

Classical integrability in the presence of a cosmological constant: analytic and machine learning results

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Abstract

We study the integrability of two-dimensional theories that are obtained by a dimensional reduction of certain four-dimensional gravitational theories describing the coupling of Maxwell fields and neutral scalar fields to gravity in the presence of a potential for the neutral scalar fields. By focusing on a certain solution subspace, we show that a subset of the equations of motion in two dimensions are the compatibility conditions for a modified version of the Breitenlohner-Maison linear system. Subsequently, we study the Liouville integrability of the 2D models encoding the chosen 4D solution subspace from a one-dimensional point of view by constructing Lax pair matrices. In this endeavour, we successfully employ a linear neural network to search for Lax pair matrices for these models, thereby illustrating how machine learning approaches can be effectively implemented to augment the identification of integrable structures in classical systems.

Keywords: Integrability, Lax pair, linear system, neural network, machine learning.

1 Introduction

It has long been established [1, 2, 3] that certain 4D gravitational theories at the two-derivative level, describing the coupling of Maxwell fields and neutral scalar fields to gravity in the absence of a cosmological constant or a potential for the scalar fields, when dimensionally reduced to two dimensions, yield 2D non-linear sigma-models coupled to gravity, whose equations of motion are classically integrable, i.e. that can be viewed as compatibility equations of a linear system. This has recently been the subject of revived interest [4, 5, 6, 7, 8, 9, 10, 11, 12]. The dimensional reduction is performed by means of a two-step procedure. In the first step one reduces over an isometry direction down to three dimensions, and then Hodge dualises all Maxwell field strengths into neutral scalar fields. One then requires the resulting target space for the enlarged set of neutral scalar fields to be a symmetric space G/H . A further reduction along a second commuting isometry direction results in a 2D gravitational model whose field equations are the compatibility equations for a linear system, called Breitenlohner-Maison (BM) linear system

[1, 3]. There exists another linear system, the Belinski-Zakharov linear system [13], which has been shown [14] to be equivalent to the Breitenlohner-Maison linear system. In this paper we will work with the latter, which has the advantage of making use of the group structure that underlies the dimensionally reduced theory.

The addition of a cosmological constant or a potential for the neutral scalar fields to these four-dimensional gravitational theories will, in general, not preserve the integrable nature of the dimensionally reduced equations of motion. Aspects of this were discussed in [15, 16]. In this paper we show that in the presence of a scalar potential, for a certain solution subspace, a subset of the equations of motion in two dimensions can still be viewed as being the compatibility conditions of a linear system, namely a modified version of the BM linear system. The BM linear system can be represented in terms of differential operators, as shown in [17]. Here, we show that the modified version of the BM linear system also admits a Lax pair description in terms of accordingly modified differential operators.

The aforementioned solution subspace for which the modified Breitenlohner-Maison linear system description exists, admits an equivalent 1D description. This enables us to discuss the Liouville integrability of this solution subspace from a one-dimensional point of view, by resorting to a description based on Lax pair matrices. Here, we use machine learning (ML) techniques to search for Lax pair matrices for these models, thereby exhibiting how machine learning approaches can be implemented effectively in the identification of classical integrability structures for 1D systems. We illustrate the search for Lax pair matrices in specific models using both analytic and ML techniques. Our ML experiments suggest conserved currents that help determine Lax pairs for the models under consideration. Finally, we discuss the interpretability of the Lax pair matrices generated in our machine learning experiments, and we compare our ML approach with prior ML approaches to integrability [18, 19].

This paper is organised as follows. In Section 2 we write down 2D non-linear sigma models and the equations of motion resulting from the dimensional reduction of specific 4D gravity theories. We identify a specific subclass of solutions amenable to a modified BM linear system. In Section 3 we write down the modified BM linear system as well as Lax pair descriptions in terms of differential operators for the chosen solution subclass, and we give two illustrative examples. In Section 4 we discuss the Liouville integrability of the systems from a 1D point of view. In Section 5 we construct analytic Lax pairs in two models. In Section 6 we set up an ML strategy to determine numerical and interpretable Lax pairs for the 1D systems studied above and then present our findings for the conserved currents suggested by our ML experiments in Section 7. Finally in Section 8 we conclude with comments on our findings and comparisons with other ML approaches. In Appendix A we present families of Lax pair matrices for certain 1D systems. Appendices B and C contain further details about our ML experiments.

2 Dimensional reduction and field equations in two dimensions

We consider a Lagrangian in four space-time dimensions describing the coupling of Maxwell fields and neutral scalar fields ϕ to gravity. We allow for the presence of a potential $V(\phi)$ for the neutral scalar fields.

We then assume that we can dimensionally reduce this theory down to two dimensions using a 2-step procedure. In the first step, we reduce to three dimensions over an isometry direction

y (which is time-like when $\lambda = 1$ and space-like when $\lambda = -1$),

$$ds_4^2 = -\lambda \Delta (dy + B d\varphi)^2 + \Delta^{-1} ds_3^2 \quad , \quad \lambda = \pm 1. \quad (1)$$

The metric factors Δ and B are taken to be independent of y . In three space-time dimensions, we Hodge dualise the Maxwell field strengths into scalar fields. The resulting three-dimensional Lagrangian then describes the coupling of an enlarged set of neutral scalar fields to three-dimensional gravity, in the presence of a potential $V(\phi)$. It takes the form [3]

$$L_3 = \sqrt{|g_3|} \left(R_3 - \frac{1}{4} g^{MN} \text{Tr}(A_M A_N) - \Delta^{-1} V(\phi) \right) \quad , \quad (2)$$

where the enlarged set of neutral scalar fields is encoded in a matrix M which we assume takes values in a symmetric space G/H , i.e. $M \in G/H$. Then $A \equiv M^{-1} dM$ denotes a matrix 1-form. The symmetric space G/H is endowed with an involutive Lie algebra automorphism \natural , which is used to decompose the Lie algebra of G into the direct sum $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$. Then $M \in G/H$ is identified with $M \in G/H \simeq \exp \mathfrak{p}$ and satisfies $M = M^\natural$.

In the second step, we further reduce the model over a second commuting isometry direction φ using

$$ds_3^2 = e^{2\Sigma} ds_2^2 + \tilde{\rho}^2 d\varphi^2 \quad . \quad (3)$$

In the above we take $\tilde{\rho} > 0$. We denote the space-time coordinates in two dimensions by (ρ, v) , which we take to satisfy $\rho > 0, v \in \mathbb{R}$. The fields $M, \Sigma, \tilde{\rho}$ are functions of (ρ, v) only. The resulting two-dimensional Lagrangian takes the form given in [3], augmented by the presence of a potential

$$\mathcal{V}(\tilde{\rho}, \Sigma, \Delta, \phi) \equiv \tilde{\rho} e^{2\Sigma} \Delta^{-1} V(\phi) \quad , \quad (4)$$

namely

$$L_2 = \sqrt{|g_2|} \left[\tilde{\rho} \left(R_2 - \frac{1}{4} g^{\mu\nu} \text{Tr}(A_\mu A_\nu) \right) + 2g^{\mu\nu} \partial_\mu \tilde{\rho} \partial_\nu \Sigma - \mathcal{V}(\tilde{\rho}, \Sigma, \Delta, \phi) \right] \quad , \quad (5)$$

where

$$A_\mu = M^{-1} \partial_\mu M \quad , \quad \mu = \rho, v \quad . \quad (6)$$

Note that \mathcal{V} depends linearly on $\tilde{\rho}$. The potential \mathcal{V} depends on the fields $\tilde{\rho}, \Sigma, \Delta, \phi$, but only the fields Δ, ϕ (and also \tilde{B} , which is defined below in (8)) are encoded in the matrix M .

We introduce the Hodge star operator \star in two dimensions, satisfying

$$\star d\rho = -\lambda dv \quad , \quad \star dv = d\rho \quad , \quad (\star)^2 = -\lambda \text{id} \quad . \quad (7)$$

Using \star , the field \tilde{B} is defined in terms of the field B in (1) by [1]

$$\rho \star d\tilde{B} = \Delta^2 dB \quad . \quad (8)$$

The equations of motion derived from (5) take the following form. First we consider the equations of motion for the scalar fields encoded in M . We will denote these scalar fields by Φ^I ; they include the scalar fields Δ, ϕ . Using

$$\begin{aligned} \delta A_\mu &= -M^{-1} \delta M A_\mu + M^{-1} \partial_\mu \delta M \\ &= -M^{-1} \delta M A_\mu + \partial_\mu (M^{-1} \delta M) + A_\mu M^{-1} \delta M \end{aligned} \quad , \quad (9)$$

we obtain

$$\frac{1}{2} \text{Tr} \left[M^{-1} \frac{\delta M}{\delta \Phi^I} \nabla^\mu (\tilde{\rho} A_\mu) \right] - \frac{\delta \mathcal{V}}{\delta \Phi^I} = 0. \quad (10)$$

Now we use the fact that $M^{-1} \frac{\delta M}{\delta \Phi^I}$ is an element of the Lie algebra \mathfrak{p} , whose generators we denote by D^I . Note that the D^I are constant matrices, and that their number equals the number of fields Φ^I . We therefore have

$$\begin{aligned} M^{-1} \frac{\delta M}{\delta \Phi^I} &= f_{IJ}(\Phi) D^J, \\ A_\mu &= \partial_\mu \Phi^I f_{IJ}(\Phi) D^J. \end{aligned} \quad (11)$$

Then, the equation of motion (10) can be written as

$$\frac{1}{2} \text{Tr} [D^K D^L] f_{IK}(\Phi) \nabla^\mu (\tilde{\rho} \partial_\mu \Phi^J f_{JL}(\Phi)) - \frac{\delta \mathcal{V}}{\delta \Phi^I} = 0. \quad (12)$$

Denoting

$$\begin{aligned} C^{KL} &= \frac{1}{2} \text{Tr} [D^K D^L] = C^{LK}, \\ T_I^L &= f_{IK}(\Phi) C^{KL}, \end{aligned} \quad (13)$$

and assuming that T_I^L is invertible with inverse $(T^{-1})_J^I$, i.e. $(T^{-1})_J^I T_I^L = \delta_J^L$, we obtain

$$\nabla^\mu (\tilde{\rho} \partial_\mu \Phi^J f_{JL}(\Phi)) - (T^{-1})_L^I \frac{\delta \mathcal{V}}{\delta \Phi^I} = 0. \quad (14)$$

Contracting this equation with D^L gives

$$\nabla^\mu (\tilde{\rho} A_\mu) = \tilde{\rho} G, \quad (15)$$

where the function G is given by

$$\tilde{\rho} G = D^L (T^{-1})_L^I \frac{\delta \mathcal{V}}{\delta \Phi^I}. \quad (16)$$

Using (11), we establish the relation

$$\frac{1}{2} \text{Tr} \left[D^L (T^{-1})_L^I \frac{\delta \mathcal{V}}{\delta \Phi^I} A_\mu \right] = \frac{\delta \mathcal{V}}{\delta \Phi^I} \partial_\mu \Phi^I. \quad (17)$$

Taking ds_2^2 to be a flat metric given by

$$ds_2^2 = \lambda d\rho^2 + dv^2, \quad (18)$$

and using (7), the field equation (15) can be written as

$$d(\tilde{\rho} \star A) = dv \wedge d\rho \tilde{\rho} \left(\lambda \partial_\rho A_\rho + \partial_v A_v + \lambda \frac{\partial \tilde{\rho}}{\partial \rho} \frac{A_\rho}{\tilde{\rho}} + \frac{\partial \tilde{\rho}}{\partial v} \frac{A_v}{\tilde{\rho}} \right) = dv \wedge d\rho \tilde{\rho} G, \quad A = M^{-1} dM. \quad (19)$$

Next, varying (5) with respect to $2\Sigma, \tilde{\rho}, g_{\mu\nu}$ gives the equations of motion

$$\begin{aligned}
\Box \tilde{\rho} &= \frac{\delta \mathcal{V}}{\delta(2\Sigma)} = -\mathcal{V}, \\
\Box(2\Sigma) &= R_2 - \frac{1}{4} g^{\mu\nu} \text{Tr}(A_\mu A_\nu) - \frac{\delta \mathcal{V}}{\delta \tilde{\rho}} = R_2 - \frac{1}{4} g^{\mu\nu} \text{Tr}(A_\mu A_\nu) - \frac{\mathcal{V}}{\tilde{\rho}}, \\
\partial_\mu \tilde{\rho} \partial_\nu(2\Sigma) + \partial_\nu \tilde{\rho} \partial_\mu(2\Sigma) - g_{\mu\nu} \partial_\sigma \tilde{\rho} \partial^\sigma(2\Sigma) + 2g_{\mu\nu} \Box \tilde{\rho} - 2\nabla_\mu \nabla_\nu \tilde{\rho} \\
&\quad - \frac{1}{2} \tilde{\rho} \left(\text{Tr} A_\mu A_\nu - \frac{1}{2} g_{\mu\nu} \text{Tr} A_\sigma A^\sigma \right) + g_{\mu\nu} \mathcal{V} = 0.
\end{aligned} \tag{20}$$

Taking ds_2^2 to be the flat metric given in (18), these field equations become

$$\begin{aligned}
\lambda \partial_\rho^2 \tilde{\rho} + \partial_v^2 \tilde{\rho} &= -\mathcal{V}, \\
\lambda \partial_\rho^2(2\Sigma) + \partial_v^2(2\Sigma) &= -\frac{1}{4} \text{Tr}(\lambda A_\rho A_\rho + A_v A_v) - \frac{\mathcal{V}}{\tilde{\rho}}, \\
(\mu, \nu) = (\rho, \rho) : \quad \partial_\rho \tilde{\rho} \partial_\rho(2\Sigma) - \lambda \partial_v \tilde{\rho} \partial_v(2\Sigma) + 2\lambda \partial_v^2 \tilde{\rho} - \frac{1}{4} \tilde{\rho} \text{Tr}[A_\rho A_\rho - \lambda A_v A_v] + \lambda \mathcal{V} &= 0, \\
(\mu, \nu) = (v, v) : \quad \partial_v \tilde{\rho} \partial_v(2\Sigma) - \lambda \partial_\rho \tilde{\rho} \partial_\rho(2\Sigma) + 2\lambda \partial_\rho^2 \tilde{\rho} - \frac{1}{4} \tilde{\rho} \text{Tr}[A_v A_v - \lambda A_\rho A_\rho] + \mathcal{V} &= 0, \\
(\mu, \nu) = (\rho, v) : \quad \partial_\rho \tilde{\rho} \partial_v(2\Sigma) + \partial_v \tilde{\rho} \partial_\rho(2\Sigma) - 2\partial_\rho \partial_v \tilde{\rho} - \frac{1}{2} \tilde{\rho} \text{Tr}[A_\rho A_v] &= 0.
\end{aligned} \tag{21}$$

Combining the third equation with the fourth equation gives the first equation. We may therefore take the first, second, third and fifth equations as independent equations.

Next, we analyse the solvability of the field equations (21). Taking the ρ -derivative of the third equation and adding it to λ times the v -derivative of the fifth equation yields

$$\begin{aligned}
\partial_\rho(\log \tilde{\rho} + 2\Sigma) \mathcal{V} &= \partial_\rho \mathcal{V} - \frac{1}{2} \tilde{\rho} \text{Tr}[G A_\rho] = \partial_\rho \mathcal{V} - \frac{1}{2} \text{Tr} \left[D^L (T^{-1})_L^I \frac{\delta \mathcal{V}}{\delta \Phi^I} A_\rho \right] \\
&= \partial_\rho \mathcal{V} - \frac{\delta \mathcal{V}}{\delta \Phi^I} \partial_\rho \Phi^I,
\end{aligned} \tag{22}$$

where we used the first two field equations given in (21) as well as (19) and (17). On the other hand, taking the v -derivative of the third equation and subtracting the ρ -derivative of the fifth equation gives

$$\begin{aligned}
\partial_v(\log \tilde{\rho} + 2\Sigma) \mathcal{V} &= \partial_v \mathcal{V} - \frac{1}{2} \tilde{\rho} \text{Tr}[G A_v] = \partial_v \mathcal{V} - \frac{1}{2} \text{Tr} \left[D^L (T^{-1})_L^I \frac{\delta \mathcal{V}}{\delta \Phi^I} A_v \right] \\
&= \partial_v \mathcal{V} - \frac{\delta \mathcal{V}}{\delta \Phi^I} \partial_v \Phi^I,
\end{aligned} \tag{23}$$

where we used the first two field equations given in (21) as well as (19) and (17).

We note that equations (22) and (23) are automatically satisfied, since they can be written as

$$\partial_\mu(\log \tilde{\rho} + 2\Sigma) \mathcal{V} = \partial_\mu \mathcal{V} - \frac{\delta \mathcal{V}}{\delta \Phi^I} \partial_\mu \Phi^I. \tag{24}$$

Here we have made use of the fact that since \mathcal{V} has the form given in (4), we have

$$\partial_\mu \mathcal{V} - \frac{\delta \mathcal{V}}{\delta \Phi^I} \partial_\mu \Phi^I = \partial_\mu(\log \tilde{\rho} + 2\Sigma) \mathcal{V}. \tag{25}$$

Summarizing, we take the four-dimensional metric to be given by

$$ds_4^2 = -\lambda \Delta (dt + Bd\phi)^2 + \Delta^{-1} (e^{2\Sigma} (\lambda d\rho^2 + dv^2) + \tilde{\rho}^2 d\phi^2) \quad , \quad \lambda = \pm 1, \quad (26)$$

and the independent field equations to be

$$d(\tilde{\rho} \star A) = dv \wedge d\rho \tilde{\rho} G \quad , \quad A = M^{-1} dM \quad (27)$$

together with

$$\begin{aligned} \lambda \partial_\rho^2 \tilde{\rho} + \partial_v^2 \tilde{\rho} &= -\mathcal{V}, \\ \lambda \partial_\rho^2 (2\Sigma) + \partial_v^2 (2\Sigma) &= -\frac{1}{4} \text{Tr} (\lambda A_\rho A_\rho + A_v A_v) - \frac{\mathcal{V}}{\tilde{\rho}}, \\ (\mu, \nu) = (\rho, \rho) : \quad \partial_\rho \tilde{\rho} \partial_\rho (2\Sigma) - \lambda \partial_v \tilde{\rho} \partial_v (2\Sigma) + 2\lambda \partial_v^2 \tilde{\rho} - \frac{1}{4} \tilde{\rho} \text{Tr} [A_\rho A_\rho - \lambda A_v A_v] + \lambda \mathcal{V} &= 0, \\ (\mu, \nu) = (\rho, v) : \quad \partial_\rho \tilde{\rho} \partial_v (2\Sigma) + \partial_v \tilde{\rho} \partial_\rho (2\Sigma) - 2\partial_\rho \partial_v \tilde{\rho} - \frac{1}{2} \tilde{\rho} \text{Tr} [A_\rho A_v] &= 0. \end{aligned} \quad (28)$$

2.1 Subspace of solutions

In the following we will focus on solutions to the field equations that have the following dependence on ρ, v ,

$$\tilde{\rho}(\rho, v) = h(\rho) g(v) \quad , \quad \Sigma = \Sigma(\rho) \quad , \quad \Phi^I = \Phi^I(\rho). \quad (29)$$

This implies

$$A_v = M^{-1} \partial_v M = 0. \quad (30)$$

Then, the field equation (27) becomes

$$\lambda \partial_\rho A_\rho + \lambda (\partial_\rho \log h) A_\rho = G, \quad (31)$$

with $\tilde{\rho} G$ given by (16). Note that G is a function of ρ only. Furthermore, we assume that

$$\partial_v^2 g(v) = -\alpha g(v) \quad , \quad \alpha = -1, 0, 1. \quad (32)$$

Then, the first equation in (28) becomes

$$\lambda \frac{\partial_\rho^2 h}{h} - \alpha = -\frac{\mathcal{V}}{\tilde{\rho}}, \quad (33)$$

while the second and third equations become

$$\lambda \partial_\rho^2 (2\Sigma) = -\frac{\lambda}{4} \text{Tr} (A_\rho A_\rho) - \frac{\mathcal{V}}{\tilde{\rho}}, \quad (34)$$

$$\frac{\partial_\rho h}{h} \partial_\rho (2\Sigma) - 2\lambda \alpha - \frac{1}{4} \text{Tr} [A_\rho A_\rho] + \lambda \frac{\mathcal{V}}{\tilde{\rho}} = 0. \quad (35)$$

Combining (33) with (34) and (35) gives

$$\partial_\rho (h \partial_\rho (2\Sigma) - 2\partial_\rho h) = 0, \quad (36)$$

which is solved by

$$2\Sigma = 2 \log h - c \int \frac{d\rho}{h}, \quad c \in \mathbb{R}. \quad (37)$$

The fourth equation in (28) becomes

$$h \partial_v g \partial_\rho \log (h^2 e^{2\Sigma}) = 0. \quad (38)$$

When $\partial_v g = 0$, this does not impose any further condition on (37). On the other hand, when $\partial_v g \neq 0$, we conclude that

$$e^{2\Sigma} = h^2, \quad (39)$$

where we have adjusted the integration constant so as to ensure compatibility with (37). Thus, 2Σ is determined by (37), with $c = 0$ when $\partial_v g \neq 0$.

Summarizing, the field equations for $\tilde{\rho}$ and Φ^I that need to be solved are (31), (33) and (35), with 2Σ determined in terms of h by (37).

When $\mathcal{V} = 0$, the field equation (31) is solved by

$$A_\rho = \frac{C}{h}, \quad (40)$$

where C denotes a constant matrix. When $\alpha = \pm 1$ we have $2\Sigma = 2 \log h$, and (35) becomes

$$2(\partial_\rho h)^2 - 2\lambda\alpha h^2 - \frac{1}{4} \text{Tr } C^2 = 0. \quad (41)$$

Writing the general solution of (33) as a linear combination of two linearly independent basis vectors h_1 and h_2 that satisfy $h_1^2 + \lambda\alpha h_2^2 = 1$,

$$h = A_1 h_1 + A_2 h_2, \quad A_1, A_2 \in \mathbb{R}, \quad (42)$$

we find that (41) imposes the following condition on the constant matrix C ,

$$2(A_1^2 - \lambda\alpha A_2^2) = \frac{1}{4} \text{Tr } C^2. \quad (43)$$

On the other hand, when $\alpha = 0$, (35) becomes

$$2(\partial_\rho h)^2 - c \partial_\rho h - \frac{1}{4} \text{Tr } C^2 = 0. \quad (44)$$

Writing the general solution of (33) as

$$h = A_1 + A_2 \rho, \quad A_1, A_2 \in \mathbb{R}, \quad (45)$$

we obtain the following condition on the constant matrix C ,

$$2A_2^2 - c A_2 - \frac{1}{4} \text{Tr } C^2 = 0. \quad (46)$$

In the next section we will introduce a linear system in two dimensions whose solvability condition is the field equation (31). To do so, we will write $\tilde{\rho}$ as a sum of two terms, $\tilde{\rho} = \tilde{\rho}_0 + \tilde{\rho}_P$, where

$$\lambda \partial_\rho^2 \tilde{\rho}_0 + \partial_v^2 \tilde{\rho}_0 = 0, \quad \lambda \partial_\rho^2 \tilde{\rho}_P + \partial_v^2 \tilde{\rho}_P = -\mathcal{V}. \quad (47)$$

We then introduce the field \tilde{v}_0 by

$$\star d\tilde{\rho}_0 = -\lambda d\tilde{v}_0 \quad , \quad \star d\tilde{v}_0 = d\tilde{\rho}_0. \quad (48)$$

We assume that $\tilde{\rho}_0$ has the form

$$\tilde{\rho}_0(\rho, v) = h_0(\rho) g_0(v). \quad (49)$$

The spectral parameter entering the linear system will be defined in terms of $(\tilde{\rho}_0, \tilde{v}_0)$.

3 A modified Breitenlohner-Maison linear system

In the absence of a potential \mathcal{V} , the field equations for M and $\tilde{\rho}$ are given by (cf. (27) and (28))

$$\begin{aligned} F &= dA + A \wedge A = 0 \iff F_{\rho v} = 0, \\ d(\tilde{\rho} \star A) &= 0 \iff \lambda \partial_\rho A_\rho + \partial_v A_v + \frac{\lambda}{\tilde{\rho}} \frac{\partial \tilde{\rho}}{\partial \rho} A_\rho + \frac{1}{\tilde{\rho}} \frac{\partial \tilde{\rho}}{\partial v} A_v = 0, \\ d \star d\tilde{\rho} &= 0 \iff \lambda \partial_\rho^2 \tilde{\rho} + \partial_v^2 \tilde{\rho} = 0. \end{aligned} \quad (50)$$

It is well known that these field equations are the compatibility conditions for an auxiliary system of first-order differential equations, called Breitenlohner-Maison linear system [1]. This linear system (a Lax pair) takes the form [3]

$$\tau (d + A) X = \star dX, \quad (51)$$

where X is a matrix function, and where τ is a function taken from the set of functions $\varphi_\omega(\rho, v)$ of the form [9]

$$\frac{-\lambda(\omega - \tilde{v}) \pm \sqrt{(\omega - \tilde{v})^2 + \lambda \tilde{\rho}^2}}{\tilde{\rho}}, \quad \omega \in \mathbb{C}, \quad (52)$$

with \tilde{v} defined as in (48). Note that

$$\star d\tilde{\rho} = -\lambda d\tilde{v} \quad , \quad \star d\tilde{v} = d\tilde{\rho} \quad \iff \quad \frac{\partial \tilde{\rho}}{\partial \rho} = \frac{\partial \tilde{v}}{\partial v} \quad , \quad \frac{\partial \tilde{\rho}}{\partial v} = -\lambda \frac{\partial \tilde{v}}{\partial \rho} \quad (53)$$

as well as

$$\frac{\partial \tau}{\partial \tilde{\rho}} = \frac{\tau(\lambda - \tau^2)}{\tilde{\rho}(\lambda + \tau^2)} \quad , \quad \frac{\partial \tau}{\partial \tilde{v}} = \frac{2\lambda\tau^2}{\tilde{\rho}(\lambda + \tau^2)}, \quad (54)$$

and hence

$$-\frac{\partial \tau}{\partial \tilde{v}} + \tau \frac{\partial \tau}{\partial \tilde{\rho}} = -\frac{\tau^2}{\tilde{\rho}} \quad , \quad \lambda \frac{\partial \tau}{\partial \tilde{\rho}} + \tau \frac{\partial \tau}{\partial \tilde{v}} = \lambda \frac{\tau}{\tilde{\rho}}. \quad (55)$$

The linear system (51) can be expressed in terms of a pair of differential operators $(\mathcal{L}, \mathcal{M})$, [17]

$$\tau (d + A) X = \star dX \iff \mathcal{L}X = \mathcal{M}X = 0, \quad (56)$$

where

$$\begin{aligned}\mathcal{L} &= -\partial_v + \tau D_\rho, \\ \mathcal{M} &= \lambda \partial_\rho + \tau D_v, \quad \lambda = \pm 1,\end{aligned}\tag{57}$$

with

$$D_\mu = \partial_\mu + A_\mu, \quad \mu = \rho, v.\tag{58}$$

For the case when $\tilde{\rho}$ is identified with the coordinate ρ , it was shown in [17] that the compatibility condition $0 = [\mathcal{L}, \mathcal{M}]X$ imply the field equations (50). Here we generalise the discussion given in [17] to the case when $\tilde{\rho}$ is a function of (ρ, v) . We compute the compatibility condition $0 = [\mathcal{L}, \mathcal{M}]X$,

$$0 = [\mathcal{L}, \mathcal{M}]X = -\frac{\lambda}{\tau} (-\partial_v \tau + \tau \partial_\rho \tau) \partial_\rho - (\lambda \partial_\rho \tau + \tau \partial_v \tau) D_\rho - \tau (\partial_v A_v + \lambda \partial_\rho A_\rho) + \tau^2 F_{\rho v},\tag{59}$$

where we used $\mathcal{M}X = 0$ once. Next, using (53) as well as (55), we establish

$$\begin{aligned}-\partial_v \tau + \tau \partial_\rho \tau &= -\frac{\tau^2}{\tilde{\rho}} \frac{\partial \tilde{\rho}}{\partial \rho} + \lambda \frac{\tau}{\tilde{\rho}} \frac{\partial \tilde{v}}{\partial \rho}, \\ \lambda \partial_\rho \tau + \tau \partial_v \tau &= \lambda \frac{\tau}{\tilde{\rho}} \frac{\partial \tilde{\rho}}{\partial \rho} - \frac{\tau^2}{\tilde{\rho}} \frac{\partial \tilde{\rho}}{\partial v}.\end{aligned}\tag{60}$$

Inserting this into (59) gives

$$\begin{aligned}0 &= [\mathcal{L}, \mathcal{M}]X = \frac{1}{\tilde{\rho}} \frac{\partial \tilde{\rho}}{\partial v} (\tau A_v + \lambda \partial_\rho + \tau^2 D_\rho) X \\ &\quad - \tau \left(\partial_v A_v + \lambda \partial_\rho A_\rho + \frac{\lambda}{\tilde{\rho}} \frac{\partial \tilde{\rho}}{\partial \rho} A_\rho + \frac{1}{\tilde{\rho}} \frac{\partial \tilde{\rho}}{\partial v} A_v \right) X + \tau^2 F_{\rho v} X.\end{aligned}\tag{61}$$

Next, using both $\mathcal{L}X = 0$ and $\mathcal{M}X = 0$ we infer the relation

$$(\lambda \partial_\rho + \tau^2 D_\rho) X = (-\tau D_v + \tau \partial_v) X = -\tau A_v X,\tag{62}$$

and hence the first line in (61) cancels out, so that

$$0 = [\mathcal{L}, \mathcal{M}]X = \left[-\tau \left(\partial_v A_v + \lambda \partial_\rho A_\rho + \frac{\lambda}{\tilde{\rho}} \frac{\partial \tilde{\rho}}{\partial \rho} A_\rho + \frac{1}{\tilde{\rho}} \frac{\partial \tilde{\rho}}{\partial v} A_v \right) + \tau^2 F_{\rho v} \right] X.\tag{63}$$

Assuming that the matrix X is invertible, we obtain

$$\partial_v A_v + \lambda \partial_\rho A_\rho + \frac{\lambda}{\tilde{\rho}} \frac{\partial \tilde{\rho}}{\partial \rho} A_\rho + \frac{1}{\tilde{\rho}} \frac{\partial \tilde{\rho}}{\partial v} A_v = 0 \quad \wedge \quad F_{\rho v} = 0,\tag{64}$$

which, together with (53), are the field equations (50). Finally, 2Σ is determined by integrating the last two equations of (28). Their solvability is a consequence of the first two equations in (28).

When $\mathcal{V} \neq 0$, the field equations for M and $\tilde{\rho}$ are modified, and in general these modified field equations will not any longer be the compatibility conditions for a linear system. Below we will show that when restricting to the subspace of solutions discussed in Section 2.1, the modified field equation (31) for M is the compatibility condition for a modified linear system. Since the field equation for $\tilde{\rho}$ ceases to be of the form given in (50), this modified linear system cannot be formulated in terms of $\tilde{\rho}$, but instead will have to be formulated in terms of a pair $(\tilde{\rho}_0, \tilde{v}_0)$ that satisfies (48). The field equation for $\tilde{\rho}$ given in (33) will therefore not arise as a compatibility condition for the modified linear system and will have to be imposed separately. Likewise, (35) will also have to be verified separately, where we recall that 2Σ is given by (37).

3.1 $\mathcal{V} \neq 0$: modified Lax pair

We focus on the solution space described in Section 2.1, in which case $A_v = 0$, and we consider the following modification of the Lax pair (57) which is induced by a non-vanishing \mathcal{V} ,

$$\begin{aligned}\mathcal{L} &= -\partial_v + \tau D_\rho = -\partial_v + \tau(\partial_\rho + A_\rho) , \\ \mathcal{M} &= \lambda \partial_\rho + \tau \partial_v + \Omega(\rho) \quad , \quad \lambda = \pm 1,\end{aligned}\tag{65}$$

where τ is a function taken from the set of functions (52) with $(\tilde{\rho}, \tilde{v})$ replaced by $(\tilde{\rho}_0, \tilde{v}_0)$, and where $\tilde{\rho}_0$ has the form given in (49). Here, $\Omega(\rho)$ describes the modification of the Lax pair (57) due to the presence of \mathcal{V} .

We impose $\mathcal{L}X = \mathcal{M}X = 0$ and study the compatibility condition

$$\mathcal{L}X = 0 \wedge \mathcal{M}X = 0 \implies [\mathcal{L}, \mathcal{M}]X = 0.\tag{66}$$

We obtain

$$0 = [\mathcal{L}, \mathcal{M}]X = [\tau(\partial_\rho \Omega - \lambda \partial_\rho A_\rho) + (-\partial_v \tau + \tau \partial_\rho \tau) \partial_v - (\lambda \partial_\rho \tau + \tau \partial_v \tau) D_\rho] X.\tag{67}$$

Using $\mathcal{M}X = 0$ we obtain

$$0 = [\mathcal{L}, \mathcal{M}]X = \left[\tau(\partial_\rho \Omega - \lambda \partial_\rho A_\rho) - \frac{1}{\tau}(-\partial_v \tau + \tau \partial_\rho \tau)(\lambda \partial_\rho + \Omega) - (\lambda \partial_\rho \tau + \tau \partial_v \tau) D_\rho \right] X.\tag{68}$$

Next, using (53) as well as (55) with $(\tilde{\rho}, \tilde{v})$ replaced by $(\tilde{\rho}_0, \tilde{v}_0)$, we establish

$$\begin{aligned}-\partial_v \tau + \tau \partial_\rho \tau &= -\frac{\tau^2}{\tilde{\rho}_0} \frac{\partial \tilde{\rho}_0}{\partial \rho} + \lambda \frac{\tau}{\tilde{\rho}_0} \frac{\partial \tilde{v}_0}{\partial \rho} = -\frac{\tau^2}{\tilde{\rho}_0} \frac{\partial \tilde{\rho}_0}{\partial \rho} - \frac{\tau}{\tilde{\rho}_0} \frac{\partial \tilde{\rho}_0}{\partial v} \\ \lambda \partial_\rho \tau + \tau \partial_v \tau &= \lambda \frac{\tau}{\tilde{\rho}_0} \frac{\partial \tilde{\rho}_0}{\partial \rho} - \frac{\tau^2}{\tilde{\rho}_0} \frac{\partial \tilde{\rho}_0}{\partial v}.\end{aligned}\tag{69}$$

Inserting this into (68) gives

$$\begin{aligned}0 &= [\mathcal{L}, \mathcal{M}]X = \tau \left(\partial_\rho f - \lambda \partial_\rho A_\rho - \frac{\lambda}{\tilde{\rho}_0} \frac{\partial \tilde{\rho}_0}{\partial \rho} A_\rho \right) X + \frac{1}{\tilde{\rho}_0} \frac{\partial \tilde{\rho}_0}{\partial v} (\lambda \partial_\rho + \tau^2 D_\rho) X \\ &\quad + \left(\frac{\tau}{\tilde{\rho}_0} \frac{\partial \tilde{\rho}_0}{\partial \rho} + \frac{1}{\tilde{\rho}_0} \frac{\partial \tilde{\rho}_0}{\partial v} \right) \Omega X.\end{aligned}\tag{70}$$

Next, using both $\mathcal{L}X = 0$ and $\mathcal{M}X = 0$ we obtain

$$(\lambda \partial_\rho + \tau^2 D_\rho) X = -\Omega X ,\tag{71}$$

and hence we get

$$\begin{aligned}0 &= [\mathcal{L}, \mathcal{M}]X = \tau \left[\left(\partial_\rho + \frac{1}{\tilde{\rho}_0} \frac{\partial \tilde{\rho}_0}{\partial \rho} \right) (\Omega - \lambda A_\rho) \right] X \\ &= \frac{\tau}{h_0} \partial_\rho [h_0 (\Omega - \lambda A_\rho)] X.\end{aligned}\tag{72}$$

Assuming that the matrix X is invertible, we infer

$$\Omega = \lambda \left(A_\rho - \frac{C}{h_0} \right) ,\tag{73}$$

where C is a constant matrix.

Demanding that when $\mathcal{V} = 0$ the modified Lax pair reduces to the original Lax pair (57), implies that $\Omega = 0$ when switching off the potential \mathcal{V} . We assume that the solution for A_ρ in the presence of $\mathcal{V} \neq 0$ reduces to the one when $\mathcal{V} = 0$, and that likewise h reduces to h_0 . It follows that when $\mathcal{V} = 0$, A_ρ takes the form

$$A_\rho|_{\mathcal{V}=0} = \frac{C}{h_0}. \quad (74)$$

Then, the modification Ω can be expressed as

$$\Omega = \lambda (A_\rho - A_\rho|_{\mathcal{V}=0}). \quad (75)$$

To relate Ω to the potential \mathcal{V} , we return to the field equation for M given in (31), which we write as

$$\partial_\rho (h A_\rho) = \lambda h G. \quad (76)$$

Both sides of this equation only depend on ρ . Integrating over ρ gives

$$A_\rho = \frac{\lambda}{h} \int h G d\rho + \frac{C}{h}, \quad (77)$$

where we have adjusted the integration constant so as to obtain $A_\rho|_{\mathcal{V}=0}$ when $G = 0$. Then, Ω can be expressed as

$$\Omega = \frac{1}{h} \int h G d\rho + \lambda \left(\frac{h_0}{h} - 1 \right) A_\rho|_{\mathcal{V}=0}. \quad (78)$$

The modified Lax pair (65) can also be rewritten as

$$\tau (d + A) X = \star dX - (\Omega dv) X = \star (d + P) X \quad , \quad P = \lambda \Omega d\rho. \quad (79)$$

At this state it is important to point out that we write the modified Lax pair only for configurations for which we can define a flat space-time limit, as we show in the examples below.

3.2 Examples

When dimensionally reducing General Relativity to two dimensions using the 2-step procedure mentioned above, the resulting coset G/H is $SL(2, \mathbb{R})/SO(2)$. The matrix M is given by [1]

$$M = \begin{pmatrix} \Delta + \tilde{B}^2/\Delta & \tilde{B}/\Delta \\ \tilde{B}/\Delta & 1/\Delta \end{pmatrix}, \quad (80)$$

where Δ is the metric factor in (26), and \tilde{B} is related to the metric factor B by (8).

In the presence of a cosmological constant $V = 2\Lambda_4 = -6/L^2$, \mathcal{V} takes the form

$$\mathcal{V} = \tilde{\rho} e^{2\Sigma} \Delta^{-1} 2\Lambda_4. \quad (81)$$

We now discuss two illustrative examples that arise in General Relativity in four dimensions in the presence of a cosmological constant, namely the AdS_4 solution and the AdS -Schwarzschild solution.

3.2.1 AdS_4 solution

We consider the AdS_4 solution in General Relativity in the presence of a cosmological constant $V = 2\Lambda_4 = -6/L^2$. Its line element reads

$$ds_4^2 = \frac{L^2}{\rho^2} (-dt^2 + d\rho^2 + dv^2 + d\varphi^2), \quad (82)$$

with Ricci scalar $R = -12/L^2$. Comparing with (26) we infer that $\lambda = 1$, $B = 0$ and

$$\Delta = e^\Sigma = \tilde{\rho} = \frac{L^2}{\rho^2}, \quad (83)$$

and hence $h(\rho) = L^2/\rho^2$, $g(v) = 1$. This satisfies (37) with $a = 0$. The matrix 1-form $A = A_\rho d\rho$ associated with this solution is given by

$$M = \begin{pmatrix} \Delta & 0 \\ 0 & \Delta^{-1} \end{pmatrix}, \quad A_\rho = M^{-1} \partial_\rho M = \Delta^{-1} \partial_\rho \Delta \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = -\frac{2}{\rho} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (84)$$

When switching off the cosmological constant (which can be achieved by sending $L \rightarrow \infty$), the Ricci scalar becomes zero. Changing the coordinate ρ from ρ/L to $e^{-\rho/L}$ and taking the limit $L \rightarrow \infty$ yields the flat space-time line element, with

$$\Delta_0 = e^{\Sigma_0} = \tilde{\rho}_0 = 1, \quad (85)$$

and hence $h_0(\rho) = 1$, $g_0(v) = 1$. The field \tilde{v}_0 is constant as well. It follows that $A_\rho|_{V=0} = 0$, and the modification Ω given in (75), when evaluated on the solution, reads

$$\Omega = A_\rho. \quad (86)$$

3.2.2 AdS -Schwarzschild solution

We consider the AdS -Schwarzschild solution in General Relativity in the presence of a cosmological constant $V = 2\Lambda_4 = -6/L^2$. In spherical coordinates, its line element reads

$$ds_4^2 = -f(r) dt^2 + \frac{1}{f(r)} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2) \quad (87)$$

with

$$f(r) = 1 - \frac{2m}{r} + \frac{r^2}{L^2}. \quad (88)$$

When switching off the cosmological constant, the line element can be brought to the form (26) (with $\lambda = 1, B = 0$) by the coordinate transformation

$$\rho = \cosh^{-1} \left(\frac{r}{m} - 1 \right), \quad v = \theta, \quad (89)$$

from which we infer

$$\Delta_0 = \frac{-1 + \cosh \rho}{1 + \cosh \rho}, \quad e^{\Sigma_0} = \tilde{\rho}_0 = m \sinh \rho \sin v, \quad \tilde{v}_0 = -m \cosh \rho \cos v. \quad (90)$$

Note that $\tilde{\rho}_0$ is of the form (49) with

$$h_0(\rho) = m \sinh \rho, \quad g_0(v) = \sin v. \quad (91)$$

The associated 1-form $A_\rho|_{\nu=0}$ is given by

$$A_\rho|_{\nu=0} = \frac{2}{\sinh \rho} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (92)$$

In the presence of the cosmological constant, the line element takes the form (26) (with $\lambda = 1, B = 0$) where

$$\rho = \int \frac{dr}{\sqrt{r^2 - 2mr + \frac{r^4}{L^2}}} \quad , \quad v = \theta. \quad (93)$$

We refrain from giving the explicit expression for $\rho(r)$, which involves an elliptic integral of the first kind. The quantities Δ, Σ and $\tilde{\rho}$ are given by

$$\Delta(\rho) = f(r) \quad , \quad e^{\Sigma(\rho)} = h(\rho) \quad , \quad \tilde{\rho}(\rho, v) = h(\rho) g(v) \quad (94)$$

with

$$h(\rho) = r \sqrt{f(r)} \quad , \quad g(v) = \sin v. \quad (95)$$

Then

$$A_\rho = \Delta^{-1} \partial_\rho \Delta \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \frac{2(L^2 m + r^3(\rho))}{L^2 h(\rho)} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (96)$$

The modification Ω given in (75), when evaluated on the solution, reads

$$\Omega = \left(A_\rho - \frac{2}{\sinh \rho} \right) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (97)$$

4 Reduction to one dimension

The class of models considered in the previous section are effectively one-dimensional, as all relevant quantities depend only on the coordinate ρ , with the exception of the metric factor $\tilde{\rho}$ (which may have an extra dependence on v through $g(v)$, see (29)). Therefore it is useful to consider a reduction of the following four-dimensional Lagrangian [20, 21, 22, 23],

$$L_{4D} = \frac{1}{2} \sqrt{-g} \left(R - 2G_{IJ}(\phi) \partial_\mu \phi^I \partial^\mu \phi^J - f_{ab}(\phi) F_{\mu\nu}^b F^{a\mu\nu} - \tilde{f}_{ab}(\phi) \tilde{F}_{\mu\nu}^b F^{a\mu\nu} - 2V(\phi) \right), \quad (98)$$

corresponding to an Einstein-Maxwell theory with N scalar fields ϕ^I . We take the four-dimensional metric to be of the form

$$ds_4^2 = -a^2(r) dt^2 + a^{-2}(r) dr^2 + b^2(r) d\Omega_\alpha^2 \quad (99)$$

with

$$d\Omega_\alpha^2 = d\theta^2 + \ell_\alpha(\theta)^2 d\varphi^2, \quad \ell_\alpha(\theta) = \begin{cases} \sin(\theta), & \alpha = 1, \\ 1, & \alpha = 0, \\ \sinh(\theta), & \alpha = -1. \end{cases} \quad (100)$$

We have adjusted the parameter α to make it coincide with the definition given in (32). The field equations read [22],

$$(a^2b^2)'' = 2(\alpha - 2b^2V), \quad (101)$$

$$\frac{b''}{b} = -(\phi')^2, \quad (102)$$

$$(a^2b^2G_{IJ}\phi^{J'})' = \frac{1}{2} \left(\frac{\partial_I V_{\text{EM}}}{b^2} + b^2\partial_I V \right), \quad (103)$$

$$-\alpha + a^2(b')^2 + \frac{1}{2}(a^2)'(b^2)' = -\frac{V_{\text{EM}}}{b^2} - b^2V + b^2a^2(\phi')^2, \quad (104)$$

with the last one being the Hamiltonian constraint. The previous equations are written in terms of $(\phi')^2 = G_{IJ}\phi^{I'}\phi^{J'}$ and an electric-magnetic potential which is given by

$$V_{\text{EM}}(\phi) = f^{ab}(Q_{e,a} - \tilde{f}_{ac}Q_m^c)(Q_{e,b} - \tilde{f}_{bd}Q_m^d) + f_{ab}Q_m^aQ_m^b, \quad (105)$$

where $Q_{e,a}$ and Q_m^a are the electric and magnetic charges carried by F^a ,

$$F^a = f^{ab}(Q_{e,a} - \tilde{f}_{ac}Q_m^c)dt \wedge dr + Q_m^a \ell_c(\theta)d\theta \wedge d\phi. \quad (106)$$

A quick comparison of the line element (99) with the line element (26) (where we take $\lambda = 1$ and use $\tilde{\rho}(\rho, v) = h(\rho)g(v)$ with $g = \ell_\alpha$),

$$ds_4^2 = -\Delta dt^2 + \Delta^{-1}(dr^2 + h^2 d\Omega_\alpha^2), \quad (107)$$

gives

$$\Delta = a^2, \quad h = ab. \quad (108)$$

Equations (101) - (104) are equivalent to (31), (33) and (35). Recall that the equation for the field A_ρ (eq. (31)) supplies the equations of motion for the warp factor Δ as well as for the scalar fields ϕ^I . Hence, (103) must come from the field equation for A_ρ . Similarly the $\square\tilde{\rho}$ equation (eq. (33)) corresponds to (101). The equation of motion for Δ (which is part of (31)) can be obtained from a combination of (101), (102) and (104). The consistency equation (35) coincides with the Hamiltonian constraint (104). In what follows, we set

$$a^2 = e^{2U}, \quad b^2 = e^{2\psi-2U}. \quad (109)$$

Then, inserting the line element (99) and the expression for the field strength (106) into the Lagrangian (98) results in the following effective one-dimensional action [23],

$$\begin{aligned} -S_{1D} &= \int dr \left[-\alpha + e^{2\psi} \left((U')^2 - (\psi')^2 + 2G_{IJ}(\phi)\phi^{I'}\phi^{J'} + 2V_{\text{EM}}(\phi) e^{2U-4\psi} + 2e^{-2U}V(\phi) \right) \right] \\ &\quad + \int dr \left[e^{2\psi} (2\psi' - U') \right]'. \end{aligned} \quad (110)$$

The 4D field equations can be derived from this effective 1D action as follows. Choosing the 1D field degrees of freedom to be $e^{\psi+U}$, $e^{\psi-U}$ and ϕ^I , we see that the equations (102) and (103) are obtained by the Euler-Lagrange variation of the action w.r.t $a^2b = e^{\psi+U}$ and ϕ^I respectively. Given that the 1D Lagrangian does not explicitly depend on ‘time’¹, its Hamiltonian \mathcal{H} is

¹Here we take the radial co-ordinate to be ‘time’.

constant on-shell. Hence the fourth 4D field equation (104) is recognizable as the Hamiltonian constraint $\mathcal{H} = 0$. The first field equation (101) is obtained by substituting this constraint into the expression obtained by the Euler-Lagrange variation w.r.t $b = e^{\psi-U}$. For the purpose of this discussion, we will not impose the Hamiltonian constraint necessary for embedding the 4D solution space in the 1D solution space, but use the more general condition of the Hamiltonian being an on-shell constant equal to say β :

$$\beta - \alpha + a^2(b')^2 + \frac{1}{2}(a^2)'(b^2)' = -\frac{V_{\text{EM}}}{b^2} - b^2V + b^2a^2(\phi')^2. \quad (111)$$

Thus we can regard the solution space of the 1D action as being generated by field configurations satisfying (101), (102) and (103), which we will refer to as the equations of motion (EOMs). The Hamiltonian constraint $\mathcal{H} = 0$ is simply enforced by $\beta = 0$. Hence, the bulk 1D action which generates the EOMs when supplemented by the Hamiltonian constancy can be written by a relabelling of the field degrees $A = a^2b$ and $B = b$ as

$$-S_{1D} = \int dr \left[\left(-A' B' + 2A B G_{IJ}(\phi) \phi^{I'} \phi^{J'} + 2 \frac{V_{\text{EM}}(\phi)}{B^2} + 2B^2 V(\phi) \right) \right]. \quad (112)$$

In the above, we have dropped the constant α as it does not affect any of the equations of motion. A classical system represented by the above action is defined to be integrable if one can identify n commuting independent² integrals of motion (i.e. possessing vanishing Poisson brackets with each other) on its phase space, where n is the number of independent fields or equivalently the dimension of the configuration space, and one can write down a Lax pair with matrices L and M which satisfy

$$\frac{dL}{dt} = [L, M] \quad (113)$$

when evaluated on-shell. Ideally, if we can identify n commuting integrals, $\{P_i | 1 \leq i \leq n\}$, one of which is the Hamiltonian, one can readily write down an $n \times n$ matrix Lax pair

$$\begin{aligned} L_{ij} &= P_i \delta_{ij}, \\ M &= I_{n \times n}, \end{aligned} \quad (114)$$

which clearly satisfies the Lax condition (113).

In the absence of a potential, $V(\phi) = 0$, we see that the EOM (101),

$$(AB)'' = 2\alpha, \quad (115)$$

simply eliminates A in terms of B as

$$A = \frac{\alpha r^2 + c_1 r + c_2}{B}, \quad (116)$$

where c_1 and c_2 are the constants of integration of the EOM. Each scalar degree of freedom ϕ^I can be associated with an integral of motion,

$$\Pi_I = P_I - 2 \int^r \frac{\partial_I V_{\text{EM}}(\phi)}{B^2} dr, \quad (117)$$

²Independence here implies that given n integrals of motion $\{P_i | 1 \leq i \leq n\}$, $dP_1 \wedge dP_2 \wedge \dots \wedge dP_n \neq 0$. Thus functions of the phase space variables that are constant throughout the phase space are not counted as integrals of motion for our purpose.

where $P_I = 4 A B G_{IJ}(\phi^J)'$ is the momentum associated with the scalar degree of freedom ϕ^I . It is easy to observe that the Poisson bracket vanishes, $\{\Pi_I, \Pi_J\} = 0$. Further, the EOM (103) guarantees $d\Pi_I/dr = 0$, verifying that these canonical momenta along with the Hamiltonian constitute a set of integrals of motion, whose cardinality is equal to the number of independent fields, enabling one to trivially write down the Lax pair for this system as in (114). We reiterate that we have not invoked the Hamiltonian constraint (104) here, which demands the vanishing of the Hamiltonian on-shell and is only required for the 1D solution space to be lifted to a 4D one. This implies that the 4D solution subspace has one less integral of motion, as the nullity of the Hamiltonian discounts it from being an independent integral of motion. However this also reduces the dimensionality of the on-shell phase space by one, and so the subset of solutions of the classical 4D theory with 2 isometries which can be reduced to solutions of a 1D theory is integrable.

A similar argument holds in the case $V(\phi) \neq 0$, where one can write down a set of integrals of motion, apart from the Hamiltonian, as

$$\Pi_I = P_I - 2 \int^r \left(\frac{\partial_I V_{\text{EM}}(\phi)}{B^2} + B^2 \partial_I V(\phi) \right) dr. \quad (118)$$

The pair (L, M) satisfies the Lax condition after imposing two of the three EOMs, namely (102) and (103). However, in this case, the remaining EOM

$$(AB)'' = 2(\alpha - 2B^2V(\phi)) \quad (119)$$

is not trivially solvable as in the $V(\phi) = 0$ case and constitutes a non-trivial constraint³ that must be imposed on the phase space, independent of the Lax pair. Here again, as in the vanishing $V(\phi) = 0$ case, if one demands that the 1D solutions space be embeddable in the solution space of the parent 4D Lagrangian, one needs to impose the null Hamiltonian constraint (104). Hence the 1D action in the presence of $V(\phi)$ is integrable provided an external constraint equation (119) is satisfied. We will refer to this property as partial integrability. Below, we will illustrate our conclusions above with specific 1D examples by writing down explicit Lax pairs for them and testing the applicability of machine learning (ML) techniques to search for Lax pairs for these systems.

5 Lax pairs for 1D systems

In the following, we consider systems described by the one-dimensional action (110). For the remainder of our discussion we will focus only on the bulk term, as the total derivative boundary term does not affect system dynamics. For the same reason, we will also drop the term proportional to α . For simplicity, we restrict the discussion to the case of one Maxwell field sourced by an electric charge Q_0 only, in which case $V_{\text{EM}}(\phi) = \frac{1}{4}f(\phi)Q_0^2$. We then consider specific choices of the coupling f and of the potential V , and we construct Lax pairs (L, M) for these systems. Systems described by the one-dimensional action (110) with $\alpha = 0$ were considered in the context of Nernst branes in [24].

Note that a system may possess a description in terms of several Lax pairs. In Appendix A we exhibit two families of Lax pairs for 1D systems described by a Hamiltonian of the form $2\mathcal{H}(p, q) = p^2 + 2V(q)$.

³This is precisely the extra constraint equation in the 2D system, corresponding to the field equation for $\tilde{\rho}$ that is trivially solvable in the case of vanishing $V(\phi)$.

5.1 No scalar field

Let us first consider the case when there is no scalar field present in (110). Then, the coupling f and the potential V are constant. Setting $2V = -\frac{3}{4}h_0^2$ and $f = 1/2$, we obtain

$$-S_{1D} = \int dr e^{2\psi} \left[(U')^2 - (\psi')^2 + \frac{1}{4}Q_0^2 e^{2U-4\psi} - \frac{3}{4}h_0^2 e^{-2U} \right]. \quad (120)$$

This system, which depends on the two parameters (Q_0, h_0) , admits a Lax pair (L, M) that satisfies (113) when evaluated on-shell. Setting $A = e^{\psi+U}$, $B = e^{\psi-U}$, the action (120) becomes

$$-S_{1D} = \int dr \left[-A'B' + \frac{Q_0^2}{4B^2} - \frac{3}{4}h_0^2 B^2 \right], \quad (121)$$

with the Hamiltonian being

$$\mathcal{H} = -P_A P_B - \frac{Q_0^2}{4B^2} + \frac{3}{4}h_0^2 B^2. \quad (122)$$

A Lax pair for this system is given in terms of the integrals of motion $P_A = -B'$ and \mathcal{H} as

$$L = \begin{pmatrix} P_A & 0 \\ 0 & \mathcal{H} \end{pmatrix}, \quad M = \mathbb{I}_2. \quad (123)$$

An alternative Lax pair is given by 3×3 matrices of the form

$$L = \begin{pmatrix} 0 & -A' & G_2 \\ B' & 0 & 0 \\ G_1 & 0 & 0 \end{pmatrix}, \quad M = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -\frac{G_1 G_2 + G_2' G_1}{B' G_2} & \frac{G_2'}{G_2} \end{pmatrix} \quad (124)$$

with

$$G_1 = -\frac{Q_0}{2B} + \frac{\sqrt{3}}{2}h_0 B, \quad G_2 = \frac{Q_0}{2B} + \frac{\sqrt{3}}{2}h_0 B, \quad (125)$$

where L satisfies $\text{Tr}(L^2) = 2\mathcal{H}$.

5.2 One scalar field

Next let us consider the case when there is one scalar field ϕ present in (110), with the coupling f and the potential V given by

$$\begin{aligned} f(\phi) &= e^{2\phi}, \\ 2V(\phi) &= -\left(h_0 h_1 + \frac{1}{2} \left(h_0 e^\phi + h_1 e^{-\phi} \right)^2 \right), \end{aligned} \quad (126)$$

where h_0, h_1 denote constants. This choice of f and V is the one that occurs in the model based on $F(X) = -iX^0 X^1$ discussed in [24]. The resulting 1D action reads

$$\begin{aligned} -S_{1D} &= \int dr e^{2\psi} \left[(U')^2 + (\phi')^2 - (\psi')^2 + \frac{1}{2} e^{2U+2\phi-4\psi} Q_0^2 \right. \\ &\quad \left. - e^{-2U} \left(h_0 h_1 + \frac{1}{2} \left(h_0 e^\phi + h_1 e^{-\phi} \right)^2 \right) \right]. \end{aligned} \quad (127)$$

We proceed to rewrite the above action as

$$\begin{aligned}
-S_{1D} = \int dr e^{2\psi} & \left[\frac{1}{2}(U' + \phi' + \psi')(U' + \phi' - \psi') + \frac{1}{2}(U' - \phi' + \psi')(U' - \phi' - \psi') \right. \\
& \left. + \frac{1}{2}e^{2U+2\phi-4\psi}Q_0^2 - e^{-2U} \left(h_0h_1 + \frac{1}{2}(h_0e^\phi + h_1e^{-\phi})^2 \right) \right]. \quad (128)
\end{aligned}$$

Labelling

$$\begin{aligned}
\log A &= U + \phi + \psi \\
\log B &= -U - \phi + \psi \\
C &= -U + \phi + \psi
\end{aligned} \quad (129)$$

we can rewrite the action as

$$\begin{aligned}
-S_{1D} = \int dr & \left[-\frac{1}{2}A'B' - \frac{1}{2}(A'B + AB')C' + \frac{1}{2}AB(C')^2 + \frac{1}{2}\frac{Q_0^2}{B^2} \right. \\
& \left. - \left(2h_0h_1Be^C + \frac{1}{2}h_0^2e^{2C} + \frac{1}{2}h_1^2B^2 \right) \right]. \quad (130)
\end{aligned}$$

This system depends on the 3 parameters (Q_0, h_0, h_1) . The canonical momenta are given by

$$\begin{aligned}
P_A + \frac{B' + BC'}{2} &= 0, \\
P_B + \frac{A' + AC'}{2} &= 0, \\
P_C + \frac{(AB)'}{2} - ABC' &= 0.
\end{aligned} \quad (131)$$

The Hamiltonian, which is conserved on-shell, takes the form

$$\begin{aligned}
\mathcal{H} = & -\frac{3}{2}P_AP_B + \frac{AP_A(AP_A-2P_C)+BP_B(BP_B-2P_C)+P_C^2}{4AB} \\
& - \frac{Q_0^2}{2B^2} + (2h_0h_1Be^C + \frac{1}{2}h_0^2e^{2C} + \frac{1}{2}h_1^2B^2). \quad (132)
\end{aligned}$$

The EOMs yield

$$\begin{aligned}
P'_A + \frac{B'C' - BC'^2}{2} &= 0, \\
P'_B + 2h_0h_1e^C + \frac{Q_0^2}{B^3} + h_1^2B + \frac{A'C' - AC'^2}{2} &= 0, \\
P'_C + h_0^2e^{2C} + 2h_0h_1Be^C &= 0.
\end{aligned} \quad (133)$$

Using these equations, one readily verifies that the following combination is a conserved quantity,

$$J = AB + \int dr [AP_A + BP_B + P_C]. \quad (134)$$

The above current was suggested by an ML experiment on this model, as we will discuss in Section 7. For the special case $h_0 = 0$, the momentum P_C becomes an additional conserved current, satisfying $\{P_C, J\} = 0$, yielding 3 integrals for a 3D configuration space, rendering the system Liouville integrable. Note that the conserved quantities J and P_C differ from the Π_I in (118) as we use different degrees of freedom to characterise the phase space here.

6 Machine Learning integrability

We will set up a ML technique to determine integrability for the 1D systems of interest by designing a neural network to search for the existence of a Lax pair (L, M) . In this context, we can frame the construction of Lax pairs for the given physical system as a neural network (NN) optimization problem. Searches of Lax pairs and r-matrices in classical and quantum integrability have been initiated in [18, 25, 19]. The efficacy of problem solving depends on the neural network architecture parameters (number of layers, neurons per layer, activation function, optimiser, etc). The mathematical principle underlying this approximation technique is the universal approximation theorem [26, 27, 28], which guarantees that a continuous function on a compact subset of \mathbb{R}^n can be arbitrarily well approximated by a neural network in the limit of infinite NN length or width. Therefore, in practice, the accuracy of the approximation is bounded above by the finite extent of the NN. And so, a vital component of problem-solving involves deciding the NN parameters and the corresponding accuracy of numerical Lax pair approximation, that we deem acceptable to decide on whether the given system is integrable. In our particular instance, we adopt an unsupervised learning approach wherein we consider a neural network that takes points in phase space as inputs and produces the entries for the matrices L and M as outputs. We do not know L and M a priori, and hence start with a randomly initialised neural network i.e. defined with randomly fixed initial parameters, and subsequently update the NN parameters⁴ θ to attain values optimised to minimise deviation from the Lax pair condition. This condition is in this context therefore referred to as a loss function and the optimization procedure follows the gradient descent method given by,

$$\theta_{i+1} = \theta_i - \ell_r \nabla_{\theta} \mathcal{L}(X, X', \theta). \quad (135)$$

Here, ℓ_r is called the learning rate and measures the rate of convergence or the convergence step size of θ towards optimal values for problem solving⁵, while $\mathcal{L}(X, X', \theta)$ is the loss function designed to produce the closest approximation to a Lax pair. The variables X and X' denote configuration space variables and their corresponding time⁶ derivatives, which together form an NN input set. For a 1D system with k degrees of freedom (a k -dimensional configuration space) described by

$$\begin{aligned} X &= (\phi_1, \phi'_1, \phi_2, \phi'_2, \dots, \phi_k, \phi'_k), \\ X' &= (\phi'_1, \phi''_1, \phi'_2, \phi''_2, \dots, \phi'_k, \phi''_k), \end{aligned} \quad (136)$$

data is sampled randomly from within a cube $\phi_i, \phi'_i \in [-\alpha, \alpha]$, $\alpha > 0$, while the expressions for ϕ''_i are supplemented by the equations of motion. At this point we mention that for all ML experiments in this paper, the phase space point sampling was performed in `Mathematica` with the ML implementations done in `Tensorflow` [29].

As mentioned before, minimisation of the loss function must correspond to finding a pair of matrices L and M corresponding to minimum deviation from the Lax pair equation (113).

⁴We collectively refer to all the NN network parameters heuristically by a single variable θ for notational simplicity.

⁵Thus it is a higher order parameter which is not associated with the model but with the evolution of the model parameters, and hence referred to as a hyperparameter.

⁶Here ‘time’ simply refers to the independent variable w.r.t which the derivatives of configuration space variables are taken to define velocities. These together with the configuration space variables form the arguments of the Lagrangian.

This motivates a choice of the loss function to be

$$\mathcal{L}_{\text{Lax},1} = \sum_{ij} |(dL_{ij}/dt) - [L, M]_{ij}|^2 \quad (137)$$

or equivalently,

$$\mathcal{L}_{\text{Lax},2} = \sum_{ij} \left| \frac{\text{Exp}(dL_{ij}/dt)}{\text{Exp}([L, M]_{ij})} - 1 \right|^2. \quad (138)$$

There are two principal caveats to this approach. Firstly, the above extremization could produce trivial solutions such as $L_{ij} = 0$. This can be avoided in one of two ways. The simplest way is to introduce a regulator term in the loss. A more systematic approach can be defined by observing that in a 1D integrable system with N degrees of freedom, an $N \times N$ Lax pair produces an infinite set of conserved quantities, $\{F_k = \text{Tr} L^k, k \in \mathbb{Z}\}$, of which at most only N are independent, i.e. have a vanishing mutual Poisson bracket. Requiring the Hamiltonian to be one of these N commuting integrals, eliminates trivial solutions. In our setup, we accordingly choose the Hamiltonian to be the trace of the square of the L -matrix, and implement this by introducing the following Hamiltonian loss constraint,

$$\mathcal{L}_H = |\text{Tr}(L^2) - \mathcal{H}|^2, \quad (139)$$

so that the total loss function is the weighted linear combination,

$$\mathcal{L} = \alpha_{\text{Lax}} \mathcal{L}_{\text{Lax}} + \alpha_{\mathcal{H}} \mathcal{L}_H, \quad (140)$$

with α_{Lax} and $\alpha_{\mathcal{H}}$ being positive coefficients. The second caveat is that the afore defined loss function minimization does not uniquely fix the Lax pair or even its matrix dimensionality. This is because, given a Lax pair, one can generate an equivalent Lax pair satisfying the Hamiltonian loss constraint by a $GL(n, \mathbb{R})$ transformation

$$L \rightarrow \alpha U L U^{-1}, \quad M \rightarrow U M U^{-1} - \frac{dU}{dt} U^{-1}. \quad (141)$$

One could argue that this equivalence is not necessarily an obstacle for our searching technique, as we are only interested in proving the existence of a Lax pair and this caveat only assumes significance if we want to search for a very specific conjectured Lax pair. However, as explained above, the results of our search algorithm are never exact, and so one must have a consistent numerical threshold tolerance, within which the extremum value of the loss function can be taken to be zero, for all intents and purposes, and the existence of this extremum can be taken as a convincing indicator of integrability. For ML implementations of this approach, see Appendix C. Developing these criteria would be all the more feasible, if the ambiguity in specifying the target Lax pairs for the searching function is reduced. Further, in certain 1D systems, even though the full system may not be integrable, there could be integrable sectors in certain decoupling limits. Isolating these sectors systematically would be greatly facilitated by choosing a certain dimensionality and a corresponding $GL(n, \mathbb{R})$ gauge. Ideally, this chosen gauge should economise on the independent components of the Lax matrices for optimising search function efficiency. For these reasons, we will now use a different ML approach. We fix the ‘canonical’ Lax pair gauge to be the ‘action-angle’ gauge, wherein the pair (L, M) is represented by $N \times N$ matrices, where N is the number of degrees of freedom, M is simply

an N -dimensional identity matrix, while L is a diagonal matrix. Considering that we will be restricting ourselves to time-independent models, where the Hamiltonian is a conserved quantity, we choose one of the diagonal elements of L , say L_{NN} , to equal \mathcal{H} . The remaining diagonal elements are the canonical momenta corresponding to action-angle variables, which, by construction, are integrals of motion. Only for an integrable system will there be N such conserved canonical momenta, including the Hamiltonian, in the action-angle frame, so that the Lax pair in the chosen gauge will exist, and as L is explicitly time independent and M is diagonal, the Lax pair equation (113) is trivially satisfied.

Thus our unsupervised ML approach to determining the integrability of a given system with a $2N$ -dimensional phase space, via determining the existence of a Lax pair, boils down to building a NN that can approximate $N - 1$ conserved currents, i.e. functions of the phase space variables whose time derivative vanishes on-shell, but which do not take the same value at all points of phase space or, in other words, are not phase space independent.

For our ML experiments in this note, we employ a deep neural network architecture with three hidden layers, each containing 350 neurons with a hyperbolic tangent activation function. For details on the performance benchmark tests run with this network on simple 1D systems such as the DFF model, see Appendix C. For the NN to generate a conserved current J_{NN} , we adopt a minimization procedure corresponding to the loss function,

$$\mathcal{L} = |dJ_{NN}/dt|^2 + (\overline{|dJ_{NN}/dX_i|^2} - 1)^2, \quad (142)$$

where the second term is a regulator that eliminates phase space-independent constants. The term $\overline{|dJ_{NN}/dX_i|^2}$ denotes an average over the all inputs ϕ_i and all batch points. We schematically represent the above protocol below:

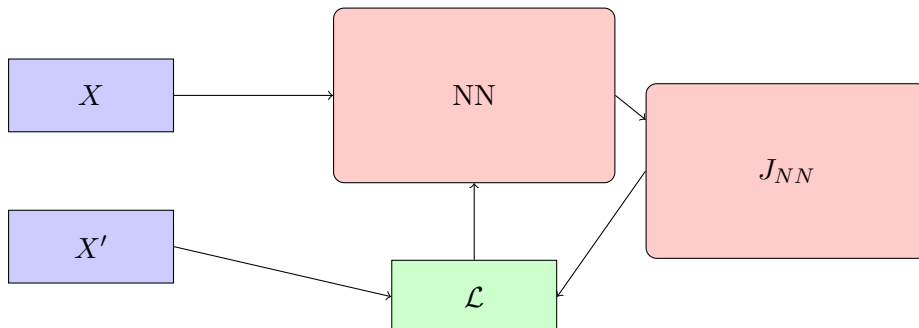


Figure 1: *Flow chart of the integrability determining algorithm. The neural network takes points in phase space as its inputs, and in training, minimises the loss function (142), using the EOMs, to output one of the $N - 1$ currents which constitutes a diagonal element of the Lax matrix L in the action-angle gauge.*

We now proceed to implement our ML approach to determining currents for some of the 1D systems studied in Section 5.

7 Machine Learning of conserved currents

In order to train a neural network to search for conserved currents for a given 1D system, we start by defining a set $B = \{b_i\}$ constituted of functions of the phase space variables. Then for

a given degree $k \in \mathbb{N}$, we define a monomial basis constructed from the elements of B ,

$$B^{(k)} = \left\{ \prod_i b_i^{n_i} \mid 0 \leq n_i \leq k \right\}. \quad (143)$$

For example, in a system with one degree of freedom one could consider $B = \{p, q\}$ and the monomial basis at degree two would be $B^{(2)} = \{1, p, q, p^2, pq, q^2, pq^2, qp^2, p^2q^2\}$. Denoting the monomials in $B^{(k)}$ as $p_m = \prod_i b_i^{n_i}$, we can use this set to write a putative current in terms of a single linear network,

$$J_{NN}^{(k)} = w_m p_m, \quad p_m \in B^{(k)}, \quad (144)$$

and train to optimise the weights w_m to minimise the loss function (142), based on the EOMs. At each k , this procedure is essentially a polynomial regression, a best fit algorithm in the space of polynomial functions of degree k . Once we have obtained all the predicted currents upto a given k , we need to extract from them the largest set of currents with mutually vanishing Poisson brackets, as this will form our complete set J ⁷ which, along with the Hamiltonian, constitutes the diagonal entries for L in the action-angle frame. Note that this is ultimately a numerical approximation scheme and hence will generically only serve to indicate the functional form of the actual current. So, at a given k , the optimisation might yield a J_{NN} with a small number of weights being orders of magnitude larger than the rest, indicating that the true J element in this case might simply be expressed in terms of the monomials corresponding to the large weights with the remaining terms set to zero. This drastically constrains the functional search space in which to look for conserved currents, and can potentially serve as a valuable guide for analytic constructions of Lax pairs.

For the simplest 1D system in Section 5, given by (121) with two degrees of freedom A and B , J is a single real function, which we analytically know to be P_A . While it is convenient to use the $\{A, B, P_A, P_B\}$ to identify the current P_A as a monomial in these basis variables, here we choose a different basis where the Hamiltonian too can be returned as a polynomial. By looking at the action (120) we note that the basis

$$B = \{U', \psi', e^U, e^\psi, e^{-U}, e^{-\psi}\} \quad (145)$$

can be used to reconstruct all monomials involved in the Hamiltonian at degree $k = 2$. There are 224 such monomials after removing the constant term, which would not yield an independent integral of motion. The linear network at degree $k = 2$ is then expressed as

$$J_{NN}^{(2)} = w_1 U' + w_2 \psi' + w_3 e^U + w_4 e^\psi + w_5 e^{-U} + w_6 e^{-\psi} + \dots + w_{224} (U' \psi')^2. \quad (146)$$

The ML results for this linear NN applied to the no scalar model are presented in Figure 2. There we observe spikes that single out 9 monomial terms from a total of 224. These are the elements with the highest weighted contribution to the ML current, and constitute a constrained set of monomials within which we can search for conserved currents. This search yields two currents that can be constructed at degree two, which we recognise as the current P_A as well as the Hamiltonian. Further, one can explore the currents appearing at degree $k = 4$, where we obtain 2024 monomials. Performing the same experiment as before produces a set of monomials that allow for the construction of H and P_A again, as well as products and powers thereof.

⁷For notational simplicity, we use J interchangeably for both a complete set of action-angle momenta as well as for its individual elements.

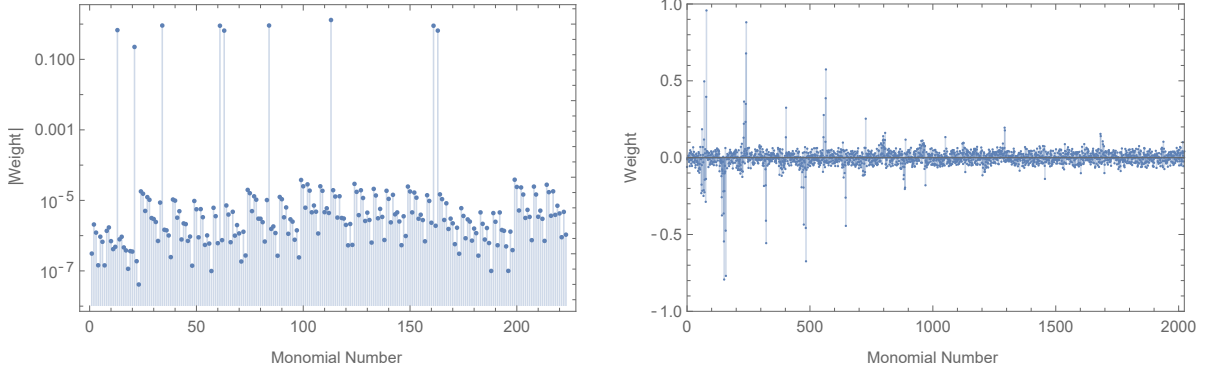


Figure 2: *Left: Absolute value of the weights for degree two monomials, one sees a clear gap separating the current components from the noise. Right: The value of the trained weights at degree four. In both graphs the basis of monomials was chosen in terms of coordinates U and ψ , given in (120).*

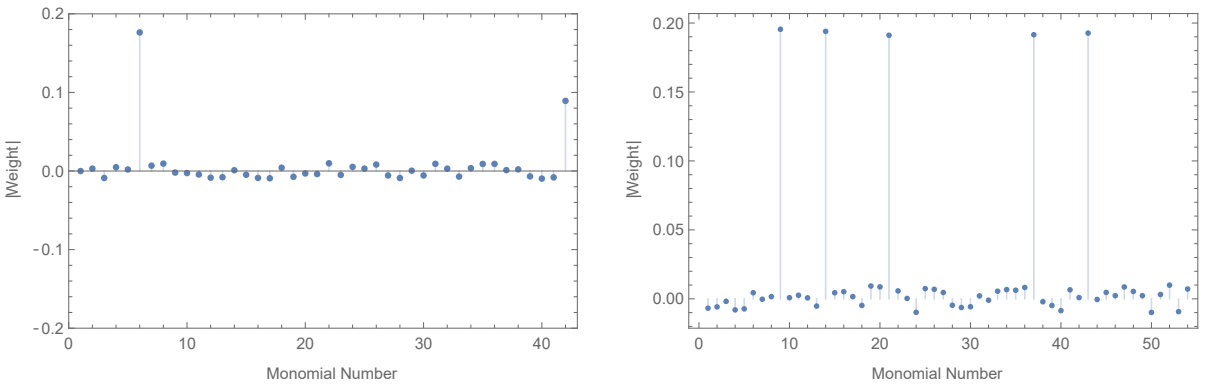


Figure 3: *Left: Learning of the current P_C for the case when $h_0 = 0$ in the one scalar model (130). Right: Learning of the $(AB)'$ current in the one scalar field model with $h_0, h_1, Q_0 \neq 0$. Both experiments are performed with degree two monomials.*

The ML approach so far has served to underline pre-existing knowledge of integrability in the systems under consideration. However it makes a decisive contribution to uncovering the presence of integrable structures in the system (130), in the most generic case with all three parameters turned on. The NN results for this system are presented in Figure 3. On the left plot in Figure 3 we have introduced the NN results for the restricted case $h_0 = 0$, where we have opted for the simple basis $B = \{A, B, C, P_A, P_B, P_C\}$ leading to 42 monomials at degree 2. For this restricted case we extract the conserved current recognisable as P_C , which is indeed conserved as it can be explicitly verified by using the expressions in (133). We observe two spikes in the plot, which we identify as P_C and P_C^2 . For the plot on the right hand side of Figure 3 we have considered the basis

$$B = \{A, B, C, P_A, P_B, P_C, A', B', C'\} . \quad (147)$$

Restricting ourselves to $k = 2$, we present the results of an implemented search for $dJ_{NN}/dt = 0$ rather than for J_{NN} . We observe five spikes corresponding to the monomials AB' , BA' , AP_A , BP_B and P_C . This motivates us to write down the following conserved current as a specific combination of these monomials,

$$J = AB + \int dr [AP_A + BP_B + P_C] . \quad (148)$$

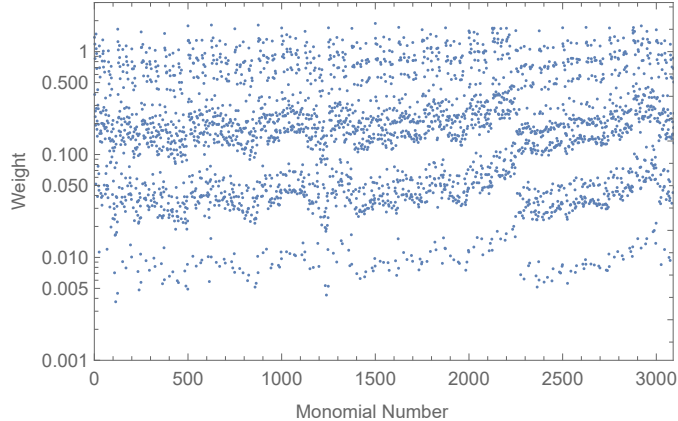


Figure 4: *Absolute value of the weights for degree 2 monomials.*

One can show that this current is indeed conserved using (131).

Finally, let us comment on the most general approach which would consist in choosing a basis analogous to (147), that also includes exponential terms as well as derivatives of the scalar field ϕ . At degree $k = 2$ this basis produces 3374 inequivalent monomials. After training, the weights are plotted in Figure 4. In contrast to all previous experiments we observe a band structure, as opposed to a clearly identifiable gap structure that could suggest a conserved current from a smaller set of monomials. Nevertheless, by restricting ourselves to monomials with weights above a chosen cutoff, enables us to reconstruct the Hamiltonian but not the current (148). This is due to the fact that that the expression for this current involves an integral which can not be accounted for by the linear network (144).

8 Conclusions

In this paper we have identified integrability structures in 2D gravitational theories resulting from a 2-step reduction of certain 4D gravitational theories describing the coupling of Maxwell fields and neutral scalar fields ϕ to gravity in the presence of a scalar potential $V(\phi)$. Restricting ourselves to the subspace of solutions given in (29), we demonstrated that a subset of the 2D equations of motion can be viewed as compatibility equations for a modified version of the Breitenlohner-Maison linear system, which in turn can be represented in terms of differential operators $(\mathcal{L}, \mathcal{M})$. Further, we studied the Liouville integrability of this subspace of solutions from a 1D point of view, by resorting to Lax pair matrices (L, M) constructed using conserved currents in these models. We used both analytic and ML techniques to construct these conserved currents. One such conserved current was suggested by an ML experiment that we performed to determine the integrability of these models.

In this note, we have illustrated, with examples, a proof-of-concept point about how ML approaches could be implemented effectively in the study of integrable structures. Our ML algorithm incorporates broad non-model dependent physics principles that help transcode the problem of searching for matrix Lax pairs to functional regression problems, rendering the approach not just universally applicable but also effective in indicating previously non-observed conserved current constructions underlying the integrability aspects of a given system. Hence, as opposed to prior literature [18, 19] in this subject, which has used ML techniques to validate

analytically known integrable structures, our ML approach does not use an a priori knowledge of the integrability of the system and augments our analytic search for integrals of motion.

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A Families of Lax pairs for 1D models

A 1D system described by the Hamiltonian

$$\mathcal{H}(p, q) = \frac{1}{2}p^2 + V(q) \quad (149)$$

admits several

Lax pairs (L, M) , i.e. pairs (L, M) satisfying

$$\frac{d}{dt}L = [L, M] \quad (150)$$

on-shell. Consider the following pair

$$L = \begin{pmatrix} p & \frac{h(p,q)f(q)}{\lambda} \\ \lambda f(q) & -p \end{pmatrix}, \quad M = \begin{pmatrix} a & \frac{ch^2(p,q)}{\lambda^2} + \frac{h(p,q)f'(q)}{2\lambda} \\ c - \frac{\lambda f'(q)}{2h(p,q)} & a - \frac{2cph(p,q)}{\lambda f(q)} + \frac{p\partial_q h(p,q)}{h(p,q)} - \frac{f(q)f'(q)\partial_p h(p,q)}{h(p,q)} \end{pmatrix}, \quad (151)$$

where a and c are functions of p and q and where $\lambda \in \mathbb{R} \setminus \{0\}$ is a parameter. One can show that this pair satisfies (150) by virtue of Hamilton's equations $\dot{q} = \partial\mathcal{H}/\partial p = p$ and $\dot{p} = -\partial\mathcal{H}/\partial q = -\partial V/\partial q$. Thus, regardless of the particular choice of the functions a and c , this pair (L, M) constitutes a Lax pair that also depends on the parameter λ . Furthermore we note that

$$\text{Tr}(L^2) = 4\mathcal{H}, \quad (152)$$

provided that f is expressed in terms of the potential V by

$$f^2(q) = 2V(q), \quad (153)$$

which we will assume to be the case in the following.

Next, let us pick specific choices for the functions a and c . First, let us choose $a = c = 0$ and let us set $h(p, q) = 1$, in which case (151) becomes

$$\text{FAMILY I: } L = \begin{pmatrix} p & \frac{f(q)}{\lambda} \\ \lambda f(q) & -p \end{pmatrix}, \quad M = \begin{pmatrix} 0 & \frac{f'(q)}{2\lambda} \\ -\frac{\lambda f'(q)}{2} & 0 \end{pmatrix}, \quad (154)$$

which describes a family of Lax pairs for the system described by (149). A different family of Lax pairs for the same system is obtained by choosing $a = 0$, $c = \frac{\lambda f'(q)}{2h(p,q)}$ and $h(p, q) = f(q)$,

$$\text{FAMILY II: } L = \begin{pmatrix} p & f^2(q)/\lambda \\ \lambda & -p \end{pmatrix}, \quad M = \begin{pmatrix} 0 & f(q)f'(q)/\lambda \\ 0 & 0 \end{pmatrix}. \quad (155)$$

Next, we will use these two specific families of Lax pairs as a guide to constructing Lax pairs for the systems described by the Lagrangians discussed in Section 5.

First, let us consider the action (120). By redefining the integration variable as $e^{-2\psi(r)} dr = dR$, this action can be recast as

$$-S_{1D} = \int dR \left[(U')^2 - (\psi')^2 + \frac{1}{4} Q_0^2 e^{2U} - \frac{3}{4} h_0^2 e^{-2U+4\psi} \right]. \quad (156)$$

We notice that when $h_0 = 0$, the two fields U and ψ decouple, in which case the action describes two decoupled systems of the form $\mathcal{L} = \frac{1}{2} \dot{q}^2 - V(q)$, each with a description in terms of a Lax pair as described above. Thus, using $f^2 = -\frac{1}{4} Q_0^2 e^{2U}$ and choosing $\lambda = -1$, a Lax pair describing these two decoupled systems is given by

$$L = \begin{pmatrix} U' & \frac{1}{4} Q_0^2 e^{2U} & 0 & 0 \\ -1 & -U' & 0 & 0 \\ 0 & 0 & 0 & \psi' \\ 0 & 0 & -\psi' & 0 \end{pmatrix}, \quad M = \begin{pmatrix} 0 & \frac{1}{4} Q_0^2 e^{2U} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad (157)$$

When switching on h_0 , we expect these 4×4 matrices to get deformed by terms proportional to h_0 . However, we were not able to obtain these deformed 4×4 matrices, and so in Section 5.1 we instead presented a Lax pair for the full system based on 3×3 matrices L and M .

Next, let us consider the action (127). By redefining the integration variable as $e^{-2\psi(r)} dr = dR$, this action can be recast as

$$-S_{1D} = \int dR \left[(U')^2 + (\phi')^2 - (\psi')^2 + \frac{1}{2} e^{2U+2\phi} Q_0^2 - e^{-2U+4\psi} \left(h_0 h_1 + \frac{1}{2} (h_0 e^\phi + h_1 e^{-\phi})^2 \right) \right]. \quad (158)$$

Let us restrict to the case when $h_1 = 0$. Redefining the fields as

$$\begin{aligned} A &= U + \phi \\ B &= U - \phi - 2\psi \\ C &= -U + \phi + \psi, \end{aligned} \quad (159)$$

the Lagrangian becomes

$$\mathcal{L}_{h_1=0} = \frac{1}{2} (A'^2 - B'^2 + 2C'^2 + e^{2A} Q_0^2 - e^{-2B} h_0^2). \quad (160)$$

The three fields A, B and C are thus decoupled, each having a description in terms of a Lax pair as described above. A Lax pair describing these three decoupled systems is given by the

following 6×6 matrices,

$$L = \begin{pmatrix} A' & Q_0^2 e^{2A} & 0 & 0 & 0 & 0 \\ -1 & -A' & 0 & 0 & 0 & 0 \\ 0 & 0 & h_0 e^{-B} & B' & 0 & 0 \\ 0 & 0 & -B' & -h_0 e^{-B} & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2}C' & 0 \\ 0 & 0 & 0 & 0 & 0 & -\sqrt{2}C' \end{pmatrix}, \quad M = \begin{pmatrix} 0 & Q_0^2 e^{2U} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2}h_0 e^{-B} & 0 & 0 \\ 0 & 0 & -\frac{1}{2}h_0 e^{-B} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \quad (161)$$

We note that L satisfies $\text{Tr}(L^2) = 4\mathcal{H}$. In Appendix C we gave evidence for the existence of a 6×6 Lax pair for the full system when $h_1 \neq 0$, by using machine learning techniques to construct numerical 6×6 Lax pairs for the full system.

We note that when $h_0 = 0$, the action (158) can be brought to a form similar to (156) by redefining the fields U and ϕ as $A = U + \phi$, $B = U - \phi$.

B Machine Learning Lax pairs

A particular feature of neural networks is their success at making predictions over large data sets. However, reverse engineering the network to obtain what it is actually doing is less trivial. The problem at hand was to build neural networks that can produce Lax pairs for a given system and therefore demonstrate their integrability. In order for the network to produce a sensible result, finding an analytic expression is unavoidable.

If a Lax pair exists one expects the Lax pair equation (150) to be satisfied identically over the entire phase space. A neural network could then be used to approximate L and M as neural network outputs. However one has to bear in mind the following. First, neural networks are better suited to make predictions over compact domains. Hence we expect a neural network to give a good approximation only over a compact domain that contains the training set. Second, since we do not work at any of the limits where the universal approximation holds, we have to rely on a neural network that is expressive enough (enough layers, enough neurons per layer, suitable activation function), such that upon training, the error in the approximation can be controlled by a parameter ϵ such that

$$\epsilon > \Delta_{ij} = \max_{(p_\alpha, q_\alpha) \in \mathcal{D}} |L_{ij} - f_{ij}^{NN}|^2, \quad (162)$$

with f_{ij}^{NN} being the NN approximation function for the ij component of L , and similarly for M . The parameter ϵ is an architecture dependent quantity and it is supposed to approach zero as one reaches either infinite width or infinite length, provided a Lax pair exists. However, we are limited to work with a finite network, and therefore must look for a suitable NN architecture for our problem, since a systematic architecture analysis is not feasible with the resources at our disposal.

Using the neural network as an approximation for a Lax pair in a system where no analytic Lax pair is known, presents us with the issue of decidability. When no analytical expressions for L and M are known, one can not estimate ϵ for a NN approximation. The only parameter at hand is the loss function which, as in (140), can be designed to measure deviations from the Lax equation (with additional regulator terms). After training, the loss is going to be fixed at a certain value different from zero, i.e. the neural network does not provide a decisive answer and therefore we must decide the parameters for the system being integrable based on the

existence of a numerical Lax pair. In the following we highlight the basic differences between the numerical Lax pair approximation by NNs and the framing of the actual Lax pair search as an optimisation problem.

- If we construct a neural network with traditional activation functions (e.g. ReLU, sigmoid, tanh) the output produced is going to be an analytic expression, given as a composition of activation functions at each network layer and decorated with weights and biases. This analytic expression is then used to generate a **numerical** Lax pair. A numerical Lax pair is a pair of matrices L and M whose entries are neural network outputs such that after training the Lax loss (i.e. a loss that measures deviations from L and M satisfying the Lax pair equation (150)) reaches values below a certain threshold value. The result is a pair of matrices with numerical values at each phase space point. Furthermore, the training is performed over a compact set in phase space, where the loss function takes a acceptably small minimum value. As we move away from this region the NN approximation will become worse. These factors limit the interpretability of the output in terms of deciding whether the system is integrable or not.
- Approximating the Lax pair by neural networks (NNs) has the inherent problem of interpretability: even in the case of small numerical deviations from the Lax pair equation, the Lax pair approximations are hard to recast into an interpretable form, i.e. analytic expressions that actually satisfy the Lax pair equation. If we take the existence of the Lax pair as the criterion for integrability of the 1D systems under consideration, one has to convert these numerical values into a meaningful functional output that exactly satisfies the Lax pair equation.
- If after training, an acceptably small loss is reached, this may hint at the integrability of the system: consider a system for which a Lax pair is known; after training one obtains a numerical Lax pair for this system. This output provides a measure of what type of loss is to be expected for the given NN architecture and the type of system under consideration. We now switch on a deformation and train again to obtain a numerical Lax pair for the deformed system. If the loss at the end of the training process is of the same order as the one obtained for the undeformed system, then we consider the deformation to have a good chance of being integrable as well [18, 19]. This type of analysis is carried out in Appendix C. In order to ensure that the deformed system is indeed integrable, the numerical ML output has to be made concrete by casting the numerical Lax pair into an analytic one, valid over the whole phase space. One possible means to achieve this is to apply symbolic regression: some options to try in this direction are packages such as PySR [30]; we leave this for future work.
- An alternative way is to use NNs that simulate the evolution of parameters in the physical problem. This requires making assumptions, such as what type of functional expressions appear in the entries of the Lax pair, and what is a suitable basis for those terms. We denote these as **interpretable** Lax pairs. Since we work with suitable non-standard activation functions, this interpretable approach can not be generalised to any other one dimensional Hamiltonian, as the choice of a basis is model dependent. While in this case one trains again over a compact dataset, there is the chance that upon training the

resulting approximation functions can be turned into actual analytical results, valid away from the training set. We explore this alternative in Section 7.

At last, let us make a comment about the dimensionality of the Lax pairs. In Appendix A we presented two families of 2×2 Lax pairs for 1D models with one degree of freedom. We then used these families to construct Lax pairs for the 1D systems discussed in Section 5, in cases where the chosen degrees of freedom are decoupled. The 1D systems discussed in Section 5 have $n + 2$ degrees of freedom, n of which correspond to n scalar fields in four dimensions. Since in these decoupling cases, the analytic Lax pair description was based on $2(n + 2) \times 2(n + 2)$ matrices, for the ML experiments in the general case, we have taken the Lax pair for these systems to also have the same matrix dimensionality.

C NN performance benchmarks for 1D integrability

In order to construct numerical Lax pairs, we start by considering the 1D Hamiltonian given in (149). For concreteness we contemplate three different potentials for our experiments: the Liouville potential $V = g^2 e^{-q}$, the de Alfaro-Fubini-Furlan (DFF) potential $V = \frac{1}{2}\omega^2 q^2 + g^2/q$ as well as a linear potential $V = g^2 q$. We start with a learning rate of 10^{-3} which we use during the first 25 epochs of training. Afterwards, the learning rate is updated to 10^{-5} for the next 25 epochs and finally set to 10^{-6} for the remainder of the training process. Sampling of phase space elements is done by setting ω and g to 1. After the training process we attain losses of around 10^{-6} , which corresponds roughly to deviations of 10^{-3} away from zero for the Lax pair condition $dL/dt - [L, M] = 0$. This is shown in Figure 5.

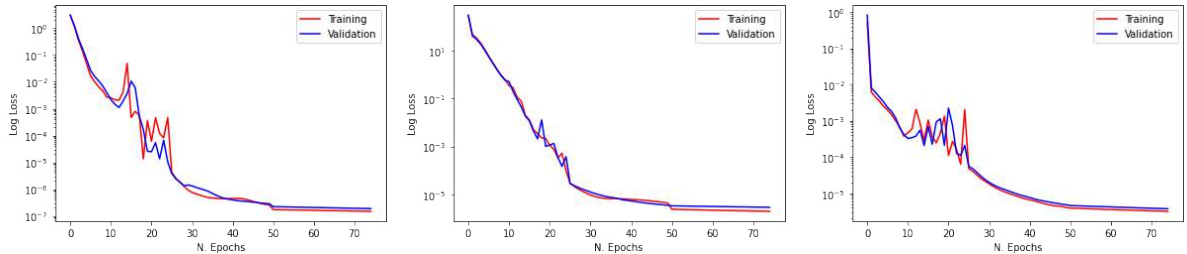


Figure 5: *Training process for three different potentials. Left: Liouville potential $V = e^{-q}$. Center: de Alfaro-Fubini-Furlan (DFF) potential $V = \frac{1}{2}q^2 + 1/q$. Right: A linear potential $V = q$.*

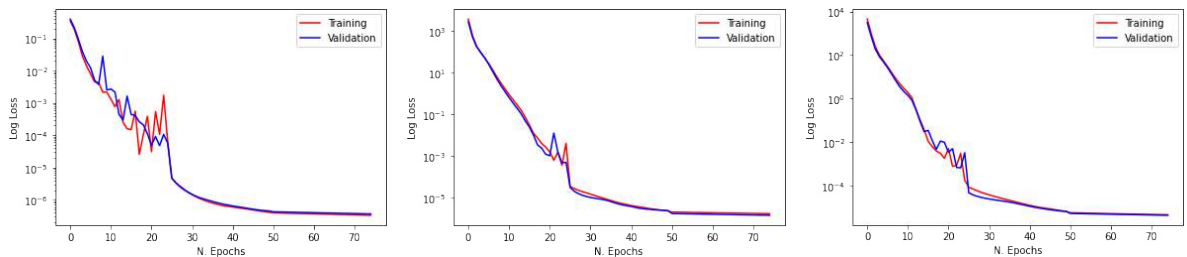


Figure 6: *Numerical Lax pair training on the no scalar field model described by (120). Left: Case (a) $h_0 = 0, Q_0 = 1$. Center: Case (b) $h_0 = 1, Q_0 = 0$ and Right: Case (c) $h_0 = Q_0 = 1$.*

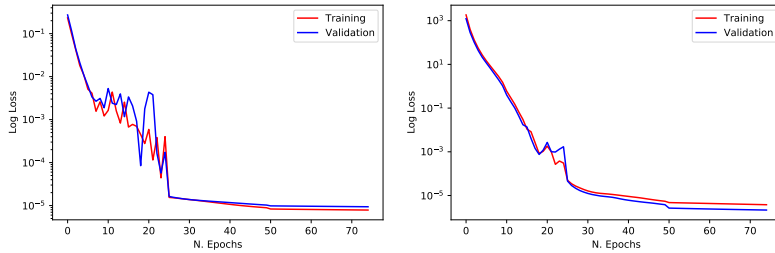


Figure 7: Numerical Lax pair training on the one scalar field model (see (127)). Left: Case $h_1 = 0$, $h_0 = Q_0 = 1$. Right: $h_0 = h_1 = Q_0 = 1$.

Having the results for the models with one degree of freedom we proceed to explore the system described by (120). As shown in Figure 6, we have considered three cases, namely, (a) $h_0 = 0, Q_0 = 1$, (b) $h_0 = 1, Q_0 = 0$ and (c) $h_0 = Q_0 = 1$. We assume that the dimensionality of the Lax matrices is 4×4 and perform the training along similar lines to the description above.

In case (a) we have that ψ describes a free particle and U describes a Liouville type model. The Lax pair can be written straightforwardly and it is given in (157). Note that in that case, after training we obtain losses in the order of 10^{-6} . In cases (b) and (c) we observe that the network training begins at much higher losses of order 10^3 and 10^4 respectively. This is suggestive of the enhanced difficulty in finding the Lax pairs for these cases. Notice that as in the case (a), after training we obtain losses in the order of 10^{-6} , just as in the undeformed case. Hence, following the arguments given in Appendix B, we take this as an indication that the deformations are integrable. This is consistent with the 2D analysis (see Section 3). We further recall that the model (b) with $Q_0 = 0$ contains the Schwarzschild AdS solution, so in particular, these results signal the integrability of that model.

Let us now consider the one scalar field model described in (127). In Appendix A we have considered the case $h_1 = 0$. For this restricted setting we gave an analytic Lax pair in (161). Once again we search for numerical Lax pairs for the system with $h_1 = 0$ and $h_1 \neq 0$. The results are displayed in Figure 7. We observe that in both cases the loss decreases down to 10^{-5} after training.

Finally, let us consider a setting in which we ML numerical approximations to conserved currents directly as opposed to Lax pairs. The setting is similar to the one described above: we consider a neural network with the same architecture as before, but with a single scalar output. If we consider the no scalar model, following a training process similar to the one described for the Lax pair, we obtain losses of the order 10^{-6} after training for the numerical neural network current. Further, knowing that the currents for this system are H and P_A we perform a polynomial regression. This procedure yields the expression,

$$J_{NN} \sim 0.14867 + 0.25717H + 0.24235P_A \quad (163)$$

with an accuracy of 92% over the whole dataset. We have also performed a similar experiment for the one scalar field model, applying a similar regression procedure for the case $h_0 = 0$, where we know that H and P_C are conserved currents for the system, to obtain

$$J_{NN} \sim -2.11911 - 0.77678H + 0.00381HP_C \quad (164)$$

with an accuracy of 40% only. This low accuracy indicates that our basis $\{H, P_C\}$ for our

polynomial regression is incomplete and this suggests the existence of an extra conserved current, in agreement with our analytic construction in Appendix A.

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