R_3}{2} + \frac{\Delta}{3}\right)$
ω_2	$J_1^3 + J_2^3 + \frac{R_1}{2} + \frac{R_3}{2} + \frac{\Delta}{3}$	$+\widetilde{\omega}_2 \cdot \left(J_1^3 + J_2^3 + \frac{R_1}{2} + \frac{R_3}{2} + \frac{\Delta}{3}\right)$
$\varphi_{\widetilde{v}}$	$-R_1 - R_2$	$+\varphi_{\widetilde{v}} \cdot (-R_1 - R_2)$
$\omega_u = -2\pi i\alpha$	$-R_3$	$+\omega_u \cdot (-R_3)$

Table 3. The chemical potentials, charges and source terms that can be potentially added to $-\mathcal{F}_{\frac{1}{16}\text{near}\frac{1}{8}}$.

Partial remarks (4.27) is predicting that in the region of charges (4.33) near-$\frac{1}{8}$-BPS black holes within the family of [69], have near vanishing horizon and entropy. This result is consistent with recent expectations [76]. Our analysis, however, only covers the case [41]

$$\frac{R_1}{R_2} \to -1\,. \tag{4.33}$$

when two out of the three independent R-charges (the ones to be indentified with electric charges of the dual gravitational solutions) Q_1, Q_2, and Q_3 (defined in (C.2)) are equal. We do not see any complication though in repeating our analysis in the more general region of charges, but we leave doing so for future work. Also, being fair, our analysis does not exclude the existence of $\frac{1}{8}$-BPS black holes which can not be continuously recovered from the $\frac{1}{16}$-BPS ones of [58, 69].

We should note also that the mass-gap of the Schwarzian mode goes to infinity in the limit

$$M = \frac{J_-}{4N^2 R_{\widetilde{v}}} \to \infty\,, \qquad R_{\widetilde{v}} \to 0\,. \tag{4.34}$$

Thus the Schwarzian becomes irrelevant signaling the vanishing of the horizon.

It would be interesting to test (4.34) in supergravity. Relatedly, it would be also interesting to visualize what happens to the horizon of the dual gravitational solutions in [69] when one approaches the vicinity

$$\Delta\,,\, \Delta_+\,,\, R_1 + R_2 = 0 \tag{4.35}$$

of their moduli space. Note that the dual gravitational solutions relevant for this analysis can not be embedded in minimally gauged supergravity in five dimensions because they have two different electric charges.

As an unsurprising consistency check, in appendix B we recover the free energy (4.9) starting from the more general near-$\frac{1}{16}$-BPS computation, which we move on to study next. Such a match confirms the selection of the gauge saddle point (4.1). [42]

[40] Top with top, and bottom with bottom.

[41] At $\delta = 0$ this covers the case of two equal R-charges $Q_1 = Q_3$ and a different third one Q_2, and the case of three equal R-charges $Q_1 = Q_2 = Q_3$ (using the definitions given in equation (C.2)).

[42] From these results ond should be able to understand whether the feature of vanishing mass gap continues to hold if one approaches the $\frac{1}{8}$ BPS locus in generic ways, not only along the index $\mathcal{I}_2$. We postpone such an analysis for future work and move on to study the generic near-BPS expansion with our proposal.

5 The near-1/16-BPS Schwarzian mass gap

Let us test the proposed RG-flow procedure against known predictions in gravity. Let us start from the maximally refined partition function (2.23)

$$Z[x\,,\, u = e^{2\pi \mathrm{i}\alpha}\,,\, v\,,\, w\,,\, t,\, y] = e^{-\mathcal{F}}\,. \tag{5.1}$$

At $\alpha = \frac{1}{2}$ this partition function reduces to the $\frac{1}{16}$-BPS index $\mathcal{I}_1$. To compute corrections of $\mathcal{F}$, again, we focus on the asymptotic behaviour of its Taylor coefficients around the point of supersymmetric cancellations $\alpha = \frac{1}{2}$

$$\mathcal{F}^{(p)} \;:=\; \frac{1}{p!}\partial_\alpha^p \mathcal{F}\Big|_{\alpha=\frac{1}{2}}\,, \qquad p \;=\; 0\,,\,1\,,\,2\,. \tag{5.2}$$

[43] In the naive zero-temperature limit $\beta\; \to\; \infty$ all the $\mathcal{F}^{(p\geq 1)}$ vanish exponentially fast. In the holographic low-temperature expansion obtained after implementing (3.52), or more precisely after following the Steps 1 to 4. in 3.1, one finds [44]

$$\begin{aligned}
\mathcal{F}^{(0)} \;&=\; -\,N^2\,\frac{\Lambda^2}{\omega_{1,0}\omega_{2,0}}\,L_3(\varphi_v,\varphi_w) + \mathcal{O}(\Lambda)\;\sim\; -\,N^2\,\frac{L_3(\varphi_v,\varphi_w)}{\omega_1\omega_2}\\
\mathcal{F}^{(1)} \;&=\; \pi \mathrm{i} N^2\,\frac{\Lambda^3}{\beta_0\omega_{1,0}\omega_{2,0}}\left(1\,+\,\mathcal{O}\Big(\frac{\omega_{1,0}}{\beta_0},\frac{\omega_{2,0}}{\beta_0}\Big)\right)L_3(\varphi_v,\varphi_w)\,+\,\mathcal{O}(\Lambda^2)\\
&\sim\; \pi \mathrm{i} N^2\,\frac{1}{\beta\,\omega_1\omega_2}\left(1\,+\,\mathcal{O}\Big(\frac{\omega_1}{\beta},\frac{\omega_2}{\beta}\Big)\right)L_3(\varphi_v,\varphi_w)\,,\\
\mathcal{F}^{(2)} \;&=\; \pi^2 N^2\frac{\Lambda^3}{\beta_0\omega_{1,0}\omega_{2,0}}\left(1+\mathcal{O}\Big(\frac{\omega_{1,0}}{\beta_0},\frac{\omega_{2,0}}{\beta_0}\Big)\right)L_{2,0}(\varphi_v,\varphi_w)\,+\,\mathcal{O}(\Lambda^2)\\
&\sim\; \pi^2 N^2\frac{1}{\beta\omega_1\omega_2}\left(1+\mathcal{O}\Big(\frac{\omega_1}{\beta},\frac{\omega_2}{\beta}\Big)\right)L_{2,0}(\varphi_v,\varphi_w)\,,
\end{aligned} \tag{5.3}$$

where

$$\begin{aligned}
L_{2,0}(\varphi_v,\varphi_w) :=&\; +2\mathrm{Li}_2^\Lambda(1)\,+\mathrm{Li}_2^\Lambda\left(e^{-\varphi_v}\right)+\mathrm{Li}_2^\Lambda\left(e^{\varphi_v}\right)+\mathrm{Li}_2^\Lambda\left(e^{-\varphi_w}\right)+\mathrm{Li}_2^\Lambda\left(e^{\varphi_w}\right)\\
&-3\mathrm{Li}_2^\Lambda\left(e^{\varphi_v+\varphi_w}\right)-3\mathrm{Li}_2^\Lambda\left(e^{-\varphi_v-\varphi_w}\right)\\
L_3(\varphi_v,\varphi_w) :=&\; \mathrm{Li}_3^\Lambda\left(e^{-\varphi_v}\right)-\mathrm{Li}_3^\Lambda\left(e^{\varphi_v}\right)+\mathrm{Li}_3^\Lambda\left(e^{-\varphi_w}\right)-\mathrm{Li}_3^\Lambda\left(e^{\varphi_w}\right)\\
&+\mathrm{Li}_3^\Lambda\left(e^{\varphi_v+\varphi_w}\right)-\mathrm{Li}_3^\Lambda\left(e^{-\varphi_v-\varphi_w}\right)\,.
\end{aligned} \tag{5.4}$$

The $\mathcal{O}(\Lambda)$ term in $\mathcal{F}^{(0)}$ is

$$\sim\; N^2\,\frac{1}{2}\,\frac{\omega_1\,+\,\omega_2}{\omega_1\omega_2}\,L_{2,1}(\varphi_v,\varphi_w) \tag{5.5}$$

and the $\mathcal{O}(\Lambda^2)$ term in $\mathcal{F}^{(1)}$ is

$$\sim\; -\,\pi \mathrm{i} N^2\frac{1}{\omega_1\omega_2}L_{2,0}(\varphi_v,\varphi_w)\;-\;\pi \mathrm{i} N^2\,\frac{\omega_1+\omega_2}{2\beta\omega_1\omega_2}\,L_{2,1}(\varphi_v,\varphi_w)\,+\,\mathcal{O}\big(\frac{1}{\beta^2}\big) \tag{5.6}$$

[43] The other relevant coefficients $p=3,4$ contribute only at order $\mathcal{O}(\Lambda^0)$ and $\mathcal{O}(\Lambda^{-1})$ and can be removed by a trivial redefinition of limits.

[44] These expansions can be computed directly by expanding (D.1). We have chosen to explain how they can be derived from scratch. They can be also derived with the shared Mathematica file.

where

$$L_{2,1}(\varphi_v,\varphi_w) := 2\,\mathrm{Li}_2^{\Lambda}(1) - \mathrm{Li}_2^{\Lambda}\left(e^{-\varphi_v}\right) - \mathrm{Li}_2^{\Lambda}\left(e^{\varphi_v}\right) - \mathrm{Li}_2^{\Lambda}\left(e^{-\varphi_w}\right) - \mathrm{Li}_2^{\Lambda}\left(e^{\varphi_w}\right) \\ + \mathrm{Li}_2^{\Lambda}\left(e^{-\varphi_v-\varphi_w}\right) + \mathrm{Li}_2^{\Lambda}\left(e^{\varphi_v+\varphi_w}\right) . \tag{5.7}$$

The functions L_3, $L_{2,0}$ and $L_{2,1}$ can be recast as combinations of periodic Bernoulli polynomials. Such representation can be straightforwardly recovered by expanding the most general answer (D.1) (evaluated at (3.16)) at large Λ.

With these expansions we find

$$\mathcal{F} \sim -N^2\,\frac{1}{\omega_1\omega_2}\,L_3(\varphi_v,\varphi_w) + \pi \mathrm{i} N^2\,C\,\frac{\left(\widetilde{\alpha}-\frac{1}{2}\right)}{\beta\,\omega_1\omega_2}\left(1 + \mathcal{O}\Big(\frac{\omega_1}{\beta},\frac{\omega_2}{\beta}\Big)\right) L_3(\varphi_v,\varphi_w)\,, \tag{5.8}$$

after substituting α by the re-parameterization choice (3.72). Subleading corrections in $\frac{1}{\Lambda}$ expansion to $\mathcal{F}$ can be recovered reinstating the contribution coming from $\mathcal{F}^{(2)}$ in (4.4), and the order $\mathcal{O}(\Lambda^2)$ contribution in $\mathcal{F}^{(1)}$ in (5.3). Or equivalently, considering only leading corrections and then applying the substitution rule

$$L_3(\varphi_v,\varphi_w) \to L_3(\varphi_v,\varphi_w) - \frac{\omega_1 + \omega_2}{2}\,L_{2,1}(\varphi_v,\varphi_w) + O(\Lambda^{-2}) \tag{5.9}$$

on (5.8). Ignoring logarithmic contributions and spurious c-numbers, the missing $\mathcal{O}(\Lambda^0)$ to the free energy, or the missing $\mathcal{O}(\Lambda^{-2})$ contributions in (5.9), have origin in

$$\begin{aligned} \mathcal{F}^{(0)}: &\qquad N^2\frac{\pi \mathrm{i}\left(\omega_1^2+\omega_2^2\right)}{12\omega_1\omega_2} + \ldots, \\ \mathcal{F}^{(1)}: &\qquad -N^2\frac{\pi^2\beta}{3\omega_1\omega_2} - N^2\frac{\pi\left(\omega_1+\omega_2\right)\left(8\pi-3i\left(\varphi_v+\varphi_w\right)\right)}{3\omega_1\omega_2} \\ &\qquad -N^2\frac{\pi^2\left(\omega_1^2+3\omega_2\omega_1+\omega_2^2\right)}{12\beta\omega_1\omega_2} + \ldots . \end{aligned} \tag{5.10}$$

where the $\ldots$ denote contributions coming from the subleading analytic corrections to the gauge saddle point values $u_i^{\star}$. The contributions at order $\mathcal{O}(\Lambda^0)$ induced by (5.10) on the onshell action $\mathcal{F}_{\infty}$ can be always removed by a convenient choice of representative in the following family of redefinitions of limits to the BPS locus $\alpha=\frac{1}{2}$, which for obvious reasons we feel inclined to call a *choice of counterterms* [45]

$$\omega_{1,2} \to \omega_1\Bigg(1 + \frac{\gamma_1\omega_1\omega_2}{2} + \frac{\gamma_2(\alpha-\frac{1}{2})\omega_1\omega_2}{2\beta} + \frac{\gamma_3\left(\alpha-\frac{1}{2}\right)\beta}{2} + \gamma_4\omega_{2,1}^2 + \frac{\gamma_5\left(\alpha-\frac{1}{2}\right)\omega_{2,1}^2}{\beta} \\ + \gamma_6\big(\alpha-\frac{1}{2}\big)\omega_{2,1} + \gamma_7\big(\alpha-\frac{1}{2}\big)\omega_{2,1}\left(\varphi_v+\varphi_w\right) + \gamma_8\,\omega_{2,1}^3 + \gamma_9\,\omega_{2,1}^2\omega_{1,2}\Bigg) . \tag{5.11}$$

This reparameterization of the infrared limit only generates changes in the asymptotic form of the free energy at order $\mathcal{O}(\Lambda^0)$ and below. It will be used to fix a convenient reference

[45] Notice for instance that it may be used to remove a Casimir energy-like term like the one explicited in (5.10).

form for the $\mathcal{O}(\Lambda^0)$ and $\mathcal{O}(\Lambda^{-1})$ contributions to $\mathcal{F}$ (up to order $\mathcal{O}(\alpha - \frac{1}{2})$)

$$\frac{\eta_1(\omega_1^2+\omega_2^2)\,+\,\eta_2\omega_1\omega_2+\eta_3(\omega_1^3+\omega_2^3)+\eta_4(\omega_1^2\omega_2+\omega_1\omega_2^2)}{\omega_1\omega_2}\left(1\,-\,\pi\mathrm{i}\,C_{0,0}\frac{(\widetilde{\alpha}-\frac{1}{2})}{\beta}\right). \tag{5.12}$$

Coming back to (5.8). Note that the first correction in temperature to the asymptotic expansion of $\mathcal{F}$ (5.8) is

$$\pi\mathrm{i}N^2\,C\,\frac{(\widetilde{\alpha}-\frac{1}{2})}{\beta\,\omega_1\omega_2}\left(1\,+\,\mathcal{O}\Big(\frac{\omega_1}{\beta},\frac{\omega_2}{\beta}\Big)\right)L_3(\varphi_v,\varphi_w)\,, \tag{5.13}$$

which is, essentially, a Schwarzian contribution in grand-canonical ensemble. Let us show this.

Chemical potential	Dual charge	Source term
β	Δ	$+\beta\Delta$
ω_1	$J_1^3-J_2^3+\frac{R_1}{2}+\frac{R_3}{2}+\frac{\Delta}{3}$	$+\omega_1\cdot\big(J_1^3-J_2^3+\frac{R_1}{2}+\frac{R_3}{2}+\frac{\Delta}{3}\big)$
ω_2	$J_1^3+J_2^3+\frac{R_1}{2}+\frac{R_3}{2}+\frac{\Delta}{3}$	$+\omega_2\cdot\big(J_1^3+J_2^3+\frac{R_1}{2}+\frac{R_3}{2}+\frac{\Delta}{3}\big)$
φ_v	$-R_1$	$+\varphi_v\cdot(-R_1)$
φ_w	$-R_2$	$+\varphi_w\cdot(-R_2)$
$\omega_u\,=\,-2\pi\mathrm{i}\alpha$	$-R_1-R_3$	$+\omega_u\cdot(-R_1-R_3)$

Table 4. The chemical potentials, charges and source terms to be added to $-\mathcal{F}$ before performing extremization.

In the mixed ensemble defined by extremizing

$$-\mathcal{F}\,+\,(\text{Some of the source terms in Table 4}) \tag{5.14}$$

with respect to the chemical potentials (φ_v,φ_w) under the condition

$$R_1\,=\,R_2\,=\,0 \tag{5.15}$$

we obtain

$$\varphi_v=\varphi_w\,\sim\,\pm\frac{2\pi\mathrm{i}}{3}\,+\,\frac{1}{3}\,(\omega_1+\omega_2)\,+\,C\,\frac{4\pi\mathrm{i}}{3}\left(\widetilde{\alpha}-\frac{1}{2}\right)\left(1-\frac{(\omega_1+\omega_2)}{4\beta}\right)\,+\,\mathcal{O}\Big((\widetilde{\alpha}-\frac{1}{2})^2\Big)\,. \tag{5.16}$$

As a result of such intermediate extremization and fixing

$$\eta_1=\mp\frac{\pi\mathrm{i}}{9}\,,\,\eta_2\,=\,\mp\frac{2\pi\mathrm{i}}{9}\,,\,\eta_3\,=\,-\frac{1}{54}\,,\,\eta_4\,=\,-\frac{1}{18}\,, \tag{5.17}$$

we obtain

$$-\mathcal{F}_\infty\,=\,-\mathcal{F}_0\,-C\,\left(\widetilde{\alpha}-\frac{1}{2}\right)\mathcal{F}_{0,1}\,+\,\pi\mathrm{i}C\frac{(\widetilde{\alpha}\,-\,\frac{1}{2})}{\beta}\mathcal{F}_0\,+\,\mathcal{O}\Big((\widetilde{\alpha}-\frac{1}{2})^2\Big) \tag{5.18}$$

where

$$\mathcal{F}_0\,=\,-\,N^2\frac{(\pm2\pi\mathrm{i}+\omega_1+\omega_2)^3}{54\,\omega_1\omega_2}\,,\qquad \mathcal{F}_{0,1}\,=\,-N^2\frac{4\pi^3\mathrm{i}}{9\omega_1\omega_2} \tag{5.19}$$

and C was defined in (3.74). Then we proceed to evaluate [28]

$$
\begin{aligned}
-I_{\mathrm{ME}}(\beta, J_-, J_+, \widetilde{\alpha}) \;&=\; S \\
&:= \underset{\varphi_v,\varphi_w,\omega_1,\omega_2}{\mathrm{ext}} \left(-\mathcal{F} \,+\, \omega_1\sqrt{2}J_- \,+\, \omega_2\sqrt{2}J_+ \right) \\
&= \underset{\omega_1,\omega_2}{\mathrm{ext}} \left(-\mathcal{F}_\infty \,+\, \omega_1\sqrt{2}J_- \,+\, \omega_2\sqrt{2}J_+ \right),
\end{aligned} \tag{5.20}
$$

by enforcing $J_\pm$ to be

$$
\begin{aligned}
\sqrt{2}J_- \;&=\; \sqrt{2}J_-^\star \,+\, \mathcal{O}(\alpha^2) \,+\, \mathcal{O}(\frac{1}{\beta^2}), \\
\sqrt{2}J_+ \;&=\; \sqrt{2}J_+^\star \,+\, \mathcal{O}(\alpha^2) \,+\, \mathcal{O}(\frac{1}{\beta^2}),
\end{aligned} \tag{5.21}
$$

where $J_\pm^\star$ are fixed values, independent of $\{\alpha, \beta\}$, as it was required in the near BPS limit flow used in [28] (see appendix D in that reference). Under this constraint, the extremization procedure (5.20) fixes (with the choice $\pm$ in (5.19))

$$
\begin{aligned}
\omega_1 \;&=\; \omega_1^* \,\mp\, C_{0,0}\,(\widetilde{\alpha} - \frac{1}{2})\omega_1^* \,+ \\
&\quad+\; \frac{(\widetilde{\alpha} - \frac{1}{2})\,\omega_1^* \left(\pm\frac{1}{6}C_{0,0}\,(\omega_1^* - 2\omega_2^* \mp 2\pi\mathrm{i}) + \frac{\mathrm{i}C_{1,0}\left(\omega_1^* - 2\omega_2^* \pm \pi\mathrm{i}\right)}{\pi} \right)}{6\beta} \\
&\quad+\; \mathcal{O}(\Lambda^{-2}), \\
\omega_2 \;&=\; \omega_2^* \,\mp\, C_{0,0}(\widetilde{\alpha} - \frac{1}{2})\omega_2^* \\
&\quad+\; \frac{\omega_2^*\,(\widetilde{\alpha} - \frac{1}{2}) \left(\pm\frac{1}{6}C_{0,0}\,(-2\omega_1^* + \omega_2^* \mp 2\pi\mathrm{i}) + \frac{\mathrm{i}C_{1,0}\left(-2\omega_1^* + \omega_2^* \pm \pi\mathrm{i}\right)}{\pi} \right)}{\beta} \\
&\quad+ \mathcal{O}(\Lambda^{-2}),
\end{aligned} \tag{5.22}
$$

where the $\omega_1^\star$ and $\omega_2^\star$ are functions of the extremal charges $J_\pm^\star$. These functions are fixed by the auxiliary *minimal supergravity-like extremization problem* (not to confuse with (5.20)) [95]

$$
\begin{aligned}
&\underset{\omega_1^\star,\omega_2^\star}{\mathrm{ext}} \left(-\mathcal{F}_0[\omega_1^\star, \omega_2^\star] \,+\, \omega_1^\star\sqrt{2}J_-^\star \,+\, \omega_2^\star\sqrt{2}J_+^\star \right) \\
&= \underset{\omega^\star{}_1,\omega^\star{}_2}{\mathrm{ext}} \left(\frac{N^2(\pm 2\pi\mathrm{i} + \omega_1^\star + \omega_2^\star)^3}{54\omega_1^\star\omega_2^\star} \,+\, \omega_1^\star\sqrt{2}J_-^\star \,+\, \omega_2\sqrt{2}J_+^\star \right).
\end{aligned} \tag{5.23}
$$

Collecting the results so far we write down the onshell value of the physical functional (5.20) in Schwarzian-like form

$$
S \;=\; S_0 \,+\, 2\pi\mathrm{i}\widetilde{\alpha}R_0 \,-\, \frac{8\pi^2}{M}\frac{(\widetilde{\alpha} - \frac{1}{2}) \,+\, \mathcal{O}\left((\widetilde{\alpha} - \frac{1}{2})^2\right)}{\beta} \,+\, \mathcal{O}\left(\frac{1}{\beta^2}\right), \tag{5.24}
$$

with

$$
\begin{aligned}
S_0 &= \mp\left(\frac{\mathrm{i}N^2\pi(2\pi\mp\mathrm{i}(\omega_1^\star+\omega_2^\star))^2}{9\omega_1^\star\omega_2^\star}\right)-\left(\frac{2\mathrm{i}N^2\pi^3}{9\omega_1^\star\omega_2^\star}\right)C_{0,0}\\
R_0 &= +\left(\frac{2N^2\pi^2}{9\omega_1^\star\omega_2^\star}\right)C_{0,0}\\
\frac{1}{M} &= \pm\frac{N^2(2\pi\mp\mathrm{i}(\omega_1^\star+\omega_2^\star))^3}{432\pi\omega_1^\star\omega_2^\star}C_{0,0}-\left(\frac{\mathrm{i}N^2\pi}{18\omega_1^\star\omega_2^\star}\right)C_{1,0}\,.
\end{aligned}
\tag{5.25}
$$

The solution for the $\omega^\star$'s as functions of $J_\pm$ can be written in the implicit form [23, 28]

$$
\begin{aligned}
\omega_1^\star &= \frac{2(1-a^\star)(b^\star\mp\mathrm{i}\sqrt{a^\star+b^\star+a^\star b^\star})\pi}{2(1+a^\star+b^\star)\sqrt{a^\star+b^\star+a^\star b^\star}\mp 2\mathrm{i}(a^\star+b^\star+a^\star b^\star)}\,,\\
\omega_2^\star &= \frac{2(1-b^\star)(a^\star\mp\mathrm{i}\sqrt{a^\star+b^\star+a^\star b^\star})\pi}{2(1+a^\star+b^\star)\sqrt{a^\star+b^\star+a^\star b^\star}\mp 2\mathrm{i}(a^\star+b^\star+a^\star b^\star)}\,,
\end{aligned}
\tag{5.26}
$$

for charges parameterized as follows [58]

$$
\begin{aligned}
\sqrt{2}J_-^\star &= -N^2\,\frac{(1+a^\star)(1+b^\star)(a^\star+b^\star)}{2(-1+a^\star)^2(-1+b^\star)}\,,\\
\sqrt{2}J_+^\star &= -N^2\,\frac{(1+a^\star)(1+b^\star)(a^\star+b^\star)}{2(-1+a^\star)(-1+b^\star)^2}\,,
\end{aligned}
\tag{5.27}
$$

with the parameters $a^\star = a^\star(\Lambda)$ and $b^\star = \mathcal{O}(\Lambda)$, being smooth real functions of Λ such that $0 \leq a^\star\,, b^\star < 1$ and $a^\star - 1 = \mathcal{O}(\frac{1}{\Lambda})\,,\, b^\star - 1 = \mathcal{O}(\frac{1}{\Lambda})$ at large Λ.

Next, we proceed to impose the three physical (reality) conditions

$$
\mathrm{Im}(S_0) = \mathrm{Im}(R_0) = \mathrm{Im}(M) = 0\,. \tag{5.28}
$$

The first two conditions fix $C_{0,0} = |C_{0,0}|e^{\mathrm{i}\eta_{0,0}}$ with

$$
\begin{aligned}
|C_{0,0}| &= \frac{9\sqrt{(1+a^\star)(1+b^\star)}(a^\star+b^\star)^2}{2(a^\star+b^\star+a^\star b^\star)\left(1+a^{\star 2}+3a^\star(1+b^\star)+b^\star(3+b^\star)\right)} = 1+\mathcal{O}(\frac{1}{\Lambda})\,,\\
\eta_{0,0} &= \arccos\left(\mp\left(\frac{-1+a^\star+b^\star+b^{\star 2}+a^{\star 2}(1+2b^\star)+a^\star b^\star(5+2b^\star)}{\sqrt{(1+a^\star)(1+b^\star)}\left(1+a^{\star 2}+3a^\star(1+b^\star)+b^\star(3+b^\star)\right)}\right)\right)\\
&= \frac{\pi}{3}+\mathcal{O}\Big(\frac{1}{\Lambda}\Big)\,,
\end{aligned}
\tag{5.29}
$$

The last condition fixes a linear relation between the real (Y_0) and imaginary (Y_1) parts of $C_{1,0} = Y_0 + \mathrm{i}Y_1$ that leads to an isomorphism relation between $M = M(Y_1)$ and Y_1. In the language of appendix D of [28] we are in a path to the BPS locus defined by

$$
\epsilon_q = \epsilon_a = \epsilon_b = 0 \tag{5.30}
$$

This means that Y_1 controls the relation between the variation created by a differential change $\epsilon_r \propto \frac{1}{\beta_g}$ upon the gravitational charges $\{R_g, \ldots\}$ (as given in equation (3.25)-(3.26) in [28]), and the variation created by our differential $\frac{1}{\beta}$ upon the field theory charges $\{R_3,\dots\}$ [46]. At this point in the analysis, and up to order $\mathcal{O}(\frac{1}{\beta}=T)$, the only non trivial variation of charges remaining to identify is

$$\delta R_3 \,:=\, R_3+\left(\frac{\partial\widetilde{\alpha}}{\partial\alpha}\right)R_0 \,=\, -\frac{1}{2\pi\mathrm{i}}\left(\frac{\partial\widetilde{\alpha}}{\partial\alpha}\right)\frac{\partial^2 S}{\partial\widetilde{\alpha}\partial T}\Big|_{T=0}T \,=\, +\left(\frac{\partial\widetilde{\alpha}}{\partial\alpha}\right)\frac{4\pi\mathrm{i}+\mathcal{O}\Big((\widetilde{\alpha}-\frac{1}{2})\Big)}{M(Y_1)}T\,. \tag{5.31}$$

In the holographic dual side, the analogous quantity (created by a variation of type (5.30)) is most easily computed by using equation (3.73) in [28]

$$-\delta R_g \,:=\, R_g^\star-R_g \,=\, \frac{1}{2\pi\mathrm{i}}\frac{\partial^2 I_{\mathrm{ME},g}}{\partial\alpha_g\partial T_g}\Big|_{T_g=0}T_g \,=\, -\frac{4\pi\mathrm{i}+\mathcal{O}\Big((\alpha_g-\frac{1}{2})\Big)}{M_{SU(1,1|1)}}T_g\,. \tag{5.32}$$

Then identifying our field theory chemical potentials [47]

$$(\widetilde{\alpha}\to-\alpha+1,\beta)\,, \tag{5.33}$$

(α,β) with the gravitational chemical potentials $(-\alpha_g,\beta_g)$. More generally,

$$(\alpha,\beta,a^\star,b^\star,\omega_{1,2}) \,\doteq\, (-\alpha_g,\beta_g,,a_g^\star,b_g^\star,-\omega_{1,2;g}) \tag{5.34}$$

and

$$\begin{aligned} S \,&\doteq\, -I_{ME,g}\,,\\ R_0 \,&\doteq\, -\left(\frac{\partial\alpha}{\partial\widetilde{\alpha}}\right)R^\star\,, \end{aligned} \tag{5.35}$$

$$\delta R_3 \,\doteq\, \delta R_g \,\Longrightarrow\, R_3 \,\doteq\, R_g\,, \tag{5.36}$$

fixes (Y_0,Y_1) to large expressions that are not entirely reported in here. For example, for the choices $\pm$ in (5.23) they are

$$\begin{aligned} Y_0 \,=\, &\pm\left(\frac{\pi(b^\star-1)}{12\sqrt{3}}-\frac{5\pi(b^\star-1)^2}{72\sqrt{3}}+\mathcal{O}\left((b^\star-1)^3\right)\right)\\ &\pm(a^\star-1)\left(\frac{\pi}{12\sqrt{3}}-\frac{2\pi(b^\star-1)}{9\sqrt{3}}+\frac{\pi(b^\star-1)^2}{8\sqrt{3}}+\mathcal{O}\left((b^\star-1)^3\right)\right)\\ &\pm(a^\star-1)^2\left(-\frac{5\pi}{72\sqrt{3}}+\frac{\pi(b^\star-1)}{8\sqrt{3}}-\frac{\pi(b^\star-1)^2}{144\sqrt{3}}+\mathcal{O}\left((b^\star-1)^3\right)\right)\\ &+\mathcal{O}\left((a^\star-1)^3\right),\\ Y_1 \,=\, &\left(\frac{7}{36}\pi(b^\star-1)-\frac{5}{72}\pi(b^\star-1)^2+\mathcal{O}\left((b^\star-1)^3\right)\right)\\ &+(a^\star-1)\left(\frac{7\pi}{36}-\frac{17}{216}\pi(b^\star-1)^2+\mathcal{O}\left((b^\star-1)^3\right)\right)\\ &+(a^\star-1)^2\left(-\frac{5\pi}{72}-\frac{17}{216}\pi(b^\star-1)+\frac{127\pi(b^\star-1)^2}{1296}+\mathcal{O}\left((b^\star-1)^3\right)\right)\\ &+\mathcal{O}\left((a^\star-1)^3\right)\,. \end{aligned}$$

[46] This will be eventually identified with the Bekenstein-Hawking temperature $\frac{1}{\beta_g}$ but for the moment it is the field theory temperature.

[47] Here we use the active transformation trick once more.

More importantly, this procedure fixes

$$\frac{1}{M} = \frac{N^2(a^\star+b^\star)^2(3+a^\star+b^\star-a^\star b^\star)}{8(-1+a^\star)(-1+b^\star)(1+a^{\star 2}+3b^\star+b^{\star 2}+3a^\star(1+b^\star))} \doteq \left(\frac{\partial\alpha}{\partial\widetilde{\alpha}}\right)\frac{1}{M_{SU(1,1|1)}}\,, \tag{5.37}$$

and

$$\begin{aligned} S_0 &= \frac{\pi N^2\,(a^*+b^*)\,\sqrt{a^*b^*+a^*+b^*}}{(a^*-1)\,(b^*-1)} \doteq S^\star\,,\\ -\left(\frac{\partial\widetilde{\alpha}}{\partial\alpha}\right)R_0 &= \frac{N^2(a^*+b^*)}{(a^*-1)(b^*-1)} \doteq R^\star\,. \end{aligned} \tag{5.38}$$

This means, summarizing, that

$$S = S_0 + \left(\frac{\partial\widetilde{\alpha}}{\partial\alpha}\right)2\pi\mathrm{i}\alpha R_0 - \left(\frac{\partial\widetilde{\alpha}}{\partial\alpha}\right)\frac{8\pi^2}{M}\frac{(\alpha-\frac{1}{2}) + \left(\frac{\partial\widetilde{\alpha}}{\partial\alpha}\right)(\alpha-\frac{1}{2})^2}{\beta} + \mathcal{O}\Big(\frac{1}{\beta^2}\Big)\,, \tag{5.39}$$

where

$$\left(\frac{\partial\widetilde{\alpha}}{\partial\alpha}\right) = -1\,. \tag{5.40}$$

Namely, S in (5.39) exactly matches the supergravity answer $-I_{ME}$ in [28] upon the identification of chemical potentials and charges summarized in table 5 below. In (5.39) we have reinstated the canonical $\mathcal{O}\big((\alpha-\frac{1}{2})^2\big)$ contribution to S, up to order $\frac{1}{\beta}$, which comes from a repetition of the procedure above reported considering $C_{1,1}\neq 0$.

Quantities	Minimally gauged gravity	Field-Theory
Angular velocities	$-\omega_{1,2;g}$ as in (3.14),(3.15) of [23]	$\omega_{1,2}$
$U(1)$ potential	$-\alpha$ of [28]	α
Charge dual to ω_1	j_1 of [28]	$\sqrt{2}J_-^\star := \sqrt{2}\,J_-\Big\vert_{\Delta=0\,,\,R_{1,2}=0}$
Charge dual to ω_2	j_2 of [28]	$\sqrt{2}J_+^\star := \sqrt{2}\,J_+\Big\vert_{\Delta=0\,,\,R_{1,2}=0}$
$U(1)$ charge	$R^\star$ (resp.R) of [28]	R_0 (resp.R_3)
Large BPS entropy	S^* of [28]	S_0
Mixed ensemble	$-I_{\mathrm{ME}}$ of [28]	S

Table 5. The identifications $\doteq$ between gravity and field theory. Working in minimally gauged supergravity corresponds to the choice of φ_v and φ_w reported in (5.16).

Recovering the perturbative expansion Once reality conditions are imposed on the infrared theory up to order $\mathcal{O}(\frac{1}{\beta})$ and its Schwarzian physical form has been recovered, [48]

[48] By this we mean the reality conditions on the expectation values of charges $\sqrt{2}J_\pm$, which are proportional to derivatives of the infrared free energy with respected to their dual chemical potentials $\omega_{1,2}$ up to, and including $\mathcal{O}(\frac{1}{\beta})$. These reality conditions correspond to restricting the initial 2-complex plane (ω_1,ω_2) to the middle-dimensional complex contour (5.26) spanned by the real values of parameters a and b ranging between 0 and 1, including 0. Once reality conditions are imposed on $\sqrt{2}J_\pm$ the only remaining freedom in reparameterization that is consistent with them is the set of real reparameterizations of the latter complex curve, which corresponds to the infinitely many choices of smooth real functions (5.42). These residual reparameterizations remain unbroken in the presence of the Schwarzian.

there remains ambiguity in the choice of implicit dependence on Λ within the auxiliary chemical potentials

$$\omega_{1,0} \,=\, \Lambda\,\omega_1(a^\star,b^\star), \qquad \omega_{2,0} \,=\, \Lambda\,\omega_2(a^\star,b^\star)\,. \tag{5.41}$$

Up to order $\mathcal{O}(\frac{1}{\beta})$ this ambiguity is parameterized by the possible choices of smooth real functions $0 \leq a_0^\star < 1$ and $0 \leq b_0^\star < 1$

$$a^\star \,=\, a_0^\star(\Lambda)\,, \qquad b^\star \,=\, b_0^\star(\Lambda)\,, \tag{5.42}$$

which at large-Λ respect the boundary conditions

$$a^\star(\Lambda) \,-\, 1 \,=\, \mathcal{O}(\Lambda^{-1}) \qquad b^\star(\Lambda) \,-\, 1 \,=\, \mathcal{O}(\Lambda^{-1})\,. \tag{5.43}$$

Inverting the parametric expressions of the physical charges (5.27) in terms of $a^\star$ and $b^\star$ and substituting the result in the parametric representation of BPS entropy and BPS R-charge (5.38), and mass gap (5.37), one obtains their unambiguous relations to physical charges and entropy, respectively. Here we only show the explicit relation at leading order at large-Λ and first order in the low-temperature expansion

$$S_0 \,=\, \sqrt{3}\pi(N^2J_+J_-)^{\frac{1}{3}}\,, \qquad R_0 \,=\, (N^2J_+J_-)^{\frac{1}{3}}\,, \qquad \frac{1}{M} \,=\, \frac{S_0}{12\sqrt{3}\pi}\,. \tag{5.44}$$

This result could have been obtained directly using only leading expressions for the free energy at large Λ (and low temperature), indeed we have performed that simpler computation independently, as a check.

The hierarchy of charges that this near-BPS RG-flow probes is

$$\begin{aligned}\sqrt{2}J_\mp \,&=\, \mathcal{O}(1)N^2\Lambda^3\,,\\ \Delta \,&=\, \mathcal{O}(1)N^2\frac{|\alpha_0|}{\beta_0^2}\Lambda^3 \,=\, \mathcal{O}(1)N^2\frac{\left(|\alpha-\frac{1}{2}|\right)}{\beta^2}\Lambda^2\,.\end{aligned} \tag{5.45}$$

Notice that up to $\mathcal{O}(\frac{1}{\beta_0})$ only states with eigenvalues $\Delta=0$ are probed (i.e. supersymmetric states). Thus, the same results could have been obtained just starting from the truncated partition function that counts $\frac{1}{16}$-BPS states (without $(-1)^F$ insertion). It also means that if one wants to recover corrections starting from order $\mathcal{O}(\frac{1}{\beta^2})$ then one has to consider a truncation of the partition function that counts states within the hierarchy (5.45) which includes states away from the BPS sector.

Four comments before concluding.

Recall that

$$J_\mp \,=\, \frac{1}{\sqrt{2}}\left(\frac{3\Delta_\mp-\Delta}{6}\right) \,=\, \left(\frac{\Delta_\mp}{2\sqrt{2}} \,+\, \mathcal{O}(\frac{1}{\beta_0^2})\right) \underset{\beta_0\to\infty}{=} \frac{\Delta_\mp}{2\sqrt{2}} \,\geq\, 0\,, \tag{5.46}$$

Thus, in the limit $\Delta_+\to 0$ the S_0 vanishes. This is consistent with the findings in the near-$\frac{1}{8}$-BPS expansion, which by definition is located at $\Delta=\Delta_+=0$, and has $S_0=0$, (4.31). [49]

[49] The $\frac{1}{8}$-BPS reference point used in the previous subsection only intersects with the $\frac{1}{16}$-BPS case in this section at $R_1=R_2=0$.

One natural extension of the analysis in this section corresponds to relaxing the minimal gauged supergravity constraint

$$R_1 = R_2 = 0 \quad \longleftrightarrow \quad Q_1 = Q_2 = Q_3 = \frac{R_3}{2}\,. \tag{5.47}$$

A gravitational mass gap has not been computed yet in such regime (as far as we understand). This would require a generalization of the analysis in [28] to non-minimally gauged supergravities.

Without loss of generality, we have chosen to focus on the Schwarzian corresponding to the winding sector $n_{\text{there}} = j = 0$ (in the notation of [28]). Our discussion trivially generalizes to other sectors $j \neq 0$, by expanding α around other integer shifts of $\alpha = \pm\frac{1}{2}$. Namely the result of those is obtained from (5.39) by substituting $\alpha \to \pm\alpha + j$. Analogously, the result obtained with the more general orbifold sectors mentioned in section 3.2 can be obtained following analogous steps to the ones before summarized. The analysis of these more general cases is left for future work.

It is necessary to compute small Yang-Mills ('t Hooft) coupling corrections to the mass-gap M, and find a different explanation of why the free-field theory computation matches the conjectured strong coupling result in the near-BPS region. As it was just explained, kinematically, the here-proposed RG-flow procedure enforces that only states with $\Delta = 0$, i.e., only BPS states contribute up to order $\mathcal{O}(T)$ to the free energy. [50]

6 Final remarks

Holographic low-temperature expansions were recovered from the free energy of four dimensional maximally supersymmetric Yang-Mills theory. The analytic part of the free energy associated to the infrared effective theory, (D.1) was computed. Assuming supergravity predictions are correctly capturing strong-coupling results in field theory, and in virtue of analyticity, the computed infrared free energy is bound to encode the Gibbons-Hawking free energy of the dual gravitational solutions even well beyond their BPS locus. The formula was tested at leading and next-to-leading order in the low-temperature expansion. Up to such order the infrared free energy was shown to reproduce the long awaited Schwarzian contributions with a small mass gap M scaling as a negative power of the energy scale Λ.

These low-temperature expansions, and in particular their Schwarzian vicinities, localize around complexified values of chemical potentials where supersymmetric cancellations happen and the physical partition function reduces to a superconformal index $\mathcal{I}_1$. If $\mathcal{I}_1$ counts states preserving four supercharges, then the corresponding Schwarzian contribution is protected against corrections in $g^2_{YM}N$. If these low-temperature expansions are appropriately directed accross the complexified space of chemical potentials [51] then all of its

[50] However, the BPS states in the free gauge theory are lifted by a small coupling [74, 96–100]. This suggests that the explanation may be more subtle than that.

[51] By this we mean that the derivatives with respect to α, at $\alpha = \frac{1}{2}$, equal traces over the Hilbert space that receive contributions only from certain supersymmetric states. For examples, this would correspond to the constraint $\gamma = 1$ in (B.14).

terms, not just the Schwarzian contribution can be exactly computed using another superconformal index, $\mathcal{I}_2 \neq \mathcal{I}_1$. The new index $\mathcal{I}_2$ counts states preserving only two supercharges out of the four ones preserved by the states counted by $\mathcal{I}_1$.

Near-$\frac{1}{8}$-BPS black holes with charges $\Delta \approx 0$, $\Delta_+ \approx 0$, $R_1 + R_2 \approx 0$ and $\Delta_- \neq 0$, were found to have near-vanishing horizon, and thus vanishing BPS entropy S_0, due to the BPS contraint $\Delta_- \geq 0$. In these cases, the low-temperature corrections to the near-$\frac{1}{8}$-BPS mixed-ensemble free energy S was found to be dominated by the protected Schwarzian contribution identified in subsection 4. This result suggests that near-$\frac{1}{8}$-BPS gravitational solutions within the family found in [69] have a near-vanishing near extremal horizon. It also suggests that their horizon vanishes in the strict and smooth $\frac{1}{8}$-BPS limit. [52]

The Schwarzian corrections already identified in the supergravity side of the duality [28] were recovered from the partition function of free $\mathcal{N} = 4$ SYM. The mass-gap predicted by the free gauge-theory computation exactly matches the answer coming from supergravity. We plan to compare higher order low-temperature corrections in the infrared free energy against dual corrections obtained out of the Gibbons-Hawking onshell action of the solutions of [58, 68, 69] about their BPS locus [59, 60] [23]. Namely, to test the formula (3.48), which we reiterate, assuming the supergravity predictions are accurate at strong coupling are trivially bound to be true, in virtue of analyticity. More importantly, it is necessary to confirm formula (3.48) strictly using field theoretic tools.

Other interesting questions become tangible. For instance, one may try to develop analytic tools to quantitatively understand the emergence of quantum chaos [78–80] in gauge theories. In the case of $\mathcal{N} = 4$ SYM one next obvious goal is to identify which is the random matrix theory emerging in the infrared theory. We will address this question in forthcoming work. Relatedly, one could also explore this possibility in simpler quantum systems, possibly even in some systems already known to be realized in nature.

In $\mathcal{N} = 4$ SYM this problem has been studied at $\beta = \infty$ [83]. The erratic behaviour there identified in the spectral form factor (even at zero temperature), is closely related to interference effects [55, 56][47] among complex phases associated to different saddle point contributions of the superconformal index [31, 40–42, 46, 50]. The contribution of all such saddles is encoded in subleading corrections within the RG-flow procedure this paper focused on. The majority of them are e^{-N^2}-suppressed at large N, but their contribution is essential at finite N. There are also instantonic e^{-N}-suppressed contributions [41, 50], which come from D3-brane instantons [50]. Our result predicts – and calls for the study and understanding of – the presence of analogous contributions away from the BPS locus. That said, it is still unclear to us whether in order to identify the ramp/plateau feature in the spectral form factor of $\mathcal{N} = 4$ SYM it will be necessary to resort to these non-perturbatively suppressed contributions.

Relatedly, it would be very interesting to promote the analytic formula for the infrared

[52] It would be interesting to understand how such near-horizon geometry meets the conclusions of [25]. Such reference identified BTZ geometries in the near-horizon region of near-$\frac{1}{8}$-BPS black hole solutions. One may then wonder how the effective one-dimensional Schwarzian entropy relates (if they happen to be related e.g. in the spirit of [101, 102]) to the conjectured two-dimensional conformal field theory computing the entropy of such BTZ geometries [25].

free energy $\mathcal{F}_\infty$ into an "dual" formula for $\mathcal{F}$ equating the latter in any domain of chemical potentials, not only around the $\frac{1}{\Lambda}$-vicinity of its leading singularities. This improvement would give us a tool useful for computing non-perturbative finite-temperature quantum gravity effects.

Acknowledgements

It is a pleasure to thank Matteo Beccaria, Nana G. Cabo Bizet, Alejandro Cabo Montes de Oca, Justin R. David, Marina David, Stefano Giusto, Alfredo Gonzalez Lezcano, Camillo Imbimbo, Vishnu Jejjala, Albrecht Klemm, Kimyeong Lee, Gabriel Lopes Cardoso, Robert de Mello Koch, Sameer Murthy, Suresh Nampuri, and Leopoldo Pando Zayas for useful conversations or discussions. The author would like to thank the Isaac Newton Institute for Mathematical Sciences, Cambridge, for support and hospitality during the programme Black holes: bridges between number theory and holographic quantum information, where work on this paper was undertaken. This work was supported by EPSRC grant EP/R014604/1, and by the INFN grants GSS and GAST.

A The saddle point for gauge potentials

Discarding *heavy enough* states, the truncated version of the partition function (2.23) is

$$Z_\Lambda[x,u,v,w,t,y] = \int [DU] e^{\sum_{n=1}^{\mathcal{O}(1)\Lambda^{n'}} \frac{1}{n}\left(f_{Bos}[x^n,u^n,v^n,\ldots]+(-1)^{n+1}f_{Fer}[x^n,u^n,v^n,\ldots]\right)\mathrm{Tr}U^n \mathrm{Tr}U^{\dagger n}}\,, \tag{A.1}$$

where the f_{Bos} and f_{Fer} were defined in (2.27).

Z_Λ is as good as Z to compute (weighted) degeneracies at charges up to order $\mathcal{O}(1)\Lambda^{n'}$.

Let $\{e^{2\pi \mathrm{i} u_i}\}_{i=1,\ldots N.}$ be the eigenvalues of U then integral (A.1) can be written in the form

$$Z_\Lambda[x,u,v,w,t,y] = \frac{1}{N!}\int_0^1 \mathrm{d}u_1 \ldots \int_0^1 \mathrm{d}u_N e^{-F_\Lambda(u_i)}\,. \tag{A.2}$$

Expanding, for instance,

$$Z_\Lambda[x, u=e^{2\pi\mathrm{i}\alpha}, e^{-2\pi\mathrm{i}(\alpha-1/2)}\widetilde{v}, w=\widetilde{v}ty\,,\, t,\, y] \;=\; \mathcal{I}_{2\Lambda}[x, e^{2\pi\mathrm{i}\alpha}, \widetilde{v}, t, y] \tag{A.3}$$

at leading order in the low-temperature expansion (3.52) with finite gauge potentials u_i's and recalling that

$$t \;=\; e^{-\frac{1}{6}(\omega_1+\omega_2)}, y \;=\; e^{\frac{\omega_1-\omega_2}{2}}, \widetilde{v} \;=\; e^{\frac{1}{3}(\omega_1+\omega_2-3\varphi_{\widetilde{v}})}\,, \tag{A.4}$$

we find that at leading order at large Λ

$$\begin{aligned} -F_\Lambda(u_i) \;=& \frac{1}{\omega_1}\sum_{i,j=1}^{N}\left(\mathrm{Li}_2^\Lambda\left(e^{2\pi\mathrm{i}\left(u_{ij}-\frac{i\varphi_{\widetilde{v}}}{2\pi}\right)}\right) + \mathrm{Li}_2^\Lambda\left(e^{2\pi\mathrm{i}\left(u_{ij}+\frac{i\varphi_{\widetilde{v}}}{2\pi}\right)}\right) - \right. \\ &\left. -2\mathrm{Li}_2^\Lambda\left(e^{2\pi\mathrm{i}(u_{ij})}\right)\right)\times\left(1+\frac{4\pi i}{\beta}c\left(\widetilde{\alpha}-\frac{1}{2}\right)+\mathcal{O}\left(\frac{\omega_{1,0}}{\beta_0^2}\right)\right) \end{aligned} \tag{A.5}$$

where $u_{ij} := u_i - u_j$ and

$$\mathrm{Li}_p^{\Lambda}(z) := \sum_{j=1}^{\mathcal{O}(1)\Lambda^n} \frac{z^j}{j^p}\,. \tag{A.6}$$

One saddle solution of (A.5) is

$$u_i \,=\, u^\star\,, \qquad i\,=\,1\,,\ldots,N\,. \tag{A.7}$$

We note that even though in the limit $\Lambda\,\to\,\infty$ $S_\Lambda(\underline{u})$ develops a cusp at (A.7) as it was pointed out recently in [76], [53] the ansatz (A.7) is a well-defined saddle point of the truncated partition function Z_Λ at any finite value of Λ. This implies, as it will be shown in the following appendix, around equation (B.21), that this saddle point is bound to capture any potential growth in the number of-$\frac{1}{8}$-BPS states with charges of order N^2 at leading $\mathcal{O}(N^2)$ order.

For example, at this saddle point

$$\begin{aligned} -F_\Lambda(u_i=u^\star) \,\sim\, & \frac{N^2\left(\mathrm{Li}_2^\Lambda\left(e^{\varphi\widetilde{v}}\right)+\mathrm{Li}_2^\Lambda\left(e^{-\varphi\widetilde{v}}\right)-2\mathrm{Li}_2^\Lambda\left(1\right)\right)}{\omega_1} \\ & \times\left(1-\pi\mathrm{i}C\frac{(\widetilde{\alpha}-\frac{1}{2})}{\beta}\,+\,\mathcal{O}\big(\frac{\omega_1}{\beta^2}\big)\right)+\mathcal{O}\Big((\widetilde{\alpha}-\frac{1}{2})^2\Big)\,. \end{aligned} \tag{A.8}$$

which is minus the low-temperature expansion of the free energy $\mathcal{F}_{\frac{1}{16}\mathrm{near}\frac{1}{8}}$ reported in (4.9).

B The generic near-1/8-BPS susceptibility is also protected

The generic refinement of the partition function

$$Z[x,u=e^{2\pi\mathrm{i}\alpha},v,w=e^{-\varphi}vty\,,\,t,\,y]\,=\,e^{-\mathcal{F}_{\mathrm{near}\frac{1}{8}}}\,. \tag{B.1}$$

flows to the 1/8-BPS index $\mathcal{I}^{4,-;2,+}$ in the limit

$$\alpha\to\frac{1}{2}\,,\qquad \varphi\,\to\,0\,. \tag{B.2}$$

If $\alpha=\frac{1}{2}$ and $\varphi\,\neq\,0$ then it reduces instead to the 1/16-BPS index $\mathcal{I}_1$. [54] For later reference we note that

$$e^{\varphi}\,=\,e^{-\varphi_w}\frac{v\,y}{t}\,,\qquad \varphi\,=\,\omega_1-\varphi_v-\varphi_w\,. \tag{B.3}$$

Our goal next is to show that the generic Schwarzian contribution (to free energy) about $(\alpha,\varphi)\,=\,(\frac{1}{2},0)$ is protected against gauge-coupling corrections, for example, if in the expansion (3.52)

$$\varphi\,=\,\frac{\varphi_0}{\Lambda}\,, \tag{B.4}$$

[53]This is because $\partial_u\mathrm{Li}_2^\Lambda(e^{2\pi\mathrm{i}u})+\partial_u\mathrm{Li}_2^\Lambda(e^{-2\pi\mathrm{i}u})\underset{\Lambda\to\infty}{\to}4\pi^2(\{u\}-\frac{1}{2})$ if $u\neq 0\,\mathrm{mod}\,1$, and 0 if $u\,=\,0\,\mathrm{mod}\,1$. $\{u\}:=u-\lfloor u\rfloor$. Here for simplicity we assumed real u, but the generalization to complex u of the previous identity can be obtained straightforwardly.

[54]There are other possible restrictions that may have been chosen, but none more general than (B.1).

with fixed φ_0 and

$$\frac{|\varphi_0|}{|\alpha_0|} = \text{finite and} \quad |\alpha_0| \quad \text{is small.} \tag{B.5}$$

The reason for such protectedness is that the first variation (susceptibility) of

$$\mathcal{F}_{\text{near}\frac{1}{8}} = \mathcal{F}[x, u = e^{2\pi \mathrm{i}\alpha}, v, w = e^{-\varphi} vty\,,\, t,\, y] \tag{B.6}$$

in the variables (α, φ) at the point $(\alpha, \varphi) = (\frac{1}{2}, 0)$,

$$\delta^{(1)}\mathcal{F}_{\text{near}\frac{1}{8}} := \Big(\mathcal{F}^{(1)}_{\text{near}\frac{1}{8}}\Big|_{\varphi=0}\Big)\big(\alpha - \frac{1}{2}\big) + \Big(\partial_\varphi \mathcal{F}^{(0)}_{\text{near}\frac{1}{8}}\Big|_{\varphi=0}\Big)\,\varphi\,, \tag{B.7}$$

which is encoded in the derivatives

$$\begin{aligned} \mathcal{F}^{(1)}_{\text{near}\frac{1}{8}} &= -\,2\pi\mathrm{i}\,\mathrm{Tr}_{\mathcal{H}}(R_1 + R_3)(-1)^F x^\Delta v^{-R_1-R_2} t^{2(H+J_1^3)-R_2} y^{2J_2^3-R_2}\,, \\ \partial_\varphi \mathcal{F}^{(0)}_{\text{near}\frac{1}{8}}\Big|_{\varphi=0} &= \mathrm{Tr}_{\mathcal{H}}\, R_2\,(-1)^F\, x^\Delta v^{-R_1-R_2} t^{2(H+J_1^3)-R_2} y^{2J_2^3-R_2}\,, \end{aligned} \tag{B.8}$$

happens to be a linear combination of two $\mathcal{Q}_1$-protected

$$-\mathrm{Tr}_{\mathcal{H}}(R_{1,2})(-1)^F x^\Delta v^{-R_1-R_2} t^{2(H+J_1^3)-R_2} y^{2J_2^3-R_2} \tag{B.9}$$

and one $\mathcal{Q}_2$-protected

$$-\mathrm{Tr}_{\mathcal{H}} R_3 (-1)^F x^\Delta v^{-R_1-R_2} t^{2(H+J_1^3)-R_2} y^{2J_2^3-R_2} \tag{B.10}$$

traces. Namely, the linear differential (B.7) is a linear combination of three indices and thus it is protected.

To compute the large-Λ expansion of the free energy (B.6), (B.8), we start from the matrix integral representation (2.23) at zero gauge coupling. We extract the leading behaviour of such integral in the expansion (3.52), assuming the condition (B.4), and keeping fixed φ_0 and α_0 , i.e., without imposing (B.5). In such expansion the leading saddle-point for the gauge-singlet condition is $U \sim e^{2\pi \mathrm{i} u^\star} \times 1_{N\times N}$, and thus, again, we can substitute $\mathrm{Tr}U^n \mathrm{Tr}U^{-n} \to N^2$ and obtain

$$\begin{aligned} \mathcal{F}^{(0)}_{\text{near}\frac{1}{8}}\Big|_{\varphi=0} &\sim -\,N^2 \frac{L_2(\varphi_v)}{\omega_1} = \mathcal{O}(\Lambda^1)\,. \\ \mathcal{F}^{(1)}_{\text{near}\frac{1}{8}} &\sim \pi \mathrm{i} N^2 \Big(\frac{-2}{\omega_1\omega_2} + \frac{\varphi + \omega_2}{\beta\,\omega_1\omega_2} + \mathcal{O}\Big(\frac{1}{{\beta_0}^2}\Big)\Big) \times L_2(\varphi_v) = \mathcal{O}(\Lambda^2)\,. \\ \partial^1_\varphi \mathcal{F}^{(0)}_{\text{near}\frac{1}{8}}\Big|_{\varphi=0} &\sim -N^2 \frac{L_2(\varphi_v)}{\omega_1\omega_2} = \mathcal{O}(\Lambda^2)\,. \\ \mathcal{F}^{(2)}_{\text{near}\frac{1}{8}}\Big|_{\varphi=0} &\sim -\,2\pi^2\, N^2\, \frac{L_2(\varphi_v)}{\beta\,\omega_1\omega_2} = \mathcal{O}(\Lambda^3)\,. \end{aligned} \tag{B.11}$$

Again, $\sim$ means that the objects to the right-hand and left-hand sides have identical leading asymptotic bevarior in the large-Λ expansion.

Using (B.11) we obtain the leading asymptotic behavior of the free energy (2.48) (at order $\mathcal{O}(\Lambda^1)$)

$$\begin{aligned}\mathcal{F}_{\text{near}\frac{1}{8}} \sim & -N^2\,\frac{\varphi+\omega_2+2\pi\mathrm{i}(\alpha-\frac{1}{2})}{\omega_1\omega_2}\,L_2(\varphi_v)\\ & +\pi\mathrm{i}N^2\,\frac{\big(\varphi+\omega_2+2\pi\mathrm{i}(\alpha-\frac{1}{2})\big)\big(\alpha-\frac{1}{2}\big)}{\beta\,\omega_1\omega_2}\,\Big(1\,+\,\mathcal{O}\Big(\frac{\omega_1}{\beta},\frac{\omega_2}{\beta}\Big)\Big)\,L_2(\varphi_v)\,.\end{aligned} \tag{B.12}$$

In the right hand-side of this asymptotic relation only the quadratic contributions in $(\varphi,\alpha-\frac{1}{2})$

$$\pi\mathrm{i}N^2\,\frac{\big(\varphi+2\pi\mathrm{i}(\alpha-\frac{1}{2})\big)\big(\alpha-\frac{1}{2}\big)}{\beta\,\omega_1\omega_2}\,\Big(1\,+\,\mathcal{O}\Big(\frac{\omega_1}{\beta},\frac{\omega_2}{\beta}\Big)\Big)\,L_2(\varphi_v) \tag{B.13}$$

may receive corrections in the gauge coupling. This follows from the protectedness argument above. If on the right-hand side of (B.12) we assume α_0 to be small and fix

$$\varphi\;=\;-2\pi\mathrm{i}\gamma(\alpha-\frac{1}{2})\;=\;\mathcal{O}\big(\frac{1}{\Lambda}\big)\,, \tag{B.14}$$

with γ being a finite constant – which is equivalent to assuming (B.5) –, then we obtain

$$\begin{aligned}\mathcal{F}_{\text{near}\frac{1}{8}} &\sim \mathcal{F}_{\frac{1}{16}\text{near}\frac{1}{8}}\,-\,2\pi\mathrm{i}N^2\frac{(\alpha-\frac{1}{2})L_2(\varphi_v)}{\omega_1\omega_2}\,-\,N^2\frac{\varphi L_2(\varphi_v)}{\omega_1\omega_2}+\mathcal{O}\big((\alpha-\frac{1}{2})\varphi,(\alpha-\frac{1}{2})^2\big)\,.\\ &\sim \mathcal{F}_{\frac{1}{16}\text{near}\frac{1}{8}}\,+\,2\pi\mathrm{i}N^2\frac{\big(\gamma-1\big)(\alpha-\frac{1}{2})L_2(\varphi_v)}{\omega_1\omega_2}+\mathcal{O}\big((\alpha-\frac{1}{2})\varphi,(\alpha-\frac{1}{2})^2\big)\,.\end{aligned} \tag{B.15}$$

This equation is implicitly saying that the first-order correction in temperature to the free energy $\mathcal{F}_{\text{near}\frac{1}{8}}$ at order $\mathcal{O}(\frac{1}{\beta_0})$ is independent of γ (resp. φ) and equals the one computed with $\mathcal{F}_{\text{near}\frac{1}{8}}$.

Chemical potential	Dual charge	Source term
β	Δ	$+\beta\Delta$
ω_1	$J_1^3-J_2^3+\frac{R_1}{2}+R_2+\frac{R_3}{2}+\frac{\Delta}{3}$	$+\omega_1\cdot\big(J_1^3-J_2^3+\frac{R_1}{2}+R_2+\frac{R_3}{2}+\frac{\Delta}{3}\big)$
ω_2	$J_1^3+J_2^3+\frac{R_1}{2}+\frac{R_3}{2}+\frac{\Delta}{3}$	$+\omega_2\cdot\big(J_1^3+J_2^3+\frac{R_1}{2}+\frac{R_3}{2}+\frac{\Delta}{3}\big)$
φ_v	$-R_1-R_2$	$+\varphi_v\cdot(-R_1-R_2)$
φ	$-R_1$	$+\varphi\cdot(-R_1)$
$\omega_u=-2\pi\mathrm{i}\alpha$	$-R_1-R_3$	$+\omega_u\cdot(-R_1-R_3)$

Table 6. The chemical potentials, charges and source terms to be potentially added to $-\mathcal{F}_{\text{near}\frac{1}{8}}$ before performing extremization.

The extra term, which is protected against coupling corrections because it is included in the protected susceptibility $\delta^{(1)}\mathcal{F}_{\text{near}\frac{1}{8}}$

$$+\,2\pi\mathrm{i}N^2\frac{\big(\gamma-1\big)(\alpha-\frac{1}{2})\,L_2(\varphi_v)}{\omega_1\omega_2}\;\in\;\delta^{(1)}\mathcal{F}_{\text{near}\frac{1}{8}}\,, \tag{B.16}$$

captures the singularity associated to the growth of $\frac{1}{16}$-BPS states, $\frac{1}{\omega_1\omega_2}$, and it is bound to match certain near-$\frac{1}{8}$-BPS (thermodynamic) susceptibility of the AdS_5 black hole solutions found in [58, 69].

The near-$\frac{1}{8}$-BPS expansion along $\mathcal{I}_2$ (4.9) is recovered from (B.12) upon the constraint $\gamma = 1$, as it has to be the case, due to consistency with the constraints that reduce $Z[x,u,v,w,t,y]$ to $\mathcal{I}_2[x, e^{2\pi\mathrm{i}\alpha}, \widetilde{v}, t, y]$

$$\varphi_v \,=\, \varphi_{\widetilde{v}} \,+\, 2\pi\mathrm{i}(\alpha - \frac{1}{2})\,, \qquad \varphi \,=\, -2\pi\mathrm{i}(\alpha - \frac{1}{2})\,. \tag{B.17}$$

B.1 Confirming the near-1/8-BPS saddle-point selection

It should be noted that the low-temperature expansion of the protected near-$\frac{1}{8}$-BPS susceptibility (B.7), which is implicit in the first-order differential in the right-hand side of (B.12), is *continuously* recovered from the low-temperature expansion of $\mathcal{F}$, (5.8), by substituting (B.4)

$$\varphi \,=\, \frac{\varphi_0}{\Lambda} \,=\, \mathcal{O}(\frac{1}{\Lambda})\,, \tag{B.18}$$

on the large-Λ expansion of the latter. Concretely, by using the relation

$$\varphi_w \,=\, \omega_1 \,-\, \varphi_v \,-\, \varphi \tag{B.19}$$

and the identities

$$L_3(\varphi_v, -\varphi_v) = 0\,, \qquad \partial_{\varphi_w} L_3(\varphi_v, \varphi_w)\Big|_{\varphi_w = -\varphi_v} \,=\, -\,L_2(\varphi_v) \tag{B.20}$$

on the first-order Taylor expansion of (5.8) around φ, $\omega_1 = 0$, and the identity

$$L_{2,1}(\varphi_v, -\varphi_v) \,=\, -\,2\,L_2(\varphi_v) \tag{B.21}$$

on the zeroeth-order Taylor expansion around φ, $\omega_1 = 0$ of the $\mathcal{O}(\Lambda)$ (and large-charge subleading) correction (5.5). In particular, this demonstrates that the free energy of the near-$\frac{1}{8}$-BPS phase, (4.9), is continuously recovered from the free energy of the $\frac{1}{16}$-BPS phase, (5.8). This check also reaffirms the assumption that the saddle-point (4.1) determines the free energy of the near-$\frac{1}{8}$-BPS sector, (B.12).

Useful equation In this subsection we will use equalities instead of $\sim$ symbols. The equalities should be always understood up to the order we used them in the analysis summarized in the main body of the paper.

$$\begin{aligned}
\mathcal{F}_{\frac{1}{16}near\frac{1}{8}} \,&=\, -\frac{L_2\big(\varphi_{\widetilde{v}}\big) + \frac{\omega_1}{2}L_1\big(\varphi_{\widetilde{v}}\big)}{\omega_1} \,+\, C\frac{\pi\mathrm{i}(\widetilde{\alpha} - \frac{1}{2})L_1\big(\varphi_{\widetilde{v}}\big)}{\omega_1} + \\
&\quad + C\frac{\pi\mathrm{i}\,(\widetilde{\alpha} - \frac{1}{2})\Big(L_2\big(\varphi_{\widetilde{v}}\big) + \frac{\omega_1}{2}L_1\big(\varphi_{\widetilde{v}}\big)\Big)}{\beta\omega_1} + C^2\frac{\pi^2(\widetilde{\alpha} - \frac{1}{2})^2 L_1[\varphi_{\widetilde{v}}]}{\beta\omega_1} \\
&= -\frac{L_2\big(\varphi_{\widetilde{v}}\big)}{\omega_1} \,+\, C\frac{\pi\mathrm{i}(\widetilde{\alpha} - \frac{1}{2})L_1\big(\varphi_{\widetilde{v}}\big)}{\omega_1} + C\frac{\pi\mathrm{i}\,(\widetilde{\alpha} - \frac{1}{2})L_2\big(\varphi_{\widetilde{v}}\big)}{\beta\omega_1} \\
&\quad + C^2\frac{\pi^2(\widetilde{\alpha} - \frac{1}{2})^2 L_1[\varphi_{\widetilde{v}}]}{\beta\omega_1} \,+\, \ldots\,,
\end{aligned} \tag{B.22}$$

The dots denote terms that do not contribute to the leading behaviour at large-Λ. The complete expression can obtained from (D.1) using the relations

$$\varphi_w = \omega_1 - \varphi_v\,, \qquad \varphi_v = \varphi_{\widetilde{v}} + 2\pi i(\alpha - \frac{1}{2})\,. \tag{B.23}$$

C BPS inequalities and conventions

In the conventions of charges we have used, the 16 semi-positivity conditions are –written for instance in there –

$$H_{\text{there}} - \sum_{I=1}^{3} s_I Q_{I\text{there}} - \sum_{i=1}^{2} t_i J_{i\text{there}} \geq 0\,, \tag{C.1}$$

where $s_I = \pm 1$, $t_i = \pm 1$ and $s_1 s_2 s_3 t_1 t_2 = 1$, are

$$\begin{aligned}
H + 2J_1^3 + \frac{3R_1}{2} + R_2 + \frac{R_3}{2} &\geq 0, & H - 2J_2^3 + \frac{R_1}{2} + R_2 + \frac{3R_3}{2} &\geq 0,\\
H + 2J_2^3 + \frac{R_1}{2} + R_2 + \frac{3R_3}{2} &\geq 0, & H - 2J_1^3 + \frac{3R_1}{2} + R_2 + \frac{R_3}{2} &\geq 0,\\
H + 2J_1^3 - \frac{R_1}{2} + R_2 + \frac{R_3}{2} &\geq 0, & H - 2J_2^3 + \frac{R_1}{2} + R_2 - \frac{R_3}{2} &\geq 0,\\
H + 2J_2^3 + \frac{R_1}{2} + R_2 - \frac{R_3}{2} &\geq 0, & H - 2J_1^3 - \frac{R_1}{2} + R_2 + \frac{R_3}{2} &\geq 0,\\
H + 2J_1^3 - \frac{R_1}{2} - R_2 + \frac{R_3}{2} &\geq 0, & H - 2J_2^3 + \frac{R_1}{2} - R_2 - \frac{R_3}{2} &\geq 0,\\
H + 2J_2^3 + \frac{R_1}{2} - R_2 - \frac{R_3}{2} &\geq 0, & H - 2J_1^3 - \frac{R_1}{2} - R_2 + \frac{R_3}{2} &\geq 0,\\
H + 2J_1^3 - \frac{R_1}{2} - R_2 - \frac{3R_3}{2} &\geq 0, & H - 2J_2^3 - \frac{3R_1}{2} - R_2 - \frac{R_3}{2} &\geq 0,\\
H + 2J_2^3 - \frac{3R_1}{2} - R_2 - \frac{R_3}{2} &\geq 0, & H - 2J_1^3 - \frac{R_1}{2} - R_2 - \frac{3R_3}{2} &\geq 0\,.
\end{aligned}$$

The relations between the charges used in here and the ones used in [35] is

$$\begin{aligned}
H_{\text{there}} &= H, \quad Q_{1\text{there}} = \frac{1}{2}(R_1 + 2R_2 + R_3), \quad Q_{2\text{there}} = \frac{1}{2}(R_1 + R_3),\\
Q_{3\text{there}} &= \frac{1}{2}(-R_1 + R_3)\,, \quad J_{1\text{there}} = J_1^3 - J_2^3\,, \quad J_{2\text{there}} = J_1^3 + J_2^3\,.
\end{aligned} \tag{C.2}$$

D The analytic effective potential for gauge variables

The complete *analytic part* of the infrared effective potential of gauge variables is [55]

$$\begin{aligned}
&\mathcal{F}_{sl}^{L} + \sum_{j=1}^{LR} \sum_{\rho\neq 0} \frac{e^{2\pi i j \rho(u)}}{j}\\
&\underset{\Lambda\to\infty}{\mapsto} \mathcal{F}_{\infty,\,sl} = - \sum_{i\leq j=1}^{N} \sum_{p=-1}^{2} \Big(V_p(u_{ij}) + V_p(u_{ji}) + \mathcal{O}(\frac{1}{\beta^2}) \Big)\,,
\end{aligned} \tag{D.1}$$

[55]There is a possible $\mathcal{O}(\Lambda^0)$ ambiguity coming from the ambiguity in choice of different branches of the logarithmic contributions, but we can always choose to work in a branch where these contributions vanish. Equivalently, we can always choose to approach the BPS point $\alpha = \frac{1}{2}$ in such a way such contribution vanishes (e.g. the selection of couterterms like (5.11)). So, we assume that we work in such a branch.

where

$$
\begin{aligned}
V_2(u) &:= \frac{4\pi^3\left(\pi\left(\alpha-\frac{1}{2}\right)+\mathrm{i}\beta\right)\left(\overline{B}_3\left[-u-\frac{\mathrm{i}\varphi_v+\mathrm{i}\varphi_w}{2\pi}\right]+\overline{B}_3\left[-u+\frac{\mathrm{i}\varphi_v}{2\pi}\right]+\overline{B}_3\left[-u+\frac{\mathrm{i}\varphi_w}{2\pi}\right]\right)}{3\beta\omega_1\omega_2}\,,\\
V_1(u) &:= \frac{\pi^2}{\beta\omega_1\omega_2}\left(\pi\left(\alpha-\frac{1}{2}\right)+\mathrm{i}\beta\right)\times\left(2\pi\left(\alpha-\frac{1}{2}\right)\left(-3\overline{B}_2\left[-u-\frac{\mathrm{i}(\varphi_v+\varphi_w)}{2\pi}\right]\right.\right.\\
&\quad\left.+\overline{B}_2\left[-u+\frac{\mathrm{i}\varphi_v}{2\pi}\right]+\overline{B}_2\left[-u+\frac{\mathrm{i}\varphi_w}{2\pi}\right]+\overline{B}_2\left[-u\right]\right)\\
&\quad+\mathrm{i}(\omega_1+\omega_2)\left(\overline{B}_2\left[-u-\frac{\mathrm{i}(\varphi_v+\varphi_w)}{2\pi}\right]\right.\\
&\quad\left.\left.-\overline{B}_2\left[-u+\frac{\mathrm{i}\varphi_v}{2\pi}\right]-\overline{B}_2\left[-u+\frac{\mathrm{i}\varphi_w}{2\pi}\right]+\overline{B}_2[-u]\right)\right)\,,\\
V_0(u) &:= \frac{8\pi^3\left(\alpha-\frac{1}{2}\right)\left(\pi\left(\alpha-\frac{1}{2}\right)+\mathrm{i}\beta\right)\left(\left(\alpha-\frac{1}{2}\right)-\frac{\mathrm{i}\omega_1+\mathrm{i}\omega_2}{4\pi}\right)\overline{B}_1\left[-\frac{2\pi u+\mathrm{i}\varphi_v+\mathrm{i}\varphi_w}{2\pi}\right]}{\beta\omega_1\omega_2}\\
&\quad+\left(\frac{\pi\left(18\mathrm{i}\pi^2\left(\alpha-\frac{1}{2}\right)^2(2\beta-\omega_1-\omega_2)+24\pi^3(\alpha-\frac{1}{2})^3-3\mathrm{i}\beta\left(\omega_1^2+3\omega_2\omega_1+\omega_2^2\right)\right)}{18\beta\omega_1\omega_2}\right.\\
&\quad\left.-\frac{\pi^2\left(\alpha-\frac{1}{2}\right)\left(12\beta^2-20\beta\left(\omega_1+\omega_2\right)+3\left(\omega_1^2+3\omega_2\omega_1+\omega_2^2\right)\right)}{18\beta\omega_1\omega_2}\right)\times\\
&\quad\times\left(\overline{B}_1\left[-u-\frac{\mathrm{i}\varphi_v}{2\pi}-\frac{\mathrm{i}\varphi_w}{2\pi}\right]+\overline{B}_1\left[-u+\frac{\mathrm{i}\varphi_v}{2\pi}\right]+\overline{B}_1\left[-u+\frac{\mathrm{i}\varphi_w}{2\pi}\right]\right)\,,\\
V_{-1}(u) &:= -\frac{2\pi\left(\alpha-\frac{1}{2}\right)\left(\pi\left(\alpha-\frac{1}{2}\right)+i\beta\right)\left(2\pi\left(\alpha-\frac{1}{2}\right)-i\omega_1\right)\left(2\pi\left(\alpha-\frac{1}{2}\right)-i\omega_2\right)}{\beta\omega_1\omega_2}\,.
\end{aligned}
\tag{D.2}
$$

$V_p(u)$ is the contribution to the potential at order $\mathcal{O}(\Lambda^p)$. We have reported here only the expansion up to order $\mathcal{O}(\frac{1}{\beta})$ but the complete expansion in terms of rational functions of β is presented in the shared Mathematica file.

References

[1] G. 't Hooft, *A Planar Diagram Theory for Strong Interactions*, *Nucl. Phys. B* **72** (1974) 461.

[2] J. M. Maldacena, *The Large N limit of superconformal field theories and supergravity*, *Adv. Theor. Math. Phys.* **2** (1998) 231–252, [hep-th/9711200].

[3] E. Witten, *Anti-de Sitter space and holography*, *Adv. Theor. Math. Phys.* **2** (1998) 253–291, [hep-th/9802150].

[4] S. S. Gubser, I. R. Klebanov, and A. M. Polyakov, *Gauge theory correlators from noncritical string theory*, *Phys. Lett. B* **428** (1998) 105–114, [hep-th/9802109].

[5] J. Polchinski, *Dirichlet Branes and Ramond-Ramond charges*, *Phys. Rev. Lett.* **75** (1995) 4724–4727, [hep-th/9510017].

[6] K. G. Wilson, *The Renormalization Group: Critical Phenomena and the Kondo Problem*, *Rev. Mod. Phys.* **47** (1975) 773.

[7] K. G. Wilson, *The renormalization group and critical phenomena*, *Rev. Mod. Phys.* **55** (1983) 583–600.

[8] M. Gell-Mann and F. E. Low, *Quantum electrodynamics at small distances*, *Phys. Rev.* **95** (1954) 1300–1312.

[9] D. J. Gross and F. Wilczek, *Ultraviolet Behavior of Nonabelian Gauge Theories*, *Phys. Rev. Lett.* **30** (1973) 1343–1346.

[10] J. D. Bekenstein, *Black holes and the second law*, *Lett. Nuovo Cim.* **4** (1972) 737–740.

[11] S. W. Hawking, *Black hole explosions*, *Nature* **248** (1974) 30–31.

[12] A. Strominger and C. Vafa, *Microscopic origin of the Bekenstein-Hawking entropy*, *Phys. Lett.* **B379** (1996) 99–104, [hep-th/9601029].

[13] C. G. Callan and J. M. Maldacena, *D-brane approach to black hole quantum mechanics*, *Nucl. Phys. B* **472** (1996) 591–610, [hep-th/9602043].

[14] G. T. Horowitz and A. Strominger, *Counting states of near extremal black holes*, *Phys. Rev. Lett.* **77** (1996) 2368–2371, [hep-th/9602051].

[15] J. C. Breckenridge, D. A. Lowe, R. C. Myers, A. W. Peet, A. Strominger, and C. Vafa, *Macroscopic and microscopic entropy of near extremal spinning black holes*, *Phys. Lett. B* **381** (1996) 423–426, [hep-th/9603078].

[16] G. T. Horowitz, J. M. Maldacena, and A. Strominger, *Nonextremal black hole microstates and U duality*, *Phys. Lett. B* **383** (1996) 151–159, [hep-th/9603109].

[17] E. Witten, *Anti-de Sitter space, thermal phase transition, and confinement in gauge theories*, *Adv. Theor. Math. Phys.* **2** (1998) 505–532, [hep-th/9803131].

[18] L. Susskind, *The World as a hologram*, *J. Math. Phys.* **36** (1995) 6377–6396, [hep-th/9409089].

[19] A. Sen, *Extremal black holes and elementary string states*, *Mod. Phys. Lett.* **A10** (1995) 2081–2094, [hep-th/9504147].

[20] G. W. Gibbons and S. W. Hawking, *Action Integrals and Partition Functions in Quantum Gravity*, *Phys. Rev.* **D15** (1977) 2752–2756.

[21] W. Chen, H. Lu, and C. N. Pope, *Mass of rotating black holes in gauged supergravities*, *Phys. Rev.* **D73** (2006) 104036, [hep-th/0510081].

[22] H. K. Kunduri and J. Lucietti, *Notes on non-extremal, charged, rotating black holes in minimal D=5 gauged supergravity*, *Nucl. Phys.* **B724** (2005) 343–356, [hep-th/0504158].

[23] A. Cabo-Bizet, D. Cassani, D. Martelli, and S. Murthy, *Microscopic origin of the Bekenstein-Hawking entropy of supersymmetric AdS_5 black holes*, *JHEP* **10** (2019) 062, [arXiv:1810.11442].

[24] F. Larsen, J. Nian, and Y. Zeng, *AdS_5 black hole entropy near the BPS limit*, *JHEP* **06** (2020) 001, [arXiv:1907.02505].

[25] K. Goldstein, V. Jejjala, Y. Lei, S. van Leuven, and W. Li, *Probing the EVH limit of supersymmetric AdS black holes*, *JHEP* **02** (2020) 154, [arXiv:1910.14293].

[26] L. V. Iliesiu and G. J. Turiaci, *The statistical mechanics of near-extremal black holes*, *JHEP* **05** (2021) 145, [arXiv:2003.02860].

[27] M. Heydeman, L. V. Iliesiu, G. J. Turiaci, and W. Zhao, *The statistical mechanics of near-BPS black holes*, *J. Phys. A* **55** (2022), no. 1 014004, [arXiv:2011.01953].

[28] J. Boruch, M. T. Heydeman, L. V. Iliesiu, and G. J. Turiaci, *BPS and near-BPS black holes in AdS_5 and their spectrum in $\mathcal{N}=4$ SYM*, arXiv:2203.01331.

[29] G. J. Turiaci, *New insights on near-extremal black holes*, arXiv:2307.10423.

[30] M. Beccaria and A. Cabo-Bizet, *Large black hole entropy from the giant brane expansion*, arXiv:2308.05191.

[31] A. Cabo-Bizet, *On the 4d superconformal index near roots of unity: bulk and localized contributions*, *JHEP* **02** (2023) 134, [arXiv:2111.14941].

[32] A. Arabi Ardehali and S. Murthy, *The 4d superconformal index near roots of unity and 3d Chern-Simons theory*, arXiv:2104.02051.

[33] A. A. Ardehali and J. Hong, *Decomposition of BPS Moduli Spaces and Asymptotics of Supersymmetric Partition Functions*, arXiv:2110.01538.

[34] V. Jejjala, Y. Lei, S. van Leuven, and W. Li, *$SL(3,\mathbb{Z})$ Modularity and New Cardy Limits of the $\mathcal{N}=4$ Superconformal Index*, arXiv:2104.07030.

[35] S. Choi, J. Kim, S. Kim, and J. Nahmgoong, *Large AdS black holes from QFT*, arXiv:1810.12067.

[36] F. Benini and E. Milan, *Black Holes in 4D $\mathcal{N}$=4 Super-Yang-Mills Field Theory*, *Phys. Rev. X* **10** (2020), no. 2 021037, [arXiv:1812.09613].

[37] M. Honda, *Quantum Black Hole Entropy from 4d Supersymmetric Cardy formula*, *Phys. Rev.* **D100** (2019), no. 2 026008, [arXiv:1901.08091].

[38] A. Arabi Ardehali, *Cardy-like asymptotics of the 4d $\mathcal{N}=4$ index and AdS_5 blackholes*, *JHEP* **06** (2019) 134, [arXiv:1902.06619].

[39] J. Kim, S. Kim, and J. Song, *A 4d $\mathcal{N}$ = 1 Cardy Formula*, *JHEP* **01** (2021) 025, [arXiv:1904.03455].

[40] A. Cabo-Bizet, D. Cassani, D. Martelli, and S. Murthy, *The asymptotic growth of states of the 4d $\mathcal{N}=1$ superconformal index*, *JHEP* **08** (2019) 120, [arXiv:1904.05865].

[41] A. Cabo-Bizet and S. Murthy, *Supersymmetric phases of 4d $\mathcal{N}$ = 4 SYM at large N*, *JHEP* **09** (2020) 184, [arXiv:1909.09597].

[42] A. Arabi Ardehali, J. Hong, and J. T. Liu, *Asymptotic growth of the 4d $\mathcal{N}$ = 4 index and partially deconfined phases*, *JHEP* **07** (2020) 073, [arXiv:1912.04169].

[43] A. Cabo-Bizet, D. Cassani, D. Martelli, and S. Murthy, *The large-N limit of the 4d $\mathcal{N}$ = 1 superconformal index*, *JHEP* **11** (2020) 150, [arXiv:2005.10654].

[44] F. Benini, E. Colombo, S. Soltani, A. Zaffaroni, and Z. Zhang, *Superconformal indices at large N and the entropy of $AdS_5 \times SE_5$ black holes*, *Class. Quant. Grav.* **37** (2020), no. 21 215021, [arXiv:2005.12308].

[45] A. González Lezcano, J. Hong, J. T. Liu, and L. A. Pando Zayas, *Sub-leading Structures in Superconformal Indices: Subdominant Saddles and Logarithmic Contributions*, arXiv:2007.12604.

[46] C. Copetti, A. Grassi, Z. Komargodski, and L. Tizzano, *Delayed deconfinement and the Hawking-Page transition*, *JHEP* **04** (2022) 132, [arXiv:2008.04950].

[47] A. Cabo-Bizet, *From multi-gravitons to Black holes: The role of complex saddles*, arXiv:2012.04815.

[48] A. Amariti, M. Fazzi, and A. Segati, *Expanding on the Cardy-like limit of the SCI of 4d $\mathcal{N} = 1$ ABCD SCFTs*, *JHEP* **07** (2021) 141, [arXiv:2103.15853].

[49] D. Cassani and Z. Komargodski, *EFT and the SUSY Index on the 2nd Sheet*, *SciPost Phys.* **11** (2021) 004, [arXiv:2104.01464].

[50] O. Aharony, F. Benini, O. Mamroud, and E. Milan, *A gravity interpretation for the Bethe Ansatz expansion of the $\mathcal{N} = 4$ SYM index*, *Phys. Rev. D* **104** (2021) 086026, [arXiv:2104.13932].

[51] S. Choi, S. Jeong, S. Kim, and E. Lee, *Exact QFT duals of AdS black holes*, *JHEP* **09** (2023) 138, [arXiv:2111.10720].

[52] O. Mamroud, *The SUSY index beyond the Cardy limit*, *JHEP* **01** (2024) 111, [arXiv:2212.11925].

[53] S. Choi, S. Kim, and J. Song, *Large N Universality of 4d $\mathcal{N} = 1$ Superconformal Index and AdS Black Holes*, arXiv:2309.07614.

[54] O. Aharony, O. Mamroud, S. Nowik, and M. Weissman, *The Bethe Ansatz for the superconformal index with unequal angular momenta*, arXiv:2402.03977.

[55] P. Agarwal, S. Choi, J. Kim, S. Kim, and J. Nahmgoong, *AdS black holes and finite N indices*, arXiv:2005.11240.

[56] S. Murthy, *The growth of the $\frac{1}{16}$-BPS index in 4d $\mathcal{N} = 4$ SYM*, arXiv:2005.10843.

[57] A. Cabo-Bizet, *Quantum phases of 4d SU(N) $\mathcal{N} = 4$ SYM*, *JHEP* **10** (2022) 052, [arXiv:2111.14942].

[58] Z. W. Chong, M. Cvetic, H. Lu, and C. N. Pope, *General non-extremal rotating black holes in minimal five-dimensional gauged supergravity*, *Phys. Rev. Lett.* **95** (2005) 161301, [hep-th/0506029].

[59] J. B. Gutowski and H. S. Reall, *Supersymmetric AdS_5 black holes*, *JHEP* **02** (2004) 006, [hep-th/0401042].

[60] J. B. Gutowski and H. S. Reall, *General supersymmetric AdS(5) black holes*, *JHEP* **04** (2004) 048, [hep-th/0401129].

[61] D. Cassani and L. Papini, *The BPS limit of rotating AdS black hole thermodynamics*, *JHEP* **09** (2019) 079, [arXiv:1906.10148].

[62] N. Bobev, K. Hristov, and V. Reys, *AdS_5 holography and higher-derivative supergravity*, *JHEP* **04** (2022) 088, [arXiv:2112.06961].

[63] D. Cassani, A. Ruipérez, and E. Turetta, *Corrections to AdS_5 black hole thermodynamics from higher-derivative supergravity*, *JHEP* **11** (2022) 059, [arXiv:2208.01007].

[64] J. Maldacena, D. Stanford, and Z. Yang, *Conformal symmetry and its breaking in two dimensional Nearly Anti-de-Sitter space*, *PTEP* **2016** (2016), no. 12 12C104, [arXiv:1606.01857].

[65] K. Jensen, *Chaos in AdS$_2$ Holography*, *Phys. Rev. Lett.* **117** (2016), no. 11 111601, [arXiv:1605.06098].

[66] A. Kitaev and S. J. Suh, *The soft mode in the Sachdev-Ye-Kitaev model and its gravity dual*, *JHEP* **05** (2018) 183, [arXiv:1711.08467].

[67] A. Almheiri and J. Polchinski, *Models of AdS$_2$ backreaction and holography*, *JHEP* **11** (2015) 014, [arXiv:1402.6334].

[68] Z. W. Chong, M. Cvetic, H. Lu, and C. N. Pope, *Five-dimensional gauged supergravity black holes with independent rotation parameters*, *Phys. Rev.* **D72** (2005) 041901, [hep-th/0505112].

[69] S.-Q. Wu, *General Nonextremal Rotating Charged AdS Black Holes in Five-dimensional $U(1)^3$ Gauged Supergravity: A Simple Construction Method*, *Phys. Lett. B* **707** (2012) 286–291, [arXiv:1108.4159].

[70] M. Cvetic, G. W. Gibbons, H. Lu, and C. N. Pope, *Rotating black holes in gauged supergravities: Thermodynamics, supersymmetric limits, topological solitons and time machines*, hep-th/0504080.

[71] E. Gava, G. Milanesi, K. S. Narain, and M. O'Loughlin, *1/8 BPS states in Ads/CFT*, *JHEP* **05** (2007) 030, [hep-th/0611065].

[72] H. Lin, O. Lunin, and J. M. Maldacena, *Bubbling AdS space and 1/2 BPS geometries*, *JHEP* **10** (2004) 025, [hep-th/0409174].

[73] L. Maoz and V. S. Rychkov, *Geometry quantization from supergravity: The Case of 'Bubbling AdS'*, *JHEP* **08** (2005) 096, [hep-th/0508059].

[74] C.-M. Chang and Y.-H. Lin, *Holographic covering and the fortuity of black holes*, arXiv:2402.10129.

[75] E. Deddo, J. T. Liu, L. A. Pando Zayas, and R. J. Saskowski, *The Giant Graviton Expansion from Bubbling Geometry*, arXiv:2402.19452.

[76] C.-M. Chang, Y.-H. Lin, and J. Wu, *On $\frac{1}{8}$-BPS black holes and the chiral algebra of $\mathcal{N} = 4$ SYM*, arXiv:2310.20086.

[77] G. Eleftheriou, *Root of unity asymptotics for Schur indices of 4d Lagrangian theories*, *JHEP* **01** (2023) 081, [arXiv:2207.14271].

[78] J. S. Cotler, G. Gur-Ari, M. Hanada, J. Polchinski, P. Saad, S. H. Shenker, D. Stanford, A. Streicher, and M. Tezuka, *Black Holes and Random Matrices*, *JHEP* **05** (2017) 118, [arXiv:1611.04650]. [Erratum: JHEP 09, 002 (2018)].

[79] P. Saad, S. H. Shenker, and D. Stanford, *A semiclassical ramp in SYK and in gravity*, arXiv:1806.06840.

[80] G. J. Turiaci and E. Witten, *$\mathcal{N} = 2$ JT Supergravity and Matrix Models*, arXiv:2305.19438.

[81] Y. Chen, *Spectral form factor for free large N gauge theory and strings*, *JHEP* **06** (2022) 137, [arXiv:2202.04741].

[82] Y. Chen and G. J. Turiaci, *Spin-Statistics for Black Hole Microstates*, arXiv:2309.03478.

[83] S. Choi, S. Kim, and J. Song, *Supersymmetric Spectral Form Factor and Euclidean Black Holes*, *Phys. Rev. Lett.* **131** (2023), no. 15 151602, [arXiv:2206.15357].

[84] J. Kinney, J. M. Maldacena, S. Minwalla, and S. Raju, *An Index for 4 dimensional super conformal theories*, *Commun. Math. Phys.* **275** (2007) 209–254, [hep-th/0510251].

[85] L. F. Alday, J. R. David, E. Gava, and K. S. Narain, *Towards a string bit formulation of N=4 super Yang-Mills*, *JHEP* **04** (2006) 014, [hep-th/0510264].

[86] O. Aharony, J. Marsano, S. Minwalla, K. Papadodimas, and M. Van Raamsdonk, *The Hagedorn - deconfinement phase transition in weakly coupled large N gauge theories*, *Adv. Theor. Math. Phys.* **8** (2004) 603–696, [hep-th/0310285].

[87] C. Romelsberger, *Counting chiral primaries in N = 1, d=4 superconformal field theories*, *Nucl. Phys.* **B747** (2006) 329–353, [hep-th/0510060].

[88] F. Dolan, *Counting BPS operators in N=4 SYM*, *Nucl. Phys. B* **790** (2008) 432–464, [arXiv:0704.1038].

[89] F. A. Dolan and H. Osborn, *Applications of the Superconformal Index for Protected Operators and q-Hypergeometric Identities to N=1 Dual Theories*, *Nucl. Phys. B* **818** (2009) 137–178, [arXiv:0801.4947].

[90] S. Nawata, *Localization of N=4 Superconformal Field Theory on $S^1 \times S^3$ and Index*, *JHEP* **11** (2011) 144, [arXiv:1104.4470].

[91] A. Gadde, L. Rastelli, S. S. Razamat, and W. Yan, *Gauge Theories and Macdonald Polynomials*, *Commun. Math. Phys.* **319** (2013) 147–193, [arXiv:1110.3740].

[92] S. Garoufalidis and D. Zagier, *Asymptotics of Nahm sums at roots of unity*, *Ramanujan J.* **55** (2021), no. 1 219–238, [arXiv:1812.07690].

[93] S. de Haro, S. N. Solodukhin, and K. Skenderis, *Holographic reconstruction of space-time and renormalization in the AdS / CFT correspondence*, *Commun. Math. Phys.* **217** (2001) 595–622, [hep-th/0002230].

[94] F. Benini, S. Soltani, and Z. Zhang, *A quantum mechanics for magnetic horizons*, *JHEP* **05** (2023) 070, [arXiv:2212.00672].

[95] S. M. Hosseini, K. Hristov, and A. Zaffaroni, *An extremization principle for the entropy of rotating BPS black holes in AdS5*, *Journal of High Energy Physics* **2017** (2017), no. 7 [arXiv:1705.05383].

[96] C.-M. Chang and Y.-H. Lin, *Words to describe a black hole*, *Journal of High Energy Physics* **2023** (Feb., 2023).

[97] S. Choi, S. Kim, E. Lee, and J. Park, *The shape of non-graviton operators for SU(2)*, arXiv:2209.12696.

[98] S. Choi, S. Kim, E. Lee, S. Lee, and J. Park, *Towards quantum black hole microstates*, *JHEP* **11** (2023) 175, [arXiv:2304.10155].

[99] C.-M. Chang, L. Feng, Y.-H. Lin, and Y.-X. Tao, *Decoding stringy near-supersymmetric black holes*, arXiv:2306.04673.

[100] K. Budzik, H. Murali, and P. Vieira, *Following Black Hole States*, arXiv:2306.04693.

[101] T. G. Mertens, G. J. Turiaci, and H. L. Verlinde, *Solving the Schwarzian via the Conformal Bootstrap*, *JHEP* **08** (2017) 136, [arXiv:1705.08408].

[102] A. Ghosh, H. Maxfield, and G. J. Turiaci, *A universal Schwarzian sector in two-dimensional conformal field theories*, *JHEP* **05** (2020) 104, [arXiv:1912.07654].