Riemann-Hilbert problems, Toeplitz operators and ergosurfaces

M. Cristina Câmara and Gabriel Lopes Cardoso

Center for Mathematical Analysis, Geometry and Dynamical Systems,
Department of Mathematics, Instituto Superior Técnico, Universidade de Lisboa,
Av. Rovisco Pais, 1049-001 Lisboa, Portugal

cristina.camara@tecnico.ulisboa.pt, gabriel.lopes.cardoso@tecnico.ulisboa.pt

Abstract

The Riemann-Hilbert approach, in conjunction with the canonical Wiener-Hopf factorisation of certain matrix functions called monodromy matrices, enables one to obtain explicit solutions to the non-linear field equations of some gravitational theories. These solutions are encoded in the elements of a matrix M depending on the Weyl coordinates ρ and v, determined by that factorisation. We address here, for the first time, the underlying question of what happens when a canonical Wiener-Hopf factorisation does not exist, using the close connection of Wiener-Hopf factorisation with Toeplitz operators to study this question. For the case of rational monodromy matrices, we prove that the non-existence of a canonical Wiener-Hopf factorisation determines curves in the (ρ, v) plane on which some elements of $M(\rho, v)$ tend to infinity, but where the space-time metric may still be well behaved. In the case of uncharged rotating black holes in four space-time dimensions and, for certain choices of coordinates, in five space-time dimensions, we show that these curves correspond to their ergosurfaces.

Keywords: General Relativity, classical integrable systems, Riemann-Hilbert problems, Wiener-Hopf factorisation, Toeplitz operators.

1 Introduction

It is a remarkable and rather surprising fact that a great variety of problems in mathematics, physics and engineering – in diffraction theory, elastodynamics, control theory, integrable systems and, more recently, in the study of various compressions of multiplication operators such as truncated and dual-band Toeplitz operators – can be reformulated as, or reduced to a Riemann-Hilbert problem [15, 26, 17, 20, 10, 14, 9, 22, 23].

One field of application of the Riemann-Hilbert approach is the study of the solution space of the Einstein field equations in D space-time dimensions. Building on the work of [4, 25], this field of application has become the object of revived interest, see for instance [21, 12, 7, 11, 2, 30, 6] and references therein. Here we study the cases of $D = 4$ and $D = 5$. The resulting PDE's, after reduction to two dimensions, form an integrable system [4, 25], i.e., they appear as a compatibility condition for an auxiliary linear system, called a Lax pair [20]. The Lax pair that underlies the work of [4, 25] is called Breitenlohner-Maison linear system. There exists another Lax pair, called Belinski-Zakharov linear system, which has been shown to be equivalent to the Breitenlohner-Maison linear system [16], and which is particularly well suited to obtain exact solutions of solitonic type from a seed solution by the inverse scattering method [3]. Even though these two linear systems are equivalent, they originate different approaches to solving the field equations. The approach based on the Breitenlohner-Maison linear system, which we will follow here, does not require knowledge of a seed solution and has the advantage of making use of the group structure that underlies the dimensionally reduced model.

The Breitenlohner-Maison linear system [4, 25] is a Lax pair depending on two real coordinates which will be denoted by (ρ, v) (Weyl coordinates). It also depends on a complex parameter τ that varies on an algebraic curve, called the spectral curve, that depends on (ρ, v). The Riemann-Hilbert approach based on this Lax pair allows for the explicit construction of solutions of the Einstein field equations by means of the canonical Wiener-Hopf factorisation of monodromy matrices with respect to a contour Γ in the complex τ-plane, as proven in [2]. In this factorisation, the coordinates (ρ, v) play the role of parameters.

This approach necessarily assumes the existence of a canonical Wiener-Hopf factorisation. But does the latter always exist? And what happens if it does not?

Here we use for the first time the close connection of Toeplitz operators with Wiener-Hopf factorisation to establish necessary and sufficient conditions for the factorisation of a rational monodromy matrix to be canonical, and we show that a canonical Wiener-Hopf factorisation may not exist on certain curves $D(\rho, v) = 0$.

Note that, due to the role of the spectral curve in the case that we are studying, the question addressed here is different from the well studied question in factorisation theory, of when does a general rational matrix in the variable τ have a canonical Wiener-Hopf factorisation. For instance, it follows from a theorem in [11] that *any* monodromy matrix which can be reduced to triangular form by multiplication by constant matrices admits a canonical Wiener-Hopf factorisation, in sharp contrast with what happens with general triangular matrices [13, 27, 24, 1, 18].

When performing the canonical Wiener-Hopf factorisation of a monodromy matrix $\mathcal{M}_{\rho,v}(\tau)$ with respect to a certain contour Γ, one of the factors that arise in the factorisation determines the space-time solution. This is the factor $\mathcal{M}^-_{\rho,v}(\tau)$, see (12). The limit of $\mathcal{M}^-_{\rho,v}(\tau)$ when $\tau \to \infty$, $M(\rho, v)$, encodes the space-time metric, which can be obtained from the elements of $M(\rho, v)$ as explained in [4, 12], see also Section 2 and Appendix A.

A natural question then is: what happens to $M(\rho, v)$ when (ρ, v) approaches the points on a curve $D(\rho, v) = 0$ where the monodromy matrix does not admit a canonical factorisation? We show that elements in the matrix $M(\rho, v)$ blow up when approaching such points. Since the matrix $M(\rho, v)$ determines the space-time metric, we study the behaviour of the latter for (ρ, v) on the curve $\mathcal{C}$ defined by $D(\rho, v) = 0$. We show that in four dimensions this corresponds to g_{tt}, the component in the line element which is proportional to dt^2, vanishing and, although some elements of $M(\rho, v)$ may tend to infinity as one approaches $\mathcal{C}$, this does not imply that the space-time metric is ill-behaved on that curve. In the case of the four-dimensional non-extremal Kerr black hole in General Relativity it can be verified that the space-time metric is well behaved for all (ρ, v) and that the locus $D(\rho, v) = 0$ corresponds to the ergosurface of the black hole, where the norm of the Killing vector $\partial/\partial t$ vanishes. This is shown in Section 4, where we also discuss the case of the five-dimensional rotating Myers-Perry black hole carrying one angular momentum. We show that in this case the space-time metric is also well behaved but now g_{tt} may or may not vanish when (ρ, v) lies on the curve $\mathcal{C}$, depending on the choice of coordinates in five dimensions.

These results moreover shed new light on the relations between the matrix $M(\rho, v)$ and the corresponding space-time metric, showing that a singular behaviour in the elements of $M(\rho, v)$ may not be reflected in the space-time metric.

The paper is organised as follows. To keep it as self-contained as possible, in Section 2 we briefly review the description of the dimensionally reduced gravitational field equations as an integrable system associated with a Lax pair (the Breitenlohner-Maison linear system), and we describe how to construct solutions to the field equations by means of canonical Wiener-Hopf factorisations, if the latter exist. In Section 3, which contains the main results of the paper, we use the close relation of canonical Wiener-Hopf factorisation with Toeplitz operators to show that the canonical factorisation of a rational monodromy matrix may not exist on certain curves in the (ρ, v) plane, and that elements of $M(\rho, v)$ blow up there. We prove this for general 2×2 monodromy matrices, but our approach can be extended to the case of $n \times n$ rational monodromy matrices in a straightforward manner. In Section 4, we illustrate the above with two examples. In the first example we discuss a 2×2 monodromy matrix, whose canonical Wiener-Hopf factorisation with respect to a suitably chosen contour Γ yields a space-time solution describing the exterior region of the four-dimensional non-extremal Kerr black hole in General Relativity. We show that in this case the curve $D(\rho, v) = 0$ on which the canonical Wiener-Hopf factorisation ceases to exist is the curve for the ergosurface of the black hole. In the second example, we consider two 3×3 monodromy matrices. In both cases, their canonical Wiener-Hopf factorisation with respect to suitably chosen contours Γ yield the same Myers-Perry solution, although written in different coordinates. This solution describes the exterior region of a five-dimensional rotating black hole with one angular momentum. In each of these cases, the canonical Wiener-Hopf factorisation ceases to exist on a certain curve. It coincides with the ergosurface of the black hole in one case. In the other case, there is no such correspondence, and we see that the choice of coordinates in five space-time dimensions allows for a matrix $M(\rho, v)$ which is well behaved on the ergosurface. Finally, in Appendix A we give a few details of the canonical Wiener-Hopf factorisation of one of the 3×3 monodromy matrices mentioned above.

2 Solving the field equations by canonical Wiener-Hopf factorisation

The field equations of gravitational theories in D space-time dimensions form a system of non-linear PDE's for the space-time metric and matter fields. Due to their non-linear nature, obtaining exact solutions to these PDE's is a highly non-trivial task. By restricting attention to the subclass of solutions that only depend on two of the D space-time coordinates, the field equations become effectively two-dimensional, and in certain cases powerful methods for constructing solutions to these field equations become available. This is the case of the gravitational theories discussed in [29, 31, 25], whose two-step reduction to two dimensions yields the following matricial non-linear field equation in two-dimensions,

$$d\,(\rho \star A) = 0 \quad , \quad \text{with } \; A = M^{-1} dM \; , \tag{1}$$

where $M \in G/H$ is a coset representative of the symmetric space G/H that arises in the two-step reduction [5]. G/H is invariant under an involution $\natural$ called *generalized transposition*, i.e. $M^\natural = M$, and M depends on two coordinates, denoted here by ρ and v, with $\rho > 0$ and $v \in \mathbb{R}$, called Weyl coordinates. In (1), $\star$ denotes the Hodge star operator in two dimensions; we have that

$$\star d\rho = -\lambda\, dv \quad , \quad \star dv = d\rho \quad , \quad (\star)^2 = -\lambda\, \mathrm{id} \; , \tag{2}$$

where $\lambda = \pm 1$ depending on whether (ρ, v) are both space-like coordinates ($\lambda = 1$) or whether one of the two is time-like ($\lambda = -1$). When $D = 4$, a solution M to (1) yields a gravitational solution whose four-dimensional space-time metric is given by

$$ds_4^2 = -\lambda\,\Delta(dt + Bd\phi)^2 + \Delta^{-1}\left(e^{\psi}\,ds_2^2 + \rho^2 d\phi^2\right)\ , \tag{3}$$

where $ds_2^2 = \sigma d\rho^2 + \varepsilon dv^2$ with $\sigma\varepsilon = \lambda$, Δ and B are functions of (ρ, v) determined by the solution $M(\rho, v)$ of (1) and $\psi(\rho, v)$ is a scalar function determined from $M(\rho, v)$ by integration [31, 25], see also eq. (2.7) in [2]. As an example, consider the two-step reduction of the four-dimensional Einstein-Hilbert action to two dimensions. The resulting coset is $G/H = SL(2,\mathbb{R})/SO(2)$, the involution $\natural$ is matrix transposition and the coset representative M takes the form

$$M = \begin{pmatrix} \Delta + \tilde{B}^2/\Delta & \tilde{B}/\Delta \\ \tilde{B}/\Delta & 1/\Delta \end{pmatrix}\ , \tag{4}$$

where $\tilde{B}$ is related with B through $\rho \star d\tilde{B} = \Delta^2\, dB$ [4].

The non-linear field equation (1) is the compatibility condition for an auxiliary linear system of differential equations (a Lax pair) given by

$$\tau\,(dX + AX) = \star\, dX \tag{5}$$

([25]) called the Breitenlohner-Maison linear system [4]. Here τ denotes a complex parameter, called spectral parameter, associated through the following algebraic relation with another complex variable ω and the coordinates (ρ, v),

$$\omega = v + \frac{\lambda}{2}\,\rho\,\frac{\lambda - \tau^2}{\tau}\quad ,\quad \tau \in \mathbb{C}\backslash\{0\}\ , \tag{6}$$

which we will refer to as the spectral relation. The significance of the linear system (5) is explained in [25, 2]. One powerful method to obtain a pair (X, A) where X is a solution to the Lax pair (5) with input $A = M^{-1}dM$, and M is a solution to the field equation (1), consists in obtaining a canonical bounded Wiener-Hopf (WH) factorisation, also known as Birkhoff factorisation, of a so-called monodromy matrix ([2]), defined as follows.

Let $\mathcal{M}(\omega)$ be a $\natural$-invariant $n \times n$ matrix function of the complex variable ω and denote by $\mathcal{M}_{\rho,v}(\tau)$ the matrix that is obtained from $\mathcal{M}(\omega)$ by composition using the spectral relation (6), i.e.,

$$\mathcal{M}_{\rho,v}(\tau) = \mathcal{M}\left(v + \frac{\lambda}{2}\,\rho\,\frac{\lambda - \tau^2}{\tau}\right)\ . \tag{7}$$

We call $\mathcal{M}_{\rho,v}(\tau)$ a monodromy matrix.

Let now Γ be a simple closed contour in $\mathbb{C}$, encircling the origin, and invariant under the involution $\tau \mapsto -\lambda/\tau$, and let $\mathbb{D}_\Gamma^+$ and $\mathbb{D}_\Gamma^-$ denote the interior and the exterior of Γ (including the point ∞), respectively.

Definition 2.1. A bounded Wiener-Hopf (WH) factorisation of $\mathcal{M}_{\rho,v}(\tau)$ with respect to Γ is a decomposition of the form

$$\mathcal{M}_{\rho,v}(\tau) = \mathcal{M}_{\rho,v}^-(\tau)\, D(\tau)\, \mathcal{M}_{\rho,v}^+(\tau)\quad ,\quad \tau \in \Gamma \tag{8}$$

such that the $n \times n$ matrix functions $\mathcal{M}_{\rho,v}^{\pm}(\tau)$, as well as their inverses, admit an analytic and bounded extension to $\mathbb{D}_\Gamma^{\pm}$, respectively, and $D(\tau)$ is a diagonal matrix with diagonal elements of the form τ^{k_i}, $i = 1, 2, \dots, n$, with $k_i \in \mathbb{Z}$. If $D(\tau) = \mathbb{I}_{n\times n}$, i.e. $k_i = 0$ for all $i = 1, 2, \dots, n$, then the factorisation

$$\mathcal{M}_{\rho,v}(\tau) = \mathcal{M}_{\rho,v}^-(\tau)\, \mathcal{M}_{\rho,v}^+(\tau)\quad ,\quad \tau \in \Gamma \tag{9}$$

is said to be *canonical.*

We will use the abbreviated forms WH factorisation and canonical WH factorisation to denote (8) and (9), respectively.

We will consider here only matrix functions which, for fixed (ρ, v) and variable τ, have elements in the algebra $C^\mu(\Gamma)$ of all Hölder continuous functions on Γ with exponent $\mu \in]0,1[$ ([27]). It is well known that every matrix function in $(C^\mu(\Gamma))^{n\times n}$, invertible in this algebra, admits a WH factorisation with factors in the same algebra. If this factorisation is canonical, then it is unique once one imposes the normalisation condition

$$\mathcal{M}^+_{\rho,v}(0) = \mathbb{I}_{n\times n} \ . \tag{10}$$

With this normalisation condition, we denote

$$\mathcal{M}^+_{\rho,v}(\tau) =: X(\tau, \rho, v) \ . \tag{11}$$

It was shown in Theorem 6.1 of [2] that, under very general assumptions, if a canonical WH factorisation of $\mathcal{M}_{\rho,v}(\tau)$ with respect to a contour Γ (satisfying the conditions above) exists, then a solution M to the field equation (1) is given by

$$M(\rho, v) := \lim_{\tau\to\infty} \mathcal{M}^-_{\rho,v}(\tau), \tag{12}$$

while $X(\tau, \rho, v)$ is a solution to the linear system (5) with input $A = M^{-1}dM$.

Two questions naturally arise at this point. The first is the question of *existence* of a canonical WH factorisation. The second is the question of the *behaviour* of the solution $M(\rho, v)$, obtained from a canonical WH factorisation, when (ρ, v) approaches a point (ρ_0, v_0) in the plane of the Weyl coordinates for which a canonical WH factorisation of $\mathcal{M}_{\rho_0,v_0}(\tau)$ does not exist. We address these two questions in the next section.

Note that, since every invertible matrix in $(C^\mu(\Gamma))^{n\times n}$, in the variable τ, with determinant equal to 1 (as will be our case), has a WH factorisation ([27]), for such a point (ρ_0, v_0) the corresponding monodromy matrix will have a *non canonical* WH factorisation (8). However, there is no point in determining the latter if one wants to answer the question of behaviour of the solutions, since the factors (8) are not the limit, when $(\rho, v) \to (\rho_0, v_0)$, of those in a canonical factorisation valid for points (ρ, v) in a neighbourhood of (ρ_0, v_0).

3 Toeplitz operators, Wiener-Hopf factorisation and ergosurfaces

To address the fundamental question of *existence* of a canonical WH factorisation for a monodromy matrix underlying all results obtained in the literature using the Breitenlohner-Maison factorisation approach, we will use the close connection of WH factorisation with Toeplitz operators. We start by briefly explaining this relation.

Let Γ be a simple closed curve Γ in the complex τ-plane encircling 0, positively oriented and invariant under the involution $\tau \mapsto -\lambda/\tau$. We call such a contour *an admissible contour.* Let us denote by S_Γ the singular integral operator with Cauchy kernel on $L^2(\Gamma)$,

$$(S_\Gamma f)(\tau) = \frac{1}{\pi i}\,\text{p.v.}\int_\Gamma \frac{f(z)}{z-\tau}\,dz \quad , \quad \tau \in \Gamma, \tag{13}$$

where p.v. denotes Cauchy's principal value. Then one can define two complementary orthogonal projections

$$P^\pm_\Gamma = \frac{1}{2}\left(I \pm S_\Gamma\right) \ . \tag{14}$$

The functions in $H^2_+ := P^+_\Gamma L^2(\Gamma)$ have an analytic extension to the interior of Γ, while the functions in $H^2_- := P^-_\Gamma L^2(\Gamma)$ have an analytic extension to the exterior of Γ and vanish at ∞. We can write

$$L^2(\Gamma) = H^2_- \oplus H^2_+ \,, \tag{15}$$

so every function in $L^2(\Gamma)$ admits a unique decomposition $f = f_- + f_+$ with $f_\pm \in H^2_\pm$. It may be shown that S_Γ maps $C^\mu(\Gamma)$ into $C^\mu(\Gamma)$ and $C^\mu_+(\Gamma) := P^+_\Gamma C^\mu(\Gamma)$, $C^\mu_-(\Gamma) := P^-_\Gamma C^\mu(\Gamma) \oplus \mathbb{C}$ are closed subalgebras of $C^\mu(\Gamma)$. For this reason, $C^\mu(\Gamma)$ is called a decomposing algebra ([27]). Every invertible element of $C^\mu(\Gamma)$ has a WH factorisation with factors in the same algebra, and this result extends to matrix functions in $(C^\mu(\Gamma))^{n\times n}$ ([27]). However, the question of determining whether or not this factorisation is canonical is a rather nontrivial one in general for matrix functions. It turns out that this question may be formulated in terms of Toeplitz operators.

Toeplitz operators are compressions of multiplication operators into the Hardy space H^2_+ (or its vectorial analogue, in the matricial case). Concretely, given an $n \times n$ matrix function G whose elements are bounded functions on Γ, the Toeplitz operator T_G is defined by

$$T_G = P^+ G P^+|_{(H^2_+)^n} : (H^2_+)^n \to (H^2_+)^n. \tag{16}$$

G is called the symbol of the operator T_G. There exists a close connection between the study of Toeplitz operators and the theory of WH factorisation. Indeed, the operator T_G is Fredholm, i.e. it has a closed range and finite dimensional kernel and cokernel, if and only if G admits a WH factorisation; T_G is invertible if and only if that factorisation is canonical (see [8]) and references therein).

It follows from the above results on WH factorisation of matrix functions in $(C^\mu(\Gamma))^{n\times n}$ that, if G is invertible in that algebra, T_G is always Fredholm [13, 27, 24]. In that case, assuming moreover that $\det G = 1$, we have that

$$\dim\ker T_G - \text{codim Im} T_G = 0, \tag{17}$$

i.e., the Fredholm index of T_G is zero. Here $\ker T_G$ denotes the kernel of T_G, Im T_G denotes its range and codim Im $T_G = \dim\big((H^2_+)^n/\text{Im}T_G\big)$. Therefore, T_G is invertible if and only if it is injective. As a consequence, G has a canonical WH factorisation if and only if $\ker T_G = \{0\}$, or, equivalently, the Riemann-Hilbert problem

$$G\phi_+ = \phi_- \quad , \quad \text{with} \quad \phi_\pm \in (H^2_\pm)^n, \tag{18}$$

whose solutions ϕ_+ constitute the kernel of T_G, admits only the zero solution. We summarize the above relations as follows.

Theorem 3.1. *If Γ is an admissible contour in the complex plane and $G \in (C^\mu(\Gamma))^{n\times n}$, $\mu \in]0,1[$, with $\det G = 1$, then G has a canonical WH factorisation if and only if the Toeplitz operator T_G in $(H^2_+)^n$ is injective, i.e., if and only if (18) admits only the trivial solution $\phi_\pm = 0$. In that case the factors $G_\pm$ in a canonical WH factorisation $G = G_- G_+$ belong to $(C^\mu_\pm)^{n\times n}$, as well as their inverses, and the i^{th} column of G_+^{-1} (respectively G_-) is given by the solution ψ_+ (respectively ψ_-) of the Riemann-Hilbert problem*

$$G\psi^i_+ = \psi^i_- \quad \textit{with} \quad \psi^i_\pm \in (C^\mu_\pm)^n \tag{19}$$

satisfying the conditions $\psi_+(0) = [\delta_{ij}]^T_{j=1,\dots,n}$.

Remark 3.2. Note that, although formally similar, (18) and (19) are different Riemann-Hilbert problems, since we seek their solutions in different spaces. In particular we have that $\phi_-(\infty) = 0$, while $\psi^i_-(\infty) \neq 0$ for every $i = 1, \dots, n$.

We now apply these results to monodromy matrices obtained from rational matrix functions $\mathcal{M}(\omega)$ of the form

$$\mathcal{M}(\omega) = \frac{1}{q(\omega)} \begin{pmatrix} p_{11}(\omega) & p_{12}(\omega) \\ p_{12}(\omega) & p_{22}(\omega) \end{pmatrix}, \tag{20}$$

where $q, p_{11}, p_{12}, p_{22}$ denote polynomials of degree $n, k_{11}, k_{12}, k_{22}$, respectively, and $\det \mathcal{M} = 1$, i.e.,

$$q^2 = p_{11}p_{22} - (p_{12})^2 \, . \tag{21}$$

Note that $\mathcal{M}$ must be a symmetric matrix, since $\natural$ is matrix transposition in this case and we must have $\mathcal{M} = \mathcal{M}^\natural$. For simplicity, and taking the applications into account, we also assume that the zeroes of q are all of order 1 and $\lambda = 1$.

Remark 3.3. It is easy to see that the approach presented in this section to address the questions formulated at the end of Section 2 can be extended to the case of $n \times n$ rational matrix functions (examples will be given in Section 4.2 for 3×3 monodromy matrices) and does not depend on the order of the zeroes of q nor on the particular value of λ.

Let us denote the right hand side of (6), with $\lambda = 1$, by $\omega_{\tau,\rho,v}$. Any non-trivial polynomial $p(\omega)$ of degree k, upon composition with $\omega = \omega_{\tau,\rho,v}$, becomes a product of k factors of the form

$$\omega_{\tau,\rho,v} - \omega_0 = \frac{-\frac{\rho}{2}\,\tau^2 + (v - \omega_0)\,\tau + \frac{\rho}{2}}{\tau} \, . \tag{22}$$

The numerator of the right hand side of (22) is a polynomial of degree 2 in τ, which does not vanish for $\tau = 0$, with simple zeroes at

$$\frac{v - \omega_0 \pm \sqrt{(v - \omega_0)^2 + \rho^2}}{\rho} \, . \tag{23}$$

These zeroes appear in pairs, of the form

$$\{\tau_0, -\frac{1}{\tau_0}\}\, ; \tag{24}$$

from each pair we will choose one of the elements. Suppose that the latter is denoted by τ_i; then the other element, $-1/\tau_i$, will be denoted by $\tilde{\tau}_i$. Note that, for any admissible contour Γ, if τ_i does not belong to Γ then neither does $\tilde{\tau}_i$, and we necessarily have one of the points in $\mathbb{D}^+_\Gamma$ and the other in $\mathbb{D}^-_\Gamma$ ([2]).

With this in mind, for any polynomial $p_k(\omega)$ of degree k, let us use the representation (upon composition with $\omega_{\tau,\rho,v}$)

$$p_k(\omega_{\tau,\rho,v}) = \frac{p_{2k}(\tau)}{\tau^k}, \tag{25}$$

where we omit the dependence of the numerator on (ρ, v). With this notation, the monodromy matrix associated with (20) takes the form

$$\mathcal{M}_{\rho,v}(\tau) = \frac{\tau^n}{q_{2n}(\tau)}\, \tilde{\mathcal{M}}_{\rho,v}(\tau) \quad \text{with} \quad \tilde{\mathcal{M}}_{\rho,v}(\tau) = \begin{pmatrix} \frac{\tilde{p}_{2k_{11}}(\tau)}{\tau^{k_{11}}} & \frac{\tilde{p}_{2k_{12}}(\tau)}{\tau^{k_{12}}} \\ \frac{\tilde{p}_{2k_{12}}(\tau)}{\tau^{k_{12}}} & \frac{\tilde{p}_{2k_{22}}(\tau)}{\tau^{k_{22}}} \end{pmatrix}, \tag{26}$$

with

$$\det \mathcal{M}_{\rho,v}(\tau) = \frac{\tilde{p}_{2k_{11}}(\tau)\,\tilde{p}_{2k_{22}}(\tau)}{\tau^{k_{11}+k_{22}}} - \frac{\tilde{p}^2_{2k_{12}}(\tau)}{\tau^{2k_{12}}} = \frac{q^2_{2n}(\tau)}{\tau^{2n}}. \tag{27}$$

Note that, since $q_{2n}(0) \neq 0$ (cf. (22)), we must have, by (21),

$$2n = \max\{k_{11} + k_{22}, 2k_{12}\}. \tag{28}$$

Let moreover, for the first and second rows of $\tilde{\mathcal{M}}_{\rho,v}(\tau)$,

$$N_1 = \max\{k_{11}, k_{12}\} \quad , \quad N_2 = \max\{k_{12}, k_{22}\}. \tag{29}$$

Remark 3.4. It is not difficult to see that the conditions (28) and (29) together imply that we cannot have simultaneously $N_1 = k_{11} > k_{12}$ and $N_2 = k_{12} > k_{22}$, nor can we have simultaneously $N_1 = k_{12} > k_{11}$ and $N_2 = k_{22} > k_{12}$.

In order to formulate our main results in this section, we still need to define the contour with respect to which the factorisation of $\mathcal{M}_{\rho,v}(\tau)$ is considered. So, from each of the n pairs of zeroes of $q_{2n}(\tau)$, of the form (23), we choose one point which we denote by τ_i ($i = 1, \dots, n$), and we denote the other by $\tilde{\tau}_i$, and we take Γ to be any admissible contour such that $\{\tau_i, i = 1, \dots, n\}$ is contained in $\mathbb{D}^+_\Gamma$. Note that Γ may depend on (ρ, v), but we omit this for simplicity of notation, unless necessary. We assume (ρ, v) to be such that $\rho > 0$ and $v \pm i\rho$ are not zeroes of q.

We can now state our main theorems. The proofs will be given in Section 3.1.

We start by addressing *the question of existence* of a canonical WH factorisation of $\mathcal{M}_{\rho,v}(\tau)$ by reducing it to the question of whether the injectivity Riemann-Hilbert problem for the Toeplitz operator with symbol $\mathcal{M}_{\rho,v}(\tau)$, (18), admits only the zero solution.

Theorem 3.5. *With the notation above, for $\mathcal{M}(\omega)$ of the form* (20) *and $\mathcal{M}_{\rho,v}(\tau)$ given by* (7)*, we have that:*
(i) if $N_1 + N_2 < 2n$, then $\mathcal{M}_{\rho,v}(\tau)$ has a canonical WH factorisation w.r.t. Γ, for all ρ, v;
(ii) if $N_1 + N_2 = 2n$, then $\mathcal{M}_{\rho,v}(\tau)$ has a canonical WH factorisation if and only if

$$D(\rho, v) \neq 0\,, \tag{30}$$

where $D(\rho, v)$ is the determinant of the matrix coefficient of the following linear system of $2n$ equations for $2n$ unknowns,

$$\begin{aligned} \left(Q_{N_1-1}\,\tau^{N_2-k_{22}}\,\tilde{p}_{2k_{22}} - Q_{N_2-1}\,\tau^{N_1-k_{12}}\,\tilde{p}_{2k_{12}}\right)(\tau_i) &= 0, \\ \left(Q_{N_1-1}\,\tau^{N_2-k_{22}}\,\tilde{p}_{2k_{22}} - Q_{N_2-1}\,\tau^{N_1-k_{12}}\,\tilde{p}_{2k_{12}}\right)'(\tau_i) &= 0 \quad , \quad i = 1, 2, \dots, n, \end{aligned} \tag{31}$$

where Q_{N_1-1} and Q_{N_2-1} are unknown polynomials of degree at most $N_1 - 1$ and $N_2 - 1$, respectively, their coefficients being the $2n = N_1 + N_2$ unknowns of the system;
(iii) the case where $N_1 + N_2 > 2n$ can be reduced to (ii).

It is easy to see from (26) that, since $\frac{\tau^n}{q_{2n}(\tau)}$ is a scalar function that, by a theorem given in [11, Section 2], admits a canonical WH factorisation with respect to any admissible contour Γ where q_{2n} does not vanish, and that this factorisation can be obtained straightforwardly, the existence of a canonical WH factorisation for $\mathcal{M}_{\rho,v}(\tau)$ is equivalent to that of $\tilde{\mathcal{M}}_{\rho,v}(\tau)$. The latter can be obtained, if it exists, as described in Theorem 3.1. For the particular case (*ii*) of Theorem 3.5, writing $\tilde{\mathcal{M}}_{\rho,v}(\tau) = \tilde{\mathcal{M}}^-_{\rho,v}(\tau)\tilde{X}(\tau, \rho, v)$ according to (9) and (11), we have the following.

Theorem 3.6. *With the same notation as in Theorem 3.5, if $N_1 + N_2 = 2n$ and $D(\rho, v) \neq 0$, each column of $\tilde{X}^{-1}(\tau, \rho, v)$ is given by $\psi_+ = (\psi_{1+}, \psi_{2+})^T$ with*

$$\psi_{1+} = \frac{Q_{N_1}\, \tau^{N_2-k_{22}}\, \tilde{p}_{2k_{22}} - Q_{N_2}\, \tau^{N_1-k_{12}}\, \tilde{p}_{2k_{12}}}{q_{2n}^2(\tau)} \quad , \quad \psi_{2+} = \frac{Q_{N_1} - \tau^{N_1-k_{11}}\, \tilde{p}_{2k_{11}}\, \psi_{1+}}{\tau^{N_1-k_{12}}\, \tilde{p}_{2k_{12}}}, \tag{32}$$

where the $2n + 2$ coefficients of Q_{N_1}, Q_{N_2} are uniquely determined by the analyticity of ψ_{1+} and ψ_{2+} in $\mathbb{D}_\Gamma^+$, together with the normalizing conditions $\psi_{1+}(0) = 1, \psi_{2+}(0) = 0$ for the 1^{st} column, and $\psi_{1+}(0) = 0, \psi_{2+}(0) = 1$ for the second column. The factor $\tilde{\mathcal{M}}^-_{\rho,v}(\tau)$ is given by $\tilde{\mathcal{M}}^-_{\rho,v}(\tau) = \tilde{\mathcal{M}}_{\rho,v}(\tau)\tilde{X}^{-1}(\tau, \rho, v)$, and both $\tilde{X}(\tau, \rho, v)$ and $\tilde{\mathcal{M}}^-_{\rho,v}(\tau)$ are C^∞ functions of (ρ, v).

The equation

$$D(\rho, v) = 0 \tag{33}$$

defines a curve (or a curve system) $\mathcal{C}$ in the plane of the Weyl coordinates ρ, v. A natural question that arises from *(ii)* in Theorem 3.5 regards the behaviour of the solution $M(\rho, v)$, obtained as in (12), for points (ρ, v) that *do not* satisfy (33), as we approach a point $(\rho_0, v_0) \in \mathcal{C}$. We address this question in the following theorem, where g_{tt} denotes the coefficient of the term proportional to dt^2 in (3).

Theorem 3.7. *With the same notation as in Theorem 3.5, let $N_1 + N_2 = 2n$ and let (ρ, v) be such that (30) holds, so that $\mathcal{M}_{\rho,v}(\tau)$ admits a canonical WH factorisation. Let moreover $M(\rho, v)$ be the solution to the field equation (1) given by (12). If (ρ_0, v_0) is a point on $\mathcal{C}$ then $g_{tt} = -\lambda\Delta(\rho, v)$ in (3) tends to 0 as (ρ, v) tends to (ρ_0, v_0).*

In the case when the space-time metric (3) describes the exterior region of a non-extremal Kerr black hole solution in four space-time dimensions, the vanishing of g_{tt} defines the ergosurface of the rotating black hole.

The previous results can be reformulated in terms of Toeplitz operators as follows.

Corollary 3.8. *Let Γ be an admissible contour and let $\mathcal{M}_{\rho,v}(\tau)$ be of the form (26). If $N_1+N_2 < 2n$, or $N_1 + N_2 = 2n$ with $D(\rho, v) \neq 0$ for all (ρ, v), then the Toeplitz operator with symbol $\mathcal{M}_{\rho,v}(\tau)$ is invertible and the space-time metric (3), corresponding to $M(\rho, v)$ defined by (12), is well behaved, for all (ρ, v).*

If $N_1 + N_2 = 2n$ and there are points in the Weyl half-plane satisfying (33), then the Toeplitz operator with symbol $\mathcal{M}_{\rho,v}(\tau)$ is not invertible for (ρ, v) on the curve $D(\rho, v) = 0$ and, in the space-time metric (3), we have $g_{tt} = 0$ on that curve.

3.1 Proofs

The proof of Theorem 3.5 is based on various lemmata which we now introduce.

Lemma 3.9. *Let the assumptions of Theorem 3.5 be satisfied and let ϕ_{1+}, ϕ_{2+} satisfy*

$$\left.\begin{aligned} p_{11}(\tau)\,\phi_{1+} + p_{12}(\tau)\,\phi_{2+} &= q_1(\tau) \\ p_{21}(\tau)\,\phi_{1+} + p_{22}(\tau)\,\phi_{2+} &= q_2(\tau) \end{aligned}\right\} \quad on \quad \Gamma, \tag{34}$$

where p_{ij}, q_i $(i, j = 1, 2)$ are polynomials such that p_{12} and p_{22} do not have common zeroes and $p_{11}\, p_{22} - p_{12}\, p_{21} \neq 0$ for all $\tau \in \Gamma$. Then if ϕ_{1+} is analytic in $\mathbb{D}_\Gamma^+$, ϕ_{2+} is also analytic in $\mathbb{D}_\Gamma^+$.

Proof. By Cramer's rule we have

$$\phi_{1+} = \frac{q_1\, p_{22} - q_2\, p_{12}}{p_{11}\, p_{22} - p_{12}\, p_{21}}, \tag{35}$$

and, from the first and the second equation in (34),

$$\frac{q_1 - p_{11}\,\phi_{1+}}{p_{12}} = \frac{q_2 - p_{21}\,\phi_{1+}}{p_{22}} = \phi_{2+}. \tag{36}$$

Let ϕ_{1+} be analytic in $\mathbb{D}_\Gamma^+$. Since p_{12} and p_{22} do not have common zeroes, in the neighbourhood of any zero of p_{12} in the interior of Γ, the right hand side of (36) must be analytic, and vice-versa. Hence, ϕ_{2+} be analytic in $\mathbb{D}_\Gamma^+$.

□

Lemma 3.10. *Let G be an $n \times n$ matrix function and let T_G denote the Toeplitz operator on $(H_+^2)^n$ with symbol G. If all elements of* $\ker T_G$ *vanish at* 0*, then* $\ker T_G = \{0\}$.

Proof. Let $\phi_+ \in (H_+^2)^n$ be an element of $\ker T_G$, i.e.

$$G\phi_+ = \phi_- \quad \text{with} \quad \phi_- \in (H_-^2)^n, \tag{37}$$

and assume that $\phi_+(0) = 0$. Then $\phi_+/\tau \in (H_+^2)^n$, and hence $\phi_+/\tau \in \ker T_G$ because

$$G\frac{\phi_+}{\tau} = \frac{\phi_-}{\tau} \in (H_-^2)^n. \tag{38}$$

Therefore ϕ_+/τ also vanishes at 0 and, by repeating the same argument, also $\phi_+/\tau^k \in (H_+^2)^n$ for all $k \in \mathbb{N}$, which is only possible if $\phi_+ = 0$.

□

Lemma 3.11. *Let the assumptions and notation of Theorem 3.5 hold. Then $N_1 + N_2 > 2n$ if and only if*

$$k_{11} > k_{12} > k_{22} \;\vee\; k_{22} > k_{12} > k_{11}. \tag{39}$$

Proof. Since $2n = \max\{k_{11} + k_{22}, 2k_{12}\}$, it is clear that (39) implies $N_1 + N_2 > 2n$. Now let us prove the converse, which is equivalent to proving

$$\sim (k_{11} > k_{12} > k_{22} \;\vee\; k_{22} > k_{12} > k_{11}) \;\Rightarrow \sim (N_1 + N_2 > 2n)\,. \tag{40}$$

Let us therefore assume that $k_{11} > k_{12} > k_{22}$ and $k_{22} > k_{12} > k_{11}$ are both false. Falsity of $k_{11} > k_{12} > k_{22}$ implies that either (i) $k_{11} \leq k_{12}$ or (ii) $k_{12} \leq k_{22}$. Let us first consider the case (i). Then $N_1 + N_2 = k_{12} + \max\{k_{12}, k_{22}\}$. If $N_2 = \max\{k_{12}, k_{22}\} = k_{12}$, then $N_1 + N_2 = 2k_{12} \geq k_{11} + k_{22}$, and hence $2n = 2k_{12} = N_1 + N_2$. If $N_2 = \max\{k_{12}, k_{22}\} = k_{22}$, then either $k_{22} > k_{12}$ or $k_{22} = k_{12}$. If $k_{22} > k_{12}$, for the condition $k_{22} > k_{12} > k_{11}$ to be false, we must have $k_{12} \leq k_{11}$, and hence $k_{12} = k_{11}$. Then $N_1 + N_2 = k_{11} + k_{22} > 2k_{12}$ and hence $N_1 + N_2 = 2n$. On the other hand, if $k_{22} = k_{12}$, then $N_1 + N_2 = 2k_{12} \geq k_{11} + k_{12}$, and hence $N_1 + N_2 = 2n$.

Now let consider the case (ii). Then $N_2 = k_{22}$ and $N_1 + N_2 = \max\{k_{11}, k_{12}\} + k_{22}$. If $N_1 = \max\{k_{11}, k_{12}\} = k_{11}$, then $N_1 + N_2 = k_{11} + k_{22} \geq 2k_{12}$, and hence $N_1 + N_2 = 2n$. If $N_1 = \max\{k_{11}, k_{12}\} = k_{12}$, then either $k_{12} > k_{11}$ or $k_{11} = k_{12}$. If $k_{12} > k_{11}$, for the condition $k_{22} > k_{12} > k_{11}$ to be false, we must have $k_{12} \geq k_{22}$, and hence $k_{12} = k_{22}$, in which case $N_1 + N_2 = 2k_{12} > k_{11} + k_{22}$ and hence $N_1 + N_2 = 2n$. On the other hand, if $k_{12} = k_{11}$, then $N_1 + N_2 = k_{11} + k_{22} \geq 2k_{12}$, and hence $N_1 + N_2 = 2n$.

□

Using the above lemmata, we proceed with the proofs of Theorems 3.5,3.6 and3.7.

Proof of Theorem 3.5:
Let us consider the Riemann-Hilbert problem (18) for the existence of a canonical Wiener-Hopf factorisation for $\mathcal{M}_{\rho,v}(\tau) = G$. Since $\tau^n/q_{2n}(\tau)$ admits a canonical Wiener-Hopf factorisation as mentioned before, we are reduced to studying (18) replacing G by $\tilde{\mathcal{M}}_{\rho,v}$. We then have, using (26),

$$\begin{aligned} \frac{\tilde{p}_{2k_{11}}}{\tau^{k_{11}}}\,\phi_{1+} + \frac{\tilde{p}_{2k_{12}}}{\tau^{k_{12}}}\,\phi_{2+} &= \phi_{1-}, \\ \frac{\tilde{p}_{2k_{12}}}{\tau^{k_{12}}}\,\phi_{1+} + \frac{\tilde{p}_{2k_{22}}}{\tau^{k_{22}}}\,\phi_{2+} &= \phi_{2-}, \end{aligned} \tag{41}$$

where $\phi_{1\pm}, \phi_{2\pm} \in H^2_{\pm}$, which is equivalent to

$$\begin{aligned} \tau^{N_1-k_{11}}\tilde{p}_{2k_{11}}\,\phi_{1+} + \tau^{N_1-k_{12}}\tilde{p}_{2k_{12}}\,\phi_{2+} &= \tau^{N_1}\,\phi_{1-}, \\ \tau^{N_2-k_{12}}\,\tilde{p}_{2k_{12}}\,\phi_{1+} + \tau^{N_2-k_{22}}\,\tilde{p}_{2k_{22}}\,\phi_{2+} &= \tau^{N_2}\,\phi_{2-}. \end{aligned} \tag{42}$$

Since, for any $N \in \mathbb{N}\cup\{0\}$, $H^2_+ \cap \tau^N H^2_- = \mathcal{P}_{N-1}$, where $\mathcal{P}_{N-1}$ is the space of all polynomials of degree at most $N-1$, we conclude that both sides of the equations (42) are equal to polynomials $Q_{N_1-1} \in \mathcal{P}_{N_1-1}$ for the first equation and $Q_{N_2-1} \in \mathcal{P}_{N_2-1}$ for the second equation. Then

$$\begin{aligned} \tau^{N_1-k_{11}}\,\tilde{p}_{2k_{11}}\,\phi_{1+} + \tau^{N_1-k_{12}}\,\tilde{p}_{2k_{12}}\,\phi_{2+} &= Q_{N_1-1}, \\ \tau^{N_2-k_{12}}\,\tilde{p}_{2k_{12}}\,\phi_{1+} + \tau^{N_2-k_{22}}\,\tilde{p}_{2k_{22}}\,\phi_{2+} &= Q_{N_2-1}, \end{aligned} \tag{43}$$

and by Cramer's rule

$$\phi_{1+} = \frac{\begin{vmatrix} Q_{N_1-1} & \tau^{N_1-k_{12}}\,\tilde{p}_{2k_{12}} \\ Q_{N_2-1} & \tau^{N_2-k_{22}}\,\tilde{p}_{2k_{22}} \end{vmatrix}}{q^2_{2n}(\tau)}\,\tau^{2n-(N_1+N_2)}\,,\quad \phi_{2+} = \frac{\begin{vmatrix} \tau^{N_1-k_{11}}\,\tilde{p}_{2k_{11}} & Q_{N_1-1} \\ \tau^{N_2-k_{12}}\,\tilde{p}_{2k_{12}} & Q_{N_2-1} \end{vmatrix}}{q^2_{2n}(\tau)}\,\tau^{2n-(N_1+N_2)}, \tag{44}$$

where we recall that q^2_{2n} is given by (27). If $N_1 + N_2 < 2n$ then ϕ_{1+} and ϕ_{2+} vanish at 0 and, by Lemma 3.10 we must have $\ker T_{\tilde{\mathcal{M}}_{\rho,v}} = \{0\}$, and hence $\tilde{\mathcal{M}}_{\rho,v}$ has a canonical Wiener-Hopf factorisation. If $N_1 + N_2 = 2n$, then ϕ_{2+} will be analytic if ϕ_{1+} is analytic in $\mathbb{D}^+_\Gamma$ by Lemma 3.9, and therefore we are reduced to studying the condition for ϕ_{1+} to be analytic. This corresponds to imposing that the zeroes of the numerator of

$$\phi_{1+} = \frac{Q_{N_1-1}\,\tau^{N_2-k_{22}}\,\tilde{p}_{2k_{22}} - Q_{N_2-1}\,\tau^{N_1-k_{12}}\,\tilde{p}_{2k_{12}}}{q^2_{2n}(\tau)} \tag{45}$$

cancel the zeroes of $q^2_{2n}(\tau)$ in the interior of Γ, as in (31). The result of (ii) in Theorem 3.5 follows from here.

Finally, if $N_1 + N_2 > 2n$, then by Lemma 3.11 we must have either $k_{11} > k_{12} > k_{22}$ or $k_{22} > k_{12} > k_{11}$. Let us consider the case $k_{11} > k_{12} > k_{22}$ (the other case can be dealt with in a similar manner). It follows that $N_1 = k_{11}, N_2 = k_{12}$. Now recall the definition of n given in (28). Let us assume that $2n = k_{11} + k_{22}$ (the other case, $2n = 2k_{12}$, can be dealt with in a similar manner). Then

$$k_{11} + k_{22} = 2n < N_1 + N_2 = k_{11} + k_{12}. \tag{46}$$

Then, (43) can be written as

$$\begin{aligned} \tilde{p}_{2k_{11}}\,\phi_{1+} + \tau^{k_{11}-k_{12}}\tilde{p}_{2k_{12}}\,\phi_{2+} &= Q_{N_1-1}, \\ \tilde{p}_{2k_{12}}\,\phi_{1+} + \tau^{k_{12}-2n+k_{11}}\,\tilde{p}_{2k_{22}}\,\phi_{2+} &= Q_{N_2-1}, \end{aligned} \tag{47}$$

and by Cramer's rule

$$\begin{aligned}\phi_{1+} &= \frac{\begin{vmatrix} Q_{N_1-1} & \tau^{k_{11}-k_{12}}\,\tilde{p}_{2k_{12}} \\ Q_{N_2-1} & \tau^{k_{11}-2n+k_{12}}\,\tilde{p}_{2k_{22}} \end{vmatrix}}{\tau^{k_{11}-2n+k_{12}}\,\tilde{p}_{2k_{11}}\,\tilde{p}_{2k_{22}} - \tau^{k_{11}-k_{12}}\,\tilde{p}^2_{2k_{12}}} = \frac{\begin{vmatrix} Q_{N_1-1} & \tau^{2n-2k_{12}}\,\tilde{p}_{2k_{12}} \\ Q_{N_2-1} & \tilde{p}_{2k_{22}} \end{vmatrix}}{\tilde{p}_{2k_{11}}\,\tilde{p}_{2k_{22}} - \tau^{2n-2k_{12}}\,\tilde{p}^2_{2k_{12}}} \\ &= \frac{\begin{vmatrix} Q_{N_1-1} & \tau^{2n-2k_{12}}\,\tilde{p}_{2k_{12}} \\ Q_{N_2-1} & \tilde{p}_{2k_{22}} \end{vmatrix}}{q^2_{2n}(\tau)}, \end{aligned} \tag{48}$$

where we used (27). Note that $2n \geq 2k_{12}$, and hence the powers of τ in (48) are all non-negative. Therefore, the analyticity of ϕ_{1+} follows by imposing that the numerator in (48) has the same zeroes as $q^2_{2n}(\tau)$ in the interior of Γ, i.e. $2n$ zeroes (counting their multiplicity), as it happens when $N_1 + N_2 = 2n$.

□

Proof of Theorem 3.6:
Let us consider the Riemann-Hilbert problem (19) which, in this case, can be written (analogously to the proof of Theorem 3.5) as

$$\begin{aligned} \tau^{N_1-k_{11}}\,\tilde{p}_{2k_{11}}\,\psi_{1+} + \tau^{N_1-k_{12}}\,\tilde{p}_{2k_{12}}\,\psi_{2+} = \tau^{N_1}\,\psi_{1-} &= Q_{N_1}, \\ \tau^{N_2-k_{12}}\,\tilde{p}_{2k_{12}}\,\psi_{1+} + \tau^{N_2-k_{22}}\,\tilde{p}_{2k_{22}}\,\psi_{2+} = \tau^{N_2}\,\psi_{2-} &= Q_{N_2}, \end{aligned} \tag{49}$$

where now $\psi_{i+} \in C^\mu_+$, $\psi_{i-} \in C^\mu_-$ and Q_{N_1}, Q_{N_2} are polynomials of degree N_1, N_2 (respectively), at most. We have, by Cramer's rule,

$$\psi_{1+} = \frac{Q_{N_1}\,\tau^{N_2-k_{22}}\,\tilde{p}_{2k_{22}} - Q_{N_2}\,\tau^{N_1-k_{12}}\,\tilde{p}_{2k_{12}}}{q^2_{2n}(\tau)}, \tag{50}$$

and ψ_{2+} is obtained from ψ_{1+} by

$$\psi_{2+} = \frac{Q_{N_1} - \tau^{N_1-k_{11}}\,\tilde{p}_{2k_{11}}\,\psi_{1+}}{\tau^{N_1-k_{12}}\,\tilde{p}_{2k_{12}}}, \tag{51}$$

where the $2n+2$ coefficients of Q_{N_1} and Q_{N_2} are determined by imposing zeroes in the numerator of the right hand side of (50), so that ψ_{1+} is analytic in the interior of Γ, together with the normalization conditions. Note that the analyticity of ψ_{2+} then follows from there according to Lemma 3.9.

Since we assumed that the zeroes of $q_{2n}(\tau)$ are simple, and denoting its zeroes in the interior of Γ by $\tau_i, i = 1, 2, \ldots, n$, the analyticity of ψ_{1+} is obtained by imposing that the numerator of ψ_{1+} in (50) has zeroes of order at least 2 at τ_i. Let us write

$$Q_{N_j} = \tau\,\tilde{Q}_{N_j-1} + A_j \quad , \quad j = 1, 2, \tag{52}$$

where the constants A_1, A_2 are determined by the normalization conditions $\psi_{1+}(0) = 1, \psi_{2+}(0) = 0$ for the 1st column of G_+^{-1} and $\psi_{1+}(0) = 0, \psi_{2+}(0) = 1$ for the 2nd column. By (49) the constants A_1 and A_2 cannot be simultaneously equal to 0, see Remark 3.4. The coefficients of $\tilde{Q}_{N_1-1}, \tilde{Q}_{N_2-1}$ will be given by the non-homogenous linear system of $2n$ equations for the $2n$ unknown coefficients,

$$\begin{aligned} &\left[\tau\left(\tilde{Q}_{N_1-1}\,\tau^{N_2-k_{22}}\,\tilde{p}_{2k_{22}} - \tilde{Q}_{N_2-1}\,\tau^{N_1-k_{12}}\,\tilde{p}_{2k_{12}}\right)\right](\tau_i) = \\ &\qquad -A_1\,\tau_i^{N_2-k_{22}}\,\tilde{p}_{2k_{22}}(\tau_i) + A_2\,\tau_i^{N_1-k_{12}}\,\tilde{p}_{2k_{12}}(\tau_i), \\ &\left[\tau\left(\tilde{Q}_{N_1-1}\,\tau^{N_2-k_{22}}\,\tilde{p}_{2k_{22}} - \tilde{Q}_{N_2-1}\,\tau^{N_1-k_{12}}\,\tilde{p}_{2k_{12}}\right)\right]'(\tau_i) = \\ &\qquad -A_1\left(\tau^{N_2-k_{22}}\,\tilde{p}_{2k_{22}}\right)'(\tau_i) + A_2\left(\tau^{N_1-k_{12}}\,\tilde{p}_{2k_{12}}\right)'(\tau_i), \end{aligned} \tag{53}$$

which is equivalent to

$$
\begin{aligned}
&\left[\left(\tilde{Q}_{N_1-1}\,\tau^{N_2-k_{22}}\,\tilde{p}_{2k_{22}} - \tilde{Q}_{N_2-1}\,\tau^{N_1-k_{12}}\,\tilde{p}_{2k_{12}}\right)\right](\tau_i) = \\
&\quad -\frac{1}{\tau_i}\left(A_1\,\tau^{N_2-k_{22}}\,\tilde{p}_{2k_{22}}(\tau_i) - A_2\,\tau_i^{N_1-k_{12}}\,\tilde{p}_{2k_{12}}(\tau_i)\right), \\
&\left[\left(\tilde{Q}_{N_1-1}\,\tau^{N_2-k_{22}}\,\tilde{p}_{2k_{22}} - \tilde{Q}_{N_2-1}\,\tau^{N_1-k_{12}}\,\tilde{p}_{2k_{12}}\right)\right]'(\tau_i) = \\
&\quad -\frac{1}{\tau_i}\left[A_1\,\left(\tau^{N_2-k_{22}}\,\tilde{p}_{2k_{22}}\right)'(\tau_i) - A_2\,\left(\tau^{N_1-k_{12}}\,\tilde{p}_{2k_{12}}\right)'(\tau_i)\right. \\
&\qquad\qquad \left.+\frac{1}{\tau_i}\left(A_1\,\tau_i^{N_2-k_{22}}\,\tilde{p}_{2k_{22}}(\tau_i) - A_2\,\tau_i^{N_1-k_{12}}\,\tilde{p}_{2k_{12}}(\tau_i)\right)\right].
\end{aligned}
\tag{54}
$$

Finally, the fact the solutions are C^∞ functions of (ρ, v) can be verified directly.

□

Proof of Theorem3.7:
Note that the matrix coefficient of the system (54) is the same as that of (31), so its determinant is given by $D(\rho, v)$ in Theorem 3.5. On the other hand, by Cramer's rule, the coefficients of the polynomials $\tilde{Q}_{N_1-1}$ and $\tilde{Q}_{N_2-1}$ are given by a quotient, where the numerator is not zero (since A_1 and A_2 cannot vanish simultaneously) and the denominator is $D(\rho, v)$. Now, $1/\Delta$ is given by the element in the second row and second column in $\lim_{\tau\to\infty} \mathcal{M}^-_{\rho,v}(\tau)$, i.e. according to (49) and (52), $1/\Delta$ is equal to the coefficient of the term of order $N_2 - 1$ in $\tilde{Q}_{N_2-1}$, which tends to ∞ when $D(\rho, v) \to 0$. Since $g_{tt} = -\Delta$, we have $g_{tt} \to 0$.

□

4 Examples in four and five space-time dimensions

As an application, we consider in this section several examples which illustrate the results of Sections 2 and 3 as well as possible generalisations.

4.1 The non-extremal Kerr monodromy matrix

Let $\mathcal{M}$ be given by (see [32, 21])

$$
\mathcal{M}(\omega) = \frac{1}{\omega^2 - c^2}\begin{pmatrix} (\omega - m)^2 + a^2 & 2am \\ 2am & (\omega + m)^2 + a^2 \end{pmatrix} \quad , \quad c = \sqrt{m^2 - a^2} > 0. \tag{55}
$$

By composition with $\omega_{\tau,\rho,v}$ we obtain a monodromy matrix of the form (26), where

$$
q_{2n}(\tau) \;=\; q_4(\tau) = \frac{1}{4}\left[\rho^2(1-\tau^2)^2 + 4(v^2 - c^2)\tau^2 + 4\rho v\tau(1-\tau^2)\right] \qquad (n = 2)\;, \tag{56}
$$

$$
\tilde{\mathcal{M}}_{\rho,v}(\tau) \;=\; \begin{pmatrix} (v - m + \rho\,\frac{(1-\tau^2)}{2\tau})^2 + a^2 & 2am \\ 2am & (v + m + \rho\,\frac{(1-\tau^2)}{2\tau})^2 + a^2 \end{pmatrix}. \tag{57}
$$

Note that this is a case considered in *(ii)* in Theorem 3.5, with

$$
k_{11} = k_{22} = 2 \quad , \quad k_{12} = 0 \quad , \quad N_1 = N_2 = 2 \quad , \quad 2n = 4 = N_1 + N_2\;. \tag{58}
$$

Let $\tau_1, \tilde{\tau}_1, \tau_2, \tilde{\tau}_2$ be the (simple) zeroes of q_4, in the notation of Section 3, and let Γ be any admissible contour with τ_1, τ_2 in its interior.

The system (31) now takes the form of a system of 4 linear equations for the unknown coefficients $\alpha_0, \alpha_1, \beta_0, \beta_1$:

$$\begin{aligned}
(\alpha_1\tau_1+\alpha_0)P_{22}(\tau_1) - 2am\tau_1^2(\beta_1\tau_1+\beta_0) &= 0,\\
\alpha_1 P_{22}(\tau_1) + (\alpha_1\tau_1+\alpha_0)P_{22}'(\tau_1) - 4am\tau_1(\beta_1\tau_1+\beta_0) - 2am\tau_1^2\beta_1 &= 0,\\
(\alpha_1\tau_2+\alpha_0)P_{22}(\tau_2) - 2am\tau_2^2(\beta_1\tau_2+\beta_0) &= 0,\\
\alpha_1 P_{22}(\tau_2) + (\alpha_1\tau_2+\alpha_0)P_{22}'(\tau_2) - 4am\tau_2(\beta_1\tau_2+\beta_0) - 2am\tau_2^2\beta_1 &= 0,
\end{aligned} \tag{59}$$

which we can write as

$$\tilde{T}\begin{pmatrix}\alpha_0\\ \alpha_1\\ \beta_0\\ \beta_1\end{pmatrix} = 0, \tag{60}$$

where $\tilde{T}$ is the 4×4 matricial coefficient of the system (59), with entries depending on the parameters (ρ, v). The determinant $D(\rho, v)$ of $\tilde{T}$ is given by

$$D(\rho, v) = f(\rho, v)\, h(\rho, v), \tag{61}$$

where

$$\begin{aligned}
f(\rho, v) &= \frac{a^2m^2}{4}\,\rho^2\,(\tau_1-\tau_2)^4 \neq 0 \quad , \quad \text{for all } \ (\rho, v)\ ,\\
h(\rho, v) &= -16(m-v)^2\tau_1^2\tau_2^2 + \rho^2\left(1+4\tau_1^3\tau_2+6\tau_1^2\tau_2^2+4\tau_1\tau_2^3+\tau_1^4\tau_2^4\right)\\
&\quad -8\rho(m-v)\tau_1\tau_2\left(-\tau_1-\tau_2+\tau_1^2\tau_2+\tau_1\tau_2^2\right).
\end{aligned} \tag{62}$$

Hence $D(\rho, v) = 0$ if and only if

$$h(\rho, v) = 0\ . \tag{63}$$

This condition describes a curve $\mathcal{C}$ in the Weyl half-plane of the coordinates $\rho > 0, v \in \mathbb{R}$, which corresponds to the values of (ρ, v) for which $\mathcal{M}_{\rho,v}(\tau)$ does not admit a canonical WH factorisation and for which we have $g_{tt} = 0$, see Theorem 3.7. This curve $\mathcal{C}$ naturally depends on the choice of the points τ_1 and τ_2 that one chooses to be in the interior of Γ, which is the contour in the complex plane of the spectral parameter τ with respect to which the canonical factorisation of $\mathcal{M}_{\rho,v}(\tau)$ is sought.

If one looks for a canonical factorisation of $\mathcal{M}_{\rho,v}(\tau)$ w.r.t. an admissible contour Γ such that

$$\begin{aligned}
\tau_1 &= \frac{v-c-\sqrt{(v-c)^2+\rho^2}}{\rho},\\
\tau_2 &= \frac{v+c-\sqrt{(v+c)^2+\rho^2}}{\rho}
\end{aligned} \tag{64}$$

lie in $\mathbb{D}_\Gamma^+$, then that factorisation, obtained for $(\rho, v) \notin \mathcal{C}$, yields by (12) a solution to the field equation (1) that describes the exterior region of the non-extremal Kerr black hole solution in General Relativity. The curve $\mathcal{C}$ defined by (63) in the Weyl coordinates half-plane describes the ergosurface of that four-dimensional solution, as follows. If we express (63) in terms of prolate spheroidal coordinates (u, y) (see [19, 21]),

$$v = u\,y \quad , \quad \rho = \sqrt{(u^2 - c^2)(1 - y^2)}, \tag{65}$$

where

$$c < u < +\infty \quad , \quad |y| < 1 \, , \tag{66}$$

we obtain

$$\begin{aligned} \rho\tau_1 &= uy - c - (u - cy), \\ \rho\tau_2 &= uy + c - (u + cy). \end{aligned} \tag{67}$$

Inserting these expressions into the expression of $h(\rho, v)$ given in (62) we get

$$h(u, y) = \frac{16}{[(u^2 - m^2 + a^2)(1 - y^2)]^3}(u^2 - c^2)^3(y - 1)^4(u^2 - m^2 + a^2y^2) \tag{68}$$

and we see that (63) is equivalent to

$$u(y) = \sqrt{m^2 - a^2y^2}, \tag{69}$$

This equation indeed defines the ergosurface of the non-extremal Kerr black hole in four dimensions, in the exterior region. Namely, in standard Boyer-Lindquist coordinates, the metric component g_{tt} of the Kerr solution is given by $g_{tt} = -(r^2 - 2mr + a^2\cos^2\theta)/(r^2 + a^2\cos^2\theta)$. Converting from Boyer-Lindquist coordinates to coordinates (u, y) using $u = r - m$, $y = \cos\theta$ [19] shows that g_{tt} vanishes precisely when (69) holds. The vanishing of g_{tt} defines the ergosurface of the black hole.

The curve $\mathcal{C}$ defined by $D(\rho, v) = 0$ is represented in Figure 1, in the Weyl half-plane $\rho > 0, v \in \mathbb{R}$. The region between the curve $\mathcal{C}$ and the axis $\rho = 0$ represents the ergosphere, the region between the ergosurface $\mathcal{C}$ and the outer horizon of the non-extremal Kerr black hole, while the complementary region describes the region outside of the ergosphere.

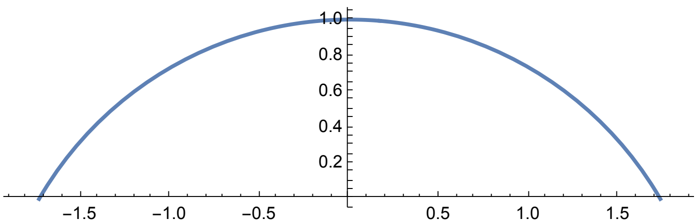

Figure 1: Curve $\mathcal{C}$ in the Weyl coordinates upper half-plane $(\rho > 0, v)$ for the values $m = 2, a = 1$. The horizontal axis represents $v \in \mathbb{R}$, while the vertical axis represents $\rho > 0$.

Remark 4.1. Different choices of the contour Γ yield different factorisations of $\mathcal{M}_{\rho,v}(\tau)$ and correspondingly different solutions $M(\rho, v)$. We can have 3 other cases: (i) $\tilde{\tau}_1$ and $\tilde{\tau}_2$ in $\mathbb{D}^+_\Gamma$, (ii) $\tilde{\tau}_1$ and τ_2 in $\mathbb{D}^+_\Gamma$, (iii) τ_1 and $\tilde{\tau}_2$ in $\mathbb{D}^+_\Gamma$. In each case we obtain a curve $\mathcal{C}$ where the metric component g_{tt} vanishes, $g_{tt} = 0$; in case (i) it coincides with the curve for the ergosurface of the non-extremal Kerr black hole, while in the other two cases we obtain a different curve, namely

$$u(y) = \frac{c}{a}\sqrt{m^2 - c^2y^2} \, . \tag{70}$$

These various solutions reduce to solutions belonging to the class of A-metrics when the rotation parameter a is set to 0 (for the class of A-metrics see [2] and references therein).

4.2 Rotating black hole in five space-time dimensions

In this section we will discuss solutions to (1) with $\lambda = 1$ that yield solutions in five space-time dimensions. Such solutions are described by a five-dimensional line element whose metric factors in Weyl coordinates are encoded in the 3×3 matrix $M(\rho, v)$ given in (98). In the notation used in (98), the metric component g_{tt} of the line element is given by $g_{tt} = -e^{2\Sigma_3} + e^{2\Sigma_2}\chi_1^2$, which is contained in the last component of the third column in (98).

In the following we will consider the factorisation of two monodromy different matrices with respect to suitably chosen contours. When these matrices have a canonical factorisation, the resulting matrix factor (12) has the form (98). In the first example considered below, the metric factor g_{tt} is given by $g_{tt} = -e^{2\Sigma_3}$ since $\chi_1 = 0$, whereas in the second example g_{tt} takes the form $g_{tt} = -e^{2\Sigma_3} + e^{2\Sigma_2}\chi_1^2$. In both cases, the resulting space-time solution describes the exterior region of a rotating Myers-Perry black hole solution in five space-time dimensions carrying one angular momentum a [28], albeit in different space-time coordinates. In one case, it yields the Myers-Perry solution in standard spherical coordinates [19], while in the other case it yields the Myers-Perry solution in the coordinates used in [12]. Even though the functional form of g_{tt} in both solutions looks different when expressed in terms of the quantities Σ_3 and χ_1, when converting to standard spherical coordinates the metric factor g_{tt} takes the form (103) in both cases.

For each monodromy matrix, there is a curve $\mathcal{C}$ in the Weyl coordinate plane $\rho > 0, v \in \mathbb{R}$ where the canonical factorisation does not exist. When approaching a point (ρ, v) on the curve, the factor $e^{2\Sigma_1}$ blows up, while the factor $e^{2\Sigma_3}$ becomes vanishing. In the first example, $g_{tt} = -e^{2\Sigma_3}$ vanishes on the curve $\mathcal{C}$, which therefore coincides with the ergosurface in the exterior region of the black hole. In the second case, when approaching the corresponding curve $\mathcal{C}$, the combination $e^{2\Sigma_2}\chi_1^2$ remains finite and g_{tt} does not vanish on this curve. However, there are terms in $M(\rho, v)$, given in (98), proportional to $e^{2\Sigma_1}$, that blow up on the curve $\mathcal{C}$.

First, let us then consider the following matrix (where $4\alpha = 2m - a^2 > 0$),

$$\mathcal{M}(\omega) = \begin{pmatrix} \frac{\omega-\alpha+m}{2(\omega+\alpha)(\omega-\alpha)} & 0 & \frac{am}{2(\omega+\alpha)(\omega-\alpha)} \\ 0 & 2(\omega+\alpha) & 0 \\ \frac{am}{2(\omega+\alpha)(\omega-\alpha)} & 0 & \frac{8(\omega-\alpha)^2(\omega+\alpha)+4a^2m^2}{8(\omega+\alpha)(\omega-\alpha)(\omega-\alpha+m)} \end{pmatrix}. \tag{71}$$

This matrix satisfies $\det \mathcal{M} = 1$ and has the property $\mathcal{M}^\natural(\omega) = \mathcal{M}(\omega)$, where $\mathcal{M}^\natural = \eta \mathcal{M}^T \eta$ with $\eta = \mathrm{diag}(1, -1, 1)$ [12].

When the rotation parameter a is set to zero, this reduces to the matrix

$$\mathcal{M}(\omega) = \begin{pmatrix} \frac{1}{2(\omega-\alpha)} & 0 & 0 \\ 0 & 2(\omega+\alpha) & 0 \\ 0 & 0 & \frac{\omega-\alpha}{\omega+\alpha} \end{pmatrix} \tag{72}$$

which, by composition with the spectral relation (6) with $\lambda = 1$, yields a monodromy matrix whose canonical Wiener-Hopf factorisation with respect to a suitably chosen admissible contour Γ gives the five-dimensional Schwarzschild black hole. This contour is defined such that

$$\begin{aligned} \tau_\alpha &= \frac{v - \alpha - \sqrt{(v-\alpha)^2 + \rho^2}}{\rho}, \\ \tau_{-\alpha} &= \frac{v + \alpha - \sqrt{(v+\alpha)^2 + \rho^2}}{\rho} \end{aligned} \tag{73}$$

lie in the interior of Γ. As as consequence, $\tilde{\tau}_\alpha = -1/\tau_\alpha$ and $\tilde{\tau}_{-\alpha} = -1/\tau_{-\alpha}$ lie in the exterior of Γ.

We compose (71) with the spectral relation (6) with $\lambda = 1$. The resulting monodromy matrix $\mathcal{M}_{\rho,v}(\tau)$ has poles in the τ-plane located at $\tau_\alpha, \tau_{-\alpha}, \tilde{\tau}_\alpha, \tilde{\tau}_{-\alpha}$ and also at

$$\tau_{\alpha-m} = \frac{v - \alpha + m - \sqrt{(v-\alpha+m)^2 + \rho^2}}{\rho} \tag{74}$$

and at $\tilde{\tau}_{\alpha-m} = -1/\tau_{\alpha-m}$. We will choose the contour Γ with respect to which we will perform the canonical factorisation of $\mathcal{M}_{\rho,v}(\tau)$ to be such that the *three* points $\tau_\alpha, \tau_{-\alpha}$ and $\tau_{\alpha-m}$ lie in its interior. The points $\tilde{\tau}_\alpha, \tilde{\tau}_{-\alpha}$ and $\tilde{\tau}_{\alpha-m}$ will then lie in the exterior of Γ.

Applying Theorem 3.1 and studying the Riemann-Hilbert problem (18) with $n = 3$, we conclude that the canonical factorisation of $\mathcal{M}_{\rho,v}(\tau)$ with respect to Γ exists for any point $(\rho > 0, v)$ in the Weyl coordinates upper half-plane so long as $D(\rho, v) \neq 0$, where $D(\rho, v)$ is the determinant of a linear system that is analogous to the one in (31) and is obtained by an entirely similar reasoning,

$$\begin{aligned} D(\rho,v) &= \rho^6 \frac{(\tau_\alpha - \tau_{-\alpha})^2 (\tau_\alpha - \tau_{\alpha-m})^2 (\tau_{-\alpha} - \tau_{\alpha-m})}{2\tau_\alpha^3 \tau_{-\alpha}\, \tau_{\alpha-m}} (1 + \tau_\alpha \tau_{\alpha-m})^3 \\ & \quad \left[2\tau_\alpha \tau_{\alpha-m} (1 + \tau_\alpha \tau_{\alpha-m})(1 + \tau_{-\alpha}\tau_{\alpha-m})^2 - \tau_{-\alpha} (\tau_\alpha - \tau_{\alpha-m})^3 \, L \right], \end{aligned} \tag{75}$$

where L denotes the ratio $L = a^2/m$ which, in view of the condition $4\alpha = 2m - a^2 > 0$, lies in the range $0 \leq L < 2$. Note that the bracket $(1 + \tau_\alpha \tau_{\alpha-m})$ can be expressed as $(\tilde{\tau}_\alpha - \tau_{\alpha-m})/\tilde{\tau}_\alpha$. Hence it cannot vanish, since $\tilde{\tau}_\alpha$ and $\tau_{\alpha-m}$ lie on different sides of the contour Γ. Therefore, $D(\rho, v)$ can only vanish if the big bracket $[\dots]$ in (75) vanishes. To infer when this happens we express the coordinates $(\rho > 0, v)$ in terms of prolate spheroidal coordinates (u, y) (see [19, 12]),

$$v = \alpha\, u\, y \quad , \quad \rho = \alpha \sqrt{(u^2-1)(1-y^2)}, \tag{76}$$

where

$$1 < u < +\infty \quad , \quad |y| < 1\,. \tag{77}$$

Then we have

$$\begin{aligned} \rho\tau_\alpha &= \alpha\,(uy - 1 - (u-y))\,, \\ \rho\tau_{-\alpha} &= \alpha\,(uy + 1 - (u+y))\,, \\ \rho\tau_{\alpha-m} &= \alpha(uy-1) + m - \sqrt{(\alpha(uy-1)+m)^2 + \alpha^2(u^2-1)(1-y^2)}\,. \end{aligned} \tag{78}$$

The big bracket $[\dots]$ in (75) becomes

$$2\tau_\alpha \tau_{\alpha-m}(1 + \tau_\alpha \tau_{\alpha-m})(1 + \tau_{-\alpha}\tau_{\alpha-m})^2 - \tau_{-\alpha}(\tau_\alpha - \tau_{\alpha-m})^3 \, L = (2 + (L-2)u - Ly)\; \Sigma(u,y)\,, \tag{79}$$

with

$$\Sigma(u,y) = f_1(u,y) + f_2(u,y)\sqrt{f_3(u,y)}\,, \tag{80}$$

where f_1, f_2, f_3 are polynomials in (u, y) given by

$$\begin{aligned} f_1(u,y) &= -4\, f_3(x,y)\,(y-1)^2 \left(4(u-1)(y-1)^2 + L^2(u-1)(y+1)^2 - 4L(3 + u + y^2(u-1)\right)\,, \\ f_2(u,y) &= -4(y-1)^2 \left[(L+2)\left(16L - (L-2)^2 u + (L-2)^2 u^2\right) \right. \\ & \qquad + (L-2)(-1+2u)\left(4(-1+u) + L^2(-1+u) - 4L(3+u)\right) y \\ & \qquad \left. + (L-2)^2 (L+2)(u-2)(u-1)y^2 - (L-2)^3(u-1)y^3 \right]\,, \\ f_3(u,y) &= L^2(u-y)^2 + 4(u+y)^2 - 4L(-2 + u^2 + y^2)\,. \end{aligned} \tag{81}$$

It can be verified that f_3 and Σ do not vanish when (u, y) is the range (77) with $0 \leq L < 2$. The combination (79) can therefore only vanish on the line

$$2 + (L-2)u - Ly = 0 \Longleftrightarrow y - 1 + (u-y)\frac{2\alpha}{m} = 0 \,. \tag{82}$$

This line defines the ergosurface of the Myers-Perry black hole solution carrying angular momentum a, where the metric component g_{tt} vanishes. Therefore, we conclude that the canonical factorisation with respect to the chosen contour Γ exists for any point (ρ, v) in the Weyl coordinates upper half-plane so long as it doesn't lie on the ergosurface of the rotating black hole.

By explicitly performing the canonical Wiener-Hopf factorisation (9) (when it exists) of $\mathcal{M}_{\rho,v}(\tau)$ with respect to Γ and using (12), we obtain for the solution $M(\rho, v)$, in the notation used in [12, 7],

$$M(\rho, v) = \begin{pmatrix} e^{2\Sigma_1} & 0 & e^{2\Sigma_1}\chi_3 \\ 0 & e^{2\Sigma_2} & 0 \\ e^{2\Sigma_1}\chi_3 & 0 & e^{2\Sigma_3} + e^{2\Sigma_1}\chi_3^2 \end{pmatrix} \,, \tag{83}$$

with

$$\begin{aligned} e^{2\Sigma_2} &= \sqrt{\rho^2 + (v+\alpha)^2} + v + \alpha \,, \\ e^{2\Sigma_3} &= \frac{\sqrt{\rho^2 + (v+\alpha)^2} + \sqrt{\rho^2 + (v-\alpha)^2}\left(1 - \frac{a^2}{m}\right) - 2\alpha}{\sqrt{\rho^2 + (v+\alpha)^2} + \sqrt{\rho^2 + (v-\alpha)^2}\left(1 - \frac{a^2}{m}\right) + 2\alpha} \,, \\ e^{2\Sigma_1} &= e^{-2\Sigma_2} e^{-2\Sigma_3} \,, \\ \chi_3 &= a \frac{\sqrt{\rho^2 + (v+\alpha)^2} - \sqrt{\rho^2 + (v-\alpha)^2} + 2\alpha}{\sqrt{\rho^2 + (v+\alpha)^2} + \sqrt{\rho^2 + (v-\alpha)^2}\left(1 - \frac{a^2}{m}\right) + 2\alpha} \,. \end{aligned} \tag{84}$$

This describes a Myers-Perry black hole solution, with one angular momentum a, in Weyl coordinates [19], which can be brought into standard form by converting the Weyl coordinates (ρ, v) into standard spherical coordinates. The g_{tt} component of the space-time metric is given by $g_{tt} = -e^{2\Sigma_3}$.

The entries of $M(\rho, v)$ in (83) have the following behaviour on the ergosurface: $e^{2\Sigma_3}$ vanishes, $e^{2\Sigma_2}$ and χ_3 are finite, while $e^{2\Sigma_1}$ blows up. Thus, various entries of $M(\rho, v)$ blow up when approaching the ergosurface, similarly to what happens in the case of the Kerr black hole in four dimensions discussed above.

Next, let us consider the matrix (where $4\alpha = 2m - a^2 > 0$)

$$\mathcal{M}(\omega) = \begin{pmatrix} -\frac{2}{\omega+\alpha} & 1 - \frac{m}{2(\omega+\alpha)} & 0 \\ -1 + \frac{m}{2(\omega+\alpha)} & -\frac{a^2 m}{4(\omega-\alpha)} + \frac{m^2}{8(\omega+\alpha)} & \frac{am}{2(\omega-\alpha)} \\ 0 & -\frac{am}{2(\omega-\alpha)} & 1 + \frac{m}{\omega-\alpha} \end{pmatrix} \,. \tag{85}$$

This matrix is a special case of the one considered in [12]. The latter depends on two angular momenta denoted by (l_1, l_2), while here we restrict ourselves to one angular momentum, namely $a = l_1$. This matrix satisfies $\det \mathcal{M} = 1$ and has the property $\mathcal{M}^{\natural}(\omega) = \mathcal{M}(\omega)$, where $\mathcal{M}^{\natural} = \eta \mathcal{M}^T \eta$ with $\eta = \text{diag}(1, -1, 1)$ [12]. We now consider the composition of (85) with the spectral relation (6) with $\lambda = 1$, and we denote the resulting monodromy matrix by $\mathcal{M}_{\rho,v}(\tau)$.

When the rotation parameter a is set to zero, we have $m = 2\alpha$ and the matrix (85) reduces to the matrix

$$\mathcal{M}(\omega) = \begin{pmatrix} -\frac{2}{\omega+\alpha} & \frac{\omega}{\omega+\alpha} & 0 \\ -\frac{\omega}{\omega+\alpha} & \frac{\alpha^2}{2(\omega+\alpha)} & 0 \\ 0 & 0 & \frac{\omega+\alpha}{\omega-\alpha} \end{pmatrix} \qquad (86)$$

which results in a monodromy matrix $\mathcal{M}_{\rho,v}(\tau)$, whose canonical Wiener-Hopf factorisation with respect to a suitably chosen admissible contour Γ gives a solution describing the exterior region of the five-dimensional Schwarzschild black hole. This contour is the one that we choose in the following. It is defined such that

$$\begin{aligned} \tau_\alpha &= \frac{v - \alpha + \sqrt{(v-\alpha)^2 + \rho^2}}{\rho}, \\ \tau_{-\alpha} &= \frac{v + \alpha + \sqrt{(v+\alpha)^2 + \rho^2}}{\rho} \end{aligned} \qquad (87)$$

lie in the interior of Γ. As as consequence, $\tilde{\tau}_\alpha = -1/\tau_\alpha$ and $\tilde{\tau}_{-\alpha} = -1/\tau_{-\alpha}$ lie in the exterior of Γ. Note that the choice of points (87) that lie in the interior of Γ differs from the choice of points (73) in the previous example. Therefore, the contour Γ with respect to which we perform the factorisation in this example differs from the one used in the previous example.

We now turn to the study of the existence of the canonical Wiener-Hopf factorisation of the monodromy $\mathcal{M}_{\rho,v}(\tau)$ associated with (85). This is done as described in the proof of Theorem 3.5. The expression for ϕ_{2+} given in (44) now reads

$$\phi_{2+} = \frac{\begin{vmatrix} \frac{4\tau}{\rho(\tau-\tilde{\tau}_{-\alpha})} & k_1 & 0 \\ -1 & 0 & -\frac{a}{2} \\ 0 & k_3 & \tau - \tau_\alpha - \frac{2m\tau}{\rho(\tau-\tilde{\tau}_\alpha)} \end{vmatrix}}{(\tau-\tau_\alpha)(\tau-\tau_{-\alpha})} = \frac{k_1\left(\tau - \tau_\alpha - \frac{2m\tau}{\rho(\tau-\tilde{\tau}_\alpha)}\right) + k_3 a \frac{2\tau}{\rho(\tau-\tilde{\tau}_{-\alpha})}}{(\tau-\tau_\alpha)(\tau-\tau_{-\alpha})}, \qquad (88)$$

where $k_1, k_3 \in \mathbb{R}$, and analogously for ϕ_{1+}. Imposing the analyticity of ϕ_{2+} at $\tau = \tau_\alpha$ and $\tau = \tau_{-\alpha}$ implies that

$$\begin{aligned} -k_1\left(\frac{2m\tau_\alpha}{\rho(\tau_\alpha - \tilde{\tau}_\alpha)}\right) + k_3 a \frac{2\tau_\alpha}{\rho(\tau_\alpha - \tilde{\tau}_{-\alpha})} &= 0\,, \\ k_1\left(\tau_{-\alpha} - \tau_\alpha - \frac{2m\tau_{-\alpha}}{\rho(\tau_{-\alpha} - \tilde{\tau}_\alpha)}\right) + k_3 a \frac{2\tau_{-\alpha}}{\rho(\tau_{-\alpha} - \tilde{\tau}_{-\alpha})} &= 0\,. \end{aligned} \qquad (89)$$

This is a linear system for the constants k_1, k_3. If the determinant of this linear system vanishes, the kernel of the associated Toeplitz operator has dimension one (c.f. the discussion around (18)), and there is no canonical factorisation. The determinant of the linear system (89) reads

$$-\frac{2am\tau_\alpha}{\rho(\tau_\alpha - \tilde{\tau}_\alpha)}\, D(\rho, v)\,, \qquad (90)$$

with $D(\rho, v)$ given by

$$D(\rho, v) = \frac{\tau_{-\alpha}}{\rho} \begin{vmatrix} 1 & -\frac{1}{m}\frac{(\tau_\alpha - \tilde{\tau}_\alpha)}{(\tau_\alpha - \tilde{\tau}_{-\alpha})} \\ -\frac{a^2}{(\tau_{-\alpha} - \tilde{\tau}_\alpha)} & \frac{2}{(\tau_{-\alpha} - \tilde{\tau}_{-\alpha})} \end{vmatrix}. \qquad (91)$$

Assuming $a \neq 0$, it follows that the vanishing of (91) occurs when $D(\rho, v) = 0$, or equivalently $D(\rho, v) = 0 \Leftrightarrow g(\rho, v) = 0$, where

$$g(\rho, v) = 2m(\tau_\alpha - \tilde{\tau}_{-\alpha})(\tau_{-\alpha} - \tilde{\tau}_\alpha) - a^2(\tau_\alpha - \tilde{\tau}_\alpha)(\tau_{-\alpha} - \tilde{\tau}_{-\alpha})\,. \qquad (92)$$

To study the condition $g(\rho, v) = 0$, we express the coordinates $(\rho > 0, v)$ in terms of the prolate spheroidal coordinates (u, y) given in (76). Then, using

$$\sqrt{(v \pm \alpha)^2 + \rho^2} = \alpha\,(u \pm y) > 0, \tag{93}$$

we obtain

$$\begin{aligned} \rho\tau_\alpha &= \alpha\,(uy - 1 + u - y)\,, \\ \rho\tau_{-\alpha} &= \alpha\,(uy + 1 + u + y)\,, \end{aligned} \tag{94}$$

and the condition $g(\rho, v) = 0$ gives

$$u^2 - y^2 = \frac{m}{2\alpha}(1 - y^2)\;. \tag{95}$$

This defines a curve $\mathcal{C}$ in the Weyl coordinates upper half-plane $(\rho > 0, v)$. On this curve, $e^{2\Sigma_3}$ vanishes and the quantities χ_1, χ_2, χ_3 and $e^{2\Sigma_2}$ remain finite, but the matrix $M(\rho, v)$ is ill behaved, since its matrix entry (104) blows up. We note, however, that the metric component g_{tt}, given in (103), stays finite and non-zero on $\mathcal{C}$: although the condition (95) resembles the condition (82) for the ergosurface of the rotating Myers-Perry black hole with one angular momentum a, the former does not equal the latter.

We have explicitly performed the canonical Wiener-Hopf factorisation (9) (assuming its existence) of $\mathcal{M}_{\rho,v}(\tau)$ with respect to Γ. The resulting expression for $M(\rho, v)$ is given in (96). The solution $M(\rho, v)$ describes the exterior region of a rotating Myers-Perry black hole solution with one angular momentum a, as can be verified by converting the Weyl coordinates (ρ, v) into the coordinates used in [12].

Summarising, in the examples discussed above we have shown that, as in the 2×2 case, the non-existence of a canonical Wiener-Hopf factorisation of the monodromy matrices occurs on simple smooth curves in the Weyl coordinates upper-half plane. However, unlike in the the 2×2 case considered in Section 3 (see Theorem 3.7), these curves may or may not correspond to ergosurfaces, which are defined by the norm of the Killing vector $\partial/\partial t$ becoming null there.

Acknowledgements

The authors would like to thank the Isaac Newton Institute for Mathematical Sciences, Cambridge, for support and hospitality during the programme *Black holes: bridges between number theory and holographic quantum information*, where part of the work on this paper was undertaken. This work was supported by EPSRC grant no EP/R014604/1. This work was partially supported by FCT/Portugal through CAMGSD, IST-ID, projects UIDB/04459/2020 and UIDP/04459/2020.

A Canonical factorisation of a 3×3 monodromy matrix

The canonical factorisation (9) (if it exists) of a monodromy matrix $\mathcal{M}_{\rho,v}(\tau)$ with respect to an admissible contour Γ is performed as in (19). Here we consider the monodromy matrix given in [12], obtained by composition of (85) with the spectral relation (6) with $\lambda = 1$. Its canonical factorisation with respect to the contour Γ specified below (86) results in a matrix (12) given by

$$M(\rho, v) = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ -1 - \frac{m}{4}A_{11} + \frac{a}{2}A_{31} & \frac{m}{4} - \frac{m}{4}A_{12} + \frac{a}{2}A_{32} & -\frac{a}{2} - \frac{m}{4}A_{13} + \frac{a}{2}A_{33} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}\,, \tag{96}$$

where

$$
\begin{aligned}
A_{11} &= \frac{4\tau_{-\alpha}\left(\tau_\alpha - \tilde{\tau}_\alpha\right)}{m\rho^2\left(\tau_{-\alpha} - \tilde{\tau}_{-\alpha}\right)\left(\tau_\alpha - \tilde{\tau}_{-\alpha}\right)D}\left(\frac{2m}{\tau_\alpha - \tilde{\tau}_\alpha} - \frac{a^2}{\tau_{-\alpha} - \tilde{\tau}_\alpha}\right) \\
A_{12} &= 1 + \frac{m}{4}A_{11} - \frac{a}{2}A_{31} \\
A_{31} &= A_{13} = -\frac{4a\tau_{-\alpha}}{\rho^2\left(\tau_{-\alpha} - \tilde{\tau}_\alpha\right)D}\left(\frac{1}{\tau_{-\alpha} - \tilde{\tau}_{-\alpha}} - \frac{1}{\tau_\alpha - \tilde{\tau}_{-\alpha}}\right) \\
A_{32} &= \frac{a}{2} + \frac{m}{4}A_{13} - \frac{a}{2}A_{33} \\
A_{33} &= \frac{1}{m\rho D}\left[\frac{2m}{(\tau_{-\alpha} - \tilde{\tau}_{-\alpha})}\left(\tau_\alpha - \frac{a^2\tau_{-\alpha}}{\rho(\tau_{-\alpha} - \tilde{\tau}_\alpha)}\right) - \frac{a^2\tau_{-\alpha}}{(\tau_{-\alpha} - \tilde{\tau}_\alpha)(\tau_\alpha - \tilde{\tau}_{-\alpha})}\left(\tau_\alpha - \tilde{\tau}_\alpha - \frac{2m}{\rho}\right)\right] ,
\end{aligned}
\tag{97}
$$

with D given by (91).

The matrix $M(\rho, v)$ is of the form [12, 7]

$$
\begin{pmatrix}
e^{2\Sigma_1} & e^{2\Sigma_1}\chi_2 & e^{2\Sigma_1}\chi_3 \\
-e^{2\Sigma_1}\chi_2 & -e^{2\Sigma_1}\chi_2^2 + e^{2\Sigma_2} & -e^{2\Sigma_1}\chi_2\chi_3 + e^{2\Sigma_2}\chi_1 \\
e^{2\Sigma_1}\chi_3 & e^{2\Sigma_1}\chi_2\chi_3 - e^{2\Sigma_2}\chi_1 & -e^{2\Sigma_2}\chi_1^2 + e^{2\Sigma_1}\chi_3^2 + e^{2\Sigma_3}
\end{pmatrix}
\tag{98}
$$

with $\Sigma_1 + \Sigma_2 + \Sigma_3 = 0$ and satisfies $M^\natural = M$, where $M^\natural = \eta M^T \eta$ with $\eta = \mathrm{diag}(1, -1, 1)$, as required for a coset representative $M \in SL(3, \mathbb{R})/SO(2, 1)$. The five-dimensional space-time metric is then expressed in terms of these matrix entries as (c.f. eqs. (A.1)-(A.6) in [12])

$$
ds_5^2 = e^{2\Sigma_1}\, ds_3^2 - e^{2\Sigma_3}\left(dt + \mathcal{A}_2\right)^2 + e^{2\Sigma_2}\left(d\psi + \chi_1\, dt + \mathcal{A}_1\right)^2
\tag{99}
$$

with

$$
ds_3^2 = f^2\left(d\rho^2 + dv^2\right) + \rho^2\, d\phi^2 .
\tag{100}
$$

The scalars $\Sigma_1, \Sigma_2, \Sigma_3, \chi_1, f$ and the 1-forms $\mathcal{A}_1$ and $\mathcal{A}_2$ are functions of ρ, v. The two 1-forms are dualised into scalars χ_2 and χ_3 using

$$
e^{-(4\Sigma_1 + 2\Sigma_3)} \star_3 \mathcal{F}_1 = d\chi_2 \quad , \quad -e^{-2(\Sigma_1 - \Sigma_3)} \star_3 \mathcal{F}_2 = d\chi_3 - \chi_1 d\chi_2 ,
\tag{101}
$$

where

$$
\mathcal{F}_1 = d\mathcal{A}_1 + \mathcal{A}_2 \wedge d\chi_1 \quad , \quad \mathcal{F}_2 = d\mathcal{A}_2 ,
\tag{102}
$$

and where $\star_3$ denotes the Hodge star operator in three dimensions. Finally, the function f^2 is determined from $M(\rho, v)$ by integration, as in eq. (2.7) in [2].

In particular, the metric component g_{tt} is given by

$$
-g_{tt} = e^{2\Sigma_3} - e^{2\Sigma_2}\chi_1^2 = A_{33} - \frac{A_{13}^2}{A_{11}} = \frac{u - y - \frac{m}{2\alpha}(1 - y)}{u - y + \frac{m}{2\alpha}(1 + y)} = 1 - \frac{2m}{r^2 + a^2\cos^2\theta} ,
\tag{103}
$$

where we converted from prolate spheroidal coordinates (u, y) to spherical coordinates (r, θ) using the relations $r^2 = 2\alpha(u + 1)$, $2\cos^2\theta = y + 1$ [19, 21].

When approaching a point on the curve $\mathcal{C}$ specified by (91) in a non-tangential manner, the matrix entry $e^{2\Sigma_1}$,

$$
e^{2\Sigma_1} = \frac{2}{\alpha}\left[\frac{u - y + \frac{m}{2\alpha}(1 + y)}{u^2 - y^2 - \frac{m}{2\alpha}(1 - y^2)}\right] ,
\tag{104}
$$

blows up, while g_{tt} remains finite. This metric component vanishes when $u - y = \frac{m}{2\alpha}(1 - y)$, which defines the ergosurface of the rotating Myers-Perry black hole.

References

[1] V. M. Adukov, N. V. Adukova, and G. Mishuris, *An explicit Wiener-Hopf factorisation algorithm for matrix polynomials and its exact realizations within ExactMPF package*, *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences* **478** (2022), no. 2263 20210941, https://doi.org/10.1098/rspa.2021.0941.

[2] P. Aniceto, M. C. Câmara, G. L. Cardoso, and M. Rosselló, *Weyl metrics and Wiener-Hopf factorisation*, *JHEP* **05** (2020) 124, 1910.10632.

[3] V. A. Belinsky and V. E. Zakharov, *Integration of the Einstein Equations by the Inverse Scattering Problem Technique and the Calculation of the Exact Soliton Solutions*, *Sov. Phys. JETP* **48** (1978) 985–994.

[4] P. Breitenlohner and D. Maison, *On the Geroch Group*, *Ann. Inst. H. Poincare Phys. Theor.* **46** (1987) 215.

[5] P. Breitenlohner, D. Maison, and G. W. Gibbons, *Four-Dimensional Black Holes from Kaluza-Klein Theories*, *Commun. Math. Phys.* **120** (1988) 295.

[6] M. C. Câmara and G. L. Cardoso, *Generating new gravitational solutions by matrix multiplication*, *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences* **480** (2024), no. 2285, https://doi.org/10.1098/rspa.2023.0857, 2211.01702.

[7] M. C. Câmara, G. L. Cardoso, T. Mohaupt, and S. Nampuri, *A Riemann-Hilbert approach to rotating attractors*, *JHEP* **06** (2017) 123, 1703.10366.

[8] M. C. Câmara, C. Diogo, and L. Rodman, *Fredholmness of Toeplitz operators and corona problems*, *J. Funct. Anal.* **259** (2010) 1273-1299.

[9] M. C. Câmara, R. O'Loughlin, and J. R. Partington, *Dual-band general Toeplitz operators*, *Mediterr. J. Math.* **19** (2022) 175, 2012.14725.

[10] M. C. Câmara and J. R. Partington, *Spectral properties of truncated Toeplitz operators by equivalence after extension*, *J. Math. Anal. Appl.* **433** (2016) 762-784.

[11] G. L. Cardoso and J. C. Serra, *New gravitational solutions via a Riemann-Hilbert approach*, *JHEP* **03** (2018) 080, 1711.01113.

[12] B. Chakrabarty and A. Virmani, *Geroch Group Description of Black Holes*, *JHEP* **11** (2014) 068, [1408.0875].

[13] K. F. Clancey and I. Gohberg, *Factorisation of Matrix Functions and Singular Integral Operators*, in *Operator Theory: Advances and Applications, vol. 3, Birkhäuser Verlag, Basel, 1981.*

[14] P. Deift, *Riemann-Hilbert problems*, in *Random matrices, pp. 1-40, IAS/Park City Math. Ser. 26, Amer. Math. Soc., Providence, RI, 2019*, 1903.08304.

[15] R. Duduchava, *Integral Equations with Fixed Singularities*, *Teubner, Leipzig, 1979.*

[16] P. Figueras, E. Jamsin, J.V. Rocha and A. Virmani, *Integrability of Five Dimensional Minimal Supergravity and Charged Rotating Black Holes*, *Class. Quant. Grav.* **27** (2010) 135011 [0912.3199].

[17] I. Gohberg, N. Manojlovic, and A. F. dos Santos, *Factorization and integrable systems : summer school in Faro, Portugal, September 2000*, 2003.

[18] G. J. Groenewald, M. A. Kaashoek, and A. C. M. Ran, *Wiener-Hopf factorisation indices of rational matrix functions with respect to the unit circle in terms of realization*, *Indag. Math.* **34** (2023), no. 2, 338 - 356, `2203.07821`.

[19] T. Harmark, *Stationary and axisymmetric solutions of higher-dimensional general relativity*, *Phys. Rev. D* **70** (2004) 124002 [`hep-th/0408141`].

[20] A. R. Its, *The Riemann-Hilbert Problem and Integrable Systems*, *Notices of the AMS* **50** (2003) 1389.

[21] D. Katsimpouri, A. Kleinschmidt, and A. Virmani, *Inverse Scattering and the Geroch Group*, *JHEP* **02** (2013) 011, `1211.3044`.

[22] A. V. Kisil, I. D. Abrahams, G. Mishuris, and S. V. Rogosin, *The Wiener-Hopf technique, its generalizations and applications: constructive and approximate methods*, *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences* **477** (2021), no. 2254 20210533, `https://doi.org/10.1098/rspa.2021.0533`.

[23] V. D. Kunz and R. C. Assier, *Diffraction by a Right-Angled No-Contrast Penetrable Wedge Revisited: A Double Wiener-Hopf Approach*, *SIAM J. Appl. Math.* **82** (2022), no. 4, 1495 - 1519, `2112.03173`.

[24] G. Litvinchuk and I. Spitkovsky, *factorisation of Measurable Matrix Functions*, in *Oper. Theory Adv. Appl., vol. 25, Birkhäuser Verlag, Basel, 1987. Translated from Russian by B. Luderer, with a foreword by B. Silbermann.*

[25] H. Lu, M. J. Perry, and C. N. Pope, *Infinite-dimensional symmetries of two-dimensional coset models coupled to gravity*, *Nucl. Phys. B* **806** (2009) 656–683, `0712.0615`.

[26] E. Meister, E. and F.-O. Speck, *Modern Wiener-Hopf methods in diffraction theory*, in *Ordinary and partial differential equations, Vol. II, Proc. 10th Dundee Conf., Dundee/UK 1988. Pitman Res. Notes Math. Ser. 216, 130-171 (1989).*

[27] S. Mikhlin and S. Prössdorf, *Singular integral operators*, *Springer-Verlag, Berlin, 1986. Translated from German by Albrecht Böttcher and Reinhard Lehmann.*

[28] R.C. Myers and M.J. Perry, *Black Holes in Higher Dimensional Space-Times*, *Annals Phys.* **172** (1986) 304.

[29] H. Nicolai, *Two-dimensional gravities and supergravities as integrable system*, *Lect. Notes Phys.* **396** (1991) 231–273.

[30] R. F. Penna, *Einstein–Rosen waves and the Geroch group*, *J. Math. Phys.* **62** (2021), no. 8 082503, `2106.13252`.

[31] J. H. Schwarz, *Classical symmetries of some two-dimensional models coupled to gravity*, *Nucl. Phys. B* **454** (1995) 427–448, `hep-th/9506076`.

[32] N. Woodhouse, *Integrability and Einstein's equations*, *Banach Center Publications* **41** (1997), no. 1 221–232.